HW 11, Math 322, Fall 2016

Nasser M. Abbasi

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Math 322 Homework 11

Due Wednesday Dec. 14, 2016

1. (a) Use the method of images to solve

$$\nabla^2 u(\mathbf{x}) = f(\mathbf{x}) \tag{1a}$$

in a semi-infinite 2D domain with boundary condition $u(x,0) = h(x)$.
(b) Use the method of images to solve

$$\nabla^2 u(\mathbf{x}) = f(\mathbf{x}) \tag{1b}$$

in a semi-infinite 2D domain with boundary condition $\partial u(x,0)/\partial y = h(x)$.
(c) (a) Use the method of images to solve

$$\nabla^2 u(\mathbf{x}) = f(\mathbf{x}) \tag{1c}$$

in a semi-infinite 3D domain with boundary condition $\partial u(x,0,z)/\partial y = h(x,z)$.

2. Using the method of images, solve

$$\nabla^2 u(\mathbf{x}) = f(\mathbf{x}) \tag{2}$$

in the 2D domain $x \geq 0$, $y \geq 0$ with boundary conditions $u(0,y) = g(y)$ and $u(x,0) = h(x)$.

The following are the general steps used in all the problems below :

1. Image points were placed to satisfy homogenous boundary conditions for Green function using the solution for infinite domain.
2. The Green formula was applied to determine the particular solution and the boundary terms.
3. Derivative of Green function was found and used in the result found above.
4. The role of $\vec{x}_0, \vec{x}$ was reversed in the final expression to express the final result as $u\left(\vec{x}\right)$ instead of $u\left(\vec{x}_0\right)$.

1.1 Problem 1

1.1.1 Part (a)

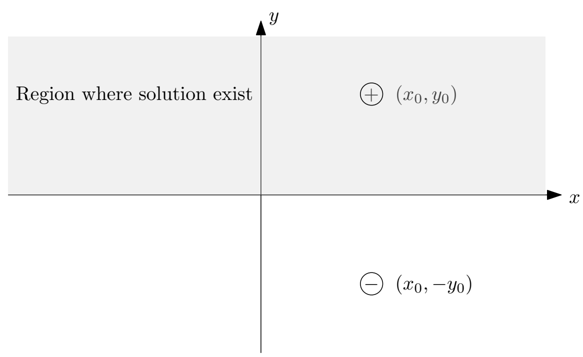
Green function on infinite domain, which is the solution to

$$\nabla^2 G\left(\vec{x}, \vec{x}_0\right) = \delta\left(\vec{x} - \vec{x}_0\right)$$

Is given by

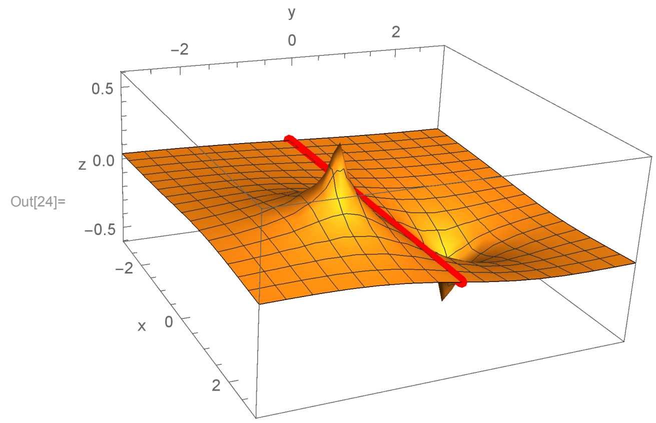
$$
\begin{aligned}
G_{\infty}\left(\vec{x},\vec{x}_0\right) &= \frac{1}{2\pi}\ln\left(r\right)\\
&= \frac{1}{2\pi}\ln\left(\left|\vec{x}-\vec{x}_0\right|\right)\\
&= \frac{1}{2\pi}\ln\left(\sqrt{\left(x-x_0\right)^2+\left(y-y_0\right)^2}\right)\\
&= \frac{1}{4\pi}\ln\left(\left(x-x_0\right)^2+\left(y-y_0\right)^2\right)
\end{aligned}
$$

By placing a negative impulse at location $\vec{x}_0^{*}=\left(x_0,-y_0\right)$, the Green function for semi-infinite domain is obtained



$$
\begin{aligned}
G\left(\vec{x},\vec{x}_0\right) &= \frac{1}{2\pi}\ln\left(r_1\right)-\frac{1}{2\pi}\ln\left(r_2\right)\\
&= \frac{1}{2\pi}\ln\left(\left|\vec{x}-\vec{x}_0\right|\right)-\frac{1}{2\pi}\ln\left(\left|\vec{x}-\vec{x}_0^{*}\right|\right)\\
&= \frac{1}{4\pi}\left(\ln\left(\left(x-x_0\right)^2+\left(y-y_0\right)^2\right)-\ln\left(\left(x-x_0\right)^2+\left(y+y_0\right)^2\right)\right)\\
&= \frac{1}{4\pi}\ln\frac{\left(x-x_0\right)^2+\left(y-y_0\right)^2}{\left(x-x_0\right)^2+\left(y+y_0\right)^2} \qquad (1)
\end{aligned}
$$

