

HW4, MAE 200A. Fall 2005. UCI

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October 20, 2005

Contents

1 Problem	2
2 Solution	3
2.1 Part 1	3
2.2 Part 2	3
2.2.1 Analytical solution	3
2.2.2 Euler solution	5
3 Part 4	14

1 Problem

Compare the integration of the standard linear, constant coefficient second-order system between the 3 methods: analytic integration, Euler integration, and the Matlab integrator ODE45.

The ODE, using the derivative notation I used for the previous assignment, is

$$x''(t) + 2\zeta\omega_n x'(t) + \omega_n^2 x(t) = 0$$

where ω_n is natural frequency and ζ is damping ratio.

On the course website I have posted 2 m-files that you can use, or you can write your own. If you use mine, please figure out what each step does so that you can learn how to use Matlab and eventually be capable of writing your own programs. The 2 m-files, called HW4mfile and f_hw4 work together.

The first is the main program and it calls the other. You run the main program using the 'save and run' command or click on the icon. When it works it will produce Figure 1 with superimposed plots of the solutions determined in the 3 ways.

Consider three sets of system parameter values

1. $\omega_n = 4$ and $\zeta = 0.5$ (the values in the files on the website)
2. $\omega_n = 1$ and $\zeta = 0.5$
3. $\omega_n = 1$ and $\zeta = 5.0$

The report you turn in should include the following

1. Problem statement
2. Give good and bad numerical integrator parameters (e.g., the step size Δt for the Euler integrator) and a figure illustrating good integration and a figure illustrating bad integration. For the 'good' integration try to use a step size that gives an accurate answer, but isn't excessively small, so that the computational burden is minimized.
3. Describe how you adjusted the step size for the Euler integration and the relative and absolute errors for ODE45 to get solutions that match well with the analytic solution. How do these adjustments depend on the parameters of the system, ζ and ω_n ?
4. If you didn't have an analytic solution, which will be the case for most nonlinear ODEs, how would you know when you have an accurate numerical solution?

2 Solution

2.1 Part 1

Please see above for the problem statement.

2.2 Part 2

2.2.1 Analytical solution

I start by solving the 2nd order ODE to determine the analytical solution.

$$x''(t) + 2\zeta\omega_n x'(t) + \omega_n^2 x(t) = 0$$

Let the solution be

$$x(t) = e^{\lambda t}$$

Substitute in the above equation we get the characteristic equation

$$\begin{aligned}\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 &= 0 \\ \lambda_{1,2} &= \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}\end{aligned}$$

Hence the solution is

$$x(t) = Ae^{(-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})t} + Be^{(-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})t} \quad (1)$$

Let initial conditions be $x(0)$ and $x'(0)$

Hence at $t = 0$ eq (1) becomes

$$x(0) = A + B \quad (2)$$

Take derivative of (1) we get

$$x'(t) = A \left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \right) e^{(-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})t} + B \left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \right) e^{(-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})t}$$

at $t = 0$ the above becomes

$$x'(0) = A \left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \right) + B \left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \right) \quad (3)$$

Now solve (2) and (3) for A,B. From (2) $A = x(0) - B$, plug into (3) we get

$$\begin{aligned}x'(0) &= (x(0) - B) \left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \right) + B \left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \right) \\ &= x(0) \left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \right) - B \left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \right) + B \left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \right) \\ &= x(0) \left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \right) - 2B \left(\omega_n\sqrt{\zeta^2 - 1} \right)\end{aligned}$$

Hence

$$B = \frac{-x'(0) + x(0) \left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \right)}{2 \left(\omega_n\sqrt{\zeta^2 - 1} \right)}$$

