HW3, MAE 200A. Fall 2005. UCI

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1 Problem 1

Due: Oct. 13 at beginning of class

Analyze the forced response of the pendulum. The forcing, as demonstrated in class, is the sinusoidal side to side movement of the attachment point.

Assume the lateral motion of the attachment point is of the form

$$\boxed{z = b\cos(\omega\ t)}$$

b is the maximum displacement. The linearized dynamics about the equilibrium at $\theta = 0$ are

$$\boxed{\theta''(t) + \frac{c}{ML}\theta'(t) + \frac{g}{L}\theta(t) = -\frac{1}{L}z''(t)}$$

1. Construct a physical pendulum. It can be as simple as the one I used in class, although a rigid rod would be preferable to an elastic one. Estimate the values of M, L and c and use these values in your calculations. $L =$ length of the rod; $M =$ mass of the pendulum bob; $c =$damping coefficient

2. Determine the natural frequency ω_n and damping ratio ζ of your pendulum.

3. Determine the general solution (homogeneous plus particular) for the linear pendulum model. You can use the result I gave in lecture if you know how to obtain it. If you don't know how to obtain it, this would be a good opportunity to learn how to.

4. Using the particular solution, construct the Bode plots theta (amplitude and phase angle as functions of the forcing frequency ω).

5. Experiment with forcing your pendulum at different frequencies and convince yourself that what you see corresponds to the predictions from the Bode plots.

6. If you move the attachment point according to

$$z = \cos(\omega_1 t) + \cos(\omega_2 t) + \cos(\omega_3 t)$$

   where, relative to the natural frequency ω_n of the pendulum, ω_1 is much less than ω_n, ω_2 is close to ω_n, and ω_3 is greater than ω_n, describe qualitatively what the θ response will look like after the homogeneous solution has died out. Your answer should be based on theory not experiment.

Please turn in all your work except for your pendulum. Bring your pendulum to class if you want to show it off, but this is optional.

2 Answer

2.1 Answer parts 1,2

A simple pendulum was used for this experiment.
The Mass M was weighted and found out to be about $.5kg$

The length L was measured to be 80 cm or $0.8m$

To determine the damping coefficient, since

$$\begin{aligned}\frac{c}{ML} &= 2\zeta\omega_n \\ c &= 2ML\zeta\omega_n\end{aligned}$$

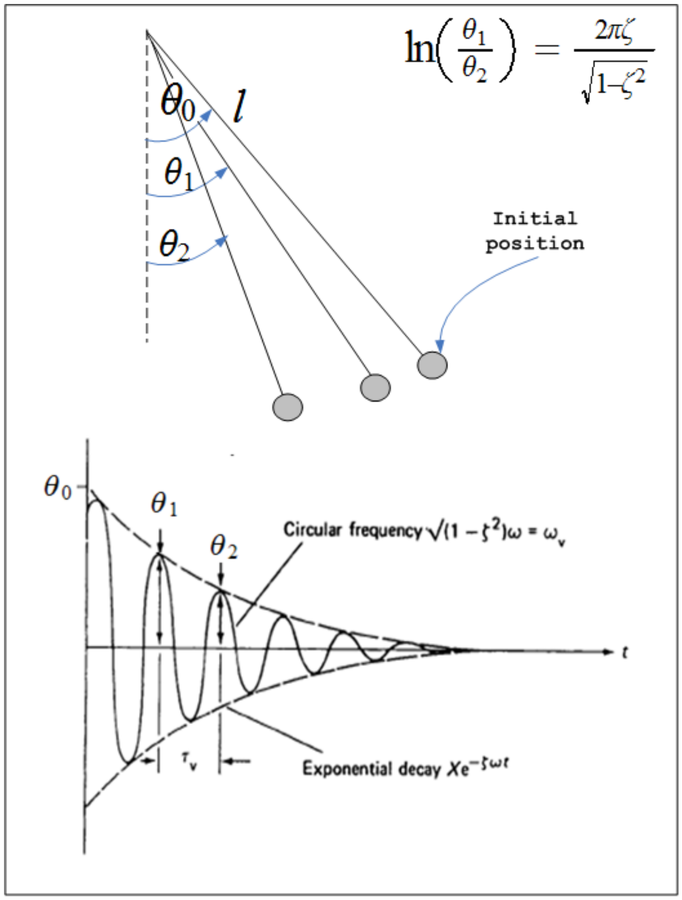
I will first find ζ.

