HW 11, Math 121 A
Spring, 2004
UC BERKELEY

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Spring, 2004 Compiled on April 17, 2021 at 10:04pm

Contents

1 chapter 15, problem 2.11

Problem Find the inverse transform of the function $F(p) = \frac{3p+2}{3p^2+5p-2}$

Solution

Need to simplify the above expression to some expressions which are shown in the table on page 636.

$$\begin{aligned} F(p) &= \frac{3p+2}{3p^2+5p-2} \\ &= \frac{2}{(3p-1)(p+2)} + \frac{3p}{(3p-1)(p+2)} \end{aligned} \tag{1}$$

Expanding in partial fractions. For the first term in (1):

$$\begin{aligned} \frac{1}{(3p-1)(p+2)} &= \frac{A}{(3p-1)} + \frac{B}{(p+2)} = \frac{A(p+2)+B(3p-1)}{(3p-1)(p+2)} \\ A(p+2)+B(3p-1) &= 1 \\ Ap+2A+3Bp-B &= 1 \end{aligned}$$

Hence $2A-B=1$ and $(A+3B)=0$ which gives $A=\frac{1}{2}+\frac{B}{2}$. Therefore $\frac{1}{2}+\frac{B}{2}+3B=0$ or $\frac{1+B+6B}{2}=0$ or $1+7B=0$ or $B=\frac{-1}{7}$. Hence $A=\frac{1}{2}+\frac{\frac{-1}{7}}{2}=\frac{1}{2}-\frac{1}{14}=\frac{7-1}{14}=\frac{6}{14}$.

Now the first term in (1) can be written as $2\left(\frac{A}{(3p-1)}+\frac{B}{(p+2)}\right)$ or $2\left(\frac{\frac{6}{14}}{(3p-1)}+\frac{\frac{-1}{7}}{(p+2)}\right)$ or

$$\frac{6}{7(3p-1)} - \frac{2}{7(p+2)} \tag{2}$$

Doing partial fraction on the second term in (1) which is $\frac{3p}{(3p-1)(p+2)}$ gives

$$\begin{aligned} \frac{p}{(3p-1)(p+2)} &= \frac{A}{(3p-1)} + \frac{B}{(p+2)} = \frac{A(p+2)+B(3p-1)}{(3p-1)(p+2)} \\ A(p+2)+B(3p-1) &= p \\ Ap+2A+3Bp-B &= p \end{aligned}$$

Hence $2A - B = 0$ and $(A + 3B) = 1$, therefore $A = \frac{B}{2}$. Hence $\left(\frac{B}{2} + 3B\right) = 1$ or $\frac{B+6B}{2} = 1$ or $7B = 2$ or $B = \frac{2}{7}$. Therefore $A = \frac{B}{2} = \frac{2}{14}$. Hence $\frac{3p}{(3p-1)(p+2)} = 3\left(\frac{A}{(3p-1)} + \frac{B}{(p+2)}\right) = 3\left(\frac{\frac{2}{14}}{(3p-1)} + \frac{\frac{2}{7}}{(p+2)}\right)$ or

$$\frac{3}{7(3p-1)} + \frac{6}{7(p+2)} \tag{3}$$

Combining (2) and (3) gives

$$\begin{aligned}
F(p) &= \frac{6}{7(3p-1)} - \frac{2}{7(p+2)} + \frac{3}{7(3p-1)} + \frac{6}{7(p+2)} \\
&= \frac{6}{7}\frac{1}{(3p-1)} - \frac{2}{7}\frac{1}{(p+2)} + \frac{2}{7}\frac{1}{(3p-1)} + \frac{6}{7}\frac{1}{(p+2)} \\
&= \frac{4}{7}\frac{1}{(p+2)} + \frac{9}{7}\frac{1}{(3p-1)} \\
&= \frac{4}{7}\frac{1}{(p+2)} + \frac{9}{21}\frac{1}{\left(p-\frac{1}{3}\right)}
\end{aligned}$$

Now we can use the table to find the inverse transform. Use property L2, which says

$$\mathscr{L}\left(e^{-at}\right) = \frac{1}{p+a}$$

Hence , setting $a = 2$, gives $\mathscr{L}\left(e^{-2t}\right) = \frac{1}{p+2}$ and setting $a = -\frac{1}{3}$ gives $\mathscr{L}\left(e^{-\frac{1}{3}t}\right)^{\frac{1}{3}t} = \frac{1}{p-\frac{1}{3}}$. Hence $f(t) = \frac{4}{7}e^{-2t} + \frac{9}{21}e^{\frac{1}{3}t}$ and the inverse Laplace transform is

$$f(t) = \frac{4}{7}e^{-2t} + \frac{3}{7}e^{\frac{1}{3}t}$$

2 chapter 15, problem 2.17

Problem Use L32 and L11 to obtain $\mathscr{L}\left(t^2\ \sin at\right)$

Solution

$$\mathscr{L}\left(t^n f(t)\right) = (-1)^n \frac{d^n F(p)}{dp^n} \tag{L32}$$

$$\mathscr{L}(t\ \sin at) = \frac{2ap}{\left(p^2 + a^2\right)^2} \tag{L11}$$

we set $f(t) = t \ \sin at$ then we can write using L32

$$\mathscr{L}\big(t\, f(t)\big) = (-1)\frac{d\mathscr{L}\big(f(t)\big)}{dp} \tag{1}$$

But $\mathscr{L}\big(f(t)\big) = \mathscr{L}(t \ \sin at) \ = \ \frac{2ap}{\left(p^2+a^2\right)^2}$ from table L11 (1) becomes

$$\begin{aligned}
\mathscr{L}\big(t\, f(t)\big) &= -\frac{d}{dp}\left(\frac{2ap}{\left(p^2+a^2\right)^2}\right) \\
\mathscr{L}(t \times t \ \sin at) &= -\left(p\, \frac{-2\times 2a}{\left(p^2+a^2\right)^3} \times 2p + \frac{2a}{\left(p^2+a^2\right)^2} \times 1\right) \\
\mathscr{L}\big(t^2 \ \sin at\big) &= \frac{8ap^2}{\left(p^2+a^2\right)^3} - \frac{2a}{\left(p^2+a^2\right)^2} \\
&= \frac{8ap^2 - 2a\left(p^2+a^2\right)}{\left(p^2+a^2\right)^3} \\
&= \frac{a\left(8p^2 - 2p^2 - 2a^2\right)}{\left(p^2+a^2\right)^3} \\
&= \frac{a\left(6p^2 - 2a^2\right)}{\left(p^2+a^2\right)^3}
\end{aligned}$$

