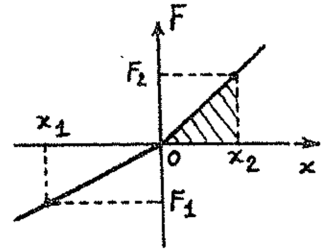
1.



T = kinetic energy at time zero $= \frac{1}{2} m (\dot{x}_0)^2$

Let x_2 = maximum displacement on right side.

V = potential energy in spring at displacement $x_2 = \frac{1}{2} k_2 x_2^2$ ($\dot{x}$ is zero at x_2)

Since $T = V$, $\quad x_2 = \sqrt{\frac{m(\dot{x}_0)^2}{k_2}} = \sqrt{\frac{m}{k_2}}\,\dot{x}_0$

Let x_1 = maximum displacement to left side. $\quad V = \frac{1}{2} k_1 x_1^2$

$T = V$ gives $\quad x_1 = \sqrt{\frac{m}{k_1}(\dot{x}_0)^2} = \sqrt{\frac{m}{k_1}}\,\dot{x}_0$

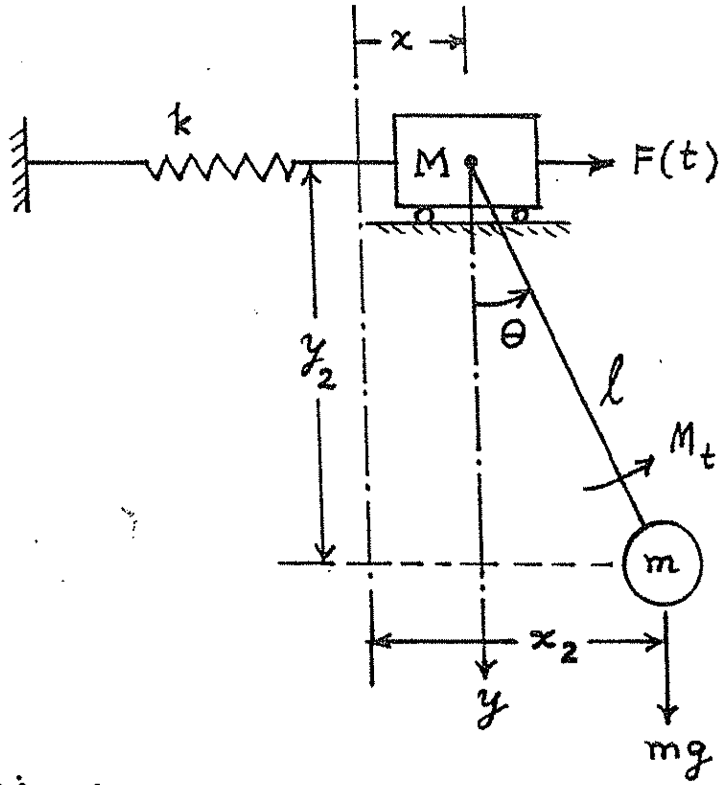
(a) Since $k_1 < k_2$, maximum deflection $= x_1 = \sqrt{\frac{m}{k_1}}\,\dot{x}_0$

(b) Period of vibration for a spring – mass system is $\tau_n = 2\pi\sqrt{\frac{m}{k}}$.

In the present case, τ_n = (time for m to go to $x = x_1$ from $x = 0$ and return to $x = 0$) + (time for m to go to $x = x_2$ from $x = 0$ and return to $x = 0$)

$$\therefore \tau_n = \pi\left(\sqrt{\frac{m}{k_1}} + \sqrt{\frac{m}{k_2}}\right)$$

2.