The following is 3D plot of the above Green function, showing the image impulse and showing that $G=0$ at the line $y=0$ (marked as red)



The Green function in (1) is now used to solve $\nabla^2 u(\vec{x}) = f(\vec{x})$, with $u(x,0) = h(x)$. Starting with Green formula for 2D

$$
\begin{aligned}
\iint u(\vec{x})\nabla^2 G(\vec{x},\vec{x}_0) - G(\vec{x},\vec{x}_0)\nabla^2 u(\vec{x})\, dA &= \oint \left(u(\vec{x})\nabla G(\vec{x},\vec{x}_0) - G(\vec{x},\vec{x}_0)\nabla u(\vec{x})\right)\cdot \hat{n}\ ds \\
&= \oint \left(u(\vec{x})\nabla G(\vec{x},\vec{x}_0) - G(\vec{x},\vec{x}_0)\nabla u(\vec{x})\right)\cdot \left(-\hat{j}\right)\ ds \\
&= \oint \left(G(\vec{x},\vec{x}_0)\nabla u(\vec{x}) - u(\vec{x})\nabla G(\vec{x},\vec{x}_0)\right)\cdot \hat{j}\ ds \\
&= \int_{-\infty}^{\infty} \left(G(\vec{x},\vec{x}_0)\frac{\partial u(\vec{x})}{\partial y} - u(\vec{x})\frac{\partial G(\vec{x},\vec{x}_0)}{\partial y}\right)_{y=0} dx
\end{aligned}
$$

But $\nabla^2 G(\vec{x},\vec{x}_0) = \delta(\vec{x},\vec{x}_0)$ and $\nabla^2 u(\vec{x}) = f(\vec{x})$, therefore the above becomes

$$
\iint u(\vec{x})\,\delta(\vec{x},\vec{x}_0)\, dA - \iint G(\vec{x},\vec{x}_0) f(\vec{x})\, dA = \int_{-\infty}^{\infty} \left(G(\vec{x},\vec{x}_0)\frac{\partial u(\vec{x})}{\partial y} - u(\vec{x})\frac{\partial G(\vec{x},\vec{x}_0)}{\partial y}\right)_{y=0} dx
$$

Since $\int\int u(\vec{x})\,\delta(\vec{x},\vec{x}_0)\, dA = u(\vec{x}_0)$, the above reduces to

$$
u(\vec{x}_0) - \iint G(\vec{x},\vec{x}_0) f(\vec{x})\, dA = \int_{-\infty}^{\infty} \left(G(\vec{x},\vec{x}_0)\frac{\partial u(\vec{x})}{\partial y} - u(\vec{x})\frac{G(\vec{x},\vec{x}_0)}{dy}\right)_{y=0} dx
$$

$$
u(\vec{x}_0) = \iint G(\vec{x},\vec{x}_0) f(\vec{x})\ dA + \int_{-\infty}^{\infty} \left(G(\vec{x},\vec{x}_0)\frac{\partial u(\vec{x})}{\partial y} - u(\vec{x})\frac{\partial G(\vec{x},\vec{x}_0)}{\partial y}\right)_{y=0} dx \quad (2)
$$

And since $G(\vec{x},\vec{x}_0) = 0$ at $y = 0$, therefore

$$
u(\vec{x}_0) = \iint G(\vec{x},\vec{x}_0) f(\vec{x})\ dA + \int_{-\infty}^{\infty} \left(-u(\vec{x})\frac{\partial G(\vec{x},\vec{x}_0)}{\partial y}\right)_{y=0} dx
$$

And since $u\left(\vec{x}\right)=h\left(x\right)$ at $y=0$, then

$$u\left(\vec{x}_0\right)=\iint G\left(\vec{x},\vec{x}_0\right)f\left(\vec{x}\right)\,dA-\int_{-\infty}^{\infty}h\left(x\right)\left(\frac{\partial G\left(\vec{x},\vec{x}_0\right)}{\partial y}\right)_{y=0}dx \tag{3}$$

$\left(\frac{dG\left(\vec{x},\vec{x}_0\right)}{dy}\right)_{y=0}$ is now evaluated to complete the solution. Using $G\left(\vec{x},\vec{x}_0\right)$ in equation (1), therefore

$$\begin{aligned}\frac{dG\left(\vec{x},\vec{x}_0\right)}{dy}&=\frac{1}{4\pi}\frac{d}{dy}\left(\ln\left((x-x_0)^2+\left(y-y_0\right)^2\right)-\ln\left((x-x_0)^2+\left(y+y_0\right)^2\right)\right)\\&=\frac{1}{4\pi}\left(\frac{2\left(y-y_0\right)}{(x-x_0)^2+\left(y-y_0\right)^2}-\frac{2\left(y+y_0\right)}{(x-x_0)^2+\left(y+y_0\right)^2}\right)\end{aligned}$$

