Problem 30.3 HW#7

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Use explicit method to solve for the temp distribution of a long thin rod with length 10 cm and $k^{'} = 0.49$ cal/(s.cm.C^0), $\Delta x = 2$ cm and $\Delta t = 0.05s$. At t=0, the temp. of the rod is 0^0C and the boundary conditions fixed for all times at $T(0) = 100^0$C and $T(10) = 50^0C$. note that the rode is aluminum with C=0.2147 cal/(g.C^0) and $\rho = 2.7$ g/cm^3. therefor k=0.49/(2.7×0.2174)=0.835 cm^2/s and $\lambda = 0.835(0.05)/2^2 = 0.0104375$.

Solution

The only difference between this problem and the example shown is in the time step. This causes λ to change from 0.020875 to 0.0104375

We need to solve a parabolic PDE using the explicit method.

Recall that the solution using the parabolic PDE is written as

$$T_i^{l+1} = T_i^l + k\frac{\Delta t}{(\Delta x)^2}\left(T_{i+1}^l - 2T_i^l + T_{i-1}^l\right)$$

Now apply the PDE solution with the above boundary conditions in place.

Apply one time step. now at time=0.05 seconds:

$T_0^1 = 100$

$T_1^1 = T_1^0 + k\frac{\Delta t}{(\Delta x)^2}\left(T_2^0 - 2T_1^0 + T_0^0\right) = 0 + 0.0104375\,(0 - 2(0) + 100) = 1.04375$

$T_2^1 = T_2^0 + k\frac{\Delta t}{(\Delta x)^2}\left(T_3^0 - 2T_2^0 + T_1^0\right) = 0 + 0.0104375\,(0 - 2(0) + 0) = 0$

$T_3^1 = T_3^0 + k\frac{\Delta t}{(\Delta x)^2}\left(T_4^0 - 2T_3^0 + T_2^0\right) = 0 + 0.0104375\,(0 - 2(0) + 0) = 0$

$T_4^1 = T_4^0 + k\frac{\Delta t}{(\Delta x)^2}\left(T_5^0 - 2T_4^0 + T_3^0\right) = 0 + 0.0104375\,(50 - 2(0) + 0) = 0.521875$

$T_5^1 = 50$

Ok, now continue. Let t=0.1 seconds, i.e. $l = 2$. Apply the same process as above:

$T_0^2 = 100$

$T_1^2 = T_1^1 + k\frac{\Delta t}{(\Delta x)^2}\left(T_2^1 - 2T_1^1 + T_0^1\right) = 1.04375 + 0.0104375\,(0 - 2(1.04375) + 100) = 2.06571171875$

$T_2^2 = T_2^1 + k\frac{\Delta t}{(\Delta x)^2}\left(T_3^1 - 2T_2^1 + T_1^1\right) = 0 + 0.0104375\,(0 - 2(0) + 1.04375) = 0.010894140625$

$T_3^2 = T_3^1 + k\frac{\Delta t}{(\Delta x)^2}\left(T_4^1 - 2T_3^1 + T_2^1\right) = 0 + 0.0104375\,(0.521875 - 2(0) + 0) = 0.0054470703125$

$T_4^2 = T_4^1 + k\frac{\Delta t}{(\Delta x)^2}\left(T_5^1 - 2T_4^1 + T_3^1\right) = 0.521875 + 0.0104375\,(50 - 2(0.521875) + 0) = 1.032855859375$

$T_5^2 = 50$

Ok, now continue. Let t=0.15 seconds, i.e. $l = 3$. Apply the same process as above:

$T_0^3 = 100$

$T_1^3 = T_1^2 + k\frac{\Delta t}{(\Delta x)^2}\left(T_2^2 - 2T_1^2 + T_0^2\right) =$

$2.06571171875 + 0.0104375\,(0.010894140625 - 2(2.06571171875) + 100) = 3.06645369421387$

$T_2^3 = T_2^2 + k\frac{\Delta t}{(\Delta x)^2}\left(T_3^2 - 2T_2^2 + T_1^2\right) =$

$0.010894140625 + 0.0104375\,(0.0054470703125 - 2(0.010894140625) + 2.06571171875) = 0.032284445300293$

$T_3^3 = T_3^2 + k\frac{\Delta t}{(\Delta x)^2}\left(T_4^2 - 2T_3^2 + T_2^2\right) =$

$0.0054470703125 + 0.0104375\,(1.032855859375 - 2(0.0054470703125) + 0.010894140625) = 0.0162275033447266$

$T_4^3 = T_4^2 + k\frac{\Delta t}{(\Delta x)^2}\left(T_5^2 - 2T_4^2 + T_3^2\right) =$

$1.032855859375 + 0.0104375\,(50 - 2(1.032855859375) + 0.0054470703125) = 1.53322684710693$

$T_5^3 = 50$

Ok, now continue. Let t=0.2 seconds, i.e. $l = 4$. Apply the same process as above:
$T_0^4 = 100$
$T_1^4 = T_1^3 + k\frac{\Delta t}{(\Delta x)^2}\left(T_2^3 - 2T_1^3 + T_0^3\right) =$
$3.06645369421387{+}0.0104375\,(0.032284445300293 - 2(3.06645369421387) + 100) = 4.04652844224498$
$T_2^4 = T_2^3 + k\frac{\Delta t}{(\Delta x)^2}\left(T_3^3 - 2T_2^3 + T_1^3\right) =$
$0.032284445300293{+}0.0104375\,(0.0162275033447266 - 2(0.032284445300293) + 3.06645369421387) =$
0.0637859925041672
$T_3^4 = T_3^3 + k\frac{\Delta t}{(\Delta x)^2}\left(T_4^3 - 2T_3^3 + T_2^3\right) =$
$0.0162275033447266{+}0.0104375\,(1.53322684710693 - 2(0.0162275033447266) + 0.032284445300293) =$
0.0322287783269058
$T_4^4 = T_4^3 + k\frac{\Delta t}{(\Delta x)^2}\left(T_5^3 - 2T_4^3 + T_3^3\right) =$
$1.53322684710693{+}0.0104375\,(50 - 2(1.53322684710693) + 0.0162275033447266) = 2.02326511123973$
$T_5^4 = 50$

Summary:

Using $\nabla t = 0.05$ seconds, this is the final value of T
$T_0^4 = 100$
$T_1^4 = 4.04652844224498$
$T_2^4 = 0.0637859925041672$
$T_3^4 = 0.0322287783269058$
$T_4^4 = 2.02326511123973$
$T_5^4 = 50$

Compare that with using $\nabla t = 1$ second:
$T_0^4 = 100$
$T_1^4 = 4.0878$
$T_2^4 = 0.043577$
$T_3^4 = 0.021788$
$T_4^4 = 2.0439$
$T_5^4 = 50$

Notice that with $\nabla t = 0.05$ sec, the values of T inside the rod are a little smaller than with $\nabla t = 1$ second.

Both solutions are stable (i.e. with $\nabla t = 1$ second, and $\nabla t = 0.05$ seconds) since in both cases $\lambda \leq 1/2$, hence the above difference means that for $\nabla t = 0.05$ it is more accurate than for $\nabla t = 1$