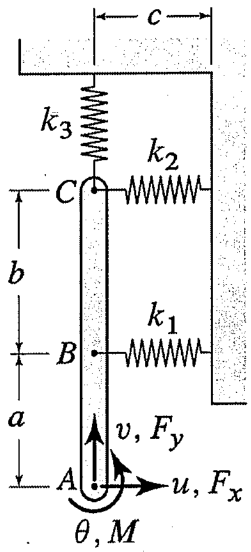
EMA 605 **Sem. I, 2009-10** **Quiz #1** name Solution

Problems 1-3 are equally weighted. Closed book and notes. Show all work. Have a good time.

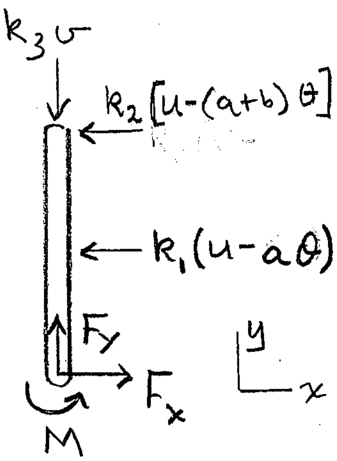
1a) [50%] Bar *ABC* is rigid and weightless. When there are no forces applied, springs k_1 and k_2 are horizontal and spring k_3 and bar *ABC* are vertical. Use the equilibrium approach (show all FBDs) to determine the equilibrium equations **[K]{D}={F}** where **[K]** is symmetric, in terms of d.o.f. u, v, and θ. Assume small displacements. Express your answer in terms of parameters like a, b, c, k_1, k_2, k_3, etc.



Could use unit displ. method, but we will activate all dof simultaneously.

$$\Sigma F_x = 0: \quad F_x - k_1(u - a\theta) - k_2[u - (a+b)\theta] = 0$$

$$k_1(u - a\theta) + k_2[u - (a+b)\theta] = F_x$$



$$\Sigma F_y = 0: \quad F_y - k_3 v = 0$$

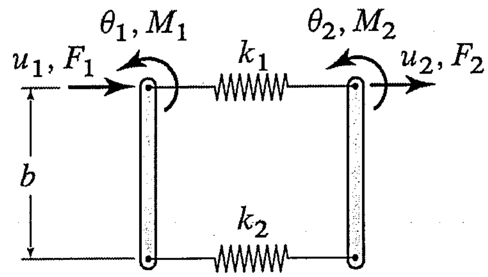
$$k_3 v = F_y$$

$$\Sigma M_A = 0: \quad M + a k_1(u - a\theta) + (a+b)k_2[u - (a+b)\theta] = 0$$

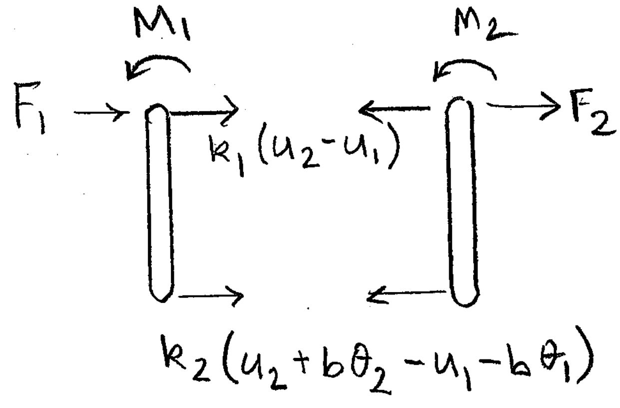
$$-a k_1(u - a\theta) - (a+b)k_2[u - (a+b)\theta] = M$$

$$\begin{bmatrix} k_1 + k_2 & 0 & -ak_1 - (a+b)k_2 \\ 0 & k_3 & 0 \\ -ak_1 - (a+b)k_2 & 0 & a^2 k_1 + (a+b)^2 k_2 \end{bmatrix} \begin{Bmatrix} u \\ v \\ \theta \end{Bmatrix} = \begin{Bmatrix} F_x \\ F_y \\ M \end{Bmatrix}$$

b) [50%] The plane structure shown consists of rigid weightless bars connected by linear springs. Degrees of freedom are horizontal translations u_1 and u_2, and small rotations θ_1 and θ_2. Vertical motion and out-of-plane motion are not allowed. Determine the potential energy for the structure. Using this expression, determine the equilibrium equations [K]{D}={F} where [K] is symmetric, in terms of d.o.f. u_1, θ_1, u_2, and θ_2.