The method of logarithmic decrement was used. The logarithmic decrement Λ is the natural logarithm of the ratio of any two successive amplitudes in the same direction. The pendulum mass is held initially at an angle $\theta_0 = 45^0$ in the positive direction, and then released. The mass will then make one full cycle by swinging to left and then back to the same side as it started and stop before starting its second cycle and so on. The angle the mass reach at the end of its first cycle was estimated to be $\theta_1 = 40^0$ and the angle it reached at the end of it second full cycle was estimated to be $\theta_2 = 35^0$

Hence

$$\ln\left(\frac{\theta_1}{\theta_2}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

The following diagram showing the process and the derivation of the above equation



Measuring pendulum position using 2 successive full rotation to estimate zeta

To derive the equation

$$\ln\left(\frac{\theta_1}{\theta_2}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

We utilize the above plot of $\theta(t)$ for a damped second order system.

In the above diagram

$$\theta_1 = \theta_0 e^{-\zeta\omega_n t}$$

and

$$\theta_2 = \theta_0 e^{-\zeta\omega_n(t+\tau_\nu)}$$

where τ_ν is the time it takes to make one full swing between these 2 successive oscillations.

$$\ln\left(\frac{\theta_2}{\theta_1}\right) = \ln\left(\frac{\theta_0 e^{-\zeta\omega_n t}}{\theta_0 e^{-\zeta\omega_n(t+\tau_\nu)}}\right) = \ln\left(e^{\zeta\omega_n\tau_v}\right) = \zeta\omega_n\tau_v$$

But

$$\frac{2\pi}{\tau_v} = \omega_d$$

where ω_d is the damped natural frequency. Hence $\tau_v = \frac{2\pi}{\omega_d}$ hence

$$\ln\left(\frac{\theta_2}{\theta_1}\right) = 2\pi\zeta\frac{\omega_n}{\omega_d}$$

But $\omega_d = \omega_n\sqrt{1-\zeta^2}$ hence the above equation becomes

$$\ln\left(\frac{\theta_2}{\theta_1}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

Now that this equation is derived, it can be used to estimate ζ, Once ζ is found then c can be easily found.

$$\ln\frac{40^0}{35^0} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

square each side and solve for ζ

$$\begin{aligned}
\left(\ln\frac{40}{35}\right)^2 &= \frac{4\pi^2\zeta^2}{1-\zeta^2} \\
0.01783\,1 - 0.01783\,1\,\zeta^2 &= 4\pi^2\zeta^2 \\
\zeta^2 &= \frac{0.01783\,1}{4\pi^2 + 0.01783\,1}
\end{aligned}$$

$$\boxed{\zeta = 0.02}$$

Next, since $\omega_n = \sqrt{\frac{g}{L}}$Hence, then

$$\boxed{\omega_n = \sqrt{\frac{9.8}{0.8}} = 3.5 \text{ rad/sec}}$$

and since $c = 2ML\zeta\omega_n$, then

$$\begin{aligned}
c &= 2\times 0.5\,(kg)\times 0.8\,(m)\times 0.02\times 3.5\,(rad/\sec) \\
&= 2.8 \quad \text{N s/rad}
\end{aligned}$$

2.2 Answer part 3

First we find an analytical solution $\theta(t)$.

$$\theta(t) = \theta_h(t) + \theta_p(t)$$

where $\theta_h(t)$ is the homogenous solution (due to initial conditions) and $\theta_p(t)$ is the particular solution (due to the forcing function). To find $\theta_h(t)$, looking at the ODE

$$\theta''(t) + \frac{c}{ML}\theta'(t) + \frac{g}{L}\theta(t) = 0$$

We see this is a standard second order system. Let $\frac{c}{ML} = 2\zeta\omega_n$, $\frac{g}{L} = \omega_n^2$, hence we can write the above as

$$\theta''(t) + 2\zeta\omega_n\ \theta'(t) + \omega_n^2\ \theta(t) = 0$$

the solution is

$$\theta_h(t) = \ e^{-\zeta\omega_n t}\left[A\sin\left(\omega_n t\sqrt{1-\zeta^2}\right) + B\cos\left(\omega_n t\sqrt{1-\zeta^2}\right)\right]$$

where the damped natural frequency be $\omega_d = \omega_n\sqrt{1-\zeta^2}$ hence the above can be written as

$$\boxed{\theta_h(t) = \ e^{-\zeta\omega_n t}\left[A\sin(\omega_d t) + B\cos(\omega_d t)\right]} \tag{1}$$

