HW2, EGME 511 (Advanced Mechanical Vibration)

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Contents

1 Problem 1

Find the equation of motion for the following system

Solution

Assume initial conditions are $x(0) = x_0$ and $\dot{x}(0) = 0$. Assume that x_0 was positive (i.e. to the right of the static equilibrium position, and also assume that $kx_0 > N\ \mu_{static}$). This second requirement is needed to enable the mass to undergo motion by overcoming static friction. The normal force N is given by

$$N = mg\cos\theta$$

And the dynamic friction force f_c due to the dynamic friction is defined as follows

$$f_c = \begin{cases} -\mu N & \dot{x} > 0 \\ 0 & \dot{x} = 0 \\ \mu N & \dot{x} < 0 \end{cases}$$

But since $N = mg\cos\theta$, then the above becomes

$$f_c = \begin{cases} -\mu mg\cos\theta & \dot{x} > 0 \\ 0 & \dot{x} = 0 \\ \mu mg\cos\theta & \dot{x} < 0 \end{cases} \tag{1}$$

Where μ is the coefficient of dynamic friction. Now we can obtain the Lagrangian

$$L = T - U$$
$$T = \frac{1}{2}m\dot{x}^2$$
$$U = \frac{1}{2}kx^2$$

Hence

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

and

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}$$
$$\frac{\partial L}{\partial x} = -kx$$

Then the EQM is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = f_c$$
$$m\ddot{x} + kx = f_c$$

Where f_c is given by (1). Since f_c sign depends in the mass is moving to the left or to the right, we will generate 2 equation of motions, one for each case.

When mass is moving to the left, EQM 1 is

$$m\ddot{x} + kx = \mu mg\cos\theta \tag{2}$$

When mass is moving to the right, EQM 2 is

$$m\ddot{x} + kx = -\mu mg\cos\theta \tag{3}$$

So, for the first move, starting from x_0 and moving to the left, we have

$$\ddot{x} + \frac{k}{m}x = \mu g\cos\theta$$
$$\ddot{x} + \omega_n^2 x = \mu g\cos\theta$$

$$x = x_h + x_p$$

Guess $x_p = X$, hence $\omega_n^2 X = \mu g\cos\theta$ or $X = \frac{\mu g\cos\theta}{\omega_n^2}$, and $x_h = A\cos\omega_n t + B\sin\omega_n t$, therefore, the solution to EQM 1 is

$$x(t) = A\cos\omega_n t + B\sin\omega_n t + \frac{\mu g\cos\theta}{\omega_n^2}$$

$x(0) = x_0 = A + \frac{\mu g \cos\theta}{\omega_n^2}$ hence $A = x_0 - \frac{\mu g \cos\theta}{\omega_n^2}$, then

$$x(t) = \left(x_0 - \frac{\mu g \cos\theta}{\omega_n^2}\right)\cos\omega_n t + B\sin\omega_n t + \frac{\mu g \cos\theta}{\omega_n^2}$$

and

$$\dot{x}(t) = -\omega_n\left(x_0 - \frac{\mu g \cos\theta}{\omega_n^2}\right)\sin\omega_n t + \omega_n B\cos\omega_n t$$

$$\dot{x}(0) = v_0 = 0 = \omega_n B$$

Hence $B = 0$, then EQM is (for $0 < t < \frac{\pi}{\omega_n}$)

$$x_{left}(t) = \left(x_0 - \frac{\mu g \cos\theta}{\omega_n^2}\right)\cos\omega_n t + \frac{\mu g \cos\theta}{\omega_n^2} \tag{4}$$

The mass will move according to the above equation (4) until the velocity is zero, then it will turn and start moving to the right. To find the time this happens:

$$\dot{x}(t) = -\omega_n\left(x_0 - \frac{\mu g \cos\theta}{\omega_n^2}\right)\sin\omega_n t$$

Now solve for t when $\dot{x}(t) = 0$, i.e.,

$$0 = -\omega_n\left(x_0 - \frac{\mu g \cos\theta}{\omega_n^2}\right)\sin\omega_n t \tag{5}$$

Hence $\omega_n t = n\pi$, where $n = 0, \pm1, \pm2, \cdots$The case for $n = 0$ do not apply since this implies $t = 0$, then consider the next time this can happen, which is $n = 1$, which implies

$$t_1 = \frac{\pi}{\omega_n} \tag{6}$$

Now we need to determine $x(t)$ at this time t_1 since this will become the initial x for the second equation of motion going to the right in the second leg of the journey. Using (4) and (6) we obtain

$$\begin{aligned} x\left(\frac{\pi}{\omega_n}\right) &= \left(x_0 - \frac{\mu g \cos\theta}{\omega_n^2}\right)\cos\omega_n\frac{\pi}{\omega_n} + \frac{\mu g \cos\theta}{\omega_n^2} \\ &= \frac{2\mu g \cos\theta}{\omega_n^2} - x_0 \end{aligned}$$

Notice that in the above equation, x_0 is a positive number, since we assumed that the initial conditions x_0 was to the right of the static equilibrium position, and we are assume the right of the static equilibrium position to be positive. This also implied that $x\left(\frac{\pi}{\omega_n}\right)$ will be negative number (which is what we expect, as the mass will by the end of its first trip be on the left of the static equilibrium position).

Now we can use right equation of motion (EQM 2) to solve for the mass moving to the right. Notice that the initial conditions for this motion are $x_1 = \frac{2\mu g \cos\theta}{\omega_n^2} - x_0$ and $t_1 = \frac{\pi}{\omega_n}$

The equation of motion is now

$$m\ddot{x} + kx = -\mu mg\cos\theta$$
$$\ddot{x} + \omega_n^2 x = -\mu g\cos\theta$$

With the general solution

$$x(t) = A\cos\omega_n t + B\sin\omega_n t - \frac{\mu g\cos\theta}{\omega_n^2} \tag{7}$$

At $t = \frac{\pi}{\omega_n}$, $x(t) = \frac{2\mu g\cos\theta}{\omega_n^2} - x_0$, hence from the above

$$\begin{aligned}\frac{2\mu g\cos\theta}{\omega_n^2} - x_0 &= A\cos\omega_n\frac{\pi}{\omega_n} + B\sin\omega_n\frac{\pi}{\omega_n} - \frac{\mu g\cos\theta}{\omega_n^2}\\ &= -A - \frac{\mu g\cos\theta}{\omega_n^2}\\ A &= x_0 - \frac{3\mu g\cos\theta}{\omega_n^2}\end{aligned}$$

Hence (7) becomes

$$x(t) = \left(x_0 - \frac{3\mu g\cos\theta}{\omega_n^2}\right)\cos\omega_n t + B\sin\omega_n t - \frac{\mu g\cos\theta}{\omega_n^2}$$

And

$$\dot{x}(t) = -\omega_n\left(x_0 - \frac{3\mu g\cos\theta}{\omega_n^2}\right)\sin\omega_n t + \omega_n B\cos\omega_n t$$

But $\dot{x}(t) = 0$ at $t = \frac{\pi}{\omega_n}$, hence the above becomes

$$\begin{aligned}0 &= -\omega_n\left(x_0 - \frac{3\mu g\cos\theta}{\omega_n^2}\right)\sin\omega_n\frac{\pi}{\omega_n} + \omega_n B\cos\omega_n\frac{\pi}{\omega_n}\\ &= -\omega_n B\end{aligned}$$

Hence $B = 0$, then the EQM for the right move is, for $\frac{\pi}{\omega_n} < t < \frac{2\pi}{\omega_n}$

$$\boxed{x_{right}(t) = \left(x_0 - \frac{3\mu g\cos\theta}{\omega_n^2}\right)\cos\omega_n t - \frac{\mu g\cos\theta}{\omega_n^2}}$$

This diagram below summarize this

Now, we would like to have one equation to express the motion with for any time instance when the mass is moving to the left, or to the right. Looking at the above 2 equation of motion, we see immediately that we can write the equation of motion as follows

$$\boxed{x_n(t) = \left(x_0 - \frac{(2n-1)\mu g\cos\theta}{\omega_n^2}\right)\cos\omega_n t \ + (-1)^{n+1}\,\frac{\mu g\cos\theta}{\omega_n^2}}$$

Where n above is the number of the trip. So, the first trip, going from x_0 and moving to the left, will have $n = 1$, and then second trip, moving from x_1 and going to the right will have $n = 2$, and so on. As for the time during which trip travels, this is found by the following equation

$$\frac{(n-1)\pi}{\omega_n} < t_n < \frac{n\pi}{\omega_n}$$

What the above is saying is that for first trip $(n = 1)$, we have

$$0 < t < \frac{\pi}{\omega_n}$$

And for the second trip, we have

$$\frac{\pi}{\omega_n} < t < \frac{2\pi}{\omega_n}$$

etc...

Now that we have one equation, and we have the time during which each equation is valid, we can now plot the equation of motion vs. time. The following is a plot for some values for k, g, m. Please see the appendix for the Matlab code which generated this simulation.

Observation found on this problem: Changing the angle of inclination θ causes no change in results. In other words, the same oscillation will occur for flat plane $(\theta = 0)$ or for $\theta = 45^0$ or any other angle. The reason is because x_0, the initial position, is measured from the static equilibrium position, and this static equilibrium position will be different as the angle changes, but the effect of the angle change is already accounted for by this change and will not be reflected in the actual displacement $x(t)$.

