# HW 11, Math 121 A
# Spring, 2004
# UC BERKELEY

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## Contents



## 1 chapter 15, problem 2.11

**Problem** Find the inverse transform of the function $F(p) = \frac{3p+2}{3p^2+5p-2}$

**Solution**

Need to simplify the above expression to some expressions which are shown in the table on page 636.

$$\begin{aligned} F(p) &= \frac{3p+2}{3p^2+5p-2} \\ &= \frac{2}{(3p-1)(p+2)} + \frac{3p}{(3p-1)(p+2)} \end{aligned} \tag{1}$$

Expanding in partial fractions. For the first term in (1):

$$\begin{aligned} \frac{1}{(3p-1)(p+2)} &= \frac{A}{(3p-1)} + \frac{B}{(p+2)} = \frac{A(p+2)+B(3p-1)}{(3p-1)(p+2)} \\ A(p+2)+B(3p-1) &= 1 \\ Ap+2A+3Bp-B &= 1 \end{aligned}$$

Hence $2A-B=1$ and $(A+3B)=0$ which gives $A=\frac{1}{2}+\frac{B}{2}$. Therefore $\frac{1}{2}+\frac{B}{2}+3B=0$ or $\frac{1+B+6B}{2}=0$ or $1+7B=0$ or $B=\frac{-1}{7}$. Hence $A=\frac{1}{2}+\frac{\frac{-1}{7}}{2}=\frac{1}{2}-\frac{1}{14}=\frac{7-1}{14}=\frac{6}{14}$.

Now the first term in (1) can be written as $2\left(\frac{A}{(3p-1)}+\frac{B}{(p+2)}\right)$ or $2\left(\frac{\frac{6}{14}}{(3p-1)}+\frac{\frac{-1}{7}}{(p+2)}\right)$ or

$$\frac{6}{7(3p-1)} - \frac{2}{7(p+2)} \tag{2}$$

Doing partial fraction on the second term in (1) which is $\frac{3p}{(3p-1)(p+2)}$ gives

$$\begin{aligned} \frac{p}{(3p-1)(p+2)} &= \frac{A}{(3p-1)} + \frac{B}{(p+2)} = \frac{A(p+2)+B(3p-1)}{(3p-1)(p+2)} \\ A(p+2)+B(3p-1) &= p \\ Ap+2A+3Bp-B &= p \end{aligned}$$

Hence $2A-B=0$ and $(A+3B)=1$, therefore $A=\frac{B}{2}$. Hence $\left(\frac{B}{2}+3B\right)=1$ or $\frac{B+6B}{2}=1$ or $7B=2$ or $B=\frac{2}{7}$. Therefore $A=\frac{B}{2}=\frac{2}{14}$. Hence $\frac{3p}{(3p-1)(p+2)}=3\left(\frac{A}{(3p-1)}+\frac{B}{(p+2)}\right)=3\left(\frac{\frac{2}{14}}{(3p-1)}+\frac{\frac{2}{7}}{(p+2)}\right)$ or

$$\frac{3}{7(3p-1)} + \frac{6}{7(p+2)} \tag{3}$$

Combining (2) and (3) gives

$$\begin{aligned} F(p) &= \frac{6}{7(3p-1)} - \frac{2}{7(p+2)} + \frac{3}{7(3p-1)} + \frac{6}{7(p+2)} \\ &= \frac{6}{7}\frac{1}{(3p-1)} - \frac{2}{7}\frac{1}{(p+2)} + \frac{2}{7}\frac{1}{(3p-1)} + \frac{6}{7}\frac{1}{(p+2)} \\ &= \frac{4}{7}\frac{1}{(p+2)} + \frac{9}{7}\frac{1}{(3p-1)} \\ &= \frac{4}{7}\frac{1}{(p+2)} + \frac{9}{21}\frac{1}{\left(p-\frac{1}{3}\right)} \end{aligned}$$

Now we can use the table to find the inverse transform. Use property L2, which says

$$\mathscr{L}\left(e^{-at}\right) = \frac{1}{p+a}$$

Hence , setting $a = 2$, gives $\mathcal{L}\left(e^{-2t}\right) = \frac{1}{p+2}$ and setting $a = -\frac{1}{3}$ gives $\mathcal{L}\left(e^{-\frac{1}{3}t}\right)^{\frac{1}{3}t} = \frac{1}{p-\frac{1}{3}}$. Hence $f(t) = \frac{4}{7}e^{-2t} + \frac{9}{21}e^{\frac{1}{3}t}$ and the inverse Laplace transform is

$$f(t) = \frac{4}{7}e^{-2t} + \frac{3}{7}e^{\frac{1}{3}t}$$

## 2 chapter 15, problem 2.17

**Problem** Use L32 and L11 to obtain $\mathcal{L}\left(t^2 \sin at\right)$

**Solution**

$$\mathcal{L}\left(t^n f(t)\right) = (-1)^n \frac{d^n F(p)}{dp^n} \tag{L32}$$

$$\mathcal{L}(t \sin at) = \frac{2ap}{\left(p^2 + a^2\right)^2} \tag{L11}$$

we set $f(t) = t \sin at$ then we can write using L32

$$\mathcal{L}\left(t\, f(t)\right) = (-1)\frac{d\mathcal{L}\left(f(t)\right)}{dp} \tag{1}$$

But $\mathcal{L}\left(f(t)\right) = \mathcal{L}(t \sin at) = \frac{2ap}{\left(p^2+a^2\right)^2}$ from table L11 (1) becomes

$$\begin{aligned}
\mathcal{L}\left(t\, f(t)\right) &= -\frac{d}{dp}\left(\frac{2ap}{\left(p^2 + a^2\right)^2}\right) \\
\mathcal{L}(t \times t \sin at) &= -\left(p\frac{-2 \times 2a}{\left(p^2 + a^2\right)^3} \times 2p + \frac{2a}{\left(p^2 + a^2\right)^2} \times 1\right) \\
\mathcal{L}\left(t^2 \sin at\right) &= \frac{8ap^2}{\left(p^2 + a^2\right)^3} - \frac{2a}{\left(p^2 + a^2\right)^2} \\
&= \frac{8ap^2 - 2a\left(p^2 + a^2\right)}{\left(p^2 + a^2\right)^3} \\
&= \frac{a\left(8p^2 - 2p^2 - 2a^2\right)}{\left(p^2 + a^2\right)^3} \\
&= \frac{a\left(6p^2 - 2a^2\right)}{\left(p^2 + a^2\right)^3}
\end{aligned}$$

