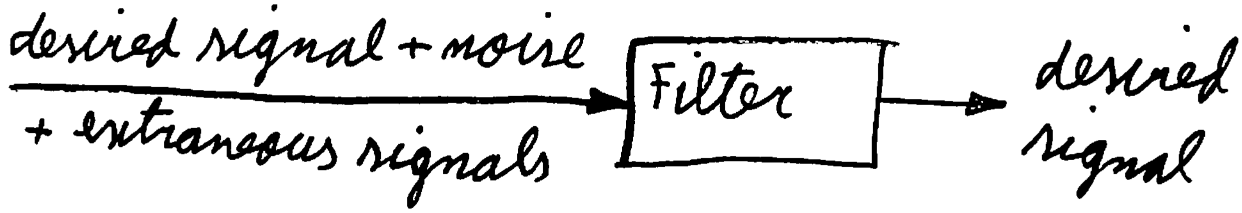
Handout G 4/14/2010

Digital filter design techniques

why do we need filters?



noise & distortion could have been obtained during the transmission process

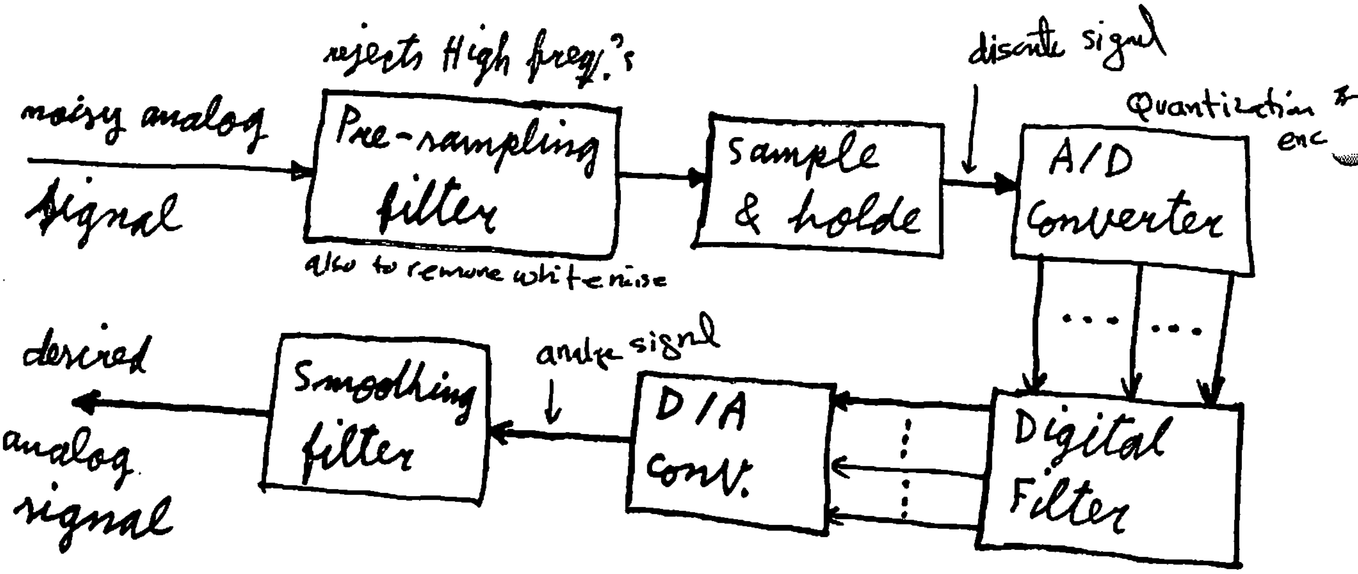
By **filtering** we try to remove the extraneous signals and noise so as to obtain the desired signal