2 Problem 2

Given $\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}\ddot{\boldsymbol{X}}+\begin{bmatrix}3 & -1\\ -1 & 1\end{bmatrix}\boldsymbol{X}=\begin{bmatrix}0\\ 0\end{bmatrix}$, $m:kg, k:N/m$, use modal analysis to calculate the solution of this given $\boldsymbol{X}(0)=\begin{bmatrix}0\\ 1\end{bmatrix} mm,\ \dot{\boldsymbol{X}}(0)=\begin{bmatrix}0\\ 0\end{bmatrix} mm/\sec$ also calculate the eigenvalues of the system and the normalized eigenvectors.

Answer

Since this is a 2 ODE's that are coupled, we use modal analysis to de-couple the system first in order to obtain 2 separate ODE's which we can then solve easily.

Let

$M=\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}$ and let $K=\begin{bmatrix}3 & -1\\ -1 & 1\end{bmatrix}$, then the above system becomes

$$M\ddot{\boldsymbol{X}}+K\boldsymbol{X}=0 \tag{1}$$

Let $\boldsymbol{X}=M^{-\frac{1}{2}}\boldsymbol{q}$, then $\ddot{\boldsymbol{X}}=M^{-\frac{1}{2}}\ddot{\boldsymbol{q}}$ and the above equation becomes

$$MM^{-\frac{1}{2}}\ddot{\boldsymbol{q}}+KM^{-\frac{1}{2}}\boldsymbol{q}=0$$

premultiply by $M^{-\frac{1}{2}}$ we obtain

$$\begin{aligned}M^{-\frac{1}{2}}MM^{-\frac{1}{2}}\ddot{\boldsymbol{q}}+M^{-\frac{1}{2}}KM^{-\frac{1}{2}}\boldsymbol{q}&=0\\ I\ddot{\boldsymbol{q}}+\tilde{K}\boldsymbol{q}&=0\end{aligned} \tag{2}$$

Where $\tilde{K}=M^{-\frac{1}{2}}KM^{-\frac{1}{2}}$

Let $\boldsymbol{q}=\boldsymbol{v}e^{i\omega t}$, then $\ddot{\boldsymbol{q}}=-\omega^2\boldsymbol{v}e^{i\omega t}$ and (2) becomes

$$\begin{aligned}-\omega^2e^{i\omega t}I\boldsymbol{v}+\tilde{K}\boldsymbol{v}e^{i\omega t}&=0\\ \left(\tilde{K}-\omega^2I\right)\boldsymbol{v}&=0\end{aligned}$$

Let $\lambda=\omega^2$ then we have

$$\left(\tilde{K}-\lambda I\right)\boldsymbol{v}=0 \tag{3}$$

For $\boldsymbol{v}\neq 0$, we requires that $\left|\tilde{K}-\lambda I\right|=0$ But

$$\begin{aligned}\tilde{K}&=M^{-\frac{1}{2}}KM^{-\frac{1}{2}}\\ &=\begin{bmatrix}1^{\frac{-1}{2}} & 0\\ 0 & 4^{\frac{-1}{2}}\end{bmatrix}\begin{bmatrix}3 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1^{\frac{-1}{2}} & 0\\ 0 & 4^{\frac{-1}{2}}\end{bmatrix}\\ &=\begin{bmatrix}3 & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{4}\end{bmatrix}\end{aligned}$$

Hence

$$
\begin{aligned}
\left|\tilde{K}-\lambda I\right| &= 0\\
\left|\begin{bmatrix} 3 & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{4}\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right| &= 0\\
\left|\begin{bmatrix} 3-\lambda & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{4}-\lambda\end{bmatrix}\right| &= 0\\
(3-\lambda)\left(\frac{1}{4}-\lambda\right)-\frac{1}{4} &= 0\\
\lambda^2-\frac{13}{4}\lambda+\frac{1}{2} &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda &= \frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}\\
&= \frac{13}{8}\pm\frac{\sqrt{\left(\frac{13}{4}\right)^2-2}}{2}\\
&= \frac{13}{8}\pm\frac{1}{8}\sqrt{137}
\end{aligned}
$$

Hence

$$
\boxed{\lambda_{1,2} = \left\{\frac{13-\sqrt{137}}{8},\frac{13+\sqrt{137}}{8}\right\} = \{0.161\,91, 3.088\,1\}}
$$

From (3) we then have

$$
\begin{aligned}
\left(\tilde{K}-\lambda I\right)v &= 0\\
\left(\begin{bmatrix} 3 & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{4}\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right)v &= 0
\end{aligned}
$$

When $\lambda = \lambda_1 = 0.161\,9$ we obtain

$$
\begin{aligned}
\left(\begin{bmatrix} 3.0 & -0.5\\ -0.5 & 0.25\end{bmatrix} - \begin{bmatrix}0.161\,9 & 0\\ 0 & 0.161\,9\end{bmatrix}\right)\begin{bmatrix}a\\ b\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}\\
\begin{bmatrix} 2.838\,1 & -0.5\\ -0.5 & 0.088\,1\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} &= \begin{bmatrix}0\\ 0\end{bmatrix}
\end{aligned}
$$

Hence

$$
\begin{aligned}
2.838\,1a-0.5b &= 0\\
-0.5a-0.094\,1b &= 0
\end{aligned}
$$

Let $a = 1$, then $b = \frac{-2.838\,1}{-0.5} = 5.676\,2$, hence the second eigenvector is

$$
v_1 = \begin{bmatrix}1\\ 5.676\,2\end{bmatrix}
$$

$\|v_1\| = \sqrt{1 + 5.6762^2} = 5.7636$, hence normalized v_1 is

$$v_1 = \frac{1}{5.7636}\begin{bmatrix} 1 \\ 5.6762 \end{bmatrix}$$

$$\boxed{v_1 = \begin{bmatrix} 0.1735 \\ 0.98484 \end{bmatrix}}$$

When $\lambda = \lambda_2 = 3.0881$ we obtain

$$\left(\begin{bmatrix} 3.0 & -0.5 \\ -0.5 & 0.25 \end{bmatrix} - \begin{bmatrix} 3.0881 & 0 \\ 0 & 3.0881 \end{bmatrix}\right)\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -0.0881 & -0.5 \\ -0.5 & -2.8381 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence

$$-0.0881a - 0.5b = 0$$
$$-0.5a - 2.8381b = 0$$

Let $a = 1$ in the first equation above, then $b = \frac{-0.0881}{0.5} = -0.1762$, hence the first eigenvector is

$$v_2 = \begin{bmatrix} 1 \\ -0.1762 \end{bmatrix}$$

$\|v_2\| = \sqrt{1 + 0.1762^2} = 1.0154$, hence normalized v_2 is

$$v_2 = \frac{1}{1.0154}\begin{bmatrix} 1 \\ -0.1762 \end{bmatrix}$$

$$\boxed{v_2 = \begin{bmatrix} 0.98483 \\ -0.17353 \end{bmatrix}}$$

Then the P matrix

$$[P] = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$
$$= \boxed{\begin{bmatrix} 0.1735 & 0.98483 \\ 0.98484 & -0.17353 \end{bmatrix}}$$

Now let $\boldsymbol{q} = P\boldsymbol{r}$, then equation (2) above becomes

$$I\ddot{\boldsymbol{q}} + \tilde{K}\boldsymbol{q} = 0$$
$$IP\ddot{\boldsymbol{r}} + \tilde{K}P\boldsymbol{r} = 0$$

Premultiply by P^T

$$P^T IP\ddot{\boldsymbol{r}} + P^T\tilde{K}P\boldsymbol{r} = 0$$
$$I\ddot{\boldsymbol{r}} + P^T\tilde{K}P\boldsymbol{r} = 0$$

Let $\Lambda = P^T \tilde{K} P$ then the above becomes

$$I\ddot{\boldsymbol{r}} + \Lambda \boldsymbol{r} = 0 \tag{4}$$

Now find Λ[1]

$$\begin{aligned}\Lambda &= P^T \tilde{K} P\\ &= \begin{bmatrix} 0.1735 & 0.98483\\ 0.98484 & -0.17353\end{bmatrix}^T \begin{bmatrix} 3.0 & -0.5\\ -0.5 & 0.25\end{bmatrix}\begin{bmatrix} 0.1735 & 0.98483\\ 0.98484 & -0.17353\end{bmatrix}\\ &= \boxed{\begin{bmatrix} 0.16191 & 0\\ 0 & 3.0881\end{bmatrix}}\end{aligned}$$

Hence (4) becomes

$$I\ddot{\boldsymbol{r}} + \begin{bmatrix} 0.16191 & 0\\ 0 & 3.0881\end{bmatrix}\boldsymbol{r} = 0$$

Which can be written as 2 equations

$$\begin{bmatrix}\ddot{r}_1\\ \ddot{r}_2\end{bmatrix} + \begin{bmatrix} 0.16191 r_1\\ 3.0881 r_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

or

$$\begin{aligned}\ddot{r}_1 + 0.16191 r_1 &= 0\\ \ddot{r}_2 + 3.0881 r_2 &= 0\end{aligned} \tag{5}$$

With IC given as

$$\boldsymbol{X}(0) = \begin{bmatrix}0\\1\end{bmatrix}$$

and

$$\dot{\boldsymbol{X}}(0) = \begin{bmatrix}0\\0\end{bmatrix}$$

Now $\mathbf{X} = M^{-\frac{1}{2}}\mathbf{q}$ and $\mathbf{q} = P\mathbf{r}$, hence $\mathbf{X} = M^{-\frac{1}{2}}P\mathbf{r}$, then

$$\begin{aligned}\boldsymbol{r}(0) &= P^T M^{\frac{1}{2}} \boldsymbol{X}(0)\\ \begin{bmatrix} r_1(0)\\ r_2(0)\end{bmatrix} &= \begin{bmatrix} 0.1735 & 0.98483\\ 0.98484 & -0.17353\end{bmatrix}^T \begin{bmatrix} 1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\\ \begin{bmatrix} r_1(0)\\ r_2(0)\end{bmatrix} &= \begin{bmatrix} 1.9697\\ -0.34706\end{bmatrix}\end{aligned}$$

now need to find $\ddot{\boldsymbol{r}}(0)$,but since $\ddot{\boldsymbol{X}}(0) = 0$, then $\ddot{\boldsymbol{r}}(0) = 0$ as well.