Or

$$\mathcal{L}\left(t^2 \sin at\right) = \frac{6ap^2 - 2a^3}{\left(p^2 + a^2\right)^3}$$

## 3 chapter 15, problem 2.18

**Problem** Use L31 to derive L21

**Solution**

$$\mathscr{L}\left(\frac{f(t)}{t}\right) = \int_{u=p}^{\infty} F(u)du \tag{L31}$$

$$\mathscr{L}\left(\frac{e^{-at}-e^{-bt}}{t}\right) = \ln\frac{p+b}{p+a} \tag{L21}$$

we set $f(t) = e^{-at} - e^{-bt}$ then we can write using L31

$$\begin{aligned}\mathscr{L}\left(\frac{f(t)}{t}\right) &= \int_{u=p}^{\infty} F(u)du \\ &= \int_{u=p}^{\infty} \mathscr{L}\big[f(t)\big]\ du\end{aligned}$$

but $\mathscr{L}\big[f(t)\big] = \mathscr{L}\big(e^{-at} - e^{-bt}\big) = \mathscr{L}\big(e^{-at}\big) - \mathscr{L}\big(e^{-bt}\big) = \frac{1}{p+a} - \frac{1}{p+b}$ By using L2. But since we are using $u$ in place of $p$ in integral, we need to call $p = u$. Hence

$$\begin{aligned}\mathscr{L}\left(\frac{f(t)}{t}\right) &= \int_{u=p}^{\infty}\left(\frac{1}{u+a} - \frac{1}{u+b}\right)\ du \\ &= \int_{u=p}^{\infty}\frac{1}{u+a}\ du - \int_{u=p}^{\infty}\frac{1}{u+b}\ du \\ &= [\ln(u+a)]_p^{\infty} - [\ln(u+b)]_p^{\infty} \\ &= \big[\ln(\infty+a) - \ln\big(p+a\big)\big] - \big[\ln(\infty+b) - \ln\big(p+b\big)\big] \\ &= \ln(\infty) - \ln\big(p+a\big) - \ln(\infty) + \ln\big(p+b\big) \\ &= \ln\big(p+b\big) - \ln\big(p+a\big)\end{aligned}$$

but $\ln A - \ln B = \ln\frac{A}{B}$, therefore

$$\begin{aligned}\mathscr{L}\left(\frac{f(t)}{t}\right) &= \ln\big(p+b\big) - \ln\big(p+a\big) \\ \mathscr{L}\left(\frac{e^{-at}-e^{-bt}}{t}\right) &= \ln\left(\frac{p+b}{p+a}\right)\end{aligned}$$

## 4 chapter 15, problem 2.2

**Problem** Use relation L2 to find L7 and L8 in laplace table.

**Solution**

$$\mathscr{L}\big(e^{-at}\big) = \frac{1}{p+a} \tag{L2}$$

for $\Re\big(p+a\big) > 0$

$$\mathscr{L}\frac{e^{-at}-e^{-bt}}{b-a} = \frac{1}{\big(p+a\big)\big(p+b\big)} \tag{L7}$$

$$\mathscr{L}\frac{ae^{-at}-be^{-bt}}{a-b} = \frac{p}{\big(p+a\big)\big(p+b\big)} \tag{L8}$$

For $\Re(p+a) > 0$ and $\Re(p+b) > 0$. Where $\mathscr{L}f(t)$ is the laplace transform of $f(t)$ defined as $\mathscr{L}f(t) = F(p) = \int_0^\infty e^{-pt} f(t)dt$.

From the linearity property of the $\mathscr{L}$ operator, expand the LHS of L7, we get

$$\mathscr{L}\frac{e^{-at}-e^{-bt}}{b-a} = \frac{1}{b-a}\mathscr{L}\left(e^{-at}-e^{-bt}\right) = \frac{1}{b-a}\left(\mathscr{L}\left(e^{-at}\right)-\mathscr{L}\left(e^{-bt}\right)\right)$$

Now applying L2 gives for $\Re(p+a) > 0$ and $\Re(p+b) > 0$

$$\begin{aligned}
\mathscr{L}\frac{e^{-at}-e^{-bt}}{b-a} &= \frac{1}{b-a}\left(\frac{1}{p+a}-\frac{1}{p+b}\right)\\
&= \frac{1}{b-a}\left(\frac{p+b-(p+a)}{(p+a)(p+b)}\right)\\
&= \frac{1}{b-a}\left(\frac{b-a}{(p+a)(p+b)}\right)\\
&= \frac{1}{(p+a)(p+b)}
\end{aligned}$$

For Which is L7 as required to show. Similarly for L8, expand the LHS of L8 we get for $\Re(p+a) > 0$ and $\Re(p+b) > 0$

$$\begin{aligned}
\mathscr{L}\frac{ae^{-at}-be^{-bt}}{a-b} &= \frac{1}{a-b}\mathscr{L}\left(ae^{-at}-be^{-bt}\right)\\
&= \frac{1}{a-b}\left(\mathscr{L}\left(e^{-at}\right)-\mathscr{L}\left(e^{-bt}\right)\right)\\
&= \frac{1}{a-b}\left(a\mathscr{L}\left(e^{-at}\right)-b\mathscr{L}\left(e^{-bt}\right)\right)\\
&= \frac{1}{a-b}\left(a\frac{1}{p+a}-b\frac{1}{p+b}\right)\\
&= \frac{1}{a-b}\left(\frac{a(p+b)-b(p+a)}{(p+a)(p+b)}\right)\\
&= \frac{1}{a-b}\left(\frac{ap+ab-bp-ba}{(p+a)(p+b)}\right)\\
&= \frac{1}{a-b}\left(\frac{ap-bp}{(p+a)(p+b)}\right)\\
&= \frac{1}{a-b}\left(\frac{p(a-b)}{(p+a)(p+b)}\right)\\
&= \frac{p}{(p+a)(p+b)}
\end{aligned}$$

Which is L8.

