i) $\ddot{x} + 2 z W_n \dot{x} + W_n^2 x = 0$ I.Cs $x(0) = 1$, $\dot{x}(0) = -5$

$$\left.\begin{array}{l} x_1 = x \\ x_2 = \dot{x} \end{array}\right\} \rightarrow \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2 z W_n x_2 - W_n^2 x_1 \end{array} \rightarrow$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -W_n^2 & -2 z W_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

ii) Eigen Values :

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -W_n^2 & -2 z W_n - \lambda \end{vmatrix}$$

$$= (-\lambda)(-2 z W_n - \lambda) + W_n^2 = 0 \Rightarrow$$

$$\lambda^2 + 2 z W_n \lambda + W_n^2 = 0 \Rightarrow$$

$$\lambda_1 = - z W_n + W_n \sqrt{z^2 - 1}$$

$$\lambda_2 = - z W_n - W_n \sqrt{z^2 - 1}$$

Eigen Vectors :

System 1 : $W_n = 1.0$; $z = 5.0 \Rightarrow \lambda_{1,2} = -5 \pm 2\sqrt{6}$

$$[A - \lambda_1 I] V_1 = 0 \Rightarrow \begin{bmatrix} 5 - 2\sqrt{6} & 1 \\ -1 & -10 + 5 - 2\sqrt{6} \end{bmatrix} V_1 = 0$$

$$\Rightarrow V_1 = \begin{pmatrix} 1 \\ 2\sqrt{6} - 5 \end{pmatrix}$$

$$(A - \lambda_2 I) V_2 = 0 \Rightarrow \begin{vmatrix} 5 + 2\sqrt{6} & 1 \\ -1 & -5 + 2\sqrt{6} \end{vmatrix} V_2 = 0$$

$$\text{so} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 e^{(-5+2\sqrt{6})t} \begin{pmatrix} 1 \\ 2\sqrt{6}-5 \end{pmatrix} + C_2 e^{(-5-2\sqrt{6})t} \begin{pmatrix} 1 \\ -5-2\sqrt{6} \end{pmatrix}$$

System 2: $W_n = 1.0$, $Z = 10.0 \Longrightarrow$

$$\lambda_1 = -10 + 3\sqrt{11} \quad ; \quad \lambda_2 = -10 - 3\sqrt{11}$$

$$|A - \lambda_1 I|\, V_1 = 0 \Longrightarrow \begin{vmatrix} 10 - 3\sqrt{11} & 1 \\ -1 & -20 + 10 - 3\sqrt{11} \end{vmatrix} V_1 = 0 \Longrightarrow$$

$$V_1 = \begin{pmatrix} 1 \\ 3\sqrt{11} - 10 \end{pmatrix}$$

$$|A - \lambda_2 I|\, V_2 = 0 \Longrightarrow \begin{vmatrix} 10 + 3\sqrt{11} & 1 \\ -1 & -20 + 10 + 3\sqrt{11} \end{vmatrix} V_2 = 0 \Longrightarrow$$

$$V_2 = \begin{pmatrix} 1 \\ -10 - 3\sqrt{11} \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 e^{(-10+3\sqrt{11})t} \begin{pmatrix} 1 \\ 3\sqrt{11} - 10 \end{pmatrix} + C_2 e^{(-10-3\sqrt{11})t} \begin{pmatrix} 1 \\ -10 - 3\sqrt{11} \end{pmatrix}$$

System 3: $W_n = 1.0$; $Z = 0.5 \Longrightarrow$

$$\lambda_1 = -0.5 - \frac{\sqrt{3}}{2} j \qquad \lambda_2 = -0.5 + \frac{\sqrt{3}}{2} j$$

$$|A - \lambda_1 I|\, V_1 = 0 \Longrightarrow \begin{vmatrix} 0.5 + \frac{\sqrt{3}}{2} j & 1 \\ -1 & -1 + 0.5 + \frac{\sqrt{3}}{2} j \end{vmatrix} V_1 = 0$$