Now we can find the particular solution. Since the forcing function is a sinusoidal, we can try

$$\boxed{\theta_p = G_1\sin\omega t + G_2\cos\omega t} \tag{2}$$

This particular solution will take care of the case when the forcing function is out of phase with the response, that is why both a sin and a cos function are present. Substitute θ_p into the dynamic equation that represents the linearized pendulum given by

$$\theta''(t) + 2\zeta\omega_n\ \theta'(t) + \omega_n^2\ \theta(t) = -\frac{1}{L}z''(t)$$

and since

$$z(t) = b\cos(\omega t)$$

then $z''(t) = -b\omega^2\cos(\omega t)$ hence the above equation becomes

$$\theta_p''(t) + 2\zeta\omega_n\ \theta_p'(t) + \omega_n^2\ \theta_p(t) = \frac{b\omega^2}{L}\cos(\omega t) \tag{4}$$

Now

$$\begin{aligned}
\theta_p'(t) &= G_1\omega\cos\omega t - G_2\omega\sin\omega t \\
\theta_p''(t) &= -G_1\omega^2\sin\omega t - G_2\omega^2\cos\omega t
\end{aligned}$$

Hence (4) becomes

$$\begin{aligned}
-G_1\omega^2\sin\omega t - G_2\omega^2\cos\omega t + 2\zeta\omega_n\ (G_1\omega\cos\omega t - G_2\omega\sin\omega t) + \omega_n^2\ (G_1\sin\omega t + G_2\cos\omega t) &= \frac{b\omega^2}{L}\cos(\omega t) \\
\cos(\omega t)\left(-G_2\omega^2 + 2\zeta\omega_n G_1\omega + \omega_n^2 G_2\right) + \sin\omega t\left(-G_1\omega^2 - 2\zeta\omega_n G_2\omega + G_1\omega_n^2\right) &= \frac{b\omega^2}{L}\cos(\omega t)
\end{aligned}$$

compare coefficients of $\cos(\omega t)$ we get

$$\begin{aligned}
-G_2\omega^2 + 2\zeta\omega_n\omega G_1 + \omega_n^2 G_2 &= \frac{b\omega^2}{L} \\
G_2\left(\omega_n^2 - \omega^2\right) + 2\zeta\omega_n\omega G_1 &= \frac{b\omega^2}{L} \qquad (5)
\end{aligned}$$

and

$$\begin{aligned}
-G_1\omega^2 - 2\zeta\omega_n G_2\omega + G_1\omega_n^2 &= 0 \\
G_1\left(\omega_n^2 - \omega^2\right) - 2\zeta\omega_n G_2\omega &= 0 \qquad (6)
\end{aligned}$$

Need to solve (5) and (6) for G_1, G_2 , first divide (5) and (6) by ω_n^2 and call the ratio $\frac{\omega}{\omega_n} = \beta$ which is the response ratio, to obtain new expressions for (5) and (6)

$$G_2\left(1-\beta^2\right) + 2\zeta\beta G_1 = \frac{b}{L}\beta^2 \qquad (5a)$$

$$G_1\left(1-\beta^2\right) - 2\zeta G_2\beta = 0 \qquad (6a)$$

from (6a)

$$G_1 = \frac{2\zeta G_2\beta}{\left(1-\beta^2\right)} \qquad (7)$$

Plug (7) into (5a) we obtain

$$\begin{aligned}
G_2\left(1-\beta^2\right) + 2\zeta\beta\frac{2\zeta G_2\beta}{\left(1-\beta^2\right)} &= \frac{b}{L}\beta^2 \\
G_2\left[\left(1-\beta^2\right) + \frac{\left(2\zeta\beta\right)^2}{\left(1-\beta^2\right)}\right] &= \frac{b}{L}\beta^2 \\
G_2 &= \frac{b}{L}\frac{\beta^2\left(1-\beta^2\right)}{\left(1-\beta^2\right)^2 + \left(2\zeta\beta\right)^2} \qquad (8)
\end{aligned}$$

Substitute (8) into (7) to solve for G_1

$$\begin{aligned}
G_1 &= \frac{2\zeta\beta}{\left(1-\beta^2\right)}\frac{b}{L}\frac{\beta^2\left(1-\beta^2\right)}{\left(1-\beta^2\right)^2 + \left(2\zeta\beta\right)^2} \\
&= \frac{2b}{L}\frac{\zeta\beta^3}{\left(1-\beta^2\right)^2 + \left(2\zeta\beta\right)^2} \qquad (9)
\end{aligned}$$

