Solution Key HW #4

Section 2.1 # 2, 3, 5, 8, 9, 24, 28

#2 We need to check closure under vector addition & scalar multiplication

a) $(0, b_2, b_3) + (0, b_2', b_3') = (0, b_2 + b_2', b_3 + b_3')$ ✓ closed under vector addition

$c(0, b_2, b_3) = (c\cdot 0, c\cdot b_2, c\cdot b_3) = (0, cb_2, cb_3)$ ✓ closed under scalar multiplicati.

$\boxed{\text{yes}}$ $S = \{\vec{v} \in \mathbb{R}^3 : \vec{v} = (0, b_2, b_3) \text{ with } b_2, b_3 \in \mathbb{R}\}$ is a subspace

b) $(1, b_2, b_3) + (1, b_2', b_3') = (2, b_2 + b_2', b_3 + b_3')$ has 2 in the first position!

$\boxed{\text{no}}$ $S = \{\vec{v} \in \mathbb{R}^3 : \vec{v} = (1, b_2, b_3),\ b_2, b_3 \in \mathbb{R}\}$ is not closed under vector addition and hence is not a subspace.

c) $S = \{\vec{v} \in \mathbb{R}^3 : \vec{v} = (b_1, b_2, b_3) \text{ with } b_2 b_3 = 0,\ b_1, b_2, b_3 \in \mathbb{R}\}$

$= \{\vec{v} \in \mathbb{R}^3 : \vec{v} = (b_1, b_2, 0) \cup \vec{v} = (b_1, 0, b_3),\ b_1, b_2, b_3 \in \mathbb{R}\}$

$(b_1, b_2, 0) + (b_1', 0, b_3') = (b_1' + b_1, b_2, b_3')$ not necessarily in $S$, for example with $b_2, b_3' = 1$.

$\boxed{\text{no.}}$

d) $S = \{\vec{v} \in \mathbb{R}^3 : \vec{v} = c_1(1,1,0) + c_2(2,0,1),\ c_1, c_2 \in \mathbb{R}\}$

$\vec{v}_1 = c_1(1,1,0) + c_2(2,0,1)$, $\vec{v}_2 = c_1'(1,1,0) + c_2'(2,0,1)$

$\vec{v}_1 + \vec{v}_2 = (c_1 + c_1')(1,1,0) + (c_2' + c_2)(2,0,1)$ ✓

$c\vec{v}_1 = c\cdot c_1(1,1,0) + c\cdot c_2(2,0,1)$ ✓

$\boxed{\text{yes}}$

e) $S = \{\vec{v} \in \mathbb{R}^3 : \vec{v} = (b_1, b_2, b_3) \text{ with } b_3 - b_2 + 3b_1 = 0,\ b_1, b_2, b_3 \in \mathbb{R}\}$

if we solve for $b_3 = b_2 - 3b_1$

$S = \{\vec{v} \in \mathbb{R}^3 : \vec{v} = (b_1, b_2, b_2 - 3b_1),\ b_1, b_2, b_3 \in \mathbb{R}\}$

$\vec{v}_1 = (b_1, b_2, b_2 - 3b_1)$, $\vec{v}_2 = (b_1', b_2', b_2' - 3b_1')$

$\vec{v}_1 + \vec{v}_2 = (b_1 + b_1', b_2 + b_2', b_2 + b_2' - 3(b_1 + b_1'))$

if we let $b_1 + b_1' = a_1$ and $b_2 + b_2' = a_2$

$= (a_1, a_2, a_2 - 3a_1)$ ✓

$c\vec{v}_1 = (cb_1, cb_2, cb_2 - 3cb_1)$ ✓ $\boxed{\text{yes}}$

#3 We Describe the column space and nullspace of: (2)

$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ column space is the space generated by the linear combinations of the columns. $c\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ which is the x-axis ($y=0$).

The nullspace $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x - y = 0$

$\boxed{y = x}$ or equivalently $c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$B = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ The column space is given by $c_1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} 3 \\ 3 \end{bmatrix}$

Since $2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, and $3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, The column space is

$a\begin{bmatrix} 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, with $a, b \in \mathbb{R}$

i.e. the X-Y plane.

The nullspace $B\vec{x} = 0$, $\begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$ $3z = 0 \Rightarrow z = 0$

$x + 2y + 3z = 0$

$x = -2y$, $z = 0$

$\begin{bmatrix} -2y \\ y \\ 0 \end{bmatrix} = y\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ a line passing through $(-2, 1, 0)$.

$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Column space $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ origin in $\mathbb{R}^2$, Rowspace $[0\ 0\ 0]$ origin in $\mathbb{R}^3$

#5 We re-define addition in $\mathbb{R}^2$ as $(a,b)$ "+" $(c,d) = (a+c+1, b+d+1)$, and scalar multiplication unchanged $s(a,b) = (sa, sb)$ with $s \in \mathbb{R}$.

a) Let's check the 8 rules.