[1] This can also be found more quickly by noting that $\Lambda = diag(\lambda_1, \lambda_2)$

Now we can solve for $r_1(t)$ and $r_2(t)$ since we have the IC. From (5) above

$$\ddot{r}_1 + 0.161\,91 r_1 = 0$$
$$r_1(t) = A\cos\omega_{n_1}t + B\sin\omega_{n_1}t$$

At $t = 0, r_1(0) = 1.969\,6$, hence $1.969\,6 = A$, then

$$r_1(t) = 1.969\,6\cos\omega_{n_1}t + B\sin\omega_{n_1}t$$
$$\dot{r}_1(t) = -1.969\,6\omega_{n_1}\sin\omega_{n_1}t + \omega_{n1}B\cos\omega_{n_1}t$$

At $t = 0$

$$\dot{r}_1(t) = 0 = \omega_{n_1}B$$

Hence $B = 0$, then

$$r_1(t) = 1.969\,6\cos\omega_{n_1}t$$

But $\omega_{n_1} = \sqrt{0.161\,91} = 0.402\,38$, hence

$$\boxed{r_1(t) = 1.969\,6\cos(0.402\,38t)}$$

Similarly we find $r_2(t)$

$$\ddot{r}_2 + 3.088\,1 r_2 = 0$$
$$r_2(t) = A\cos\omega_{n_2}t + B\sin\omega_{n_2}t$$

At $t = 0, r_2(0) = -0.346\,98$, hence $-0.346\,98 = A$, then

$$r_2(t) = -0.346\,98\cos\omega_{n_2}t + B\sin\omega_{n_2}t$$
$$\dot{r}_2(t) = 0.346\,98\omega_{n_2}\sin\omega_{n_2}t + \omega_{n_2}B\cos\omega_{n_2}t$$

At $t = 0$

$$\dot{r}_2(t) = 0 = \omega_{n_2}B$$

Hence $B = 0$, then

$$r_2(t) = -0.346\,98\cos\omega_{n_2}t$$

But $\omega_{n_2} = \sqrt{3.088\,1} = 1.757\,3$, hence

$$\boxed{r_2(t) = -0.346\,98\cos(1.757\,3t)}$$

Now that we found the solution in the r space, we switch back to the original x space

$$\boldsymbol{X}(t) = M^{-\frac{1}{2}}P\boldsymbol{r}(t)$$

Then

$$\boldsymbol{X}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0.173\,5 & 0.984\,83 \\ 0.984\,84 & -0.173\,53 \end{bmatrix} \begin{bmatrix} 1.969\,6\cos(0.402\,38t) \\ -0.346\,98\cos(1.757\,3t) \end{bmatrix}$$

Hence

$$\boxed{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0.341\,73\cos 0.402\,38t - 0.341\,72\cos 1.757\,3t \\ 0.969\,87\cos 0.402\,38t + 0.03010\,6\cos 1.757\,3t \end{bmatrix}}$$

This is a plot of the solutions

Observation on final result: Notice that power of the harmonic $\omega_n = 1.7573$ rad/sec. in the motion $x_2(t)$ is small (amplitude is only 0.03) hence the dominant harmonic present in $x_2(t)$ is $\omega_n = 0.40238$ rad/sec. and this reflects in the plot where it appears that $x_2(t)$ contain one harmonic. In the case of $x_1(t)$ we see from the solution that both frequencies contribute equal amount of power, hence the plot for $x_1(t)$ reflects this.

3 Problem 3

Solution Use as generalized coordinates θ_1, θ_2,. Assume that the spring remain horizontal, and assume that $\theta_2 > \theta_1$

$$
\begin{aligned}
L &= T - U \\
T &= \frac{1}{2}m_1\left(L\dot{\theta}_1\right)^2 + \frac{1}{2}m_2\left(L\dot{\theta}_2\right)^2 \\
U_{gravity} &= m_1 gL\left(1-\cos\theta_1\right) + m_2 gL\left(1-\cos\theta_2\right) \\
U_{spring} &= \frac{1}{2}k\left(a\sin\theta_2 - a\sin\theta_1\right)^2
\end{aligned}
$$

Hence

$$
L = \frac{1}{2}m_1\left(L\dot{\theta}_1\right)^2 + \frac{1}{2}m_2\left(L\dot{\theta}_2\right)^2 - \left(m_1 gL\left(1-\cos\theta_1\right) + m_2 gL\left(1-\cos\theta_2\right) + \frac{1}{2}k\left(a\sin\theta_2 - a\sin\theta_1\right)^2\right)
$$

Now determine the Lagrangian equation

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}_1} &= m_1 L^2 \dot{\theta}_1 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} &= m_1 L^2 \ddot{\theta}_1 \\
\frac{\partial L}{\partial \dot{\theta}_2} &= m_2 L^2 \dot{\theta}_2 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} &= m_2 L^2 \ddot{\theta}_2 \\
\frac{\partial L}{\partial \theta_1} &= -m_1 g L \sin\theta_1 + ak\left(a\sin\theta_2 - a\sin\theta_1\right)\cos\theta_1 \\
\frac{\partial L}{\partial \theta_2} &= -m_1 g L \sin\theta_2 - ak\left(a\sin\theta_2 - a\sin\theta_1\right)\cos\theta_2
\end{aligned}$$

Hence the EQM for m_1 is

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} &= 0 \\
m_1 L^2 \ddot{\theta}_1 + m_1 g L \sin\theta_1 - ak\left(a\sin\theta_2 - a\sin\theta_1\right)\cos\theta_1 &= 0
\end{aligned}$$

Now apply small angle approximation. $\sin\theta \approx \theta$ and $\cos\theta \approx 1$hence

$$\begin{aligned}
m_1 L^2 \ddot{\theta}_1 + m_1 g L \theta_1 - ak\left(a\theta_2 - a\theta_1\right) &= 0 \\
m_1 L^2 \ddot{\theta}_1 + m_1 g L \theta_1 - a^2 k \theta_2 + a^2 k \theta_1 &= 0
\end{aligned}$$

$$m_1 L^2 \ddot{\theta}_1 + \left(m_1 g L + a^2 k\right)\theta_1 - a^2 k \theta_2 = 0 \tag{1}$$

And the EQM for m_2 is

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} &= 0 \\
m_2 L^2 \dot{\theta}_2 + m_1 g L \sin\theta_2 + ak\left(a\sin\theta_2 - a\sin\theta_1\right)\cos\theta_2 &= 0
\end{aligned}$$

Now apply small angle approximation. $\sin\theta \approx \theta$ and $\cos\theta \approx 1$hence

$$\begin{aligned}
m_2 L^2 \dot{\theta}_2 + m_1 g L \theta_2 + ak\left(a\theta_2 - a\theta_1\right) &= 0 \\
m_2 L^2 \dot{\theta}_2 + m_1 g L \theta_2 + a^2 k \theta_2 - a^2 k \theta_1 &= 0
\end{aligned}$$

Therefore

$$m_2 L^2 \ddot{\theta}_2 + \theta_2\left(m_2 g L + a^2 k\right) - a^2 k \theta_1 = 0$$

Now we write the system as $M\ddot{\boldsymbol{\theta}} + K\boldsymbol{\theta} = 0$

$$\begin{bmatrix} m_1 L^2 & 0 \\ 0 & m_2 L^2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} m_1 g L + a^2 k & -a^2 k \\ -a^2 k & m_2 g L + a^2 k \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Substitute numerical values for the above quantities, we obtain

$$\begin{bmatrix}10\times 0.5^2 & 0\\ 0 & 10\times 0.5^2\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix}+\begin{bmatrix}10\times 9.8\times 0.5+0.1^2\times 20 & -a^2\times 20\\ -0.1^2\times 20 & 10\times 9.8\times 0.5+0.1^2\times 20\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}2.5 & 0\\ 0 & 2.5\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix}+\begin{bmatrix}49.2 & -0.2\\ -0.2 & 49.2\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

