Range of A: all possible linear combinations of all columns of A.

Rank A ~~Ra~~ # of linearly independent columns in A. or dimension of Range(A).

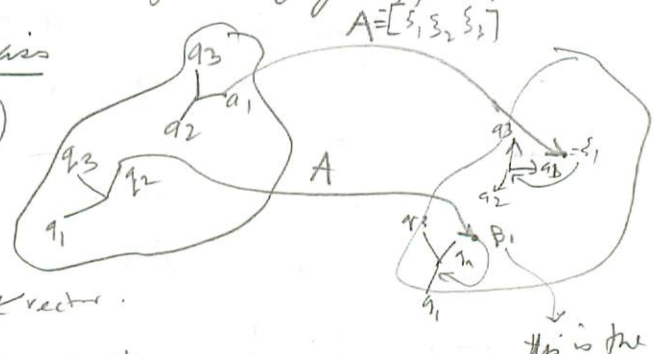
given $Ax=y$, then a solution x exist if $\text{Rank}(A)=\text{Rank}([A\ y])$.

Given $Ax=y$, solution x exist for every y iff A has rank m (full row rank).

To Find rep of A under new Basis

so given A, and (q_1, q_2, q_3)

we can find $\bar{A}=Q^{-1}AQ$ as follows:



~~So for~~

for each q_i, find $Aq_i=h_i$ vector.

~~Then write~~ Then write $Q[?]=h_i$

and find ?, this is the ith column.

Midterm.

To Find eigenvector for A, find the λ's

Then solve for $(A-\lambda I)x=0$

Find A^{100} $\quad A=\begin{pmatrix}0 & 1\\ -1 & -2\end{pmatrix} \rightarrow$ let $h(\lambda)=\beta_0+\beta_1\lambda \Rightarrow f(-1)=h(-1) \Rightarrow (-1)^{100}=\beta_0-\beta_1$

$\lambda=-1,-1$ $\qquad f'(-1)=h'(-1) \Rightarrow 100(-1)^{99}=\beta_1$

$\Rightarrow h(\lambda)=-99-100\lambda \Rightarrow A^{100}=-99I-100A \Rightarrow \begin{pmatrix}-99 & -100\\ 100 & 101\end{pmatrix}$

$e^{At}=I+tA+\frac{t^2}{2!}A^2+\cdots=$, $\boxed{e^0=I}$, $\boxed{e^{A(t_1+t_2)}=e^{At_1}e^{At_2}}$ $\quad [e^{At}]^{-1}=e^{-At}$

$\boxed{e^{(A+B)t}\neq e^{At}e^{Bt}}$ ← holds only if $AB=BA$. $\qquad \boxed{\frac{d}{dt}e^{At}=Ae^{At}=e^{At}A}$

Consider $n\times n$ matrix A. let A has λ with multiplicity m. suppose $\text{rank}(A)=3 \Rightarrow$ nullity $=1$

so need 3 more. find q_2, q_3, q_4 s.t. $(A-\lambda I)^2 q_2=0$

$(A-\lambda I)^3 q_3=0$

$(A-\lambda I)^4 q_4=0$

$f(B^T)=f^T(B)$. $\qquad \boxed{(A^{-1})^T=(A^T)^{-1}}$

A has same λ's as A^T, $B=AQ$

$\bar{A}=Q^{-1}AQ \quad$ & $\quad A=Q\bar{A}Q^{-1}$

if A diagonal, i.e. $A=\begin{bmatrix}\lambda_1 & & 0\\ & \lambda_2 & \\ 0 & & \lambda_3\end{bmatrix} \Rightarrow e^{At}\begin{bmatrix}e^{\lambda_1 t} & & 0\\ & e^{\lambda_2 t} & \\ 0 & & e^{\lambda_3 t}\end{bmatrix}$

if A Jordan Form $A=\begin{bmatrix}\lambda_1 & 1 & 0\\ 0 & \lambda_1 & 1\\ & & \lambda_1\end{bmatrix} \Rightarrow e^{A}=\begin{bmatrix}e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{t^2}{2!}e^{\lambda_1 t}\\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t}\\ & & e^{\lambda_1 t}\end{bmatrix}$

$\boxed{G(s) = C(sI-A)^{-1}B + D}$ $g(t,\tau) = 0$ for $t < \tau$ for Causal system.