$$\Pi_p = \frac{1}{2} k_1 (u_2 - u_1)^2 + \frac{1}{2} k_2 (u_2 + b\theta_2 - u_1 - b\theta_1)^2 - u_1 F_1 - \theta_1 M_1 - u_2 F_2 - \theta_2 M_2$$



$$\frac{\partial \Pi_p}{\partial u_1} = 0 = k_1(u_2-u_1)(-1) + k_2(u_2+b\theta_2-u_1-b\theta_1)(-1) - F_1$$

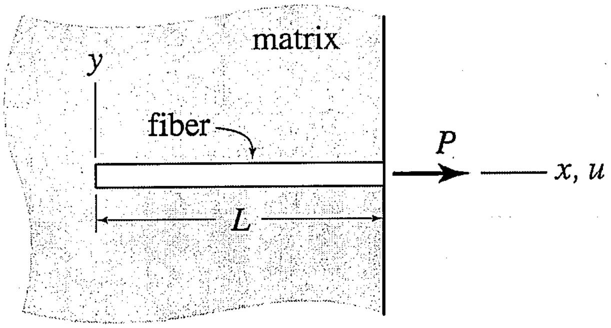
$$\frac{\partial \Pi_p}{\partial \theta_1} = 0 = k_2(u_2+b\theta_2-u_1-b\theta_1)(-b) - M_1$$

$$\frac{\partial \Pi_p}{\partial u_2} = 0 = k_1(u_2-u_1)(1) + k_2(u_2+b\theta_2-u_1-b\theta_1)(1) - F_2$$

$$\frac{\partial \Pi_p}{\partial \theta_2} = 0 = k_2(u_2+b\theta_2-u_1-b\theta_1)(b) - M_2$$

$$\begin{bmatrix} k_1+k_2 & bk_2 & -k_1-k_2 & -bk_2 \\ bk_2 & b^2k_2 & -bk_2 & -b^2k_2 \\ -k_1-k_2 & -bk_2 & k_1+k_2 & bk_2 \\ -bk_2 & -b^2k_2 & bk_2 & b^2k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix}$$

2a) [90%] An elastic fiber with length L, uniform elastic modulus E and cross sectional area A is embedded in a deformable matrix. To model the effect of the matrix on the fiber, we assume the matrix applies an axial force to the fiber that is proportional to the displacement of the fiber, and is in the direction opposite the displacement of the fiber. The right-hand end of the fiber is subjected to a force P, and we assume the fiber is sufficiently long so that the left-hand end of the fiber has zero displacement. The potential energy of the fiber is



$$\pi_p = \int_0^L \left(\tfrac{1}{2} AE\, u_{,x}^2 + \tfrac{1}{2}\gamma u^2\right) dx - \{D\}^T\{F\}$$

where γ is the stiffness of the fiber-matrix interface per unit length. Use the Rayleigh-Ritz method with a one d.o.f. trial function (approximate solution) to determine an approximate deflection field $u(x)$. Express your answer in terms of parameters like A, E, γ, P, etc.

$u = a_0 + a_1 x + \dots$

Kinematic admissibility requires

$u(0) = 0 \rightarrow a_0 = 0$

Thus, a 1 dof trial function is $u = a_1 x$.

$u = a_1 x$

$u_{,x} = a_1$

$$\pi_p = \int_0^L \left(\frac{AE}{2} a_1^2 + \frac{\gamma}{2} a_1^2 x^2\right) dx - P(a_1 L)$$

$$= \frac{AEL}{2} a_1^2 + \frac{\gamma}{2} a_1^2 \frac{L^3}{3} - P a_1 L$$

$$\frac{\partial \pi_p}{\partial a_1} = 0 = AEL\, a_1 + \frac{\gamma L^3}{6} 2 a_1 - PL$$

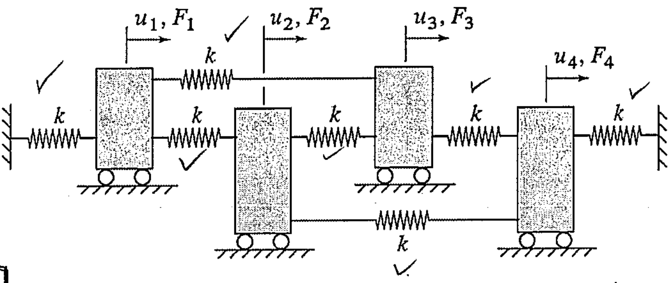
Divide by L, solve for a_1 to obtain

$$a_1 = \frac{P}{AE + \gamma L^2/3} \quad \rightarrow \quad \boxed{u = \frac{P}{AE + \gamma L^2/3}\, x}$$

b) [10%] How can you determine if the solution obtained in part (a) is exact or approximate?