Evaluating the above at $y=0$ gives

$$\begin{aligned}\left(\frac{dG\left(\vec{x},\vec{x}_0\right)}{dy}\right)_{y=0}&=\frac{1}{4\pi}\left(\frac{-2y_0}{(x-x_0)^2+y_0^2}-\frac{2y_0}{(x-x_0)^2+y_0^2}\right)\\&=\frac{-1}{\pi}\left(\frac{y_0}{(x-x_0)^2+y_0^2}\right)\end{aligned}$$

Replacing the above into (3) gives

$$u\left(\vec{x}_0\right)=\iint G\left(\vec{x},\vec{x}_0\right)f\left(\vec{x}\right)dA+\frac{y_0}{\pi}\int_{-\infty}^{\infty}\frac{h\left(x\right)}{(x-x_0)^2+y_0^2}\,dx$$

Using the expression for $G\left(\vec{x},\vec{x}_0\right)$ from (1), the above result becomes

$$u\left(x_0,y_0\right)=\frac{1}{4\pi}\int_{x=-\infty}^{\infty}\int_{y=0}^{\infty}\ln\frac{(x-x_0)^2+\left(y-y_0\right)^2}{(x-x_0)^2+\left(y+y_0\right)^2}f\left(x,y\right)dydx+\frac{y_0}{\pi}\int_{-\infty}^{\infty}\frac{h\left(x\right)}{(x-x_0)^2+y_0^2}\,dx$$

And finally, order of $\vec{x},\vec{x}_0$ is reversed giving

$$u\left(x,y\right)=\frac{1}{4\pi}\int_{x_0=-\infty}^{\infty}\int_{y_0=0}^{\infty}\ln\frac{(x_0-x)^2+\left(y_0-y\right)^2}{(x_0-x)^2+\left(y_0+y\right)^2}f\left(x_0,y_0\right)dy_0dx_0+\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{h\left(x_0\right)}{(x_0-x)^2+y^2}\,dx_0$$

1.1.2 Part (b)

This is similar to part (a), and the image is placed on the same location as shown above, but now the boundary conditions are different. Starting from equation (2) in part (a)

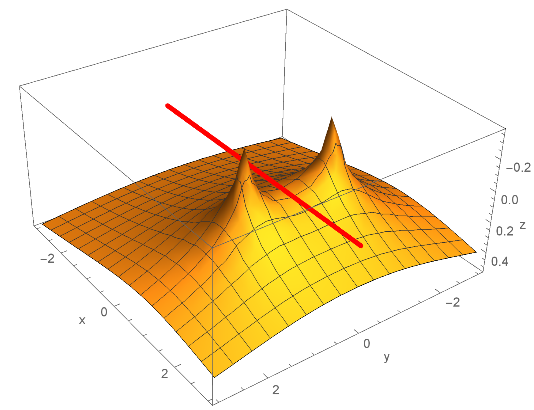
$$u\left(\vec{x}_0\right)=\iint G\left(\vec{x},\vec{x}_0\right)f\left(\vec{x}\right)dA+\int_{-\infty}^{\infty}\left(G\left(\vec{x},\vec{x}_0\right)\frac{\partial u\left(\vec{x}\right)}{\partial y}-u\left(\vec{x}\right)\frac{\partial G\left(\vec{x},\vec{x}_0\right)}{\partial y}\right)_{y=0}dx \tag{1}$$

But now $\frac{\partial G(\vec{x},\vec{x}_0)}{\partial y} = 0$ at $y = 0$ and not $G(\vec{x},\vec{x}_0) = 0$ as in part (a). This means the image is a positive impulse and not negative as in part (a). Therefore $G(\vec{x},\vec{x}_0)$ becomes the following

$$\begin{aligned} G(\vec{x},\vec{x}_0) &= \frac{1}{2\pi}\ln(r_1) + \frac{1}{2\pi}\ln(r_2) \\ &= \frac{1}{2\pi}\ln\left(\left|\vec{x}-\vec{x}_0\right|\right) + \frac{1}{2\pi}\ln\left(\left|\vec{x}-\vec{x}_0^*\right|\right) \\ &= \frac{1}{4\pi}\left(\ln\left((x-x_0)^2+(y-y_0)^2\right) + \ln\left((x-x_0)^2+(y+y_0)^2\right)\right) \end{aligned} \tag{2}$$

The following is 3D plot of the above Green function, showing that showing that $\frac{\partial G(\vec{x},\vec{x}_0)}{\partial y} = 0$ at $y = 0$ (marked as red)