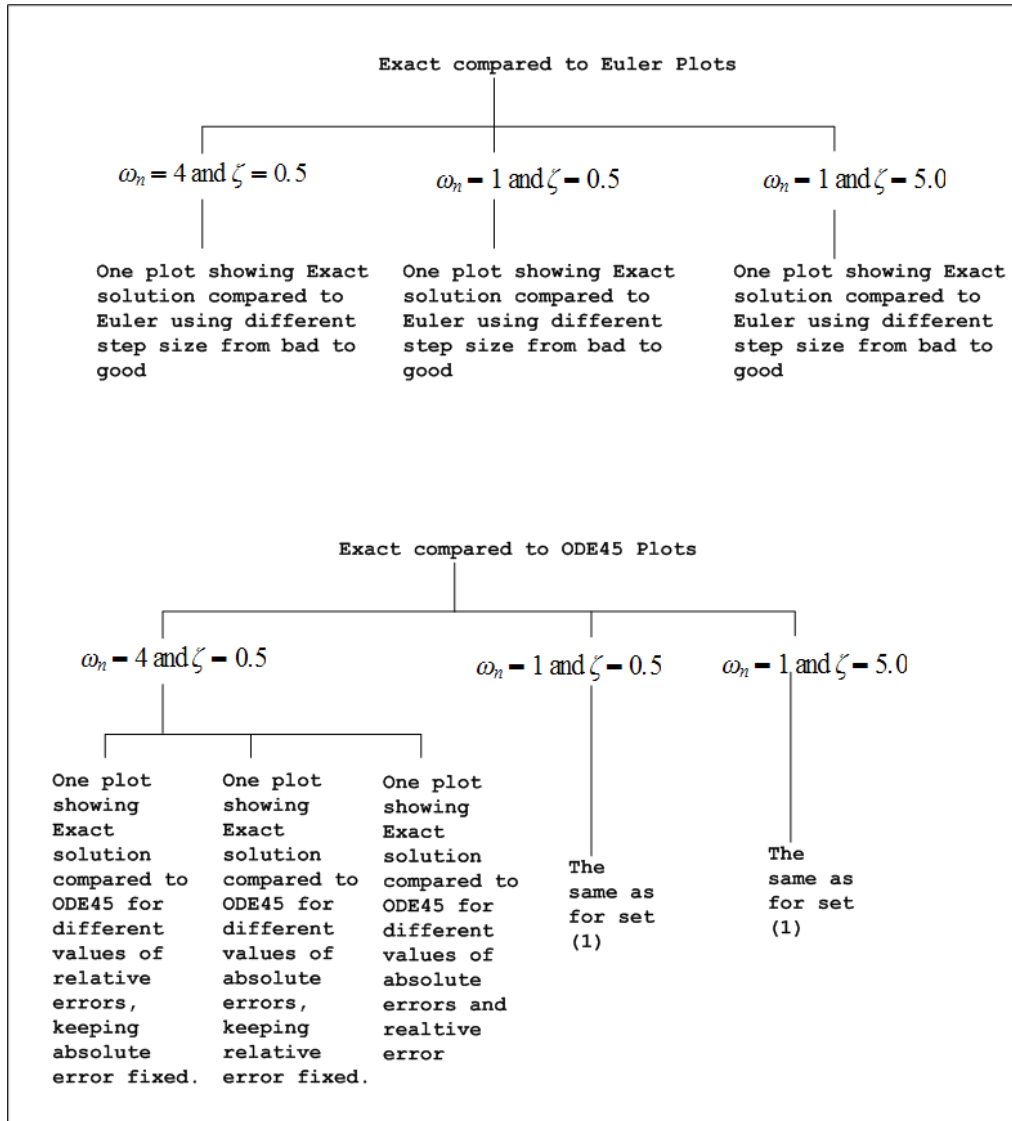
Hence from (2) we obtain A

$$\begin{aligned} A &= x(0) - B \\ &= x(0) - \frac{-x'(0) + x(0) \left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \right)}{2 \left(\omega_n\sqrt{\zeta^2 - 1} \right)} \\ &= \frac{2x(0) \left(\omega_n\sqrt{\zeta^2 - 1} \right) + x'(0) - x(0) \left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \right)}{2 \left(\omega_n\sqrt{\zeta^2 - 1} \right)} \\ &= \frac{2x(0) \left(\omega_n\sqrt{\zeta^2 - 1} \right) + x'(0) + x(0) \zeta\omega_n - x(0) \omega_n\sqrt{\zeta^2 - 1}}{2 \left(\omega_n\sqrt{\zeta^2 - 1} \right)} \\ &= \frac{x'(0) + x(0) \left(\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \right)}{2 \left(\omega_n\sqrt{\zeta^2 - 1} \right)} \end{aligned}$$

Let $\omega_n\sqrt{\zeta^2 - 1} = \omega_d$ hence

$$\begin{aligned} A &= \frac{x'(0) + x(0) (\zeta\omega_n + \omega_d)}{2\omega_d} \\ B &= \frac{-x'(0) + x(0) (-\zeta\omega_n + \omega_d)}{2\omega_d} \end{aligned}$$

Now that we have the full analytical solution we can implement the Euler integration and compare with the analytical solution. This will be done for the 3 sets of values given. For each set try a good and a bad Euler step. The following diagram describes the types of plots that will be generated



2.2.2 Euler solution

To obtain the numerical solution using the Euler method, we start by converting the 2nd order ODE to a set of 2 first order ODE as follows.