$$\Longrightarrow V_1 = \begin{pmatrix} 1 \\ -0.5 - \frac{\sqrt{3}}{2} j \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 e^{(-0.5-\frac{\sqrt{3}}{2}j)t} \begin{pmatrix} 1 \\ -0.5-\frac{\sqrt{3}}{2}j \end{pmatrix} + C_2 e^{(-0.5+\frac{\sqrt{3}}{2}j)t} \begin{pmatrix} 1 \\ -0.5+\frac{\sqrt{3}}{2}j \end{pmatrix}$$

iii) Find C_1 & C_2

system 1 : $x(0)=1$, $\dot{x}(0)=-5$

$$\Longrightarrow \begin{pmatrix} 1 \\ -5 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 2\sqrt{6}-5 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -5-2\sqrt{6} \end{pmatrix}$$

$$\Longrightarrow \begin{cases} C_1 + C_2 = 1 \\ (2\sqrt{6}-5)C_1 + (-5-2\sqrt{6})C_2 = -5 \end{cases} \Longrightarrow C_1 = C_2 = 0.5$$

system 2 : $x(0)=1$, $\dot{x}(0)=-5$

$$\Longrightarrow \begin{pmatrix} 1 \\ -5 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 3\sqrt{11}-10 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -10-3\sqrt{11} \end{pmatrix}$$

$$\Longrightarrow \begin{cases} C_1 + C_2 = 1 \\ (3\sqrt{11}-10)C_1 + (-10-3\sqrt{11})C_2 = -5 \end{cases} \Longrightarrow \cancel{C_1 = C_2 = 0.5}$$

$$C_2 = \frac{3\sqrt{11}-5}{6\sqrt{11}} \quad C_1 = \frac{5+3\sqrt{11}}{6\sqrt{11}}$$

system3 : $x(0)=1$, $\dot{x}(0)=-5$

$$\Longrightarrow \begin{pmatrix} 1 \\ -5 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -0.5-\frac{\sqrt{3}}{2}j \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -0.5+\frac{\sqrt{3}}{2}j \end{pmatrix}$$

$$\Longrightarrow \begin{cases} C_1 + C_2 = 1 \\ (-0.5-\frac{\sqrt{3}}{2}j)C_1 + (-0.5+\frac{\sqrt{3}}{2}j)C_2 = -5 \end{cases}$$

$$\Longrightarrow C_1 = \frac{\frac{\sqrt{3}}{2} - 4.5j}{\sqrt{3}}$$

$$\frac{\sqrt{3}}{2} + 4.5j$$

iv) From general solution of case c, we know that

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{-0.5t}\left(\cos\left(-\frac{\sqrt{3}}{2}t\right) + i\sin\left(-\frac{\sqrt{3}}{2}t\right)\right)\begin{pmatrix} 1 \\ -0.5-\frac{\sqrt{3}}{2}i \end{pmatrix}$$

is a solution. Therefore, its real part and imaginary part are solutions too.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{-0.5t}\begin{pmatrix} \cos\frac{\sqrt{3}}{2}t - i\sin\frac{\sqrt{3}}{2}t \\ -0.5\cos\frac{\sqrt{3}}{2}t - \sin\frac{\sqrt{3}}{2}t + 0.5\sin\frac{\sqrt{3}}{2}t\,i - \frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t\,i \end{pmatrix}$$

seperate the real and imaginary part.

general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1' e^{-0.5t}\begin{pmatrix} \cos\frac{\sqrt{3}}{2}t \\ -\frac{1}{2}\cos\frac{\sqrt{3}}{2}t - \sin\frac{\sqrt{3}}{2}t \end{pmatrix} + c_2' e^{-0.5t}\begin{pmatrix} -\sin\frac{\sqrt{3}}{2}t \\ 0.5\sin\frac{\sqrt{3}}{2}t - \frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t \end{pmatrix}$$

2. Predict the maximum step size. $\dot{x} = ax$

Euler:
$$x(t+\Delta t) - x(t) = \Delta t \cdot \dot{x} = \Delta t\, a x(t)$$

$$\Rightarrow x(t+\Delta t) = (1+\Delta t\, a)\, x(t)$$

$$|1+\Delta t\, a| < 1 \longleftrightarrow \text{stable solution}$$

Backward Euler:

$$x(t+\Delta t) - x(t) = \Delta t \cdot \dot{x}(t+\Delta t) = \Delta t\, a\, x(t+\Delta t)$$

$$\Rightarrow x(t+\Delta t) = \frac{1}{1-\Delta t\, a}\, x(t)$$

$$\left|\frac{1}{1-\Delta t a}\right| < 1 \longleftrightarrow \text{stable solution}$$

since $a<0$ for stable system, it is ALWAYS satisfied.