## 5 chapter 15, problem 2.21

**Problem** Use L29 and L11 to obtain $\mathscr{L}\left(te^{-at}\sin bt\right)$

**Solution**

$$\mathscr{L}\left(e^{-at}f(t)\right) = F(p+a) \tag{L29}$$

$$\mathscr{L}(t\sin at) = \frac{2ap}{\left(p^2+a^2\right)^2} \tag{L11}$$

Then from L11 we get

$$\mathscr{L}(t\ \sin bt) = \frac{2bp}{\left(p^2+b^2\right)^2}$$

Now, let $p = \left(p+a\right)$ then from L29, the above becomes

$$\mathscr{L}\left(e^{-at}\ t\ \sin bt\right) = \frac{2b\left(p+a\right)}{\left(\left(p+a\right)^2+b^2\right)^2}$$

## 6 chapter 15, problem 2.22

**Problem** similar to problem 2.21, Use L29 and L12 to obtain $\mathscr{L}\left(te^{-at}\ \cos bt\right)$

**Solution**

$$\mathscr{L}\left(e^{-at} f(t)\right) = F\left(p+a\right) \qquad \text{(L29)}$$

$$\mathscr{L}(t\ \cos at) = \frac{p^2-a^2}{\left(p^2+a^2\right)^2} \qquad \text{(L12)}$$

then from L12 we get

$$\mathscr{L}(t\ \cos bt) = \frac{p^2-b^2}{\left(p^2+b^2\right)^2}$$

Now, let $p = \left(p+a\right)$ then from L29, the above becomes

$$\mathscr{L}\left(e^{-at}\ t\ \cos bt\right) = \frac{\left(p+a\right)^2-b^2}{\left(\left(p+a\right)^2+b^2\right)^2}$$

## 7 chapter 15, problem 2.23

**Problem** use result obtained in problem 2.21 and 2.22 to find inverse transform for $\frac{p^2+2p-1}{\left(p^2+4p+5\right)^2}$

**Solution**

Recall, from 2.21 we showed that $\mathscr{L}\left(te^{-at}\ \sin bt\right) = \frac{2b\left(p+a\right)}{\left(\left(p+a\right)^2+b^2\right)^2}$ and from 2.22 $\mathscr{L}\left(te^{-at}\ \cos bt\right) = \frac{\left(p+a\right)^2-b^2}{\left(\left(p+a\right)^2+b^2\right)^2}$ Hence

$$\begin{aligned}\mathscr{L}\left(te^{-at}\ \cos bt\right) - \mathscr{L}\left(te^{-at}\ \sin bt\right) &= \frac{\left(p+a\right)^2-b^2}{\left(\left(p+a\right)^2+b^2\right)^2} - \frac{2b\left(p+a\right)}{\left(\left(p+a\right)^2+b^2\right)^2}\\ &= \frac{\left(p+a\right)^2+b^2\ -2b\left(p+a\right)}{\left(\left(p+a\right)^2+b^2\right)^2}\end{aligned}$$

Now Let $a = 2$ and let $b = 1$ we get

$$\mathcal{L}\left(te^{-2t}\cos t\right) - \mathcal{L}\left(te^{-2t}\sin t\right) = \frac{\left(p+2\right)^2 - 1^2 - 2\left(p+2\right)}{\left(\left(p+2\right)^2 + 1^2\right)^2}$$

$$\mathcal{L}\left(te^{-2t}\cos t - te^{-2t}\sin t\right) = \frac{p^2+4+4p-1-2p-4}{\left(\left(p+2\right)^2+1^2\right)^2} = \frac{p^2+2p-1}{\left(p^2+4p+5\right)^2}$$

Hence inverse transform of $\frac{p^2+2p-1}{\left(p^2+2p+5\right)^2}$ is $te^{-2t}\cos t - te^{-2t}\sin t$ or $te^{-2t}(\cos t - \sin t)$

## 8 chapter 15, problem 2.3

**Problem** Using either relation L2 or L3 and L4, verify L9 and L10 in laplace table.

**Solution**

$$\mathcal{L}\left(e^{-at}\right) = \frac{1}{p+a} \qquad \operatorname{Re}\left(p+a\right) > 0 \tag{L2}$$

$$\mathcal{L}\sin at = \frac{a}{p^2+a^2} \qquad \operatorname{Re}(p) > |\operatorname{Im} a| \tag{L3}$$

$$\mathcal{L}\cos at = \frac{p}{p^2+a^2} \qquad \operatorname{Re}(p) > |\operatorname{Im} a| \tag{L4}$$

$$\mathcal{L}\sinh at = \frac{a}{p^2-a^2} \qquad \operatorname{Re}(p) > |\operatorname{Re}(a)| \tag{L9}$$

$$\mathcal{L}\cosh at = \frac{p}{p^2-a^2} \qquad \operatorname{Re}(p) > |\operatorname{Re}(a)| \tag{L10}$$

Where $\mathcal{L}\left(f(t)\right)$ is the laplace transform of $f(t)$ defined as $\mathcal{L}\left(f(t)\right) = Y(s) = \int_0^\infty e^{-pt} f(t)dt$. To derive L9, use the relation that

$$i\sinh(x) = \sin(ix)$$

Hence, using L3, we get

$$\mathcal{L}(i\sinh at) = \mathcal{L}(\sin iat)$$
$$i\mathcal{L}(\sinh at) = \frac{ia}{p^2+(ia)^2} \qquad \operatorname{Re}(p) > |\operatorname{Im} a|$$
$$\mathcal{L}(\sinh at) = \frac{a}{p^2-a^2} \qquad \operatorname{Re}(p) > |\operatorname{Re}(a)|$$

which is L9. To find L10, use the relation

$$\cosh(x) = \cos(ix)$$

And using L4, we get

$$\begin{aligned}
\mathscr{L}(\cosh(at)) &= \mathscr{L}(\cos(iat)\,) \\
\mathscr{L}(\cosh at) &= \frac{p}{p^2+(ia)^2} && \operatorname{Re}(p) > |\operatorname{Im} a| \\
\mathscr{L}(\sinh at) &= \frac{p}{p^2-a^2} && \operatorname{Re}(p) > |\operatorname{Re}(a)|
\end{aligned}$$

Which is L10.

