Homework Assignment #9

Part 2 - Analyzing spacecraft motion in the Earth-Moon system using the circular, restricted 3-body model

1. Plot the zero velocity curves for different values of the constant alpha (or C). Show the different situations that were presented in the 11/22 lecture.

We defined the relative energy (Jacobi integral) as

Erel = v^2/2 - U

which is constant in the relative coordinate frame. The term U referrs to the relative potential energy, defined as

U= (x^2+y^2)/2 + (1-μ)/r1 + μ/r2

with r1= sqrt((x-x1)^2 + y^2), r2=sqrt((x-x2)^2+y^2), x1=-μ, x2=1-μ.

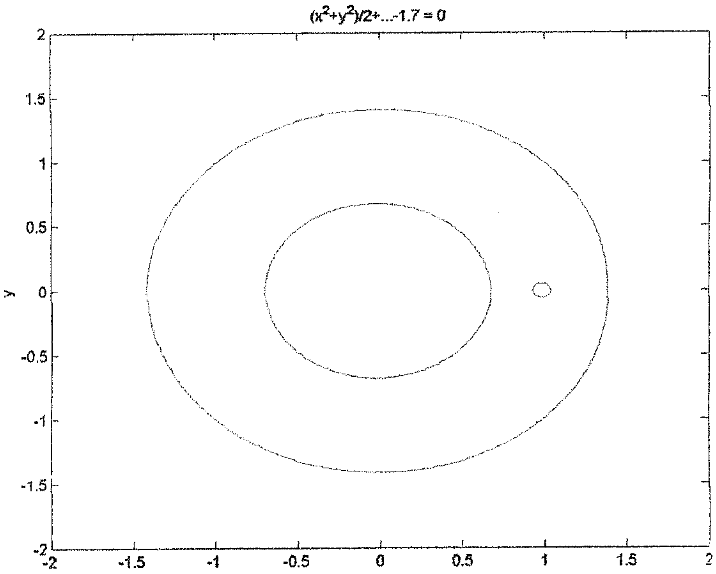
If we want to plot zero velocity curves for different values of the constant alpha, that means that we have the equation

U = α

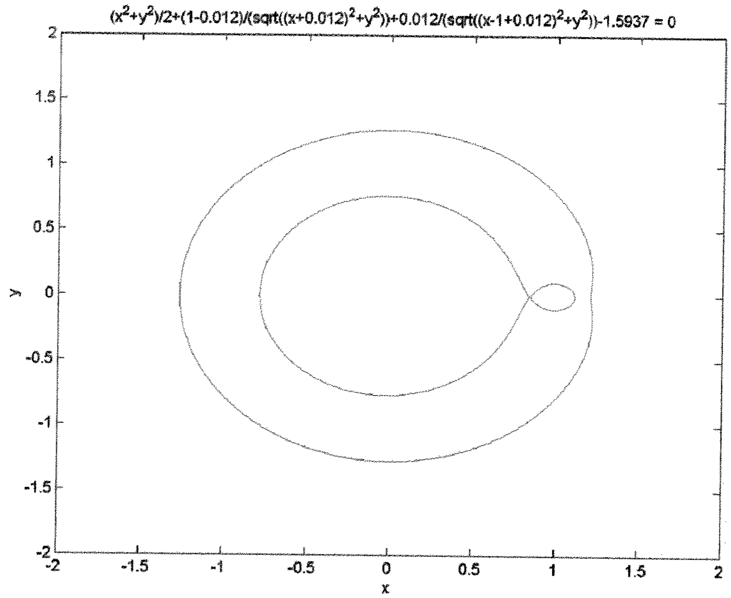
Below we have plots in the x-y space for different values of α, which have been plotted using the following command:

ezplot('(x^2+y^2)/2+(1-0.012)/(sqrt((x+0.012)^2+y^2))+0.012/(sqrt((x-1+0.012)^2+y^2))-alpha,[-2,2,-2,2])