Or

$$\mathscr{L}\big(t^2 \ \sin at\big) = \frac{6ap^2 - 2a^3}{\left(p^2+a^2\right)^3}$$

3 chapter 15, problem 2.18

Problem Use L31 to derive L21

Solution

$$\mathscr{L}\left(\frac{f(t)}{t}\right) = \int_{u=p}^{\infty} F(u)du \tag{L31}$$

$$\mathscr{L}\left(\frac{e^{-at} - e^{-bt}}{t}\right) = \ln\frac{p+b}{p+a} \tag{L21}$$

we set $f(t) = e^{-at} - e^{-bt}$ then we can write using L31

$$\begin{aligned}\mathscr{L}\left(\frac{f(t)}{t}\right) &= \int_{u=p}^{\infty} F(u)du \\ &= \int_{u=p}^{\infty} \mathscr{L}\left[f(t)\right]\ du\end{aligned}$$

but $\mathscr{L}\left[f(t)\right] = \mathscr{L}\left(e^{-at} - e^{-bt}\right) = \mathscr{L}\left(e^{-at}\right) - \mathscr{L}\left(e^{-bt}\right) = \frac{1}{p+a} - \frac{1}{p+b}$ By using L2. But since we are using u in place of p in integral, we need to call $p = u$. Hence

$$\begin{aligned}\mathscr{L}\left(\frac{f(t)}{t}\right) &= \int_{u=p}^{\infty}\left(\frac{1}{u+a} - \frac{1}{u+b}\right)\ du \\ &= \int_{u=p}^{\infty} \frac{1}{u+a}\ du - \int_{u=p}^{\infty} \frac{1}{u+b}\ \ du \\ &= [\ln(u+a)]_p^{\infty} - [\ln(u+b)]_p^{\infty} \\ &= \left[\ln(\infty + a) - \ln\left(p+a\right)\right] - \left[\ln(\infty + b) - \ln\left(p+b\right)\right] \\ &= \ln(\infty) - \ln\left(p+a\right) - \ln(\infty) + \ln\left(p+b\right) \\ &= \ln\left(p+b\right) - \ln\left(p+a\right)\end{aligned}$$

but $\ln A - \ln B = \ln \frac{A}{B}$, therefore

$$\begin{aligned}\mathscr{L}\left(\frac{f(t)}{t}\right) &= \ \ln\left(p+b\right) - \ln\left(p+a\right) \\ \mathscr{L}\left(\frac{e^{-at} - e^{-bt}}{t}\right) &= \ln\left(\frac{p+b}{p+a}\right)\end{aligned}$$

4 chapter 15, problem 2.2

Problem Use relation L2 to find L7 and L8 in laplace table.

Solution

$$\mathscr{L}\left(e^{-at}\right) = \frac{1}{p+a} \tag{L2}$$

for $\Re\left(p+a\right) > 0$

$$\mathscr{L}\frac{e^{-at} - e^{-bt}}{b-a} = \frac{1}{\left(p+a\right)\left(p+b\right)} \tag{L7}$$

$$\mathscr{L}\frac{ae^{-at}-be^{-bt}}{a-b}=\frac{p}{\left(p+a\right)\left(p+b\right)} \tag{L8}$$

For $\Re\left(p+a\right)>0$ and $\Re\left(p+b\right)>0$. Where $\mathscr{L}f(t)$ is the laplace transform of $f(t)$ defined as $\mathscr{L}f(t)=F(p)=\int_0^\infty e^{-pt}f(t)dt$.

From the linearity property of the $\mathscr{L}$ operator, expand the LHS of L7, we get

$$\mathscr{L}\frac{e^{-at}-e^{-bt}}{b-a}=\frac{1}{b-a}\mathscr{L}\left(e^{-at}-e^{-bt}\right)=\frac{1}{b-a}\left(\mathscr{L}\left(e^{-at}\right)-\mathscr{L}\left(e^{-bt}\right)\right)$$

Now applying L2 gives for $\Re\left(p+a\right)>0$ and $\Re\left(p+b\right)>0$

$$\begin{aligned}\mathscr{L}\frac{e^{-at}-e^{-bt}}{b-a}&=\frac{1}{b-a}\left(\frac{1}{p+a}-\frac{1}{p+b}\right)\\&=\frac{1}{b-a}\left(\frac{p+b-\left(p+a\right)}{\left(p+a\right)\left(p+b\right)}\right)\\&=\frac{1}{b-a}\left(\frac{b-a}{\left(p+a\right)\left(p+b\right)}\right)\\&=\frac{1}{\left(p+a\right)\left(p+b\right)}\end{aligned}$$

For Which is L7 as required to show. Similarly for L8, expand the LHS of L8 we get for $\Re\left(p+a\right)>0$ and $\Re\left(p+b\right)>0$

$$
\begin{aligned}
\mathscr{L}\frac{ae^{-at}-be^{-bt}}{a-b} &= \frac{1}{a-b}\mathscr{L}\left(ae^{-at}-be^{-bt}\right)\\
&= \frac{1}{a-b}\left(\mathscr{L}\left(e^{-at}\right)-\mathscr{L}\left(e^{-bt}\right)\right)\\
&= \frac{1}{a-b}\left(a\mathscr{L}\left(e^{-at}\right)-b\mathscr{L}\left(e^{-bt}\right)\right)\\
&= \frac{1}{a-b}\left(a\frac{1}{p+a}-b\frac{1}{p+b}\right)\\
&= \frac{1}{a-b}\left(\frac{a\left(p+b\right)-b\left(p+a\right)}{\left(p+a\right)\left(p+b\right)}\right)\\
&= \frac{1}{a-b}\left(\frac{ap+ab-bp-ba}{\left(p+a\right)\left(p+b\right)}\right)\\
&= \frac{1}{a-b}\left(\frac{ap-bp}{\left(p+a\right)\left(p+b\right)}\right)\\
&= \frac{1}{a-b}\left(\frac{p(a-b)}{\left(p+a\right)\left(p+b\right)}\right)\\
&= \frac{p}{\left(p+a\right)\left(p+b\right)}
\end{aligned}
$$

Which is L8.

