HW1, MAE 200A. Fall 2005. UCI

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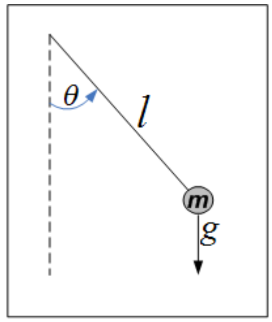

Pendulum

1 Problem 1

Given this simple pendulum, compute the equilibrium points and determine the linearized dynamics at each equilibrium point

Answer
part(1)
The system equation is given by

$$ml^2\ddot{\theta} = -c\dot{\theta} - mgl\sin\theta$$

Since this is a second order ODE, there are 2 state variables. Convert this to state space formulation: Let $x_1 = \theta$ and $x_2 = \dot{\theta}$, hence

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{-cx_2}{ml^2} - \frac{g}{l}\sin x_1 \end{bmatrix}$$

At the equilibrium points we must have,

$$\begin{aligned} \dot{x} &= 0 \\ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= 0 \end{aligned}$$

Hence we obtain 2 equations

$$\begin{aligned} x_2 &= 0 \\ \frac{-cx_2}{ml^2} - \frac{g}{l}\sin x_1 &= 0 \end{aligned}$$

Solving these equations, we obtain

$$\frac{g}{l}\sin x_1 = 0$$

or $x_1 = n\pi$ for $n = 0, 1, 2, ...$

Since the period is 2π, then $x_1 = 0$ or π, but x_1 is state variable that represents the angle θ hence $\boxed{\text{equilibrium occurs at } \theta = 0 \text{ and } \theta = \pi}$
note: equilibrium at $\theta = 0$ is stable, while at $\theta = \pi$ is unstable.

part(2)
starting with the nonlinear system equation

$$ml^2\ddot{\theta} = -c\dot{\theta} - mgl\sin\theta$$

Near the equilibrium points, we express the nonlinear term in taylor series.
Suppose the penulium is at angle θ near the angle θ_{eq} so it is a distance $\triangle\theta = \theta - \theta_{eq}$
Hence now

$$\begin{aligned}
\sin\theta &= \sin\theta_{eq} + (\theta - \theta_{eq}) \left.\frac{d\sin\theta}{d\theta}\right|_{\theta=\theta_{eq}} + \frac{(\theta-\theta_{eq})^2}{2!} \left.\frac{d^2\sin\theta}{d\theta^2}\right|_{\theta=\theta_{eq}} \\
&\quad + \frac{(\theta-\theta_{eq})^3}{3!} \left.\frac{d^3\sin\theta}{d\theta^3}\right|_{\theta=\theta_{eq}} + \frac{(\theta-\theta_{eq})^4}{4!} \left.\frac{d^4\sin\theta}{d\theta^4}\right|_{\theta=\theta_{eq}} + \cdots \\
&= \sin\theta_{eq} + (\theta-\theta_{eq}) \cos\theta|_{\theta=\theta_{eq}} - \frac{(\theta-\theta_{eq})^2}{2!} \sin\theta|_{\theta=\theta_{eq}} \\
&\quad - \frac{(\theta-\theta_{eq})^3}{3!} \cos\theta|_{\theta=\theta_{eq}} + \frac{(\theta-\theta_{eq})^4}{4!} \sin\theta|_{\theta=\theta_{eq}} + \cdots \qquad (1)
\end{aligned}$$

For the first equilibrium point, $\theta_{eq} = 0$ so the above becomes

$$\begin{aligned}
\sin\theta &= 0 + \theta - 0 - \frac{\theta^3}{3!} + \cdots \\
&= \theta - \frac{\theta^3}{3!} - \cdots
\end{aligned}$$

For the first equilibrium point, $\theta_{eq} = \pi$ so equation (1) becomes

$$\begin{aligned}
\sin\theta &= \sin\pi + (\theta-\pi) \cos\theta|_{\theta=\pi} - \frac{(\theta-\pi)^2}{2!} \sin\theta|_{\theta=\pi} - \frac{(\theta-\pi)^3}{3!} \cos\theta|_{\theta=\pi} + \frac{(\theta-\pi)^4}{4!} \sin\theta|_{\theta=\pi} + \cdots \\
&= 0 - (\theta-\pi) + \frac{(\theta-\theta_\pi)^3}{3!} + \cdots \\
&= -(\theta-\pi) + \frac{(\theta-\pi)^3}{3!} + \cdots
\end{aligned}$$

Hence near $\theta_{eq} = 0$, the linearized system equation is

$$ml^2\ddot{\theta} = -c\dot{\theta} - mgl\left(\theta - \frac{\theta^3}{3!} - \cdots\right)$$

and near $\theta_{eq} = \pi$ the system equation is

$$ml^2\ddot{\theta} = -c\dot{\theta} - mgl\left(-(\theta-\pi) + \frac{(\theta-\pi)^3}{3!} + \cdots\right)$$