The above can be written as

$$M\ddot{\boldsymbol{\theta}}+K\boldsymbol{\theta}=0$$

Let $\boldsymbol{\theta}=M^{-\frac{1}{2}}\boldsymbol{q}$, then $\ddot{\boldsymbol{\theta}}=M^{-\frac{1}{2}}\ddot{\boldsymbol{q}}$ and the above equation becomes

$$MM^{-\frac{1}{2}}\ddot{\boldsymbol{\theta}}+KM^{-\frac{1}{2}}\boldsymbol{\theta}=0$$

premultiply by $M^{-\frac{1}{2}}$ we obtain

$$M^{-\frac{1}{2}}MM^{-\frac{1}{2}}\ddot{\boldsymbol{\theta}}+M^{-\frac{1}{2}}KM^{-\frac{1}{2}}\boldsymbol{\theta}=0$$

$$I\ddot{\boldsymbol{\theta}}+\tilde{K}\boldsymbol{\theta}=0 \tag{2}$$

Where $\tilde{K}=M^{-\frac{1}{2}}KM^{-\frac{1}{2}}$

Let $\boldsymbol{q}=\boldsymbol{v}e^{i\omega t}$, then $\ddot{\boldsymbol{q}}=-\omega^2\boldsymbol{v}e^{i\omega t}$ and (2) becomes

$$-\omega^2 e^{i\omega t}I\boldsymbol{v}+\tilde{K}\boldsymbol{v}e^{i\omega t}=0$$

$$\left(\tilde{K}-\omega^2 I\right)\boldsymbol{v}=0$$

Let $\lambda=\omega^2$ then we have

$$\left(\tilde{K}-\lambda I\right)\boldsymbol{v}=0 \tag{3}$$

For $\boldsymbol{v}\neq 0$, we requires that $\left|\tilde{K}-\lambda I\right|=0$ But

$$\tilde{K}=M^{-\frac{1}{2}}KM^{-\frac{1}{2}}$$

$$=\begin{bmatrix}2.5^{-\frac{1}{2}} & 0\\ 0 & 2.5^{-\frac{1}{2}}\end{bmatrix}\begin{bmatrix}49.2 & -0.2\\ -0.2 & 49.2\end{bmatrix}\begin{bmatrix}2.5^{-\frac{1}{2}} & 0\\ 0 & 2.5^{-\frac{1}{2}}\end{bmatrix}$$

$$=\boxed{\begin{bmatrix}19.68 & -0.08\\ -0.08 & 19.68\end{bmatrix}}$$

Hence

$$\left|\tilde{K}-\lambda I\right|=0$$

$$\left\|\begin{bmatrix}19.68 & -0.08\\ -0.08 & 19.68\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right\|=0$$

$$\left\|\begin{bmatrix}19.68-\lambda & -0.08\\ -0.08 & 19.68-\lambda\end{bmatrix}\right\|=0$$

$$(19.68-\lambda)(19.68-\lambda)-0.08^2=0$$

Hence the characteristic equation is

$$\boxed{\lambda^2 - 39.36\ \lambda + 387.30 = 0}$$

Hence

$$\boxed{\lambda_{1,2} = 19.6, 19.76}$$

Hence the natural frequencies are

$$\begin{aligned}\omega_n &= \left\{\sqrt{19.6}, \sqrt{19.76}\right\} \\ &= \boxed{\{4.4272, 4.4452\} \text{ rad/sec}}\end{aligned}$$

From (3) we then have

$$\begin{aligned}\left(\tilde{K} - \lambda I\right) v &= 0 \\ \left(\begin{bmatrix} 19.68 & -0.08 \\ -0.08 & 19.68 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) v &= 0\end{aligned}$$

When $\lambda = \lambda_1 = 19.6$ we obtain

$$\begin{aligned}\left(\begin{bmatrix} 19.68 & -0.08 \\ -0.08 & 19.68 \end{bmatrix} - \begin{bmatrix} 19.6 & 0 \\ 0 & 19.6 \end{bmatrix}\right)\begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0.08 & -0.08 \\ -0.08 & 0.08 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}\end{aligned}$$

Hence

$$\begin{aligned}0.08a - 0.08b &= 0 \\ -0.08a + 0.08b &= 0\end{aligned}$$

Hence $a = b$ then

$$v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.70711 \\ 0.70711 \end{bmatrix}$$

When $\lambda = \lambda_2 = 19.76$ we obtain

$$\begin{aligned}\left(\begin{bmatrix} 19.68 & -0.08 \\ -0.08 & 19.68 \end{bmatrix} - \begin{bmatrix} 19.76 & 0 \\ 0 & 19.76 \end{bmatrix}\right)\begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -0.08 & -0.08 \\ -0.08 & -0.08 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}\end{aligned}$$

Hence $a = -b$, then

$$v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.70711 \\ 0.70711 \end{bmatrix}$$

Now that we have obtained the eigenvectors of the de-coupled system, we can plot the mode shapes[2]. I will use a diagram similar to that shown in the textbook Engineering Vibration

[2]The book also calls the S matrix as the shape matrix, so I better show this as well, which is defined as

by Inman on page 313)

$S = M^{-\frac{1}{2}}P$, hence

$$
\begin{aligned}
P &= \begin{bmatrix} v_1 & v_2 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
S &= \begin{bmatrix} 2.5 & 0 \\ 0 & 2.5 \end{bmatrix}^{-\frac{1}{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 2.5^{-\frac{1}{2}} & 0 \\ 0 & 2.5^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \sqrt{2} \begin{bmatrix} 0.31623 & -0.31623 \\ 0.31623 & 0.31623 \end{bmatrix} \\
&= \boxed{\begin{bmatrix} 0.44722 & -0.44722 \\ 0.44722 & 0.44722 \end{bmatrix}}
\end{aligned}
$$

4 Problem 4

$$M\ddot{\boldsymbol{x}} + K\boldsymbol{x} = 0$$

Where $K = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}, M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$ Let $\boldsymbol{X} = M^{-\frac{1}{2}}\boldsymbol{q}$, then $\ddot{\boldsymbol{X}} = M^{-\frac{1}{2}}\ddot{\boldsymbol{q}}$ and the above equation becomes

$$MM^{-\frac{1}{2}}\ddot{\boldsymbol{q}} + KM^{-\frac{1}{2}}\boldsymbol{q} = 0$$

premultiply by $M^{-\frac{1}{2}}$ we obtain

$$M^{-\frac{1}{2}}MM^{-\frac{1}{2}}\ddot{\boldsymbol{q}} + M^{-\frac{1}{2}}KM^{-\frac{1}{2}}\boldsymbol{q} = 0$$

$$I\ddot{\boldsymbol{q}} + \tilde{K}\boldsymbol{q} = 0 \tag{2}$$

Where $\tilde{K} = M^{-\frac{1}{2}}KM^{-\frac{1}{2}}$

Let $\boldsymbol{q} = \boldsymbol{v}e^{i\omega t}$, then $\ddot{\boldsymbol{q}} = -\omega^2\boldsymbol{v}e^{i\omega t}$ and (2) becomes

$$-\omega^2 e^{i\omega t}I\boldsymbol{v} + \tilde{K}\boldsymbol{v}e^{i\omega t} = 0$$

$$\left(\tilde{K} - \omega^2 I\right)\boldsymbol{v} = 0$$

Let $\lambda = \omega^2$ then we have

$$\left(\tilde{K} - \lambda I\right)\boldsymbol{v} = 0 \tag{3}$$

For $v \neq 0$, we requires that $\left|\tilde{K} - \lambda I\right| = 0$ But

$$\begin{aligned}\tilde{K} &= M^{-\frac{1}{2}} K M^{-\frac{1}{2}} \\ &= \begin{bmatrix} 9^{-\frac{1}{2}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 9^{-\frac{1}{2}} & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3.0 & -1.0 \\ -1.0 & 3.0 \end{bmatrix}\end{aligned}$$

Hence

$$\begin{aligned}\left|\tilde{K} - \lambda I\right| &= 0 \\ \left\| \begin{bmatrix} 3.0 & -1.0 \\ -1.0 & 3.0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\| &= 0 \\ \left\| \begin{bmatrix} 3 - \lambda & -1 \\ -1 & 3 - \lambda \end{bmatrix} \right\| &= 0 \\ (3 - \lambda)^2 - 1 &= 0\end{aligned}$$

Hence the characteristic equation is

$$\boxed{\lambda^2 - 6\lambda + 8 = 0}$$

Hence

$$\boxed{\lambda_{1,2} = \{2, 4\}}$$

Then the natural frequencies are

$$\omega_n = \left\{\sqrt{2}, 2\right\}$$

From (3) we then have

$$\begin{aligned}\left(\tilde{K} - \lambda I\right) v &= 0 \\ \left(\begin{bmatrix} 3.0 & -1.0 \\ -1.0 & 3.0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) v &= 0\end{aligned}$$

When $\lambda = \lambda_1 = 2$ we obtain

$$\begin{aligned}\left(\begin{bmatrix} 3.0 & -1.0 \\ -1.0 & 3.0 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}\end{aligned}$$