Hence

$$\begin{aligned}
\theta_p &= G_1\sin\omega t + G_2\cos\omega t \\
&= \sqrt{G_1^2 + G_2^2}\cos\left(\omega t - \tan^{-1}\frac{G_1}{G_2}\right)
\end{aligned}$$

Where G_1, G_2 are as given above. Hence

$$\theta_p = \sqrt{\left(\frac{2b}{L}\frac{\zeta\beta^3}{\left(1-\beta^2\right)^2+\left(2\zeta\beta\right)^2}\right)^2+\left(\frac{b}{L}\frac{\beta^2\left(1-\beta^2\right)}{\left(1-\beta^2\right)^2+\left(2\zeta\beta\right)^2}\right)^2}\cos\left(\omega t-\tan^{-1}\frac{\frac{2b}{L}\frac{\zeta\beta^3}{\left(1-\beta^2\right)^2+\left(2\zeta\beta\right)^2}}{\frac{b}{L}\frac{\beta^2\left(1-\beta^2\right)}{\left(1-\beta^2\right)^2+\left(2\zeta\beta\right)^2}}\right)$$

$$\begin{aligned}\theta_p &= \frac{b}{L}\sqrt{\frac{\beta^4\left(2\zeta\beta\right)^2}{\left(\left(1-\beta^2\right)^2+\left(2\zeta\beta\right)^2\right)^2}+\frac{\beta^4\left(1-\beta^2\right)^2}{\left(\left(1-\beta^2\right)^2+\left(2\zeta\beta\right)^2\right)^2}}\cos\left(\omega t-\tan^{-1}\frac{2\zeta\beta}{1-\beta^2}\right)\\
&= \frac{b}{L}\sqrt{\frac{\beta^4\left(\left(1-\beta^2\right)^2+\left(2\zeta\beta\right)^2\right)}{\left(\left(1-\beta^2\right)^2+\left(2\zeta\beta\right)^2\right)^2}}\cos\left(\omega t-\tan^{-1}\frac{2\zeta\beta}{1-\beta^2}\right)\\
&= \frac{b}{L}\sqrt{\frac{\beta^4}{\left(1-\beta^2\right)^2+\left(2\zeta\beta\right)^2}}\cos\left(\omega t-\tan^{-1}\frac{2\zeta\beta}{1-\beta^2}\right)\\
&= \frac{b}{L}\frac{\beta^2}{\sqrt{\left(1-\beta^2\right)^2+\left(2\zeta\beta\right)^2}}\cos\left(\omega t-\tan^{-1}\frac{2\zeta\beta}{1-\beta^2}\right)\end{aligned}$$

Hence the general solution $\theta\left(t\right)=\theta_h\left(t\right)+\theta_p\left(t\right)$ is

$$\boxed{\theta\left(t\right)=e^{-\zeta\omega_n t}\left(A\sin\left(\omega_d t\right)+B\cos\left(\omega_d t\right)\right)+\frac{b}{L}\frac{\beta^2}{\sqrt{\left(1-\beta^2\right)^2+\left(2\zeta\beta\right)^2}}\cos\left(\omega t-\tan^{-1}\frac{2\zeta\beta}{1-\beta^2}\right)}$$

Where A, B, can now be determined from initial conditions. However, since we are interested in steady state solution (not the transient solution), we do not need to find these constants for the purpose of this solution.

The first term in the solution (the $\theta_h\left(t\right)$) term, will damp down quickly with time since it has the inverse exponential term in it. What is left then is the particular solution.

2.3 Answer part 4

Since

$$\boxed{\theta_p(t) = \frac{b}{L}\frac{\beta^2}{\sqrt{\left(1-\beta^2\right)^2+(2\zeta\beta)^2}}\cos\left(\omega t - \tan^{-1}\frac{2\zeta\beta}{1-\beta^2}\right)}$$

Hence in the above equation we can write it as

$$\theta_p(t) = A\cos(\omega t + \phi)$$

Where amplitude

$$A = \frac{b}{L}\frac{\beta^2}{\sqrt{\left(1-\beta^2\right)^2+(2\zeta\beta)^2}}$$