## 9 chapter 15, problem 2.4

**Problem** by differentiating the appropriate formulas w.r.t. 'a', verify L12

**Solution**

L12 is

$$\mathscr{L}(t\ \cos t) = \frac{p^2-a^2}{\left(p^2+a^2\right)^2} \qquad \operatorname{Re}\left(p\right) > |\operatorname{Im} a| \tag{L12}$$

Where $\mathscr{L}\left(f(t)\right)$ is the laplace transform of $f(t)$ defined as $\mathscr{L}\left(f(t)\right) = F(p) = \int_0^\infty e^{-pt} f(t) dt$. To derive this, I start with L3, which says

$$\mathscr{L}(\sin at) = \frac{a}{p^2+a^2} \qquad \operatorname{Re}(p) > |\operatorname{Im} a|$$

The above can be rewritten in the full definition of the transform to make it easier to see

$$\int_0^\infty e^{-pt}\ \sin at\ \ dt = \frac{a}{p^2+a^2}$$

Taking derivative of both sides w.r.t. $a$ gives

$$\begin{aligned}
\frac{d}{da}\left[\int_0^\infty e^{-pt}\ \sin at\ \ dt\right] &= \frac{d}{da}\left[\frac{a}{p^2+a^2}\right] \\
\int_0^\infty \frac{d}{da}\left[e^{-pt}\ \sin at\right]\ dt &= a\left(\frac{-2a}{\left(p^2+a^2\right)^2}\right) + \frac{1}{p^2+a^2} \times 1 \\
\int_0^\infty e^{-pt}\ t\cos at\ \ dt &= \frac{-2a^2}{\left(p^2+a^2\right)^2} + \frac{1}{p^2+a^2} \\
\mathscr{L}(t\ \cos at) &= \frac{-2a^2+\left(p^2+a^2\right)}{\left(p^2+a^2\right)^2} \\
&= \frac{p^2-a^2}{\left(p^2+a^2\right)^2}
\end{aligned}$$

Which is L12

## 10 chapter 15, problem 2.5

**Problem** by integrating the appropriate formulas w.r.t. 'a', verify L19

**Solution** L19 is

$$\mathscr{L}\left(\frac{\sin at}{t}\right) = \arctan\frac{a}{p} \qquad \operatorname{Re}(p) > |\operatorname{Im} a| \tag{L19}$$

Where $\mathscr{L}\big(f(t)\big)$ is the laplace transform of $f(t)$ defined as $\mathscr{L}\big(f(t)\big) = F(p) = \int_0^\infty e^{-pt} f(t) dt$. To derive this, we start with L4, which says

$$\mathscr{L}(\cos at) = \frac{p}{p^2 + a^2} \qquad \operatorname{Re}(p) > |\operatorname{Im} a|$$

The above can be rewritten in the full definition of the transform to make it easier to see

$$\int_0^\infty e^{-pt} \cos at \ dt = \frac{p}{p^2 + a^2}$$

Integrating both sides w.r.t. $a$ gives

$$\begin{aligned} \int \left( \int_{t=0}^{t=\infty} e^{-pt} \cos at \ dt \right) da &= \int \frac{p}{p^2 + a^2} da \\ \int_{t=0}^{t=\infty} \left( \int e^{-pt} \cos at \ da \right) dt &= \arctan \frac{a}{p} + K \\ \int_{t=0}^{t=\infty} e^{-pt} \left( \int \cos at \ da \right) dt &= \arctan \frac{a}{p} + K \end{aligned}$$

Now, since $\int \cos at \, da = \frac{\sin at}{t} + k$ and choose zero for values of the K's, the constants of integration gives

$$\int_0^\infty e^{-pt} \left( \frac{\sin at}{t} \right) dt = \arctan \frac{a}{p}$$

Hence

$$\mathscr{L}\left( \frac{\sin at}{t} \right) = \arctan \frac{a}{p}$$

Which is L19

## 11 chapter 15, problem 2.9

**Problem** Find the inverse transform of the function $F\big(p\big) = \frac{5-2p}{p^2+p-2}$

**Solution**

Need to simplify the above expression to some expressions which are shown in the table on page 636.

$$\begin{aligned} F(p) &= \frac{5 - 2p}{p^2 + p - 2} \\ &= \frac{5}{p^2 + p - 2} - \frac{2p}{p^2 + p - 2} \\ &= \frac{5}{\big(p-1\big)\big(p+2\big)} - \frac{2p}{\big(p-1\big)\big(p+2\big)} \end{aligned} \tag{1}$$

From table, L7, we see that $\mathscr{L}\left(\frac{e^{-at}-e^{-bt}}{b-a}\right) = F(p) = \frac{1}{\big(p+a\big)\big(p+b\big)}$. Setting $a = -1, b = 2$ we get the first expression in $(1)$, that is

$$\mathscr{L}\left( \frac{e^t - e^{-2t}}{3} \right) = \frac{1}{\big(p-1\big)\big(p+2\big)} \tag{2}$$

also from table, we see that L8 is $\mathscr{L}\left(\frac{ae^{-at}-be^{-bt}}{a-b}\right) = F(p) = \frac{p}{\big(p+a\big)\big(p+b\big)}$. Hence, letting $a = -1, b = 2$ we get the second expression in $(1)$, that is

$$\mathscr{L}\left(\frac{-e^{+t}-2e^{-2t}}{-3}\right)=\frac{p}{(p-1)(p+2)} \tag{3}$$