1. $(a,b)$"+"$(c,d) = (a+c+1, b+d+1) = (c+a+1, d+b+1) = (c,d)$"+"$(a,b)$ ✓

2. $(a,b)$"+"$((c,d)$"+"$(e,f)) = (a,b)$"+"$(c+e+1, d+f+1)$

$= (a+c+e+2, b+d+f+2)$

$= (a+c+1+(e+1), b+d+1+(f+1))$

$= (a+c+1, b+d+1)$"+"$(e,f)$

$= ((a,b)$"+"$(c,d))$"+"$(e,f)$ ✓

3. $(a,b)$"+"$\vec{0} = (a,b)$, $\vec{0} = (-1,-1)$ so $(a,b)$"+"$(-1,-1) = (a+1-1, b+1-1)$

$= (a,b)$ ✓

4. $(a,b) + "(-\vec{x})" = "0"$ , $(a,b) + (-a-2, -b-2) = (-1,-1)$ ✓

5. $1\vec{x} = \vec{x}$
6. $(c_1 c_2) x = c_1 (c_2 x)$ } scalar multiplication are unchanged ✓.

7. $c(x+y) = c\left( (a,b) + (d,e) \right) = c(a+d+1, b+e+1)$

$= ((a+d+1)c, (b+e+1)c)$

$cx + cy = (ca, cb) + (cd, ce) = (ca + cd + 1, cb + ce + 1)$ Not the same!

8. $(c_1 + c_2) x = ((c_1 + c_2) a, (c_1 + c_2) b)$

$c_1 x + c_2 x = (c_1 a, c_1 b) + (c_2 a, c_2 b) = (c_1 a + c_2 a + 1, c_1 b + c_2 b + 1)$

Not the same!

b) $x "+" y = xy$

$"cx" = x^c$

1. $x "+" y = xy = yx = y "+" x$ ✓
2. $x "+" (y "+" z) = x "+" (yz) = xyz = (x "+" y) z = (x "+" y) "+" z$ ✓
3. $x "+" "0" = x \Rightarrow x "0" = x \Rightarrow "0" = 1!$
4. $x "+" \frac{1}{x} = 1 = "0"$ ! we have to rule out $x=0$!
5. $"1" x = x^{"1"} = x$ $"1" = 1$ ✓
6. $(c_1 c_2) x = x^{c_1 c_2} = (x^{c_2})^{c_1} = c_1 x^{c_2} = c_1 (c_2 x)$ ✓
7. $c(x "+" y) = (x "+" y)^c = (xy)^c = x^c y^c = x^c "+" y^c = cx "+" cy$ ✓
8. $(c_1 + c_2) x = x^{c_1 + c_2} = x^{c_1} \cdot x^{c_2} = x^{c_1} "+" x^{c_2} = c_1 x "+" c_2 x$ ✓

c) $(x_1, x_2) "+" (y_1, y_2) = (x_1 + y_2, x_2 + y_1)$

1. $(x_1, x_2) "+" (y_1, y_2) = (x_1 + y_2, x_2 + y_1)$
$(y_1, y_2) "+" (x_1, x_2) = (y_1 + x_2, y_2 + x_1)$ Not the same!

2. $(x_1, x_2) "+" ((y_1, y_2) "+" (z_1, z_2)) = (x_1, x_2) "+" (y_1 + z_2, y_2 + z_1) = (x_1 + y_2 + z_1, x_2 + y_1 + z_2)$
$((x_1, x_2) "+" (y_1, y_2)) "+" (z_1, z_2) = (x_1 + y_2, x_2 + y_1) "+" (z_1, z_2) = (x_1 + y_2 + z_2, x_2 + y_1 + z_1)$
Not the same!

3. $(x_1, x_2) "+" (0,0) = (x_1, y_1)$ ✓

4. $(x_1, x_2) "+" (-x_2, -x_1) = (0,0)$ ✓

5. $1(x_1, x_2) = (x_1, x_2)$ ✓

6. $(c_1c_2)(x_1,x_2) = (c_1c_2x_1, c_1c_2x_2) = c_1(c_2x_1, c_2x_2) = c_1(c_2(x_1,x_2))$ ✓

Note scalar multiplication was not re-defined so it should work.

