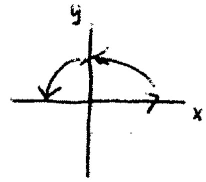
Math 307 Solution Key #6

Section 2.6 # 1, 4, 6, 7, 14, 18, 22

#1 We build matrices by seeing how the linear transformations act on the basis vectors.

Rotation by $90°$



$$\begin{bmatrix}1\\0\end{bmatrix} \to \begin{bmatrix}0\\1\end{bmatrix}, \quad \begin{bmatrix}0\\1\end{bmatrix} \to \begin{bmatrix}-1\\0\end{bmatrix}$$

If we then think of matrix multiplication one column at a time,

$$M\begin{bmatrix}1\\0\end{bmatrix} = 1c_1 + 0c_2 = \begin{bmatrix}0\\1\end{bmatrix} \Rightarrow c_1 = \begin{bmatrix}1\\0\end{bmatrix}, \text{ equivalently } c_2 = \begin{bmatrix}-1\\0\end{bmatrix}$$

matrix representing transformation

columns of m

Hence, $M = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$

Projection onto x-axis $\begin{bmatrix}1\\0\end{bmatrix} \to \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \to \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow M = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$

Projection onto y-axis $\begin{bmatrix}1\\0\end{bmatrix} \to \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \to \begin{bmatrix}0\\1\end{bmatrix} \Rightarrow M = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

Now, rotating (x,y) and then projecting onto the x-axis can be represented by

$$\underbrace{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}}_{\text{then project to x-axis}}\underbrace{\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}}_{\text{First Rotate by }90°}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0 & -1\\0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

Projection onto x-axis followed by projection onto y axis

$$\underbrace{\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}}_{\text{Projection onto y}}\underbrace{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}}_{\text{projection onto x}}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

#4 The half way point between two vectors $\vec{x}, \vec{y}$ is $\vec{z} = \dfrac{\vec{x}+\vec{y}}{2}$

Since $A\left(\dfrac{\vec{x}+\vec{y}}{2}\right) = \dfrac{A\vec{x} + A\vec{y}}{2}$ then $A\vec{z} = \frac{1}{2}\left[A\vec{x} + A\vec{y}\right]$ hence $A\vec{z}$ is halfway between $A\vec{x}$ and $A\vec{y}$.

#6 [Note, this problem was on the board but not on the email. I won't take off points in the HW if you didn't do it but you will be responsible for the material in quizzes + tests]

a) Find the matrix that projects a vector in $\mathbb{R}^3$ onto the x-y plane.

We see how it acts on the basis vectors

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \to \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \begin{bmatrix}0\\1\\0\end{bmatrix} \to \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\1\end{bmatrix} \to \begin{bmatrix}0\\0\\0\end{bmatrix} \quad \text{so } M = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$$

using the column view of matrix multiplication.

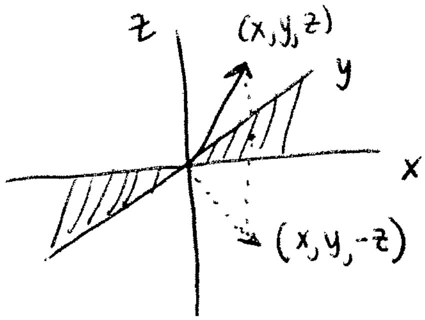
Note that this projection is not unique even though it does seem the most natural.

For example, we could have done

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \to \begin{bmatrix}1/\sqrt{2}\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\1\\0\end{bmatrix} \to \begin{bmatrix}0\\1/\sqrt{2}\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\1\end{bmatrix} \to \begin{bmatrix}0\\0\\0\end{bmatrix} \quad \text{with } M = \begin{bmatrix}1/\sqrt{2}&0&0\\0&1/\sqrt{2}&0\\0&0&0\end{bmatrix}$$

and that would also be a projection.

b) Let's look geometrically what reflecting across the x-y plane means..



It means that the z coordinate changes sign, while x, y stay the same.

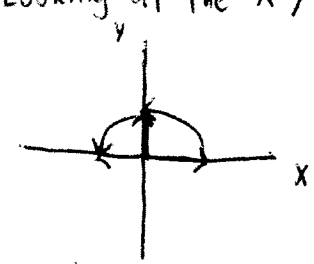
$$\begin{bmatrix}1\\0\\0\end{bmatrix} \to \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \begin{bmatrix}0\\1\\0\end{bmatrix} \to \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\1\end{bmatrix} \to \begin{bmatrix}0\\0\\-1\end{bmatrix}$$

hence

$$M = \begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$$

c) Rotate x-y plane leaving the z axis alone?

Looking at the x-y plane



$$\begin{bmatrix}1\\0\\0\end{bmatrix} \to \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\1\\0\end{bmatrix} \to \begin{bmatrix}-1\\0\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\1\end{bmatrix} \to \begin{bmatrix}0\\0\\1\end{bmatrix} \quad M = \begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}$$

d) using same approach

$$\begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&0&-1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix} = \begin{bmatrix}0&0&1\\0&-1&0\\1&0&0\end{bmatrix}$$

e)

$$\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\begin{bmatrix}-1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}!$$

#7

any element of P_3 can be written as $a+bt+ct^2+dt^3$ and hence can be mapped to $(a,b,c,d) \in \mathbb{R}^4$.