5 chapter 15, problem 2.21

Problem Use L29 and L11 to obtain $\mathscr{L}\left(te^{-at}\ \sin bt\right)$

Solution

$$
\mathscr{L}\left(e^{-at}f(t)\right) = F\left(p+a\right) \tag{L29}
$$

$$
\mathscr{L}(t\ \sin at) = \frac{2ap}{\left(p^2+a^2\right)^2} \tag{L11}
$$

Then from L11 we get

$$
\mathscr{L}(t\ \sin bt) = \frac{2bp}{\left(p^2+b^2\right)^2}
$$

Now, let $p=\left(p+a\right)$ then from L29, the above becomes

$$\mathcal{L}\left(e^{-at}\ t\ \sin bt\right) = \frac{2b\left(p+a\right)}{\left(\left(p+a\right)^2+b^2\right)^2}$$

6 chapter 15, problem 2.22

Problem similar to problem 2.21, Use L29 and L12 to obtain $\mathcal{L}\left(te^{-at}\ \cos bt\right)$

Solution

$$\mathcal{L}\left(e^{-at}\ f(t)\right) = F\left(p+a\right) \tag{L29}$$

$$\mathcal{L}(t\ \cos at) = \frac{p^2-a^2}{\left(p^2+a^2\right)^2} \tag{L12}$$

then from L12 we get

$$\mathcal{L}(t\ \cos bt) = \frac{p^2-b^2}{\left(p^2+b^2\right)^2}$$

Now, let $p=\left(p+a\right)$ then from L29, the above becomes

$$\mathcal{L}\left(e^{-at}\ t\ \cos bt\right) = \frac{\left(p+a\right)^2-b^2}{\left(\left(p+a\right)^2+b^2\right)^2}$$

7 chapter 15, problem 2.23

Problem use result obtained in problem 2.21 and 2.22 to find inverse transform for $\frac{p^2+2p-1}{\left(p^2+4p+5\right)^2}$

Solution

Recall, from 2.21 we showed that $\mathcal{L}\left(te^{-at}\ \sin bt\right) = \frac{2b\left(p+a\right)}{\left(\left(p+a\right)^2+b^2\right)^2}$ and from 2.22 $\mathcal{L}\left(te^{-at}\ \cos bt\right) =$

$\frac{(p+a)^2-b^2}{((p+a)^2+b^2)^2}$ Hence

$$
\begin{aligned}
\mathscr{L}\left(te^{-at}\ \cos bt\right) - \mathscr{L}\left(te^{-at}\ \sin bt\right) &= \frac{(p+a)^2 - b^2}{\left((p+a)^2+b^2\right)^2} - \frac{2b(p+a)}{\left((p+a)^2+b^2\right)^2} \\
&= \frac{(p+a)^2 + b^2\ - 2b(p+a)}{\left((p+a)^2+b^2\right)^2}
\end{aligned}
$$

Now Let $a = 2$ and let $b = 1$ we get

$$
\begin{aligned}
\mathscr{L}\left(te^{-2t}\ \cos t\right) - \mathscr{L}\left(te^{-2t}\ \sin t\right) &= \frac{(p+2)^2 - 1^2 - 2(p+2)}{\left((p+2)^2+1^2\right)^2} \\
\mathscr{L}\left(te^{-2t}\ \cos t - te^{-2t}\ \sin t\right) &= \frac{p^2+4+4p-1-2p-4}{\left((p+2)^2+1^2\right)^2} \\
&= \frac{p^2+2p-1}{\left(p^2+4p+5\right)^2}
\end{aligned}
$$

Hence inverse transform of $\frac{p^2+2p-1}{(p^2+2p+5)^2}$ is $te^{-2t}\ \cos t - te^{-2t}\ \sin t$ or $te^{-2t}\ (\cos t - \sin t)$

8 chapter 15, problem 2.3

Problem Using either relation L2 or L3 and L4, verify L9 and L10 in laplace table.

Solution

$$\mathscr{L}\left(e^{-at}\right) = \frac{1}{p+a} \qquad \operatorname{Re}(p+a) > 0 \tag{L2}$$

$$\mathscr{L}\sin at = \frac{a}{p^2+a^2} \qquad \operatorname{Re}(p) > |\operatorname{Im} a| \tag{L3}$$

$$\mathscr{L}\cos at = \frac{p}{p^2+a^2} \qquad \operatorname{Re}(p) > |\operatorname{Im} a| \tag{L4}$$

$$\mathscr{L} \sinh at = \frac{a}{p^2 - a^2} \qquad \text{Re}(p) > |\,\text{Re}(a)| \tag{L9}$$

$$\mathscr{L} \cosh at = \frac{p}{p^2 - a^2} \qquad \text{Re}(p) > |\,\text{Re}(a)| \tag{L10}$$

Where $\mathscr{L}\big(f(t)\big)$ is the laplace transform of $f(t)$ defined as $\mathscr{L}\big(f(t)\big) = Y(s) = \int_0^\infty e^{-pt} f(t) dt$. To derive L9, use the relation that

$$i \sinh(x) = \sin(ix)$$

Hence, using L3, we get

$$\begin{aligned} \mathscr{L}(i \sinh at) &= \mathscr{L}(\sin iat\ ) \\ i\mathscr{L}(\sinh at) &= \frac{ia}{p^2 + (ia)^2} \qquad \text{Re}(p) > |\text{Im}\, a| \\ \mathscr{L}(\sinh at) &= \ \frac{a}{p^2 - a^2} \qquad \text{Re}(p) > |\,\text{Re}(a)| \end{aligned}$$

which is L9. To find L10, use the relation

$$\cosh(x) = \cos(ix)$$

And using L4, we get

$$\begin{aligned} \mathscr{L}(\cosh(at)) &= \mathscr{L}(\cos(iat)\ ) \\ \mathscr{L}(\cosh at) &= \frac{p}{p^2 + (ia)^2} \qquad \text{Re}(p) > |\text{Im}\, a| \\ \mathscr{L}(\sinh at) &= \ \frac{p}{p^2 - a^2} \qquad \text{Re}(p) > |\,\text{Re}(a)| \end{aligned}$$

Which is L10.