④

$c((x_1,x_2) \text{“}+\text{”} (y_1,y_2)) = c(x_1+y_2, x_2+y_1) = (cx_1+cy_2, cx_2+cy_1) = c(x_1,x_2)+c(y_1$ ✓

8. $(c_1+c_2)(x_1,x_2) = ((c_1+c_2)x_1, (c_1+c_2)x_2) = (c_1x_1+c_2x_1, c_1x_2+c_2x_2)$

$= (c_1x_1, c_1x_2) + (c_2x_2, c_2x_1)$

$= c_1(x_1,x_2) + c_2(x_2,x_1)$

Not the same!

#8

$$Ax = \begin{bmatrix}1&1&1\\1&0&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \sim \begin{bmatrix}1&1&1\\0&-1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Two equations 3 unknowns $\Rightarrow$ a line in $\mathbb{R}^3$.

a) no, b) yes, c) no, d) since it is the nullspace of A, yes, e) yes, f) no!

#9

$A_1 = \begin{bmatrix}1&0\\0&1\end{bmatrix}$, $A_1 = \begin{bmatrix}1&0\\0&1\end{bmatrix}$ are certainly invertible but $A_1 - A_2 = \begin{bmatrix}0&0\\0&0\end{bmatrix}$ is not!

$A_1 = \begin{bmatrix}1&0\\0&0\end{bmatrix}$, $A_2 = \begin{bmatrix}0&0\\0&1\end{bmatrix}$ are singular but $A_1 + A_2 = \begin{bmatrix}1&0\\0&1\end{bmatrix}$ is invertible!

Note: to disprove that a set is a subspace, all you need IS one of the properties to fail.

#24

$$\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

Three pivots, three unknowns $\Rightarrow$ Unique solution, always!

$$\begin{bmatrix}1&1&1\\0&1&1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

2 pivots, three unknowns, there is a solution only if $b_3 = 0$.

#28

a) Let $A = \begin{bmatrix}1&0\\0&0\end{bmatrix}$, the column space $C(A)$ is spanned by $\begin{bmatrix}1\\0\end{bmatrix}$, if you take $\begin{bmatrix}0\\1\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ we have two vectors not in the nullspace which when subtracted give a vector in the column space! False!

b) Proof by Contradiction: Assume one element $a_{ij} \neq 0$, then the column space contains $c\begin{bmatrix}a_{ij}\end{bmatrix}$ which is non-zero, hence a contradiction

True!

28)



c) Column space of $A = c_1\vec{c}_1 + c_2\vec{c}_2 + \dots + c_n\vec{c}_n$ with $\{c_n\} \in \mathbb{R}$ & $\vec{c}_n$ the columns of $A$.

Column space of $2A = 2b_1\vec{c}_1 + 2b_2\vec{c}_2 + \dots + 2b_n\vec{c}_n$ with $\{b_n\} \in \mathbb{R}$ & $\vec{c}_n$ the columns of $A$.

let $2b_1 = c_1$. True

d) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $A - I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\Rightarrow$ False.

Section 2.2 #1, 2, 7, 30, 39, 45, 50

#1

$\left.\begin{matrix} x+y+z=0 \\ x+y+z=1 \end{matrix}\right\}$ No solution! $x+y+z=0 \Rightarrow z=-x-y \Rightarrow \begin{bmatrix} x \\ y \\ -x-y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

#2

$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$

$x, y$ are pivot variables
$z, w$ are free variables
rank = 2

$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$ $1x + 2y + w = 0$, $x = 2l - w$
$y + z = 0$, $y = -l$
$w = t$, $z = l$

All solutions $\begin{bmatrix} 2l-t \\ -l \\ l \\ t \end{bmatrix}$ with $l, t \in \mathbb{R}$.

Special solutions, setting one free variable to zero and the other to 1:

$\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \underset{\text{through Gaussian elimination}}{\sim} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$z$ is free variable
$x, y$ are pivot variables
rank = 2

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$ $x = -2y - 3z = t$
$-3y - 6z = 0 \Rightarrow y = -2t$
$z = t$

all solutions are $t\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

also special solution.

#7

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & 3 & -1 & 5 \\ 3 & 4 & 1 & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -5 & 1 \\ 0 & 1 & -5 & c-6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & c-7 \end{array}\right]$$

For there to be a solution, we need $c-7=0 \Rightarrow c=7$

if $c=7$, then

$$\begin{aligned} x+y+2z&=2 & x&=2-y-2z=2-1-5t-2t=+1-7t \\ y-5z&=1 & y&=1+5z=1+5t \\ z&=t \end{aligned}$$

$$\begin{bmatrix} -1-7t \\ 1+5t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

#30

$$A=\begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

1. $$\begin{bmatrix} 2 & 4 & 6 & 4 & b_1 \\ 2 & 5 & 7 & 6 & b_2 \\ 2 & 3 & 5 & 2 & b_3 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2-b_1 \\ 0 & -1 & -1 & -2 & b_3-b_1 \end{bmatrix} \sim \left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2-b_1 \\ 0 & 0 & 0 & 0 & b_3+b_2-2b_1 \end{array}\right]$$

$$U \qquad \vec{c}$$

2. $b_3+b_2-2b_1=0$

3. The column space is all the linear combinations of $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}$ (i.e. the pivot columns)

or all vectors $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_3+b_2-2b_1=0$.