case A: $\quad \underline{X}(t) = C_1 e^{-0.101t} v_1 + C_2 e^{-9.89t} v_2$

$\lambda_1 = -0.101 \qquad \lambda_2 = -9.89$

$$\Rightarrow \left.\begin{cases} -1 < 1+\Delta t(-0.101) < 1 & \Rightarrow \Delta t < 1.98 \\ -1 < 1+\Delta t(-9.89) < 1 & \Rightarrow \Delta t < 0.202 \end{cases}\right\} \Rightarrow \Delta t < 0.202$$

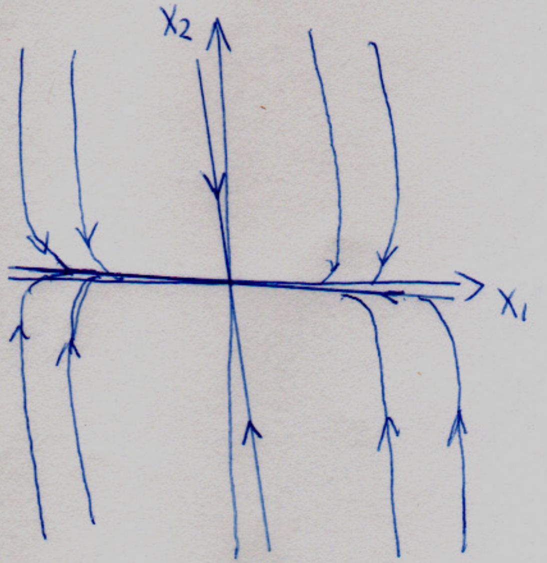
case B: $\quad \mathbf{X}(t) = c_1 e^{-0.05t} v_1 + c_2 e^{-19.95t} v_2$

$$\lambda_1 = -0.05 \quad ; \quad \lambda_2 = -19.95$$

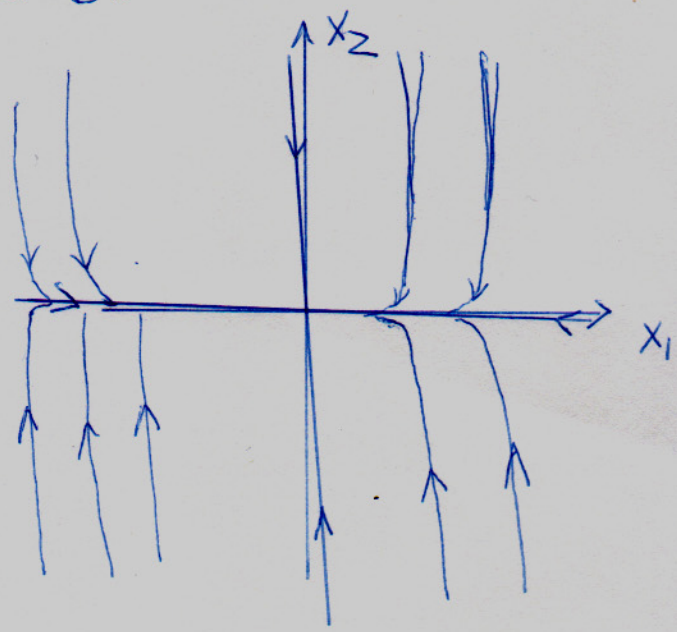
$$\Rightarrow \left\{ \begin{array}{ll} -1 < 1 + \Delta t(-0.05) < 1 & \Rightarrow \Delta t < 40 \\ -1 < 1 + \Delta t(-19.95) < 1 & \Rightarrow \Delta t < 0.1 \end{array} \right\} \Rightarrow \Delta t < 0.1$$