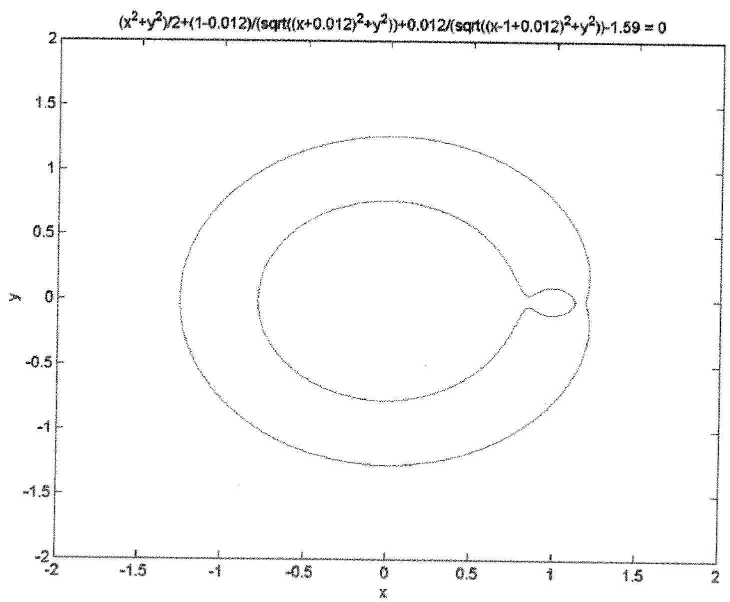
a) α **= 1.7**



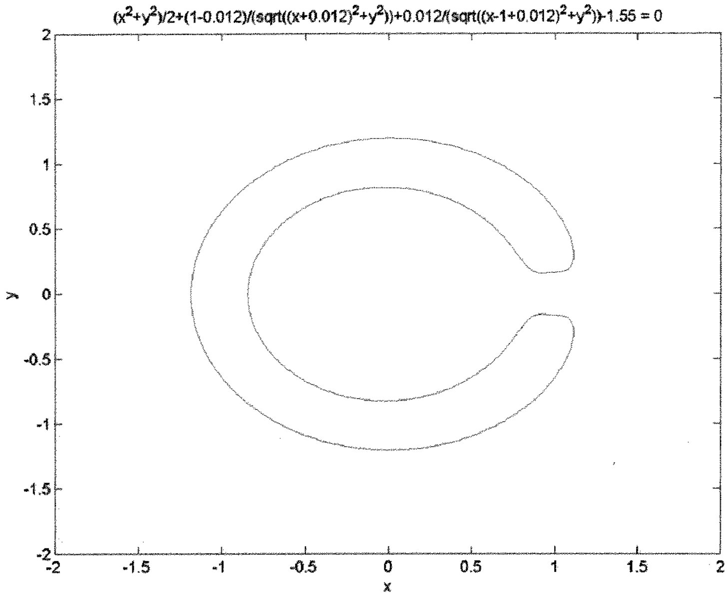
b) α = 1.5937



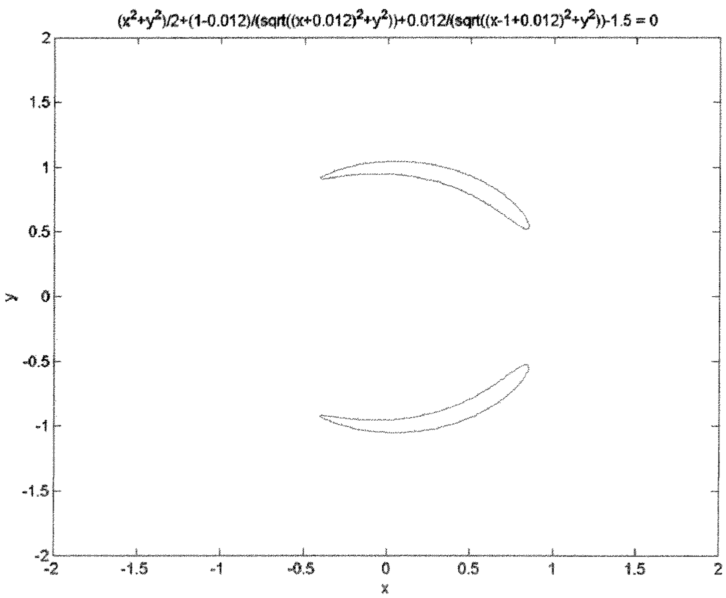
c) α = 1.59



d) $\alpha = 1.55$



e) $\alpha = 1.5$



2. Based on the zero-velocity curve analysis and the linear analysis you did in Part 1, characterize the spacecraft motion in the vicinity of L1 and in the vicinity of L4. Be clear about what each type of analysis let's you conclude.

L1: The linear analysis is conclusive that L1 is unstable (a saddle point). The local state space structure has the 3 invariant subspaces E^u, E^s, and E^c as defined in HW 8. The behavior predicted by the linear analysis (superposition of 3 modes in general) is valid in a sufficiently small neighborhood of L1 for sufficiently small times. The zero velocity curves show that the possibility exists for orbits beginning in the neighborhood of L1 to connect to the vicinity of the earth and the vicinity of the moon. It cannot be concluded, however, from the zero velocity curves that connecting trajectories exist.

L4: The linear analysis predicts, in general, a superposition of 2 periodic modes, and that periodic orbits can exist for certain initial conditions (ones that excite only one of the two modes). The zero velocity curves cannot rule out escape from the neighborhood of L4 because L4 is a local maximum of - U. Therefore, the asymptotic behavior is undecided from the linear analysis and the zero-velocity curves. It could be investigated via numerical simulation. One could also use a Lyapunov function approach, but this approach has not been covered in our class. L4 and L5 have been proven to be locally stable equilibria.

3 Consider the problem of transferring a spacecraft from a lunar orbit to an earth orbit. Assume a value for the Jacobi integral just large enough that the corresponding zero velocity curve indicates a pathway between lunar and earth orbits. Using backward and forward time integration from a point in the vicinity of L1, construct a potential transfer trajectory. The transfer strategy would then be to use a propulsive burn (energy increase) to get on this trajectory from lunar orbit and then a second propulsive burn (energy decrease) to get off it into an earth orbit. You only need to construct the potential transfer orbit.

Here is the rationale for picking up a starting point. For backward integration trajectories will approach the stable manifold. One could take an initial condition near L1 and its stable manifold, picking the branch of the stable manifold that with backward integration will produce a trajectory leading to the moon. Then with forward integration the trajectory will approach the unstable manifold and hopefully lead toward the earth. Some experimentation may be necessary to find a good initial condition.