9 chapter 15, problem 2.4

Problem by differentiating the appropriate formulas w.r.t. 'a', verify L12

Solution

L12 is

$$\mathscr{L}(t\ \cos t) = \frac{p^2 - a^2}{\left(p^2 + a^2\right)^2} \qquad \operatorname{Re}\left(p\right) > |\operatorname{Im} a| \tag{L12}$$

Where $\mathscr{L}\left(f(t)\right)$ is the laplace transform of $f(t)$ defined as $\mathscr{L}\left(f(t)\right) = F(p) = \int_0^\infty e^{-pt} f(t) dt$. To derive this, I start with L3, which says

$$\mathscr{L}(\sin at) = \frac{a}{p^2 + a^2} \qquad \operatorname{Re}(p) > |\operatorname{Im} a|$$

The above can be rewritten in the full definition of the transform to make it easier to see

$$\int_0^\infty e^{-pt}\ \sin at\ \ dt = \frac{a}{p^2 + a^2}$$

Taking derivative of both sides w.r.t. a gives

$$\begin{aligned}
\frac{d}{da}\left[\int_0^\infty e^{-pt}\ \sin at\ \ dt\right] &= \frac{d}{da}\left[\frac{a}{p^2 + a^2}\right] \\
\int_0^\infty \frac{d}{da}\left[e^{-pt}\ \sin at\right]\ dt &= a\left(\frac{-2a}{\left(p^2 + a^2\right)^2}\right) + \frac{1}{p^2 + a^2} \times 1 \\
\int_0^\infty e^{-pt}\ t \cos at\ \ dt &= \frac{-2a^2}{\left(p^2 + a^2\right)^2} + \frac{1}{p^2 + a^2} \\
\mathscr{L}(t\ \cos at) &= \frac{-2a^2 + \left(p^2 + a^2\right)}{\left(p^2 + a^2\right)^2} \\
&= \frac{p^2 - a^2}{\left(p^2 + a^2\right)^2}
\end{aligned}$$

Which is L12

10 chapter 15, problem 2.5

Problem by integrating the appropriate formulas w.r.t. 'a', verify L19

Solution L19 is

$$\mathscr{L}\left(\frac{\sin at}{t}\right) = \arctan \frac{a}{p} \qquad \operatorname{Re}(p) > |\operatorname{Im} a| \tag{L19}$$

Where $\mathscr{L}\big(f(t)\big)$ is the laplace transform of $f(t)$ defined as $\mathscr{L}\big(f(t)\big) = F(p) = \int_0^\infty e^{-pt} f(t)dt$. To derive this, we start with L4, which says

$$\mathscr{L}(\cos at) = \frac{p}{p^2 + a^2} \qquad\qquad \text{Re}(p) > |\text{Im}\, a|$$

The above can be rewritten in the full definition of the transform to make it easier to see

$$\int_0^\infty e^{-pt} \ \cos at \ \ dt = \frac{p}{p^2 + a^2}$$

Integrating both sides w.r.t. a gives

$$\int\left(\int_{t=0}^{t=\infty} e^{-pt} \ \cos at \ \ dt\right)da = \int \frac{p}{p^2 + a^2}da$$
$$\int_{t=0}^{t=\infty}\left(\int e^{-pt} \ \cos at \ da\right) dt = \arctan\frac{a}{p} + K$$
$$\int_{t=0}^{t=\infty} e^{-pt} \left(\int \cos at \ da\right) dt = \arctan\frac{a}{p} + K$$

Now, since $\int \cos at \ da = \frac{\sin at}{t} + k$ and choose zero for values of the K's, the constants of integration gives

$$\int_0^\infty e^{-pt} \left(\frac{\sin at}{t}\right) dt = \arctan\frac{a}{p}$$

Hence

$$\mathscr{L}\left(\frac{\sin at}{t}\right) = \arctan\frac{a}{p}$$

Which is L19

11 chapter 15, problem 2.9

Problem Find the inverse transform of the function $F\big(p\big) = \frac{5-2p}{p^2+p-2}$

Solution

Need to simplify the above expression to some expressions which are shown in the table on page 636.

$$
\begin{aligned}
F(p) &= \frac{5-2p}{p^2+p-2} \\
&= \frac{5}{p^2+p-2} - \frac{2p}{p^2+p-2} \\
&= \frac{5}{(p-1)(p+2)} - \frac{2p}{(p-1)(p+2)}
\end{aligned} \tag{1}
$$

From table, L7, we see that $\mathscr{L}\left(\frac{e^{-at}-e^{-bt}}{b-a}\right) = F(p) = \frac{1}{(p+a)(p+b)}$. Setting $a = -1, b = 2$ we get the first expression in (1), that is

$$
\mathscr{L}\left(\frac{e^{t}-e^{-2t}}{3}\right) = \frac{1}{(p-1)(p+2)} \tag{2}
$$

also from table, we see that L8 is $\mathscr{L}\left(\frac{ae^{-at}-be^{-bt}}{a-b}\right) = F(p) = \frac{p}{(p+a)(p+b)}$. Hence, letting $a = -1, b = 2$ we get the second expression in (1), that is

$$
\mathscr{L}\left(\frac{-e^{+t}-2e^{-2t}}{-3}\right) = \frac{p}{(p-1)(p+2)} \tag{3}
$$