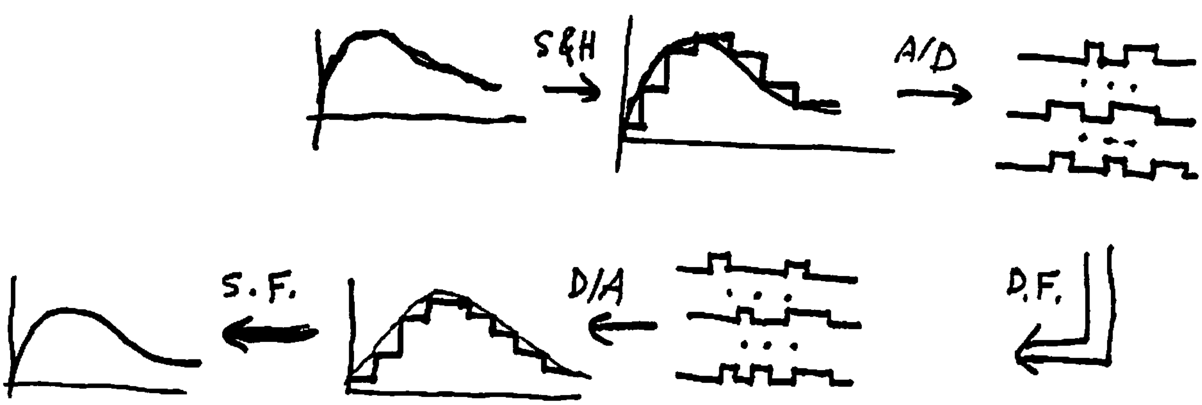
Filters

only in real time

a) Analog filter : does the filtering process for analog signals

b) Digital filters : does the filtering process for digital signals





4 Kinds of filters

Digital filter design

3 steps:

1) Specification of the desired system
2) Approximation of the specified system (using a causal discrete-time sys.)
3) realization of the system.

because we want it to work in real time

We focus our attention on the **second step**.

1) **Specification** (given in db, convert to straight scale).

We want to minimize some error.

- Freq.-domain: $\sum_{i=1}^{N} \left| H(w_i) - \underbrace{H_d(w_i)}_{\text{desired}} \right|^2 \longleftarrow \text{Min.}$

desired → ideal low pass

- time-domain: $\sum_{i=1}^{M} \left| Y(n) - \sum_j a_j\, x(n) \right|^2 \leftarrow \text{Min.}$

- Optimization : performance func.
- Analytical : tolerance limits to be specified

Ex. a low-pass filter ⟸ Normalized: so that 1 is hight.

