

Chapter 14 Systems of Particles

07/31/2006

Lecture notes

§ 14.1.

Objective:

- | How to describe the motion of Particle systems (kinematics)
- | How to relate ~~the~~ resultant force to motion (kinetics) from energy.

§ 14.2. Motion of Particle systems (kinematics)

In a Newton frame, N particles compose a system. The position vectors for ~~each~~ the k th particle is \vec{r}_k .

The position vector of the mass center is defined by

$$\left(\sum_{k=1}^N m_k \right) \vec{r} = \sum_{k=1}^N m_k \vec{r}_k \Rightarrow \vec{r} = \frac{\sum_{k=1}^N m_k \vec{r}_k}{\sum_{k=1}^N m_k}$$

properties:

$$\textcircled{1} \text{ velocity } \frac{d\vec{r}}{dt} = \vec{v} = \frac{\sum_{k=1}^N m_k \frac{d\vec{r}_k}{dt}}{\sum_{k=1}^N m_k} = \frac{\sum_{k=1}^N m_k \vec{v}_k}{\sum_{k=1}^N m_k}$$

$$\textcircled{2} \text{ acceleration } \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \frac{\sum_{k=1}^N m_k \frac{d^2\vec{r}_k}{dt^2}}{\sum_{k=1}^N m_k} = \frac{\sum_{k=1}^N m_k \vec{a}_k}{\sum_{k=1}^N m_k}$$

$$\textcircled{3} \text{ momentum. } \sum_{k=1}^N m_k \vec{v} = \sum_{k=1}^N m_k \vec{v}_k = m \frac{d\vec{r}}{dt}$$

\textcircled{4} angular momentum.

$$\sum_{k=1}^N (\vec{r}_k \times m_k \vec{v}_k) = \sum_{k=1}^N m_k (\vec{r}_{kc} \times \vec{v}_{kc}) + m (\vec{r} \times \vec{v})$$

$$\begin{aligned} \text{proof: } \sum_{k=1}^N (\vec{r}_k \times m_k \vec{v}_k) &= \sum_{k=1}^N [(\vec{r}_{kc} + \vec{r}) \times m_k (\vec{v}_{kc} + \vec{v})] \\ &= \sum_{k=1}^N \vec{r}_{kc} \times m_k \vec{v}_{kc} + \sum_{k=1}^N \vec{r}_{kc} \times m_k \vec{r} + \sum_{k=1}^N \vec{r} \times m_k \vec{v}_{kc} + \sum_{k=1}^N \vec{r} \times m_k \vec{v} \\ &= \sum_{k=1}^N m_k (\vec{r}_{kc} \times \vec{v}_{kc}) + \cancel{\sum_{k=1}^N (\vec{r}_{kc} \times \vec{r})} + \cancel{\sum_{k=1}^N (m_k \vec{r}_{kc}) \times \vec{r}} + \cancel{\sum_{k=1}^N (m_k \vec{v}) \times \vec{r}} \\ &= \sum_{k=1}^N m_k (\vec{r}_{kc} \times \vec{v}_{kc}) + m (\vec{r} \times \vec{v}). \end{aligned}$$

Special case:

$$\vec{v}_{kc} = \vec{\omega} \times \vec{r}_{kc} \quad \text{and} \quad \vec{\omega} \perp \vec{r}_{kc}. \Rightarrow \sum_{k=1}^N (\vec{r}_k \times m_k \vec{v}_k) = m (\vec{r} \times \vec{v}) + \left(\sum_{k=1}^N m_k r_{kc}^2 \right) \vec{\omega}$$

$$\begin{aligned} \text{proof: } \sum_{k=1}^N (\vec{r}_k \times m_k \vec{v}_k) &= m (\vec{r} \times \vec{v}) + \sum_{k=1}^N m_k (\vec{r}_{kc} \times \vec{\omega} \times \vec{r}_{kc}) \\ &= m (\vec{r} \times \vec{v}) + \sum_{k=1}^N m_k (\vec{r}_{kc} \cdot \vec{r}_{kc}) \vec{\omega} - \sum_{k=1}^N m_k (\vec{\omega} \cdot \vec{r}_{kc}) \vec{r}_{kc} \\ &= m (\vec{r} \times \vec{v}) + \left(\sum_{k=1}^N m_k (r_{kc})^2 \right) \vec{\omega} - \sum_{k=1}^N m_k (\vec{\omega} \cdot \vec{r}_{kc}) \vec{r}_{kc}. \end{aligned}$$

