Solution Key Math 307 HW #9

Section 5.1 # 1, 2, 4, 11, 24, 25

#1 Solution in text.

• #2 $\frac{d\vec{u}}{dt} = A\vec{u}$ with $u(0) = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$

will have solutions of the form $c_1 e^{\lambda_1 t}\vec{x}_1 + c_2 e^{\lambda_2 t}\vec{x}_2$

and we know the eigenvalues and eigenvectors from #1.

$$\vec{u} = c_1 e^{2t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{3t}$$

we use $u(0) = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$ to find the values of the constants: $\begin{bmatrix} 0 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

[Note, you could also solve this system by Gaussian Elimination]

Hence $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \left(\begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 \\ 6 \end{bmatrix}$

$c_1 = -6$, $c_2 = 6$

Hence $\vec{u} = -6e^{2t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 6e^{3t}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, the pure exponential solutions are $c_1 e^{2t}$ and $c_2 e^{3t}$.

#4 $\frac{d\vec{u}}{dt} = P\vec{u}$ with $u(0) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

This problem is the same as #2 but because P is a projection matrix we will be able to see how the geometric interpretation of P shows up in the solution to the DE.

$\lambda_1 = 0$ $\vec{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\lambda_2 = 1$ $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$u = c_1 e^{0t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Note that the components of the solution with $\lambda_1 = 0$ will stay the same size (they represent the vectors orthogonal to the space where P projects to).

$$u(0) = \begin{bmatrix} 5 \\ 3 \end{bmatrix} = c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$\Rightarrow c_1 = -1$, $c_2 = 4$

vectors in the direction of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, column space of P will grow exponentially!

$$u(t) = 4e^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

#11 one of the properties of determinants is that $|A| = |A^T|$ so

$$\det(A-\lambda I) = \det\left[(A-\lambda I)^T\right] = \det(A^T - \lambda I) \quad \blacksquare.$$

Example: $A=\begin{bmatrix}1&1\\0&0\end{bmatrix}$, $A^T=\begin{bmatrix}1&0\\1&0\end{bmatrix}$ aigenvectors of A are $\underset{\lambda=1}{\begin{bmatrix}1\\0\end{bmatrix}}, \underset{\lambda=0}{\begin{bmatrix}1\\-1\end{bmatrix}}$ for A^T are $\underset{\lambda=1}{\begin{bmatrix}1\\1\end{bmatrix}}$ [λ

#24

a) $AA\vec{x} = \lambda A\vec{x} \Rightarrow A^2\vec{x} = \lambda^2\vec{x}$

b) $A^{-1}A\vec{x} = \lambda A^{-1}x \Rightarrow \frac{\vec{x}}{\lambda} = A^{-1}\vec{x} \Rightarrow A^{-1}\vec{x} = \lambda^{-1}\vec{x}$

c) $A\vec{x} + I\vec{x} = \lambda\vec{x} + \vec{x} \Rightarrow (A+I)\vec{x} = (\lambda+1)\vec{x}$

#25 solution in text.

5.2 #1, 2, 8, 19, 22, 33

#1 Solution in text

#2

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 18 \\ -3 & 10 \end{bmatrix}$$

$$S \quad \Lambda \quad S^{-1}$$

The key idea here is that the matrix is uniquely determined by the eigenvalues and eigenvectors.

#8 a) $A\vec{u} = \vec{u}\vec{v}^T\vec{u} = \vec{u}(\vec{v}^T\vec{u})$ note that $\vec{v}^T\vec{u}$ is the inner product of $\vec{v}$ and $\vec{u}$ so it is a scalar!

$$= (\vec{v}^T\vec{u})\vec{u} = \lambda\vec{u}.$$

b) Since A is rank 1, the only non-zero eigenvalue is $\vec{v}^T\vec{u}$, otherwise the column space would have dimension greater than 1. That means that all other eigenvalues must be zero and the nullspace be dimension $n-1$.

c)

$$A = \vec{u}\vec{v}^T = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}(v_1 \ v_2 \ldots v_n) = \begin{bmatrix} u_1v_1 & u_1v_2 \ldots & u_1v_n \\ u_2v_1 & u_2v_2 \ldots & u_2v_n \\ \vdots & & \\ u_nv_1 & \ldots & u_nv_n \end{bmatrix}.$$

$$\text{trace}(A) = u_1v_1 + u_2v_2 + \ldots + u_nv_n = \vec{v}^T\vec{u}\,!$$

#19

a) False, one of the λ's could be zero. Ex. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ it is already diagonal, but not invertible.

b) True, $A = S\Lambda S^{-1}$, S^{-1} exists since it's columns are linearly independent.

c) True b)

d) False, S is invertible but it may not be diagonalizable, we would need to know it's eigenvectors! ex. $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is invertible but not diagonalizable.

• #22

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$S \quad \Lambda \quad S^{-1}$$

this is really the same question as #2, if you know the eigenvectors you just need 2 parameters (λ_1, λ_2) to uniquely determine the matrix.

33 Solution in text.

5.3 #1, 2, 3, 8, 10, 17

#1 Solution in text

#2

$A = \begin{bmatrix} 0 & 0 & 6 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}$, $A^3 = I$ by multiplication. This implies that the population repeats itself every 3 years.

Year 1	Year 2	Year 3	Year 4	Year 5	Year 6	Year 7
3000	1500	500	6000	1500	500	6000
beginning population	1/2 survive	1/3 survive	they reproduce w/ 6 children & die			

In vector form, we look at the evolution of the beetle population distribution:

$$\begin{bmatrix} \text{beetles in their first year} \\ \text{beetles in their second year} \\ \text{beetles in their third year} \end{bmatrix} = \underset{\text{at year } k.}{\overset{\vec{u}_k}{\begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix}}} \quad \overset{k=1}{\begin{bmatrix} 3000 \\ 0 \\ 0 \end{bmatrix}} \overset{k=2}{\begin{bmatrix} 0 \\ 1500 \\ 0 \end{bmatrix}} \overset{k=3}{\begin{bmatrix} 0 \\ 0 \\ 500 \end{bmatrix}} \overset{k=4}{\begin{bmatrix} 6000 \\ 0 \\ 0 \end{bmatrix}} \overset{k=5}{\begin{bmatrix} 0 \\ 1500 \\ 0 \end{bmatrix}} \overset{k=6}{\begin{bmatrix} 0 \\ 0 \\ 500 \end{bmatrix}} \cdots$$

#3 Solution in text

• #8 A has eigenvalues $\lambda = 1, 3/4, 1/2$. We know that at steady state, i.e. when $t \to \infty$, the components of the solution with $|\lambda| < 1$ go to zero.

the eigenvector for $\lambda = 1$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, so unfortunately, everybody dies.

#10 For $\lambda = 1$, $y_{k \to \infty} = 3$, $z_{k \to \infty} = 2$; $\quad y_k = 3 - \frac{3}{2^k}$

$$z_k = 2 + \frac{3}{2^k}$$

#17 Solution in text.