Let

$$\begin{aligned} x_1 &= x(t) \\ x_2 &= x'(t) \end{aligned}$$

Hence one state variable is the position and the second state variable is the speed. Differentiate we obtain

$$x_1' = x_2$$

$$\begin{aligned} x_2' &= x''(t) \\ &= -2\zeta\omega_n x'(t) - \omega_n^2 x(t) \\ &= -2\zeta\omega_n x_2 - \omega_n^2 x_1 \end{aligned}$$

We are given the initial position $x_1(0)$ and the initial speed $x_2(0)$. Hence to find to the position at the next time instance (i.e. to integrate x_1) we write

$$x_1(i) = x_1(i-1) + \Delta t x_1'(i-1)$$

But $x_1'(i-1) = x_2(i-1)$ hence the above becomes

$$x_1(i) = x_1(i-1) + \Delta t x_2(i-1) \quad (4)$$

Similarly, to integrate the state variable x_2 we write

$$x_2(i) = x_2(i-1) + \Delta t x_2'(i-1)$$

But $x_2' = -2\zeta\omega_n x_2 - \omega_n^2 x_1$ hence the above becomes

$$x_2(i) = x_2(i-1) + \Delta t (-2\zeta\omega_n x_2(i-1) - \omega_n^2 x_1(i-1)) \quad (5)$$

Hence equation (4) and (5) gives us the needed equation to determine the solution (which is x_1)

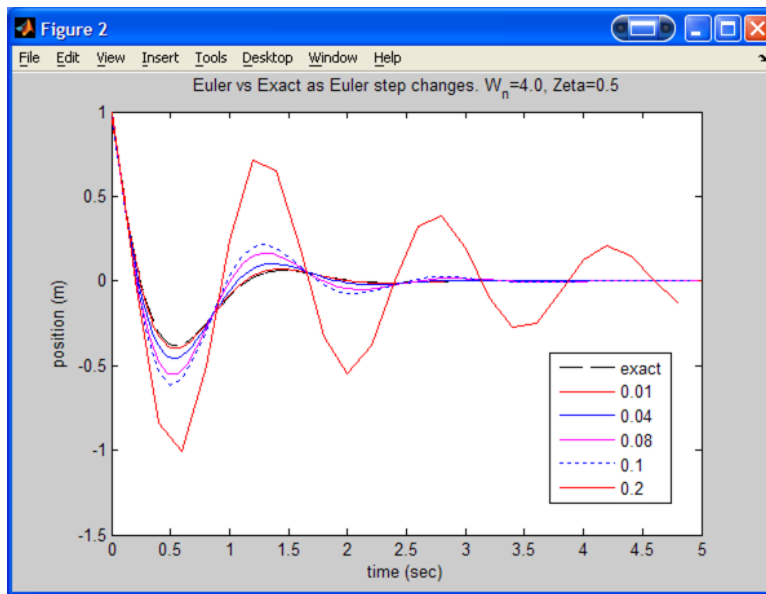
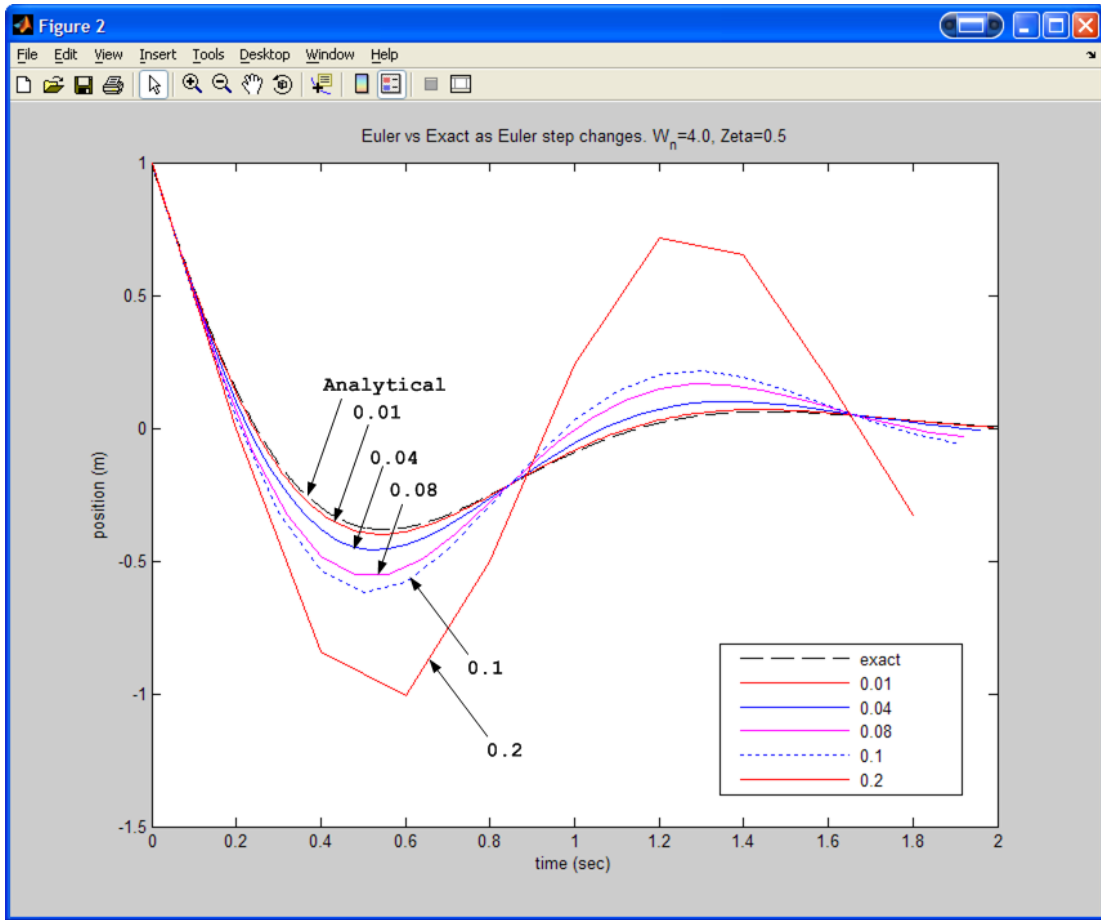
We start the process by finding $x_1(1)$ from eq(4) in terms of $x_2(0)$ and $x_1(0)$ which are the initial conditions. Next we find $x_2(1)$ in terms of $x_2(0)$ and $x_1(0)$ which are the initial conditions.

