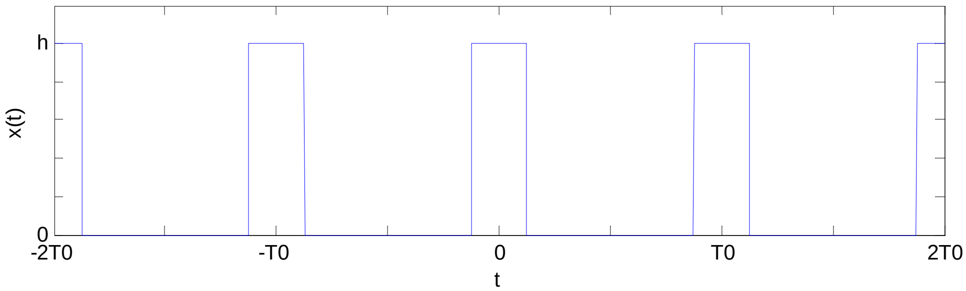
FOURIER SERIES REPRESENTATION OF COMMON SIGNALS

Rectangular Pulse Train



τ = pulse width ($-\tau/2$ to $\tau/2$)
d = duty cycle = τ/T_0.
ω_0 = fundamental frequency = $2\pi/T_0$
sinc(x) = sin(πx)/(πx)

$$X_n = \frac{h\tau}{T_0}\text{sinc}(nd) = h\,d\,\text{sinc}(nd) = \begin{cases} hd, & n = 0 \\ h\dfrac{\sin(n\pi d)}{n\pi}, & n \neq 0 \end{cases}$$

$$x(t) = hd + \sum_{n=1}^{\infty} 2hd\,\text{sinc}(nd)\cos(n\omega_0 t)$$

$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \theta_n)$$

$$c_0 = hd = \frac{h\tau}{T_0}, \quad c_n = |2hd\,\text{sinc}(nd)|, \quad \theta_n = \begin{cases} \pi, & 2hd\,\text{sinc}(nd) < 0 \\ 0, & \textit{otherwise} \end{cases}$$

If $\tau = T_0/2$, d = 1/2, and the equations given above becomes

$$X_n = \frac{h}{2}\text{sinc}\left(\frac{\text{n}}{2}\right) = \begin{cases} \dfrac{h}{2}, & n = 0 \\ h\dfrac{\sin\left(\frac{\text{n}\pi}{2}\right)}{n\pi}, & n \neq 0 \end{cases}$$

$$x(t) = \frac{h}{2} + \sum_{n=1}^{\infty} h\,\text{sinc}\left(\frac{\text{n}}{2}\right)\cos(n\omega_0 t)$$

$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos\left(n\omega_0 t + \theta_n\right)$$

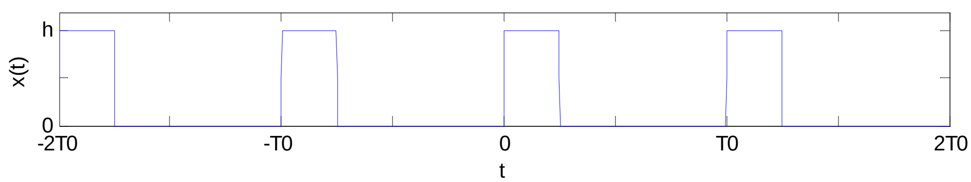
$$c_0 = \frac{h}{2}, \quad c_n = \left|h\,\text{sinc}\left(\frac{\text{n}}{2}\right)\right|, \quad \theta_n = \begin{cases} \pi, & h\,\text{sinc}\left(\frac{\text{n}}{2}\right) < 0 \\ 0, & otherwise \end{cases}$$

Let y(t) = x(t – T_0/2). Then,

$$Y_n = X_n e^{-jn\frac{2\pi}{T_0}\frac{T_0}{2}} = X_n e^{-jn\pi} = X_n \cos\left(n\pi\right) = \begin{cases} \dfrac{h}{2}, & n = 0 \\ h\dfrac{(-1)^{\text{n}} \sin\left(\dfrac{\text{n}\pi}{2}\right)}{n\pi}, & n \neq 0 \end{cases}$$