So combine (2) and (3) we get

$$
5\mathscr{L}\left(\frac{e^{+t}-e^{-2t}}{2+1}\right) - 2\mathscr{L}\left(\frac{-e^{+t}-2e^{-2t}}{-1-2}\right) = \frac{5}{(p-1)(p+2)} - \frac{2p}{(p-1)(p+2)}
$$

which is (1). Hence

$$
\begin{aligned}
f(t) &= 5\frac{e^{+t}-e^{-2t}}{3} - 2\frac{-e^{+t}-2e^{-2t}}{-3} \\
&= 5\frac{e^{+t}-e^{-2t}}{3} + 2\frac{-e^{+t}-2e^{-2t}}{3} \\
&= \frac{5e^{+t}-5e^{-2t}-2e^{+t}-4e^{-2t}}{3} \\
&= \frac{3e^{t}-9e^{-2t}}{3} \\
&= e^{t}-3e^{-2t}
\end{aligned}
$$

Hence the inverse transform of $F(p)$ is $e^{t}-3e^{-2t}$

12 chapter 15, problem 3.11

Problem Use laplace transform to solve $y'' - 4y = 4e^{2t}$,$y_0 = 0, y_0' = 1$

Solution

Let $\mathscr{L}\left(y(t)\right) = Y(p)$, and taking the laplace transform of both sides, noting first that $\mathscr{L}\left(y''\right) = p^2 Y - py_0 - y_0'$ then we get

$$\begin{aligned}
\mathscr{L}\left(y''\right) - 4\mathscr{L}\left(y(t)\right) &= \mathscr{L}\left(4e^{2t}\right) \\
\left(p^2 Y - py_0 - y_0'\right) - 4Y &= 4\mathscr{L}\left(e^{2t}\right)
\end{aligned}$$

L2 from table on page 636 : $\mathscr{L}\left(e^{-at}\right) = \frac{1}{p+a}$, hence $\mathscr{L}\left(e^{2t}\right) = \frac{1}{p-2}$, hence, after applying boundary conditions, we get

$$\begin{aligned}
\left(p^2 Y - 1\right) - 4Y &= 4\frac{1}{p-2} \\
Y\left(p^2 - 4\right) - 1 &= 4\frac{1}{p-2} \\
Y &= 4\frac{1}{\left(p^2-4\right)\left(p-2\right)} + \frac{1}{\left(p^2-4\right)} \\
Y &= 4\frac{1}{\left(p-2\right)\left(p+2\right)\left(p-2\right)} + \frac{1}{\left(p^2-4\right)} \\
Y &= 4\frac{1}{\left(p+2\right)\left(p-2\right)^2} + \frac{1}{\left(p^2-4\right)}
\end{aligned}$$

Doing partial fractions, repeated roots, gives

$$\begin{aligned}
\frac{1}{\left(p+2\right)\left(p-2\right)^2} &= \frac{A}{\left(p+2\right)} + \frac{B}{\left(p-2\right)} + \frac{C}{\left(p-2\right)^2} \\
1 &= A\left(p-2\right)^2 + B\left(p+2\right)\left(p-2\right) + C\left(p+2\right) \\
1 &= A\left(p^2 - 4p + 4\right) + B\left(p^2 - 4\right) + C\left(p+2\right) \\
1 &= p^2(A+B) + p(-4A + C) + 4A - 4B + 2C
\end{aligned}$$

Hence

$$\begin{aligned}
A + B &= 0 \\
-4A + C &= 0 \\
4A - 4B + 2C &= 1
\end{aligned}$$

hence $A=-B$, then $-4B-4B+2C=1$ or $-8B+2C=1$ or $C=\frac{1+8B}{2}$, therefore $4B+\frac{1+8B}{2}=0$ then $B=-\frac{1}{16}$, then $A=\frac{1}{16}, C=\frac{1+8\left(-\frac{1}{16}\right)}{2}=\frac{1-\left(\frac{1}{2}\right)}{2}=\frac{1}{4}$. Hence

$$
\begin{aligned}
Y &= 4\frac{1}{(p+2)(p-2)^2}+\frac{1}{(p^2-4)}\\
&= 4\left(\frac{A}{(p+2)}+\frac{B}{(p-2)}+\frac{C}{(p-2)^2}\right)+\frac{1}{(p^2-4)}\\
&= 4\left(\frac{\frac{1}{16}}{(p+2)}+\frac{-\frac{1}{16}}{(p-2)}+\frac{\frac{1}{4}}{(p-2)^2}\right)+\frac{1}{(p^2-4)}\\
&= \frac{1}{4}\frac{1}{(p+2)}-\frac{1}{4}\frac{1}{(p-2)}+\frac{1}{(p-2)^2}+\frac{1}{(p^2-4)}\\
&= \frac{1}{4}\frac{1}{(p+2)}-\frac{1}{4}\frac{1}{(p-2)}+\frac{1}{(p-2)^2}+\left(\frac{1}{4}\frac{1}{(p-2)}-\frac{1}{4}\frac{1}{(p+2)}\right)\\
&= \frac{1}{(p-2)^2}
\end{aligned}
$$

$\frac{1}{(p-2)^2}\rightarrow$ using L6, we have $\mathscr{L}\left(t^k e^{-at}\right)=\frac{k!}{(p+a)^{k+1}}$, here $a=-2, k=1$ and $\frac{1}{(p-2)^2}\rightarrow te^{2t}$. Hence ,

$$f(t)=te^{2t}$$

13 chapter 15, problem 3.24

Problem Use laplace transform to solve $y''-2y'+y=2\cos t$,$y_0=5, y_0'=-2$

Solution

Let $\mathscr{L}(y(t))=Y(p)$, Take the laplace transform of both sides, noting first that $\mathscr{L}(y'')=p^2Y-py_0-y_0'$, $\mathscr{L}(y')=pY-y_0$then we get

$$
\begin{aligned}
\mathscr{L}(y'')-2\mathscr{L}(y')+\mathscr{L}(y(t)) &= \mathscr{L}(2\cos t)\\
\left(p^2Y-py_0-y_0'\right)-2\left(pY-y_0\right)+Y &= 2\frac{p}{p^2+1}
\end{aligned}
$$