Next we find $x_1(2)$ from eq(4) in terms of $x_2(1)$ and $x_1(1)$, and we again update $x_2(1)$ to find $x_2(2)$.

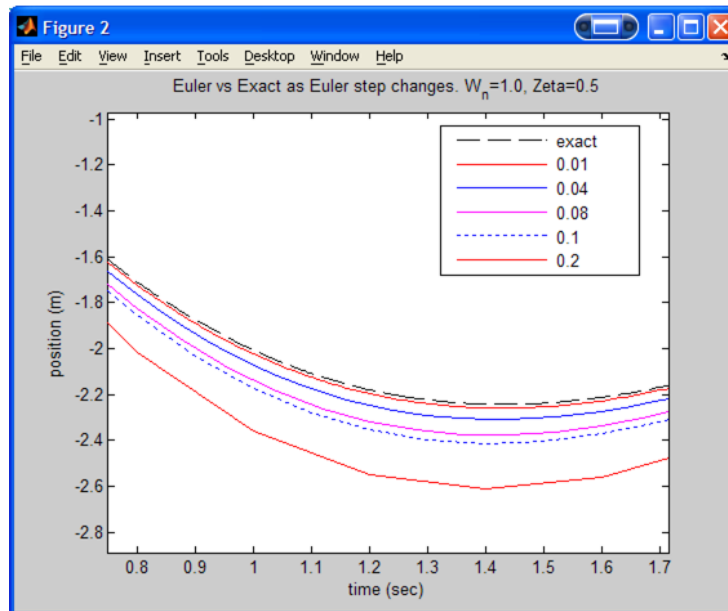
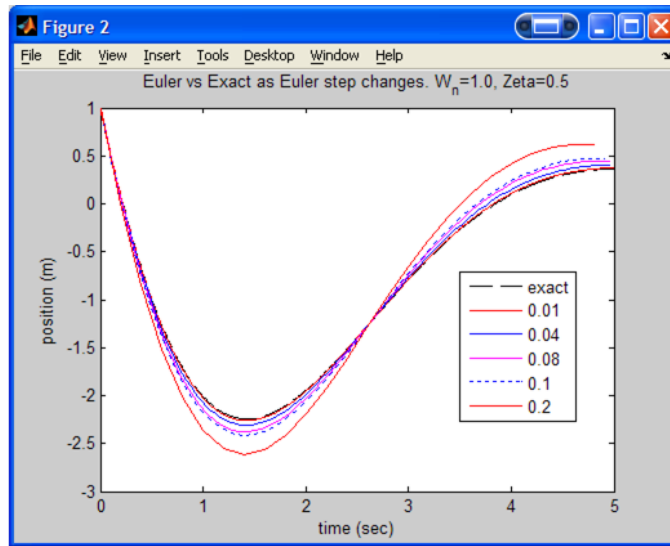
This process continues until the end of the time span.