So combine (2) and (3) we get

$$5\mathscr{L}\left(\frac{e^{+t}-e^{-2t}}{2+1}\right)-2\mathscr{L}\left(\frac{-e^{+t}-2e^{-2t}}{-1-2}\right)=\frac{5}{(p-1)(p+2)}-\frac{2p}{(p-1)(p+2)}$$

which is (1). Hence

$$\begin{aligned} f(t) &= 5\frac{e^{+t}-e^{-2t}}{3}-2\frac{-e^{+t}-2e^{-2t}}{-3} \\ &= 5\frac{e^{+t}-e^{-2t}}{3}+2\frac{-e^{+t}-2e^{-2t}}{3} \\ &= \frac{5e^{+t}-5e^{-2t}-2e^{+t}-4e^{-2t}}{3} \\ &= \frac{3e^{t}-9e^{-2t}}{3} \\ &= e^{t}-3e^{-2t} \end{aligned}$$

Hence the inverse transform of $F(p)$ is $e^t-3e^{-2t}$

## 12 chapter 15, problem 3.11

**Problem** Use laplace transform to solve $y''-4y=4e^{2t}$ ,$y_0=0, y_0'=1$

**Solution**

Let $\mathscr{L}(y(t))=Y(p)$, and taking the laplace transform of both sides, noting first that $\mathscr{L}(y'')=p^2Y-py_0-y_0'$ then we get

$$\begin{aligned} \mathscr{L}(y'')-4\mathscr{L}(y(t)) &= \mathscr{L}(4e^{2t}) \\ (p^2Y-py_0-y_0')-4Y &= 4\mathscr{L}(e^{2t}) \end{aligned}$$

L2 from table on page 636 : $\mathscr{L}(e^{-at})=\frac{1}{p+a}$ , hence $\mathscr{L}(e^{2t})=\frac{1}{p-2}$, hence, after applying boundary conditions, we get

$$\begin{aligned} (p^2Y-1)-4Y &= 4\frac{1}{p-2} \\ Y(p^2-4)-1 &= 4\frac{1}{p-2} \\ Y &= 4\frac{1}{(p^2-4)(p-2)}+\frac{1}{(p^2-4)} \\ Y &= 4\frac{1}{(p-2)(p+2)(p-2)}+\frac{1}{(p^2-4)} \\ Y &= 4\frac{1}{(p+2)(p-2)^2}+\frac{1}{(p^2-4)} \end{aligned}$$

Doing partial fractions, repeated roots, gives

$$\frac{1}{(p+2)(p-2)^2} = \frac{A}{(p+2)} + \frac{B}{(p-2)} + \frac{C}{(p-2)^2}$$
$$1 = A(p-2)^2 + B(p+2)(p-2) + C(p+2)$$
$$1 = A(p^2-4p+4) + B(p^2-4) + C(p+2)$$
$$1 = p^2(A+B) + p(-4A+C) + 4A - 4B + 2C$$

Hence

$$A + B = 0$$
$$-4A + C = 0$$
$$4A - 4B + 2C = 1$$

hence $A = -B$, then $-4B - 4B + 2C = 1$ or $-8B + 2C = 1$ or $C = \frac{1+8B}{2}$, therefore $4B + \frac{1+8B}{2} = 0$ then $B = -\frac{1}{16}$, then $A = \frac{1}{16}$, $C = \frac{1+8\left(-\frac{1}{16}\right)}{2} = \frac{1-\left(\frac{1}{2}\right)}{2} = \frac{1}{4}$. Hence

$$\begin{aligned}
Y &= 4\frac{1}{(p+2)(p-2)^2} + \frac{1}{(p^2-4)} \\
&= 4\left(\frac{A}{(p+2)} + \frac{B}{(p-2)} + \frac{C}{(p-2)^2}\right) + \frac{1}{(p^2-4)} \\
&= 4\left(\frac{\frac{1}{16}}{(p+2)} + \frac{-\frac{1}{16}}{(p-2)} + \frac{\frac{1}{4}}{(p-2)^2}\right) + \frac{1}{(p^2-4)} \\
&= \frac{1}{4}\frac{1}{(p+2)} - \frac{1}{4}\frac{1}{(p-2)} + \frac{1}{(p-2)^2} + \frac{1}{(p^2-4)} \\
&= \frac{1}{4}\frac{1}{(p+2)} - \frac{1}{4}\frac{1}{(p-2)} + \frac{1}{(p-2)^2} + \left(\frac{1}{4}\frac{1}{(p-2)} - \frac{1}{4}\frac{1}{(p+2)}\right) \\
&= \frac{1}{(p-2)^2}
\end{aligned}$$

$\frac{1}{(p-2)^2} \to$ using L6, we have $\mathscr{L}\left(t^k e^{-at}\right) = \frac{k!}{(p+a)^{k+1}}$, here $a = -2, k = 1$ and $\frac{1}{(p-2)^2} \to te^{2t}$. Hence ,

$$f(t) = te^{2t}$$

## 13 chapter 15, problem 3.24

**Problem** Use laplace transform to solve $y'' - 2y' + y = 2\cos t$ ,$y_0 = 5, y_0' = -2$

**Solution**

Let $\mathscr{L}(y(t)) = Y(p)$, Take the laplace transform of both sides, noting first that $\mathscr{L}(y'') = p^2Y - py_0 - y_0'$ , $\mathscr{L}(y') = pY - y_0$then we get

$$\mathscr{L}(y'') - 2\mathscr{L}(y') + \mathscr{L}(y(t)) = \mathscr{L}(2\cos t)$$
$$\left(p^2Y - py_0 - y_0'\right) - 2\left(pY - y_0\right) + Y = 2\frac{p}{p^2+1}$$

$$\begin{aligned}
\left(p^2Y-5p+2\right)\ -2\left(pY-5\right)+Y &= 2\frac{p}{p^2+1}\\
Y\left(p^2-2p+1\right)+12-5p &= 2\frac{p}{p^2+1}\\
Y\left(p^2-2p+1\right) &= \frac{2p}{p^2+1}-12+5p\\
Y &= \frac{\frac{2p}{p^2+1}-12+5p}{\left(p^2-2p+1\right)}\\
Y &= \frac{2p}{\left(p^2+1\right)\left(p^2-2p+1\right)}+\frac{5p-12}{\left(p^2-2p+1\right)}\\
Y &= \frac{2p}{\left(p^2+1\right)\left(p-1\right)^2}+\frac{5p-12}{\left(p-1\right)^2}
\end{aligned}$$

