Solution Key for HW #3 Math 307 Spring 2007

Section 1.6 # 1, 2, 11, 17, 25, 28, 29, 49, 50, 56
Section 1.7 # 1, 3, 10

Section 1.6

#1

$$A_1: \left[\begin{array}{cc:cc} 0 & 2 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc:cc} 3 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc:cc} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 1/2 & 0 \end{array}\right] \Rightarrow A_1^{-1} = \begin{bmatrix} 0 & 1/3 \\ 1/2 & 0 \end{bmatrix}$$

$$A_2: \left[\begin{array}{cc:cc} 2 & 0 & 1 & 0 \\ 4 & 2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc:cc} 2 & 0 & 1 & 0 \\ 0 & 2 & -2 & 1 \end{array}\right] \to \left[\begin{array}{cc:cc} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -1 & 1/2 \end{array}\right] \Rightarrow A_2^{-1} = \begin{bmatrix} 1/2 & 0 \\ -1 & 1/2 \end{bmatrix}$$

$$A_3: \left[\begin{array}{cc:cc} \cos\theta & -\sin\theta & 1 & 0 \\ \sin\theta & \cos\theta & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc:cc} \cos\theta & -\sin\theta & 1 & 0 \\ 0 & \cos\theta + \frac{\sin^2\theta}{\cos\theta} & -\frac{\sin\theta}{\cos\theta} & 1 \end{array}\right]$$

$-\frac{\sin\theta}{\cos\theta}$ Row 1 + Row 2

Simplify: $\frac{\cos^2\theta + \sin^2\theta}{\cos\theta} = \frac{1}{\cos\theta}$ $\to \left[\begin{array}{cc:cc} \cos\theta & -\sin\theta & 1 & 0 \\ 0 & \frac{1}{\cos\theta} & -\frac{\sin\theta}{\cos\theta} & 1 \end{array}\right]$

Multiply Row 2 by $\cos\theta$ $\to \left[\begin{array}{cc:cc} \cos\theta & -\sin\theta & 1 & 0 \\ 0 & 1 & -\sin\theta & \cos\theta \end{array}\right]$

$\sin\theta$ Row 2 + Row 1 $\to \left[\begin{array}{cc:cc} \cos\theta & 0 & 1-\sin^2\theta & \sin\theta\cos\theta \\ 0 & 1 & -\sin\theta & \cos\theta \end{array}\right]$

Simplify $\cos^2\theta + \sin^2\theta = 1$
$\Rightarrow \cos^2\theta = 1 - \sin^2\theta$ $\to \left[\begin{array}{cc:cc} \cos\theta & 0 & \cos^2\theta & \sin\theta\cos\theta \\ 0 & 1 & -\sin\theta & \cos\theta \end{array}\right]$

Divide equation 1 by $\cos\theta$ $\to \left[\begin{array}{cc:cc} 1 & 0 & \cos\theta & \sin\theta \\ 0 & 1 & -\sin\theta & \cos\theta \end{array}\right] \Rightarrow A_3^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

You may remember this matrix as the matrix that rotated a vector in the plane by θ degrees, hence the inverse would rotate a vector $-\theta$ degrees!

#2 a) Inverses $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$; $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

b) $(PP^T)_{ij} = (\text{row } i)\cdot(\text{row } j)$

$= \begin{cases} 1 & \text{if } i=j \Rightarrow \text{diagonals} = 1. \\ 0 & \text{if } i \neq j \Rightarrow \text{off-diagonals} = 0. \end{cases}$

hence $PP^T = I$. Equivalently for $P^TP = I$. The key is to use that the rows of a permutation matrix are the rows of the identity.

#11

a) Let $A = I$, $B = -I$, $A + B = I - I = 0$.

b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

c) $A = I$, $B = I$, $A + B = 2I$.

$$C = A^{-1} + B^{-1} = A^{-1}(A+B)B^{-1}$$

$$= I\left(\frac{I}{2}\right)I = \frac{I}{2}$$, which is invertible.

to find C^{-1}, we need to find a matrix D such that $CD = I$.

$$A^{-1}(A+B)B^{-1}\underbrace{B} = A^{-1}(A+B)$$

$$A^{-1}(A+B)\underbrace{(A+B)^{-1}} = A^{-1}$$

$$A^{-1}\underbrace{A} = I \quad \Rightarrow \quad \boxed{D = B(A+B)^{-1}A.}$$

Let's also check $DC = I$.

$$B(A+B)^{-1}A\left(A^{-1}(A+B)B^{-1}\right) = B(A+B)^{-1}(A+B)B^{-1}$$

$$= BB^{-1}$$

$$= I \checkmark$$

#17

a) b) $A = L_1 D_1 U_1$ and $A = L_2 D_2 U_2$, we want to show that LDU factorization is unique when A is invertible.

$$L_1^{-1}L_1D_1U_1 = L_1^{-1}L_2D_2U_2 \Rightarrow D_1U_1 = L_1^{-1}L_2D_2U_2$$

$$\Rightarrow \underbrace{D_1U_1U_2^{-1}} = \underbrace{L_1^{-1}L_2D_2}$$

The product of two upper triangular matrices is upper triangular.

same for lower triangular.

$D_1U_1U_2^{-1} = D_1$

$D_2L_1^{-1}L_2 = D_2$

The only matrices that are both upper & lower triangular matrices are diagonal. This means that the off-diagonal matrix elements must be zero.

