Part e

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MAE 200A

HW # 7

population dynamics .

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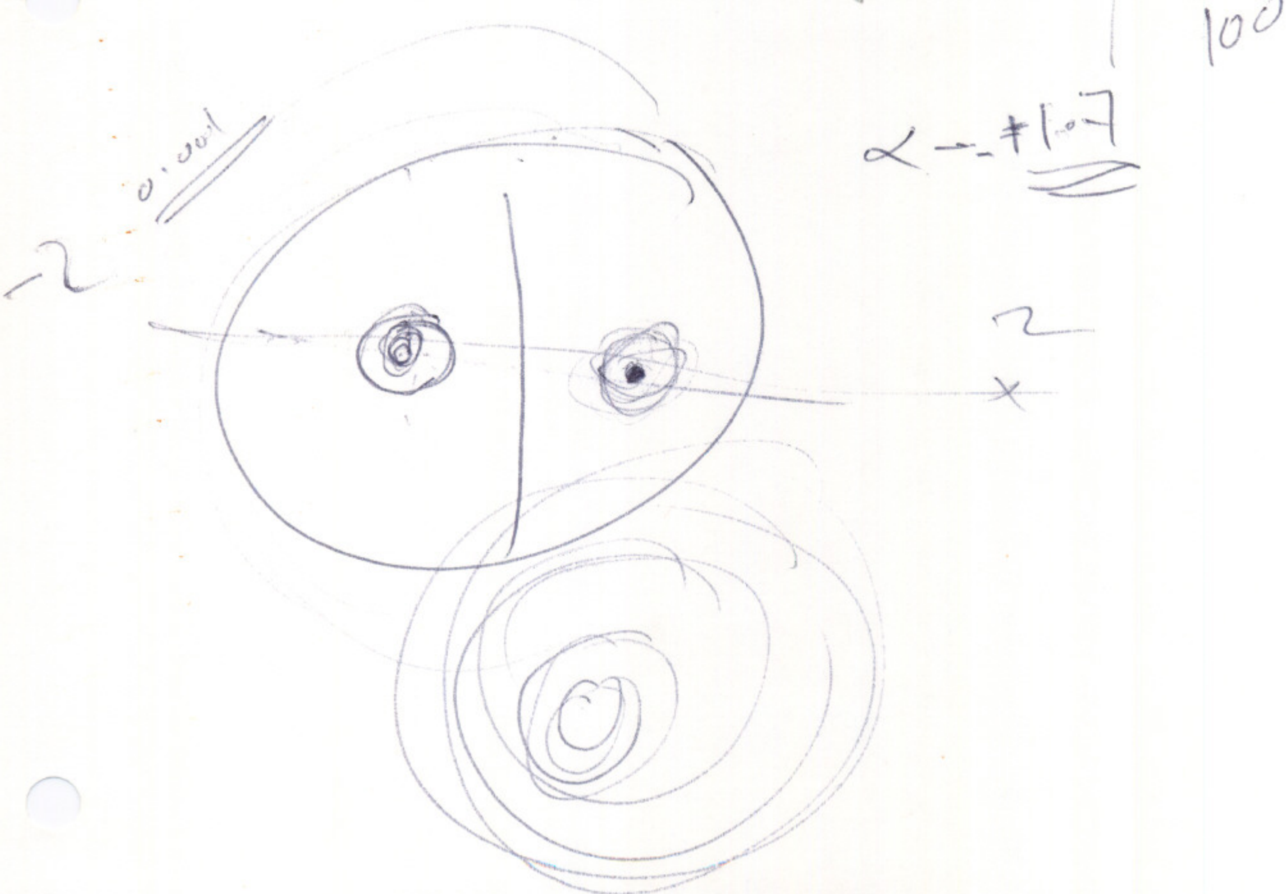


$$\dot{x}_1 = x_1 - x_1^2 - x_1 x_2 \quad \text{——— (1)}$$
$$\dot{x}_2 = 0.5x_2 - 0.25x_2^2 - 0.75\, x_1 x_2 \quad \text{——— (2)}$$

a). To find equilibrium points, solve for $\dot{x}_1 = 0$, $\dot{x}_2 = 0$

so $0 = x_1 - x_1^2 - x_1 x_2$

$0 = 0.5x_2 - 0.25x_2^2 - 0.75x_1x_2$

divide first equation by x_1 and second equation by $x_2 \Rightarrow$

$0 = 1 - x_1 - x_2$

$0 = 0.5 - 0.25x_2 - 0.75x_1$

from first equation $\Rightarrow$ $x_1 = 1 - x_2$, plug into second eq $\Rightarrow$

