my post-mortem EE420 first midterm solution
California State University, Fullerton
Spring 2010

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Contents

1 Question 1

(20) 1. The input output relation of a system is given below, is this system

(6) a) Linear
(6) b) Shift Invariant
(4) c) Causal
(4) d) Stable

$$y(n) = \sum_{k=n-5}^{n+N} x(k)$$

Note: Prove your answers, just stating an answer has no credit

n+N n+N

(a) Let operator $T[\cdot] \equiv \sum_{n-5}^{n+N} x(k)$, hence we need to show that, for any α, β, the following

$$\alpha T[x_1(n)] + \beta T[x_2(n)] \overset{?}{=} T\left[\alpha x_1(n) + \beta x_2(n)\right] \tag{1}$$

Start by finding the RHS of (1). Let

$$\hat{x}(n) = \alpha x_1(n) + \beta x_2(n)$$

Hence

$$\begin{aligned} T[\hat{x}(n)] &= \sum_{n-5}^{n+N} \hat{x}(k) \\ &= \sum_{n-5}^{n+N} \left(\alpha x_1(k) + \beta x_2(k)\right) \\ &= \sum_{n-5}^{n+N} \alpha x_1(k) + \sum_{n-5}^{n+N} \beta x_2(k) \qquad \text{By linearity of summation} \\ &= \alpha \sum_{n-5}^{n+N} x_1(k) + \beta \sum_{n-5}^{n+N} x_2(k) \qquad \text{since } \alpha, \beta \text{ do not depend on } k \end{aligned} \tag{2}$$

Now consider LHS of (1), which is

$$\alpha T[x_1(n)] + \beta T[x_2(n)] = \alpha \sum_{n-5}^{n+N} x_1(k) + \beta \sum_{n-5}^{n+N} x_2(k)$$

We see that is the same as (2). Hence LHS is the same as RHS in (1). Hence $T[\cdot]$ is linear

(2) Delay the input $x(n)$ by an amount β and see if a delayed output by the same amount is the same or not.

Let delayed input be $x\left(n-\beta\right)$, hence the output is

$$T\left[x\left(n-\beta\right)\right] = \sum_{k=n-5}^{n+N} x\left(k-\beta\right)$$

Let $k' = k - \beta$, so when $k = n-5$, then $k' = (n-5) - \beta$, and when $k = n+N$, then $k' = (n+N) - \beta$, hence the above becomes

$$T\left[x\left(n-\beta\right)\right] = \sum_{k'=(n-5)-\beta}^{(n+N)-\beta} x(k')$$

Since k' is dummy variable, we can rename it to be k

$$T\left[x\left(n-\beta\right)\right] = \sum_{k=(n-5)-\beta}^{(n+N)-\beta} x(k) \tag{1}$$

Now let us consider what a delayed output will be. The output due to $x(n)$ is $T[x(n)] = \sum_{k=n-5}^{n+N} x(k)$, hence a delayed output is where we delay any occurrence of n in the RHS of the above to become $n-\beta$, hence a delayed output is

$$\sum_{k=\left(n-\beta\right)-5}^{\left(n-\beta\right)+N} x(k) \tag{2}$$

We see that (1) and (2) are the same, hence shift invariant. (3)A system is causal if its output at time n does not depend on future values of the input. i.e on values larger than n. From the definition

$$y(n) = \sum_{n-5}^{n+N} x(k)$$

We see that $y(n)$ will depend on future values on the input $x(n)$ only if $N > 0$. Therefore, the system is causal for $N \leq 0$, and not causal otherwise.

(4)To show stability, use the BIBO approach. Let M the absolute value of the largest possible value of the input which will be finite. Hence the output magnitude

$$\begin{aligned}
|y(n)| &= \left|\sum_{n-5}^{n+N} x(k)\right| \\
&\leq \sum_{n-5}^{n+N} |x(k)| \\
&\leq \sum_{n-5}^{n+N} M \\
&= M \sum_{n-5}^{n+N} 1 \\
&= M((N+n) - (n-5) + 1) \\
&= M(N+6)
\end{aligned}$$

Hence if $N < \infty$, then $|y(n)| < \infty$ and the system is BIBO stable.

