Problem 4, HW 4. MAE 185.

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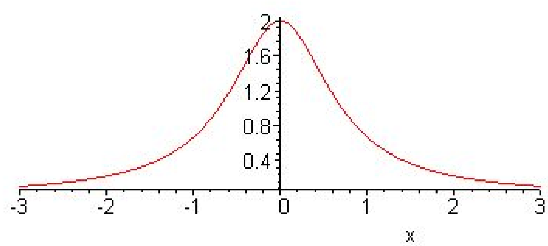
problem Employee a two-through six-points Gauss-Legendre formulas to solve $\int_{-3}^{3} \frac{2}{1+2x^2} dx$
Answer
$f(x) = \frac{2}{1+2x^2}$ This function is an even function:



f(x)

hence

$$\begin{aligned} \mathbf{I} &= \int_{-3}^{3} \frac{2}{1+2x^2} dx \\ &= 2\int_{0}^{3} \frac{2}{1+2x^2} dx \\ &= 4\int_{0}^{3} \frac{1}{1+2x^2} dx \end{aligned}$$

First transform the integral to be inside the limits of -1 to $+1$. Let $x = k_1 y + k_2$

$$\begin{aligned} (0) &= k_1(-1) + k_2 \\ (+3) &= k_1(+1) + k_2 \end{aligned}$$

Solve for k_1, k_2 . Adding the above 2 equations gives $2k_2 = 3 \Longrightarrow \boxed{k_2 = 1.5}$
Hence, $0 = -k_1 + 1.5 \Longrightarrow \boxed{k_1 = 1.5}$
hence the transformation is $\boxed{x = 1.5 + 1.5y}$ and $\boxed{dx = 1.5\ dy}$
Using the above transformation, the integral becomes

$$
\begin{aligned}
\mathbf{I} &= \int_{-3}^{3} \frac{2}{1+2x^2} dx \\
&= 4\int_{0}^{+3} \frac{1}{1+2x^2} dx \\
&= 4\int_{-1}^{+1} \frac{1}{1+2(1.5+1.5y)^2} (1.5\, dy) \\
&= 6\int_{-1}^{+1} \frac{1}{5.5+9y+4.5y^2} dy
\end{aligned}
$$

Hence $g(y) = \frac{1}{5.5+9y+4.5y^2}$

2 points Gauss

$c_0 = 1, c_1 = 1 \;\; ,x_0 = -\frac{1}{\sqrt{3}}, x_1 = +\frac{1}{\sqrt{3}}$

$$
\begin{aligned}
g(x_0) &= \frac{1}{5.5+9\left(-\sqrt{\frac{1}{3}}\right)+4.5\left(-\sqrt{\frac{1}{3}}\right)^2} = \frac{1}{1.804} = 0.554 \\
g(x_1) &= \frac{1}{5.5+9\left(\sqrt{\frac{1}{3}}\right)+4.5\left(\sqrt{\frac{1}{3}}\right)^2} = \frac{1}{12.196} = 0.0820
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathbf{I} &= 6\;\left[c_0\, g(x_0) + c_1\, g(x_1)\right] \\
&= 3.818
\end{aligned}
$$

3 points Gauss

$c_0 = 5/9,\; c_1 = 8/9,\; c_2 = 5/9, x_0 = -\sqrt{0.6},\; x_1 = 0.0,\; x_2 = \sqrt{0.6}$

$$
\begin{aligned}
g(x_0) &= \frac{1}{5.5+9\left(-\sqrt{0.6}\right)+4.5\left(-\sqrt{0.6}\right)^2} = \frac{1}{1.223} = 0.814 \\
g(x_1) &= \frac{1}{5.5+9(0)+4.5(0)^2} = \frac{1}{5.5} = 0.182 \\
g(x_2) &= \frac{1}{5.5+9\left(\sqrt{0.6}\right)+4.5\left(\sqrt{0.6}\right)^2} = \frac{1}{15.171} = 0.066
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathbf{I} &= 6\;\left[c_0\, g(x_0) + c_1\, g(x_1) + c_2\, g(x_2)\right] \\
&= 6\left[(5/9)\,0.814 + (8/9)\,0.182 + (5/9)\,0.066\,\right] \\
&= 3.902
\end{aligned}
$$