Doing partial fractions $\frac{2p}{\left(p-1\right)^2\left(p^2+1\right)} = \frac{A}{\left(p-1\right)}+\frac{B}{\left(p-1\right)^2}+\frac{Cp+D}{\left(p^2+1\right)}$. Solving, we get $A=0, B=1, C=0, D=-1$ Hence

$$\begin{aligned}
Y &= \frac{1}{\left(p-1\right)^2}-\frac{1}{\left(p^2+1\right)}+\frac{5p-12}{\left(p-1\right)^2}\\
&= \frac{1}{\left(p-1\right)^2}-\frac{1}{\left(p^2+1\right)}+\frac{5p}{\left(p-1\right)^2}-12\frac{1}{\left(p-1\right)^2}\\
&= \frac{-11}{\left(p-1\right)^2}-\frac{1}{\left(p^2+1\right)}+\frac{5p}{\left(p-1\right)^2}
\end{aligned}$$

Hence , using table, we get inverse laplace transform $\frac{1}{\left(p-1\right)^2} \rightarrow te^t$ and $\frac{1}{\left(p^2+1\right)} \rightarrow \sin t$ and $\frac{p}{\left(p-1\right)^2} \rightarrow e^t+te^t$, hence

$$\begin{aligned}
f(t) &= -11te^t-\sin t+5\left(e^t+te^t\right)\\
&= -11te^t-\sin t+5e^t+5te^t\\
&= 5e^t-6te^t-\sin t
\end{aligned}$$

## 14 chapter 15, problem 3.25

**Problem** Use laplace transform to solve $y''\ +4y'+5y=2e^{-2t}\cos t$, ,$y_0=0, y_0'=3$

**Solution**

Let $\mathscr{L}\left(y(t)\right) = Y(p)$, Taking the laplace transform of both sides, noting first that $\mathscr{L}\left(y''(t)\right) = p^2Y-py_0-y_0'$ , $\mathscr{L}\left(y'(t)\right)=pY-y_0$then we get

$$\begin{aligned}
\mathscr{L}\left(y''(t)\right)+4\mathscr{L}\left(y'(t)\right)+5\mathscr{L}\left(y(t)\right) &= \mathscr{L}\left(2e^{-2t}\cos t\right)\\
\left(p^2Y-py_0-y_0'\right)\ +4\left(pY-y_0\right)+5Y &= 2\mathscr{L}\left(e^{-2t}\cos t\right)
\end{aligned}$$

Applying initial conditions

$$\left(p^2Y\ -3\right)\ +4\left(pY\right)+5Y = 2\mathscr{L}\left(e^{-2t}\cos t\right)$$

From table using L14 $\mathscr{L}\left(e^{-2t}\cos t\right) = \frac{p+2}{\left(p+2\right)^2+1}$. Hence

$$\begin{aligned}
\left(p^2Y-3\right)+4\left(pY\right)+5Y &= 2\frac{p+2}{\left(p+2\right)^2+1}\\
Y\left(p^2+4p+5\right)-3 &= 2\frac{p+2}{\left(p+2\right)^2+1}\\
Y &= \frac{2\frac{p+2}{\left(p+2\right)^2+1}+3}{\left(p^2+4p+5\right)}\\
Y &= 2\frac{p+2}{\left(\left(p+2\right)^2+1\right)\left(p^2+4p+5\right)}+3\frac{1}{\left(p^2+4p+5\right)}
\end{aligned}$$

but $\left(p^2+4p+5\right)=\left(p+2\right)^2+1$, therefore

$$\begin{aligned}
Y &= 2\frac{p+2}{\left(\left(p+2\right)^2+1\right)^2}+\frac{3}{\left(p+2\right)^2+1}\\
&= 2\frac{p+2}{\left(\left(p+2\right)^2+1\right)\left(\left(p+2\right)^2+1\right)}+\frac{3}{\left(p+2\right)^2+1}
\end{aligned}$$

But inverse transform of $\frac{p+2}{\left(\left(p+2\right)^2+1\right)^2}=\frac{1}{2}te^{-2t}\sin t$ and inverse transform of $\frac{1}{\left(p+2\right)^2+1}=e^{-2t}\sin t$. Hence

$$\begin{aligned}
f(t) &= 2\left(\frac{1}{2}te^{-2t}\sin t\right)+3\left(e^{-2t}\sin t\right)\\
&= te^{-2t}\sin t+3e^{-2t}\sin t\\
&= (t+3)e^{-2t}\sin t
\end{aligned}$$

## 15 chapter 15, problem 3.29

**Problem** Use laplace transform to solve $y'+z'-2y=1, y_0=z_0=1, z-y'=t$

**Solution**

Taking laplace transform of both equations, then we get 2 equations in $Y$ and $Z$, then solve for them. Let $\mathscr{L}\left(y(t)\right)=Y(p)$, $\mathscr{L}(z(t))=Z(p)$

$$\begin{aligned}
\mathscr{L}\left(y'(t)\right)+\mathscr{L}(z'(t))-2\mathscr{L}\left(y(t)\right) &= \mathscr{L}(1)\\
\mathscr{L}(z(t))-\mathscr{L}\left(y'(t)\right) &= \mathscr{L}(t)
\end{aligned}$$

Then we get

$$\begin{aligned}
\left(pY-y_0\right)+\left(pZ-z_0\right)-2Y &= \frac{1}{p}\\
Z-\left(pY-y_0\right) &= \frac{1}{p^2}
\end{aligned}$$