$0 = 0.5 - 0.25x_2 - 0.75(1 - x_2) \Rightarrow$

$0 = 0.5 - 0.25x_2 - 0.75 + 0.75x_2$

so $0.5x_2 - 0.25 = 0 \Rightarrow \boxed{x_2 = \frac{1}{2}}$

so $x_1 = 1 - x_2 = 1 - \frac{1}{2} = \frac{1}{2}$

also $x_1 = x_2 = 0$ is an equilibrium point

hence equilibrium are $\boxed{\left(\frac{1}{2}, \frac{1}{2}\right) \;\&\; (0, 0)}$

we don't need to handle this state since

another point that makes equation (1) $= 0$ is $x_1 = 0$, $x_2 = $ Any. now try to find this x_2 value from using equation (2) as follows:

$0 = 0.5x_2 - 0.25x_2^2 \Rightarrow x_2 = \frac{-0.5}{-0.25} = 2 \Rightarrow \boxed{(0, 2)}$ is another equilibrium.

Similarly, $x_2 = 0, x_1 = $ Any will make equation (2) $= 0$.
now use equation (1) to find x_1:

$0 = x_1 - x_1^2 \Rightarrow x_1 = 1 \Rightarrow \boxed{(1, 0)}$ is another equilibrium

$\Rightarrow \boxed{(0,0),\ \left(\frac{1}{2}, \frac{1}{2}\right),\ (0,2),\ (1,0)}$ are the 4 equilibrium points.

b) Are these equilibrium states hyperbolic?

an equilibrium point is hyperbolic is the Jacobian evaluated at the point has no eigenvalue on imaginary axis.

from the equations

$$\dot{x}_1 = x_1 - x_1^2 - x_1 x_2$$

$$\dot{x}_2 = 0.5x_2 - 0.25x_2^2 - 0.75x_1x_2$$

so Jacobian $= \begin{pmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 1-2x_1-x_2 & -x_1 \\ -0.75x_2 & 0.5-0.5x_2-0.75x_1 \end{pmatrix}$

at point $\underline{(0.5, 0.5)}$, the Jacobian is

$$\begin{pmatrix} 1-1-\frac{1}{2} & -\frac{1}{2} \\ -0.75\times0.5 & 0.5-0.5^2-0.75\times0.5 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -0.375 & -0.125 \end{pmatrix}$$

find eigenvalues: $|A-\lambda I| = 0 \Rightarrow (-\frac{1}{2}-\lambda)(-0.125-\lambda) - (-\frac{1}{2}\times -0.375) = 0$

$$\Rightarrow +0.0625 + \tfrac{1}{2}\lambda + 0.125\lambda + \lambda^2 - 0.1875 = 0$$

so $\lambda^2 + 0.625\lambda - 0.125 = 0 \Rightarrow \lambda = \dfrac{-0.625 \pm \sqrt{0.625^2 - 4\times(-0.125)}}{2}$

$$\frac{-0.625 \pm \sqrt{0.890625}}{2} = \frac{-0.625 \pm 0.94373}{2} \Longrightarrow = \boxed{0.15936 \text{ or } -0.7845}$$

so this point is $\underline{\text{hyperbolic}}$, point $(\frac{1}{2}, \frac{1}{2})$

now do for $(0,0) \longrightarrow$

Jacobian $\begin{pmatrix} 1-2x_1-x_2 & -x_1 \\ -0.75x_2 & 0.5-0.5x_2-0.75x_1 \end{pmatrix}\Bigg|_{\substack{x_1=0\\x_2=0}} = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}$

unstable, unsable

so since upper triangle, eigenvalues are $1, 0.5$.
hence this point is hyperbolic as well

→ points $(0,2)$ $(1,0)$ next page →

Significance of hyperbolicity

if an equilibrium point is hyper, then this means we can saftely linearize the equation of motion near that point, because there are linear terms that dominate near this equilibrium, and hence the linearized equation does represent the behaviour of the system with good accuracy near the equilibrium point.

if the point was not hyper. then this would mean that the nonliner terms remain dominant near the equilibrium, and the linearized equation will not produce an accurate liner desciption. i.e the equations are "not linearizable".