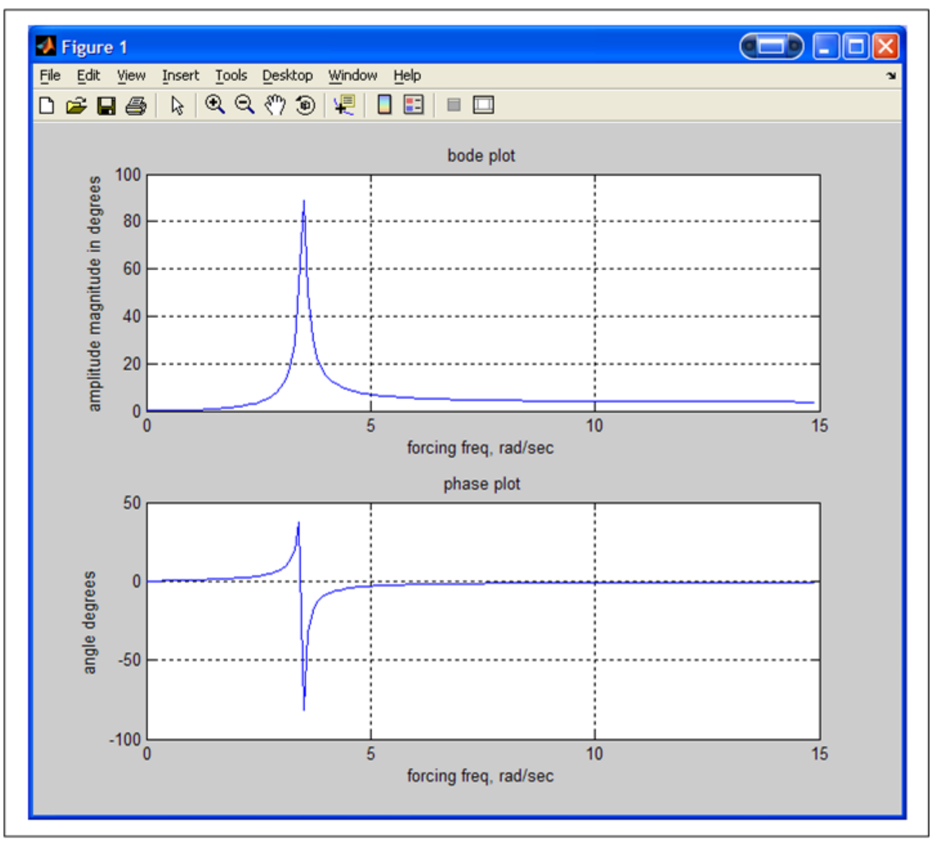
and phase

$$\phi = -\tan^{-1}\frac{2\zeta\beta}{1-\beta^2}$$

To make amplitude in degrees instead of radians, convert the above by multiplying by $180/\pi$

and $\beta = \frac{\omega}{\omega_n} = \frac{\omega}{3.5}$, $L = 0.8, \zeta = 0.02$, and for let $b = 0.05$ meters.

Now we are ready to generate the needed plots. I will show the following plot where the x-axis will show the input frequency ω in rad/sec, and the y-axis will show the amplitude A in angles.



Bode plot theta generated from plotting the particular solution (steady state solution).

2.4 Part 5

Now I changed the forcing frequency to be close to the natural frequency of the pendulum (3.5 radians per second), and observed largest oscillation at that frequency. (note: input force is lateral, at the joint, so input frequency needs to be converted from cycles per second to radiance per second).

This is in agreement with the above plot.

2.5 Part 6

Now assume $z = \cos(\omega_1 t) + \cos(\omega_2 t) + \cos(\omega_3 t)$.

For a linear system, the total response will be the same as the sum of the response to each of the above signals individually. (at each instance of time).

In addition, the response will have the same frequency but different amplitude and phase.

Hence the steady state amplitude will be the sum of the individual amplitudes, and the steady state phase shift will be the sum of the individual phase shifts. (at each instance of time).

This can be answered by looking at the analytical solution found above for $\theta_p(t)$ (steady state solution) and taking the limit of $\frac{\omega}{\omega_n}$ as it approaches 0 or 1 or ∞ and see what happens to the amplitude and the phase in each case.

