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HW # 8
MAE 200A

Restricted 3 body problem.

NASSER ABBASI

Given

$$\ddot{x} - 2\dot{y} - x = -(1-\mu)\left(\frac{x-x_1}{r_1^3}\right) - \mu\left(\frac{x-x_2}{r_2^3}\right)$$

$$\ddot{y} - 2\dot{x} - y = -\left(\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}\right)y$$

$$r_1 = \sqrt{(x-x_1)^2 + y^2} \quad , \quad r_2 = \sqrt{(x-x_2)^2 + y^2}$$

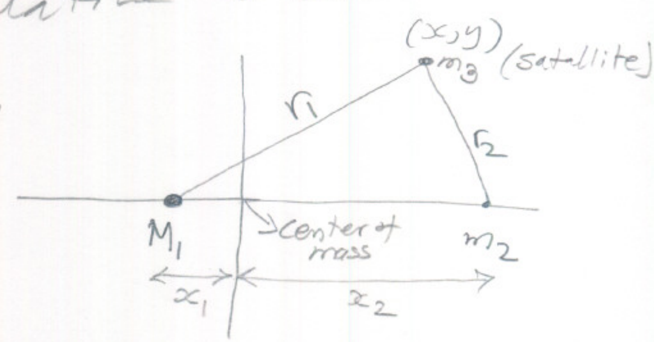
$\mu = 0.012$, $x_1 = -0.012$, $x_2 = 1-\mu = 1-0.012 = 0.988$

Find The 5 Lagrange points, which are the stable points in the state portrait of the restricted 3 body problem.

Solution

use as frame of reference the rotating (synodic) frame, where m_1, m_2 are stationary relative to each others.

we assume that m_3 has negligibal mass, so that it has no effect on M_1, m_2 motion.



relative potential Energy V of spacecraft is given by

$$V(x,y) = -\frac{(x^2+y^2)}{2} - \frac{(1-\mu)}{r_1} - \frac{\mu}{r_2}$$

and its K.E. is

$$KE = \frac{\dot{x}^2 + \dot{y}^2}{2}$$

hence Total Energy $= \dfrac{\dot{x}^2+\dot{y}^2}{2} + V(x,y)$

so Manifold is $F(x,y,\dot{x},\dot{y}) = \dot{x}^2 + \dot{y}^2 + 2V(x,y)$

To find singularities set the above to zero $\rightarrow$

$$F(x, y, \dot{x}, \dot{y}) = 0$$

where $\boxed{\frac{\partial F}{\partial x} = 0 \;,\; \frac{\partial F}{\partial y} = 0 \;,\; \frac{\partial F}{\partial \dot{x}} = 0 \;,\; \frac{\partial F}{\partial \dot{y}} = 0}$ (1)

eq(1) is equivalent to saying

$$\underbrace{\frac{-\partial V}{\partial x} = 0 \;,\; \frac{-\partial V}{\partial y} = 0}_{\text{solve these 2 eq to find } L_1, \dots, L5} \;,\; \underbrace{\dot{x} = 0 \;,\; \dot{y} = 0}_{\text{means that relative speed of spacecraft is zero at manifold singularities}}$$

$\Downarrow$

$$\frac{-\partial V}{\partial x} = 0 = -\frac{\partial}{\partial x}\left[-\frac{(x^2+y^2)}{2} - \frac{(1-\mu)}{r_1} - \frac{\mu}{r_2}\right]$$

and

$$\frac{\partial V}{\partial y} = 0 = -\frac{\partial}{\partial y}\left[-\frac{(x^2+y^2)}{2} - \frac{(1-\mu)}{r_1} - \frac{\mu}{r_2}\right]$$

solve these for x, y to find Lagrange points.

$\Downarrow$

hence

$$0 = x - \frac{(1-\mu)(x-x_1)}{r_1^3} - \frac{\mu(x-x_2)}{r_2^3} \quad (*)$$

and

$$0 = y\left(1 - \frac{(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3}\right)$$

solve for x, y.

Start by finding Lagrange points on x-axis (line connecting m_1, m_2. On this line $y=0$ hence we have one equation left which is equation marked (*) above:

$$0 = x - \frac{(1-\mu)(x-x_1)}{r_1^3} - \frac{\mu(x-x_2)}{r_2^3} \Rightarrow$$

method of solution for the collinear lagrange points;

I divide the solution into 3 Cases.

<u>Case 1</u> when spacecraft is to the left of M_1

<u>Case 2</u> when Spacecraft is between M_1 and m_2.