< relation with stability analysis >

For state $(0,2)$

Jacobian is $\begin{pmatrix} 1-2x_1-x_2 & -x_1 \\ -0.75x_2 & 0.5-0.5x_2-0.75x_1 \end{pmatrix}$

$$= \begin{pmatrix} 1-2 & 0 \\ -0.75(2) & 0.5-0.5(2) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1.5 & -0.5 \end{pmatrix}$$

so $(-1-\lambda)(-0.5-\lambda)=0 \Rightarrow \lambda=-1,\ \lambda=-0.5.$ both are stable

this point is hyperbolic.

for state $(1,0)$.

Jacobian $\begin{pmatrix} 1-2 & -1 \\ 0 & 0.5-0.75 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & -0.25 \end{pmatrix}$

so $(-1-\lambda)(-0.25-\lambda)=0 \rightarrow \lambda=-1,\ \lambda=-0.25$. both are stable

and hyperbolic.

Conclusion

State	eigenvalues	stable?	hyperbolic?
$(0,0)$	$1, 0.5$	NO, NO	yes
$(\frac{1}{2},\frac{1}{2})$	$0.15936, -0.784$	NO, yes	yes
$(0,2)$	$-1, -0.5$	yes, yes	yes
$(1,0)$	$-1, -0.25$	yes, yes	yes

$\longrightarrow$

c) describe the behavior in the neighborhood of each equilibrium point.

State $(\frac{1}{2}, \frac{1}{2})$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -0.375 & -0.125 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\lambda_1 = 0.15936, \quad \lambda_2 = -0.784$

λ_1

$(A - \lambda_1 I) v_1 = 0$

$$\begin{pmatrix} -\frac{1}{2} - 0.15936 & -\frac{1}{2} \\ -0.375 & -0.125 - 0.1593 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -0.65936 & -\frac{1}{2} \\ -0.375 & -0.2843 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Rank $= 1 \Rightarrow$ nullity $= 2 - 1 = 1$ hence can find normal eigenvector.

so $-0.65936\, a_1 - \frac{1}{2} a_2 = 0$

let $a_1 = 1 \Rightarrow \quad a_2 = -\frac{0.65936}{0.5} = -1.31872$

So $v_1 = \begin{pmatrix} 1 \\ -1.31872 \end{pmatrix}$

$\longrightarrow$

$\lambda_2 = -0.784$ ⇝ stable

$(A - \lambda_2 I) v_2 = 0$

$$\begin{pmatrix} -0.5+0.784 & -0.5 \\ -0.375 & -0.125+0.784 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