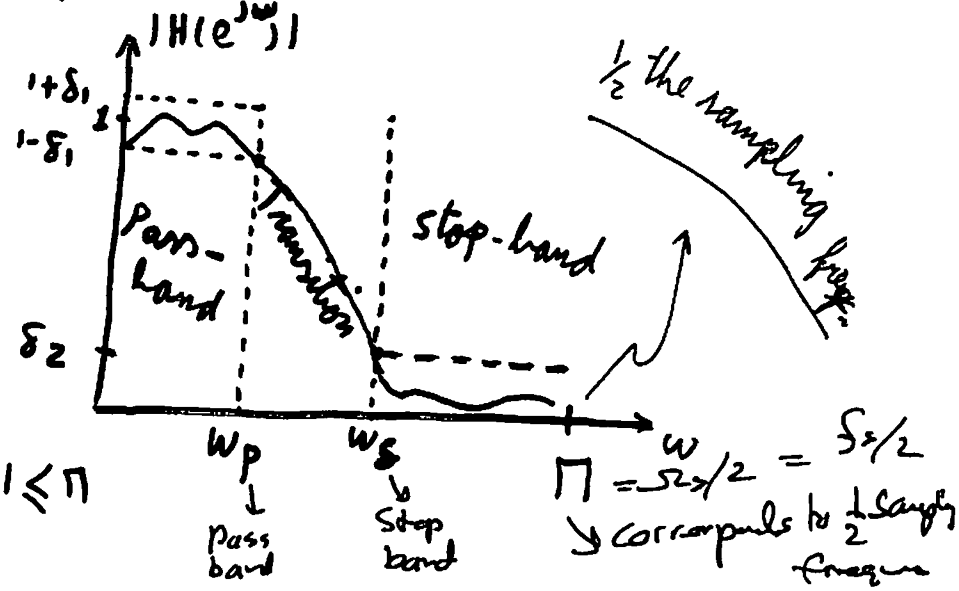
Passband

$1-\delta_1 \leqslant |H(e^{jw})| \leqslant 1+\delta_1$

$|w| \leqslant w_p$

stopband

$|H(e^{jw})| \leqslant \delta_2, \; w_s \leqslant |w| \leqslant \pi$



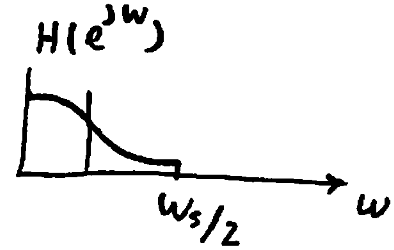
Note

Analog freq.'s $\longrightarrow$ Hz

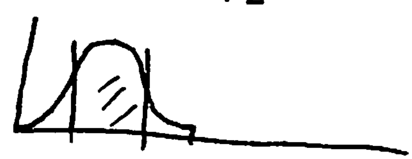
Digital freq.'s $\longrightarrow$ radian freq. <u>or</u> angle around the unit circle with $z = -1$ corresponding to half the sampling freq.

Types of filters

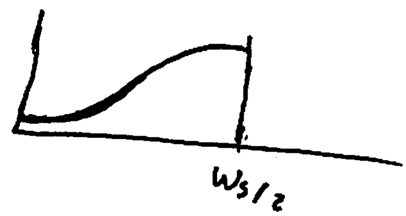
1 - low pass



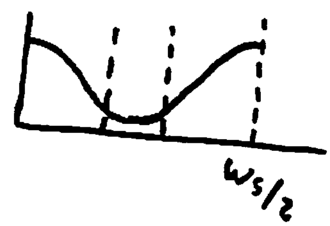
2 - band pass



3 - high pass



4 - notch filter <u>or</u> band stop
< if very narrow



To design the filter

① <u>IIR filter</u> : recursive ; approximate the desired freq. response by a rational function. poles & zeros.

② <u>FIR filter</u> : nonrecursive : polynomial approximation

$$\frac{Y(z)}{X(z)} = \text{i.e } H(z) = 1 - 2z^{-1} + 0.3z^{-2}$$

(I) Design of IIR filters from analog filters

1- transformation : Analog $\longrightarrow$ digital
(Laplace) $\longrightarrow$ (Z)

2- Geometric approach : pole-zero pattern in the Z-plane.

<u>Analog sys. func.</u> $H_a(s) = \mathcal{L}\{h_a(t)\}$

$$H_a(s) = \frac{\sum_{k=0}^{M} d_k s^k}{\sum_{k=0}^{N} c_k s^k} = \frac{Y_a(s)}{X_a(s)}$$

where $X_a(s) = \mathcal{L}\{x_a(t)\}$ & $Y_a(s) = \mathcal{L}\{y_a(t)\}$
$\downarrow$ input $\qquad \downarrow$ output

$$y_a(t) = \int_{-\infty}^{\infty} x_a(\tau)\, h_a(t-\tau)\, d\tau \qquad \underline{\text{conv. integral}}$$

<u>Analog sys.</u> with sys. func. $H_a(s)$:

$$\sum_{k=0}^{N} c_k \frac{d^k y_a(t)}{dt^k} = \sum_{k=0}^{M} d_k \frac{d^k x_a(t)}{dt^k}$$

Corresponding sys. func. for digital filter is

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{Y(z)}{X(z)}$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad \underline{\text{conv. sum}}$$

&

$$\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{k=0}^{M} b_k\, x(n-k)$$

$\begin{cases} \text{Transform analog} \longrightarrow \text{digital} \\ \quad \Leftrightarrow \text{obtain } H(z) \text{ or } h(n) \text{ from the analog filter} \end{cases}$

