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HW # 5

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HW#5

1. solve analytically $\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 0$

$x(0) = 1, \quad \dot{x}(0) = -5$

Solution

let $\left.\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned}\right\} \rightarrow \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2\xi\omega_n\dot{x} - \omega_n^2 x \\ &= -2\xi\omega_n x_2 - \omega_n^2 x_1 \end{aligned}$

hence $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Find λ's :

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -\omega_n^2 & -2\xi\omega_n - \lambda \end{vmatrix} = 0$

$-\lambda(-2\xi\omega_n - \lambda) + \omega_n^2 = 0 \Rightarrow \boxed{\lambda^2 + 2\xi\omega_n\lambda + \omega_n^2 = 0}$

roots $\lambda_{1,2} = \dfrac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2} = \boxed{-\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}}$

so $\boxed{\begin{aligned} \lambda_1 &= -\xi\omega_n + \omega_n\sqrt{\xi^2-1} \\ \lambda_2 &= -\xi\omega_n - \omega_n\sqrt{\xi^2-1} \end{aligned}}$

For each of these eigenvalues we obtain an eigenvector. say V_1, V_2. hence general solution is

$$\underset{2\times1}{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = c_1 e^{\lambda_1 t} \underset{2\times1}{\begin{pmatrix} V_1 \end{pmatrix}} + c_2 e^{\lambda_2 t} \underset{2\times1}{\begin{pmatrix} V_2 \end{pmatrix}}$$

→

to find V_1:

$$Av_1 = \lambda_1 v_1 \qquad \text{let } v_1 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\underset{2\times 2}{\begin{pmatrix} 0 & 1 \\ -w_n^2 & -2\xi w_n \end{pmatrix}} \underset{2\times 1}{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}} = \lambda_1 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

so
$$\begin{pmatrix} u_2 \\ -w_n^2 u_1 - 2\xi w_n u_2 \end{pmatrix} = \lambda_1 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

so
$$\left.\begin{aligned} u_2 &= -\xi w_n u_1 + w_n u_1\sqrt{\xi^2-1} \quad &— (1) \\ -w_n^2 u_1 - 2\xi w_n u_2 &= -\xi w_n u_2 + w_n u_2\sqrt{\xi^2-1} \quad &— (2) \end{aligned}\right\}$$

divid eq (1) by u_1 and eq (2) by u_2.

$$\left.\begin{aligned} \frac{u_2}{u_1} &= -\xi w_n + w_n\sqrt{\xi^2-1} \quad &— (3) \\ -w_n^2 \frac{u_1}{u_2} - 2\xi w_n &= -\xi w_n + w_n\sqrt{\xi^2-1} \quad &— (4) \end{aligned}\right\}$$

$$(3)-(4) \Rightarrow \frac{u_2}{u_1} + w_n^2\frac{u_1}{u_2} + 2\xi w_n = 0$$

multiply by $u_2 u_1$ $\Rightarrow$

$$\boxed{u_2^2 + w_n^2 u_1^2 + 2\xi w_n u_1 u_2 = 0.}$$

solutions are

$$u_2 = -u_1\xi w_n - \sqrt{-u_1^2 w_n^2 + u_1^2\xi^2 w_n^2}$$

$$u_2 = -u_1\xi w_n + \sqrt{-u_1^2 w_n^2 + u_1^2\xi^2 w_n^2}$$

$\longrightarrow$

hence to find V_1 I pick first solution.
to find V_2 I pick second solution.

for V_1

$$u_2 = -u_1 \zeta w_n - \sqrt{-u_1^2 w_n^2 + u_1^2 \zeta^2 w_n^2}$$

let $u_1 = 1$

so $u_2 = -\zeta w_n - \sqrt{-w_n^2 + \zeta^2 w_n^2}$

Case A $\quad w_n = 1, \ \zeta = 5 \Rightarrow w_n^2 = 1, \ (\zeta w_n)^2 = 25$

$$u_2 = -5 - \sqrt{-1+25} = -5 - \sqrt{24} = -5 - 2\sqrt{6} = -9.899$$

so $V_1 = \begin{pmatrix} 1 \\ -5-2\sqrt{6} \end{pmatrix}$

Case B $\quad w_n = 1, \ \zeta = 10 \Rightarrow w_n^2 = 1, \ (\zeta w_n)^2 = 100.$