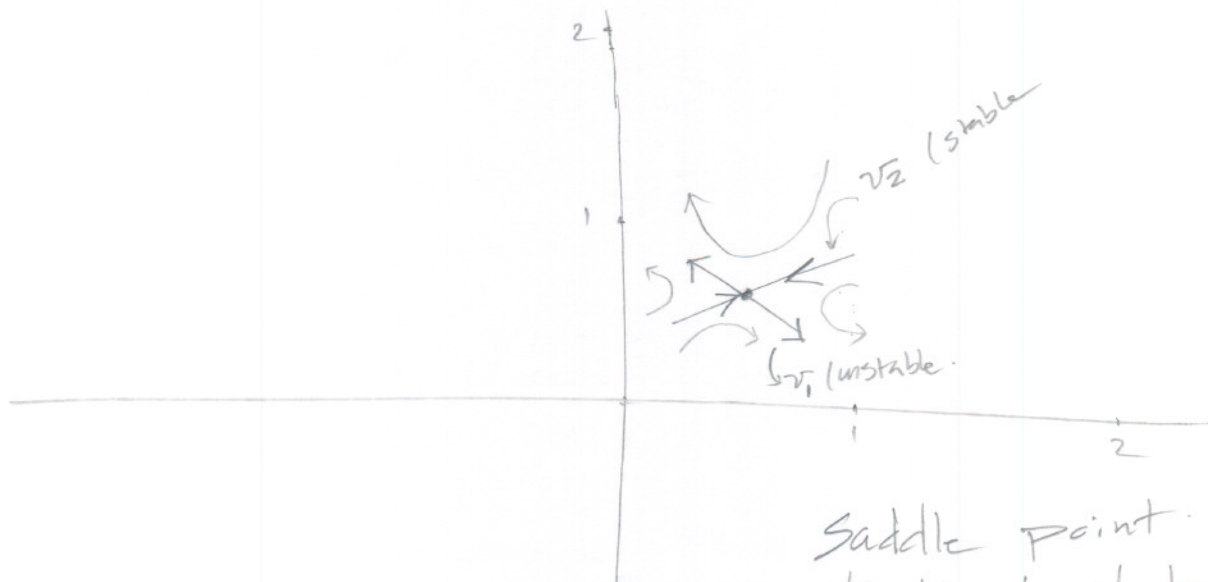
$$\begin{pmatrix} 0.284 & -0.5 \\ -0.375 & 0.659 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

↗ Rank $=1 \Rightarrow$ nullity $=1 \Rightarrow$ can find eigenvectors.

$0.284\, a_1 - 0.5\, a_2 = 0$

let $a_1 = 1 \Rightarrow a_2 = \dfrac{-0.284}{-0.5} = 0.568$

so $v_2 = \begin{pmatrix} 1 \\ 0.568 \end{pmatrix}$



Saddle point
$(\frac{1}{2}, \frac{1}{2})$. Local behavior.

For state $(0,0)$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

for $\lambda_1 = 1$ → unstable.

$$\begin{pmatrix} 0 & 0 \\ 0 & -0.5 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Rank $= 1 \Rightarrow$ nullity $= 1$. So can find one eigenvector

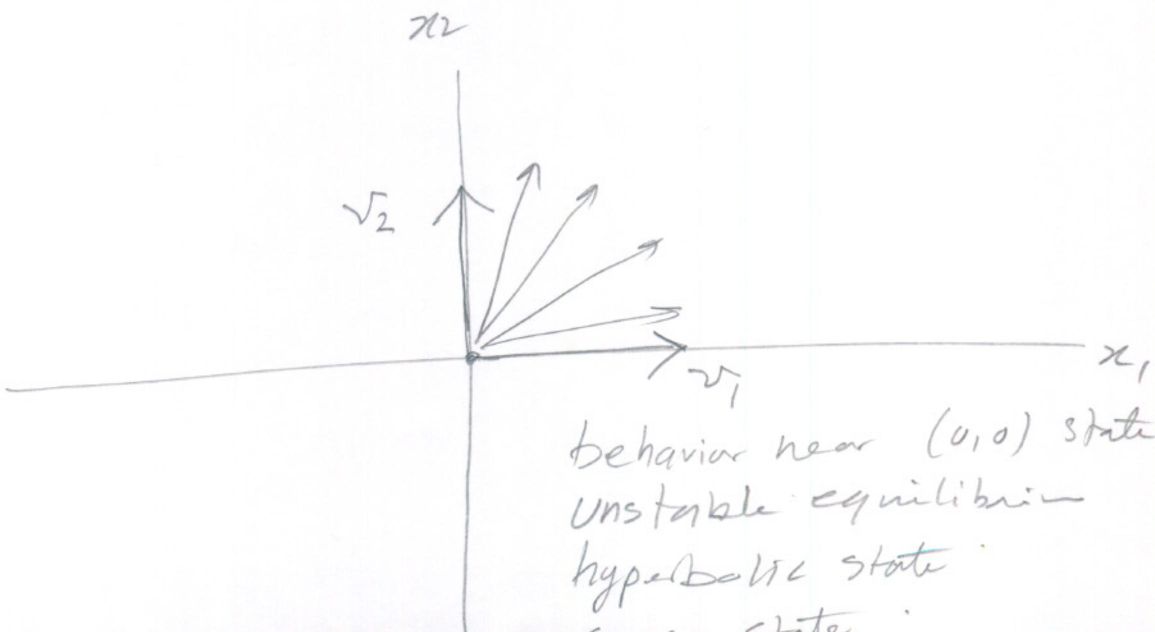
let $a_1 = 1 \Rightarrow a_2 = 0$. $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