Out[3]=



Green function, semi-infinite 2D, x0 at (1,1) and image at (1,-1) part(b)

Since $\frac{\partial G(\vec{x},\vec{x}_0)}{\partial y} = 0$ at $y = 0$ then (1) becomes

$$u(\vec{x}_0) = \iint G(\vec{x},\vec{x}_0) f(\vec{x})\, dA + \int_{-\infty}^{\infty} \left(G(\vec{x},\vec{x}_0)\frac{\partial u(\vec{x})}{\partial y}\right)_{y=0} dx$$

But $\frac{\partial u(\vec{x})}{\partial y} = h(x)$ at $y = 0$, hence the above reduces to

$$u(\vec{x}_0) = \iint G(\vec{x},\vec{x}_0) f(\vec{x})\, dA + \int_{-\infty}^{\infty} G(\vec{x},\vec{x}_0)_{y=0}\, h(x)\; dx \tag{3}$$

Evaluating $G(\vec{x},\vec{x}_0)$ at $y = 0$ gives

$$\begin{aligned} G(\vec{x},\vec{x}_0)_{y_0=0} &= \frac{1}{4\pi}\left(\ln\left((x-x_0)^2+(y-y_0)^2\right) + \ln\left((x-x_0)^2+(y+y_0)^2\right)\right)_{y=0} \\ &= \frac{1}{4\pi}\left(\ln\left((x-x_0)^2+y_0^2\right) + \ln\left((x-x_0)^2+y_0^2\right)\right) \\ &= \frac{1}{4\pi}\ln\left(\left((x-x_0)^2+y_0^2\right)^2\right) \\ &= \frac{1}{2\pi}\ln\left((x-x_0)^2+y_0^2\right) \end{aligned}$$

Substituting the above in RHS of (3) gives

$$u\left(\vec{x}_0\right) = \int_{x=-\infty}^{\infty}\int_{y=0}^{\infty} G\left(\vec{x},\vec{x}_0\right) f\left(x,y\right) dydx + \frac{1}{2\pi}\int_{-\infty}^{\infty} \ln\left((x-x_0)^2 + y_0^2\right) h\left(x\right)\ dx$$

Reversing the role of $\vec{x}_0, \vec{x}$ gives

$$u\left(\vec{x}\right) = \int_{x_0=-\infty}^{\infty}\int_{y_0=0}^{\infty} G\left(\vec{x},\vec{x}_0\right) f\left(x_0,y_0\right) dy_0 dx_0 + \frac{1}{2\pi}\int_{-\infty}^{\infty} \ln\left((x_0-x)^2 + y^2\right) h\left(x_0\right)\ dx_0$$

1.1.3 Part (c)

In infinite 3D domain, the Green function for Poisson PDE is given by

$$G\left(\vec{x},\vec{x}_0\right) = \frac{-1}{4\pi r}$$

Where r is given by

$$r = \sqrt{(x-x_0)^2 + \left(y-y_0\right)^2 + (z-z_0)^2}$$

$\vec{x}_0 = \left(x_0, y_0, z_0\right)$ is the location of the impulse. Since $\frac{\partial G}{\partial y} = 0$, then the same sign impulse is located at $x_0^* = \left(x_0, -y_0, z_0\right)$, and the Green function becomes

$$\begin{aligned} G\left(\vec{x},\vec{x}_0\right) &= \frac{-1}{4\pi r_0} - \frac{1}{4\pi r_0^*} \\ &= \frac{1}{4\pi}\left(-\frac{1}{\sqrt{(x-x_0)^2 + \left(y-y_0\right)^2 + (z-z_0)^2}} - \frac{1}{\sqrt{(x-x_0)^2 + \left(y+y_0\right)^2 + (z-z_0)^2}}\right) \end{aligned} \tag{1}$$

Using Green formula in 3D gives

$$\begin{aligned} \iiint u\left(\vec{x}\right)\nabla^2 G\left(\vec{x},\vec{x}_0\right) - G\left(\vec{x},\vec{x}_0\right)\nabla^2 u\left(\vec{x}\right) dV &= \iint \left(u\left(\vec{x}\right)\nabla G\left(\vec{x},\vec{x}_0\right) - G\left(\vec{x},\vec{x}_0\right)\nabla u\left(\vec{x}\right)\right)\cdot \hat{n}\ dxdz \\ &= \iint \left(u\left(\vec{x}\right)\nabla G\left(\vec{x},\vec{x}_0\right) - G\left(\vec{x},\vec{x}_0\right)\nabla u\left(\vec{x}\right)\right)\cdot \left(-\hat{j}\right)\ dxdz \\ &= \iint \left(G\left(\vec{x},\vec{x}_0\right)\nabla u\left(\vec{x}\right) - u\left(\vec{x}\right)\nabla G\left(\vec{x},\vec{x}_0\right)\right)\cdot \hat{j}\ dxdz \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(G\left(\vec{x},\vec{x}_0\right)\frac{\partial u\left(x,y,z\right)}{\partial y} - u\left(\vec{x}\right)\frac{\partial G\left(x,y,z,\vec{x}_0\right)}{\partial y}\right)_{y=0} dxdz \end{aligned}$$