2 Question 2

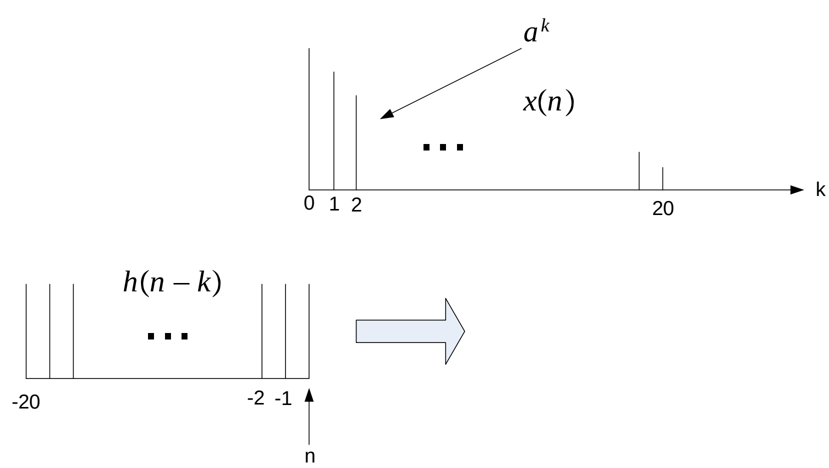
(30) 2. The input, x(n), and the unit sample response, h(n), of a LSI system are given below. 'a' is a real number less than one.

$x(n) = a^n[U(n)-U(n-21)]$ $|a|<1$

$$h(n) = \begin{cases} 1 & 0 \leq n \leq 20 \\ 0 & otherwise \end{cases}$$

Find the output of the system, y(n), **as a function of 'n'**, using the convolution sum $y(n) = x(n)*h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$

Use graphical approach



First region: $n < 0 \rightarrow y(n) = 0$ since no overlapping

second region: $0 \leq n \leq 20$ i.e. partial overlap from the left end of $x(n)$

$$y(n) = \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$$

Third region: $20 \leq n \leq 40$

$$\begin{aligned}
y(n) &= \sum_{k=n-20}^{20} a^k = \sum_{k=0}^{20} a^k - \sum_{k=0}^{n-19} a^k \\
&= \frac{1-a^{21}}{1-a} - \frac{1-a^{n-20}}{1-a} \\
&= \frac{\left(1-a^{21}\right) - \left(1-a^{n-20}\right)}{1-a} \\
&= \frac{a^{n-20} - a^{21}}{1-a}
\end{aligned}$$

Fourth region: $n > 40 \rightarrow y(n) = 0$ since no overlapping.

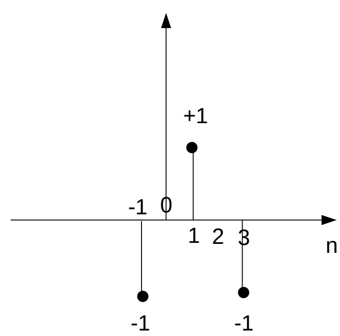
3 Question 3

(25) 3. Consider a discrete-time LSI system with unit-sample response

$$h(n) = \begin{cases} 1 & n = 1 \\ -1 & n = -1 \text{ and } 3 \\ 0 & \text{otherwise} \end{cases}$$

Compute H($e^{j\omega}$) and **plot** its **magnitude** and **phase** for $\omega \varepsilon [-\pi, \pi]$. On the plots, label all frequencies and amplitudes of importance, and keep the phase between π and $-\pi$.