<u>How to get $H(z)$</u> such that : loosly speaking $\begin{cases} \text{1- map imaginary axis of the s-plane into the unit circle of the z-plane} \\ \text{2- preserve stability} \end{cases}$

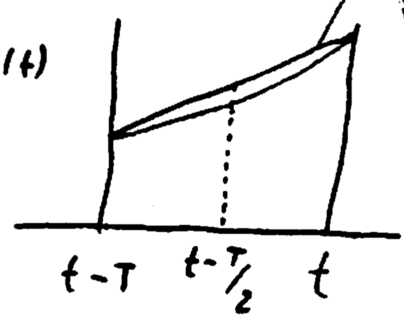
<u>(1) Derivatives $\longrightarrow$ Differences</u>

$$s \longrightarrow \frac{d}{dt} \qquad z^{-1} \longrightarrow D \ \text{(delay)}$$

$$\frac{d}{dt} y(t) \sim \frac{y(t) - y(t-T)}{T} = \frac{1-D}{T}\, y(t)$$

T: sampling interval

approximation if the signal is smooth enough



$$\frac{d y_a(t)}{dt} \longrightarrow \Delta^{(1)}[y(n)] \text{ where } \Delta^{(1)}[y(n)] = \frac{y(n) - y(n-1)}{T}$$

$$\frac{d^2 y_a(t)}{dt^2} \longrightarrow \Delta^{(2)}[y(n)] \text{ where}$$

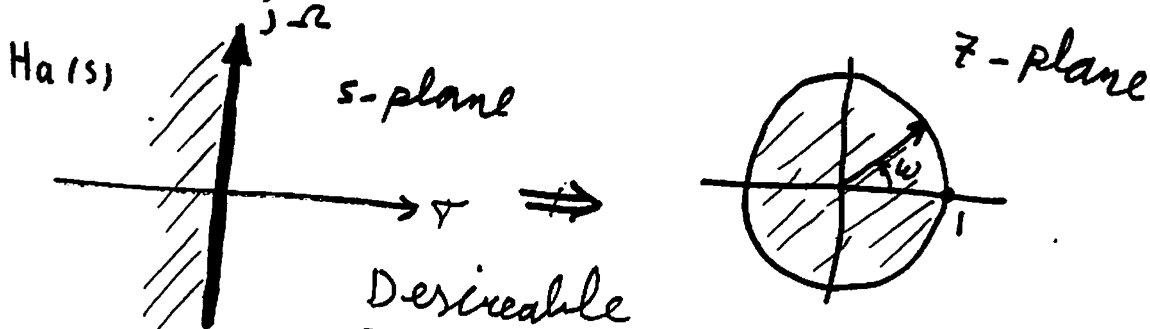
$$\Delta^{(2)}[y(n)] = \Delta^{(1)}\left[\Delta^{(1)} y(n)\right] = \Delta^{(1)}\left[\frac{y(n) - y(n-1)}{T}\right]$$

$$= \frac{y(n) - 2y(n-1) + y(n-2)}{T^2}$$

$$\frac{d\,y_a(t)}{dt} \longrightarrow \Delta^{(1)}[y(n)] = \frac{y(n) - y(n-1)}{T} = \frac{1-D}{T}\,y(n)$$

$$\left.\begin{aligned} \mathcal{L}\left[\tfrac{d}{dt} y_a(t)\right] &= s\,Y_a(s) \\ \mathcal{Z}\left[\Delta^{(1)} y(n)\right] &= \frac{1-z^{-1}}{T}\,Y(z) \end{aligned}\right\} \quad s \longleftrightarrow \frac{1-z^{-1}}{T}$$