The diagonal elements of both sides are D_1, D_2 [since L_1 & L_2 & U_1, & U_2 have ones on the diagonal]. Hence $D_1 = D_2$, $U_1U_2^{-1} = I$, $L_1^{-1}L_2 = I \Rightarrow$

$$U_1 = U_2, \; L_2 = L_1 \quad \square.$$

#25 A has row 1 + row 2 = row 3.

a) $A = \begin{bmatrix} \text{Row 1} \\ \text{Row 2} \\ \text{row 1} + \text{row 2} \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} \text{Row 1} \cdot \vec{x} \\ \text{Row 2} \cdot \vec{x} \\ \text{Row 1} \cdot \vec{x} + \text{row 2} \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

From Row 2, we get Row $2 \cdot \vec{x} = 0$, plugging into Row 3 we get Row $1 \cdot \vec{x} = 0$ which contradicts Row 1! $\Rightarrow$ There can be no solution

b) For the equations to be consistent as we carry on elimination steps, we need $b_1 + b_2 = b_3$.

c) Row 3 is not really a new equation, it can be obtained from Row 1 & Row 2 by elementary row operations, so it will be canceled out by Gaussian elimination & become a row of zeros.

#28 $M = ABC$ The objective is to find a matrix D, so that

$MC^{-1} = AB$ $DB = I$ & $BD = I \Rightarrow D = B^{-1}$.

$M^{-1}MC^{-1} = M^{-1}AB$

$CC^{-1} = CM^{-1}AB$

$I = CM^{-1}AB \Rightarrow \boxed{B^{-1} = CM^{-1}A}$, we also check that it is a right inverse:

$$\begin{aligned} BB^{-1} &= BCM^{-1}A \\ &= BC(ABC)^{-1}A \\ &= BCC^{-1}B^{-1}A^{-1}A \\ &= BB^{-1} \\ &= I \checkmark \end{aligned}$$

#29 If we think of matrix multiplication as an operation on the columns, then

$$AB = \begin{bmatrix} | & | & \\ Ab_1 & Ab_2 & \cdots \\ | & | & \end{bmatrix}$$ hence if $b_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ AB will also have a column of zeros.

Therefore there is no A such that $AB = I$.

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#49 $A^T = \begin{bmatrix} 1 & 9 \\ 0 & 3 \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1/3 \end{bmatrix}$ $\quad (A^{-1})^T = (A^T)^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1/3 \end{bmatrix}$

$A^T = A = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}$ $\quad A^{-1} = \frac{1}{c^2}\begin{bmatrix} 0 & c \\ c & -1 \end{bmatrix} = (A^{-1})^T = (A^T)^{-1}$.

#50 $(AB)^T = B^T A^T$ which is not $A^T B^T$ unless A & B commute, i.e. $AB = BA$.
To prove that $B^T A^T = A^T B^T$ in that case, we take the transpose of both AB & BA $\Rightarrow$

$$B^T A^T = A^T B^T = (AB)^T \; \blacksquare.$$

#56 Let $A = A^T$ and $B = B^T$

(a) $(A^2 - B^2)^T = (A^2)^T - (B^2)^T = (A \cdot A)^T - (B \cdot B)^T = A^T A^T - B^T B^T = A^2 - B^2$ ✓ yep.

(b) $[(A+B)(A-B)]^T = (A-B)^T (A+B)^T = (A^T - B^T)(A^T + B^T) = (A-B)(A+B)$

This is not true in general, only if $(A+B)$ and $(A-B)$ commute. Pick your favorite counter example.

(c) $(ABA)^T = (A(BA))^T = (BA)^T A^T = A^T B^T A^T = ABA$ ✓ yep.

(d) $(ABAB)^T = [(AB)(AB)]^T = (AB)^T (AB)^T = B^T A^T B^T A^T = BABA$

Also not true in general, only if A & B commute.

Section 1.7 #1

Using Gaussian Elimination or using the example in text:

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 4/3 & 0 \\ 0 & 0 & 0 & 5/4 \end{bmatrix} \begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 1 & -3/4 \\ 0 & 0 & 0 & 1 \end{bmatrix} = LDL^T$$

note since A is symmetric $U = L^T$!, the product of the pivots, A.K.A. the determinant $= 5$.

#3 $A_0 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} c \\ c \\ c \\ c \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Note that the first equation has a different boundary condition than in the notes so:

$u_0 - 2u_1 + u_2 \Rightarrow u_1 - 2u_1 + u_2 \Rightarrow -u_1 + u_2$ and hence the first row of A_0.

$u_0 = u_1$ since $\frac{du}{dx}(0) = 0$.

Since a constant gives a zero RHS, A_0 is not invertible and is consistent with the continuous problem where the solution has a free constant.

10. Why partial pivoting rocks!

Regular Gaussian Elimination

$$\begin{bmatrix} 0.001 & 0 \\ 1 & 1000 \end{bmatrix} \rightarrow \begin{bmatrix} 0.001 & 0 \\ 0 & 1000 \end{bmatrix}$$

This matrix is very close to singular (change a_{11} to 0) and has pivots that are 10^6 times different. This will lead to large roundoff errors.

Partial pivoting

$$\begin{bmatrix} 1 & 1000 \\ 0.001 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1000 \\ 0 & -1 \end{bmatrix}$$

The pivots have the same magnitude and this matrix is not close to singular at all. No problem with roundoff errors.