3. Sketches

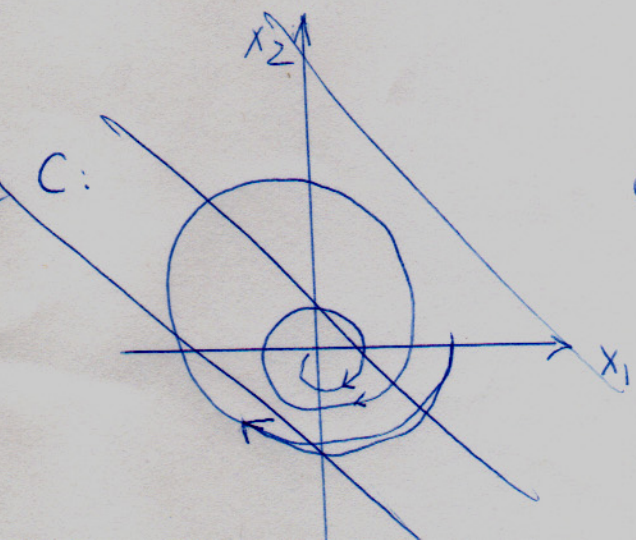
case A.



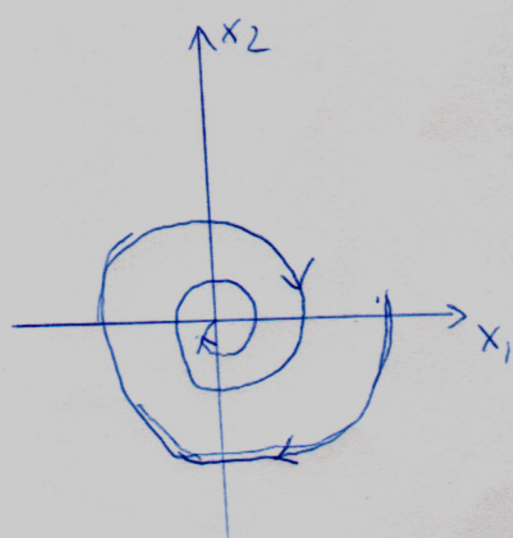
case B:



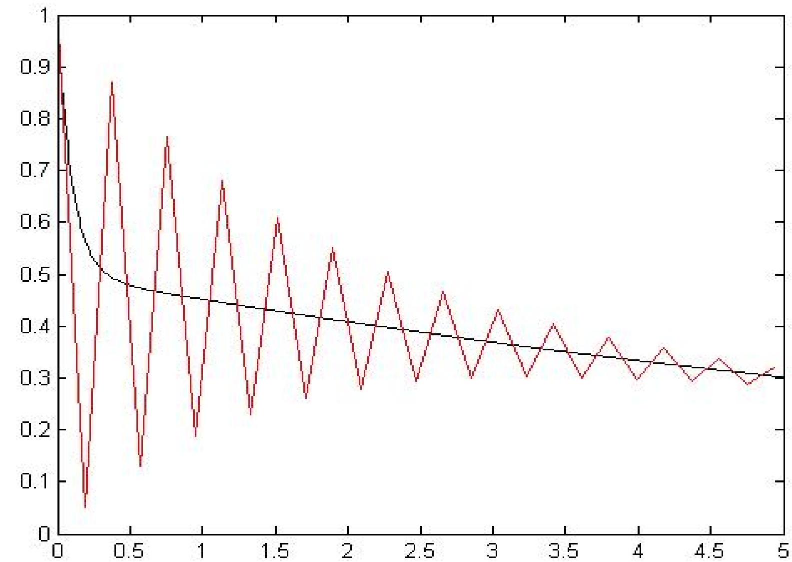
case C:



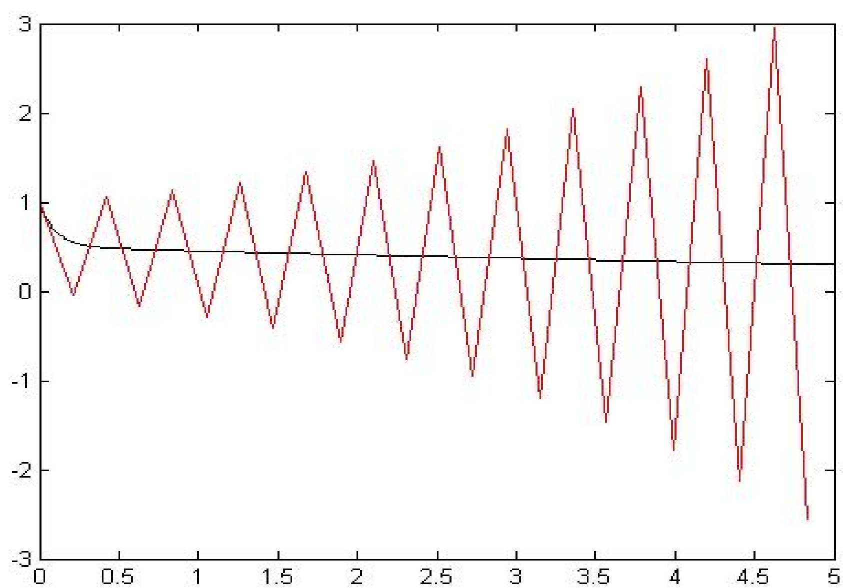
case C



Case A: wn=1.0; z=5.0. Predicted maximum step size for stability is dtE=0.202. (Black-Analytical; Red-Euler Method)

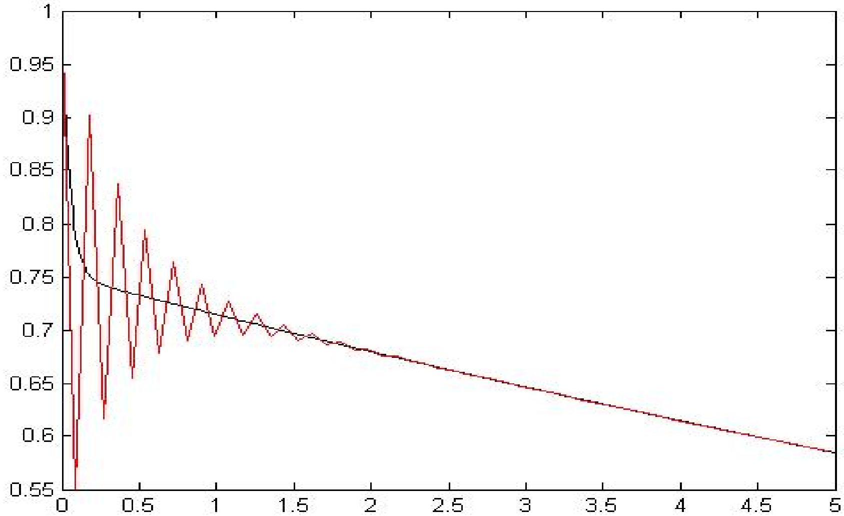


Stable, dtE=0.19

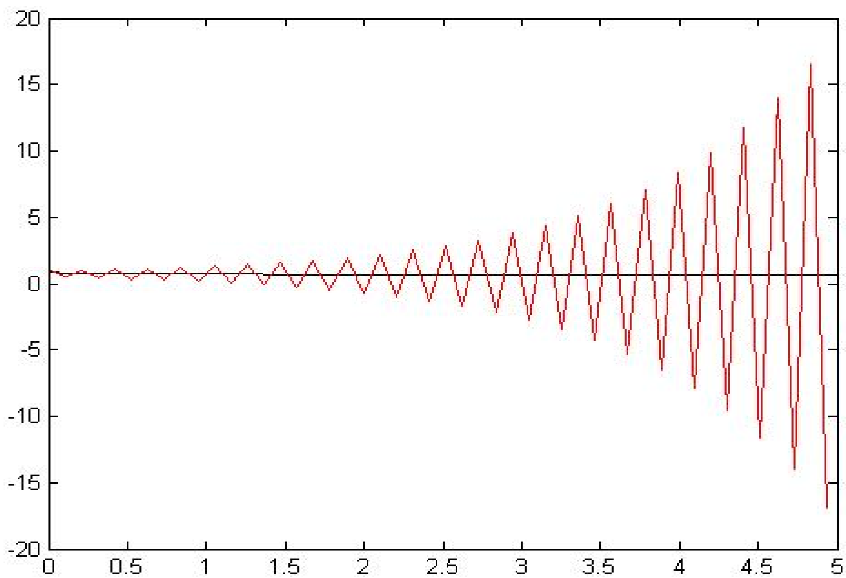


Unstable, dtE=0.21

Case B: wn=1.0; z=10.0. Predicted maximum step size for stability is dtE=0.1(Black-Analytical; Red-Euler Method)