for $\lambda_2 = 0.5$

$$\begin{pmatrix} 0.5 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\Rightarrow 0.5 a_1 + 0 \cdot a_2 = 0$$

let $a_2 = 1 \Rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.



behavior near $(0,0)$ state.
unstable equilibrium
hyperbolic state.
source state.

for state $(0, 2)$

$\lambda_1 = -1 \rightarrow$ stable.

so state matrix is $(A-\lambda I)v = \begin{pmatrix} -1+1 & 0 \\ -1.5 & -0.5+1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

so $\begin{pmatrix} 0 & 0 \\ -1.5 & .5 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$\Rightarrow \quad -1.5 a_1 + 0.5 a_2 = 0$.

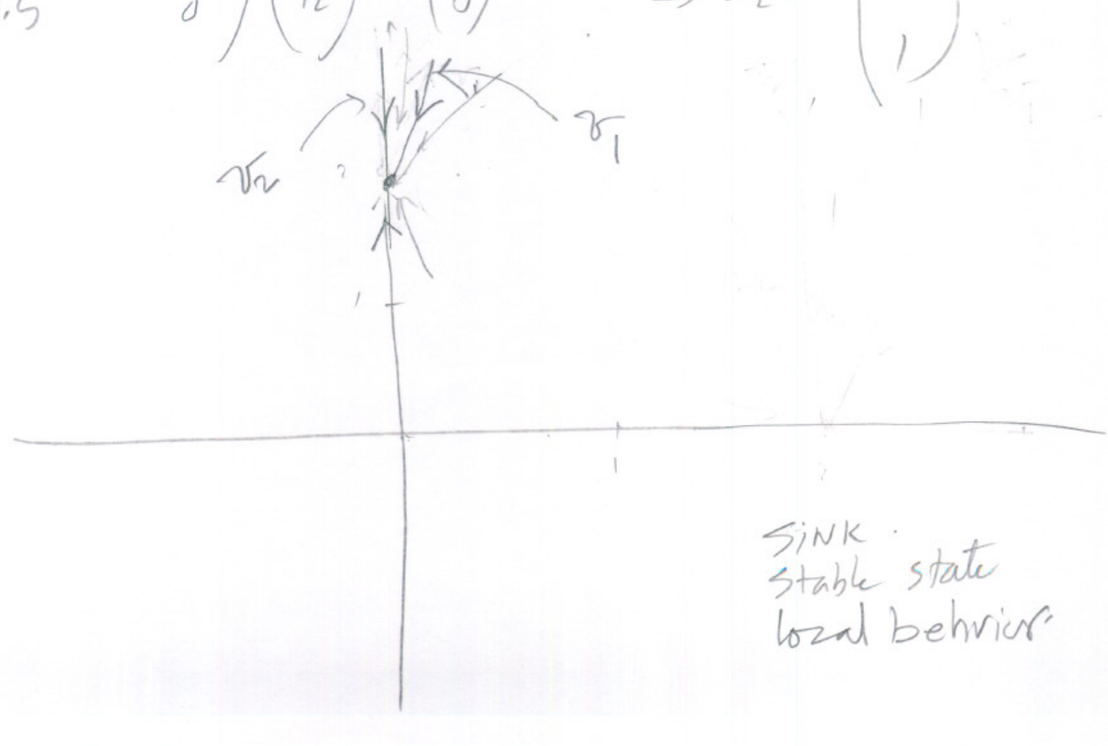
let $a_1 = 1 \Rightarrow a_2 = \frac{1.5}{0.5} = 3 \Rightarrow v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\lambda_2 = -0.5$