Putting initial conditions gives

$$\begin{aligned}
\left(pY-1\right)+\left(pZ-1\right)-2Y &= \frac{1}{p}\\
Z-\left(pY-1\right) &= \frac{1}{p^2}
\end{aligned}$$

$$\begin{aligned} Y\left(p-2\right)+\ pZ-2\ &= \frac{1}{p} &(1)\\ Z\ -pY+1\ &= \frac{1}{p^2} &(2)\end{aligned}$$

Solving for $Y$,From $(1)$, $Z=\frac{\frac{1}{p}+2-Y\left(p-2\right)}{p}$ , and substituting into (2) gives

$$\begin{aligned} \frac{\frac{1}{p}+2-Y\left(p-2\right)}{p}-pY+1\ &=\ \frac{1}{p^2}\\ \frac{1}{p}+2-Y\left(p-2\right)-p^2Y+p &= \frac{1}{p}\\ 2-Y\left(p-2\right)-p^2Y\ &= -p\\ Y\left(-p+2-p^2\right)\ &=\ -p-2\\ Y\ &=\ \frac{-p-2}{\left(-p+2-p^2\right)}\\ Y\ &=\ -\frac{p+2}{\left(-p+1\right)\left(p+2\right)}\\ Y &= -\frac{1}{\left(-p+1\right)}\end{aligned}$$

Hence $Y=\frac{1}{p-1}$ so from L2

$$y(t)=e^t$$

Now, that we have $Y$, we solve for $Z$. From (2)

$$\begin{aligned} Z\ -pY+1\ &=\ \frac{1}{p^2}\\ Z-p\left(\frac{1}{p-1}\right)+1 &= \frac{1}{p^2}\\ Z &= \frac{1}{p^2}-1+\frac{p}{p-1}\\ Z &= \frac{p-1-p^2\left(p-1\right)+p^3}{p^2\left(p-1\right)}\\ Z &= \frac{p-1\ +p^2}{p^2\left(p-1\right)}\\ Z &= \frac{p-1\ +p^2}{p^2\left(p-1\right)}\end{aligned}$$

Doing partial fraction on the above, we get $Z=\frac{1}{p^2}+\frac{1}{p-1}$, Hence

$$z(t)=t+e^t$$

## 16 chapter 15, problem 3.30

**Problem** Use laplace transform to solve $y'\ +2z=1, y_0=0,\ 2y-z'=2t, z_0=1$

**Solution**

Take laplace transform of both equations, then we get 2 equations in $Y$ and $Z$, then solve for them.

Let $\mathscr{L}\big(y(t)\big) = Y(p)$, $\mathscr{L}(z(t)) = Z(p)$

$$\begin{aligned} \mathscr{L}\big(y'(t)\big) + 2Z &= \mathscr{L}(1) \\ 2Y - \mathscr{L}(z'(t)) &= \mathscr{L}(2t) \end{aligned}$$

$$\begin{aligned} pY - y_0 + 2Z &= \mathscr{L}(1) \\ 2Y - \left(pZ - z_0\right) &= \mathscr{L}(2t) \end{aligned}$$

Then we get, by putting $z_0 = 1, y_0 = 0$

$$pY + 2Z = \frac{1}{p} \tag{1}$$

$$2Y - \left(pZ - 1\right) = \frac{2}{p^2} \tag{2}$$

Obtain Z from first equation and sub into the second to solve for $Y$, $Z = \frac{\frac{1}{p} - pY}{2}$, Hence

$$\begin{aligned} 2Y - \left(p\left(\frac{\frac{1}{p} - pY}{2}\right) - 1\right) &= \frac{2}{p^2} \\ 2Y - \frac{p}{2}\left(\frac{1}{p} - pY\right) + 1 &= \frac{2}{p^2} \\ 2Y - \frac{p}{2}\left(\frac{1 - p^2Y}{p}\right) &= \frac{2}{p^2} - 1 \\ 2Y - \frac{1}{2}\left(1 - p^2Y\right) &= \frac{2 - p^2}{p^2} \\ 2Y - \frac{1}{2} + \frac{1}{2}p^2Y &= \frac{2 - p^2}{p^2} \\ Y\left(2 + \frac{p^2}{2}\right) &= \frac{2 - p^2}{p^2} + \frac{1}{2} \\ Y &= \frac{\frac{4 - p^2}{2p^2}}{\left(\frac{4 + p^2}{2}\right)} \\ Y &= \frac{4 - p^2}{p^2\left(4 + p^2\right)} \end{aligned}$$

Hence , using partial fraction gives

$$\begin{aligned} Y &= \frac{4 + p^2}{p^2\left(4 + p^2\right)} \\ &= \frac{Ap + B}{p^2} + \frac{Cp + D}{\left(4 + p^2\right)} \end{aligned}$$

Then

$$\begin{aligned} \left(Ap + B\right)\left(4 + p^2\right) + \left(Cp + D\right)p^2 &= 4 - p^2 \\ 4Ap + Ap^3 + 4B + Bp^2 + Cp^3 + Dp^2 &= 4 - p^2 \\ p^3(A + C) + p^2(B + D) + p(4A) + 4B &= 4 - p^2 \end{aligned}$$

Hence, $4B=4\rightarrow B=1$ and $4A=0\rightarrow A=0$ and $B+D=-1$ and $A+C=0\rightarrow C=0$, therefore $D=-1-B\rightarrow D=-2$. Hence

$$\begin{aligned} Y &= \frac{Ap+B}{p^2}+\frac{Cp+D}{\left(4+p^2\right)} \\ &= \frac{1}{p^2}-\frac{2}{\left(4+p^2\right)} \end{aligned}$$

Using tables for inverse transform gives

$$y(t)=t-\sin 2t$$

Now, to find $z(t)$, subtituting value we found for Y into equation (1) above.

$$\begin{aligned} pY \quad +2Z &= \frac{1}{p} \\ p\left(\frac{1}{p^2}-\frac{2}{\left(4+p^2\right)}\right) +2Z &= \frac{1}{p} \\ \frac{1}{p}-\frac{2p}{\left(4+p^2\right)} \quad +2Z &= \frac{1}{p} \\ Z &= \frac{p}{\left(4+p^2\right)} \\ Z &= \frac{p}{\left(4+p^2\right)} \end{aligned}$$