$g(t,\tau)$: t = time response observed, τ = time response applied. plug in $t=4$, $\tau=5$ and see if $y \neq 0$

- Linear system: $L[a\, f(t) + b\, g(t)] = a\, L[f(t)] + b\, L[g(t)]$
- time invariant: shift input by some δt will shift output by δt.
- Causal: apply input at t, observe response at $t_0 < t$. must be zero for all $t_0 < t$.
- System output: output due to zero input response (due to initial conditions) + output due to zero state response.
- Jordan Form for A and $\bar{A}$ is same. $A = P^{-1}\bar{A}P$
- Cayley–Hamilton: $A^n = a_0 I + a_1 A + a_2 A^2 + \cdots + a_{n-1} A^{n-1}$ also A satisfies its own char. eq. i.e. $\lambda^3 + 2\lambda^2 + \lambda + 1 = 0 \Longrightarrow A^3 + 2A^2 + A + 1 = 0$
- To find basis for nullspace, solve $Ax = 0$
- A solution $Ax = y$ is unique if $null(A) =$ empty. $= 0$.
- Jordan Form can be found as $\bar{A} = Q^{-1}AQ$ where Q is $[q_1 | q_2 | q_3]$ the eigenvectors for A.
- $\mathcal{L}[e^{At}] = (sI-A)^{-1}$. $\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] = e^{t}$, $\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$, $\mathcal{L}^{-1}\left[\frac{1}{s(s-1)}\right] = -1 + e^{t}$

  $\mathcal{L}^{-1}\left[\frac{1}{s(s-1)^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s-1} + \frac{1}{(s-1)^2}\right] = 1 - e^{t} + te^{t}$. $\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t$
- input-output description of system $y(t) = \int_{-\infty}^{\infty} g(t,\tau)\, u(\tau)\, d\tau$

  for causal; Time inv. $y(t) = \int_0^t g(t-\tau)\, u(\tau)\, d\tau$

- state space
$$\dot{x} = Ax(t) + B\,u(t)$$
$$y = Cx(t) + D\,u(t)$$

- To linearize a system, find desire dynamics of nonlinear and write $y = x_1$ say ($f(x_1, x_2, u)$)

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} h_1(x_1, x_2, u, t) \\ h_2(x_1, x_2, u, t) \end{pmatrix}$$
next decide on X_{eq} to linearize around. say $x_1 = 0$, $x_2 = 0$, $u = 0$.

$\Rightarrow$ linearize system is
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{\partial h_1}{\partial u} \\ \frac{\partial h_2}{\partial u} \end{pmatrix} u$$

$$y = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{\partial f}{\partial u} \end{pmatrix} u$$

next evaluate all these at $x_1 = 0$, $x_2 = 0$, $u = 0$ or whatever the solution near is.
to find $\boxed{G(s) = C(sI-A)^{-1}B + D}$

Inverse example
$$\begin{pmatrix} s & -1 \\ \frac{g}{L} & s \end{pmatrix}^{-1} \Rightarrow \frac{\begin{pmatrix} s & -\frac{g}{L} \\ 1 & s \end{pmatrix}'}{s^2 + \frac{g}{L}} = \frac{\begin{pmatrix} s & 1 \\ -\frac{g}{L} & s \end{pmatrix}}{s^2 + \frac{g}{L}}$$

a vector x is normalized if its Norm $= 1$ or $\bar{x}^T\bar{x} = 1$. 2 vectors x_1, x_2 are orthogonal if $x_1^T x_2 = 0 = x_2^T x_1$

given set of independent vectors we can obtain orthonormal set using Schmidt procedure using $T^{-1} = [q_1\ q_2 \cdots q_n]$ obtain from controllability matrix (add to it as needed)

$$\Rightarrow \dot{\bar{x}} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \bar{x} + \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} u$$

where $\dot{\bar{x}} = \bar{A}_c \bar{x}_1 + \bar{B}_1 u + \bar{A}_{12}\bar{x}_2$ is controllable.

using $T = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}$ where the first $n - n_1$ L.I. rows obtained from Obs. matrix.

$$\Rightarrow \begin{bmatrix} \dot{\bar{x}}_o \\ \dot{\bar{x}}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{B}_o \\ \bar{B}_{\bar{o}} \end{bmatrix} u$$

$$y = [\bar{C}_o \ \ 0] \begin{bmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{bmatrix} + Du$$

$\rightarrow$ $\dot{\bar{x}}_o = \bar{A}_o \bar{x}_o + \bar{B}_o u$, $\bar{y} = \bar{C}_o \bar{x}_o + Du$ is the observable system.

- if the A matrix is in Jordan Form, we can easily check controllability and observability. if the last rows in B associated with Jordan block are L.I. (or not zero if one row) $\Rightarrow$ controllable. For observability, look at ~~first~~ first columns in C instead of B.

