

§ 15.

Change of a vector with respect to a rotary frame.

Lab frame $OXYZ$ } with the same origin.
 rotating frame $Oxyz$ } and angular velocity of $Oxyz$
 is $\vec{\omega}$.

In Lab frame: $\vec{r} = r_x \vec{i} + r_y \vec{j} + r_z \vec{k}$.
 position vector

$$\frac{d\vec{r}}{dt} = \frac{dr_x}{dt} \vec{i} + \frac{dr_y}{dt} \vec{j} + \frac{dr_z}{dt} \vec{k} = \vec{v}$$



In the rotating frame $Oxyz$ } $\vec{r} = r'_x (\vec{i}') + r'_y (\vec{j}') + r'_z (\vec{k})'$

$$\begin{aligned} \frac{d\vec{r}}{dt} &= \frac{dr'_x}{dt} (\vec{i}') + \frac{dr'_y}{dt} (\vec{j}') + \frac{dr'_z}{dt} (\vec{k})' \\ &\quad + r'_x \frac{d(\vec{i}')}{dt} + r'_y \frac{d(\vec{j}')}{dt} + r'_z \frac{d(\vec{k}')}{dt}. \end{aligned}$$

relative Velocity
in rotary frame

The first order!

So, we get:

$\frac{d\vec{r}}{dt} = \left(\frac{d\vec{r}}{dt} \right)' + \vec{\omega} \times \vec{r}$

instant angular velocity
of the rotary
frame.

\downarrow
absolute
velocity
in Lab frame.

\downarrow
relative velocity
in the rotary
frame.

\downarrow
the position vector.

addition velocity
due to the rotation
of frame.

It equals to
 $\vec{\omega} \times \vec{r}$

The time derivative operation in Lab frame.

equals to { the time derivative operation in the rotary frame.

$$\vec{\omega} \times \vec{r}$$

or:

$$\text{absolute velocity} = \text{relative velocity} + \vec{\omega} \times \vec{r}$$

\rightarrow the absolute velocity
coinciding with \vec{r} .

The second order!

$$\ddot{\vec{r}} = \frac{d}{dt} \cdot \left(\frac{d\vec{r}}{dt} \right) = \frac{d}{dt} \cdot \left(\left(\frac{d\vec{r}}{dt} \right)' + \vec{v} \times \vec{r} \right)$$

$$= \underbrace{\frac{d\vec{v}}{dt} \times \vec{r}}_{\text{in the Lab frame.}} + \vec{v} \times \frac{d\vec{r}}{dt} + \underbrace{\frac{d}{dt} \left(\frac{d\vec{r}}{dt} \right)'}_{\text{the time derivative operation in the rotating frame.}}$$

$$= \frac{d\vec{v}}{dt} \times \vec{r} + \vec{v} \times \frac{d\vec{r}}{dt} + \left(\frac{d^2\vec{r}}{dt^2} \right)' + \vec{v} \times \left(\frac{d\vec{r}}{dt} \right)'. \quad \begin{matrix} \text{the time derivative} \\ \text{operation in} \\ \text{the Lab frame.} \end{matrix}$$

$$= \frac{d\vec{v}}{dt} \times \vec{r} + \vec{v} \times \left(\frac{d\vec{r}}{dt} \right)' + \vec{v} \times \vec{v} \times \vec{r} + \left(\frac{d^2\vec{r}}{dt^2} \right)' + \vec{v} \times \left(\frac{d\vec{r}}{dt} \right)'. \quad \begin{matrix} \text{operation in} \\ \text{the Lab frame.} \end{matrix}$$

$$= \frac{d\vec{v}}{dt} \times \vec{r} + \vec{v} \times \vec{v} \times \vec{r} + 2\vec{v} \times \left(\frac{d\vec{r}}{dt} \right)' + \left(\frac{d^2\vec{r}}{dt^2} \right)'. \quad \begin{matrix} \text{relative acceleration} \\ \text{in the rotating frame} \end{matrix}$$

~~The second order~~ \downarrow \downarrow \downarrow
absolute acceleration. Coriolis [K]: $v' \omega \times v$
~~at the point of \vec{r} .~~ coinciding with. relative acceleration.

~~The second order~~ absolute acceleration = The absolute acceleration coinciding with \vec{r} .
+ Coriolis acceleration

+ relative acceleration

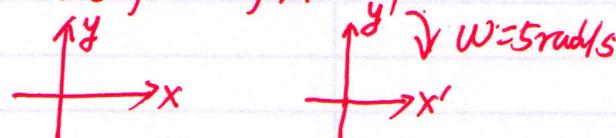
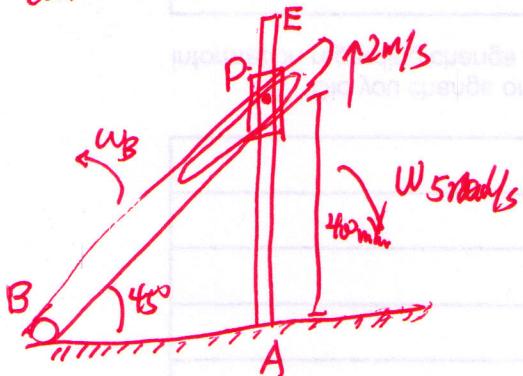
Sample problem #1.