NOTE: DTFT: $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n}$



$$\begin{aligned} H\left(e^{j\omega}\right) &= \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n} \\ &= h(-1)\, e^{-j\omega(-1)} + h(1)\, e^{-j\omega(1)} + h(3)\, e^{-j\omega(3)} \\ &= -e^{j\omega} + e^{-j\omega} - e^{-3j\omega} \end{aligned}$$

Collect complex exponential with same coefficients

$$\begin{aligned} H\left(e^{j\omega}\right) &= \left(-e^{j\omega} - e^{-3j\omega}\right) + e^{-j\omega} \\ &= -e^{-j\omega}\left(e^{2j\omega} + e^{-2j\omega}\right) + e^{-j\omega} \\ &= -e^{-j\omega}(2\cos 2\omega) + e^{-j\omega} \\ &= e^{-j\omega}(1 - 2\cos 2\omega) \end{aligned}$$

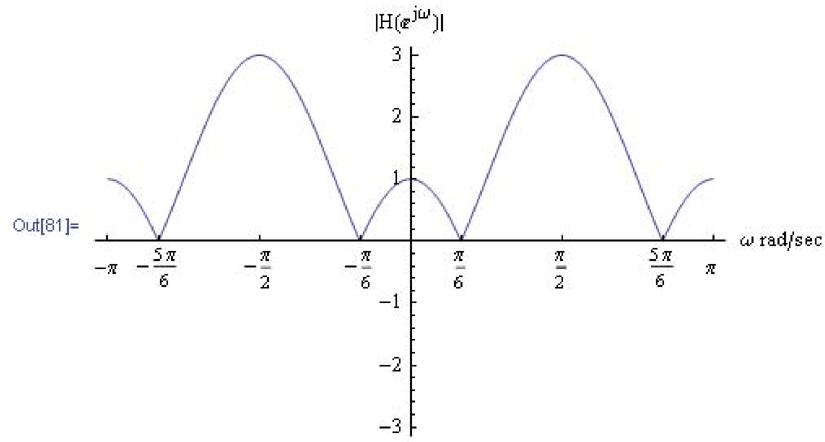
Hence

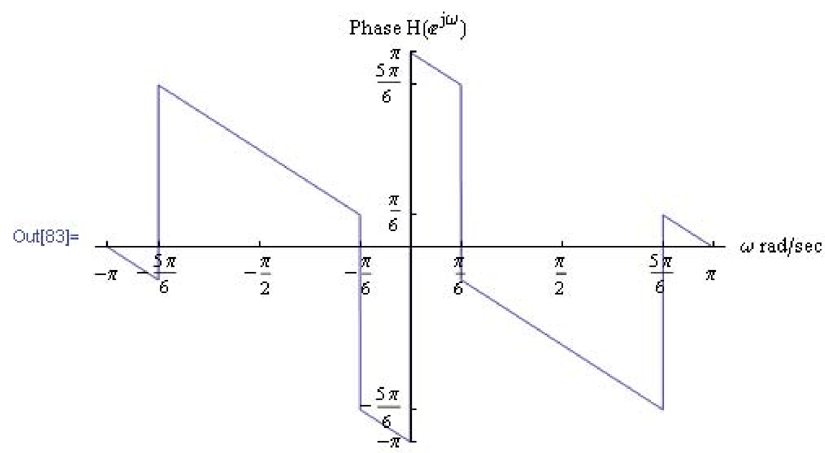
$$\left|H\left(e^{j\omega}\right)\right| = |1 - 2\cos 2\omega|$$