Hence

$$\begin{aligned}a - b &= 0 \\ -a + b &= 0\end{aligned}$$

Then $a = b$, hence

$$\boldsymbol{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0.70711\\0.70711\end{bmatrix}$$

When $\lambda = \lambda_2 = 4$ we obtain

$$\left(\begin{bmatrix}3.0 & -1.0\\-1.0 & 3.0\end{bmatrix} - \begin{bmatrix}4 & 0\\0 & 4\end{bmatrix}\right)\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$
$$\begin{bmatrix}-1 & -1\\-1 & -1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Hence $a = -b$, then

$$\boldsymbol{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}0.70711\\-0.70711\end{bmatrix}$$

Then the matrix

$$\begin{aligned}[P] &= \begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2\end{bmatrix}\\ &= \boxed{\begin{bmatrix}0.70711 & 0.70711\\0.70711 & -0.70711\end{bmatrix}}\end{aligned}$$

Now let $\boldsymbol{q} = P\boldsymbol{r}$, then equation (2) above becomes

$$\begin{aligned}I\ddot{\boldsymbol{q}} + \tilde{K}\boldsymbol{q} &= 0\\ IP\ddot{\boldsymbol{r}} + \tilde{K}P\boldsymbol{r} &= 0\end{aligned}$$

Premultiply by P^T

$$\begin{aligned}P^T IP\ddot{\boldsymbol{r}} + P^T\tilde{K}P\boldsymbol{r} &= 0\\ I\ddot{\boldsymbol{r}} + P^T\tilde{K}P\boldsymbol{r} &= 0\end{aligned}$$

Let $\Lambda = P^T\tilde{K}P$ then the above becomes

$$I\ddot{\boldsymbol{r}} + \Lambda\boldsymbol{r} = 0 \tag{4}$$

Now find Λ

$$\begin{aligned}\Lambda &= P^T\tilde{K}P\\ &= \frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}^T\begin{bmatrix}3 & -1\\-1 & 3\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\\ &= \boxed{\begin{bmatrix}2 & 0\\0 & 4\end{bmatrix}}\end{aligned}$$

Hence (4) becomes

$$I\ddot{\boldsymbol{r}}+\begin{bmatrix}2 & 0\\ 0 & 4\end{bmatrix}\boldsymbol{r}=0$$

Which can be written as 2 equations

$$\begin{bmatrix}\ddot{r}_1\\ \ddot{r}_2\end{bmatrix}+\begin{bmatrix}2r_1\\ 4r_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

or

$$\ddot{r}_1+2r_1=0 \tag{5}$$
$$\ddot{r}_2+4r_2=0 \tag{6}$$

With IC given as $\boldsymbol{X}(0)=\frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}$, $\dot{\boldsymbol{X}}(0)=\begin{bmatrix}0\\ 0\end{bmatrix}$, but

$\boldsymbol{X}=M^{-\frac{1}{2}}\boldsymbol{q}$ and $\boldsymbol{q}=P\boldsymbol{r}$, hence $\boldsymbol{X}=M^{-\frac{1}{2}}P\boldsymbol{r}$, then

$$\boldsymbol{r}(0)=P^TM^{\frac{1}{2}}\boldsymbol{X}(0)$$
$$\boldsymbol{r}(0)=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}^T\begin{bmatrix}9^{\frac{1}{2}} & 0\\ 0 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}$$
$$\begin{bmatrix}r_1(0)\\ r_2(0)\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix}$$

And since $\dot{\boldsymbol{X}}(0)=0$, then $\dot{\boldsymbol{r}}(0)=0$, now we have found IC for $r(t)$ we can solve the ODEs

$$r_1(t)=A_1\cos\sqrt{2}t+B_1\sin\sqrt{2}t$$
$$r_2(t)=A_2\cos 2t+B_2\sin 2t$$

$r_1(0)=1$ hence $A_1=1$, and $B_1=0$, similarly, $A_2=0$, and $B_1=0$, hence

$$r_1(t)=\cos\sqrt{2}t$$
$$r_2(t)=0$$

But

$$\boldsymbol{X}(t)=M^{-\frac{1}{2}}P\boldsymbol{r}(t)$$

Then

$$\boldsymbol{X}(t)=\begin{bmatrix}9^{-\frac{1}{2}} & 0\\ 0 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\boldsymbol{r}(t)$$
$$=\frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{3} & \frac{1}{3}\\ 1 & -1\end{bmatrix}\begin{bmatrix}\cos\sqrt{2}t\\ 0\end{bmatrix}$$

Hence

$$\boxed{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{3\sqrt{2}}\left(\cos\sqrt{2}t\right) \\ \frac{1}{\sqrt{2}}\left(\cos\sqrt{2}t\right) \end{bmatrix}}$$

Here is a plot of the solution

5 Problem 5 (not correct, left here to check something)

$m = 3, c = 6, k = 12$, hence $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{12}{3}} = 2$ rad/sec and $\xi = \frac{c}{c_{cr}} = \frac{c}{2\omega_n m} = \frac{6}{2\times2\times3} = \frac{1}{2}$, hence the system is underdamped and $\omega_d = \omega_n\sqrt{1-\xi^2} = 2\sqrt{1-\frac{1}{2}^2} = \sqrt{3}$ rad/sec

Let the response to $3\delta(t)$ be $x_{p_1}(t)$ and let the response to $\delta(t-1)$ be $x_{p_2}(t)$ hence the response of the system becomes

$$x(t) = x_h(t) + x_{p_1}(t) - x_{p_2}(t) \tag{1}$$

Where

$$x_h = e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) \tag{2}$$

And

$$x_{p_1}(t) = \frac{3}{m\omega_d}e^{-\xi\omega_n t}\sin\omega_d t \tag{3}$$

and

$$x_{p_2}(t) = \frac{1}{m\omega_d}e^{-\xi\omega_n(t-1)}\sin\omega_d(t-1)\Phi(t-1)$$

Hence, substitute (2),(3) into (1)

$$\begin{aligned} x(t) = e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + \frac{3}{m\omega_d}e^{-\xi\omega_n t}\sin\omega_d t \\ + \frac{1}{m\omega_d}e^{-\xi\omega_n(t-1)}\sin\omega_d(t-1)\Phi(t-1) \end{aligned} \tag{4}$$

Now using IC to find A, B. Note, we use only $x(t) = x_h(t) + x_{p_1}(t)$ for the purpose of finding A, B from I.C's since the response to the delayed impulse is not active at $t = 0$. We find

$$x(0) = \frac{1}{100} = A$$

And for the derivative

$$\begin{aligned}\dot{x}(t) &= \dot{x}_h(t) + \dot{x}_{p_1}(t)\\ &= -\xi\omega_n e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + e^{-\xi\omega_n t}(-A\omega_d\sin\omega_d t + B\omega_d\cos\omega_d t)\\ &+ \frac{3}{m\omega_d}e^{-\xi\omega_n t}\omega_d\cos\omega_d t - \frac{3\xi\omega_n}{m\omega_d}e^{-\xi\omega_n t}\sin\omega_d t\end{aligned}$$

Hence

$$\begin{aligned}\dot{x}(0) = 1 &= -\xi\omega_n A + B\omega_d + \frac{3}{m}\\ 1 &= -\frac{1}{100} + B\omega_d + 1\end{aligned}$$

Hence

$$B = \frac{1}{100\sqrt{3}}$$

Therefore the solution is, by substituting values found for A, B into the general solution from above equation (4), we obtain

$$x(t) = \frac{e^{-t}}{100}\left(\cos\sqrt{3}t + \frac{1}{100\sqrt{3}}\sin\sqrt{3}t\right) + \frac{1}{\sqrt{3}}e^{-t}\sin\sqrt{3}t - \left(\frac{1}{3\sqrt{3}}e^{-(t-1)}\sin\sqrt{3}(t-1)\Phi(t-1)\right) \tag{5}$$

The following is a plot of the solution for up to $t = 6$

6 Problem 5 (again, correct solution)

$m = 3, c = 6, k = 12$, hence $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{12}{3}} = 2$ rad/sec and $\xi = \frac{c}{c_{cr}} = \frac{c}{2\omega_n m} = \frac{6}{2\times2\times3} = \frac{1}{2}$, hence the system is underdamped and $\omega_d = \omega_n\sqrt{1-\xi^2} = 2\sqrt{1-\frac{1}{2}^2} = \sqrt{3}$ rad/sec

Let the response to $3\delta(t)$ be $x_{p_1}(t)$ and let the response to $\delta(t-1)$ be $x_{p_2}(t)$ hence the response of the system becomes

$$x(t) = x_h(t) + x_{p_1}(t) - x_{p_2}(t) \tag{1}$$

Where

$$x_h = e^{-\xi\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right) \tag{2}$$

And

$$x_{p_1}(t) = \frac{3}{m\omega_d}e^{-\xi\omega_n t}\sin\omega_d t \tag{3}$$

and

$$x_{p_2}(t) = \frac{1}{m\omega_d}e^{-\xi\omega_n(t-1)}\sin\omega_d(t-1)\Phi(t-1)$$