$$\left(p^2Y - 5p + 2\right) - 2\left(pY - 5\right) + Y = 2\frac{p}{p^2+1}$$
$$Y\left(p^2 - 2p + 1\right) + 12 - 5p = 2\frac{p}{p^2+1}$$
$$Y\left(p^2 - 2p + 1\right) = \frac{2p}{p^2+1} - 12 + 5p$$
$$Y = \frac{\frac{2p}{p^2+1} - 12 + 5p}{\left(p^2 - 2p + 1\right)}$$
$$Y = \frac{2p}{\left(p^2+1\right)\left(p^2 - 2p + 1\right)} + \frac{5p - 12}{\left(p^2 - 2p + 1\right)}$$
$$Y = \frac{2p}{\left(p^2+1\right)\left(p-1\right)^2} + \frac{5p - 12}{\left(p-1\right)^2}$$

Doing partial fractions $\frac{2p}{\left(p-1\right)^2\left(p^2+1\right)} = \frac{A}{\left(p-1\right)} + \frac{B}{\left(p-1\right)^2} + \frac{Cp+D}{\left(p^2+1\right)}$. Solving, we get $A = 0, B = 1, C = 0, D = -1$ Hence

$$Y = \frac{1}{\left(p-1\right)^2} - \frac{1}{\left(p^2+1\right)} + \frac{5p-12}{\left(p-1\right)^2}$$
$$= \frac{1}{\left(p-1\right)^2} - \frac{1}{\left(p^2+1\right)} + \frac{5p}{\left(p-1\right)^2} - 12\frac{1}{\left(p-1\right)^2}$$
$$= \frac{-11}{\left(p-1\right)^2} - \frac{1}{\left(p^2+1\right)} + \frac{5p}{\left(p-1\right)^2}$$

Hence , using table, we get inverse laplace transform $\frac{1}{\left(p-1\right)^2} \to te^t$ and $\frac{1}{\left(p^2+1\right)} \to \sin t$ and $\frac{p}{\left(p-1\right)^2} \to e^t + te^t$, hence

$$f(t) = -11te^t - \sin t + 5\left(e^t + te^t\right)$$
$$= -11te^t - \sin t + 5e^t + 5te^t$$
$$= 5e^t - 6te^t - \sin t$$

14 chapter 15, problem 3.25

Problem Use laplace transform to solve $y'' + 4y' + 5y = 2e^{-2t}\cos t$, ,$y_0 = 0, y_0' = 3$

Solution

Let $\mathscr{L}\left(y(t)\right) = Y(p)$, Taking the laplace transform of both sides, noting first that $\mathscr{L}\left(y''(t)\right) = p^2Y - py_0 - y_0'$, $\mathscr{L}\left(y'(t)\right) = pY - y_0$then we get

$$\mathscr{L}\left(y''(t)\right) + 4\mathscr{L}\left(y'(t)\right) + 5\mathscr{L}\left(y(t)\right) = \mathscr{L}\left(2e^{-2t}\cos t\right)$$
$$\left(p^2Y - py_0 - y_0'\right) + 4\left(pY - y_0\right) + 5Y = 2\mathscr{L}\left(e^{-2t}\cos t\right)$$

Applying initial conditions

$$\left(p^2Y - 3\right) + 4\left(pY\right) + 5Y = 2\mathscr{L}\left(e^{-2t}\cos t\right)$$

From table using L14 $\mathscr{L}\left(e^{-2t}\cos t\right) = \frac{p+2}{(p+2)^2+1}$. Hence

$$\begin{aligned}
\left(p^2Y - 3\right) + 4\left(pY\right) + 5Y &= 2\frac{p+2}{(p+2)^2+1} \\
Y\left(p^2 + 4p + 5\right) - 3 &= 2\frac{p+2}{(p+2)^2+1} \\
Y &= \frac{2\frac{p+2}{(p+2)^2+1} + 3}{\left(p^2+4p+5\right)} \\
Y &= 2\frac{p+2}{\left((p+2)^2+1\right)\left(p^2+4p+5\right)} + 3\frac{1}{\left(p^2+4p+5\right)}
\end{aligned}$$

but $\left(p^2 + 4p + 5\right) = (p+2)^2 + 1$, therefore

$$\begin{aligned}
Y &= 2\frac{p+2}{\left((p+2)^2+1\right)^2} + \frac{3}{(p+2)^2+1} \\
&= 2\frac{p+2}{\left((p+2)^2+1\right)\left((p+2)^2+1\right)} + \frac{3}{(p+2)^2+1}
\end{aligned}$$

But inverse transform of $\frac{p+2}{\left((p+2)^2+1\right)^2} = \frac{1}{2}te^{-2t}\sin t$ and inverse transform of $\frac{1}{(p+2)^2+1} = e^{-2t}\sin t$. Hence

$$\begin{aligned}
f(t) &= 2\left(\frac{1}{2}te^{-2t}\sin t\right) + 3\left(e^{-2t}\sin t\right) \\
&= te^{-2t}\sin t + 3e^{-2t}\sin t \\
&= (t+3)e^{-2t}\sin t
\end{aligned}$$

15 chapter 15, problem 3.29

Problem Use laplace transform to solve $y' + z' - 2y = 1, y_0 = z_0 = 1, z - y' = t$

Solution

Taking laplace transform of both equations, then we get 2 equations in Y and Z, then solve for them. Let $\mathscr{L}\left(y(t)\right) = Y(p)$, $\mathscr{L}(z(t)) = Z(p)$

$$\begin{aligned} \mathscr{L}\left(y'(t)\right) + \mathscr{L}(z'(t)) - 2\mathscr{L}\left(y(t)\right) &= \mathscr{L}(1) \\ \mathscr{L}(z(t)) - \mathscr{L}\left(y'(t)\right) &= \mathscr{L}(t) \end{aligned}$$