$$\boxed{H(z) = H_a(s)\Big|_{s \to \frac{1-z^{-1}}{T}}}$$



Desireable

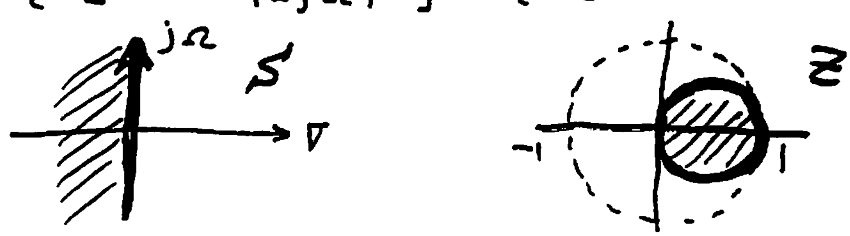
This mapping is not useful

1- jΩ axis does not map to the unit circle in z-plane

2- stability of filter is not in general preserved

$$z = \frac{1}{1-sT} \qquad s = j\Omega \Rightarrow z = \frac{1}{1-j\Omega T} \Rightarrow |z| \neq 1 \ \forall \Omega$$

$$z = \frac{1}{2}\left[1 + \frac{1+j\Omega T}{1-j\Omega T}\right] = \frac{1}{2}\left[1 + e^{j2\tan^{-1}(\Omega T)}\right]$$



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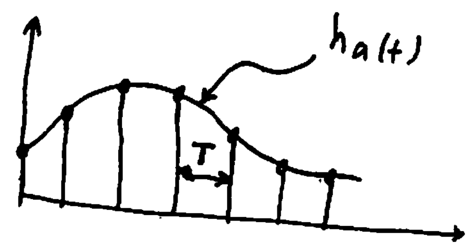
② Impulse Invariance

Let $\quad h(n) = h_a(nT)$

Impulse response fnc. or unit-sample " " $\Big\}\ h_a(t)$

$\Rightarrow \quad h(n) \quad$: Impulse resp. seq.

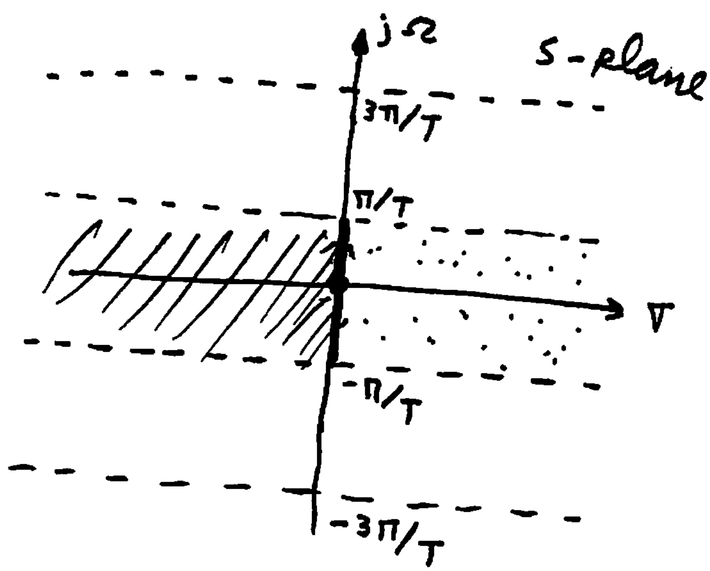


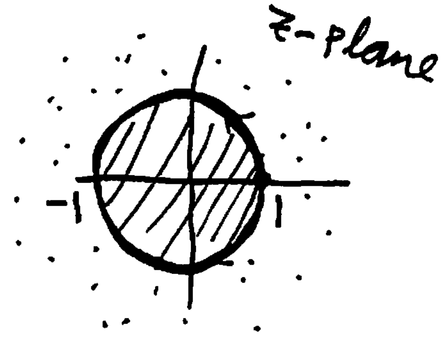
Then

$$H(z)\Big|_{z=e^{sT}} = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a\left(s + j\,\frac{2\pi}{T}k\right)$$

In sampling of continuous signals we had $\underbrace{X(e^{j\Omega T})}_{\text{seq.}} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(j\Omega + j\,\frac{2\pi k}{T}\right)$