$x_2 = x + \ell \sin\theta$; $\dot{x}_2 = \dot{x} + \ell\,\dot{\theta}\cos\theta$

$y_2 = \ell\cos\theta$; $\dot{y}_2 = -\ell\,\dot{\theta}\sin\theta$

$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m\,(\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \left[(\dot{x} + \ell\,\dot{\theta}\cos\theta)^2 + (-\ell\,\dot{\theta}\sin\theta)^2\right]$$

$$= \frac{1}{2}(M+m)\,\dot{x}^2 + \frac{1}{2} m\,\ell^2\,\dot{\theta}^2 + m\,\ell\,\dot{x}\,\dot{\theta}\cos\theta$$

$$V = \frac{1}{2} k\,x^2 + m\,g\,\ell\,(1-\cos\theta)$$

$Q_x = F(t)$; $Q_\theta = M_t(t)$

Equations of motion:

$$(M+m)\,\ddot{x} + m\,\ell\,\ddot{\theta}\cos\theta - m\,\ell\,\dot{\theta}^2\sin\theta + k\,x = F(t) \qquad (1)$$

$$m\,\ell^2\,\ddot{\theta} + m\,\ell\,\ddot{x}\cos\theta - m\,\ell\,\dot{x}\,\dot{\theta}\sin\theta + m\,g\,\ell\sin\theta = M_t(t) \qquad (2)$$

Using the approximations

$$\cos\theta \approx 1 - \frac{\theta^2}{2} \;\; ; \;\; \sin\theta \approx \theta - \frac{\theta^3}{6}$$

Eqs. (1) and (2) can be expressed as

$$(M + m)\,\ddot{x} + m\,\ell\,\ddot{\theta} - \frac{1}{2}\,m\,\ell\,\theta^2\,\ddot{\theta} - m\,\ell\,\theta\,\dot{\theta}^2 + \frac{1}{6}\,m\,\ell\,\theta^3\,\dot{\theta}^2 + k\,x = F(t) \tag{3}$$

$$m\,\ell^2\,\ddot{\theta} + m\,\ell\,\ddot{x} - \frac{1}{2}\,m\,\ell\,\theta^2\,\ddot{x} - m\,\ell\,\theta\,\dot{\theta}\,\dot{x} + \frac{1}{6}\,m\,\ell\,\theta^3\,\dot{\theta}\,\dot{x} + m\,g\,\ell\,\theta - \frac{1}{6}\,m\,g\,\ell\,\theta^3 = M_t(t) \tag{4}$$

By neglecting the nonlinear terms, the linearized equations of motion can be written as

$$(M + m)\,\ddot{x} + m\,\ell\,\ddot{\theta} + k\,x = F(t) \tag{5}$$

$$m\,\ell^2\,\ddot{\theta} + m\,\ell\,\ddot{x} + m\,g\,\ell\,\theta = M_t(t) \tag{6}$$

3.

$$\ddot{\theta} + \omega_0^2 \left(\theta - \frac{1}{6}\theta^3\right) = 0 \qquad (E_1)$$

This equation is similar to Eq. (13.9) with

$x = \theta$, $a = \omega_0$, $F(x) = F(\theta) = \theta - \frac{1}{6}\theta^3$.

Eq. (E_1) can be rewritten as

$$\frac{d}{d\theta}\left(\dot{\theta}^2\right) + 2\omega_0^2\left(\theta - \frac{1}{6}\theta^3\right) = 0 \qquad (E_2)$$

which upon integration gives

$$\dot{\theta}^2 = 2\omega_0^2 \int_{\theta}^{\theta_0} F(\eta)\cdot d\eta = 2\omega_0^2 \int_{\theta}^{\theta_0} \left(\eta - \frac{1}{6}\eta^3\right)\cdot d\eta$$

$$= 2\omega_0^2 \left(\frac{1}{2}\eta^2 - \frac{1}{24}\eta^4\right)_{\theta}^{\theta_0} = \omega_0^2\left(\theta_0^2 - \frac{1}{12}\theta_0^4 - \theta^2 + \frac{1}{12}\theta^4\right) \qquad (E_3)$$

$$= \omega_0^2\left(\theta_0^2 - \theta^2\right)\left\{1 - \frac{1}{12}\left(\theta_0^2 + \theta^2\right)\right\} \qquad (E_4)$$