Exact vs Euler Results These 3 plots below show the effect of changing the Euler step. Euler step was changed from good value (0.01) to bad value (1.2) by incrementing it and observing how the Euler starts to produce solutions that are further and further away from the analytical solution. Each plot was generated for different set of values of ω_n and ζ

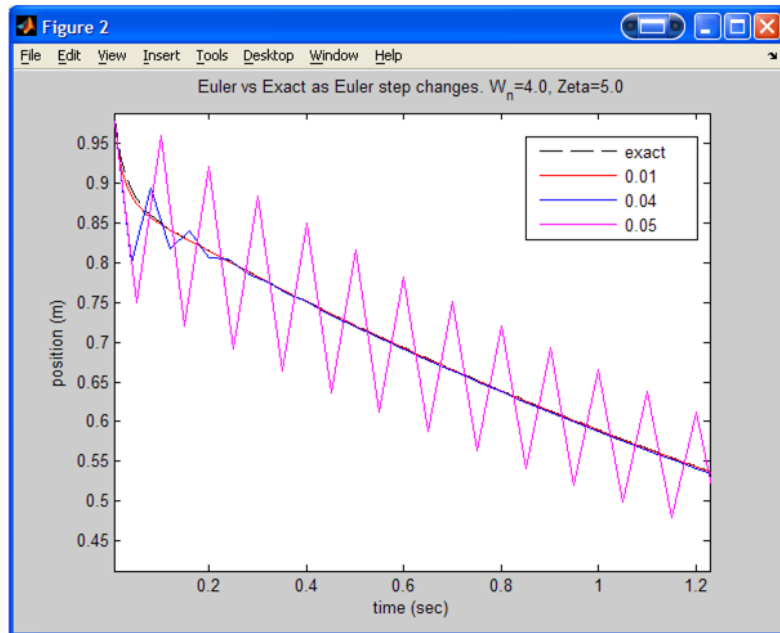
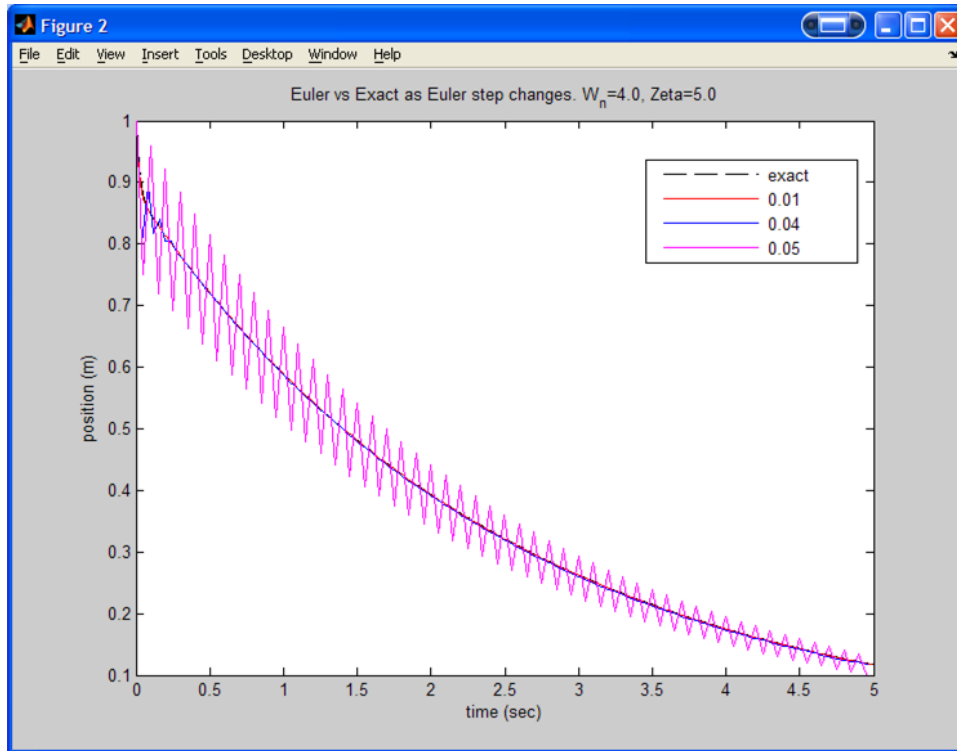
Results for $\omega_n = 4, \zeta = 0.5$ (for both small run time and longer run time).



Results for $\omega_n = 1, \zeta = 0.5$. (With a zoom in to better illustrate the effect of changing Euler step).



Results for $\omega_n = 1, \zeta = 5$. (With a zoom in to better illustrate the effect of changing Euler step). This plot was done only for Euler steps of 0.01, 0.04 and 0.05.



ODE45 solution Now we compare the ODE45 Matlab numerical solution with the analytical solution. This is done for different values of relative and absolute error. These parameters are defined as follows (this is from Matlab help)

Error Tolerance Properties		
Property	Value	Description
RelTol	Positive scalar {1e-3}	A relative error tolerance that applies to all components of the solution vector. The estimated error in each integration step satisfies $ e(i) \leq \max(\text{RelTol} * \text{abs}(y(i)), \text{AbsTol}(i))$
AbsTol	Positive scalar or vector {1e-6}	The absolute error tolerance. If scalar, the tolerance applies to all components of the solution vector. Otherwise the tolerances apply to corresponding components.