§14.3 kinetics

§14.3.1 force method. (instant value \rightarrow instant value)

$$\text{① Translation} \Rightarrow \sum \vec{F}_k^e = m \frac{d^2 \vec{r}}{dt^2} \dots \dots \quad \textcircled{1}$$

Proof: for each particle:

$$\sum \vec{F}_k = \sum_{j=1}^N \vec{f}_{kj} + \vec{F}_k^e = m_k \vec{a}_k \quad (\text{Newton 2nd Law})$$

~~$$\sum \vec{F}_k = \sum_{j=1}^N \vec{f}_{kj} + \vec{F}_k^e = m_k \vec{a}_k$$~~

$$\begin{aligned} \sum_{k=1}^N m_k \vec{a}_k &= \sum_{k=1}^N \left(\sum_{j=1}^N \vec{f}_{kj} + \vec{F}_k^e \right) \\ &= \sum_{k=1}^N \sum_{j=1}^N \vec{f}_{kj} + \sum_{k=1}^N \vec{F}_k^e = \dots \dots \quad \textcircled{1} \end{aligned}$$

Let $\vec{f}_{kk} = 0$ and

$$\begin{aligned} \text{we get } \sum_{k=1}^N m_k \vec{a}_k &= \sum_{k=1}^N \sum_{j=1}^N \vec{f}_{kj} + \sum_{k=1}^N \vec{F}_k^e = \frac{1}{2} \left(\sum_{k=1}^N \sum_{j=1}^{k-1} \vec{f}_{kj} + \sum_{j=1}^N \sum_{k=j+1}^N \vec{f}_{kj} \right) + \sum_{k=1}^N \vec{F}_k^e \\ &= \frac{1}{2} \sum_{k=1}^N \sum_{j=1}^{k-1} (\vec{f}_{kj} + \vec{f}_{jk}) + \sum_{k=1}^N \vec{F}_k^e \end{aligned}$$

By $\vec{f}_{kj} = -\vec{f}_{jk}$.

$$\Rightarrow \sum_{k=1}^N m_k \vec{a}_k = \sum_{k=1}^N \vec{F}_k^e \dots \dots \quad \textcircled{2}$$

$$\text{Because } \sum_{k=1}^N m_k \vec{a}_k = \sum_{k=1}^N m_k \frac{d^2 \vec{r}_k}{dt^2} = \frac{d^2}{dt^2} \left(\sum_{k=1}^N m_k \vec{r}_k \right) = \frac{d^2}{dt^2} m \vec{r} = m \ddot{\vec{r}}$$

$$\Rightarrow \sum_{k=1}^N \vec{F}_k^e = m \frac{d^2 \vec{r}}{dt^2}$$

$$\text{② Rotation.} \Rightarrow \sum_{k=1}^N (\vec{r}_k \times \vec{F}_k^e) = \sum_{k=1}^N (\vec{r}_k \times m \vec{a}'_k)$$

$$\text{Proof: } \vec{r}_k \times (\vec{F}_k^e + \sum_{j=1}^N \vec{f}_{kj}) = \vec{r}_k \times m \vec{a}'_k.$$

$$\Rightarrow \sum_{k=1}^N (\vec{r}_k \times (\vec{F}_k^e + \sum_{j=1}^N \vec{f}_{kj})) = \sum_{k=1}^N (\vec{r}_k \times m \vec{a}'_k)$$

$$\Rightarrow \sum_{k=1}^N \vec{r}_k \times \vec{F}_k^e + \sum_{k=1}^N (\vec{r}_k \times \sum_{j=1}^N \vec{f}_{kj}) = \sum_{k=1}^N (\vec{r}_k \times m \vec{a}'_k) \quad \leftarrow \text{let } \vec{f}_{kk} = 0$$

$$\Rightarrow \sum_{k=1}^N \vec{r}_k \times \vec{F}_k^e + \sum_{k=1}^N \sum_{j=1}^{k-1} (\vec{r}_k \times \vec{f}_{kj}) = \sum_{k=1}^N (\vec{r}_k \times m \vec{a}'_k)$$

$$\Rightarrow \sum_{k=1}^N \vec{r}_k \times \vec{F}_k^e + \frac{1}{2} \sum_{k=1}^N \sum_{j=1}^{k-1} [\vec{r}_k \times \vec{f}_{kj} + \vec{r}_j \times \vec{f}_{jk}] = \sum_{k=1}^N (\vec{r}_k \times m \vec{a}'_k) \dots \dots \quad \textcircled{3}$$