To find A, B use only $x_h(t)$ .At $t = 0$. We find

$$x(0) = \frac{1}{100} = A$$

And for the derivative

$$\begin{aligned}\dot{x}(t) &= \dot{x}_h(t)\\ &= -\xi\omega_n e^{-\xi\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right) + e^{-\xi\omega_n t}\left(-A\omega_d\sin\omega_d t + B\omega_d\cos\omega_d t\right)\end{aligned}$$

Hence

$$\dot{x}(0) = 1 = -\xi\omega_n A + B\omega_d$$
$$1 = -\frac{1}{100} + B\omega_d$$

Hence

$$B = \frac{1+\frac{1}{100}}{\sqrt{3}} = \frac{101}{100\sqrt{3}}$$

Therefore the solution is, by substituting values found for A, B into the general solution from above equation (4), we obtain

$$x(t) = \frac{e^{-t}}{100}\left(\cos\sqrt{3}t + \frac{101}{100\sqrt{3}}\sin\sqrt{3}t\right) + \frac{1}{\sqrt{3}}e^{-t}\sin\sqrt{3}t - \left(\frac{1}{3\sqrt{3}}e^{-(t-1)}\sin\sqrt{3}(t-1)\,\Phi(t-1)\right) \quad (5)$$

The following is a plot of the solution for up to $t = 6$

7 Problem 6

Let the response by $x(t)$. Hence $x(t) = x_h(t) + x_p(t)$, where $x_p(t)$ is the particular solution, which is the response due the the above forcing function. Using convolution

$$x_p(t) = \int_0^t f(\tau)\, h(t-\tau)\, d\tau$$

Where $h(t)$ is the unit impulse response of a second order underdamped system which is

$$h(t) = \frac{1}{m\omega_d} e^{-\xi\omega_n t} \sin \omega_d t$$

hence

$$\begin{aligned} x_p(t) &= \frac{F_0}{m\omega_d} \int_0^t \sin(\tau)\, e^{-\xi\omega_n(t-\tau)} \sin(\omega_d(t-\tau))\, d\tau \\ &= \frac{F_0 e^{-\xi\omega_n t}}{m\omega_d} \int_0^t e^{\xi\omega_n\tau} \sin(\tau) \sin(\omega_d(t-\tau))\, d\tau \end{aligned}$$

Using $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$ then

$$\sin(\tau)\sin(\omega_d(t-\tau)) = \frac{1}{2}[\cos(\tau - \omega_d(t-\tau)) - \cos(\tau + \omega_d(t-\tau))]$$

Then the integral becomes

$$x_p(t) = \frac{F_0 e^{-\xi\omega_n t}}{2m\omega_d} \left(\int_0^t e^{\xi\omega_n\tau} \cos(\tau - \omega_d(t-\tau))\, d\tau - \int_0^t e^{\xi\omega_n\tau} \cos(\tau + \omega_d(t-\tau))\, d\tau \right)$$

Consider the first integral I_1 where

$$I_1 = \int_0^t e^{\xi\omega_n\tau}\cos\left(\tau - \omega_d\left(t-\tau\right)\right)d\tau$$

Integrate by parts, where $\int u dv = uv - \int v du$, Let $dv = e^{\xi\omega_n\tau} \rightarrow v = \frac{e^{\xi\omega_n\tau}}{\xi\omega_n}$ and let $u = \cos\left(\tau - \omega_d\left(t-\tau\right)\right) \rightarrow du = -\left(1+\omega_d\right)\sin\left(\tau - \omega_d\left(t-\tau\right)\right)$, hence

$$\begin{aligned}
I_1 &= \left[\cos\left(\tau - \omega_d\left(t-\tau\right)\right)\frac{e^{\xi\omega_n\tau}}{\xi\omega_n}\right]_0^t - \int_0^t \frac{e^{\xi\omega_n\tau}}{\xi\omega_n}\left[-\left(1+\omega_d\right)\sin\left(\tau - \omega_d\left(t-\tau\right)\right)\right]d\tau \\
&= \left[\cos\left(t - \omega_d\left(t-t\right)\right)\frac{e^{\xi\omega_n t}}{\xi\omega_n} - \cos\left(0 - \omega_d\left(t-0\right)\right)\frac{1}{\xi\omega_n}\right] + \frac{\left(1+\omega_d\right)}{\xi\omega_n}\int_0^t e^{\xi\omega_n\tau}\sin\left(\tau - \omega_d\left(t-\tau\right)\right)d\tau \\
&= \frac{1}{\xi\omega_n}\left[\cos\left(t\right)e^{\xi\omega_n t} - \cos\left(\omega_d t\right)\right] + \frac{\left(1+\omega_d\right)}{\xi\omega_n}\int_0^t e^{\xi\omega_n\tau}\sin\left(\tau - \omega_d\left(t-\tau\right)\right)d\tau \qquad (1)
\end{aligned}$$

Integrate by parts again the last integral above, where $\int u dv = uv - \int v du$, Let $dv = e^{\xi\omega_n\tau} \rightarrow v = \frac{e^{\xi\omega_n\tau}}{\xi\omega_n}$ and let $u = \sin\left(\tau - \omega_d\left(t-\tau\right)\right) \rightarrow du = \left(1+\omega_d\right)\cos\left(\tau - \omega_d\left(t-\tau\right)\right)$, hence

$$\begin{aligned}
\int_0^t e^{\xi\omega_n\tau}\sin\left(\tau - \omega_d\left(t-\tau\right)\right)d\tau &= \left[\sin\left(\tau - \omega_d\left(t-\tau\right)\right)\frac{e^{\xi\omega_n\tau}}{\xi\omega_n}\right]_0^t - \int_0^t \frac{e^{\xi\omega_n\tau}}{\xi\omega_n}\left(1+\omega_d\right)\cos\left(\tau - \omega_d\left(t-\tau\right)\right)d\tau \\
&= \frac{1}{\xi\omega_n}\left[\sin\left(t\right)e^{\xi\omega_n t} + \sin\left(\omega_d t\right)\right] - \frac{\left(1+\omega_d\right)}{\xi\omega_n}\int_0^t e^{\xi\omega_n\tau}\cos\left(\tau - \omega_d\left(t-\tau\right)\right)d\tau \qquad (2)
\end{aligned}$$

Substitute (2) into (1) we obtain

$$\begin{aligned}
I_1 &= \frac{1}{\xi\omega_n}\left[\cos\left(t\right)e^{\xi\omega_n t} - \cos\left(\omega_d t\right)\right] + \\
&\quad \frac{\left(1+\omega_d\right)}{\xi\omega_n}\left(\frac{1}{\xi\omega_n}\left[\sin\left(t\right)e^{\xi\omega_n t} + \sin\left(\omega_d t\right)\right] - \frac{\left(1+\omega_d\right)}{\xi\omega_n}\int_0^t e^{\xi\omega_n\tau}\cos\left(\tau - \omega_d\left(t-\tau\right)\right)d\tau\right) \\
&= \frac{1}{\xi\omega_n}\left[\cos\left(t\right)e^{\xi\omega_n t} - \cos\left(\omega_d t\right)\right] + \frac{\left(1+\omega_d\right)}{\left(\xi\omega_n\right)^2}\left[\sin\left(t\right)e^{\xi\omega_n t} + \sin\left(\omega_d t\right)\right] - \frac{\left(1+\omega_d\right)^2}{\left(\xi\omega_n\right)^2}\int_0^t e^{\xi\omega_n\tau}\cos\left(\tau - \omega_d\left(t-\tau\right)\right)d\tau \\
&= \frac{1}{\xi\omega_n}\left[\cos\left(t\right)e^{\xi\omega_n t} - \cos\left(\omega_d t\right)\right] + \frac{\left(1+\omega_d\right)}{\left(\xi\omega_n\right)^2}\left[\sin\left(t\right)e^{\xi\omega_n t} + \sin\left(\omega_d t\right)\right] - \frac{\left(1+\omega_d\right)^2}{\left(\xi\omega_n\right)^2}I_1
\end{aligned}$$

Hence

$$I_1 + \frac{(1+\omega_d)^2}{(\xi\omega_n)^2} I_1 = \frac{1}{\xi\omega_n}\left[\cos(t)\, e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1+\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\, e^{\xi\omega_n t} + \sin(\omega_d t)\right]$$

$$I_1\left(\frac{(\xi\omega_n)^2 + (1+\omega_d)^2}{(\xi\omega_n)^2}\right) = \frac{1}{\xi\omega_n}\left[\cos(t)\, e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1+\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\, e^{\xi\omega_n t} + \sin(\omega_d t)\right]$$

$$I_1 = \left(\frac{(\xi\omega_n)^2}{(\xi\omega_n)^2 + (1+\omega_d)^2}\right)\left(\frac{1}{\xi\omega_n}\left[\cos(t)\, e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1+\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\, e^{\xi\omega_n t} + \sin(\omega_d t)\right]\right)$$

$$= \frac{\xi\omega_n\left[\cos(t)\, e^{\xi\omega_n t} - \cos(\omega_d t)\right] + (1+\omega_d)\left[\sin(t)\, e^{\xi\omega_n t} + \sin(\omega_d t)\right]}{(\xi\omega_n)^2 + (1+\omega_d)^2}$$

