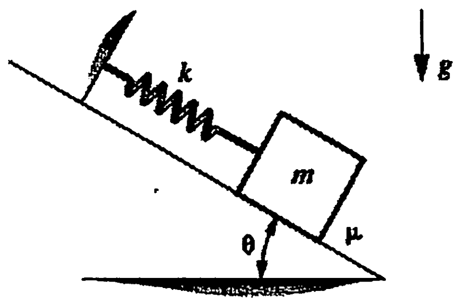
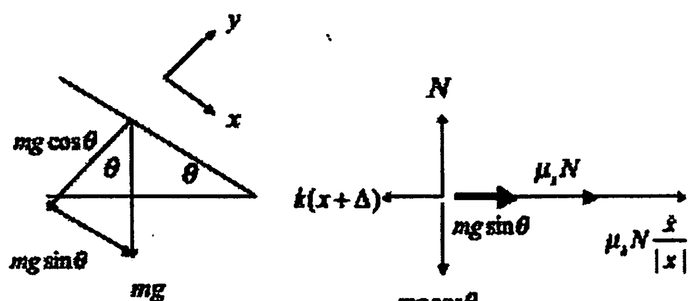
EGME 511

HW #2
SOLUTIONS

1.



Solution: Choose the x y coordinate system to be along the incline and perpendicular to it. Let μ_s denote the static friction coefficient, μ_k the coefficient of kinetic friction and Δ the static deflection of the spring. A drawing indicating the angles and a free-body diagram is given in the figure:



For the static case

$$\sum F_x = 0 \Rightarrow k\Delta = \mu_s N + mg\sin\theta, \text{ and } \sum F_y = 0 \Rightarrow N = mg\cos\theta$$

For the dynamic case

$$\sum F_x = m\ddot{x} = -k(x+\Delta) + \mu_s N + mg\sin\theta - \mu_k N\frac{\dot{x}}{|\dot{x}|}$$

Combining these three equations yields

$$m\ddot{x} + \mu_k mg\cos\theta\frac{\dot{x}}{|\dot{x}|} + kx = 0$$

Note that as the angle θ goes to zero the equation of motion becomes that of a spring mass system with Coulomb friction on a flat surface as it should.

2

Solution: Given:

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Calculate eigenvalues:

$$\det(\tilde{K} - \lambda I) = 0$$

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.25 \end{bmatrix}$$

$$\begin{vmatrix} 3-\lambda & -0.5 \\ -0.5 & 0.25-\lambda \end{vmatrix} = \lambda^2 - 3.25\lambda + 0.5 = 0$$

$$\lambda_{1,2} = 0.162, 3.088$$

The spectral matrix is

$$\Lambda = \text{diag}(\lambda_i) = \begin{bmatrix} 0.162 & 0 \\ 0 & 3.088 \end{bmatrix}$$

Calculate eigenvectors and normalize them:

$$\lambda_1 = 0.162$$

$$\begin{bmatrix} 2.838 & -0.5 \\ -0.5 & 0.088 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = 0 \Rightarrow v_{11} = 1.762v_{21}$$

$$\|v_1\| = \sqrt{v_{11}^2 + v_{21}^2} = \sqrt{(0.1762)^2 v_{21}^2 + v_{21}^2} = 1.015v_{21} = 1$$

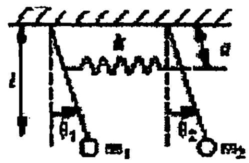
$$v_{21} = 0.9848 \text{ and } v_{11} = 0.1735 \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 0.1735 \\ 0.9848 \end{bmatrix}$$

$$\lambda_2 = 3.088$$

$$\begin{bmatrix} -0.088 & -0.5 \\ -0.5 & -2.838 \end{bmatrix}\begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = 0 \Rightarrow v_{12} = 1.762v_{22}$$

$$\|v_2\| = \sqrt{v_{12}^2 + v_{22}^2} = \sqrt{(-5.676)^2 v_{22}^2 + v_{22}^2} = 5.764v_{22} = 1$$

$$\Rightarrow v_{22} = 0.1735 \text{ and } v_{12} = -0.9848 \Rightarrow \mathbf{v}_2 = \begin{bmatrix} -0.9848 \\ 0.1735 \end{bmatrix}$$



3 **Solution:** Given:

$$k = 20 \text{ N/m} \quad m_1 = m_2 = 10 \text{ kg}$$
$$a = 0.1 \text{ m} \qquad l = 0.5 \text{ m}$$