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The spacecraft begins orbiting the moon, which means that its energy has to be lower than case 1.b), that is, its energy has to be lower than -1.5937. That means that there is no path between the Earth and the Moon and the spacecraft stays orbiting the Moon.

By using a propulsive burn, we can increase the energy of the spacecraft (value greater than -1.5937), so now we are in a case like 1.c), i.e., there exists a path between the Earth and the Moon. In this path, the spacecraft enters into the surroundings of Lagrange point L1.

Finally, when the spacecraft has moved from the right side of the L1 to its left side, we have to decrease its energy in order to close the path Earth-Moon. The result is that the spacecraft stays then orbiting the Earth.

If we want to obtain the potential transfer orbit, we have to start from Lagrange point L1. We know that the connection path is in its surroundings, concretely, changing orbits means that the spacecraft is first attracted by L1 (negative eigenvalue) and then repelled (positive eigenvalue). Consequently, we can obtain a potential transfer orbit starting from one point in the surroundings of L1 and then integrating backward (direction to the Moon) and forward (direction to the Earth).

However, we have to make sure to pick a correct starting point, so that integration backwards goes to the Moon and integration forward to the Earth.

From the HW8, we know that:

L1 --> x=0.8376 y=0

Eigenvalues and eigenvectors:

λ = 2.93 --> v1=[0.2934; -0.1351; 0.8596; -0.3958]

λ = -2.93 --> v2=[0.2934; 0.1351; -0.8596; -0.3958]

The starting point is a perturbation from L1, according to the directions of the eigenvalues:

x0=[0.83765;0;0;0]+c1(2.6*10^(-6))*[0.2934;-0.1351;0.8596;-0.3958]+c2*(2.6*10^(-6))*[0.2934;0.1351;-0.8596;-0.3958]*

To make sure that when integrating backwards we go to the Moon and that when integrating forward we go to the Earth, we have to pick

c1 = -1 (perturbation in the unstable direction)
c2 = 10 (perturbation in the stable direction)

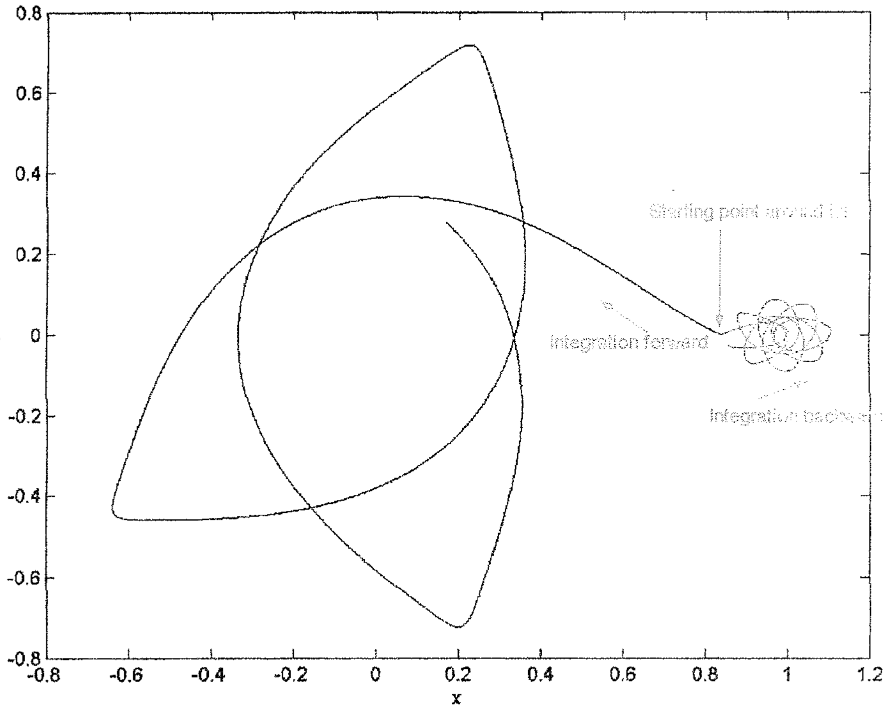
The program used:

```
% Starting point
c1=-1;
c2=10;
x0=[0.83765;0;0;0]+c1*(2.6*10^(-6))*[0.2934;-0.1351;0.8596;-
0.3958]+c2*(2.6*10^(-6))*[0.2934;0.1351;-0.8596;-0.3958];

% Forward integration
T=12;
[t,x45]=ode45('f_hw9',[0 T],x0,ODESET('RelTol',1e-10,'AbsTol',1e-10));
plot(x45(:,1),x45(:,2),'k');
xlabel('x');
ylabel('y');
hold on

%Backward integration
T=-12;
[t,x45]=ode45('f_hw9',[0 T],x0,ODESET('RelTol',1e-10,'AbsTol',1e-10));
plot(x45(:,1),x45(:,2),'b');
```

Plotting x versus y, we obtain



where we can perfectly see the path followed from one orbit to the other orbit. ✓

Notice that, if ~~we don't decrease the energy of the spacecraft it will return~~ to the Moon sooner or later, as the bottle neck is still opened. (Not Correct)

The zero velocity curves do not guarantee this. The trajectory may or may not return to the Moon; the zero velocity curve only allows one to conclude that a return to the Moon can not be ruled out.