$(A-\lambda I)v \Rightarrow \begin{pmatrix} -1+0.5 & 0 \\ -1.5 & -0.5+0.5 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

so $\begin{pmatrix} -0.5 & 0 \\ -1.5 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$ $-0.5 a_1 + 0 \cdot a_2 = 0$

$\Rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



SINK.
Stable state
local behavior

for state $(1,0)$

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$\lambda_1 = -1$

so $(A-\lambda_1 I)v_1 = 0$

i.e. $\begin{pmatrix} -1+1 & -1 \\ 0 & -0.25+1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 & -1 \\ 0 & 0.75 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

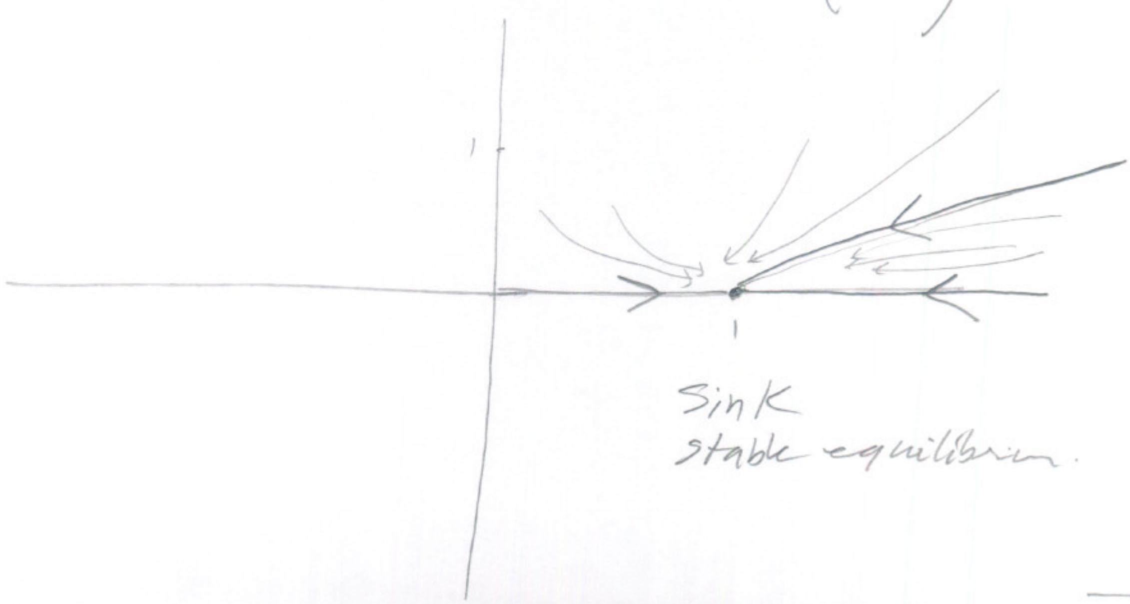
$\Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = -0.25$

$\begin{pmatrix} -1+0.25 & -1 \\ 0 & -0.25+0.25 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