If the soln. from part (a) satisfies the governing diff. eqn, plus all BCs (both essential and natural BCs), then it is the exact soln.

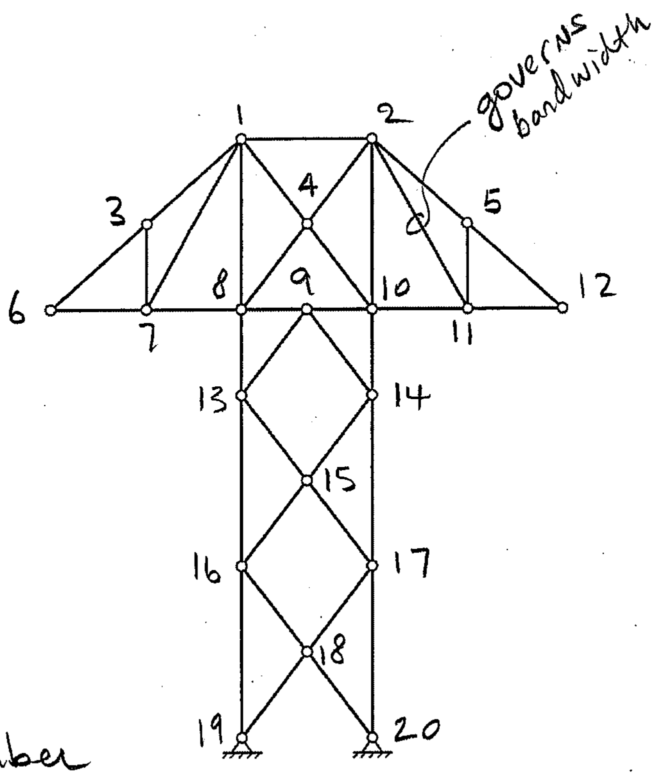
3) [30%] The four blocks can undergo horizontal motion only. All springs have the same stiffness k. By inspection, determine the stiffness matrix. If you are not capable of using inspection, then you may use the method of your choice.



$$\underset{\sim}{K} = k \begin{bmatrix} 1+1+1 & -1 & -1 & \\ -1 & 1+1+1 & -1 & -1 \\ -1 & -1 & 1+1+1 & -1 \\ & -1 & -1 & 1+1+1 \end{bmatrix} = k \begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 3 \end{bmatrix}$$

b) [30%] The transmission tower shown has 20 nodes, 37 elements, and is to be modeled using two-dimensional bar finite elements having two d.o.f. per node. Number the nodes so that the semi-bandwidth of the stiffness matrix is minimal, and determine the semi-bandwidth.

NOTE: your node numbering scheme does not need to provide the minimum bandwidth possible, but it should be reasonably close.



Strategy: number nodes across the thin dimension of the structure.

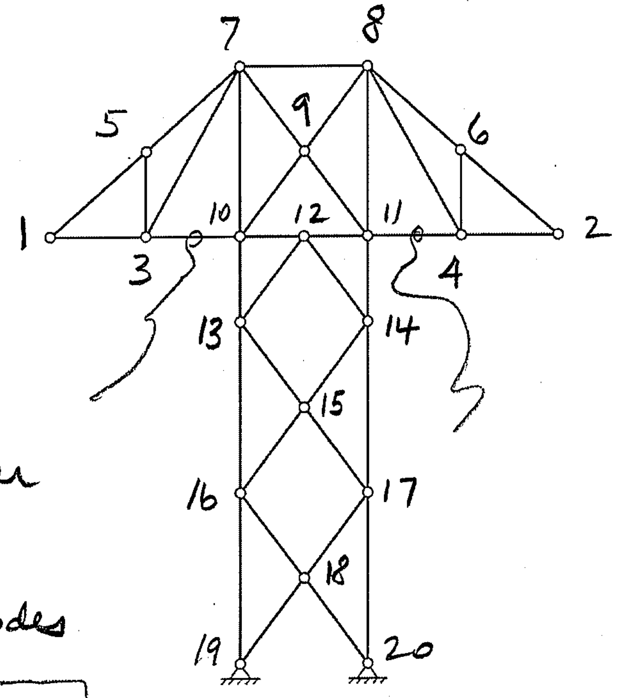
Bandwidth: the maximum node number difference in any element is $11-2=9$ nodes.

Semi bandwidth $= b = (11-2+1)2 = \boxed{20 \text{ eqns}}$

2 dof/node

Can still be improved.