For gravity use $g = 9.81$ m/s^2. For a mass on a pendulum, the inertia is: $I = ml^2$
Calculate mass and stiffness matrices (for small θ). The equations of motion are:

$$\begin{aligned} I_1\ddot{\theta}_1 &= ka^2(\theta_2-\theta_1) - m_1gl\theta_1 \\ I_2\ddot{\theta}_2 &= -ka^2(\theta_2-\theta_1) - m_2gl\theta_2 \end{aligned} \Rightarrow ml^2\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} + \begin{bmatrix} mgl+ka^2 & -ka^2 \\ -ka^2 & mgl+ka^2\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Substitution of the given values yields:

$$\begin{bmatrix} 2.5 & 0 \\ 0 & 2.5\end{bmatrix}\ddot{\theta} + \begin{bmatrix} 49.05 & -0.2 \\ -0.2 & 49.05\end{bmatrix}\theta = 0$$

Natural frequencies:

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 19.7 & -0.08 \\ -0.08 & 19.7\end{bmatrix}$$

$$\Rightarrow \lambda_1 = 19.54 \text{ and } \lambda_2 = 19.7 \Rightarrow \omega_1 = 4.42 \text{ rad/s and } \omega_2 = 4.438 \text{ rad/s}$$

Eigenvectors:

$$\lambda_1 = 19.54$$

$$\begin{bmatrix} 0.08 & -0.08 \\ -0.08 & 0.08\end{bmatrix}\begin{bmatrix}v_{11}\\ v_{12}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \quad \mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

$$\lambda_2 = 19.7$$

$$\begin{bmatrix} -0.08 & -0.08 \\ -0.08 & -0.08\end{bmatrix}\begin{bmatrix}v_{21}\\ v_{22}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

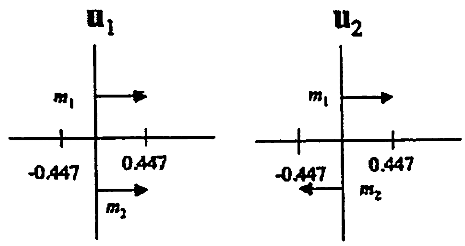
Now, $P = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$

Mode shapes:

$$\mathbf{u}_1 = M^{-1/2}\mathbf{v}_1 = \begin{bmatrix} 0.4472 \\ 0.4472 \end{bmatrix}$$

$$\mathbf{u}_2 = M^{-1/2}\mathbf{v}_2 = \begin{bmatrix} 0.4472 \\ -0.4472 \end{bmatrix}$$

A plot of the mode shapes is simply



This shows the first mode vibrates in phase and in the second mode the masses vibrate out of phase.

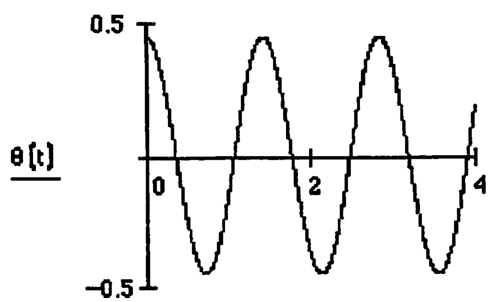
$$\theta(0) = \begin{bmatrix} 0.4472 \\ 0.4472 \end{bmatrix} \qquad \dot{\theta}(0) = 0, \; S = M^{-1/2}P = \begin{bmatrix} 0.4472 & 0.4472 \\ 0.4472 & -0.4472 \end{bmatrix}$$

$$S^{-1} = P^T M^{1/2} = \begin{bmatrix} 1.118 & 1.118 \\ 1.118 & -1.118 \end{bmatrix}, \; \mathbf{r}(0) = S^{-1}\theta(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \dot{\mathbf{r}}(0) = 0$$

$$r_1(t) = \sin\left(4.42t + \frac{\pi}{2}\right) = \cos 4.45t, \; r_2(t) = 0$$

Convert to physical coordinates: $\theta(t) = S\mathbf{r}(t) = \begin{bmatrix} 0.4472\cos 4.42t \\ 0.4472\cos 4.42t \end{bmatrix} \text{rad}$

θ (t) := 0.4472·cos (4.429·t)



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Solution:

$\omega_1 = \sqrt{\lambda_1} = 1.414$ rad/s, $\omega_2 = \sqrt{\lambda_2} = 2$ rad/s

$$P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow S = M^{-1/2}P = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 1 & -1 \end{bmatrix} \text{ and } S^{-1} = P^T M^{1/2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix}$$