<u>Case 3</u> when spacecraft is to the right of m_2.

for each Case above, I find r_1 and r_2 so the sign changes depending on case. for each Case, solve the resulting 5^{th} order equation in x using CAS system using Solve() function. I discard all the imaginary solutions and keep the real solution. for each Case I get <u>one</u> real solution which is x, the location of spacecraft on the real line. The next page shows this method and the solution. and on page after it a print out of the code using Solve showing the Calculations made

$\Longrightarrow$

```
In[55]:= (*Solving for the 3 collinear Lagrange points*)
         (*HW8, Nasser Abbasi, MAE 200 A*)

         Clear["Global`*"]
         μ = 0.012;
```

$$\text{Solve}\left[x+\frac{1-\mu}{(x+\mu)^2}+\frac{\mu}{(x-1+\mu)^2}==0,\ x\right]$$

$$\text{Solve}\left[x-\frac{1-\mu}{(x+\mu)^2}+\frac{\mu}{(x-1+\mu)^2}==0,\ x\right]$$

$$\text{Solve}\left[x-\frac{1-\mu}{(x+\mu)^2}-\frac{\mu}{(x-1+\mu)^2}==0,\ x\right]$$

```
Out[57]= {{x → -1.005}, {x → 0.491999 - 0.858984 i}, {x → 0.491999 + 0.858984 i},
          {x → 0.986501 - 0.0782147 i}, {x → 0.986501 + 0.0782147 i}}

Out[58]= {{x → -0.506677 - 0.86138 i}, {x → -0.506677 + 0.86138 i},
          {x → 0.837659}, {x → 1.06385 - 0.145146 i}, {x → 1.06385 + 0.145146 i}}

Out[59]= {{x → -0.505323 - 0.863689 i}, {x → -0.505323 + 0.863689 i},
          {x → 0.903772 - 0.130525 i}, {x → 0.903772 + 0.130525 i}, {x → 1.1551}}
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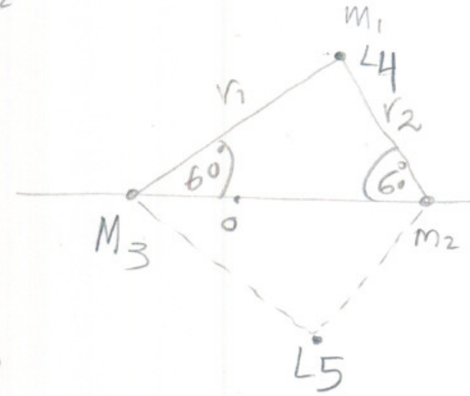
now need to find Lagrange points L_4 and L_5.

recall that the 2 equations to solve now are

$$0 = x - \frac{(1-\mu)(x-x_1)}{r_1^3} - \frac{\mu(x-x_2)}{r_2^3}$$

$$0 = y\left(1 - \frac{(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3}\right)$$

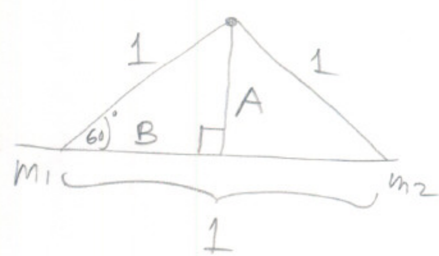


to find L_4 and L_5, let $\boxed{r_1 = r_2 = 1}$
since in this case we obtain equilateral triangles formed, and since the distance between M_3, M_2 is normalized to 1.

so now substitute $r_1 = r_2 = 1$ into the above diagram, we get

so $B = 1 \cos 60° = 0.5$

$A = 1 \sin 60° = 0.866$

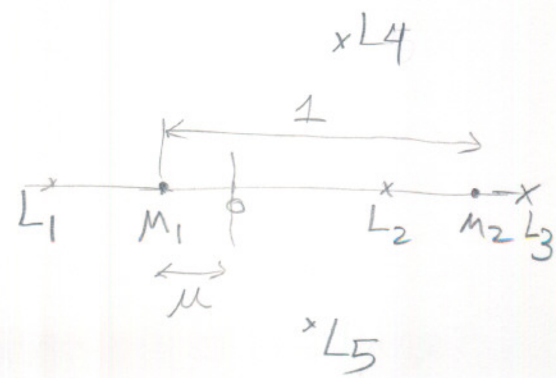


hence $L_4 = (0.5, 0.866)$

$L_5 = (0.5, -0.866)$.

Conclusion The 5 Lagrange points are

$$\boxed{\begin{aligned} L_4 &= (0.5, 0.866) \\ L_5 &= (0.5, -0.866) \\ L_1 &= (-1.005, 0) \\ L_2 &= (0.8376, 0) \\ L_3 &= (1.1551, 0) \end{aligned}}$$



Part 2 of HW #8

Characterize the local behavior in the neighborhood of L_2 and L_4 using linear analysis.

Answer

let $U = \frac{x^2+y^2+z^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$

Using U, we can write the equations of motion →

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x}$$
$$\ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y}$$
$$\ddot{z} = \frac{\partial U}{\partial z}.$$

I will only use x, y from now on and not z since I'll assume we are in in x-y plane and hence there is no z movement.