But $\nabla^2 G\left(\vec{x},\vec{x}_0\right) = \delta\left(\vec{x},\vec{x}_0\right)$ and $\nabla^2 u\left(\vec{x}\right) = f\left(\vec{x}\right)$, and the above becomes

$$\iiint u\left(\vec{x}\right)\delta\left(\vec{x},\vec{x}_0\right) dV - \iiint G\left(\vec{x},\vec{x}_0\right) f\left(\vec{x}\right) dv = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(G\left(\vec{x},\vec{x}_0\right)\frac{\partial u\left(\vec{x}\right)}{\partial y} - u\left(\vec{x}\right)\frac{\partial G\left(\vec{x},\vec{x}_0\right)}{\partial y}\right)_{y=0} dxdz$$

But $\iiint u\left(\vec{x}\right)\delta\left(\vec{x},\vec{x}_0\right) dV = u\left(\vec{x}_0\right)$, hence

$$u\left(\vec{x}_0\right) - \iiint G\left(\vec{x},\vec{x}_0\right) f\left(\vec{x}\right) dV = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(G\left(\vec{x},\vec{x}_0\right)\frac{\partial u\left(\vec{x}\right)}{\partial y} - u\left(\vec{x}\right)\frac{\partial G\left(\vec{x},\vec{x}_0\right)}{\partial y}\right)_{y=0} dxdz$$

Rearranging

$$u\left(\vec{x}_0\right) = \iiint G\left(\vec{x},\vec{x}_0\right) f\left(\vec{x}\right) dV + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(G\left(\vec{x},\vec{x}_0\right) \frac{\partial u\left(\vec{x}\right)}{\partial y} - u\left(\vec{x}\right) \frac{\partial G\left(\vec{x},\vec{x}_0\right)}{\partial y}\right)_{y=0} dxdz \tag{2}$$

But $\left(\frac{\partial u(\vec{x})}{\partial y}\right)_{y=0} = h\left(x,z\right)$ and we impose $\left(\frac{\partial G(\vec{x},\vec{x}_0)}{\partial y}\right)_{y=0} = 0$, therefore the above becomes

$$u\left(\vec{x}_0\right) = \iiint G\left(\vec{x},\vec{x}_0\right) f\left(\vec{x}\right) dV + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G\left(\vec{x},\vec{x}_0\right)_{y=0} h\left(x,z\right)\ dxdz \tag{3}$$

Evaluating $G\left(\vec{x},\vec{x}_0\right)_{y=0}$ gives

$$\begin{aligned} G\left(\vec{x},\vec{x}_0\right)_{y=0} &= \frac{1}{4\pi}\left(-\frac{1}{\sqrt{\left(x-x_0\right)^2+\left(y-y_0\right)^2+\left(z-z_0\right)^2}} - \frac{1}{\sqrt{\left(x-x_0\right)^2+\left(y+y_0\right)^2+\left(z-z_0\right)^2}}\right)_{y=0} \\ &= \frac{1}{4\pi}\left(\frac{-1}{\sqrt{\left(x-x_0\right)^2+y_0^2+\left(z-z_0\right)^2}} - \frac{1}{\sqrt{\left(x-x_0\right)^2+y_0^2+\left(z-z_0\right)^2}}\right) \\ &= -\frac{1}{2\pi\sqrt{\left(x-x_0\right)^2+y_0^2+\left(z-z_0\right)^2}} \end{aligned}$$

Using the above in (3) results in

$$u\left(\vec{x}_0\right) = \iiint G\left(\vec{x},\vec{x}_0\right) f\left(\vec{x}\right) dV - \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{\left(x-x_0\right)^2+y_0^2+\left(z-z_0\right)^2}} h\left(x,z\right)\ dxdz$$

And finally reversing the role of $\vec{x}_0,\vec{x}$ gives the final answer

$$u\left(\vec{x}\right) = \iiint G\left(\vec{x},\vec{x}_0\right) f\left(\vec{x}_0\right) dV_0 - \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{\left(x_0-x\right)^2+y^2+\left(z_0-z\right)^2}} h\left(x_0,z_0\right)\ dx_0dz_0$$