so $u_2 = -10 - \sqrt{-1+100} = -10 - 9.9498 = -19.9498$

so $V_1 = \begin{pmatrix} 1 \\ -19.9498 \end{pmatrix}$

Case C $\quad w_n = 1, \ \zeta = 0.5 \Rightarrow \zeta w_n = 0.5, \ w_n^2 = 1, \ (\zeta w_n)^2 = 0.25$

$$u_2 = -0.5 - \sqrt{-1+0.25} = -0.5 - j0.866$$

so $V_1 = \begin{pmatrix} 1 \\ -0.5 - j\,0.866 \end{pmatrix}$

for V_2 $\quad u_2 = -u_1 \zeta w_n + \sqrt{-u_1^2 w_n^2 + u_1^2 \zeta^2 w_n^2}$, let $u_1 = 1$

Case A $\quad u_2 = -5 + 4.899 = -0.101 \Rightarrow V_2 = \begin{pmatrix} 1 \\ -0.101 \end{pmatrix}$

Case B $\quad u_2 = -10 + 9.9498 = -0.0501 \Rightarrow V_2 = \begin{pmatrix} 1 \\ -0.0501 \end{pmatrix}$

Case C $\quad u_2 = -0.5 + j\,0.866 \Rightarrow V_2 = \begin{pmatrix} 1 \\ -0.5 + j\,0.866 \end{pmatrix}$

$\longrightarrow$

Summary

Case A

$$V_1 = \begin{pmatrix} 1 \\ -9.899 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 \\ -0.101 \end{pmatrix}$$

Case B

$$V_1 = \begin{pmatrix} 1 \\ -19.945 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 \\ -0.0501 \end{pmatrix}$$

Case C

$$V_1 = \begin{pmatrix} 1 \\ -0.5 - j\,0.866 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 \\ -0.5 + j\,0.866 \end{pmatrix}$$

General solutions

Case A $\quad \xi = 5,\ w_n = 1 \Rightarrow \lambda_1 = -5 + \sqrt{25-1} = -0.101$

$\lambda_2 = -5 - \sqrt{25-1} = -9.8989$

so $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 e^{+0.101t} \begin{pmatrix} 1 \\ -9.899 \end{pmatrix} + C_2 e^{-9.8989t} \begin{pmatrix} 1 \\ -0.101 \end{pmatrix}$$

Case B $\quad \xi = 10,\ w_n = 1 \Rightarrow \lambda_1 = -10 + 1\sqrt{10^2-1} = -0.05$

$\lambda_2 = -10 - \sqrt{10^2-1} = -19.9498$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 e^{-0.05t} \begin{pmatrix} 1 \\ -19.945 \end{pmatrix} + C_2 e^{-19.95t} \begin{pmatrix} 1 \\ -0.0501 \end{pmatrix}$$

Case C $\quad \xi = .5,\ w_n = 1 \Rightarrow \lambda_1 = -0.5 + 1\sqrt{.25-1} = -0.5 + 0.866j$

$\lambda_2 = -0.5 - 0.866j$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 e^{(-0.5+0.866j)t} \begin{pmatrix} 1 \\ -0.5 - j\,0.866 \end{pmatrix} + C_2 e^{(-0.5-0.866j)t} \begin{pmatrix} 1 \\ -.5 + j\,0.866 \end{pmatrix}$$

$\rightarrow$

(iii)
how to find C_1, C_2 using initial conditions.