Convert to state space using

$$S_1 = x$$
$$S_2 = y$$
$$S_3 = \dot{x}$$
$$S_4 = \dot{y}$$

state variables.

so
$$\dot{S}_1 = \dot{x} = S_3$$
$$\dot{S}_2 = \dot{y} = S_4$$
$$\dot{S}_3 = \ddot{x} = U_x + 2\dot{y} = U_x + 2S_4$$
$$\dot{S}_4 = \ddot{y} = U_y - 2\dot{x} = U_y - 2S_3$$

→

but $U_x = x - \dfrac{(1-\mu)(x-x_1)}{r_1^{\,3}} - \dfrac{\mu(x-x_2)}{r_2^{\,3}}$

so $U_x = s_1 - \dfrac{(1-\mu)(s_1-x_1)}{\left(\sqrt{(s_1-x_1)^2+s_2^2}\right)^3} - \dfrac{\mu(s_1-x_2)}{\left(\sqrt{(s_1-x_2)^2+s_2^2}\right)^3}$

and $U_y = y\left(1 - \dfrac{(1-\mu)}{r_1^{\,3}} - \dfrac{\mu}{r_2^{\,3}}\right)$

$= s_2\left(1 - \dfrac{(1-\mu)}{\left(\sqrt{(s_1-x_1)^2+s_2^2}\right)^3} - \dfrac{\mu}{\left(\sqrt{(s_1-x_2)^2+s_2^2}\right)^3}\right)$

so

$$\begin{bmatrix} \dot s_1 \\ \dot s_2 \\ \dot s_3 \\ \dot s_4 \end{bmatrix} = \begin{bmatrix} s_3 \xleftarrow{\quad} h_1(s_1, s_2, s_3, s_4) \\ s_4 \xleftarrow{\quad} h_2(s_1, s_2, s_3, s_4) \\ \underbrace{s_1 - \dfrac{(1-\mu)(s_1-x_1)}{\left(\sqrt{(s_1-x_1)^2+s_2^2}\right)^3} - \dfrac{\mu(s_1-x_2)}{\left(\sqrt{(s_1-x_2)^2+s_2^2}\right)^3} + 2s_4}_{h_3} \\ \underbrace{s_2\left(1 - \dfrac{(1-\mu)}{\left(\sqrt{(s_1-x_1)^2+s_2^2}\right)^3} - \dfrac{\mu}{\left(\sqrt{(s_1-x_2)^2+s_2^2}\right)^3}\right) - 2s_3}_{h_4} \end{bmatrix}$$

So Linearized equations are

Jacobian evaluated at L_1 and L_4

$$\begin{bmatrix} \Delta\dot s_1 \\ \Delta\dot s_2 \\ \Delta\dot s_3 \\ \Delta\dot s_4 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial h_1}{\partial s_1} & \dfrac{\partial h_1}{\partial s_2} & \dfrac{\partial h_1}{\partial s_3} & \dfrac{\partial h_1}{\partial s_4} \\ \dfrac{\partial h_2}{\partial s_1} & \dfrac{\partial h_2}{\partial s_2} & \dfrac{\partial h_2}{\partial s_3} & \dfrac{\partial h_2}{\partial s_4} \\ \dfrac{\partial h_3}{\partial s_1} & \dfrac{\partial h_3}{\partial s_2} & \dfrac{\partial h_3}{\partial s_3} & \dfrac{\partial h_3}{\partial s_4} \\ \dfrac{\partial h_4}{\partial s_4} & \dfrac{\partial h_4}{\partial s_4} & \dfrac{\partial h_4}{\partial s_4} & \dfrac{\partial h_4}{\partial s_4} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} \rightarrow$$

$$\frac{\partial h_1}{\partial s_1} = 0, \quad \frac{\partial h_1}{\partial s_2} = 0, \quad \frac{\partial h_1}{\partial s_3} = 1, \quad \frac{\partial h_1}{\partial s_4} = 0$$

$$\frac{\partial h_2}{\partial s_2} = 0, \quad \frac{\partial h_2}{\partial s_2} = 0, \quad \frac{\partial h_2}{\partial s_3} = 0, \quad \frac{\partial h_2}{\partial s_4} = 1$$

$$\frac{\partial h_3}{\partial s_1} = 1 - \frac{(1-\mu)}{r_1^3} + \frac{3(1-\mu)(s_1-x_1)^2}{r_1^5} - \frac{\mu}{r_2^3} + \frac{3\mu(s_1-x_2)^2}{r_2^5}$$

$$\frac{\partial h_3}{\partial s_2} = \frac{3 s_2 (1-\mu)(s_1-x_1)}{r_1^5} + \frac{3 s_2 \mu (s_1-x_2)}{r_2^5}$$

$$\frac{\partial h_3}{\partial s_3} = 0, \quad \frac{\partial h_3}{\partial s_4} = 2$$