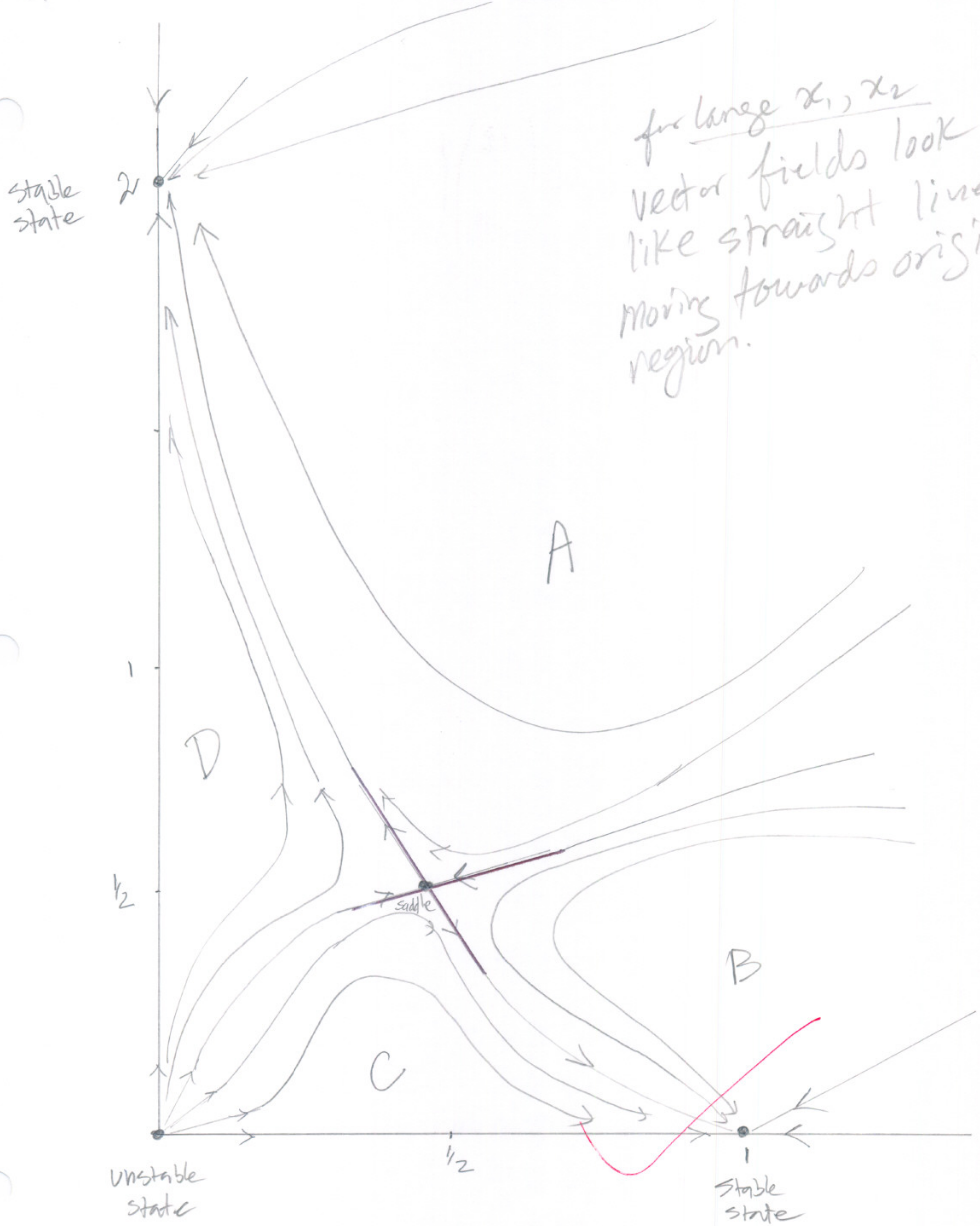
$0.75\, a_1 - a_2 = 0$ let $a_1 = 1 \Rightarrow a_2 = \frac{-0.75}{-1} = 0.75$

so $v_2 = \begin{pmatrix} 1 \\ 0.75 \end{pmatrix}$



Sink
stable equilibrium.

$\rightarrow$



for large x_1, x_2 vector fields look like straight lines moving towards origin region.

the state portrait shows that starting from any initial condition, trajorctories will evantually move to the stable state (1,0) or (0,2).

then shows the no species will grow indefinity.