And

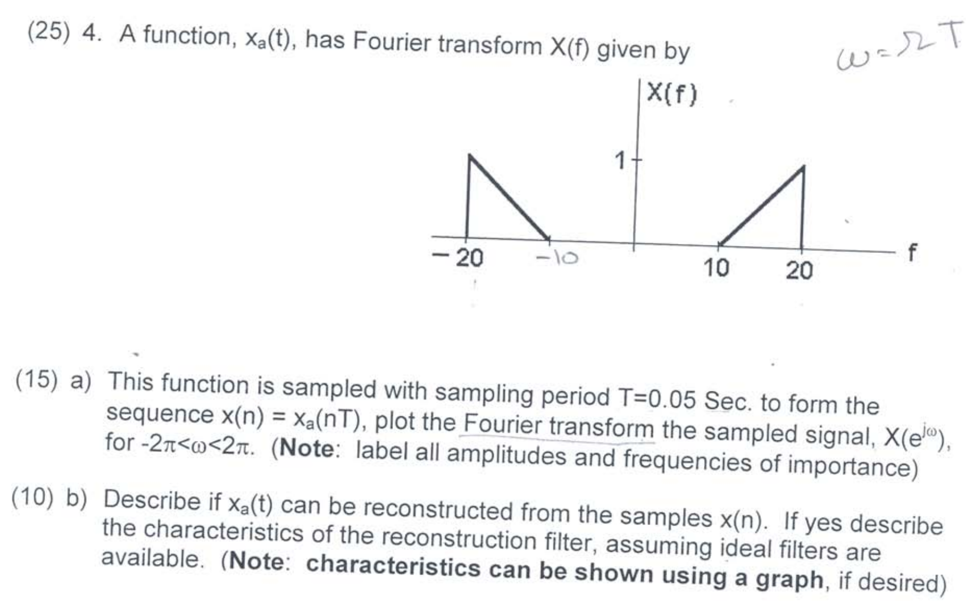
$$phase\left(H\left(e^{j\omega}\right)\right) = -\omega$$

Now carefully plotting these we obtain





4 Question 4


(25) 4. A function, $x_a(t)$, has Fourier transform X(f) given by

$\omega = \Omega T$

(15) a) This function is sampled with sampling period T=0.05 Sec. to form the sequence $x(n) = x_a(nT)$, plot the Fourier transform the sampled signal, $X(e^{j\omega})$, for $-2\pi<\omega<2\pi$. (**Note**: label all amplitudes and frequencies of importance)

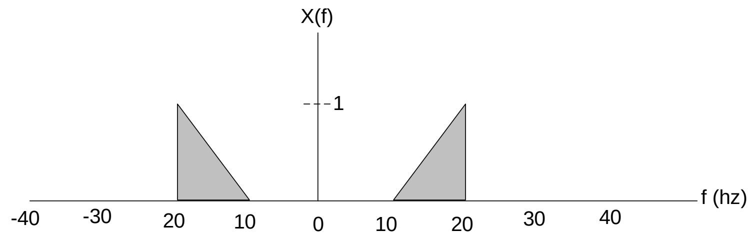
(10) b) Describe if $x_a(t)$ can be reconstructed from the samples x(n). If yes describe the characteristics of the reconstruction filter, assuming ideal filters are available. (**Note**: **characteristics can be shown using a graph**, if desired)

Maximum frequency in the signal can be seen to be $20Hz$. Hence Nyquist frequency is twice this, which is $40Hz$.

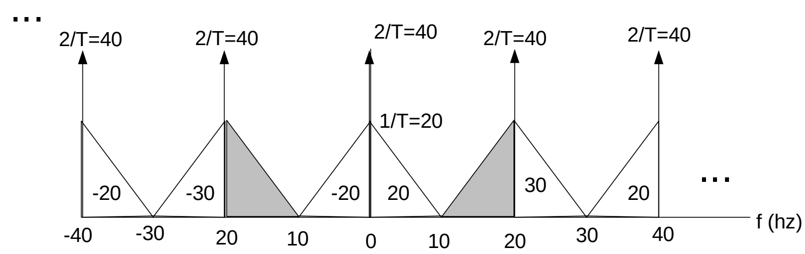
However we are told that sampling frequency $f_s = \frac{1}{T} = 20Hz$. Hence this system is undersampled.

Another observation to make: Since $X\left(f\right)$ is not zero at the maximum frequency $20Hz$. one should use a sampling frequency slightly over Nyquist.

Now, to plot the DTFT of the sampled signal. Make copies of $X\left(f\right)$ centered at multiplies of the the sampling frequency. We obtain the following diagram



Duplicate at each multiple of sampling frequency, and scale



DTFT of sampled signal x(n)

Bandpass filter required to
reconstruct xa(t) is