4. Compute a 'halo orbit' about L4. A halo orbit is a periodic orbit about a Lagrange point. Pick an initial state near L4 and numerically integrate the equations of motion. See if you get an almost periodic orbit, meaning that it should eventually repeat itself by coming back to the initial state. 'Almost' is used because numerically it is difficult to integrate accurately enough that an exact return to the initial state is achieved. The linear analysis at L4 should suggest what period to expect. What is the period?

In the hw#8 we found that:

L4 --> x= 0.488 y= 0.866

Eigenvalues and eigenvectors

λ = 0.955i --> v11=[0.585; 0.559i; -0.235+0.354i; -0.338-0.224i]

λ = -0.955i --> v12=[0.585; -0.559i; -0.235-0.354i; -0.338+0.224i]

λ = 0.296i --> v21=[-0.823; -0.244i; 0.446-0.208i; 0.062+0.132i]

λ = -0.296i --> v22=[0.823; +0.244i; 0.446+0.208i; 0.062-0.132i]

From the linear analysis we can see that the spacecraft will follow two periodic motions, one corresponding to the center created by $\lambda = \pm 0.955i$ and the other corresponding to the center of $\lambda = \pm 0.296i$. The value of the complex part of the eigenvalue tells us the frequency of the motion, so we should expect the following periods (when computing the 'halo orbit'):

$\lambda = \pm 0.955i$ --> $\omega = 0.955$ --> $T = 2\pi/\omega = 6.58$

$\lambda = \pm 0.296i$ --> $\omega = 0.296$ --> $T = 2\pi/\omega = 21.22$

Now, if we want to compute a 'halo orbit' around L4, we need to start from a point around L4, that is, to perturbate the equilibrium. Ten, by numerical integration of the equations of motion we can find out what happens.

The starting point we'll consider is:

x0=[61/125;3^0.5/2;0;0]+c2(2.6*10^(-6))*[0.8227;-0.4462;0;-0.0617]+c1*(2.6*10^(-6))*[0.5854;-0.2347;0;-0.3378];*

where

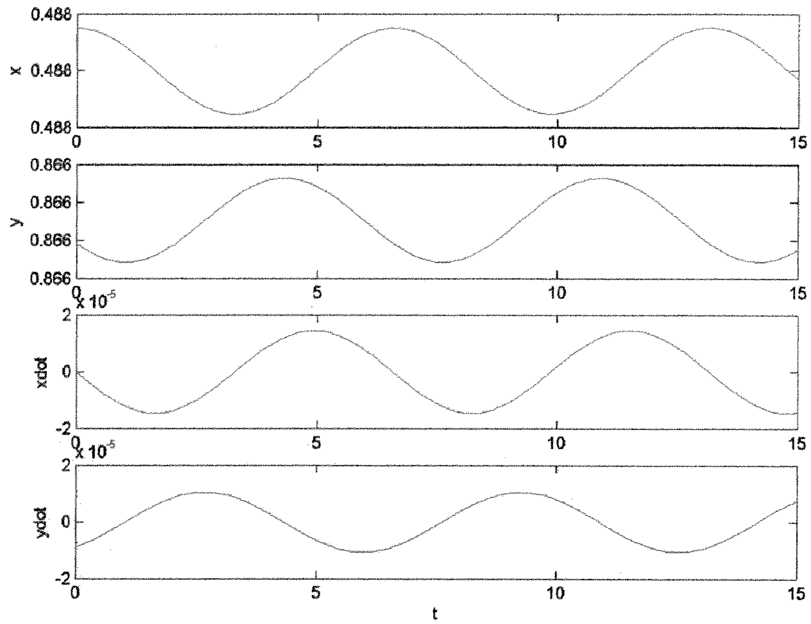
C1 = perturbation in the direction of $\lambda = \pm 0.296i$
C2 = perturbation in the direction of $\lambda = \pm 0.955i$

We will now start our analysis by perturbating only in the direction of $\lambda = \pm 0.296i$:

C1 = 10
C2 = 0

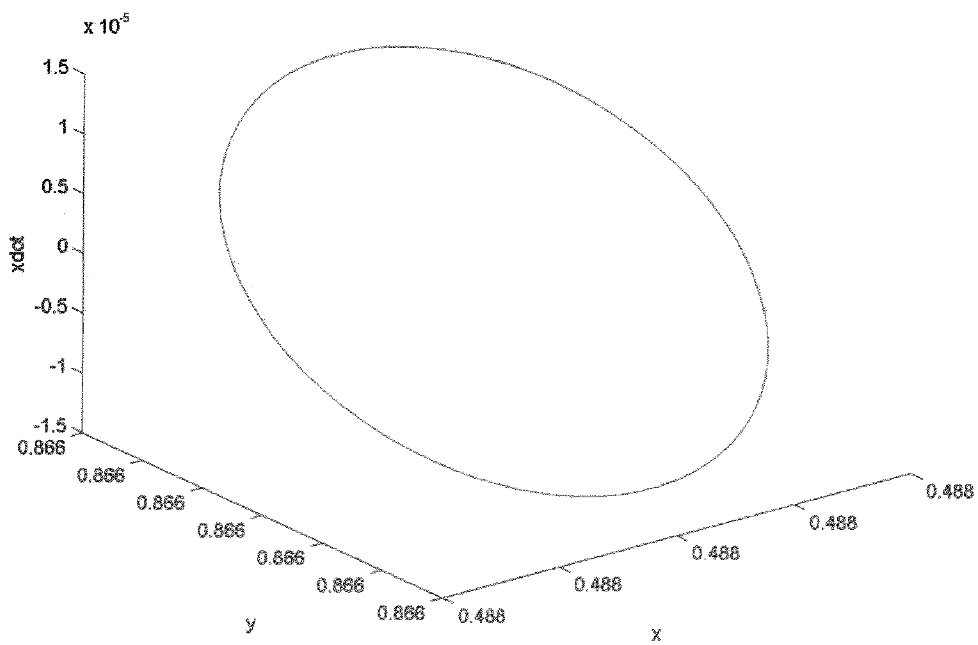
The results obtained are:

Plot of x vs t, y vs t, xdot vs t, ydot vs t:

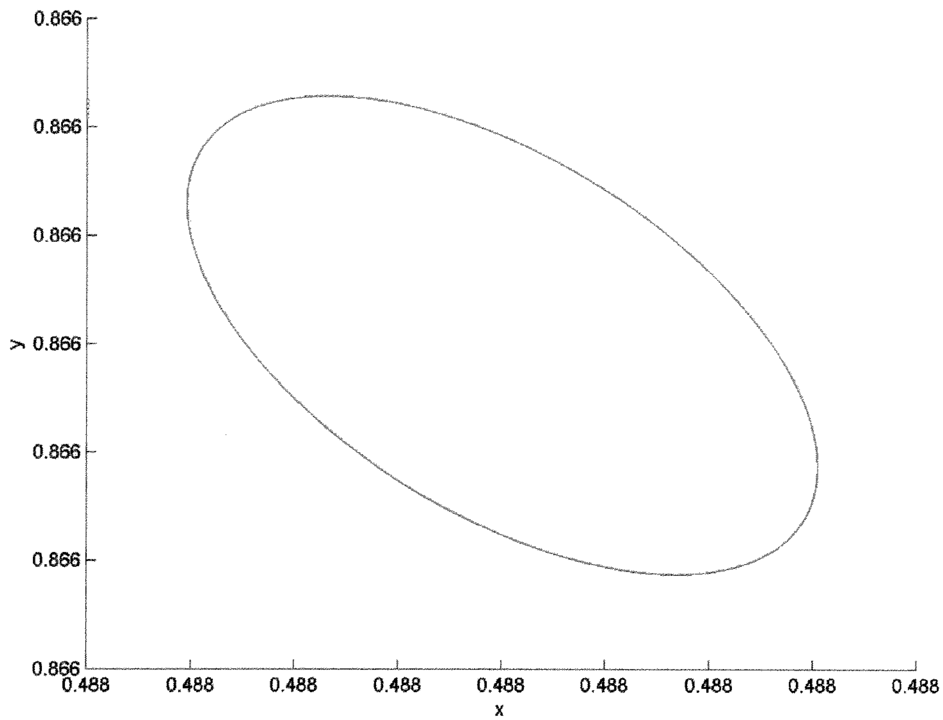


where we can see that the period of the orbit is around 6, as expected.

3D plot of x,y and xdot



Projecting this plot into x-y axis, we obtain the following picture:



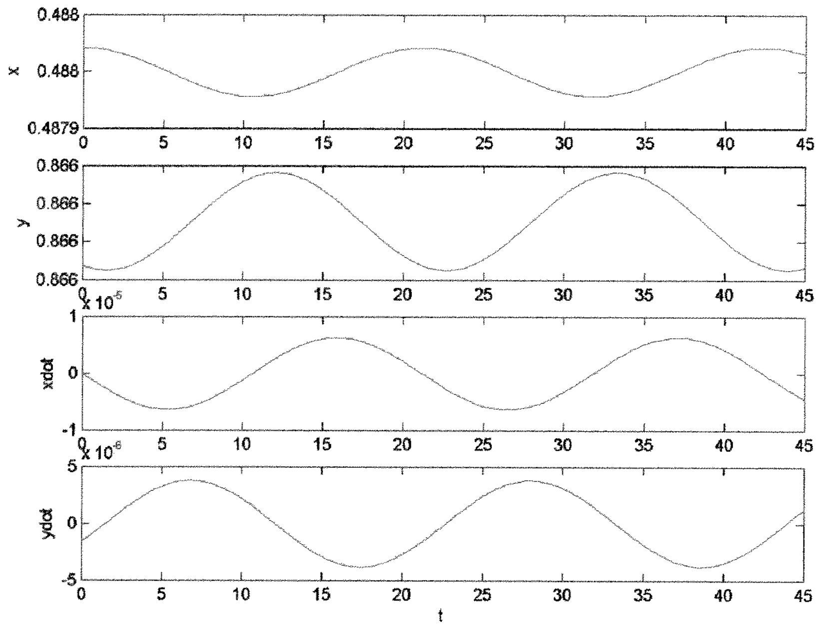
where we can see the orbit of the spacecraft about L4.

Now we can study the trajectory of the spacecraft when only perturbated in the direction of $\lambda = \pm 0.955i$.