Now consider the second integral I_2 where

$$I_2 = \int_0^t e^{\xi\omega_n \tau} \cos(\tau + \omega_d(t-\tau))\, d\tau$$

Integrate by parts, where $\int u dv = uv - \int v du$, Let $dv = e^{\xi\omega_n\tau} \rightarrow v = \frac{e^{\xi\omega_n\tau}}{\xi\omega_n}$ and let $u = \cos(\tau + \omega_d(t-\tau)) \rightarrow du = -(1-\omega_d)\sin(\tau + \omega_d(t-\tau))$, hence

$$I_2 = \left[\cos(\tau + \omega_d(t-\tau))\frac{e^{\xi\omega_n\tau}}{\xi\omega_n}\right]_0^t - \int_0^t \frac{e^{\xi\omega_n\tau}}{\xi\omega_n}\left[-(1-\omega_d)\sin(\tau + \omega_d(t-\tau))\right] d\tau$$

$$= \left[\cos(t + \omega_d(t-t))\frac{e^{\xi\omega_n t}}{\xi\omega_n} - \cos(0 + \omega_d(t-0))\frac{1}{\xi\omega_n}\right] + \frac{(1-\omega_d)}{\xi\omega_n}\int_0^t e^{\xi\omega_n\tau}\sin(\tau + \omega_d(t-\tau))\, d\tau$$

$$= \frac{1}{\xi\omega_n}\left[\cos(t)\, e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1-\omega_d)}{\xi\omega_n}\int_0^t e^{\xi\omega_n\tau}\sin(\tau + \omega_d(t-\tau))\, d\tau \quad (3)$$

Integrate by parts again the last integral above, where $\int u dv = uv - \int v du$, Let $dv = e^{\xi\omega_n\tau} \rightarrow v = \frac{e^{\xi\omega_n\tau}}{\xi\omega_n}$ and let $u = \sin(\tau + \omega_d(t-\tau)) \rightarrow du = (1-\omega_d)\cos(\tau + \omega_d(t-\tau))$, hence

$$\int_0^t e^{\xi\omega_n\tau}\sin(\tau + \omega_d(t-\tau))\, d\tau = \left[\sin(\tau + \omega_d(t-\tau))\frac{e^{\xi\omega_n\tau}}{\xi\omega_n}\right]_0^t - \int_0^t \frac{e^{\xi\omega_n\tau}}{\xi\omega_n}(1-\omega_d)\cos(\tau + \omega_d(t-\tau))\, d\tau$$

$$= \frac{1}{\xi\omega_n}\left[\sin(t)\, e^{\xi\omega_n t} - \sin(\omega_d t)\right] - \frac{(1-\omega_d)}{\xi\omega_n}\int_0^t e^{\xi\omega_n\tau}\cos(\tau + \omega_d(t-\tau))\, d\tau \quad (4)$$

Substitute (4) into (3) we obtain

$$I_2 = \frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t} - \cos(\omega_d t)\right] +$$

$$\frac{(1-\omega_d)}{\xi\omega_n}\left(\frac{1}{\xi\omega_n}\left[\sin(t)\,e^{\xi\omega_n t} - \sin(\omega_d t)\right] - \frac{(1-\omega_d)}{\xi\omega_n}\int_0^t e^{\xi\omega_n\tau}\cos(\tau + \omega_d(t-\tau))\,d\tau\right)$$

$$= \frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1-\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\,e^{\xi\omega_n t} - \sin(\omega_d t)\right] - \frac{(1-\omega_d)^2}{(\xi\omega_n)^2}\int_0^t e^{\xi\omega_n\tau}\cos(\tau + \omega_d(t-\tau))\,d\tau$$

$$= \frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1-\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\,e^{\xi\omega_n t} + \sin(\omega_d t)\right] - \frac{(1-\omega_d)^2}{(\xi\omega_n)^2}I_2$$

Hence

$$I_2 + \frac{(1-\omega_d)^2}{(\xi\omega_n)^2}I_2 = \frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1-\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\,e^{\xi\omega_n t} + \sin(\omega_d t)\right]$$

$$I_2\left(\frac{(\xi\omega_n)^2 + (1-\omega_d)^2}{(\xi\omega_n)^2}\right) = \frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1-\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\,e^{\xi\omega_n t} + \sin(\omega_d t)\right]$$

$$I_2 = \left(\frac{(\xi\omega_n)^2}{(\xi\omega_n)^2 + (1-\omega_d)^2}\right)\left(\frac{1}{\xi\omega_n}\left[\cos(t)\,e^{\xi\omega_n t} - \cos(\omega_d t)\right] + \frac{(1-\omega_d)}{(\xi\omega_n)^2}\left[\sin(t)\,e^{\xi\omega_n t} + \sin(\omega_d t)\right]\right)$$

$$= \frac{\xi\omega_n\left[\cos(t)\,e^{\xi\omega_n t} - \cos(\omega_d t)\right] + (1-\omega_d)\left[\sin(t)\,e^{\xi\omega_n t} + \sin(\omega_d t)\right]}{(\xi\omega_n)^2 + (1-\omega_d)^2}$$

Using the above expressions for I_1, I_2, we find (and multiplying the solution by $(\Phi(t) - \Phi(t-\pi))$ since the force is only active from $t = 0$ to $t = \pi$, we obtain

$$x_p(t) = \frac{F_0 e^{-\xi\omega_n t}}{2m\omega_d}(I_1 - I_2)(\Phi(t) - \Phi(t-\pi))$$

$$= (\Phi(t) - \Phi(t-\pi)) *$$

$$\frac{F_0 e^{-\xi\omega_n t}}{2m\omega_d}\frac{\xi\omega_n\left[\cos(t)\,e^{\xi\omega_n t} - \cos(\omega_d t)\right] + (1+\omega_d)\left[\sin(t)\,e^{\xi\omega_n t} + \sin(\omega_d t)\right]}{(\xi\omega_n)^2 + (1+\omega_d)^2}$$

$$- \frac{F_0 e^{-\xi\omega_n t}}{2m\omega_d}\frac{\xi\omega_n\left[\cos(t)\,e^{\xi\omega_n t} - \cos(\omega_d t)\right] + (1-\omega_d)\left[\sin(t)\,e^{\xi\omega_n t} + \sin(\omega_d t)\right]}{(\xi\omega_n)^2 + (1-\omega_d)^2} \quad (5)$$

Hence $x_p(t) = (\Phi(t) - \Phi(t-\pi))$

$$\left[\frac{F_0 e^{-\xi\omega_n t}}{2m\omega_d}\left(\frac{\xi\omega_n\left[\cos(t)e^{\xi\omega_n t} - \cos(\omega_d t)\right] + (1+\omega_d)\left[\sin(t)e^{\xi\omega_n t} + \sin(\omega_d t)\right]}{(\xi\omega_n)^2 + (1+\omega_d)^2} - \frac{\xi\omega_n\left[\cos(t)e^{\xi\omega_n t} - \cos(\omega_d t)\right] + (1-\omega_d)\left[\sin(t)e^{\xi\omega_n t} + \sin(\omega_d t)\right]}{(\xi\omega_n)^2 + (1-\omega_d)^2}\right)\right]$$

And

$$x_h(t) = e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t)$$

Hence the overall solution is

$$x(t) = e^{-\xi\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right) + x_p(t)$$

The above solution is a bit long due to integration by parts. I will not solve the same problem using Laplace transformation method. The differential equation is

$$\ddot{x}(t) + 2\xi\omega_n\dot{x}(t) + \omega_n^2 x(t) = f(t)$$

Take Laplace transform, we obtain (assuming $x(0) = x_0$ and $\dot{x}(0) = v_0$)

$$\left(s^2X - sx(0) - \dot{x}(0)\right) + 2\xi\omega_n\left(sX - x(0)\right) + \omega_n^2 X = F(s)$$
$$\left(s^2X - sx_0 - v_0\right) + 2\xi\omega_n\left(sX - x_0\right) + \omega_n^2 X = F(s) \tag{7}$$

Now we find Laplace transform of $f(t)$

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)\,dt \\
&= \int_0^\pi e^{-st} F_0 \sin t\ dt \\
&= F_0\left[\int_0^\pi e^{-st}\sin t\ dt\right]
\end{aligned}$$

Integration by parts gives

$$F(s) = F_0\left[\frac{1 + e^{-\pi s}}{1 + s^2}\right] \tag{8}$$

Substitute (8) into (7) we obtain

$$\begin{aligned}
\left(s^2X - sx_0 - v_0\right) + 2\xi\omega_n\left(sX - x_0\right) + \omega_n^2 X &= F_0\left[\frac{1 + e^{-\pi s}}{1 + s^2}\right] \\
X\left(s^2 + 2\xi\omega_n s + \omega_n^2\right) - sx_0 - v_0 - 2\xi\omega_n x_0 &= \frac{F_0\left(1 + e^{-\pi s}\right)}{1 + s^2} \\
X\left(s^2 + 2\xi\omega_n s + \omega_n^2\right) &= \frac{F_0\left(1 + e^{-\pi s}\right)}{1 + s^2} + sx_0 + v_0 + 2\xi\omega_n x_0 \\
&= \frac{F_0\left(1 + e^{-\pi s}\right) + \left(1 + s^2\right)sx_0 + v_0\left(1 + s^2\right) + 2\xi\omega_n x_0\left(1 + s^2\right)}{1 + s^2}
\end{aligned}$$