Case A $\quad x_1(0) = 1, \quad x_2(0) = -5.$

so $\begin{pmatrix} 1 \\ -5 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -9.899 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -0.101 \end{pmatrix}$

so $\left.\begin{aligned} 1 &= C_1 + C_2 \\ -5 &= -9.899 C_1 - 0.101 C_2 \end{aligned}\right\}$ $\quad C_1 = 1 - C_2 \quad \longleftrightarrow -5 = -9.899(1-C_2) - 0.101 C_2$

so $-5 = -9.899 + 9.899 C_2 - 0.101 C_2 = -9.899 + 9.798 C_2$

so $4.899 = 9.798 C_2 \Rightarrow \boxed{C_2 = 0.5} \Longrightarrow \boxed{C_1 = 0.5}$

so $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.5\, e^{-0.101t} \begin{pmatrix} 1 \\ -9.899 \end{pmatrix} + 0.5\, e^{-9.8989t} \begin{pmatrix} 1 \\ -0.101 \end{pmatrix}$

Case B

$\begin{pmatrix} 1 \\ -5 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -19.945 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -0.055 \end{pmatrix}$

$\left.\begin{aligned} 1 &= C_1 + C_2 \\ -5 &= -19.945 C_1 - 0.055 C_2 \end{aligned}\right\}$ $\quad C_1 = 1 - C_2 \quad \rightarrow -5 = -19.945(1-C_2) - 0.055 C_2$

$-5 = -19.945 + 19.945 C_2 - 0.055 C_2 \rightarrow -5 + 19.945 = C_2(19.945 - 0.055)$

$C_2 = \dfrac{14.945}{19.89} = \boxed{0.7514} \Rightarrow C_1 = 1 - 0.7514 = \boxed{0.2486}$

so $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.2486\, e^{-0.05t} \begin{pmatrix} 1 \\ -19.945 \end{pmatrix} + 0.7514\, e^{-19.95t} \begin{pmatrix} 1 \\ -0.0501 \end{pmatrix}$

Case C $\quad \begin{pmatrix} 1 \\ -5 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -0.5 - j0.866 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -0.5 + j0.866 \end{pmatrix}$

$\left.\begin{aligned} 1 &= C_1 + C_2 \quad \rightarrow C_1 = 1 - C_2 \\ -5 &= -0.5 C_1 - j0.866 C_1 - 0.5 C_2 + j0.866 C_2 \end{aligned}\right\}$

$-5 = -0.5(1-C_2) - j0.866(1-C_2) - 0.5 C_2 + j0.866 C_2 \longrightarrow$

$$-5 = -0.5 + \cancel{0.5 c_2} - j0.866 + j0.866 c_2 - \cancel{0.5 c_2} + j0.866 c_2$$

$$-4.5 = -j0.866 + c_2(1.732j)$$

$$c_2 = \frac{-4.5 + j0.866}{1.732j} \times \frac{1.732j}{1.732j} = \frac{-7.794j - 1.5}{-3} = \boxed{2.598j + 0.5}$$

$$c_1 = 1 - c_2$$

$$= 1 - (2.598j + 0.5) = \boxed{0.5 - 2.598j}$$

So

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (0.5 - 2.598j)\, e^{(-0.5+0.866j)t} \begin{pmatrix} 1 \\ -0.5 - j0.866 \end{pmatrix} + (0.5 + 2.598j)\, e^{(-0.5-0.866j)t} \begin{pmatrix} 1 \\ -0.5 + j0.866 \end{pmatrix}}$$

Case C general solution

$$\text{so } x_1 = 0.5\, e^{(-0.5+0.866j)t} + 0.5\, e^{(-0.5-0.86j)t} - 2.598j\, e^{(-0.5+0.866j)t} + 2.598j\, e^{(-0.5-0.866j)t}$$

$$x_1 = e^{-0.5t}\left[0.5\, e^{0.866jt} + 0.5\, e^{-0.866jt} - j2.598 e^{0.866jt} + j2.598 e^{-0.866jt}\right]$$

$$\text{but } \sin\alpha = \frac{e^{j\alpha} - e^{-j\alpha}}{2j}, \quad \cos\alpha = \frac{e^{j\alpha} + e^{-j\alpha}}{2}$$

$$\text{So } x_1(t) = e^{-0.5t}\left[0.5\,(2\cos 0.866t) - j2.598\,(2j \sin 0.866t)\right]$$