Since the maximum value of θ is Θ_0, we assume

$$\theta(t) = \Theta_0 \sin\beta \qquad (E_5)$$

Thus $$\theta_0^2 - \theta^2 = \theta_0^2 - \theta_0^2 \sin^2\beta = \theta_0^2 \cos^2\beta \qquad (E_6)$$

$$\theta_0^2 + \theta^2 = \theta_0^2\left(1 + \sin^2\beta\right) \qquad (E_7)$$

and $$\dot{\theta} = A_0 \cos\beta \frac{d\beta}{dt} \qquad (E_8)$$

Substitution of Eqs. (E_6) to (E_8) into (E_4) gives

$$\theta_0^2 \cos^2\beta \left(\frac{d\beta}{dt}\right)^2 = \omega_0^2\, \theta_0^2 \cos^2\beta \left\{1 - \frac{1}{12}\theta_0^2\left(1 + \sin^2\beta\right)\right\}$$

i.e.,

$$\left(\frac{d\beta}{dt}\right)^2 = \omega_0^2\left(1 - \frac{1}{12}\theta_0^2\right)\left\{1 - \frac{\theta_0^2 \sin^2\beta}{12\left(1 - \frac{1}{12}\theta_0^2\right)}\right\} \qquad (E_9)$$

Defining

$$a^2 = \frac{\theta_0^2}{12\left(1 - \frac{1}{12}\theta_0^2\right)} \qquad (E_{10})$$

Eq. (E_9) can be used to express (taking positive root):

$$\frac{d\beta}{dt} = \omega_0\left(1 - \frac{1}{12}\theta_0^2\right)^{\frac{1}{2}}\left(1 - a^2 \sin^2\beta\right)^{\frac{1}{2}} \qquad (E_{11})$$

i.e.,

$$\omega_0\left(1 - \frac{1}{12}\theta_0^2\right)^{\frac{1}{2}} dt = \int \frac{d\beta}{\sqrt{1 - a^2 \sin^2\beta}} \qquad (E_{12})$$

Integration of (E_{12}) yields

$$\omega_0\left(1 - \frac{1}{12}\theta_0^2\right)^{\frac{1}{2}}(t - t_0) = \int_{\beta_0}^{\beta} \frac{d\beta}{\sqrt{1 - a^2 \sin^2\beta}} \qquad (E_{13})$$

Using the initial conditions $\beta_0 = 0$ at $t_0 = 0$, Eq. (E_{13}) can be reduced to

$$\omega_0 \left(1 - \tfrac{1}{12}\,\theta_0^2\right)^{\frac{1}{2}} \cdot t = \int_0^{\beta} \frac{d\beta}{\sqrt{1 - a^2 \sin^2\beta}} = F(a, \beta) \qquad (E_{14})$$

where $F(a, \beta)$ is an incomplete elliptic integral of the first kind. Using $\beta = \frac{\pi}{2}$ when $\theta = \theta_0$ and $\beta = 0$ when $\theta = 0$, we get for one-quarter period,

$$\frac{\tau}{4} = t = \frac{1}{\omega_0 \left(1 - \tfrac{1}{12}\,\theta_0^2\right)^{\frac{1}{2}}} \cdot F\left(a, \frac{\pi}{2}\right) \qquad (E_{15})$$

Thus the time period of the pendulum is given by

$$\tau = \frac{4}{\omega_0 \left(1 - \tfrac{1}{12}\,\theta_0^2\right)^{\frac{1}{2}}} \cdot F\left(a, \frac{\pi}{2}\right) \qquad (E_{16})$$

4.

$$\ddot{x} + 0.1\,(x^2-1)\,\dot{x} + x = 0 \qquad \text{or} \qquad \ddot{x} = -\left[0.1\,(x^2-1)\,\dot{x} + x\right]$$

Let $x = x_1$. $\quad \dot{x}_1 = x_2 = f_1(x_1, x_2)$

$$\dot{x}_2 = -\left[0.1\,(x_1^2-1)\,x_2 + x_1\right] = f_2(x_1, x_2)$$

For equilibrium,

$$f_1 = 0 \Rightarrow x_2 = 0\ ; \quad f_2 = 0 \Rightarrow x_1 = 0$$

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

where

$$a_{11} = \left.\frac{\partial f_1}{\partial x_1}\right|_{(0,0)} = 0, \qquad a_{12} = \left.\frac{\partial f_1}{\partial x_2}\right|_{(0,0)} = 1,$$

$$a_{21} = \left.\frac{\partial f_2}{\partial x_1}\right|_{(0,0)} = -\left(0.2\,x_1\,x_2 + 1\right)\Big|_{(0,0)} = -1,$$