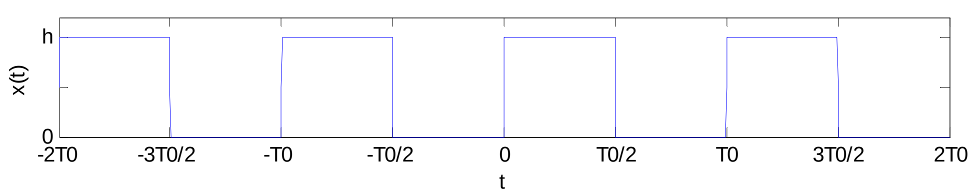
Rectangular Pulse Train with Time Shifting

$t_o = \tau/2$.



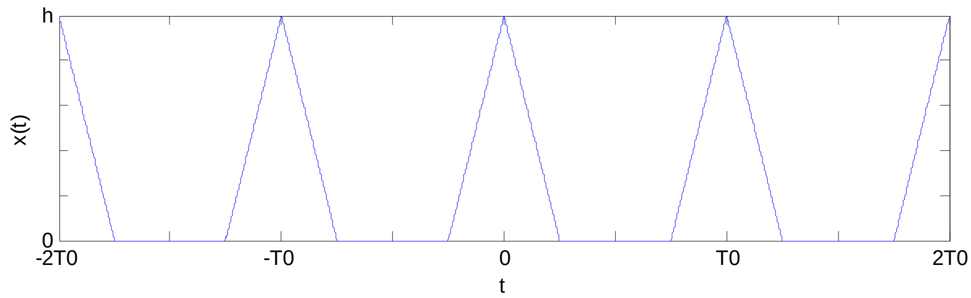
$$X_n = \frac{h\tau}{T_0}\text{sinc}(nd)e^{-jn\frac{2\pi}{T_o}t_o} = h\,d\,\text{sinc}(nd)e^{-jn\frac{2\pi}{T_o}t_o} = h\,d\,\text{sinc}(nd)e^{-jn\frac{2\pi}{T_o}\frac{\tau}{2}} = h\,d\,\text{sinc}(nd)e^{-jn\pi\frac{\tau}{T_o}}$$

If $\tau = T_o/2$, we have



$$X_n = \frac{h}{2}\text{sinc}\left(\frac{n}{2}\right)e^{-jn\pi} = \frac{h}{2}\text{sinc}\left(\frac{n}{2}\right)e^{-jn\pi} = \frac{h}{2}\text{sinc}\left(\frac{n}{2}\right)\cos\left(n\pi\right)$$

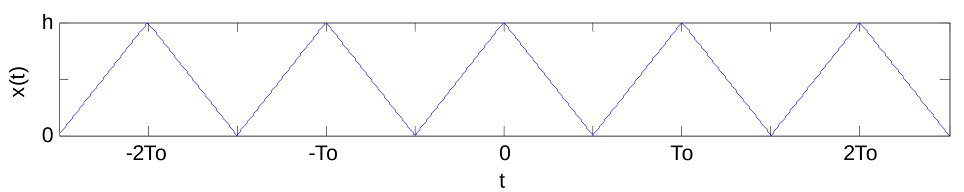
Triangular Pulse Train



τ = half of the base of the triangle ($-\tau \le t \le \tau$)
d = duty cycle = τ/T_0.
ω_0 = fundamental frequency = $2\pi/T_0$

$$X_n = hd\,\text{sinc}^2(nd) = \frac{h\tau}{T_0}\text{sinc}^2\left(\frac{n\omega_0\tau}{2\pi}\right)$$

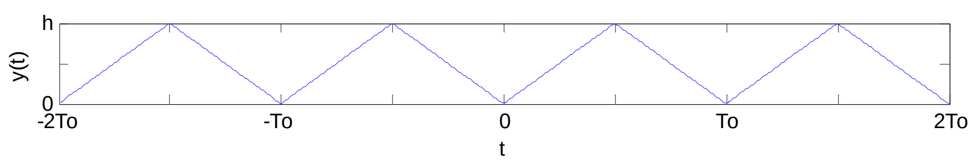
If $\tau = T_O/2$, then the pulse train looks like



and

$$X_n = \frac{h}{2}\text{sinc}^2\left(\frac{n}{2}\right) = \begin{cases} \dfrac{h}{2}, & n = 0 \\ 0, & n = even \\ \dfrac{2h}{n^2\pi^2}, & n = odd \end{cases}$$