$$\therefore \vec{r}_k \times \vec{f}_{kj} + \vec{r}_j \times \vec{f}_{jk}$$

$$= \vec{r}_k \times (\vec{f}_{kj} + \vec{f}_{jk}) + (\vec{r}_j - \vec{r}_k) \times \vec{f}_{jk}$$

$$= 0 + \underbrace{(\vec{r}_j - \vec{r}_k) \times \vec{f}_{jk}}_{\text{parallel}} = 0$$

$$\xrightarrow{\text{by } ③} \sum_{k=1}^N \vec{r}_k \times \vec{F}_k^e = \sum_{k=1}^N (\vec{r}_k \times m_k \vec{a}_k)$$

③ Impulse and Momentum.

$$\int_{t_1}^{t_2} \left(\sum_{k=1}^N \vec{F}_k^e \right) dt = (m \vec{v})_2 - (m \vec{v})_1.$$

Proof: for each particle

$$\int_{t_1}^{t_2} (\vec{F}_k^e + \sum_{j=1}^N \vec{f}_{kj}) dt = (m_k \vec{v}_k)_2 - (m_k \vec{v}_k)_1.$$

$$\sum_{k=1}^N \int_{t_1}^{t_2} (\vec{F}_k^e + \sum_{j=1}^N \vec{f}_{kj}) dt = \sum_{k=1}^N (m_k \vec{v}_k)_2 - \sum_{k=1}^N (m_k \vec{v}_k)_1.$$

$$\Rightarrow \sum_{k=1}^N \int_{t_1}^{t_2} \vec{F}_k^e dt + \int_{t_1}^{t_2} \left(\sum_{k=1}^N \sum_{j=1}^N \vec{f}_{kj} \right) dt = 0(m \vec{v})_2 - (m \vec{v})_1.$$

$$\Rightarrow \int_{t_1}^{t_2} \left(\sum_{k=1}^N \vec{F}_k^e \right) dt = (m \vec{v})_2 - (m \vec{v})_1.$$

④ Force moment and moment of Momentum.

$$\sum_{k=1}^N \vec{r}_k \times \vec{F}_k^e = \frac{d}{dt} \sum_{k=1}^N (\vec{r}_k \times m_k \vec{a}_k) = \frac{d}{dt} [m(\vec{r} \times \vec{v}) + \sum_{k=1}^N m_k (\vec{r}_{kc} \times \vec{v}_{kc})]$$

$$\text{Proof: } \because \sum_{k=1}^N \vec{r}_k \times \vec{F}_k^e = \sum_{k=1}^N (\vec{r}_k \times m_k \vec{a}_k) = \sum_{k=1}^N (\vec{r}_k \times m_k \frac{d\vec{v}_k}{dt})$$

$$= \sum_{k=1}^N (\vec{r}_k \times m_k \frac{d\vec{v}_k}{dt} + \cancel{(\frac{d\vec{r}_k}{dt} \times m_k \vec{v}_k)})$$

$$= \sum_{k=1}^N \left(\frac{d}{dt} (\vec{r}_k \times m_k \vec{v}_k) \right) = \frac{d}{dt} \left(\sum_{k=1}^N (\vec{r}_k \times m_k \vec{v}_k) \right)$$

$$= \frac{d}{dt} [m \cdot (\vec{r} \times \vec{v}) + \sum_{k=1}^N m_k (\vec{r}_{kc} \times \vec{v}_{kc})]$$

special case: $\vec{r}_{kc} = \vec{w} \times \vec{r}_{rc}$ and $\vec{w} \perp \vec{r}_{rc}$

$$\Rightarrow \sum_{k=1}^N \vec{r}_k \times \vec{F}_k^e = m \cdot \vec{r} \times \vec{a} + \left(\sum_{k=1}^N m_k r_{kc}^2 \right) \frac{d\vec{w}}{dt}$$

\downarrow
acceleration

\Rightarrow angular acceleration

~~When $\vec{r} = 0$ (frame is set at)~~

⑤ Some Laws in the frame centered at mass center.