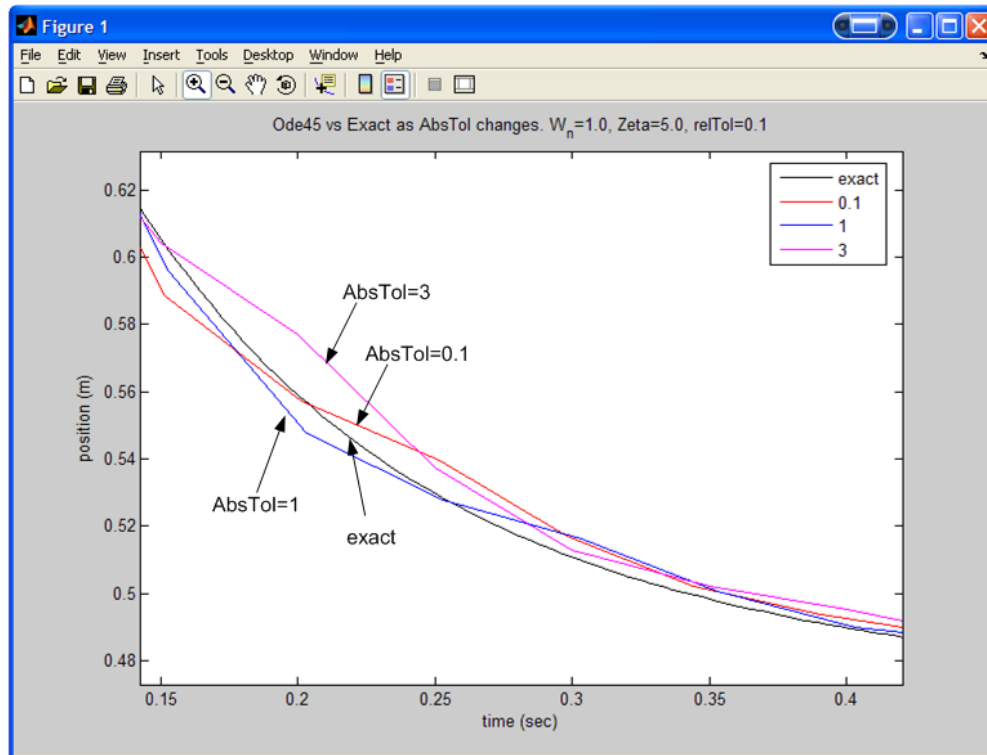
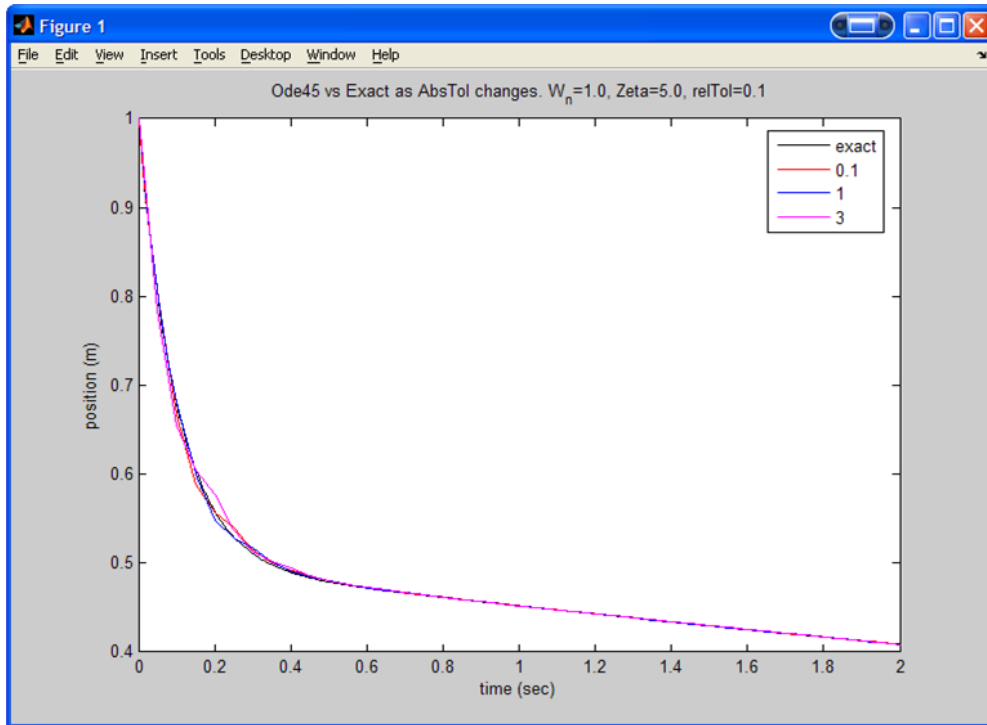
Conclusion: When the relative tolerance is made larger (while ABS. tolerance is fixed), the solution generate by ode45 was worst. In addition, as ζ increased, the solution generated by ode45 became even worst for the same relative tolerance used. (while ω_n is kept fixed) i.e. Relative tolerance is sensitive to the value of ζ .