Then we get

$$\begin{aligned} \left(pY - y_0\right) + \left(pZ - z_0\right) - 2Y &= \frac{1}{p} \\ Z - \left(pY - y_0\right) &= \frac{1}{p^2} \end{aligned}$$

Putting initial conditions gives

$$\begin{aligned} \left(pY - 1\right) + \left(pZ - 1\right) - 2Y &= \frac{1}{p} \\ Z - \left(pY - 1\right) &= \frac{1}{p^2} \end{aligned}$$

$$Y\left(p - 2\right) + pZ - 2 = \frac{1}{p} \tag{1}$$

$$Z - pY + 1 = \frac{1}{p^2} \tag{2}$$

Solving for Y,From (1), $Z = \frac{\frac{1}{p} + 2 - Y\left(p-2\right)}{p}$, and substituting into (2) gives

$$
\begin{aligned}
\frac{\frac{1}{p}+2-Y\left(p-2\right)}{p}-pY+1 &= \frac{1}{p^2}\\
\frac{1}{p}+2-Y\left(p-2\right)-p^2Y+p &= \frac{1}{p}\\
2-Y\left(p-2\right)-p^2Y &= -p\\
Y\left(-p+2-p^2\right) &= -p-2\\
Y &= \frac{-p-2}{\left(-p+2-p^2\right)}\\
Y &= -\frac{p+2}{\left(-p+1\right)\left(p+2\right)}\\
Y &= -\frac{1}{\left(-p+1\right)}
\end{aligned}
$$

Hence $Y=\frac{1}{p-1}$ so from L2

$$y(t)=e^t$$

Now, that we have Y, we solve for Z. From (2)

$$
\begin{aligned}
Z-pY+1 &= \frac{1}{p^2}\\
Z-p\left(\frac{1}{p-1}\right)+1 &= \frac{1}{p^2}\\
Z &= \frac{1}{p^2}-1+\frac{p}{p-1}\\
Z &= \frac{p-1-p^2\left(p-1\right)+p^3}{p^2\left(p-1\right)}\\
Z &= \frac{p-1\ +p^2}{p^2\left(p-1\right)}\\
Z &= \frac{p-1\ +p^2}{p^2\left(p-1\right)}
\end{aligned}
$$

Doing partial fraction on the above, we get $Z=\frac{1}{p^2}+\frac{1}{p-1}$, Hence

$$z(t)=t+e^t$$

16 chapter 15, problem 3.30

Problem Use laplace transform to solve $y' + 2z = 1, y_0 = 0, 2y - z' = 2t, z_0 = 1$

Solution

Take laplace transform of both equations, then we get 2 equations in Y and Z, then solve for them. Let $\mathscr{L}\left(y(t)\right) = Y(p)$, $\mathscr{L}(z(t)) = Z(p)$

$$\begin{aligned} \mathscr{L}\left(y'(t)\right) + 2Z &= \mathscr{L}(1) \\ 2Y - \mathscr{L}(z'(t)) &= \mathscr{L}(2t) \end{aligned}$$

$$\begin{aligned} pY - y_0 + 2Z &= \mathscr{L}(1) \\ 2Y - \left(pZ - z_0\right) &= \mathscr{L}(2t) \end{aligned}$$

Then we get, by putting $z_0 = 1, y_0 = 0$

$$pY + 2Z = \frac{1}{p} \tag{1}$$

$$2Y - \left(pZ - 1\right) = \frac{2}{p^2} \tag{2}$$

Obtain Z from first equation and sub into the second to solve for Y, $Z = \frac{\frac{1}{p} - pY}{2}$, Hence

$$2Y - \left(p\left(\frac{\frac{1}{p} - pY}{2} \right) - 1 \right) = \frac{2}{p^2}$$
$$2Y - \frac{p}{2}\left(\frac{1}{p} - pY \right) + 1 = \frac{2}{p^2}$$
$$2Y - \frac{p}{2}\left(\frac{1 - p^2Y}{p} \right) = \frac{2}{p^2} - 1$$
$$2Y - \frac{1}{2}\left(1 - p^2Y \right) = \frac{2 - p^2}{p^2}$$
$$2Y - \frac{1}{2} + \frac{1}{2}p^2Y = \frac{2 - p^2}{p^2}$$
$$Y\left(2 + \frac{p^2}{2} \right) = \frac{2 - p^2}{p^2} + \frac{1}{2}$$
$$Y = \frac{\frac{4-p^2}{2p^2}}{\left(\frac{4+p^2}{2} \right)}$$
$$Y = \frac{4 - p^2}{p^2\left(4 + p^2 \right)}$$

Hence , using partial fraction gives

$$Y = \frac{4 + p^2}{p^2\left(4 + p^2 \right)}$$
$$= \frac{Ap + B}{p^2} + \frac{Cp + D}{\left(4 + p^2 \right)}$$

Then

$$\left(Ap + B \right)\left(4 + p^2 \right) + \left(Cp + D \right)p^2 = 4 - p^2$$
$$4Ap + Ap^3 + 4B + Bp^2 + Cp^3 + Dp^2 = 4 - p^2$$
$$p^3(A + C) + p^2(B + D) + p(4A) + 4B = 4 - p^2$$

Hence, $4B = 4 \to B = 1$ and $4A = 0 \to A = 0$ and $B + D = -1$ and $A + C = 0 \to C = 0$, therefore $D = -1 - B \to D = -2$. Hence

$$Y = \frac{Ap + B}{p^2} + \frac{Cp + D}{\left(4 + p^2 \right)}$$
$$= \frac{1}{p^2} - \frac{2}{\left(4 + p^2 \right)}$$

Using tables for inverse transform gives

$$y(t) = t - \sin 2t$$

Now, to find $z(t)$, subtituting value we found for Y into equation (1) above.

$$\begin{aligned} pY \quad + 2Z &= \frac{1}{p} \\ p\left(\frac{1}{p^2} - \frac{2}{\left(4+p^2\right)}\right) \quad + 2Z &= \frac{1}{p} \\ \frac{1}{p} - \frac{2p}{\left(4+p^2\right)} \quad + 2Z &= \frac{1}{p} \\ Z &= \frac{p}{\left(4+p^2\right)} \\ Z &= \frac{p}{\left(4+p^2\right)} \end{aligned}$$