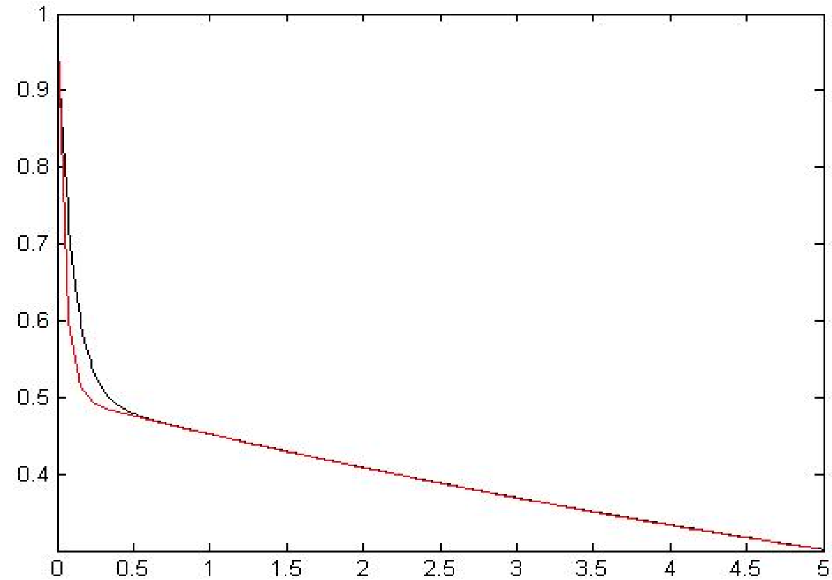
Stable,dtE=0.09



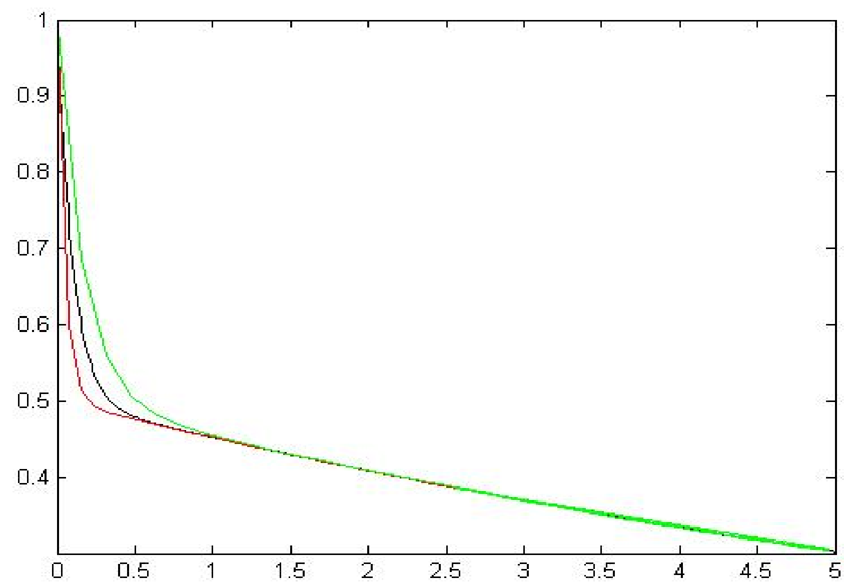
Unstable, dtE=0.105

The prediction of the maximum stepfor case A & B is validated by the four figures above.

Good Integration: Case A: wn=1.0; z=5.0. (Black-Analytical; Red-Euler Method; Green-Backward Euler Method)



Euler Method; dtE=0.08, 40% of maximum step size for stability.



Euler Method dtE=0.08; Backward Euler Method dtE=0.16 Almost same accuracy.

For Euler Method, step size of 40%-50% percent of the maximum step size is needed to get good match. For Backward Euler Method, the step size can be two times of the one used in the Euler Method to achieve a similar accuracy.

The things we learned: For Euler integration of the stiff system, the maximum step size is determined by the fater-decaying term, which has a larger negative eigenvalue. Backward Euler method is inherently stable.

Backward Euler method can have the same accuracy as Euler method with a larger step size, therefore performs better in this perspective.