The saddle point divides the region into 4 parts. Labeled A, B, C, D

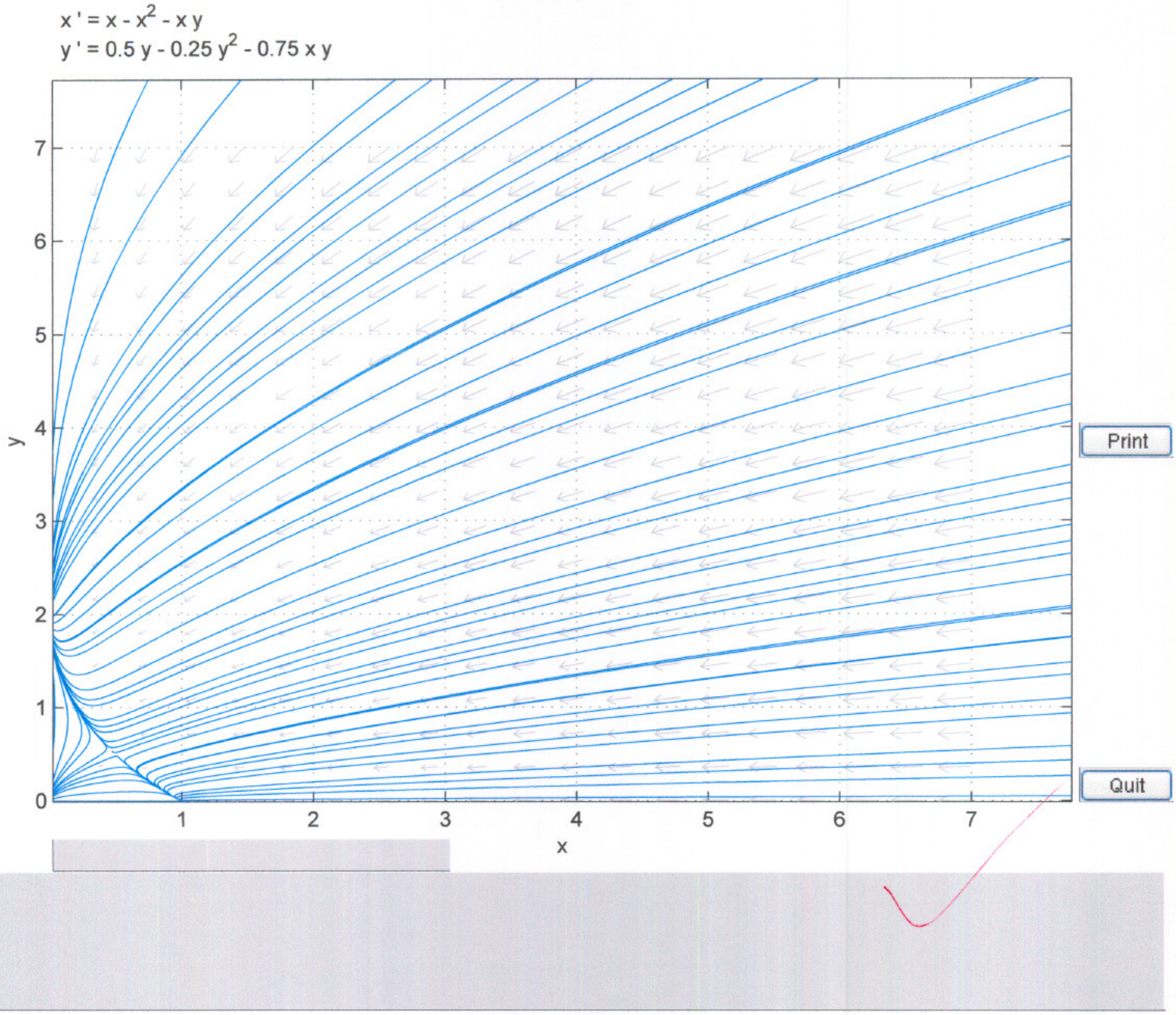
if initial conditions start in part A or D then eventually we will reach equilibrium at state (2,0). i.e. species x_2 will win and x_1 will vanish.

if initial Conditions start in region C or B then equilibrim will reach stat (1,0). i.e species x_1 will win and x_2 will die.

this shows that Saddle point divides the state space solution into 2 distinct regions.

For the required numerical validation, I used PPlane7. Please see next →

output of MatLab using
pplane7 program



Part e

We are given that the rate of change of one population x_1 depends on the size of the population x_1 and also on size of another population x_2. Similarly, the rate of change of population of x_2 depends on size of population x_2 and x_1.

Hence we have 2 different populations whose rate of growth or decline are interdependent.

one population can grow or decline depending on the size of the second population. and as that population change in size it can in turn cause the second population size to change as well.

We have found that No matter what the initial size of each population (as long as not both zero) then at the end eventually one population will survive and the second will die. we found that there are 2 sets of values that represent the initial population sizes where if we start from one set, the same speices will always survive and the second die. and if we start from the second set, then the other species will survive and the other die.