Next compute the modal initial conditions

$$\mathbf{r}(0) = S^{-1}\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ and } \dot{\mathbf{r}}(0) = S^{-1}\dot{\mathbf{x}}(0) = \mathbf{0}$$

Modal solution :ror

$$\mathbf{r}(t) = \begin{bmatrix} \cos 1.414t \\ 0 \end{bmatrix}$$

Note that the second coordinate modal coordinate has zero initial conditions and is hence not vibrating. Convert this solution back into physical coordinates:

$$\mathbf{x}(t) = S\mathbf{r}(t) = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \cos 1.414t \\ 0 \end{bmatrix}$$

$$\Rightarrow \mathbf{x}(t) = \begin{bmatrix} 0.236\cos 1.414t \\ 0.707\cos 1.414t \end{bmatrix}$$

The unique feature about the solution is that both masses are vibrating at only one frequency. That is the frequency of the first mode shape. This is because the system is excited with a position vector equal to the first mode of vibration.

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Solution: First compute the natural frequency and damping ratio:

$$\omega_n = \sqrt{\frac{12}{3}} = 2 \text{ rad/s}, \ \zeta = \frac{6}{2\cdot 2\cdot 3} = 0.5, \ \omega_d = 2\sqrt{1-0.5^2} = 1.73 \text{ rad/s}$$

so that the system is underdamped. Next compute the responses to the two impulses:

$$x_1(t) = \frac{\hat{F}}{m\omega_d} e^{-\zeta\omega_n t}\sin\omega_d t = \frac{3}{3(1.73)} e^{-(t-1)}\sin 1.73(t-1) = 0.577e^{-t}\sin 1.73t, t>0$$

$$x_2(t) = \frac{\hat{F}}{m\omega_d} e^{-\zeta\omega_n (t-1)}\sin\omega_d (t-1) = \frac{1}{3(1.73)} e^{-t}\sin 1.73t = 0.193e^{-(t-1)}\sin 1.73(t-1), t>1$$

Now compute the response to the initial conditions

$$x_h(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

$$A = \sqrt{\frac{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}}, \quad \phi = \tan^{-1}\left[\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right] = 0.071 \text{ rad}$$

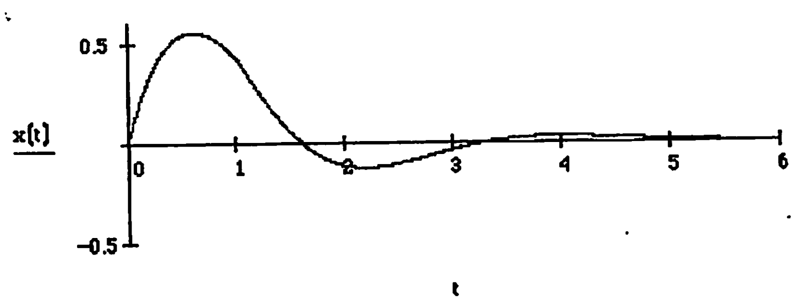
$$\Rightarrow x_h(t) = 0.5775e^{-t}\sin(t + 0.017)$$

Using the Heaviside function the total response is

$$x(t) = 0.577e^{-t}\sin 1.73t + 0.583e^{-t}\sin(t+0.017) + 0.193e^{-(t-1)}\sin 1.73(t-1)\Phi(t-1)$$

This is plotted below in Mathcad:

$$x(t) := \left(\frac{e^{-\zeta\cdot\omega n\cdot t}}{\omega d}\cdot\sin(\omega d\cdot t) + A\cdot e^{-\zeta\cdot\omega n\cdot t}\cdot\sin(\omega d\cdot t + \phi)\right) + \left[\frac{e^{-\zeta\cdot\omega n\cdot(t-1)}}{-3\cdot\omega d}\cdot\sin[\omega d\cdot(t-1)]\right]\cdot\Phi(t-1)$$



Note the slight bump in the response at $t = 1$ when the second impact occurs.