We construct a matrix that represents $\frac{d^2}{dt^2}$ by acting on the basis elements.

$$\frac{d^2}{dt^2}(1) = 0 \quad \text{so} \quad \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \to \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \quad \frac{d^2}{dt^2}(t) = 0 \quad \text{so} \quad \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \to \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\frac{d^2}{dt^2}(t^2) = 2 \quad \text{so} \quad \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \to \begin{bmatrix}2\\0\\0\\0\end{bmatrix}, \quad \frac{d^2}{dt^2}(t^3) = 6t \quad \text{so} \quad \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \to \begin{bmatrix}0\\6\\0\\0\end{bmatrix}$$

Hence $M = \begin{bmatrix}0&0&2&0\\0&0&0&6\\0&0&0&0\\0&0&0&0\end{bmatrix}$

the nullspace is spanned by $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$ which means that any linear function has zero second derivative

The column space is by coincidence the same as the nullspace since the second derivative of cubic functions are linear!

#14 If T is linear then $T(k\vec{v}) = kT(\vec{v})$ where $\vec{v}$ is a vector & k is a scalar and $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$ where $\vec{v}, \vec{w}$ are vectors.

$$T^2(k\vec{v}) = T(T(k\vec{v})) = T(kT(\vec{v})) = kT^2(\vec{v}) \quad \checkmark$$

$$T^2(\vec{v}+\vec{w}) = T(T(\vec{v}+\vec{w})) = T(T(\vec{v})+T(\vec{w})) = T^2(\vec{v}) + T^2(\vec{w}) \quad \checkmark$$

#18 The subset of polynomials with $\int_0^1 p(x)\,dx = 0$ satisfy

$$\int_0^1 a_0 + a_1 x + a_2 x^2 + a_3 x^3\,dx = a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4} \Big|_0^1 = 0$$

$$\Rightarrow a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} = 0$$

$$\Rightarrow a_0 = -\frac{a_1}{2} - \frac{a_2}{3} - \frac{a_3}{4}$$

$$\begin{bmatrix}-a_1/2 - a_2/3 - a_3/4\\a_1\\a_2\\a_3\end{bmatrix} = \underbrace{a_1\begin{bmatrix}-1/2\\1\\0\\0\end{bmatrix} + a_2\begin{bmatrix}-1/3\\0\\1\\0\end{bmatrix} + a_3\begin{bmatrix}-1/4\\0\\0\\1\end{bmatrix}}_{\text{basis}}$$

To show it is a vector space

$$\int_0^1 k\,p(x)\,dx = k\int_0^1 p(x)\,dx = 0$$

Closed under scalar multiplication.

$$\int_0^1 p_1(x) + p_2(x)\,dx = \int_0^1 p_1(x)\,dx + \int_0^1 p_2(x)\,dx = 0+0 \quad \checkmark$$

Closed under vector addition.

#22

a) $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \to \begin{bmatrix} v_2 \\ v_1 \end{bmatrix}$: $\begin{bmatrix} \alpha v_1 \\ \alpha v_2 \end{bmatrix} \to \begin{bmatrix} \alpha v_2 \\ \alpha v_1 \end{bmatrix} = \alpha \begin{bmatrix} v_2 \\ v_1 \end{bmatrix}$, $\begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix} \to \begin{bmatrix} v_2 + w_2 \\ v_1 + w_1 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_1 \end{bmatrix} + \begin{bmatrix} w_2 \\ w_1 \end{bmatrix}$

yep, linear

b) $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \to \begin{bmatrix} v_1 \\ v_1 \end{bmatrix}$: $\begin{bmatrix} \alpha v_1 \\ \alpha v_2 \end{bmatrix} \to \begin{bmatrix} \alpha v_1 \\ \alpha v_1 \end{bmatrix} = \alpha \begin{bmatrix} v_1 \\ v_1 \end{bmatrix}$, $\begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix} \to \begin{bmatrix} v_1 + w_1 \\ v_1 + w_1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_1 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_1 \end{bmatrix}$

yep, linear

c) $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \to \begin{bmatrix} 0 \\ v_1 \end{bmatrix}$: $\begin{bmatrix} \alpha v_1 \\ \alpha v_2 \end{bmatrix} \to \begin{bmatrix} 0 \\ \alpha v_1 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ v_1 \end{bmatrix}$, $\begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix} \to \begin{bmatrix} 0 \\ v_1 + w_1 \end{bmatrix} = \begin{bmatrix} 0 \\ v_1 \end{bmatrix} + \begin{bmatrix} 0 \\ w_1 \end{bmatrix}$

yep, linear

d) $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}$: $\begin{bmatrix} \alpha v_1 \\ \alpha v_2 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Not Linear

Not linear, again

Section 3.1 (1, 2, 11, 12, 26, 46)

#1. $\vec{x} = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 2 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 2 \\ -2 \\ 1 \\ 3 \end{bmatrix}$ $\quad \vec{x}^T\vec{y} = 2 - 8 + 0 + 6 = 0$ They are orthogonal!