- state $\frac{A \mid b}{c \mid d}$ is minimal realization of proper rational $g(s)$ $\Leftrightarrow$ (A, b) controllable and (c, A) observable.

- state $\frac{A \mid b}{c \mid d}$ is minimal realization of $g(s)$ $\Leftrightarrow$ $\dim A = \deg g(s)$

do not cancel factors between numerator & denominator to find $g(s)$ unless asked to find a minimal realization.

solution of LTI state equation: $x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau$

$$y(t) = C e^{At} x(0) + C \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau + Du(t)$$

$e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$

state feedback ① Pick F_1, v ② Find $A_1 = A - BF_1$, $b = Bv$ ③ Find $U_1 = [b\ A_1 b \cdots]$ (check controllability for (A_1, b)) ④ Find char eq. for A_1, $\alpha_1, \alpha_2 \cdots$ ⑤ Find $W = \begin{bmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & 1 & \alpha_1 \\ 0 & 0 & 1 \end{bmatrix}$ ⑥ Find $T^{-1} = UW$, $\rightarrow T$ ⑦ find $f = [\bar{\alpha}_1 - \alpha_1 \cdots]$ (note we use the α's for char eq of $A - BF_1$) to set $\bar{\alpha}$ construct char eq. from desired poles. ⑧ Find $F = F_1 + v f T$

if the uncontrollable part is stable, then system is called stabilizable.

Zeros of transfer function are not affected by state feedback.

solution of time varying $\dot{x} = a(t) x$ is $x(t) = e^{\int_0^t a(\tau) d\tau} x(0)$

Fundamental Matrix: contains as columns, n L.I. solutions to state eq. using different initial condition. $X(t)$ or $\Psi(t)$. Used For LTV. | $\Psi(t)$ is nonsingular for all t.

State transition Matrix $\Phi \Rightarrow \boxed{\frac{\partial}{\partial t}\Phi(t,t_0)=A(t)\Phi(t,t_0)}$

$\Phi(t,t_0)=\Psi(t)\Psi^{-1}(t_0)$ Properties: $\Phi(t,t)=I$, $\Phi^{-1}(t,t_0)=\Phi(t_0,t)$

$\Phi(t,t_0)=\Phi(t,t_1)\Phi(t_1,t_0)$

Jordan Form $\begin{bmatrix} \lambda_1 & 1 & & \\ 0 & \lambda_1 & & \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & & \lambda_2 \end{bmatrix}$ $B\begin{bmatrix} \phi \\ 0 \\ 0 \\ 1 \end{bmatrix}$ ← must be ≠ 0 to be controllable ← ok

So solution to LTV state eq. using Φ is

$$x(t)=\Phi(t,t_0)x_0+\int_{t_0}^{t}\Phi(t,\tau)B(\tau)u(\tau)\,d\tau$$

$c=[a_1 \quad a_2 \, a_3 \, a_4]$ must be ≠ 0. For observability

$$y(t)=C(t)\Phi(t,t_0)x_0+C(t)\int_{t_0}^{t}\Phi(t,\tau)B(\tau)u(\tau)\,d\tau+D(t)u(t)$$

Given $Ax=b$, solution exist if $\mathrm{Rank}(A)=\mathrm{Rank}[A \; b]$

Find A^{100} $A=\begin{pmatrix}0&1\\-1&-2\end{pmatrix} \to$ let $h(\lambda)=\beta_0+\beta_1\lambda \Rightarrow f(-1)=h(-1)\Rightarrow(-1)^{100}=\beta_0-\beta_1$

$\lambda=-1,-1$ $f'(-1)=h'(-1)\Rightarrow 100(-1)^{99}=\beta_1$

$\Rightarrow h(\lambda)=-99-100\lambda \Rightarrow A^{100}=-99I-100A \Rightarrow \begin{pmatrix}-99&-100\\100&101\end{pmatrix}$

$e^{At}=I+tA+\frac{t^2}{2!}A^2+\dots$

$e^{0}=I$

$e^{A(t_1+t_2)}=e^{At_1}e^{At_2}$

$[e^{At}]^{-1}=e^{-At}$

$e^{(A+B)t}\neq e^{At}e^{Bt}$ (only if $AB=BA$ then $=$ ok).