2nd try

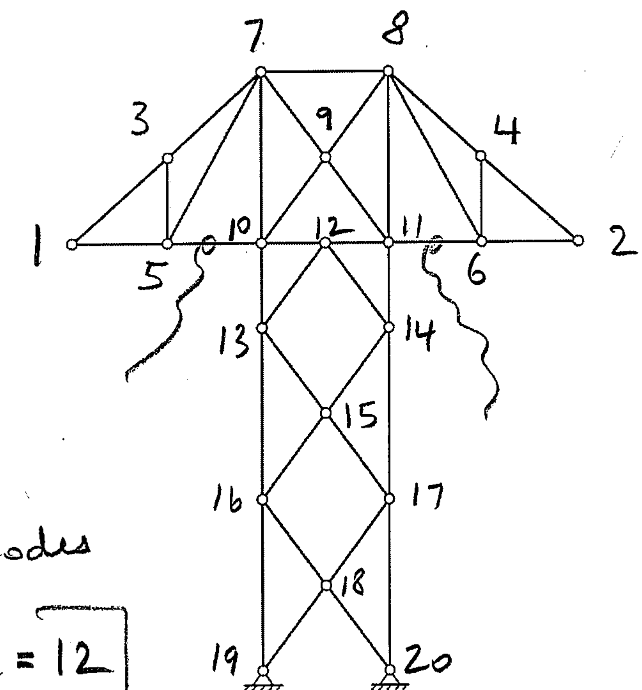


max node number difference

$= 10 - 3 = 7$ nodes

$b = (10-3+1)2 = \boxed{16}$

3rd try

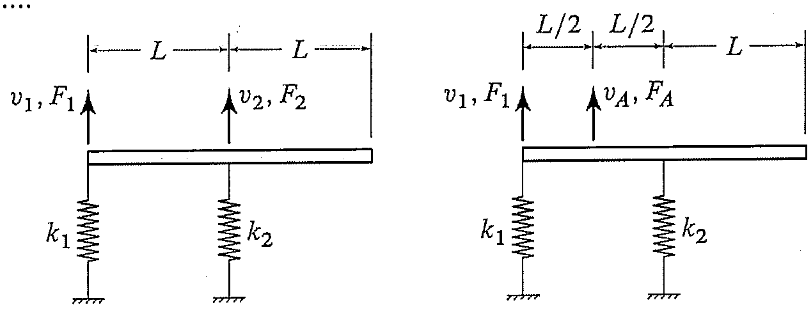


max node number diff

$= 10 - 5 = 5$ nodes

$b = (10-5+1)2 = \boxed{12}$

c) [40%] Recall the home work problem shown below. In part (a) of this HW problem, the d.o.f. are v_1 and v_2, and the equilibrium equations are given. In part (b) the d.o.f. are v_1 and v_A. Use transformation concepts to obtain the stiffness matrix for part (b) using the results of part (a). Hint: Let the equilibrium equations in part (a) be denoted by [**k**']{**d**'}={**f**'}. Determine the transformation matrix in the expression {**d**'}=[**T**]{**d**} where {**d**} are the d.o.f. for part (b). Beginning with [**k**']{**d**'}={**f**'}, substitute for {**d**'}, and then premultiply the result by $[\mathbf{T}]^T$, ...



(a) (b)

$$\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\underset{\sim}{k}' \underset{\sim}{d}' = \underset{\sim}{f}'$$

$$\underset{\sim}{k}\,\underset{\sim}{d} = \underset{\sim}{f}$$

$$\underset{\sim}{k} \begin{Bmatrix} v_1 \\ v_A \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_A \end{Bmatrix}.$$

$$\underset{\sim}{d}' = \underset{\sim}{T}\,\underset{\sim}{d}$$

by inspection, $v_A = \dfrac{v_1 + v_2}{2}$

$\rightsquigarrow v_2 = 2v_A - v_1$

$$\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_A \end{Bmatrix}$$

$$\underset{\sim}{k}' \underset{\sim}{d}' = \underset{\sim}{f}'$$

$$\underset{\sim}{k}' \underset{\sim}{T}\,\underset{\sim}{d} = \underset{\sim}{f}'$$

$$\underset{\sim}{T}^T \underset{\sim}{k}' \underset{\sim}{T}\,\underset{\sim}{d} = \underset{\sim}{T}^T \underset{\sim}{f}'$$

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}}_{\begin{bmatrix} k_1 + k_2 & -2k_2 \\ -2k_2 & 4k_2 \end{bmatrix}} \begin{Bmatrix} v_1 \\ v_A \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_A \end{Bmatrix}$$