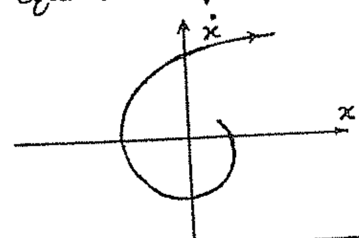
$$a_{22} = \left.\frac{\partial f_2}{\partial x_2}\right|_{(0,0)} = -\left[0.1\,(x_1^2-1)\right]\Big|_{(0,0)} = 0.1$$

we find $p = 0.1$, $q = 1$

$\lambda_1, \lambda_2 = \frac{1}{2}\left(0.1 \pm \sqrt{0.01-4}\right)$ = complex with positive real parts

Since $p > 0$, the system is unstable at the equilibrium point $(x, \dot{x}) = (0, 0)$.

Hence the phase-plane trajectory in the neighborhood of the equilibrium position appears as shown in the figure.



[EQ 4A]

[EQ 4B]

[EQ 4C]

5

Equation of motion: $\ddot{x} + f\frac{\dot{x}}{|\dot{x}|} + \omega_n^2 x = 0$ (E_1)

i.e. $\ddot{x} + \omega_n^2 (x + a) = 0$ for $\dot{x} > 0$ (E_2)

and $\ddot{x} + \omega_n^2 (x - a) = 0$ for $\dot{x} < 0$ (E_3)

where $a = f/\omega_n^2$ (E_4)

Multiplying by $2\dot{x}$ and integrating, (E_2) and (E_3) yield

$$\dot{x}^2 + \omega_n^2 (x+a)^2 = R_j^2 \quad \text{for } \dot{x} > 0 \qquad (E_5)$$

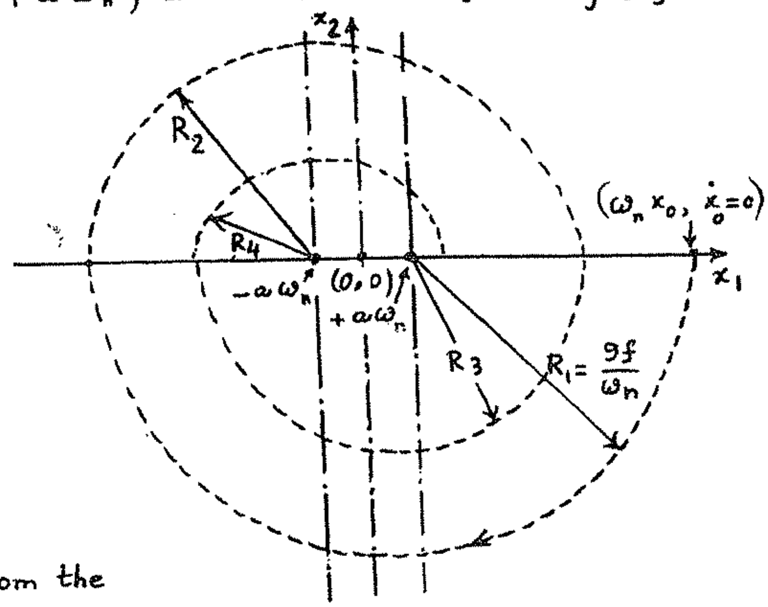
$$\dot{x}^2 + \omega_n^2 (x-a)^2 = R_{j+1}^2 \quad \text{for } \dot{x} < 0 \qquad (E_6)$$

where R_j^2 and R_{j+1}^2 are integration constants which are to be computed at each switching of sign of $\dot{x}$.

We can plot the trajectories of a representative point whose coordinates are

$$x_1 = \omega_n x \quad , \quad x_2 = \dot{x} \tag{E7}$$

Eqs. (E5) and (E6) show that the trajectory is made of semicircles whose centers are located at $x = -a$ (or $x_1 = -a\omega_n$) and $x = +a$ (or $x_1 = +a\omega_n$) as shown in the following figure.