By ~~all the frame, Newton or Non Newton~~

$$\sum_{k=1}^N (\vec{r}_k \times \vec{F}_{k\text{e}}) = \sum_{k=1}^N (\vec{r}_k \times m_k \vec{a}_k) = \frac{d}{dt} [m(\vec{r} \times \vec{v})] + \frac{d}{dt} \sum_{k=1}^N [m_k (\vec{r}_{kc} \times \vec{v}_{kc})]$$

$$\Rightarrow \sum_{k=1}^N (\vec{r} + \vec{r}_{kc}) \times \vec{F}_{k\text{e}} = \frac{d}{dt} [m(\vec{r} \times \vec{v})] + \frac{d}{dt} \left(\sum_{k=1}^N m_k (\vec{r}_{kc} \times \vec{v}_{kc}) \right)$$

$$\Rightarrow \vec{r} \times \sum_{k=1}^N \vec{F}_{k\text{e}} + \sum_{k=1}^N \vec{r}_{kc} \vec{F}_{k\text{e}} = \frac{d}{dt} m(\vec{r} \times \vec{v}) + \frac{d}{dt} \left(\sum_{k=1}^N m_k (\vec{r}_{kc} \times \vec{v}_{kc}) \right)$$

$$\Rightarrow \vec{r} \times \sum_{k=1}^N \vec{F}_{k\text{e}} = m \vec{r} \times \frac{d\vec{v}}{dt} + \frac{d}{dt} \left(\sum_{k=1}^N m_k (\vec{r}_{kc} \times \vec{v}_{kc}) \right)$$

$$\Rightarrow \sum_{k=1}^N \vec{r}_{kc} \times \vec{F}_{k\text{e}} = \frac{d}{dt} \left[\sum_{k=1}^N m_k (\vec{r}_{kc} \times \vec{v}_{kc}) \right] \rightarrow \underline{\text{Spin.}}$$

\downarrow
force moment
around mass center

\downarrow
moment of momentum
relative to mass center.

⑥ Conservation Law.

$$\sum \vec{F}_{k\text{e}} = 0 \Rightarrow m\vec{v} = \text{const.}$$

$$\sum \vec{r}_{kc} \times \vec{F}_{k\text{e}} = 0 \Rightarrow \frac{d}{dt} \left[\sum_{k=1}^N m_k (\vec{r}_{kc} \times \vec{v}_{kc}) \right] = 0, \text{ no spin.}$$

$$\sum U_{12} = 0 \Rightarrow T = \text{const.}$$

the system can be modelled by a particle at the mass center

⑦ Energy. (Principle of WQE)

$$\begin{aligned} T &= \frac{1}{2} \sum_{k=1}^N \frac{1}{2} m_k V_k^2 = \cancel{\frac{1}{2} \sum_{k=1}^N} \sum_{k=1}^N \frac{1}{2} m_k \vec{V}_k \cdot \vec{V}_k = \sum_{k=1}^N \frac{1}{2} m_k (\vec{V} + \vec{V}_{kc}) \cdot (\vec{V} + \vec{V}_{kc}) \\ &= \sum_{k=1}^N \frac{1}{2} m_k \vec{V} \cdot \vec{V} + \sum_{k=1}^N \frac{1}{2} m_k \vec{V}_{kc} \cdot \vec{V}_{kc} \\ &= \frac{1}{2} M V^2 + \sum_{k=1}^N \frac{1}{2} m_k V_{kc}^2. \end{aligned}$$

$$\sum U_{12} = \cancel{\sum_{k=1}^N F_{k\text{e}} \cdot r_k} \cdot T_2 - T_1.$$

\rightarrow external force and internal force! $\sum U_{12} = 0$.

Summary:

resultant ~~of~~ external forces \Rightarrow acceleration at mass center

motion of particle system = momentum at mass center

If the system is modelled as a particle at the mass center.

\rightarrow translation.

2. ~~the~~ spin \rightarrow moment of external forces around mass center.

3. ~~Work~~ Work: Should consider the work ~~of~~ of internal forces!
(relative displacement)

Step by step procedure:

① how many particles and what is the system

② Coordinate

3.1 mass center.

③ 3.2 translation at the mass center

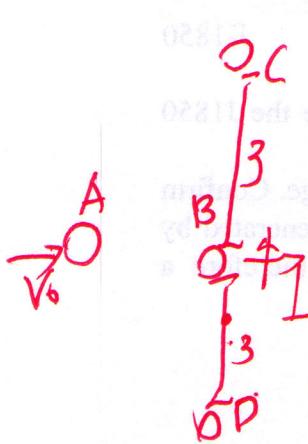
3.3 Spin. { if moment is given: \Rightarrow force Moment and moment of momentum
geometry is given:
if no relative displacement \Rightarrow Principle of W & E

④ Solution.

Sample Problem #1.