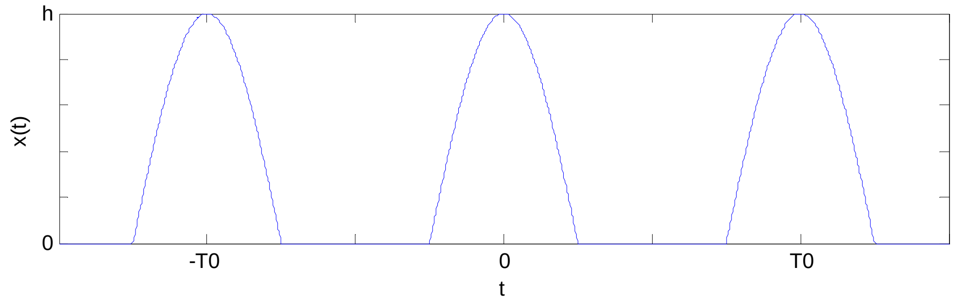
Let y(t) = x(t – T_0/2).



Then,

$$Y_n = X_n e^{-jn\frac{2\pi}{T_0}\frac{T_0}{2}} = X_n e^{-jn\pi} = X_n \cos(n\pi) = \begin{cases} \frac{h}{2}, & n = 0 \\ 0, & n = even \\ \frac{-2h}{n^2\pi^2}, & n = odd \end{cases}$$

Half-Wave Rectified Cosine

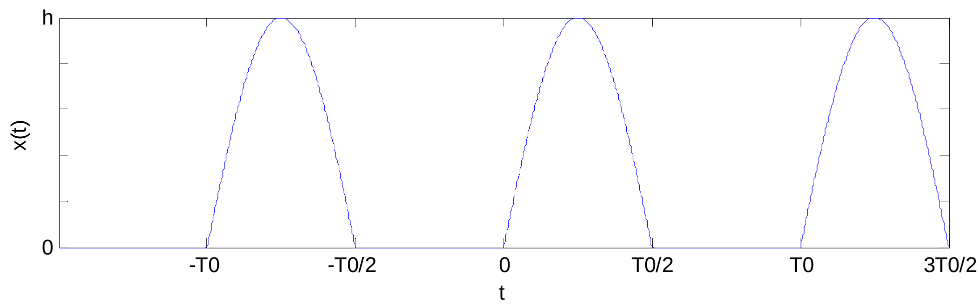


$$X_1 = \frac{h}{4}$$

$$X_{-1} = \frac{h}{4}$$

$$X_n = \frac{h}{\pi} \frac{\cos\left(\frac{n\pi}{2}\right)}{1-n^2}, \quad n \neq \pm 1$$