$$\boxed{x_1(t) = e^{-0.5t}\left[\cos 0.866t + 5.196 \sin 0.866t\right]}$$

Case C — alternative solution for x_1

now find alternative solution for $x_2(t)$ for case c

$x_2(t) =$

$$(0.5-2.598j)e^{(-0.5+0.866j)t}(-0.5-j0.866)$$

$$+ (0.5-2.598j)e^{(-0.5-0.866j)t}(-0.5+j0.866)$$

$$= e^{-0.5t}\left[(0.5e^{0.866jt} - 2.598j e^{0.866jt})(-0.5-j0.866) + 0.5e^{-0.866jt} - 2.598j e^{-0.866jt}(-0.5+j0.866)\right]$$

$$= e^{-0.5t}\left[-0.25e^{0.866jt} - j0.433e^{0.866jt} + 1.3je^{0.866jt} - 2.25e^{0.866jt} - 0.25e^{-8.66jt} + j0.433e^{-0.866jt} + 1.3je^{-0.866jt} + 2.25e^{-0.866jt}\right]$$

$$= e^{-0.5t}\left[-0.25(2\cos 0.866t) - j\,0.433(2j\sin 0.866t) + 1.3j(2j\sin 0.866t) - 2.25(2\cos 0.866t)\right]$$

$$= e^{-0.5t}\left[-0.5\cos 0.866t + 0.866\sin 0.866t - 2.6\sin 0.866t - 4.5\cos 0.866t\right]$$

$$= e^{-0.5t}\left[-5\cos 0.866t - 1.734\sin 0.866t\right]$$

hence for Case C, let $0.866 = \omega$

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-0.5t}\begin{pmatrix} \cos\omega t + 5.196\sin\omega t \\ -5\cos\omega t - 1.734\sin\omega t \end{pmatrix}}$$

Part 2

Euler Numerical integration

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -w_n^2 & -2\zeta w_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 w_n^2 - 2x_2 \zeta w_n$$

Euler equations

Euler

$$\begin{cases} x_1(i) = x_1(i-1) + \Delta t \, x_2(i-1) \\ x_2(i) = x_2(i-1) + \Delta t\left(-2\zeta w_n x_2(i-1) - w_n^2 x_1(i-1)\right) \end{cases}$$

The following is the result running matlab program to show this.

for backward Euler: $\dot{x} = \lambda x$

$$x(t+\Delta t) = x(t) + \Delta t\left[\lambda x(t+\Delta t)\right]$$

$$x(t+\Delta t) = \left(\frac{x(t)}{1-\Delta t \lambda}\right)$$

so $$x_1(i) = \frac{x_1(i-1)}{1-\Delta t \, x'(i-1)} = \frac{x_1(i-1)}{1-\Delta t \, x_2(i-1)}$$

and $$x_2(i) = \frac{x_2(i-1)}{1-\Delta t \, x_2'(i-1)} = \frac{x_2(i-1)}{1-\Delta t\left[-x_1(i-1)w_n^2 - 2\zeta w_n x_2(i-1)\right]}$$

backward Euler

$$x_2(i) = \frac{x_2(i-1)}{1+\Delta t \, x_1(i-1) w_n^2 + 2\Delta t \zeta w_n x_2(i-1)}$$

$\dot{x} = ax$

To determine max step size for A-stable

we need $|1 + a\Delta t| < 1$

or $\boxed{-1 < 1 + a\Delta t < 1}$

where λ is from $\dot{x} = ax$

Case A

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -w_n^2 & -2\zeta w_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\zeta = 5, w_n = 1$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\boxed{\dot{\underline{x}} = A \underline{x}}$$

for x_1 we found solution $x_1(t) = 0.5e^{-0.10t} + 0.5e^{-9.89t}$

hence $\lambda_1 = -0.101$, $\lambda_2 = -9.89$.

so $-1 < 1 + (-0.101)\Delta t < 1$

$-1 < 1 - 0.101\Delta t < 1$

so $\Delta t < \frac{2}{1.01} < 1.98$ for λ_1

$\lambda_2 = -9.89$

$-1 < 1 + (-9.89\Delta t) < 1$

so $\Delta t < \frac{2}{9.89} < 0.202$

hence max step size for A-stable is $\boxed{0.202 \text{ sec}}$

$\longrightarrow$

Case B

we found. $x_1(t) = 0.2486\, e^{-0.05t} + 0.7514\, e^{-19.95t}$.