As ω_n is increased (while keeping everything else fixed), ODE45 solution started to become less accurate compare to the analytical solution. Hence to improve the ODE45 solution the absolute tolerance was decreased. Decreasing the absolute tolerance in this case had better result on the accuracy of the solution as compared to decreasing the relative error while keeping the absolute error fixed.

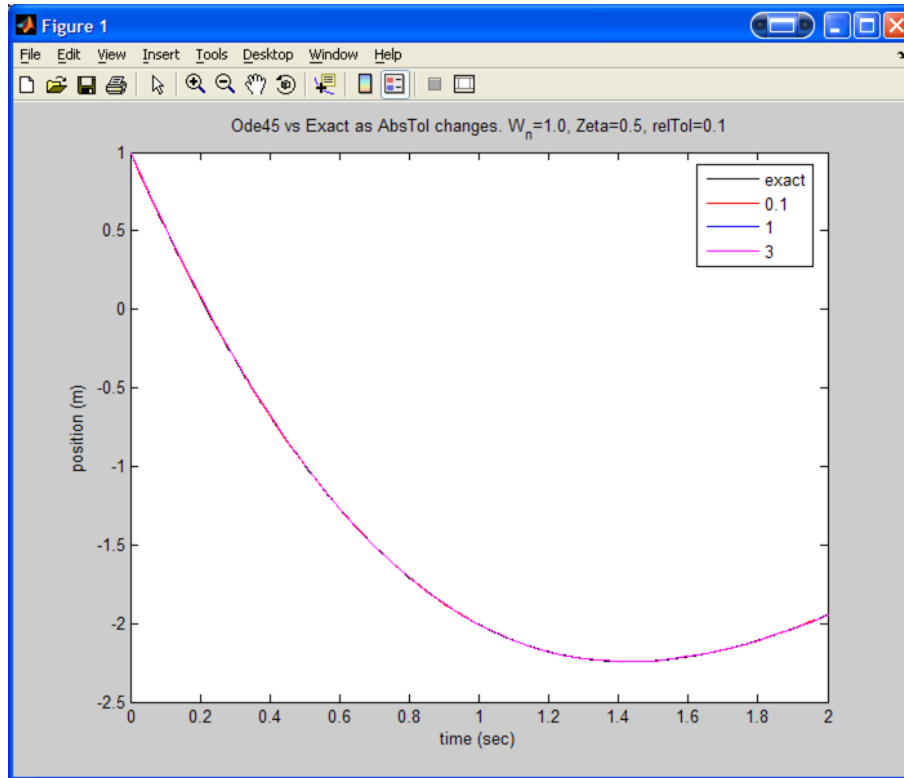
The absolute and relative tolerance values were adjust by using the Matlab function odeset.

Below are few plots showing the effect of changing the absolute and relative tolerance.

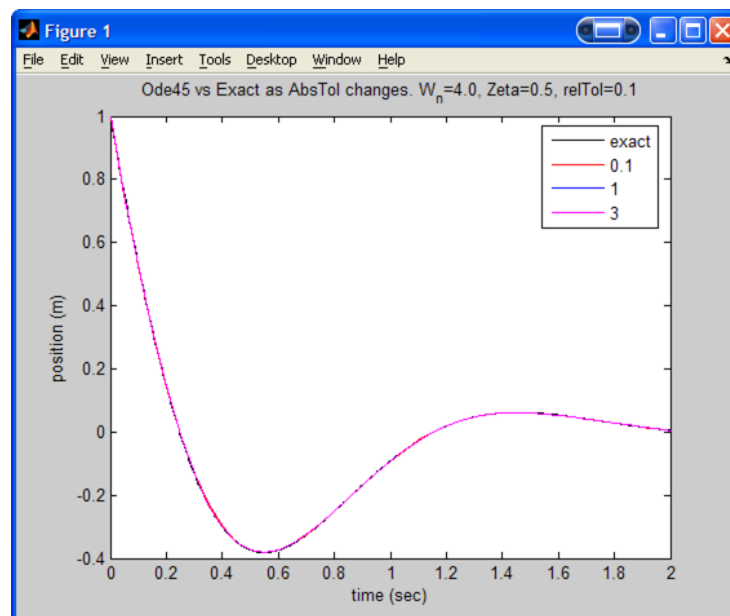
First, fix relative tolerance at 0.1 and solve for absolute tolerance for values 0.01,0.1,1 and 10. Results for $\omega_n = 1, \zeta = 5$ (With a zoom in to better illustrate the effect of changing ABS. Tolerance)



