Impulse response of second order system which is not under-damped

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Abstract

The impulse response $h(t)$ for second order single degree of freedom system which is under-damped is well known. In this note, the derivation to the impulse response of critically damped and over-damped systems are given.

1 Impulse response for over-damped system

Given the system

$$\ddot{x}(t) + 2\xi\omega_n\dot{x}(t) + \omega_n^2 x(t) = \delta(t) \tag{1}$$

Where $\delta(t)$ is an impulse. We seek to find $x(t)$, the response of the above system to this impulse.

Assume the system is initially at rest. Due to the action of this impulse, the system will obtain an initial speed which is found as follows. Let $\delta(t) \equiv \hat{F} = F\Delta t$ where Δt is the duration of the impulse and F is the magnitude (in Newtons) of the impulse (hence units of $\hat{F}$ is N sec). This impulse will impart a momentum on the mass being hit which we use to determine the initial speed

$$\hat{F} = mv_0$$

$$v_0 = \frac{\hat{F}}{m}$$

Hence, the system will now have initial conditions of $x(0) = 0$ and $\dot{x}(0) = v_0 = \frac{\hat{F}}{m}$. Now, the response of (1), when $\xi > 1$ is known and given by

$$x(t) = e^{-\xi\omega_n t}\left(Ae^{\omega_n\sqrt{\xi^2-1}t} + Be^{-\omega_n\sqrt{\xi^2-1}t}\right) \tag{2}$$

Apply $x(0) = 0$, we obtain that $0 = A + B$ or $B = -A$. Now

$$\begin{aligned}\dot{x}(t) &= -\xi\omega_n e^{-\xi\omega_n t}\left(Ae^{\omega_n\sqrt{\xi^2-1}t} + Be^{-\omega_n\sqrt{\xi^2-1}t}\right)\\ &\quad + e^{-\xi\omega_n t}\left(A\omega_n\sqrt{\xi^2-1}e^{\omega_n\sqrt{\xi^2-1}t} - B\omega_n\sqrt{\xi^2-1}e^{-\omega_n\sqrt{\xi^2-1}t}\right)\end{aligned}$$

Apply $\dot{x}(0) = \frac{\hat{F}}{m}$ to the above, we obtain

$$\frac{\hat{F}}{m} = \left(A\omega_n\sqrt{\xi^2-1} - B\omega_n\sqrt{\xi^2-1}\right)$$

But $B = -A$, hence $\frac{\hat{F}}{m} = 2A\omega_n\sqrt{\xi^2-1}$ or $A = \frac{\hat{F}}{2m\omega_n\sqrt{\xi^2-1}}$

Hence (2) becomes

$$\begin{aligned}x(t) &= e^{-\xi\omega_n t}\left(\frac{\hat{F}}{2m\omega_n\sqrt{\xi^2-1}}e^{\omega_n\sqrt{\xi^2-1}t} - \frac{\hat{F}}{2m\omega_n\sqrt{\xi^2-1}}e^{-\omega_n\sqrt{\xi^2-1}t}\right)\\ &= \frac{\hat{F}}{2m\omega_n\sqrt{\xi^2-1}}e^{-\xi\omega_n t}\left(e^{\omega_n\sqrt{\xi^2-1}t} - e^{-\omega_n\sqrt{\xi^2-1}t}\right)\end{aligned}$$

When the magnitude of the impulse is unity, i.e. a unit impulse, hence $\hat{F} = 1$, then we obtain the unit impulse response

$$\boxed{h(t) = \frac{1}{2m\omega_n\sqrt{\xi^2-1}}e^{-\xi\omega_n t}\left(e^{\omega_n\sqrt{\xi^2-1}t} - e^{-\omega_n\sqrt{\xi^2-1}t}\right)}$$

2 Impulse response for critically damped system

The response of (1), when $\xi = 1$ is given by

$$x(t) = Ae^{-\xi\omega_n t} + Bte^{-\xi\omega_n t} \tag{3}$$

Apply $x(0) = 0$, we obtain that $0 = A$ Now

$$\dot{x}(t) = Be^{-\xi\omega_n t} - \xi\omega_n Bte^{-\xi\omega_n t}$$

Apply $\dot{x}(0) = \frac{\hat{F}}{m}$ to the above, we obtain

$$\frac{\hat{F}}{m} = B$$

Hence (3) becomes

$$x(t) = \frac{\hat{F}}{m}te^{-\xi\omega_n t}$$

When the magnitude of the impulse is unity, i.e. a unit impulse, hence $\hat{F} = 1$, then we obtain the unit impulse response

$$\boxed{h(t) = \frac{1}{m} t e^{-\xi \omega_n t}}$$