C1 = 0
C2 = 10

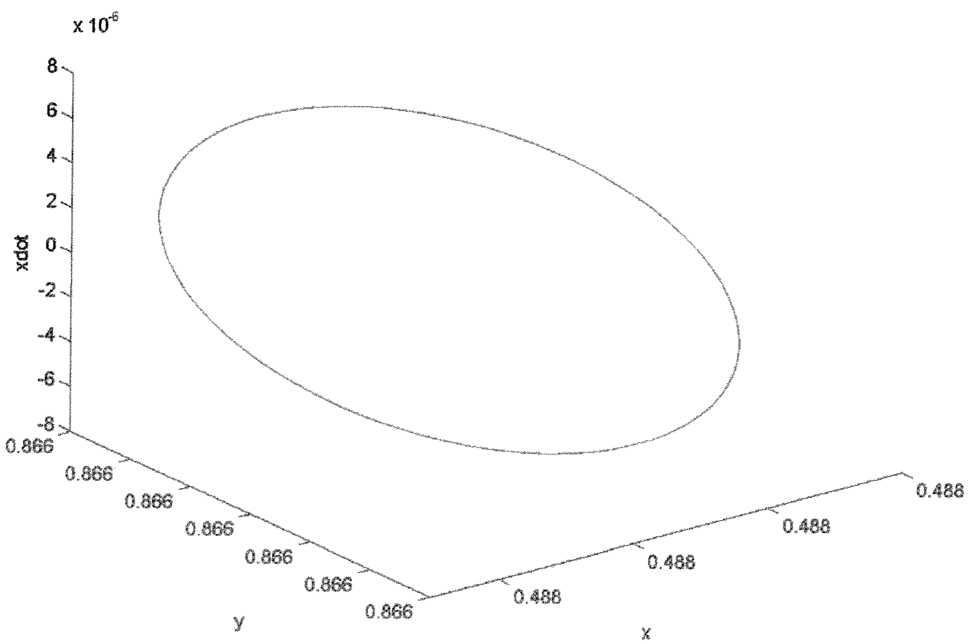
The results obtained are:

Plot of x vs t, y vs t, xdot vs t, ydot vs t:

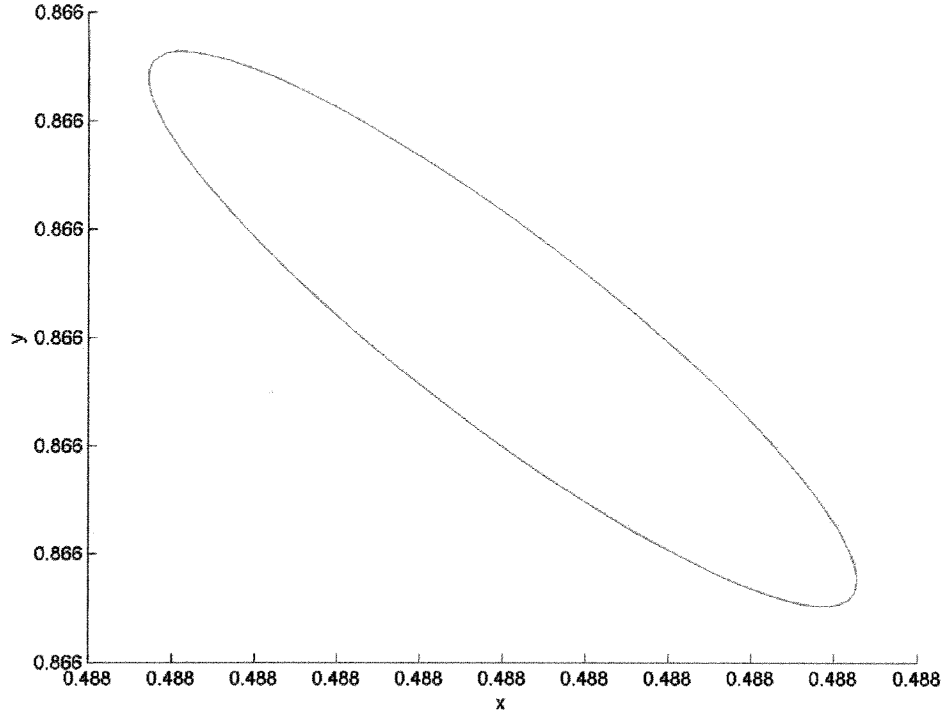


where we can see that the period is apporximately 21, as expected.

3D plot of x,y and xdot



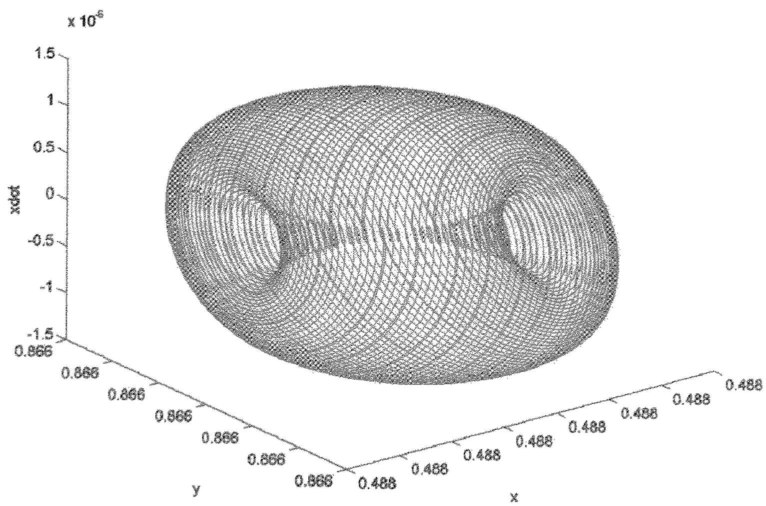
Projecting this plot into x-y axis, we obtain the following picture:



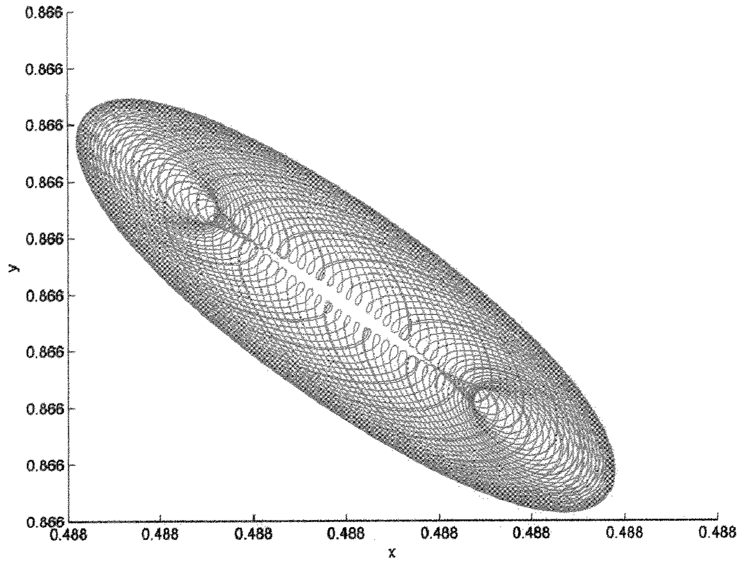
where we can see the trajectory of the spacecraft.

Now it's time to study the combination of the two periodic motions, that is, to perturbate the spacecraft in both directions (of $\lambda = \pm 0.955i$ and $\lambda = \pm 0.296i$). We'll consider different values of C1 and C2 in order to understand the motion of the spacecraft:

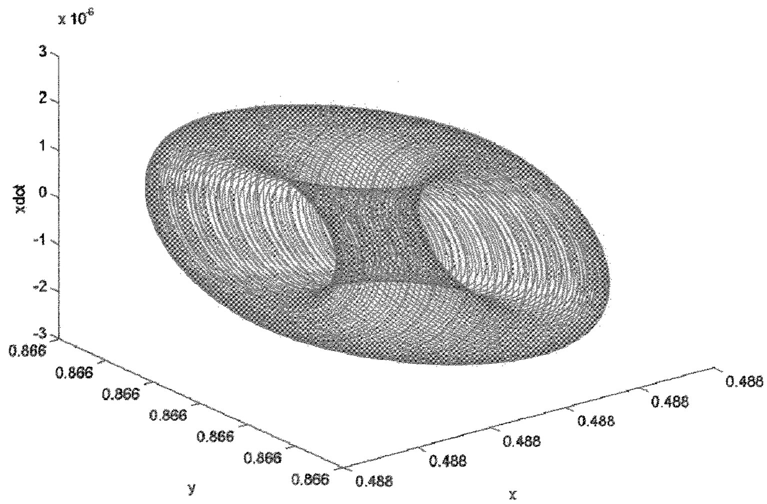
C1=0.5 C2=1



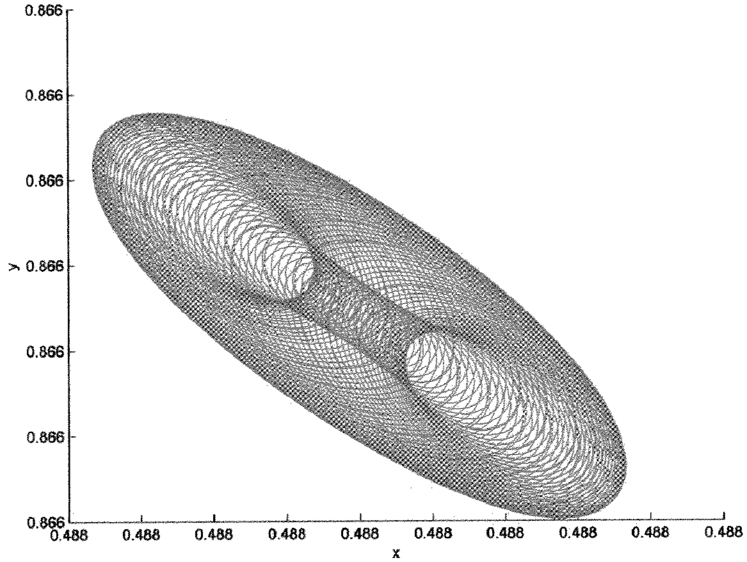
Projecting onto x-y plane



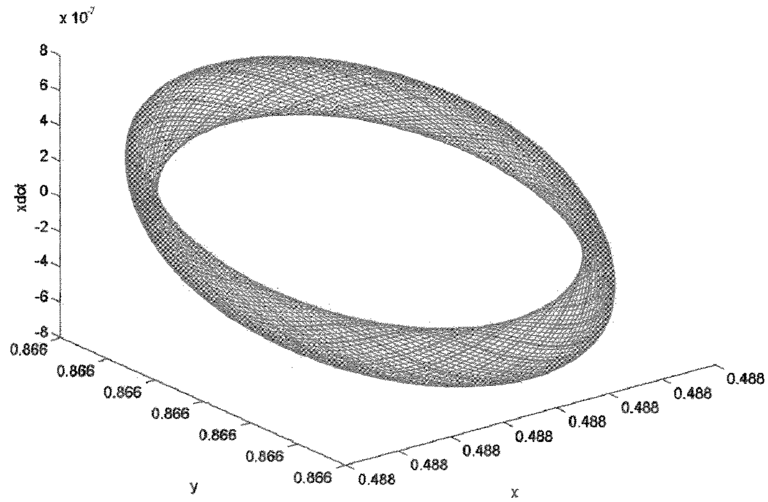
<u>C1=1 C2=1</u>



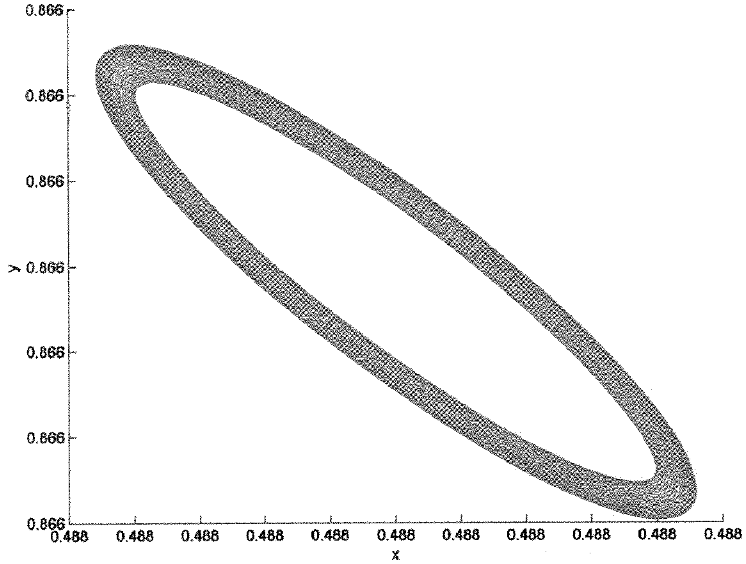
Projecting onto x-y plane



C1=0.1 C2=1



Projecting onto x-y plane

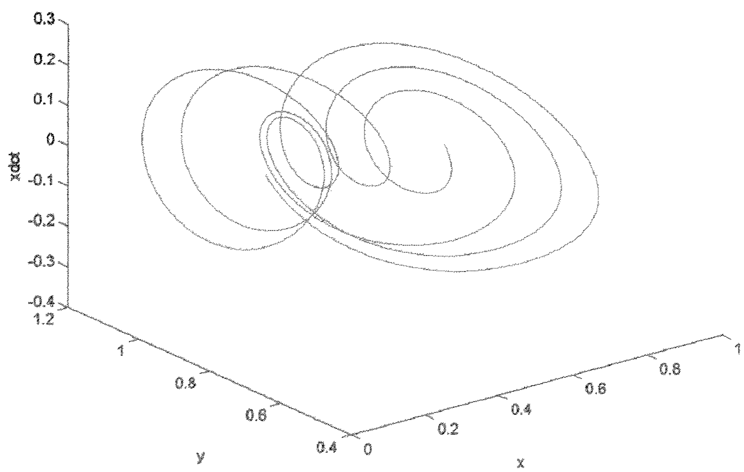


As it can be seen in the different plots, the motion of the spacecraft is the union of two circular motions, which result in a torus. Depending on how far we are from the equilibrium point, the torus has different shape. We can also realize that there are two periods present in the torus.

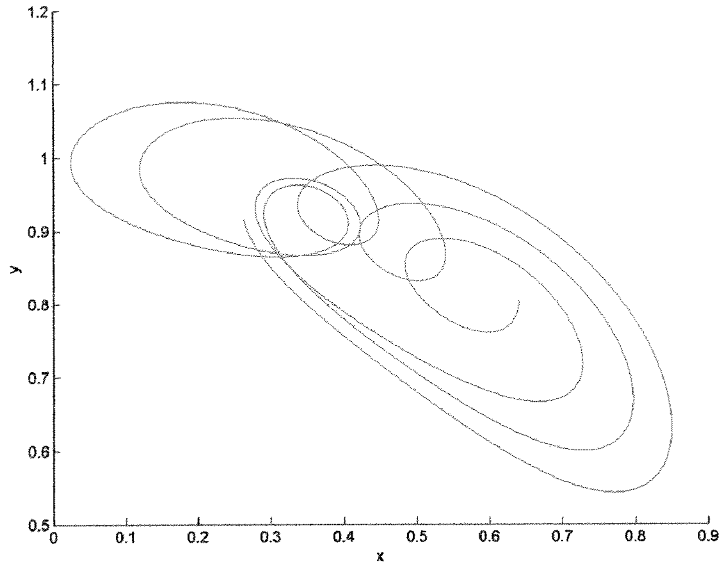
We can actually also see what happens if we perturbate too far from the equilibrium point L4. For example, using

C1= 100000
C2= 1

we obtain



Projecting onto x-y plane



and we can see that it no longer follows the results we expected from the linear analysis.