4. The nullspace $U\vec{x}=\vec{0} \Rightarrow$

$$\begin{aligned} 2x_1+4x_2+6x_3+4x_4&=0 & x_1&=-2(-t-2V)-3t-2V = -t+2V \\ x_2+x_3+2x_4&=0 & \Rightarrow x_2&=-t-2V \\ x_3&=t \\ x_4&=V \end{aligned}$$

So the nullspace is: $$\begin{bmatrix} -t+2V \\ -t-2V \\ t \\ V \end{bmatrix} = t\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + V\begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

special solutions

5. We find a particular solution for $b_1 = 4, b_2 = 3, b_3 = 5$

$$2x_1 + 4x_2 + 6x_3 + 4x_4 = 4$$
$$x_2 + x_3 + 2x_4 = -1$$

then we let the free variables $x_3, x_4 = 0$

$$2x_1 + 4x_2 = 4 \quad \Rightarrow \quad x_1 = 2x_1 + 2 = 4$$
$$x_2 = -1$$

so a particular solution $\vec{x}_p = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

The complete solution $\vec{x} = \vec{x}_p + \vec{x}_n = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$, $c_1, c_2 \in \mathbb{R}$.

6. If we reduce it all the way to $R$:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

From $R$ we can read off the particular solution $\vec{x}_p = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ and the special solutions $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

by looking at the RHS.

From First Free Column

From Second Free Column

Strang likes computing the nullspace & particular solution from $R$, I prefer $U$.

#39

a) $A(\vec{x}_p + \vec{x}_n) = A\vec{x}_p + A\vec{x}_n = \vec{b} + \vec{0} = \vec{b}$ but $A(2\vec{x}_p + \vec{x}_n) = 2\vec{b}$!

b) take one particular solution $\vec{x}_p$, add an element of nullspace $\vec{x}_p + \vec{x}_n$, $A(\vec{x}_p + \vec{x}_n) = \vec{b}$ is also a particular solution!

c) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\qquad x + y = 1$, $y = t$ ← Free variable

solution $\begin{bmatrix} 1-t \\ t \end{bmatrix}$ Has length squared:

$$(1-t)^2 + t^2$$
$$1 - 2t + t^2 + t^2 = 1 - 2t + 2t^2$$

Looking for a minimum $\frac{d}{dt}(1 - 2t + 2t^2) = -2 + 4t = 0$

$t = 1/2$!

if we let $t = 1/2$

The length$^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

if we set it to zero, length$^2 = 1$!

#39. Continued

d) $A\vec{x}=\vec{0}$ always has $\vec{x}=0$ as a solution!

#45 $A$ is $m \times n$ with rank $r$ ($r$ pivots) and $A\vec{x}=\vec{b}$

a) no solution $\Rightarrow$ there must be less pivots than rows to create a row of zeros that would prevent a solution for some $\vec{b}$.

$$r<m$$

then the only other constraint is that there cannot be more pivots than columns.

$$r \leq n.$$

b) infinitely many solutions for all $\vec{b}$ $\Rightarrow$

There must be more variables than pivots (which leads to free variables)

$$r<n$$

and there must be a solution for every $\vec{b}$

$$r=m.$$

c) exactly one solution for some $b$, no solution for other $\vec{b}$.

Since there are no solutions for som $\vec{b}$ we need

$$r<m$$

But we need exactly one solution for some $\vec{b}$

$$r=n$$

d) Yeah, invertible! $r=m=n$.

#50 $A\vec{x}=\begin{bmatrix}1\\3\end{bmatrix}$ $\vec{x}=\begin{bmatrix}1\\0\end{bmatrix}+c\begin{bmatrix}0\\1\end{bmatrix}$. Find A.

We know $A\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\3\end{bmatrix} \Rightarrow 1c_1+0c_2=\begin{bmatrix}1\\3\end{bmatrix}$

column 1 of A is $\begin{bmatrix}1\\3\end{bmatrix}$

We know the nullspace is $\begin{bmatrix}0\\1\end{bmatrix}$

$A\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ so $0c_1+1c_2=\begin{bmatrix}0\\0\end{bmatrix}$

column 2 of A $=\begin{bmatrix}0\\0\end{bmatrix}$

$$\boxed{A=\begin{bmatrix}1 & 0\\3 & 0\end{bmatrix}}$$

Think of Matrix Multiplication one column at a time.