From tables, using L4

$$z(t) = \cos 2t$$

17 chapter 15, problem 3.4

Problem Use laplace transform to solve $y'' + y = \sin t$,$y_0 = 1, y_0' = 0$

Solution

Let $\mathscr{L}\big(y(t)\big) = Y(p)$, Take the laplace transform of both sides, noting first that $\mathscr{L}\big(y''(t)\big) = p^2 Y - p y_0 - y_0'$ then we get

$$\begin{aligned} \mathscr{L}\big(y''(t)\big) + \mathscr{L}\big(y(t)\big) &= \mathscr{L}(\sin t) \\ \left(p^2 Y - p y_0 - y_0'\right) + Y \ &= \frac{1}{p^2+1} \end{aligned}$$

Where I used L3 from table on page 636 which says that $\mathscr{L}(\sin at) = \frac{a}{p^2+a^2}$. Now solving for Y, noting that $y_0 = 1$ and $y_0' = 0$ gives

$$\begin{aligned}
\left(p^2Y - p\right) + Y &= \frac{1}{p^2+1}\\
Y\left(p^2+1\right) - p &= \frac{1}{p^2+1}\\
Y\left(p^2+1\right) &= \frac{1}{p^2+1} + p\\
Y &= \frac{1}{\left(p^2+1\right)^2} + \frac{p}{\left(p^2+1\right)} \qquad (1)
\end{aligned}$$

From table using L12, $\frac{p}{\left(p^2+1\right)} \to \cos t$ and using L17, $\frac{1}{\left(p^2+1\right)^2} \to \frac{1}{2}(\sin t - t\cos t)$. Hence, putting these together into (1) gives

$$f(t) = \cos t + \frac{1}{2}(\sin t - t\cos t)$$

This is the particular solution to the ODE.

18 chapter 15, problem 3.6

Problem Use laplace transform to solve $y'' - 6y' + 9y = te^{3t}$,$y_0 = 0, y_0' = 5$

Solution

Let $\mathscr{L}\left(y(t)\right) = Y(p)$, Take the laplace transform of both sides, noting first that $\mathscr{L}\left(y''(t)\right) = p^2Y - py_0 - y_0'$ and $\mathscr{L}\left(y'(t)\right) = pY - y_0$ then we get

$$\begin{aligned}
\mathscr{L}\left(y''(t)\right) - 6\mathscr{L}\left(y'(t)\right) + 9\mathscr{L}\left(y(t)\right) &= \mathscr{L}\left(t\, e^{3t}\right)\\
\left(p^2Y - py_0 - y_0'\right) - 6\left(pY - y_0\right) + 9Y &= \mathscr{L}\left(t\, e^{3t}\right)
\end{aligned}$$

I use L6 from table on page 636 which says that $\mathscr{L}\left(t^k\, e^{-at}\right) = \frac{k!}{(p+a)^{k+1}}$, hence for $k = 1, a = -3$, we get $\mathscr{L}\left(t^k\, e^{-at}\right) = \frac{1}{(p-3)^2}$

$$\left(p^2Y - py_0 - y_0'\right) - 6\left(pY - y_0\right) + 9Y = \frac{1}{\left(p-3\right)^2}$$

Applying boundary conditions gives

$$
\begin{aligned}
\left(p^2Y-5\right)-6\left(pY\right)+9Y &= \frac{1}{\left(p-3\right)^2}\\
Y\left(p^2-6p+9\right)-5 &= \frac{1}{\left(p-3\right)^2}\\
Y\left(p^2-6p+9\right) &= \frac{1}{\left(p-3\right)^2}+5\\
Y &= \frac{1}{\left(p^2-6p+9\right)\left(p-3\right)^2}+\frac{5}{\left(p^2-6p+9\right)}\\
Y &= \frac{1}{\left(p-3\right)^2\left(p-3\right)^2}+\frac{5}{\left(p-3\right)^2}\\
Y &= \frac{1}{\left(p-3\right)^4}+\frac{5}{\left(p-3\right)^2} \qquad (1)
\end{aligned}
$$

Now using table, from L6, $\frac{1}{\left(p-3\right)^2}$, let $a=-3,k=1$ hence $\frac{1}{\left(p-3\right)^2}\to te^{3t}$ And using L6 again, $\frac{1}{\left(p-3\right)^4}$, let $k=3,a=-3$ then $\frac{6}{\left(p-3\right)^4}\to t^3e^{3t}$, therefore (1) becomes

$$
\begin{aligned}
f(t) &= \frac{1}{6}t^3e^{3t}+5\,te^{3t}\\
&= e^{3t}\left(\frac{1}{6}t^3+5t\right)
\end{aligned}
$$

19 chapter 15, problem 3.8

Problem Use laplace transform to solve $y''\ +16y=8\cos 4t$,$y_0=0,y_0'=0$

Solution

Let $\mathscr{L}\left(y(t)\right)=Y(p)$, Taking the laplace transform of both sides, noting first that $\mathscr{L}\left(y''(t)\right)=p^2Y-py_0-y_0'$ results in

$$
\begin{aligned}
\mathscr{L}\left(y''(t)\right)\ +16\mathscr{L}\left(y(t)\right) &= \mathscr{L}(8\cos 4t)\\
\left(p^2Y-py_0-y_0'\right)\ +16Y &= 8\mathscr{L}(\cos 4t)\\
Y\left(p^2+16\right) &= 8\frac{p}{p^2+16}
\end{aligned}
$$

I used L6 from table on page 636 : $\mathscr{L}(\cos at)=\frac{p}{p^2+a^2}$

$$
Y=8\frac{p}{\left(p^2+16\right)^2}
$$

Now looking at L11, which says $\mathscr{L}(t\ \sin at) = \frac{2ap}{\left(p^2+a^2\right)^2}$, hence letting $a = 4$ gives the solution

$$f(t) = t\ \sin 4t$$