so $$1 < 1 - 0.05\,\Delta t < 1$$

so $$\Delta t < \frac{2}{0.05} < 40$$

or

$$1 < 1 - 19.95\,\Delta t < 1$$

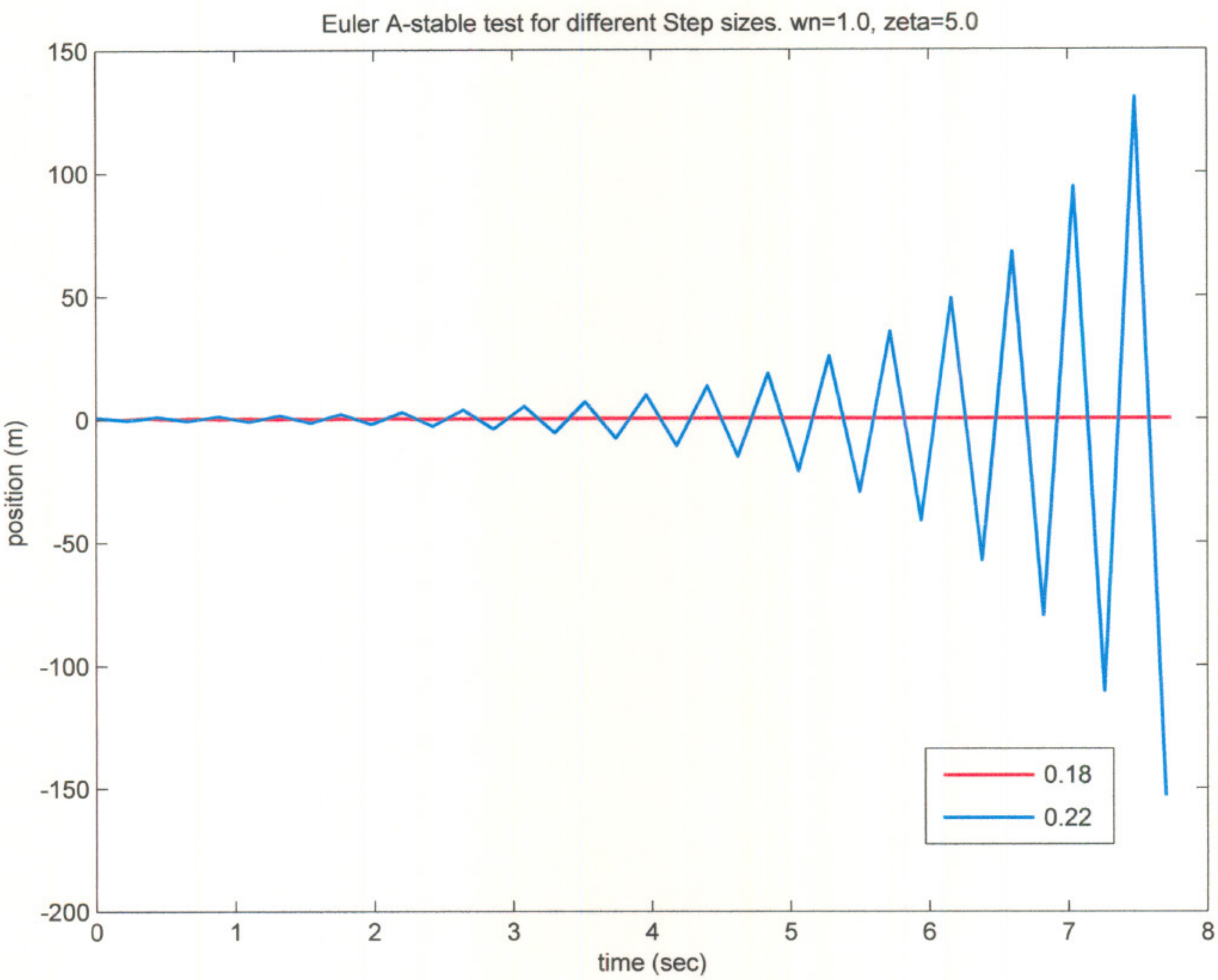
so $$\Delta t < \frac{2}{19.95} < 0.1$$

so $\boxed{\Delta t < 0.1}$ for A-stable

For stiff problem, the step size is limited by the step size needed to keep the solution which changes most rapidly stable.