1.2 Problem 2

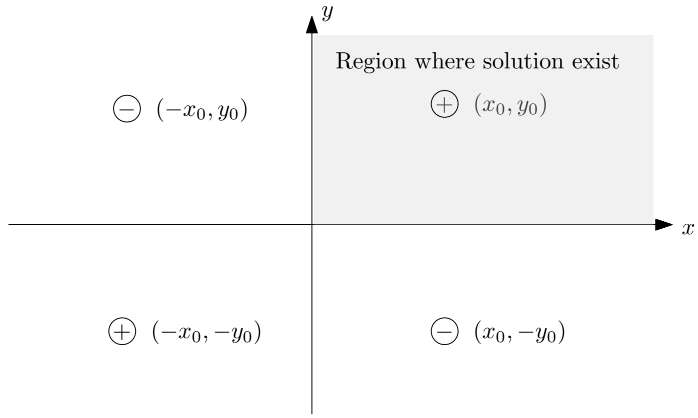
Green function in 2D on infinite domain, which is the solution to

$$\nabla^2 G\left(\vec{x},\vec{x}_0\right) = \delta\left(\vec{x}-\vec{x}_0\right)$$

Is given by

$$G_\infty\left(\vec{x},\vec{x}_0\right) = \frac{1}{2\pi}\ln\left(r\right)$$

A negative impulse is placed $\vec{x}_1 = \left(x_0,-y_0\right)$ and another negative impulse at $\vec{x}_2 = \left(-x_0,y_0\right)$ and positive one at $\vec{x}_3 = \left(-x_0,-y_0\right)$. The following is a diagram showing the placement of images.



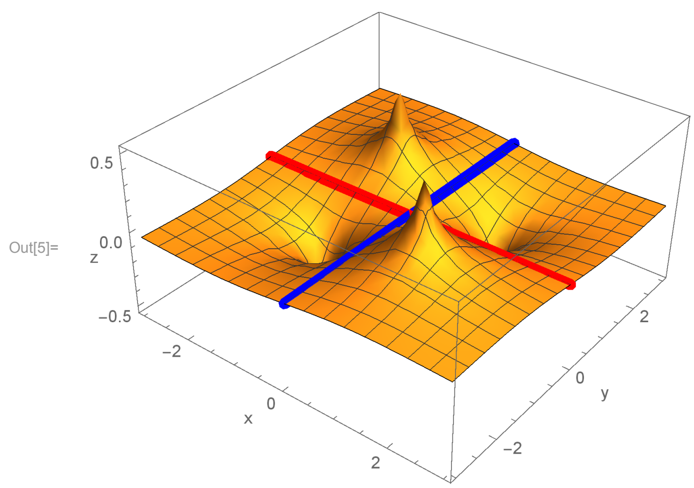
The resulting Green function becomes

$$G(\vec{x}, \vec{x}_0) = \frac{1}{2\pi}\ln(r) - \frac{1}{2\pi}\ln(r_1) - \frac{1}{2\pi}\ln(r_2) + \frac{1}{2\pi}\ln(r_3)$$

Or

$$\begin{aligned} G(\vec{x}, \vec{x}_0) = & \frac{1}{4\pi}\ln\left((x-x_0)^2 + (y-y_0)^2\right) - \frac{1}{4\pi}\ln\left((x-x_0)^2 + (y+y_0)^2\right) \\ & - \frac{1}{4\pi}\ln\left((x+x_0)^2 + (y-y_0)^2\right) + \frac{1}{4\pi}\ln\left((x+x_0)^2 + (y+y_0)^2\right) \end{aligned} \tag{1}$$

The following is 3D plot of the above Green function, showing the image impulse and showing that $G = 0$ at the line $y = 0$ and also at line $x = 0$. (Lines marked as red and blue)



Now that the Green function is found, it is used to solve $\nabla^2 u(\vec{x}) = f(\vec{x})$, with $u(x,0) = h(x)$, $u(0,y) = g(y)$ .Starting with Green formula for 2D

$$\begin{aligned} \iint u(\vec{x})\nabla^2 G(\vec{x},\vec{x}_0) - G(\vec{x},\vec{x}_0)\nabla^2 u(\vec{x})\, dA = & \oint_{s_1} \left(u(\vec{x})\nabla G(\vec{x},\vec{x}_0) - G(\vec{x},\vec{x}_0)\nabla u(\vec{x})\right)\cdot\hat{n}\, ds \\ & + \oint_{s_2} \left(u(\vec{x})\nabla G(\vec{x},\vec{x}_0) - G(\vec{x},\vec{x}_0)\nabla u(\vec{x})\right)\cdot\hat{n}\, ds \end{aligned}$$

To simplify the notation, from now on, G is used of $G(\vec{x},\vec{x}_0)$, and also u instead of $u(\vec{x})$. The line s_1

in the above is the line $x>0, y=0$ and s_2 is the line $x=0, y>0$. Therefore the above becomes

$$\iint u\nabla^2 G - G\nabla^2 u \; dA = \oint_{s_1} (u\nabla G - G\nabla u)\cdot\left(-\hat{\jmath}\right)\; ds + \oint_{s_2} (u\nabla G - G\nabla u)\cdot(-\hat{\imath})\; ds$$