Pin P is attached to the collar shown; the motion of the pin is guided by a slot cut in bar BD and by the collar that slides on rod AE. Rod AE rotates with a constant angular velocity of 5 rad/s clockwise and the distance from A to P increases at a constant rate of 2 m/s. Determine at the instant shown:

- angular velocity of BD ω_B
- the relative velocity of Pin P to BD
- the angular acceleration of bar BD
- the relative acceleration of Pin P with respect to bar BD

Solution:

- the absolute velocity, acceleration of P. ~~standn.~~
the frame of AE



$$\begin{aligned} \vec{v}_P &= (\vec{v}_P)' + \vec{\omega} \times \vec{r}_{Ap} \rightarrow \text{absolute velocity} \text{ at coincidence} \\ &\quad \xrightarrow{\text{relative velocity}} \text{with } P \\ &= 2\vec{j} + \omega |AP| \vec{x} \\ &= 2\vec{j} + 2\vec{x} \quad \dots \quad (1) \end{aligned}$$

the $\vec{a}_P = (\vec{a}_P)' + \vec{a}_{Pl} + \vec{a}_c \rightarrow \text{Coriolis acceleration.}$

$\xrightarrow{\text{absolute acceleration coincide with } P}$
 $\xrightarrow{\text{relative acceleration}}$

$$\vec{a}_c = 2\vec{\omega} \times \left(\frac{d\vec{r}_{Ap}}{dt} \right)' = 2 \cdot 5 \cdot 2 \vec{x} = 20 \text{ m/s}^2 \vec{x}$$

$\xrightarrow{\text{relative velocity}}$

$$\vec{a}_{Pl} = \vec{\omega} \times \vec{\omega} \times \vec{r}_{Ap} + \frac{d\vec{\omega}}{dt} \times \vec{r}_{Ap} = -\omega^2 \cdot r_{Ap} \cdot \vec{j} = -10 \text{ m/s}^2 \vec{j}$$

$$(\vec{a}_P)' = 0$$

$$\Rightarrow \vec{a}_P = 20\vec{x} \oplus -10\vec{j} \quad (2)$$

⑥ The absolute velocity, acceleration, of P in the frame of BP.



$$\vec{v}_p = (\vec{v}_p)' + \vec{v}_{p'} = (\vec{v}_p)' + \vec{w}_B \times \vec{r}_{BP} \rightarrow \text{absolute velocity at coincident point.}$$

$$= (\vec{v}_p)' \vec{j}' + w_B r_{BP} \vec{i}'$$

$$= [(v_p)' \cos 45^\circ + w_B r_{BP} \cos 45^\circ] \vec{x} + [(v_p)' \sin 45^\circ - w_B r_{BP} \sin 45^\circ] \vec{y}. \quad \dots \quad ③$$

By ① and ③

$$(v_p)' \cos 45^\circ + w_B r_{BP} \cos 45^\circ = 2$$

$$(v_p)' \sin 45^\circ - w_B r_{BP} \sin 45^\circ = 2$$

$$\Rightarrow [w_B = 0 \text{ rad/s} \quad \& \quad (v_p)' = 2\sqrt{2} \text{ m/s}] \quad \dots \quad \checkmark$$

~~$$\vec{a}_p = (\vec{a}_p)' + \vec{a}_{p'} + \vec{a}_c$$~~

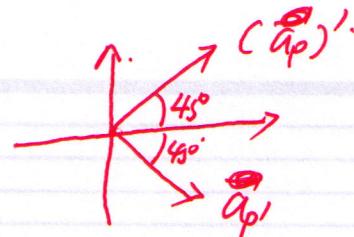
\vec{a}_p' → Coriolis acceleration.
 $\vec{a}_{p'}$ → relative acceleration at coincident point
 \vec{a}_c → relative acceleration.

$$\vec{a}_c = 2 \vec{w}_B \times (\frac{d\vec{r}}{dt})' = 0$$

$$(\vec{a}_{p'})' = ? \quad \cancel{\text{parallel to } \vec{BD}} \quad \text{but parallel to } \vec{BD}$$

$$\vec{a}_{p'} = \vec{a}_B \times \vec{a}_B \times \vec{r} + \frac{d\vec{w}_B}{dt} \times \vec{r}$$

$$= \frac{d\vec{w}_B}{dt} \times \vec{r}$$



~~$$(\vec{a}_{p'})' \cos 45^\circ + a_{p'} \cos 45^\circ =$$~~

$$\vec{a}_p = (\vec{a}_p)' \vec{j}' + (\vec{a}_{p'})' \vec{x}'$$

$$= [(a_p)' \cos 45^\circ + (\vec{a}_{p'})' \cos 45^\circ] \vec{x} + [(a_p)' \sin 45^\circ - a_{p'} \sin 45^\circ] \vec{y}$$

∴ by ② and ③

$$(a_p)' \cos 45^\circ + a_{p'} \cos 45^\circ = 20$$

$$(a_p)' \sin 45^\circ - a_{p'} \sin 45^\circ = -10$$

$$\Rightarrow \frac{d\vec{w}_B}{dt} r_{BP} = a_{p'} = 30\sqrt{2} \Rightarrow \left[\frac{d\vec{w}_B}{dt} = 37.5 \text{ m/s}^2 \right]$$

$$\therefore |(\vec{a}_p)'| = 10\sqrt{2} = 14.1 \text{ m/s}^2 \quad \dots \quad ④$$