4 points Gauss

$c_0 = 0.3478548,\; c_1 = 0.6521452,\; c_2 = 0.6521452,\; c_3 = 0.3478548$

$x_0 = -0.861136312,\; x_1 = -0.339981044,\; x_2 = 0.339981044,\; x_3 = 0.861136312$

$$g(x_0) = \frac{1}{5.5+9(-0.861)+4.5(-0.861)^2} = \frac{1}{1.0868} = 0.920$$
$$g(x_1) = \frac{1}{5.5+9(-0.340)+4.5(-0.340)^2} = \frac{1}{2.960} = 0.338$$
$$g(x_2) = \frac{1}{5.5+9(0.340)+4.5(0.340)^2} = \frac{1}{9.080} = 0.110$$
$$g(x_4) = \frac{1}{5.5+9(0.861)+4.5(0.861)^2} = \frac{1}{16.587} = 0.060$$

so

$$\begin{aligned}\mathbf{I} &= 6[c_0\, g(x_0)+c_1\, g(x_1)+c_2\, g(x_2)+c_3\, g(x_3)]\\ &= 6*(0.3478548\,)0.920+(0.6521452)\; 0.338+\;(0.6521452)0.110+(0.348)0.060\\ &= 3.780\end{aligned}$$

5 points Gauss

$c_0 = 0.2369269,\ c_1 = 0.4786287,\ c_2 = 0.568889,\ c_3 = 0.4786287,\ c_4 = 0.2369269$
$x_0 = -0.906179846,\ x_1 = -0.538469310,\ x_2 = 0.0,\ x_3 = 0.53846931,\ x_4 = 0.906179846$

$$g(x_0) = \frac{1}{5.5+9(-0.906)+4.5(-0.906)^2} = \frac{1}{1.0396} = 0.962$$
$$g(x_1) = \frac{1}{5.5+9(-0.538)+4.5(-0.538)^2} = \frac{1}{1.959} = 0.511$$
$$g(x_2) = \frac{1}{5.5+9(0)+4.5(0)^2} = \frac{1}{5.5} = 0.182$$
$$g(x_3) = \frac{1}{5.5+9(0.538)+4.5(0.538)^2} = \frac{1}{11.651} = 0.0858$$
$$g(x_4) = \frac{1}{5.5+9(0.906)+4.5(0.906)^2} = 0.0576$$

Hence,

$$\begin{aligned}\mathbf{I} &= 6[c_0\, g(x_0)+c_1\, g(x_1)+c_2\, g(x_2)+c_3\, g(x_3)+c_4\, g(x_4)]\\ &= 6\; *0.237(0.962\,)\; +0.479(0.511)\; +0.569(0.182)+0.479(0.0858)+0.237(0.0576)\\ &= 3.783\end{aligned}$$

6 points Gauss

$c_0 = 0.171,\ c_1 = 0.361,\ c_2 = 0.468,\ c_3 = 0.468,\ c_4 = 0.3607616,\ c_5 = 0.1713245$
$x_0 = -0.932,\ x_1 = -0.661,\ x_2 = -0.239,\ x_3 = 0.239,\ x_4 = 0.661x_5 = 0.93$

$$g(x_0) = \frac{1}{5.5+9(-0.932)+4.5(-0.932)^2} = 0.980$$
$$g(x_1) = \frac{1}{5.5+9(-0.661)+4.5(-0.661)^2} = 0.660$$
$$g(x_2) = \frac{1}{5.5+9(-0.239)+4.5(-0.239)^2} = 0.277$$
$$g(x_3) = \frac{1}{5.5+9(0.239)+4.5(0.237)^2} = 0.127$$
$$g(x_4) = \frac{1}{5.5+9(0.661)+4.5(0.661)^2} = 0.075$$
$$g(x_5) = \frac{1}{5.5+9(0.932)+4.5(0.932)^2} = 0.056$$

Hence

$$\begin{aligned}\mathbf{I} &= 6\{c_0\, g(x_0) + c_1\, g(x_1) + c_2\, g(x_2) + c_3\, g(x_3) + c_4\, g(x_4) + c_5\, g(x_5)\}\\ &= 6 * 0.171(0.980) + 0.361(0.660) + 0.468(0.277) + 0.468(0.127) + 0.361(0.075) + 0.171(0.056)\\ &= 3.787\end{aligned}$$

summary: This table shows the relative error normalized to the true value for value of number of points n, starting with n=2 up to n=6. Analytically,