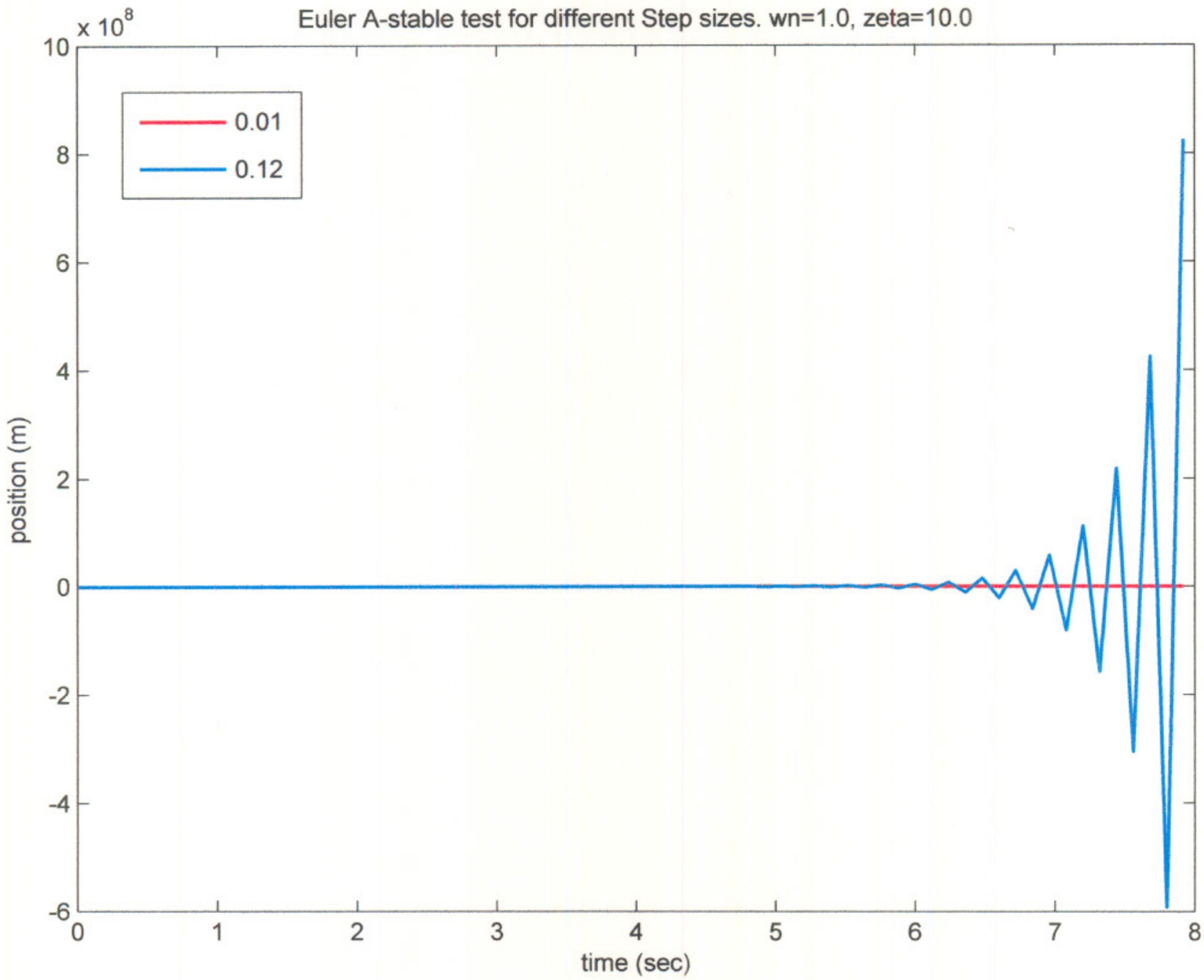
so if we have 2 eigenvectors, hence 2 solutions, and 2 eigenvalues, we check stability of both, and use the smaller step size found for the overall solution even if one of them has converged.