R_1 can be obtained from the initial conditions using Eq. (E6) as:

$$R_1^2 = 0^2 + \omega_n^2 \left(\frac{10f}{\omega_n^2} - \frac{f}{\omega_n^2} \right)^2 = \left(\frac{9f}{\omega_n} \right)^2 \quad ; \quad R_1 = \frac{9f}{\omega_n} \tag{E8}$$

Notice that the radii of the circles $R_1, R_2, \ldots$ decrease according to the relation

$$R_j = R_{j-1} - 2a\omega_n \quad ; \quad j = 1, 2, \ldots$$

and the system will stop when

$$R_k \leq 2a\omega_n$$

Here $R_1 = \dfrac{9f}{\omega_n}$, $R_2 = R_1 - \dfrac{2f}{\omega_n} = \dfrac{7f}{\omega_n}$, $R_3 = R_2 - \dfrac{2f}{\omega_n} = \dfrac{5f}{\omega_n}$,

$R_4 = R_3 - \dfrac{2f}{\omega_n} = \dfrac{3f}{\omega_n}$, $R_5 = R_4 - \dfrac{2f}{\omega_n} = \dfrac{f}{\omega_n}$,

and the motion stops at this point (after five half-cycles) since $R_6 < 2a\omega_n = \dfrac{2f}{\omega_n}$.

6

$\ddot{\theta} + c\dot{\theta} + \sin\theta = 0$ or $\ddot{\theta} = -c\dot{\theta} - \sin\theta$

Let $x = \theta$ and $y = \frac{dx}{dt} = \dot{\theta}$

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -cy - \sin x \qquad (E_1)$$

Equilibrium or critical point (where $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$) of this system is $(x=0, y=0)$. Linearization of Eqs. (E_1) about the equilibrium point (origin) leads to

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -cy - x$$

or
$$\begin{Bmatrix} dx/dt \\ dy/dt \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} \qquad (E_2)$$

The eigenvalues of this system are given by

$$\left| \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

i.e. $\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda - c \end{vmatrix} = 0$ i.e., $\lambda^2 + \lambda c + 1 \equiv \lambda^2 + p\lambda + q = 0$

i.e., $\lambda_{1,2} = -\frac{c}{2} \pm \sqrt{\left(\frac{c}{2}\right)^2 - 1}$ (E_3)

If $c = 0$: $p = 0$; $q = 1$; $\lambda_{1,2} = \pm\sqrt{-1}$
The origin will be a center.

If $0 < c < 2$: $p > 0$; $q > 0$; $\lambda_{1,2} =$ complex conjugates
The origin will be a stable focal point (spiral point).

If $c = 2$: $p > 0$; $q > 0$; $\lambda_{1,2} =$ negative and equal.
The origin will be a stable nodal point.

If $c > 2$: $p > 0$; $q > 0$; If $\lambda_{1,2} =$ negative real, the origin will be a stable nodal point.

If $-2 < c < 0$: $p < 0$; $q > 0$; $\lambda_{1,2} =$ complex conjugates.
The origin will be an unstable focal point (spiral point).

7

Equation of motion: $\ddot{\theta} + 0.5\dot{\theta} + \sin\theta = 0.8$ (E_1)

Let $x = \theta$ and $y = dx/dt$

$\therefore \frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\sin x - 0.5y + 0.8$ (E_2)

$$\frac{dy}{dx} = \frac{-\sin x - 0.5y + 0.8}{y} \qquad (E_3)$$

At $(x = \sin^{-1} 0.8,\ y = 0)$, $\frac{dy}{dx} = \frac{0}{0}$ and hence it will be an equilibrium point. To investigate the nature of singularity, we rewrite Eqs. (E_2) in linearized form as

$$\left.\begin{aligned} \frac{dx}{dt} &= (0)x + (1)y \\ \frac{dy}{dt} &= (0)x - 0.5y \end{aligned}\right\} \qquad (E_4)$$