Half-Wave Rectified Sine



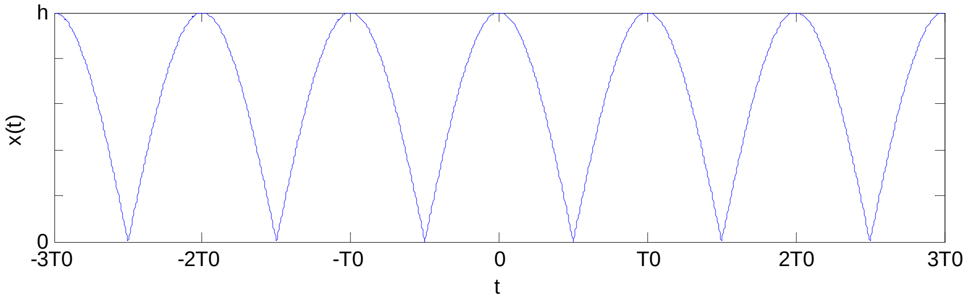
h = amplitude, $j = \sqrt{-1}$

$$X_1 = \frac{-jh}{4}$$

$$X_{-1} = \frac{jh}{4}$$

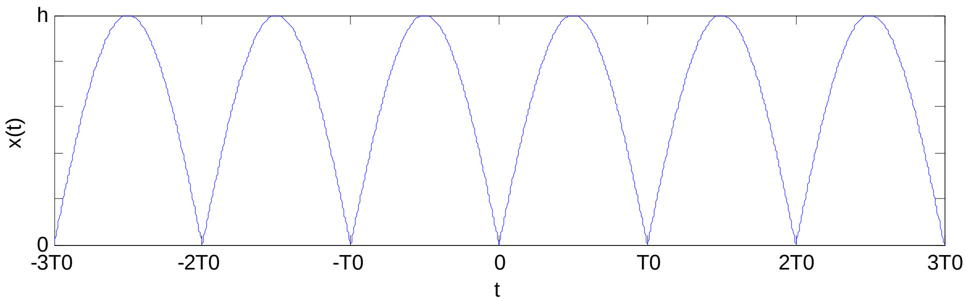
$$X_n = \frac{h}{\pi}\frac{\cos^2\left(\frac{n\pi}{2}\right)}{1-n^2} = \begin{cases} \frac{h}{\pi}, & n=0 \\ 0, & n=\pm 3, \pm 5, \pm 7, ... \\ \frac{h}{\pi}\frac{1}{1-n^2}, & n=\pm 2, \pm 4, \pm 6, ... \end{cases}$$

Full-Wave Rectified Cosine



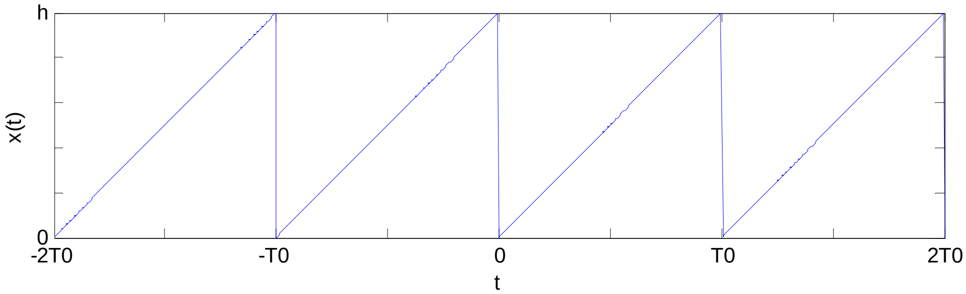
$$X_n = \frac{2h}{\pi}\frac{\cos(n\pi)}{1-4n^2} = \frac{2h}{\pi}\frac{(-1)^n}{1-4n^2}$$

Full-Wave Rectified Sine



$$X_n = \frac{2h}{\pi}\frac{\cos^2(n\pi)}{1-4n^2} = \frac{2h}{\pi}\frac{1}{1-4n^2}$$

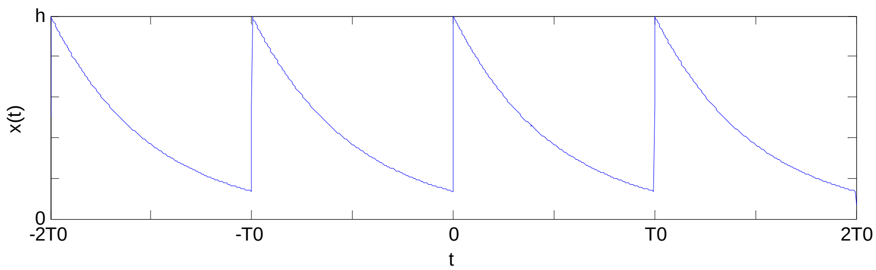
Sawtooth



$$X_0 = \frac{h}{2}$$

$$X_n = \frac{jh}{2\pi n}, \quad n \neq 0, \quad j = \sqrt{-1}$$

Exponential Decay



$$X_n = \frac{h}{T_0} \frac{1 - e^{-aT_0}}{a + jn\dfrac{2\pi}{T_0}}$$