$z = e^{sT} \qquad s = \sigma + j\Omega$





Each strip of width $\frac{2\pi}{T}$ in s-plane maps into the entire z-plane as shown above

∴ Each horizontal strip maps onto the z-plane

$\Rightarrow$ <u>Impulse Invariance</u> method is not a simple algebraic mapping of the s-plane to the z-plane

<u>Fourier Domain</u> : $s \rightarrow \frac{j\omega}{T}$

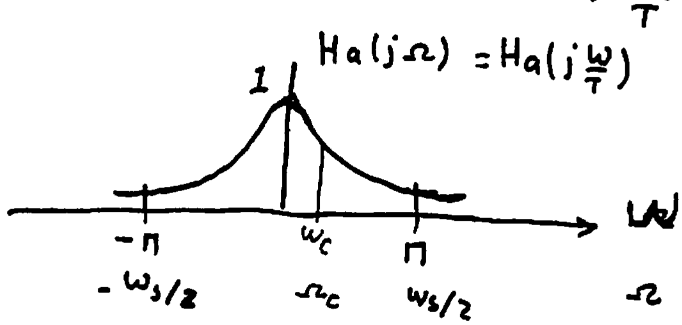
$$H(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a\Big(j \underbrace{\frac{\omega}{T}}_{\Omega} + j \underbrace{\frac{2\pi}{T}}_{\omega_s} k \Big)$$

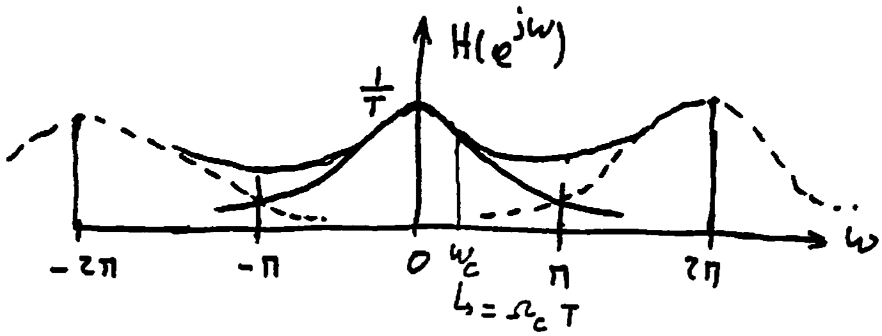
$\downarrow$

Digital freq. resp.

$\Omega = \frac{\omega}{T}$

$$\begin{cases} H(e^{j\omega}) = \frac{1}{T} H_a\left(j\frac{\omega}{T}\right) & |\omega| \leq \pi \\ \text{iff} \quad H_a(j\Omega) = 0 & |\Omega| \geq \frac{\pi}{T} \end{cases}$$





$\Rightarrow$ aliasing

Practical analog filter is not usually bandlimited $\Rightarrow$ Aliasing

ω_c : cutoff freq. smaller $T \Rightarrow$ larger Ω_c such that $\Omega_c T = cte$. ∴ T is an irrelevant parameter in design of impulse invariant filters. usually assumed to be 1. $(T=1)$

Derivation of $h(n)$ or $H(z)$ from $h_a(t)$ or $H_a(s)$

Let sys. fnc. $\boxed{H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}}$ (I)

then Impulse resp. $h_a(t) = \sum_{k=1}^{N} A_k e^{s_k t} u_a(t)$ → unit step fnc.