Or

$$\iint u\nabla^2 G\; dA - \iint G\nabla^2 u\; dA = \int_0^\infty \left(G\frac{\partial u}{\partial y} - u\frac{\partial G}{\partial y}\right)_{y=0} dx + \int_0^\infty \left(G\frac{\partial u}{\partial x} - u\frac{\partial G}{\partial x}\right)_{x=0} dy$$

But $\nabla^2 G\left(\vec{x},\vec{x}_0\right) = \delta\left(\vec{x},\vec{x}_0\right)$ and $\nabla^2 u\left(\vec{x}\right) = f\left(\vec{x}\right)$, hence the above reduces to

$$\iint u\left(\vec{x}\right)\delta\left(\vec{x},\vec{x}_0\right)\; dA - \iint Gf\left(\vec{x}\right)\; dA = \int_0^\infty \left(G\frac{\partial u}{\partial y} - u\frac{\partial G}{\partial y}\right)_{y=0} dx + \int_0^\infty \left(G\frac{\partial u}{\partial x} - u\frac{\partial G}{\partial x}\right)_{x=0} dy$$

But $\iint u\left(\vec{x}\right)\delta\left(\vec{x},\vec{x}_0\right) dA = u\left(\vec{x}_0\right)$ therefore

$$u\left(\vec{x}_0\right) - \iint Gf\left(\vec{x}\right)\; dA = \int_0^\infty \left(G\frac{\partial u}{\partial y} - u\frac{\partial G}{\partial y}\right)_{y=0} dx + \int_0^\infty \left(G\frac{\partial u}{\partial x} - u\frac{\partial G}{\partial x}\right)_{x=0} dy$$

Or

$$u\left(\vec{x}_0\right) = \iint Gf\left(\vec{x}\right)\; dA + \int_0^\infty \left(G\frac{\partial u}{\partial y} - u\frac{\partial G}{\partial y}\right)_{y=0} dx + \int_0^\infty \left(G\frac{\partial u}{\partial x} - u\frac{\partial G}{\partial x}\right)_{x=0} dy$$

Since $G\left(\vec{x},\vec{x}_0\right) = 0$ at $y=0$, and $G\left(\vec{x},\vec{x}_0\right) = 0$ at $x=0$, the above becomes

$$u\left(\vec{x}_0\right) = \iint Gf\left(\vec{x}\right)\; dA - \int_0^\infty \left(u\frac{\partial G}{\partial y}\right)_{y=0} dx - \int_0^\infty \left(u\frac{\partial G}{\partial x}\right)_{x=0} dy$$

Since $u\left(\vec{x}\right) = h(x)$ at $y=0$ and $u\left(\vec{x}\right) = g\left(y\right)$ at $x=0$ then

$$u\left(x_0,y_0\right) = \iint Gf\left(\vec{x}\right)\; dA - \int_0^\infty h(x)\left(\frac{\partial G}{\partial y}\right)_{y=0} dx - \int_0^\infty g\left(y\right)\left(\frac{\partial G}{\partial x}\right)_{x=0} dy \tag{2}$$

$\left(\frac{dG}{dy}\right)_{y=0}$ and $\left(\frac{\partial G}{\partial x}\right)_{x=0}$ are now evaluated to complete the solution. Using $G\left(\vec{x},\vec{x}_0\right)$ in equation (1) gives

$$\begin{aligned}\frac{\partial G}{\partial y} &= \frac{1}{4\pi}\left(\frac{2\left(y-y_0\right)}{(x-x_0)^2+\left(y-y_0\right)^2}\right) - \frac{1}{4\pi}\left(\frac{2\left(y+y_0\right)}{(x-x_0)^2+\left(y+y_0\right)^2}\right)\\ &\quad - \frac{1}{4\pi}\left(\frac{2\left(y-y_0\right)}{(x+x_0)^2+\left(y-y_0\right)^2}\right) + \frac{1}{4\pi}\left(\frac{2\left(y+y_0\right)}{(x+x_0)^2+\left(y+y_0\right)^2}\right)\end{aligned}$$

Evaluating the above at $y=0$ results in

$$\begin{aligned}\left(\frac{\partial G}{\partial y}\right)_{y=0} &= \frac{1}{4\pi}\left(\frac{-2y_0}{(x-x_0)^2+y_0^2}\right) - \frac{1}{4\pi}\left(\frac{2y_0}{(x-x_0)^2+y_0^2}\right)\\ &\quad - \frac{1}{4\pi}\left(\frac{-2y_0}{(x+x_0)^2+y_0^2}\right) + \frac{1}{4\pi}\left(\frac{2y_0}{(x+x_0)^2+y_0^2}\right)\end{aligned}$$

Or

$$\left(\frac{\partial G}{\partial y}\right)_{y=0} = \frac{y_0}{\pi}\left(\frac{1}{(x+x_0)^2+y_0^2} - \frac{1}{(x-x_0)^2+y_0^2}\right) \tag{3}$$