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Solution:

$$x(t)=\frac{1}{m\omega_d}e^{-\zeta\omega_n t}\int_0^t\left[F(\tau)e^{\zeta\omega_n\tau}\sin\omega_d(t-\tau)\right]d\tau$$

$$F(t)=F_0\sin(t) \quad , t<\pi \quad \text{(From Figure P3.16)}$$

$$\text{For } t\le\pi,\qquad x(t)=\frac{F_0}{m\omega_d}e^{-\zeta\omega_n t}\int_0^t\left(\sin\tau e^{\zeta\omega_n\tau}\sin\omega_d(t-\tau)\right)d\tau$$

$$x(t)=\frac{F_0}{m\omega_d}e^{-\zeta\omega_n t}\times$$

$$\left[\frac{1}{2\left[1+2\omega_d+\omega_n^2\right]}\left\{e^{\zeta\omega_n t}\left[(\omega_d-1)\sin t-\zeta\omega_n\cos t\right]-(\omega_d-1)\sin\omega_d t-\zeta\omega_n\cos\omega_d t\right\}\right.$$

$$\left.+\frac{1}{2\left[1+2\omega_d+\omega_n^2\right]}\left\{e^{\zeta\omega_n t}\left[(\omega_d-1)\sin t-\zeta\omega_n\cos t\right]+(\omega_d-1)\sin\omega_d t-\zeta\omega_n\cos\omega_d t\right\}\right]$$

$$\text{For } \tau>\pi,:\int_0^t f(\tau)h(t-\tau)d\tau=\int_0^\pi f(\tau)h(t-\tau)d\tau+\int_\pi^t(0)h(t-\tau)d\tau$$

$$x(t)=\frac{F_0}{m\omega_d}e^{-\zeta\omega_n t}\int_0^{\pi}\left(\sin\tau e^{\zeta\omega_n\tau}\sin\omega_d(t-\tau)\right)d\tau$$

$$=\frac{F_0}{m\omega_d}e^{-\zeta\omega_n t}\times$$

$$\left[\frac{1}{2\left[1+2\omega_d+\omega_n^{2}\right]}\left\{\begin{array}{r}e^{\zeta\omega_n t}\left[(\omega_d-1)\sin\left[\omega_d(t-\pi)\right]-\zeta\omega_n\cos\left[\omega_d(t-\pi)\right]\right]\\ -(\omega_d-1)\sin\omega_d t-\zeta\omega_n\cos\omega_d t\end{array}\right\}\right.$$

$$\left.+\frac{1}{2\left[1+2\omega_d+\omega_n^{2}\right]}\left\{\begin{array}{r}e^{\zeta\omega_n t}\left[(\omega_d+1)\sin\left[\omega_d(t-\tau)\right]+\zeta\omega\cos\left[\omega_d(t-\pi)\right]\right]\\ +(\omega_d-1)\sin\omega_d t-\zeta\omega_n\cos\omega_d t\end{array}\right\}\right]$$

Alternately, one could take a Laplace Transform approach and assume the under-damped system is a mass-spring-damper system of the form

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=F(t)$$

The forcing function given can be written as

$$F(t)=F_0\left(H(t)-H(t-\pi)\right)\sin(t)$$

Normalizing the equation of motion yields

$$\ddot{x}(t)+2\zeta\omega_n\dot{x}(t)+\omega_n^2x(t)=f_0\left(H(t)-H(t-\pi)\right)\sin(t)$$

where $f_0=\frac{F_0}{m}$ and m, c and k are such that $0<\zeta<1$.

Assuming initial conditions, transforming the equation of motion into the Laplace domain yields

$$X(s)=\frac{f_0\left(1+e^{-\pi s}\right)}{\left(s^2+1\right)\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}$$

The above expression can be converted to partial fractions

$$X(s)=f_0\left(1+e^{-\pi s}\right)\left(\frac{As+B}{s^2+1}\right)+f_0\left(1+e^{-\pi s}\right)\left(\frac{Cs+D}{s^2+2\zeta\omega_n s+\omega_n^2}\right)$$

where A, B, C, and D are found to be

$$A=\frac{-2\zeta\omega_n}{\left(1-\omega_n^2\right)^2+\left(2\zeta\omega_n\right)^2}$$

$$B=\frac{\omega_n^2-1}{\left(1-\omega_n^2\right)^2+\left(2\zeta\omega_n\right)^2}$$

$$C=\frac{2\zeta\omega_n}{\left(1-\omega_n^2\right)^2+\left(2\zeta\omega_n\right)^2}$$

$$D=\frac{\left(1-\omega_n^2\right)+\left(2\zeta\omega_n\right)^2}{\left(1-\omega_n^2\right)^2+\left(2\zeta\omega_n\right)^2}$$

Notice that $X(s)$ can be written more attractively as

$$\begin{aligned}X(s)&=f_0\left(\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+2\zeta\omega_n s+\omega_n^2}\right)+f_0e^{-\pi s}\left(\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+2\zeta\omega_n s+\omega_n^2}\right)\\&=f_0\left(G(s)+e^{-\pi s}G(s)\right)\end{aligned}$$

Performing the inverse Laplace Transform yields

$$x(t)=f_0\left(g(t)+H(t-\pi)g(t-\pi)\right)$$

where $g(t)$ is given below

$$g(t)=A\cos(t)+B\sin(t)+Ce^{-\zeta\omega_n t}\cos(\omega_d t)+\left(\frac{D-C\zeta\omega_n}{\omega_d}\right)e^{-\zeta\omega_n t}\sin(\omega_d t)$$

ω_d is the damped natural frequency, $\omega_d=\omega_n\sqrt{1-\zeta^2}$.