$\|x\| = \sqrt{\vec{x}^T\vec{x}} = \sqrt{1+16+0+4} = \sqrt{21}$

$\|y\| = \sqrt{4+4+1+9} = \sqrt{18}$

• #2. Linearly independent but not orthogonal

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are linearly independent since the only c's $c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\det A \neq 0}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

that solve this equation are $c_1 = c_2 = 0$ since $\det A \neq 0$.

and they are not orthogonal since

$$\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \neq 0 .$$

Orthogonal vectors that are not independent

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \Rightarrow$ orthogonal

$0 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ dependent, actually $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is linearly dependent to all vectors!

#11 The Fredholm Alternative says that one and only one of the following systems has a solution:

i) $A\vec{x} = \vec{b}$ ($\vec{b}$ is in the column space of A)
ii) $A^T\vec{y} = \vec{0}$ with $\vec{y}^T b \neq 0$ (there is an element of the left nullspace that is not orthogonal to $\vec{b}$)

The reason why only one of these systems has a solution is because the column space is the orthogonal complement to the left nullspace.

Answer: if i) has a solution then there is an $\vec{x}$ such that $A\vec{x} = \vec{b}$
ii) has a solution then there is a $\vec{y}$ such that $A^T\vec{y} = \vec{0}$ with $\vec{y}^T\vec{b} = 0$

We take the $\vec{x}, \vec{y}$ that solve i) and ii) respectively

Assume i)
$$A\vec{x} = \vec{b}$$
$$\vec{y}^T A\vec{x} = \vec{y}^T\vec{b}$$
$$(\vec{y}^T A\vec{x})^T = (\vec{y}^T\vec{b})^T$$
$$\vec{x}^T A^T\vec{y} = \vec{b}^T\vec{y}$$
$$0 = \vec{b}^T\vec{y}$$
which contradicts ii)

Assume ii) $A^T\vec{y} = \vec{0}$
$$\vec{x}^T A^T\vec{y} = 0$$
$$(\vec{x}^T A^T\vec{y})^T = 0$$
$$\vec{y}^T A\vec{x} = 0$$
$$\vec{v}^T\vec{b} = 0$$
i) $A\vec{x} = \vec{b}$

which contradicts $\vec{v}^T\vec{b} \neq 0$.

#12 The orthogonal complement of the row space is the nullspace

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 1 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{array}{l} x+2z=0 \\ y+2z=0 \end{array} \quad \begin{array}{l} x=-2z \\ y=-2z \end{array}$$

$$\begin{bmatrix} -2z \\ -2z \\ z \end{bmatrix} = z\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$$

The nullspace is spanned by $\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$ and the row space by $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

We could project onto the rowspace, but we have not covered that yet, so we write

$$\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = c_1\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad \text{and find } \vec{c}.$$

$$\left[\begin{array}{ccc|c} -2 & 1 & 0 & 3 \\ -2 & 0 & 1 & 3 \\ 1 & 2 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} -2 & 1 & 0 & 3 \\ 0 & -1 & 1 & 0 \\ 1 & 2 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & -1 & 1 & 0 \\ -2 & 1 & 0 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & -1 & 1 & 0 \\ 0 & 5 & 4 & 9 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 9 & 9 \end{array}\right] \quad \begin{array}{r} c_1+2c_2+2c_3=3 \\ -1c_2+c_3=0 \\ 9c_3=9 \end{array} \quad \begin{array}{l} c_1=-1 \\ c_2=1 \\ c_3=1 \end{array}$$

so $$\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}}_{\text{in nullspace}} + \underbrace{\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}}_{\text{in row space}}$$

#26 If $AB=0$ then the columns of B are in the nullspace of A because of the column view of matrix multiplication

$$AB = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_n \end{bmatrix} = \begin{bmatrix} \vec{0} & \cdots & \vec{0} \end{bmatrix}$$ with $\vec{b}_n$ the n^{th} column of B.

The rows of A are in the left nullspace of B because of the row interpretation of matrix multiplication

$$AB = \begin{bmatrix} \vec{a}_1 B \\ \vec{a}_2 B \\ \vdots \\ \vec{a}_m B \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{0} \\ \vdots \\ \vec{0} \end{bmatrix}$$ with $\vec{a}_n$ the n^{th} row of A

If A and B both have rank 2, then the column and row space of both matrices would be dimension 2. In particular the nullspace of A and the row space of A would both be dimension 2 which is impossible for a 3x3!

#46 If the columns of A are unit vectors and mutually perpendicular: $\vec{a}_i^T \vec{a}_j = \begin{matrix} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{matrix}$ with $\vec{a}_i$ the i^{th} column of A.

$$A^T A = \begin{bmatrix} - & \vec{a}_1 & - \\ - & \vec{a}_2 & - \\ - & \vec{a}_i & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_i \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \\ \vdots & & & \ddots & 1 \end{bmatrix} = I$$

since $[A^T A]_{ij} = \vec{a}_i^T \vec{a}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$