Thus the eigenvalues of the system are given by

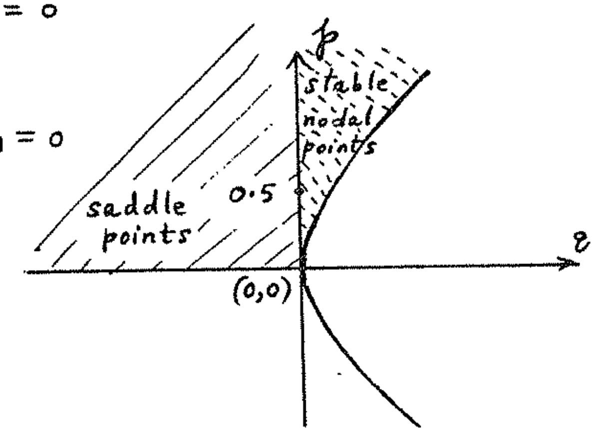
$$\left| \begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0 \quad \text{or} \quad \begin{vmatrix} -\lambda & 1 \\ 0 & -0.5-\lambda \end{vmatrix} = 0$$

i.e., $\lambda^2 + 0.5\lambda \equiv \lambda^2 + p\lambda + q = 0$

$\therefore\ \lambda_1 = 0,\ \lambda_2 =$ negative

Here $p =$ positive; $q = 0$, $\lambda_1 = 0$ and $\lambda_2 =$ negative.

Thus the equilibrium point falls on the border of saddle points and stable nodal points as shown in the adjacent figure.



8

$$\frac{dx}{dt} = (0)x + (1)y \qquad (E_1)$$

$$\frac{dy}{dt} = -1.x - cy + (0.1)x^3 \qquad (E_2)$$

Eqs. (E_1) and (E_2) are zero at $(x = 0,\ y = 0)$. Hence the origin $(0,0)$ will be equilibrium point (singularity). The eigenvalues are given by

$$\left| \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} -\lambda & 1 \\ -1 & -c-\lambda \end{vmatrix} = 0$$

i.e., $\lambda^2 + \lambda c + 1 \equiv \lambda^2 + p\lambda + q = 0$

i.e., $$\lambda_{1,2} = \left\{ \frac{-c \pm \sqrt{c^2 - 4}}{2} \right\}$$

For $c > 0$ and $c < 2$:

$p > 0$, $q > 0$ and $\lambda_{1,2}$ = complex conjugates.

Hence the origin will be a stable focus (or spiral point).

For $c \geq 2$:

$p > 0$, $q > 0$; $\lambda_{1,2}$ = negative real.

Hence the origin will be a stable nodal point.

9

Van der Pol's equation: $\ddot{x} - \alpha(1-x^2)\dot{x} + x = 0, \quad \alpha > 0$ (E_1)

Assume $x(t) = x_0(t) + \alpha\, x_1(t) + \alpha^2\, x_2(t)$ (E_2)

$\omega_0^2 = 1 = \omega^2 - \alpha\,\omega_1 - \alpha^2\,\omega_2$ (E_3)

where $\omega_0^2 = 1 =$ coefficient of x in Eq. (E_1).

Substitution of (E_2) and (E_3) into (E_1) gives

$$\alpha^0\left[\ddot{x}_0 + \omega^2 x_0\right] + \alpha^1\left[\ddot{x}_1 - \dot{x}_0 + \dot{x}_0\, x_0^2 - \omega_1 x_0 + \omega^2 x_1\right]$$
$$+ \alpha^2\left[\ddot{x}_2 - \dot{x}_1 + \dot{x}_1\, x_0^2 + 2\, x_0\, \dot{x}_0\, x_1 - \omega_2 x_0 - \omega_1 x_1 + \omega^2 x_2\right] \qquad (E_4)$$
$$+ \alpha^3[\cdots] + \cdots \quad = 0$$

Setting coefficient of α^0 in (E_4) to zero, we obtain

$\ddot{x}_0 + \omega^2 x_0 = 0$, i.e., $x_0(t) = A_1 \cos\omega t + A_2 \sin\omega t$ (E_5)