Then

$$h(n) = h_a(nT) = \sum_{k=1}^{N} A_k e^{s_k nT} u_a(n)$$

$$= \sum_{k=1}^{N} A_k \left(e^{s_k T}\right)^n u_a(n)$$

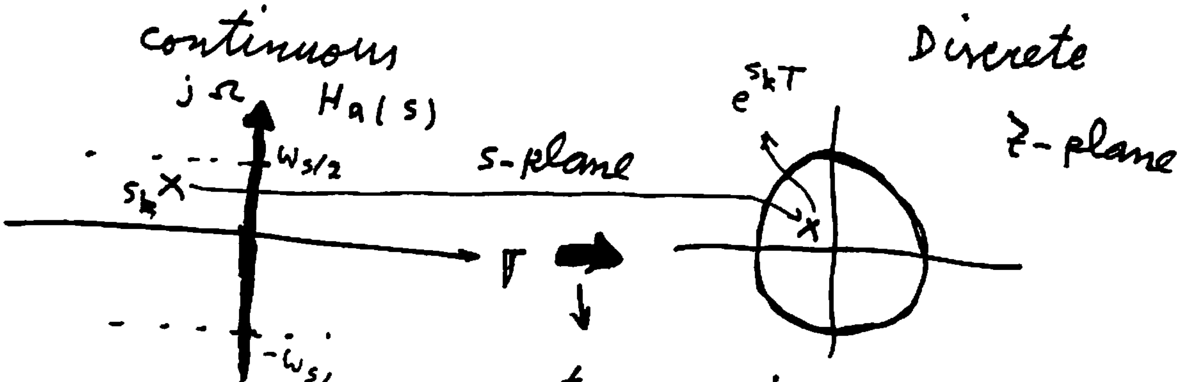
&

$$\boxed{H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - e^{s_k T} z^{-1}}} \qquad (\text{II})$$

Compare (I) to (II) $\Rightarrow$

$$\boxed{s_k \longrightarrow e^{s_k T} z^{-1}}$$

Poles: $\boxed{s = s_k \Longrightarrow z = e^{s_k T}}$



not a mapping

zeros don't map in the same way as poles

stability

$$s_k = \sigma_k + j\Omega_k \qquad \Rightarrow \qquad e^{s_k T} = e^{(\sigma_k + j\Omega_k)T}$$

stable if $\sigma_k < 0$

$$= e^{\sigma_k T} \cdot e^{j\Omega_k T}$$

$$|e^{s_k T}| = |e^{\sigma_k T}| < 1 \Rightarrow \text{stable sys.}$$

Remarks

1. Aliasing
2. Except the effect of aliasing the digital freq. resp. is a scaled replica of the analog freq. resp. i.e., the shape of analog freq. response is preserved for bandlimited filter design — good approach.
3. Sampling rate has to be observed at Nyquist rate. But for finite-dimensional $H_a(s)$ the freq. resp. often span from $-\infty$ to ∞
4. Zeros of the original analog transfer func. do not map to the z-plane in the same way that poles do.

$$\text{Pole}: \quad s_k \longrightarrow e^{s_k T}$$

$$\text{Zero}: \quad w_k \not\longrightarrow e^{w_k T}$$

Example

$\omega = \Omega T$

$H(e^{j\omega}) = \frac{1}{T} H_a(j\frac{\omega}{T})$

$\Rightarrow$ Get $H(z)$, using $T H_a(s)$

$$H_a(s) = \frac{(s+a)}{(s+a)^2+b^2}$$

$$= \frac{1/2}{s+a+jb} + \frac{1/2}{s+a-jb}$$

zero at $s = -a$

Two poles $\begin{cases} s_1 = -a - jb \\ s_2 = -a + jb \end{cases}$

Impulse Invariant digital filter is:

$$H(z) = \frac{(1/2)\,T}{1 - e^{-aT} e^{-jbT} z^{-1}} + \frac{(1/2)\,T}{1 - e^{-aT} e^{jbT} z^{-1}}$$

$$= \frac{1 - (e^{-aT}\cos bT)\, z^{-1}}{(1 - e^{-aT} e^{-jbT} z^{-1})(1 - e^{-aT} e^{jbT} z^{-1})} \times T$$

$$= \frac{z(z - e^{-aT}\cos bT)}{(z - e^{-aT} e^{-jbT})(z - e^{-aT} e^{jbT})} \times T$$

Two digital domain zeros $\begin{cases} z_1 = 0 \\ z_2 = e^{-aT}\cos bT \end{cases}$

Look at pole-zero plots and freq. responses on Pg. 202 of the Text.

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