Case A



Comparison between Euler & Backward

Case B

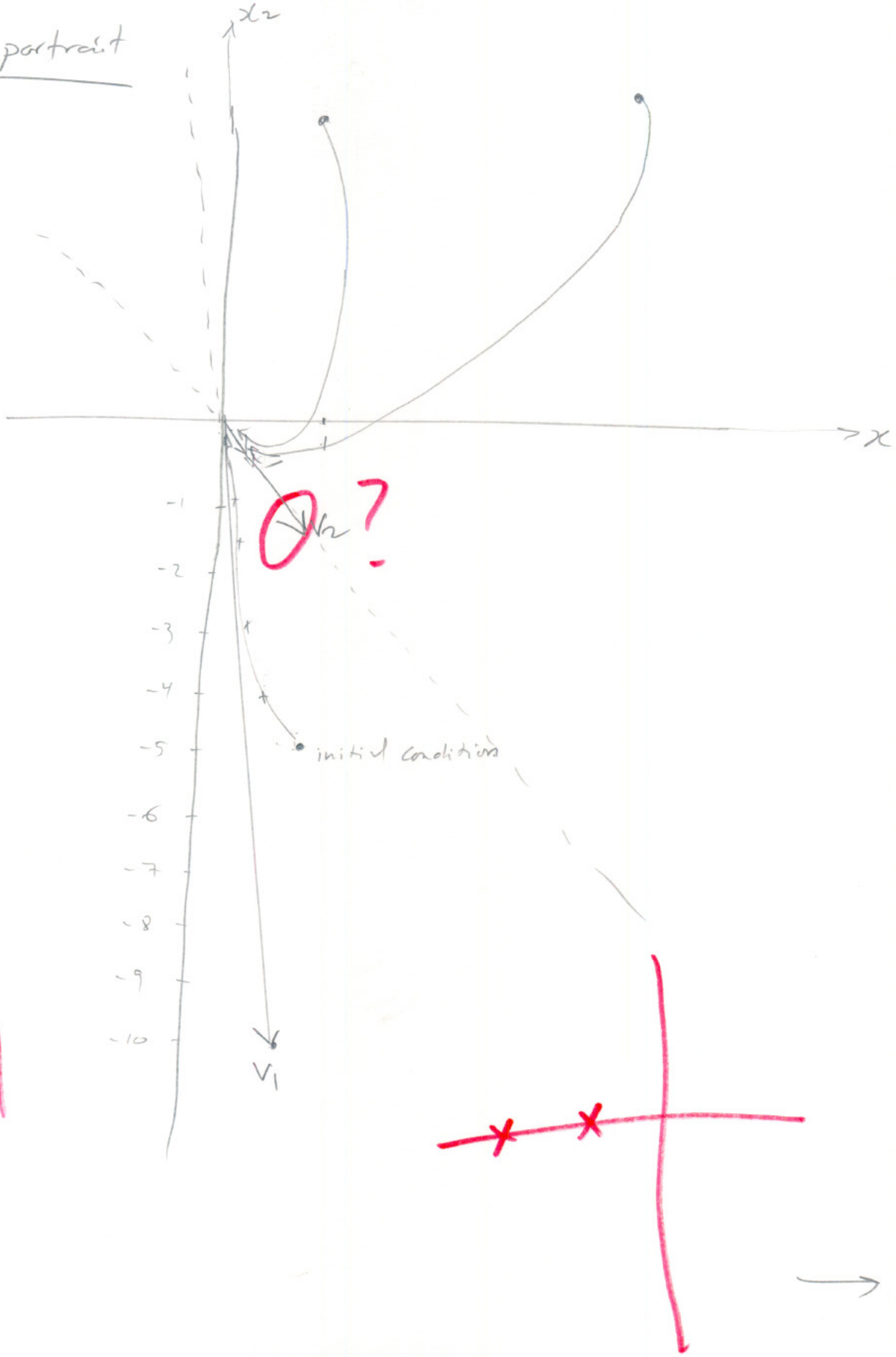


Euler A-stable test for different Step sizes. wn=1.0, zeta=10.0
x 10^8
10
8
6
4
2
0
-2
-4
-6
position (m)
0
1
2
3
4
5
6
7
8
time (sec)
0.01
0.12