Since we found

$$\theta_p(t) = \frac{b}{L} \frac{\beta^2}{\sqrt{\left(1-\beta^2\right)^2 + (2\zeta\beta)^2}} \cos\left(\omega t - \tan^{-1}\frac{2\zeta\beta}{1-\beta^2}\right)$$

Where $\beta = \frac{\omega}{\omega_n}$

Hence for the case when $\omega_1 \ll \omega_n$, then $\beta \to 0$, hence the limit of the solution will be $\theta_p(t) \to 0$, i.e. in stead state, the displacement goes to zero. For the phase we see that the phase $\tan^{-1}\frac{2\zeta\beta}{1-\beta^2}$ will also go to zero, hence response will be in-phase with the input.

For the case when $\omega_3 \gg \omega_n$, then we see β becomes very large. We see that the phase term goes to zero since it goes like $\frac{1}{\beta}$ in the limit. Hence the response in steady state when the input frequency is much larger than ω_n will be in phase. To see what happens to the amplitude, take the limit of the amplitude as $\beta \to \infty$

$$
\begin{aligned}
& \lim_{\beta\to\infty} \frac{b}{L} \frac{\beta^2}{\sqrt{\left(1-\beta^2\right)^2+(2\zeta\beta)^2}} \\
= & \lim_{\beta\to\infty} \frac{b}{L} \frac{1}{\sqrt{\frac{1+2\beta^4-2\beta^2+4\zeta^2\beta^2}{\beta^4}}} \\
= & \lim_{\beta\to\infty} \frac{b}{L} \frac{1}{\sqrt{\frac{1+2\beta^4-2\beta^2+4\zeta^2\beta^2}{\beta^4}}} \\
= & \lim_{\beta\to\infty} \frac{b}{L} \frac{1}{\sqrt{\frac{1}{\beta^4}+2-\frac{2}{\beta^2}+\frac{4}{\beta^2}}} \\
= & \frac{b}{L}\frac{1}{\sqrt{0+2-0+0}} \\
= & \frac{b}{L}\frac{1}{\sqrt{2}} \\
= & 0.707\frac{b}{L}
\end{aligned}
$$

So when forcing frequency is much larger than ω_n, the steady state response will be fixed and will be proportional to the maximum amplitude b. This agrees with the plot shown above where we see the response is steady at large ω. For the phase we see that the phase $\tan^{-1}\frac{2\zeta\beta}{1-\beta^2}$ will also go to zero, hence response will be in-phase with the input. This agrees with the phase plot shown.

For the case when $\omega \simeq \omega_n$, then $\beta \to 1$, hence

$$
\begin{aligned}
\lim_{\beta\to 1} \frac{b}{L} \frac{\beta^2}{\sqrt{\left(1-\beta^2\right)^2+(2\zeta\beta)^2}} &= \lim_{\beta\to 1} \frac{b}{L} \frac{1}{\sqrt{(1-1)^2+(2\zeta)^2}} \\
&= \frac{b}{L}\frac{1}{\zeta\sqrt{2}} = 0.707\frac{b}{L}\frac{1}{\zeta}
\end{aligned}
$$

Hence the smaller the damping ratio ζ, the larger the amplitude. This means the smaller the damping coefficient c the larger the amplitude. This is called resonance. For the phase, we see that $\lim_{\beta\to 1}\tan^{-1}\frac{2\zeta\beta}{1-\beta^2} = -\frac{\pi}{2}$ hence at resonance, the phase is 90^0 from the input. This agrees with the plot shown.

2.6 Appendix

```
%script to solve HW3, MAE 200A, by Nasser Abbasi
clear all; close all;

omega=[0.01:0.1:15];
L=0.8; zeta=0.02; b=0.05; beta=omega./3.5;

num=b*beta.^2;
den=L*sqrt( (1-beta.^2).^2+(2*zeta.*beta).^2 );
A=num./den;
A=A.*180./pi;

subplot(2,1,1);
%semilogx(omega,20*log10(A)); %second
plot(omega,A); %first
grid;
title('bode plot');
xlabel('forcing freq, rad/sec'); %first
%xlabel('forcing freq, log scale, rad/sec');
%ylabel('amplitude magnitude in db'); %second
ylabel('amplitude magnitude in degrees'); %first

subplot(2,1,2);
phase=atan( (2*zeta.*beta)./(1-beta.^2) );
plot(omega,phase*180/pi)  %first
%semilogx(omega,phase*180/pi)  %second

grid;
title('phase plot');
xlabel('forcing freq, rad/sec'); %first
%xlabel('forcing freq, log scale, rad/sec');
ylabel('angle degrees');
```