Hence

$$\begin{aligned}
X &= \frac{F_0\left(1 + e^{-\pi s}\right) + \left(1 + s^2\right)sx_0 + v_0\left(1 + s^2\right) + 2\xi\omega_n x_0\left(1 + s^2\right)}{\left(1 + s^2\right)\left(s^2 + 2\xi\omega_n s + \omega_n^2\right)} \\
&= \frac{F_0 + v_0 + \frac{F_0}{e^{\pi s}} + sx_0 + s^2 v_0 + s^3 x_0 + 2\xi\omega_n x_0 + 2s^2\xi\omega_n x_0}{\left(1 + s^2\right)\left(s^2 + 2\xi\omega_n s + \omega_n^2\right)}
\end{aligned}$$

Now we can use inverse Laplace transform on the above. It is easier to do partial fraction decomposition and use tables. I used CAS to do this and this is the result. I plot the solution $x(t)$. I used the following values to be able to obtain a plot $\xi = 0.5, \omega_n = 2, F_0 = 10, x_0 = 1, v_0 = 0$

8 Solving problem shown in class for Vibration 431, CSUF, Spring 2009

Problem

Solve $\ddot{x} + 2\dot{x} + 4x = \delta(t) - \delta(t-4)$ with the IC's $x(0) = 1mm, \dot{x}(0) = -1mm$

Answer

$m = 1, c = 2, k = 4$, hence $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{4} = 2$ rad/sec and $\xi = \frac{c}{c_{cr}} = \frac{c}{2\omega_n m} = \frac{2}{2\times2\times1} = \frac{1}{2}$, hence the system is underdamped and $\omega_d = \omega_n\sqrt{1-\xi^2} = 2\sqrt{1-\frac{1}{2}^2} = \sqrt{3}$ rad/sec

Let the response to $\delta(t)$ be $x_1(t)$ and let the response to $\delta(t-4)$ be $x_2(t)$ hence the response of the system becomes

$$x(t) = x_h(t) + x_1(t) - x_2(t) \tag{1}$$

Where

$$x_h = e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) \tag{1}$$

And

$$x_1(t) = \frac{1}{m\omega_d}e^{-\xi\omega_n t}\sin\omega_d t \tag{3}$$

and

$$x_2(t) = \frac{1}{m\omega_d}e^{-\xi\omega_n(t-4)}\sin\omega_d(t-4)\,\Phi(t-4)$$

Hence, substitute (2),(3) ,(4) into (1)

$$x(t) = e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + \frac{1}{m\omega_d}e^{-\xi\omega_n t}\sin\omega_d t - \frac{1}{m\omega_d}e^{-\xi\omega_n(t-4)}\sin(\omega_d(t-4))\,\Phi(t-4) \tag{4}$$

Now using IC to find A, B

$$x(0) = 1 = A$$

and

$$\dot{x}(t) = -\xi\omega_n e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + e^{-\xi\omega_n t}(-A\omega_d\sin\omega_d t + B\omega_d\cos\omega_d t) + \frac{1}{m\omega_d}\left(-\xi\omega_n e^{-\xi\omega_n t}\sin\omega_d t + \omega_d e^{-\xi\omega_n t}\cos\omega_d t\right) - \frac{e^{-\xi\omega_n(t-4)}}{m\omega_d}(\omega_d\cos(\omega_d(t-4))\,\Phi(t-4) + \delta(t-4)\sin(\omega_d(t-4)) - \xi\omega_n\omega_d\sin(\omega_d(t-4))\,\Phi(t-4))$$

At $t = 0, \dot{x}(0) = -1$, Hence the above becomes (terms with $\delta(t-4)$ and $\Phi(t-4)$ vanish at

$t=0$ by definition)

$$-1 = -\xi\omega_n A + B\omega_d + \frac{1}{m}$$
$$B = \frac{-1}{\sqrt{3}}$$

Hence (1) becomes

$$x(t) = e^{-\xi\omega_n t}\left(\cos\omega_d t - \frac{1}{\sqrt{3}}\sin\omega_d t\right) + \frac{1}{m\omega_d}e^{-\xi\omega_n t}\sin\omega_d t - \frac{1}{m\omega_d}e^{-\xi\omega_n (t-4)}\sin\left(\omega_d(t-4)\right)\Phi(t-4)$$

If we substitute the numerical values for the problem parameters, the above becomes

$$x(t) = e^{-t}\left(\cos\sqrt{3}t - \frac{1}{\sqrt{3}}\sin\sqrt{3}t\right) + \frac{e^{-t}}{\sqrt{3}}\sin\sqrt{3}t - \frac{1}{\sqrt{3}}e^{-(t-4)}\sin\left(\sqrt{3}(t-4)\right)\Phi(t-4)$$
$$= \boxed{e^{-t}\cos\sqrt{3}t - \frac{1}{\sqrt{3}}e^{-(t-4)}\sin\left(\sqrt{3}(t-4)\right)\Phi(t-4)}$$

Compare the above with the solution given in class, which is

$$x(t) = \boxed{e^{-t}\left(\cos\sqrt{3}t + \frac{1}{\sqrt{3}}\sin\sqrt{3}t\right) - \frac{1}{\sqrt{3}}e^{-(t-4)}\sin\left(\sqrt{3}(t-4)\right)\Phi(t-4)}$$

9 Solving problem shown in class for Vibration 431, CSUF, Spring 2009. Version 2

Problem

Solve $\ddot{x} + 2\dot{x} + 4x = \delta(t) - \delta(t-4)$ with the IC's $x(0) = 1mm, \dot{x}(0) = -1mm$

Answer

$m = 1, c = 2, k = 4$, hence $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{4} = 2$ rad/sec and $\xi = \frac{c}{c_{cr}} = \frac{c}{2\omega_n m} = \frac{2}{2\times 2\times 1} = \frac{1}{2}$, hence the system is underdamped and $\omega_d = \omega_n\sqrt{1-\xi^2} = 2\sqrt{1-\frac{1}{2}^2} = \sqrt{3}$ rad/sec

Let the response to $\delta(t)$ be $x_{p_1}(t)$ and let the response to $\delta(t-4)$ be $x_{p_2}(t)$ hence the response of the system becomes

$$x(t) = x_h(t) + x_{p_1}(t) - x_{p_2}(t) \tag{1}$$

Where

$$x_h = e^{-\xi\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right) \tag{1}$$

And

$$x_{p_1}(t) = \frac{1}{m\omega_d}e^{-\xi\omega_n t}\sin\omega_d t \tag{3}$$

and

$$x_{p_2}(t) = \frac{1}{m\omega_d} e^{-\xi\omega_n(t-4)} \sin\omega_d(t-4)\,\Phi(t-4)$$

Hence, substitute (2),(3) ,(4) into (1)

$$x(t) = e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + \frac{1}{m\omega_d} e^{-\xi\omega_n t}\sin\omega_d t - \frac{1}{m\omega_d} e^{-\xi\omega_n(t-4)}\sin(\omega_d(t-4))\,\Phi(t-4) \tag{4}$$

Now using IC to find A, B

$$x(0) = 1$$

Hence

$$\boxed{A = 1}$$

Now take the derivative of the above and evaluate at zero to find B. In doing so, we need to consider only the x_h. The reason is that the particular solution $x_{p_2}(t)$ of the delayed pulse (the second pulse) will have no effect at $t = 0$ and the first pulse particular solution $x_{p_1}(t)$ will also have no contribution, since its response is assume to occur at 0^+, i.e. an infitismal time after $t = 0$. Therefore, since we intend to evaluate $\dot{x}(t)$ at $t = 0$, we only need to take x_h derivative at this point

$$\dot{x}(t) = -\xi\omega_n e^{-\xi\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + e^{-\xi\omega_n t}(-A\omega_d\sin\omega_d t + B\omega_d\cos\omega_d t)$$

At $t = 0, \dot{x}(0) = -1$, Hence the above becomes

$$-1 = -\xi\omega_n A + B\omega_d$$
$$-1 = -1 + B\sqrt{3}$$
$$\boxed{B = 0}$$

Hence (1) becomes

$$x(t) = e^{-\xi\omega_n t}\cos\omega_d t + \frac{1}{m\omega_d} e^{-\xi\omega_n t}\sin\omega_d t - \frac{1}{m\omega_d} e^{-\xi\omega_n(t-4)}\sin(\omega_d(t-4))\,\Phi(t-4)$$

If we substitute the numerical values for the problem parameters, the above becomes

$$x(t) = e^{-t}\cos\sqrt{3}t + \frac{e^{-t}}{\sqrt{3}}\sin\sqrt{3}t - \frac{1}{\sqrt{3}} e^{-(t-4)}\sin\left(\sqrt{3}(t-4)\right)\Phi(t-4)$$
$$= \boxed{e^{-t}\left(\cos\sqrt{3}t + \tfrac{1}{\sqrt{3}}\sin\sqrt{3}t\right) - \tfrac{1}{\sqrt{3}} e^{-(t-4)}\sin\left(\sqrt{3}(t-4)\right)\Phi(t-4)}$$

Which now matches the solution given in class