Let m=1 kg, c=2 kg/sec, k=3 N/m, and F_0=2 N. The system is solved numerically. Both exact and numerical solutions are plotted below

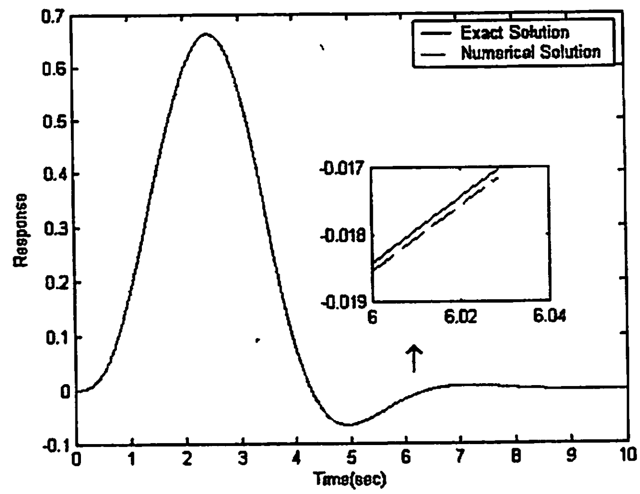

Figure 1 Analytical vs. Numerical Solutions

Below is the code used to solve this problem

```
% Establish a time vector
t=[0:0.001:10];

% Define the mass, spring stiffness and damping coefficient
m=1;
c=2;
k=3;

% Define the amplitude of the forcing function
F0=2;

% Calculate the natural frequency, damping ratio and normalized force amplitude
zeta=c/(2*sqrt(k*m));
wn=sqrt(k/m);
f0=F0/m;

% Calculate the damped natural frequency
wd=wn*sqrt(1-zeta^2);

% Below is the common denominator of A, B, C and D (partial fractions
% coefficients)
dummy=(1-wn^2)^2+(2*zeta*wn)^2;

% Hence, A, B, C, and D are given by
A=-2*zeta*wn/dummy;
B=(wn^2-1)/dummy;
C=2*zeta*wn/dummy;
```

```
D=((1-wn^2)+(2*zeta*wn)^2)/dummy;

% EXACT SOLUTION
%
**********************************************************************
*
%
**********************************************************************
*
for i=1:length(t)
   % Start by defining the function g(t)
   g(i)=A*cos(t(i))+B*sin(t(i))+C*exp(-zeta*wn*t(i))*cos(wd*t(i))+((D-
C*zeta*wn)/wd)*exp(-zeta*wn*t(i))*sin(wd*t(i));
   % Before t=pi, the response will be only g(t)
   if t(i)<pi
      xe(i)=f0*g(i);
      % d is the index of delay that will correspond to t=pi
      d=i;
   else
      % After t=pi, the response is g(t) plus a delayed g(t). The amount
      % of delay is pi seconds, and it is d increments
      xe(i)=f0*(g(i)+g(i-d));
   end;
end;

% NUMERICAL SOLUTION
%
**********************************************************************
*
%
**********************************************************************
*

% Start by defining the forcing function
for i=1:length(t)
   if t(i)<pi
      f(i)=f0*sin(t(i));
   else
      f(i)=0;
   end;
end;

% Define the transfer functions of the system
% This is given below
%        1
% ------------------------
```

```
% s^2+2*zeta*wn+wn^2

% Define the numerator and denominator
num=[1];
den=[1 2*zeta*wn wn^2];
% Establish the transfer function
sys=tf(num,den);

% Obtain the solution using lsim
xn=lsim(sys,f,t);

% Plot the results
figure;
set(gcf,'Color','White');
plot(t,xe,t,xn,'--');
xlabel('Time(sec)');
ylabel('Response');
legend('Forcing Function','Exact Solution','Numerical Solution');
text(6,0.05,'\uparrow','FontSize',18);
axes('Position',[0.55 0.3/0.8 0.25 0.25])
plot(t(6001:6030),xe(6001:6030),t(6001:6030),xn(6001:6030),'--');
```