$$\int \frac{2}{1+2x^2}\,dx = \sqrt{2}\arctan\left(\sqrt{2}x\right)$$

hence

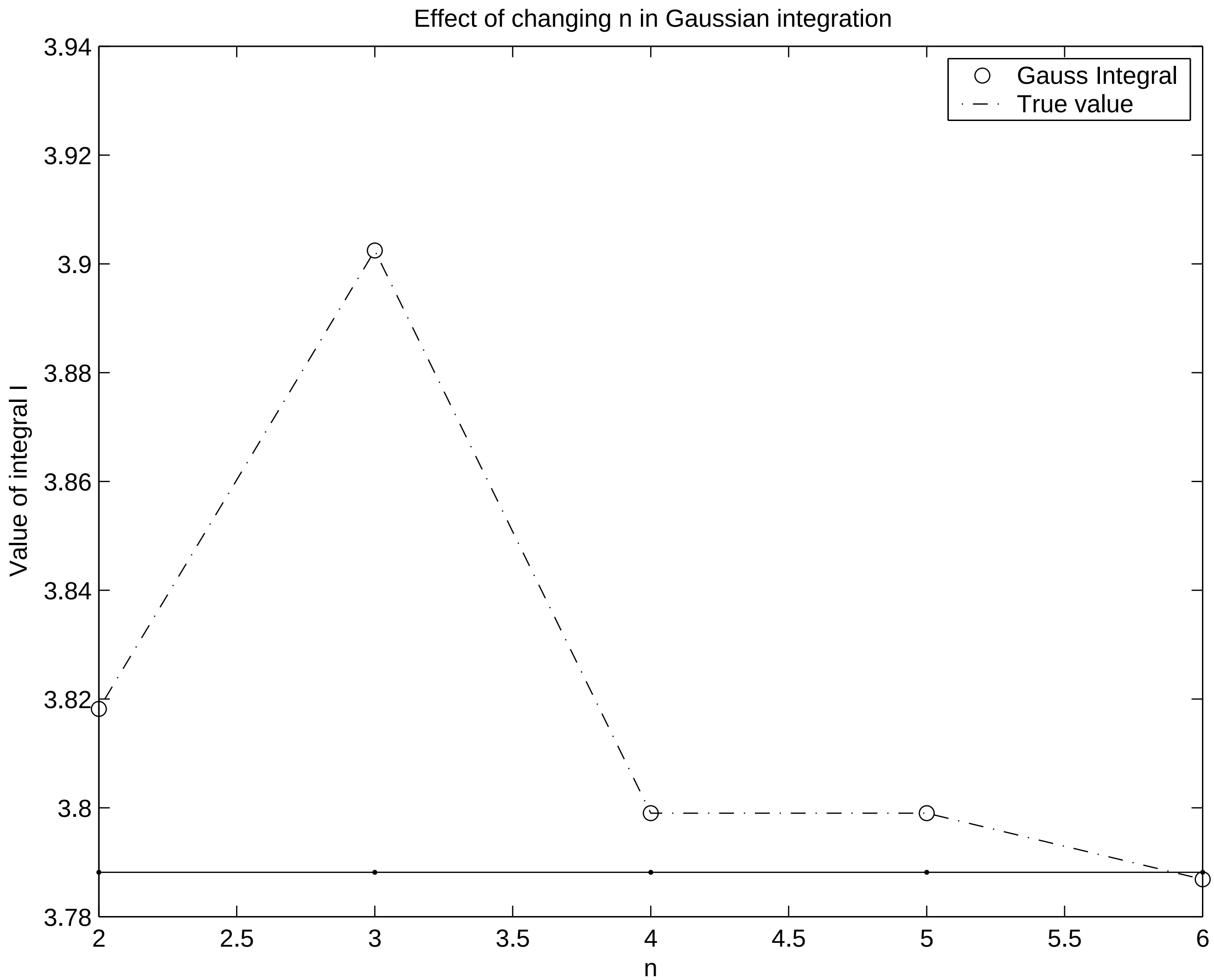
$$I_{analytical} = 2\int_0^3 \frac{2}{1+2x^2}\,dx = 2\left[\sqrt{2}\arctan\left(\sqrt{2}(3)\right) - \sqrt{2}\arctan\left(\sqrt{2}(0)\right)\right] = 2[1.894] = 3.788$$

So $\varepsilon_t = \left|\frac{true-approx}{true}\right| 100\ \%$

n	**I**	ε_t
2	3.818	$\left\lvert\frac{3.788-3.818}{3.788}\right\rvert = 0.792$
3	3.902	$\left\lvert\frac{3.788-3.902}{3.788}\right\rvert = 3.017$
4	3.799	$\left\lvert\frac{3.788-3.780}{3.788}\right\rvert = 0.287$
5	3.783	$\left\lvert\frac{3.788-3.783}{3.788}\right\rvert = 0.144$
6	3.787	$\left\lvert\frac{3.788-3.787}{3.788}\right\rvert = 0.035$

As number of points increases, the acurracy increases.

observation: I am able to find why the n=3 case seems to have some abnormality. ε_t should decrease as n increases, but for n=3 it did not. Checked the calculations but not able to see if I did some arithmitic error.


Effect of changing n in Gaussian integration
Gauss Integral
True value
3.94
3.92
3.9
3.88
3.86
3.84
3.82
3.8
3.78
Value of integral I
2
2.5
3
3.5
4
4.5
5
5.5
6
n