Four small disks A, B, C, D can slide freely on a frictionless horizontal surface. Disk B, C, and D are connected by light rods and are at rest in the position shown when disk B is struck squarely by disk A which is moving to the right with a velocity $v_0 = 38.5 \text{ ft/s}$. The weights of A, B, C = 15 lb D = 30 lb known, that the velocities of the disks after impact are $v_A = v_B = 8.25 \text{ ft/s}$, $v_C = v_D$, $v_B = v_D$.

(a) speeds v_C and v_D (b) the fraction of the initial kinetic energy of the system which is dissipated during the collision.



① how many particles? 4

② coord

③

3.1 mass center

$$\text{before} \Rightarrow \bar{x} = \frac{15 \cdot 3 + 30 \cdot 0}{75} = 0$$

$$\text{after} \quad \begin{cases} x = 0 \\ y = -\frac{3}{5}t \end{cases} \Rightarrow \bar{y} = \frac{15 \cdot 3 - 30 \cdot 3}{75} = -\frac{45}{75} = -\frac{3}{5} \text{ ft.}$$

3.2. translation.

$$\sum \vec{F}_{\text{ext}}^e = 0$$

$$\Rightarrow \vec{a} = 0 \quad \vec{v} = \text{const} = 0$$

$$v_x = \frac{m_A v_0 + m_B v_0 + m_C v_C + m_D v_D}{m_A + m_B + m_C + m_D} =$$

$$\Rightarrow \frac{15 \cdot v_0}{75} = \frac{15 \cdot v_A + 15 \cdot v_B + 30 \cdot v_C + 30 \cdot v_D}{75}$$

$$\Rightarrow 15 \cdot 38.5 = 30 \cdot 8.25 + 15 v_C + 30 v_D$$

$$\Rightarrow 15 v_C + 30 v_D = 571.5 - 247.5 = 330 \text{ ft/s.} \dots \text{ (1)}$$

3.3 Sp'm: $\Rightarrow v_C + 2v_D = 22.$

$$\cancel{\sum \vec{r}_{kC} \times \vec{F}_k^e = \frac{d}{dt} \left[\sum_{k=1}^4 m_k (\vec{r}_{kC} \times \vec{V}_{kC}) \right]} \quad \cancel{\vec{F}_k^e}$$

$$\text{before.} \quad 0 = \frac{d}{dt} \left[\sum_{k=1}^4 m_k (\vec{r}_{kC} \times \vec{V}_{kC}) \right]$$

$$m_A v_0 \cdot \frac{3}{5} = (m_A + m_B) v_A \cdot \frac{3}{5} + m_C v_C \cdot (3 + \frac{3}{5}) \quad \cancel{m_D v_D} (3 - \frac{3}{5})$$

$$④ \Rightarrow 15 \cdot 38,5 \cdot \frac{3}{5} = 30 \cdot 8,25 \cdot \frac{3}{5} + 15 \cdot V_C \cdot \frac{18}{5} + -30 \cdot V_D \cdot \frac{12}{5}$$

$$\Rightarrow 346,5 = 148,5 + 54V_C - 72V_D$$

$$198 = 54V_C - 72V_D \Rightarrow 33 = 9V_C - 12V_D \Rightarrow 11 = 3V_C - 4V_D \quad \dots \textcircled{B}$$

$$\cancel{\textcircled{A}} \xrightarrow{3} \textcircled{B} \times 3 \Rightarrow 3V_C + 6V_D + -3V_C + 4V_D = 66 - 11$$

$$10V_D = 55 \Rightarrow V_D = 5,5 \text{ f/s}$$

$$V_C = 22 - 11 = 11 \text{ f/s}$$

$$\Delta E = E_2 - E_1 = \frac{1}{2} (m_A + m_B) \cdot 8,25^2 + \frac{1}{2} m_C \cdot 11^2 + \frac{1}{2} m_D \cdot (5,5)^2 - \frac{1}{2} m_A \cdot (38,5)^2$$

$$= \frac{1}{32,2} \cdot \frac{1}{2} [30 \cdot 8,25^2 + 15 \cdot 11^2 + 30 \cdot (5,5)^2 - 15 \cdot (38,5)^2]$$

$$= \frac{1}{32,2} \cdot \frac{1}{2} \cdot (2041,875 + 1815 + 907,5 - 22233,75)$$

$$= \frac{1}{32,2} \cdot \frac{1}{2} \cdot (-17469,375)$$

$$\frac{\Delta E}{E_1} = 78,57\%$$