$Ae^{At}=e^{At}A$. $(A^{-1})^T=(A^T)^{-1}$

if A diagonal $\Rightarrow e^{At}=\begin{bmatrix}e^{\lambda_1 t} & \\ 0 & e^{\lambda_1 t} \\ & & e^{\lambda_2 t}\end{bmatrix}$ $[\lambda_1 \, \lambda_1 \, \lambda_2]$

$g(t,\tau)=0$ For $t<\tau$ For causal system

time observed; time input applied

Cayley Hamilton $A^n=a_0I+a_1A+a_2A^2+\dots+a_{n-1}A^{n-1}$

Solution $Ax=b$ is unique if $\mathrm{null}(A)=0$.

$\mathcal{L}\, e^{At}=(sI-A)^{-1}$ | $\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)=e^{t}$, $\frac{1}{s}\to 1$, $\frac{1}{s(s-1)}\to -1+e^{t}$, $\frac{1}{s(s-1)^2}\to 1-e^{t}+te^{t}$

For causal $y(t)=\int_0^t g(t-\tau)u(\tau)\,d\tau$

$\dot{x}_2=-x_2 \Rightarrow x=e^{-t}x_2(0)$

$\dot{x}_1=-x_1+e^{-t}x_2(0) \Rightarrow x_1=e^{-t}x_1(0)+x_2(0)\int_0^t e^{-(t-\tau)}e^{-\tau}d\tau$

State Feedback Closed loop is $\dot{x}=(A-BF)x+Br$

$y=Cx$

$\dot{x}_2=tx_1$

$\Rightarrow x_2=\int_0^t \tau x_1(0)\,d\tau+x_2(0)$

Controllable Form

$$\frac{s^2+5s-2}{s^3+2s^2-5s-6}\to\begin{pmatrix}-2&5&6\\1&0&0\\0&1&0\end{pmatrix}x+\begin{pmatrix}1\\0\\0\end{pmatrix}B;\; y=[1 \quad 1 \quad -2]x$$

$\dot{x}=\underset{n\times n}{A}x+\underset{n\times p}{B}u$

$y=\underset{q\times n}{C}x+\underset{q\times p}{D}u$

$$\frac{s-1}{s^3+2s^2-5s-2}\xrightarrow{\text{observable Form}}=\begin{bmatrix}-2&1&0\\1&0&1\\2&0&0\end{bmatrix}+\begin{bmatrix}0\\1\\-1\end{bmatrix}u$$

$y=[1 \quad 0 \quad 0]x$

- if 2 $\frac{A \mid B}{C \mid D}$ have same $G(s)$ $\Rightarrow$ they are zero state equivalent. $\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix} = 8$ $\quad 3 \times 5 - 2 \times 4$

- if 2 A have same λ's $\Rightarrow$ they are equivalent.

- take A^{-1} when A is 2x2 do $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \dfrac{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}{ad-bc}$

- to break system into controllable and uncontrollable parts use transformation $\bar{A} = TAT^{-1}$, $\bar{B} = TB$, $\bar{C} = CT^{-1}$, $\bar{D} = D$ and use $\bar{A}_{11}$ part only.
How to find T here? First find U, the controllability matrix for A as follow: $U = [b \;\; Ab \;\; A^2 b]$, then take the first L.I. columns. then add more columns to make it nxn matrix. this matrix will be T^{-1}. inverse it to get T. and that is it.

<u>example</u> $A = \begin{pmatrix} -1 & 4 \\ 4 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$; $y = (1 \;\; 1) x$ $\qquad \bar{A}_{11} = \bar{A}_c$

$\Rightarrow U = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \Rightarrow T^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow T = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow \bar{A} = TAT^{-1} = \begin{bmatrix} 3 & 4 \\ 0 & -5 \end{bmatrix}$ controllable part: (A_{11}, B_1)

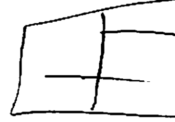
arbitrary

uncontrollable part

$\bar{A} = TAT^{-1}$, $\bar{B} = TB$, $\bar{C} = CT^{-1}$, $\bar{D} = D$, $\underbrace{\bar{x} = Tx}_{\text{equivalence transformation}}$, $\bar{U} = UT^{-1}$ (controllability matrix), $\bar{V} = TV$ (observability matrix)

The $\bar{x}$ system is obtained from the x system by the substitution $x = T^{-1}\bar{x}$

let $T^{-1} = Q = [q_1 \; q_2 \cdots q_n]$ then the ~~representation of~~ i^{th} column of $\bar{A}$ is the representation of Aq_i with r.t. $[q_1 \; q_2 \cdots q_n]$

 $= \frac{1}{s}$

$A, \bar{A}$ have same set of λ's under similarity transformation $\qquad \frac{5}{s^2+7s+2} \to 5e^{st}$

$\det(A) \det(\bar{A}) = 1$ $\qquad \boxed{\bar{G}(s) = G(s)}$ both state equation are zero state equivalent

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realization problem: given $G(s)$ find a state equation representation

Not every $G(s)$ is realizable. a distributed system can't.