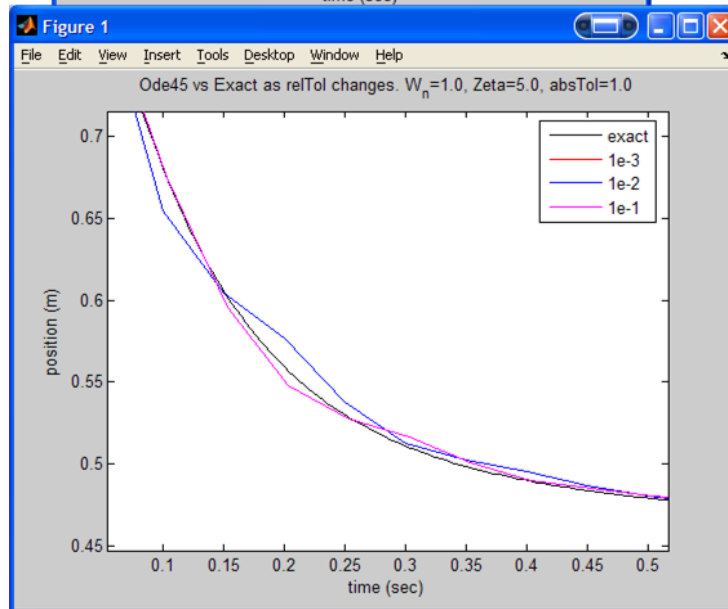
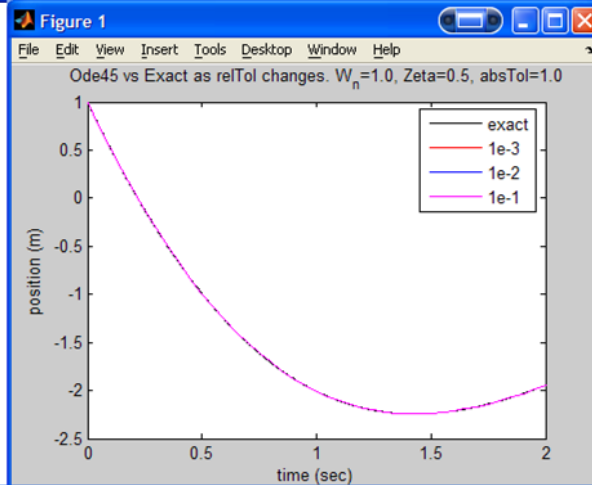
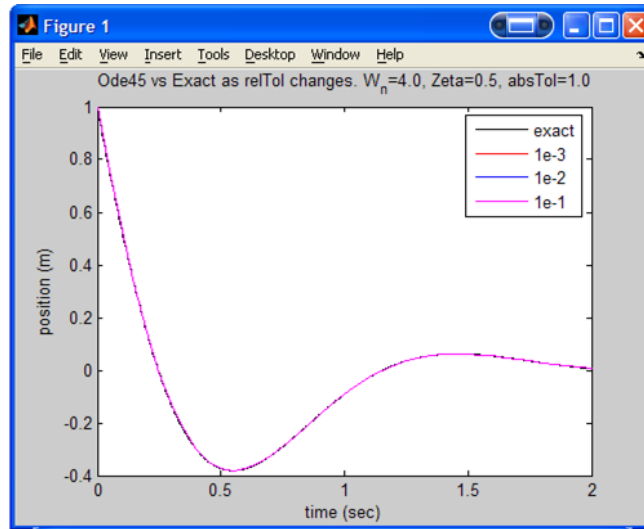
Results for $\omega_n = 1, \zeta = 0.5$ (With a zoom in to better illustrate the effect of changing ABS. Tolerance). For this case there is almost no difference as the ABS. tolerance is changed.



Results for $\omega_n = 4, \zeta = 0.5$. Again as above, we observe little effect on ode45 solution in this case as ABS. tolerance is changed.



Now we repeat the above plots but we fix the ABS. tolerance to 1 and modify the relative tolerance. Use values of $1e-3$, $1e-2$ and $1e-1$.



3 Part 4

If you didn't have an analytic solution, which will be the case for most nonlinear ODEs, how would you know when you have an accurate numerical solution?

If the solution converges to a steady solution as the time step is decreased, then we are more confident that an accurate solution was reached.

For example, in the case of Euler numerical solution, we have only one parameter at our disposal that we can modify, which is the step size. Hence we can decrease this step size until the solution no longer changes in any significantly measure. If the solution kept changing as the step size changed, this is an indication that we have not reached the correct solution yet.