Assuming the initial conditions $x(0) = A$ and $\dot{x}(0) = 0$, we get $A_1 = A$ and $A_2 = 0$. Thus (E_5) reduces to

$$x_0(t) = A \cos\omega t \qquad (E_6)$$

Setting coefficient of α^1 to zero, in Eq. (E_4),

$$\ddot{x}_1 + \omega^2 x_1 = \dot{x}_0 - \dot{x}_0\, x_0^2 + \omega_1 x_0$$
$$= -A\omega \sin\omega t + A^3 \omega \sin\omega t . \cos^2\omega t + \omega_1 A \cos\omega t$$
$$= \left(-A\omega + \tfrac{1}{4} A^3 \omega\right)\sin\omega t + \omega_1 A \cos\omega t + \frac{A^3\omega}{4} \sin 3\omega t \qquad (E_7)$$

The coefficients of $\sin\omega t$ and $\cos\omega t$ must be zero in Eq. (E_7) to avoid secular terms. This gives

$$A = \pm 2 , \quad \omega_1 = 0 \qquad (E_8)$$

Thus the particular solution of (E_7) can be expressed as

$$x_1(t) = A_3 \sin 3\omega t + A_4 \cos 3\omega t \qquad (E_9)$$

Substitution of (E_8) and (E_9) into E_7 gives

$$A_3 = \frac{1}{32}\,\frac{A^3}{\omega} \quad \text{and} \quad A_4 = 0 \qquad (E_{10})$$

Thus $\quad x_1(t) = \dfrac{1}{32}\,\dfrac{A^3}{\omega} \sin 3\omega t \qquad (E_{11})$

Finally, setting coefficient of α^2 in (E_4) to zero, we get

$$\ddot{x}_2 + \omega^2 x_2 = \dot{x}_1 - \dot{x}_1\, x_0^2 - 2\, x_0\, \dot{x}_0\, x_1 + \omega_2\, x_0 + \omega_1 x_1 \qquad (E_{12})$$

Substitution of (E_{11}), (E_6) and (E_8) into (E_{12}) leads to

$$\ddot{x}_2 + \omega^2 x_2 = \frac{3}{32} A^3 \cos 3\omega t - \left(\frac{3}{32} A^3 \cos 3\omega t\right) A^2 \cos^2\omega t$$
$$- 2\left(A\cos\omega t\right)\left(-A\omega \sin\omega t\right)\left(\frac{A^3}{32\omega}\sin 3\omega t\right) + \omega_2 A \cos\omega t$$
$$= \left(-\frac{3}{128} A^5 + \frac{1}{64} A^5 + A\omega_2\right)\cos\omega t + \left(\frac{3}{32} A^3 - \frac{3}{64} A^5\right)\cos 3\omega t$$
$$+ \left(-\frac{3}{128} A^5 - \frac{1}{64} A^5\right)\cos 5\omega t \qquad (E_{13})$$

P12

To avoid secular terms, the coefficient of $\cos \omega t$ in (E_{13}) must be zero. This gives $\omega_2 = \frac{1}{128} A^4$ (E_{14})

With this, and using $A = 2$, Eq. (E_{13}) reduces to

$$\ddot{x}_2 + \omega^2 x_2 = -\frac{3}{4} \cos 3\omega t - \frac{5}{4} \cos 5\omega t \qquad (E_{15})$$

Assuming $x_2(t) = A_5 \cos 35t + A_6 \cos 5\omega t$ (E_{16})

we find, from Eq. (E_{15}),

$$A_5 = \frac{3}{32} \cdot \frac{1}{\omega^2}, \qquad A_6 = \frac{5}{96} \cdot \frac{1}{\omega^2} \qquad (E_{17})$$

$$\therefore \; x_2(t) = \frac{3}{32\omega^2} \cos 3\omega t + \frac{5}{96\omega^2} \cos 5\omega t \qquad (E_{18})$$

Thus the complete solution, Eqs. (E_2) and (E_3), become

$$x(t) = 2 \cos \omega t + \frac{\alpha}{4\omega} \sin 3\omega t + \frac{3\alpha^2}{32\omega^2} \cos 3\omega t + \frac{5\alpha^2}{96\omega^2} \cos 5\omega t$$

and $\omega^2 = 1 + \frac{\alpha^2}{8}$.