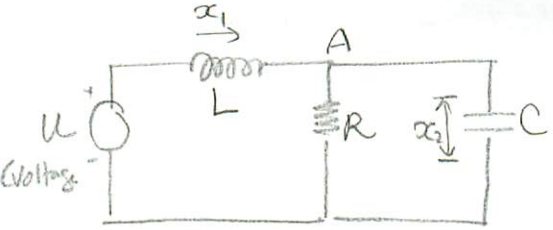
---

let Q be set of basis $[q_1 \; q_2 \cdots q_n]$, then the representation (or coordinate) of a vector b w.r.t. to Q is $Qx = b$
in the $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ basis, vector b has same representation as itself.

Companion form: $\bar{A}$ is the representation of A w.r.t. to same Q. i.e. each column of $\bar{A}$ is the rep. of Aq_i w.r.t Q.
i.e i^{th} column of $\bar{A}$ = rep. of Aq_i w.r.t. Q.

$A x = b$ (vector b)

L.I. q_i's $\qquad$ rep of b w.r.t A



For states, take current through inductor and voltage through Cap. as state variables.

$Q = CV$; $i = \frac{dQ}{dt} \Rightarrow \boxed{i = C\frac{dV}{dt}}$ capacitor $\Rightarrow i_c = C\dot{x}_2$

For inductor $\boxed{V = L\frac{di}{dt}} \Rightarrow V_L = L\dot{x}_1$

~~So current through R is~~ add currents at A $\rightarrow$ $\boxed{x_1 = \underset{x_2}{i_R} + \underset{\dot{x}_2}{i_c}}$, ohms law $\Rightarrow V = RI$

$\downarrow$

$\boxed{\dot{x}_2 = x_1 - x_2}$

and using voltage drop we get around outer loop $L\dot{x}_1 + x_2 = u$

$\Rightarrow \boxed{\dot{x}_1 = -x_2 + u}$

Companion Form of A is found as follows. $\begin{bmatrix} 0 & 0 & -\alpha_1 \\ 1 & 0 & -\alpha_2 \\ 0 & 1 & -\alpha_3 \end{bmatrix}$ $\leftarrow$ Companion Form

where if you write the char. eq for A: controllable form

$\lambda^3 + \alpha_3\lambda^2 + \alpha_2\lambda + \alpha_1 = 0$

another companion forms are $\begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_1 & -\alpha_1 & -\alpha_3 \end{bmatrix}$, $\begin{bmatrix} -\alpha_1 & 1 & 0 \\ -\alpha_2 & 0 & 1 \\ -\alpha_3 & 0 & 0 \end{bmatrix}$

a companion matrix is one which can be formed from coeff. of the char. polynomial.

Jordan Forms for similarity transformation: $\bar{A} = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix}$ where λ's are eigenvalues of A.

where here $T = Q^{-1}$ and $Q = [q_1 \ q_2 \ \cdots]$ the eigenvectors of A.

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realization. SISO $g(s) = \dfrac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3} + g(\infty) \Rightarrow A = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $C = [b_1 \ b_2 \ b_3]$, $D = g(\infty)$.

$\boxed{g(s) = C(sI - A)^{-1}B + D}$

MIMO write $G(s)$ as $\underbrace{[N_1]s^2 + [N_2]s + [N_3]}_{\text{least common multiple of denominator}} + G(s)\big|_{s=\infty}$

$s^3 + a_1 s^2 + a_2 s + a_3$

minimal canonical form called controllable

$p \times p$

so $A = \begin{bmatrix} -a_1 I_p & -a_2 I_p & -a_3 I_p \\ I_p & 0 & 0 \\ 0 & I_p & 0 \end{bmatrix}$, $B = \begin{bmatrix} I_p \\ 0 \\ 0 \end{bmatrix}$, $C = [N_1 \ N_2 \ N_3]$, $D =$

- state is controllable if we can change it from input.

$U = [B \ AB \ A^2B \cdots A^{n-1}B] \cong [A - \lambda I \quad B]$ is full row rank for any λ

controllability is invariant under similarity transformation.

- System is observable if knowledge of u and y over some t is enough to determine $x(0)$

- Duality $(A, B) \Leftrightarrow (A', B')$ observable $\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$