Finding $\frac{\partial G}{\partial x}$ gives

$$\begin{aligned}\frac{\partial G}{\partial x} = \frac{1}{4\pi}\left(\frac{2(x-x_0)}{(x-x_0)^2+(y-y_0)^2}\right) - \frac{1}{4\pi}\left(\frac{2(x-x_0)}{(x-x_0)^2+(y+y_0)^2}\right)\\ -\frac{1}{4\pi}\left(\frac{2(x+x_0)}{(x+x_0)^2+(y-y_0)^2}\right) + \frac{1}{4\pi}\left(\frac{2(x+x_0)}{(x+x_0)^2+(y+y_0)^2}\right)\end{aligned}$$

Evaluating the above at $x=0$ results in

$$\begin{aligned}\left(\frac{\partial G}{\partial x}\right)_{x=0} = \frac{1}{4\pi}\left(\frac{-2x_0}{x_0^2+(y-y_0)^2}\right) - \frac{1}{4\pi}\left(\frac{-2x_0}{x_0^2+(y+y_0)^2}\right)\\ -\frac{1}{4\pi}\left(\frac{2x_0}{x_0^2+(y-y_0)^2}\right) + \frac{1}{4\pi}\left(\frac{2x_0}{x_0^2+(y+y_0)^2}\right)\end{aligned}$$

Or

$$\begin{aligned}\left(\frac{\partial G}{\partial x_0}\right)_{x=0} &= \frac{1}{\pi}\left(\frac{x_0}{x_0^2+(y+y_0)^2}\right) - \frac{1}{\pi}\left(\frac{x_0}{x_0^2+(y-y_0)^2}\right)\\ &= \frac{x_0}{\pi}\left(\frac{1}{x_0^2+(y+y_0)^2} - \frac{1}{x_0^2+(y-y_0)^2}\right) \qquad (4)\end{aligned}$$

Substituting (3,4) into (2) gives the final answer

$$\begin{aligned}u(x_0,y_0) = \int_0^\infty\int_0^\infty G(\vec{x},\vec{x}_0)\,f(\vec{x})\;dxdy\\ -\frac{y_0}{\pi}\int_0^\infty h(x)\left(\frac{1}{(x+x_0)^2+y_0^2} - \frac{1}{(x-x_0)^2+y_0^2}\right)\;dx\\ -\frac{x_0}{\pi}\int_0^\infty g(y)\left(\frac{1}{x_0^2+(y+y_0)^2} - \frac{1}{x_0^2+(y-y_0)^2}\right)\;dy\end{aligned}$$

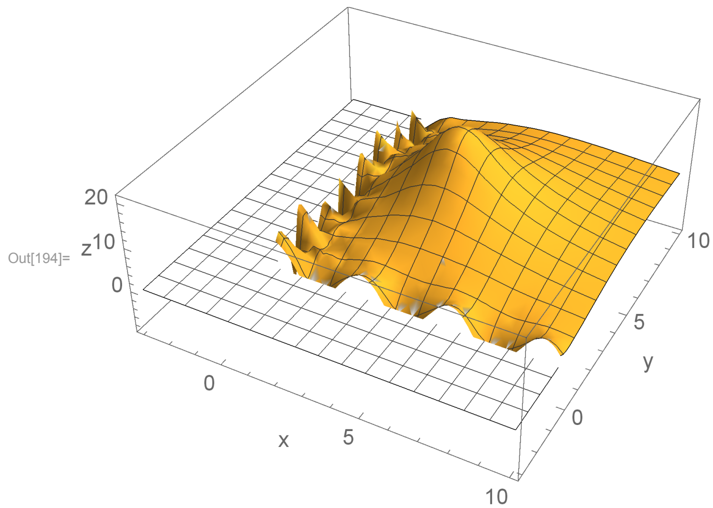
Reversing the role of $\vec{x},\vec{x}_0$ gives

$$\begin{aligned}u(x,y) = \int_0^\infty\int_0^\infty G(\vec{x},\vec{x}_0)\,f(x_0,y_0)\;dx_0dy_0\\ -\frac{y}{\pi}\int_0^\infty h(x_0)\left(\frac{1}{(x_0+x)^2+y^2} - \frac{1}{(x_0-x)^2+y^2}\right)\;dx_0\\ -\frac{x}{\pi}\int_0^\infty g(y_0)\left(\frac{1}{x^2+(y_0+y)^2} - \frac{1}{x^2+(y_0-y)^2}\right)\;dy_0\end{aligned}$$

Where $G(\vec{x},\vec{x}_0)$ is given by equation (1). This complete the solution.

The following is 3D plot of the solution (for small area is first quadrant) generated using Mathematica using

$$f(x) = -20e^{-(x-4)^2-(y-5)^2}$$
$$g(y) = 10\sin(5y)$$
$$h(x) = 5\cos(2x)$$



This is a contour plot of the above solution