$$\frac{\partial h_4}{\partial s_1} = s_2 \left(\frac{3(1-\mu)(s_1-x_1)}{r_1^5} + \frac{3\mu(s_1-x_2)}{r_2^5} \right)$$

$$\frac{\partial h_4}{\partial s_2} = 1 - \frac{(1-\mu)}{r_1^3} + s_2 \left(\frac{3 s_2 (1-\mu)}{r_1^5} + \frac{3 s_2 \mu}{r_2^5} \right) - \frac{\mu}{r_2^3}$$

$$\frac{\partial h_4}{\partial s_3} = -2, \quad \frac{\partial h_4}{\partial s_4} = 0$$

$$\begin{bmatrix} \Delta \dot{s}_1 \\ \Delta \dot{s}_2 \\ \Delta \dot{s}_3 \\ \Delta \dot{s}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\partial h_3}{\partial s_1} & \frac{\partial h_3}{\partial s_2} & 0 & 2 \\ \frac{\partial h_4}{\partial s_1} & \frac{\partial h_4}{\partial s_2} & -2 & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix}$$

$\longrightarrow$

now evaluate Jacobian at L_2 and L_4.

at L_2 (i.e L_1 in part 2) → P.S. I called L_1 as L_2 in part 1. in my diagram this is the same L we are required to solve for.

$L_2 = (0.8376, 0)$ i.e $x = 0.8376$, $y = 0$

i.e $s_1 = 0.8376$, $s_2 = 0$.

hence $\left.\dfrac{\partial h_3}{\partial s_1}\right|_{L_2} = 11.2767$

$\left.\dfrac{\partial h_3}{\partial s_2}\right|_{L_2} = 0$

$\left.\dfrac{\partial h_4}{\partial s_1}\right|_{L_2} = 0$

$\left.\dfrac{\partial h_4}{\partial s_2}\right|_{L_2} = -4.13833$

so Linearized equations at L_2

$$\Delta \dot{s} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 11.2767 & 0 & 0 & 2 \\ 0 & -4.13833 & -2 & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix}$$

$\Delta(\lambda)$ for above A matrix is $\lambda^4 - 3.13837\lambda^2 - 46.6667$

$\Rightarrow \boxed{\lambda = -2.92889,\ \lambda = -2.33239j,\ \lambda = +2.33239j,\ \lambda = 2.92889}$ (P.S. even though L_1 is non hyperbolic, it has one $\lambda > 0$ ⇒ unstable)

since one eigenvalue is positive ⟹ $\boxed{L_2 \text{ is unstable}}$ equilibrium based on Linear analysis ↑ i.e L_1 in part 2 now evaluate Jacobian at L_4

at L_4

$L_4 = (0.5, 0.866)$. i.e $x = 0.5$, $y = 0.866$

i.e $s_1 = 0.5$, $s_2 = 0.866$

$$\left.\frac{\partial h_3}{\partial s_1}\right|_{L_4} = 0.780261$$

$$\left.\frac{\partial h_3}{\partial s_2}\right|_{L_4} = 1.25961$$

$$\left.\frac{\partial h_4}{\partial s_1}\right|_{L_4} = 1.25961$$

$$\left.\frac{\partial h_4}{\partial s_2}\right|_{L_4} = 2.20229$$

Linearized equations at L_4 are

$$\Delta \dot{s} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.780261 & 1.25961 & 0 & 2 \\ 1.25961 & 2.20229 & -2 & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix}$$

$$\Delta(\lambda) = \lambda^4 + 1.01745\lambda^2 + 0.131744$$

$$\Rightarrow \boxed{\begin{array}{ll} \lambda = -0.290223j & \lambda = 0.290223j \\ \lambda = -0.930148j & \lambda = 0.930148j \end{array}}$$

$\boxed{L_4 \text{ is non-hyperbolic}}$ and it have no $\lambda > 0$ hence this is critical case. since all λ's on imaginary axis, we unable to decide on stability based on what we learned.