From tables, using L4

$$z(t)=\cos 2t$$

## 17 chapter 15, problem 3.4

**Problem** Use laplace transform to solve $y''+y=\sin t$ ,$y_0=1, y_0'=0$

**Solution**

Let $\mathscr{L}\left(y(t)\right)=Y(p)$, Take the laplace transform of both sides, noting first that $\mathscr{L}\left(y''(t)\right)=p^2Y-py_0-y_0'$ then we get

$$\begin{aligned} \mathscr{L}\left(y''(t)\right)+\mathscr{L}\left(y(t)\right) &= \mathscr{L}(\sin t) \\ \left(p^2Y-py_0-y_0'\right)+Y &= \frac{1}{p^2+1} \end{aligned}$$

Where I used L3 from table on page 636 which says that $\mathscr{L}(\sin at)=\frac{a}{p^2+a^2}$. Now solving for $Y$, noting that $y_0=1$ and $y_0'=0$ gives

$$\begin{aligned} \left(p^2Y-p\right)+Y &= \frac{1}{p^2+1} \\ Y\left(p^2+1\right) -p &= \frac{1}{p^2+1} \\ Y\left(p^2+1\right) &= \frac{1}{p^2+1}+p \\ Y &= \frac{1}{\left(p^2+1\right)^2}+\frac{p}{\left(p^2+1\right)} \end{aligned} \tag{1}$$

From table using L12, $\frac{p}{(p^2+1)} \rightarrow \cos t$ and using L17, $\frac{1}{(p^2+1)^2} \rightarrow \frac{1}{2}(\sin t - t\cos t)$. Hence, putting these together into (1) gives

$$f(t) = \cos t + \frac{1}{2}(\sin t - t\cos t)$$

This is the particular solution to the ODE.

## 18 chapter 15, problem 3.6

**Problem** Use laplace transform to solve $y'' - 6y' + 9y = te^{3t}$ ,$y_0 = 0, y_0' = 5$

**Solution**

Let $\mathscr{L}\big(y(t)\big) = Y(p)$, Take the laplace transform of both sides, noting first that $\mathscr{L}\big(y''(t)\big) = p^2Y - py_0 - y_0'$ and $\mathscr{L}\big(y'(t)\big) = pY - y_0$ then we get

$$\begin{aligned} \mathscr{L}\big(y''(t)\big) - 6\mathscr{L}\big(y'(t)\big) + 9\mathscr{L}\big(y(t)\big) &= \mathscr{L}\big(t\, e^{3t}\big) \\ \big(p^2Y - py_0 - y_0'\big) - 6\big(pY - y_0\big) + 9Y &= \mathscr{L}\big(t\, e^{3t}\big) \end{aligned}$$

I use L6 from table on page 636 which says that $\mathscr{L}\big(t^k\, e^{-at}\big) = \frac{k!}{(p+a)^{k+1}}$, hence for $k = 1, a = -3$, we get $\mathscr{L}\big(t^k\, e^{-at}\big) = \frac{1}{(p-3)^2}$

$$\big(p^2Y - py_0 - y_0'\big) - 6\big(pY - y_0\big) + 9Y = \frac{1}{(p-3)^2}$$

Applying boundary conditions gives

$$\begin{aligned} \big(p^2Y - 5\big) - 6\big(pY\big) + 9Y &= \frac{1}{(p-3)^2} \\ Y\big(p^2 - 6p + 9\big) - 5 &= \frac{1}{(p-3)^2} \\ Y\big(p^2 - 6p + 9\big) &= \frac{1}{(p-3)^2} + 5 \\ Y &= \frac{1}{\big(p^2 - 6p + 9\big)(p-3)^2} + \frac{5}{\big(p^2 - 6p + 9\big)} \\ Y &= \frac{1}{(p-3)^2(p-3)^2} + \frac{5}{(p-3)^2} \\ Y &= \frac{1}{(p-3)^4} + \frac{5}{(p-3)^2} \end{aligned} \tag{1}$$

Now using table, from L6, $\frac{1}{(p-3)^2}$, let $a = -3, k = 1$ hence $\frac{1}{(p-3)^2} \rightarrow te^{3t}$ And using L6 again, $\frac{1}{(p-3)^4}$, let $k = 3, a = -3$ then $\frac{6}{(p-3)^4} \rightarrow t^3e^{3t}$, therefore (1) becomes

$$\begin{aligned} f(t) &= \frac{1}{6}t^3e^{3t} + 5\, te^{3t} \\ &= e^{3t}\left(\frac{1}{6}t^3 + 5t\right) \end{aligned}$$

## 19 chapter 15, problem 3.8

**Problem** Use laplace transform to solve $y'' + 16y = 8\cos 4t$ ,$y_0 = 0, y'_0 = 0$

**Solution**

Let $\mathscr{L}\big(y(t)\big) = Y(p)$, Taking the laplace transform of both sides, noting first that $\mathscr{L}\big(y''(t)\big) = p^2Y - py_0 - y'_0$ results in

$$\begin{aligned}
\mathscr{L}\big(y''(t)\big) + 16\mathscr{L}\big(y(t)\big) &= \mathscr{L}(8\cos 4t) \\
\left(p^2Y - py_0 - y'_0\right) + 16Y &= 8\mathscr{L}(\cos 4t) \\
Y\left(p^2 + 16\right) &= 8\frac{p}{p^2 + 16}
\end{aligned}$$

I used L6 from table on page 636 : $\mathscr{L}(\cos at) = \frac{p}{p^2+a^2}$

$$Y = 8\frac{p}{\left(p^2 + 16\right)^2}$$

Now looking at L11, which says $\mathscr{L}(t\ \sin at) = \frac{2ap}{\left(p^2+a^2\right)^2}$, hence letting $a = 4$ gives the solution

$$f(t) = t\ \sin 4t$$