part 3

State portrait

Case A



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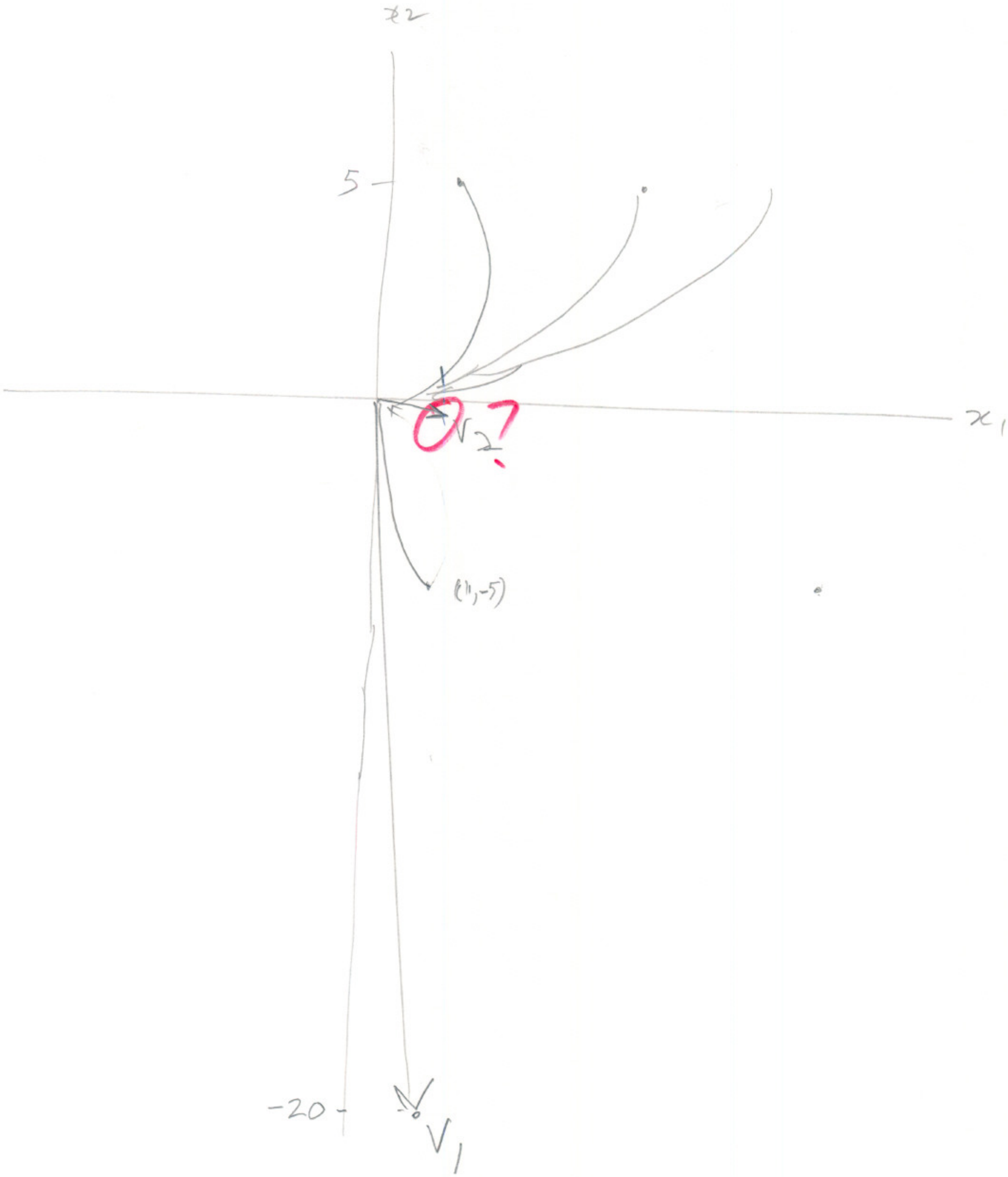
Case B



~1 case C?