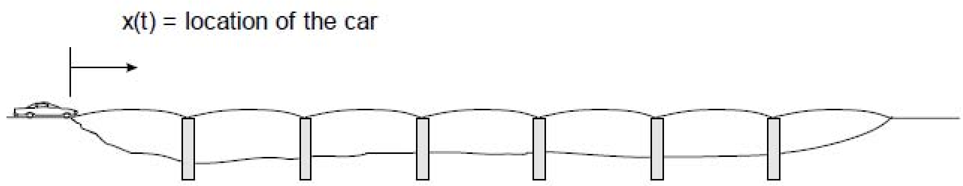
HW 3 CEE 744, Spring 2013

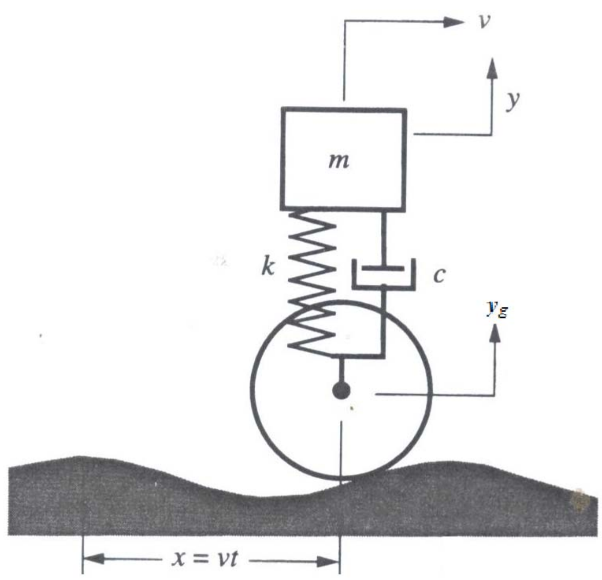
Nasser M. Abbasi

Spring 2013 Compiled on August 20, 2021 at 9:15pm

Contents

0.1 Mathematical model





The equation of motion of the car is

$$my'' + cy' + ky = cy'_g + ky_g$$

Let $y - y_g = u$ which is the distance between m and the ground. Hence the equation of motion now becomes

$$\begin{aligned} m\left(u'' + y''_g\right) + c\left(u' + y'_g\right) + k\left(u + y_g\right) &= cy'_g + ky_g \\ mu'' + cu' + ku &= -my''_g \end{aligned}$$

0.2 Summary of results found

0.2.1 Bridge data

	imperial	SI
span length λ	70 ft	$70 \times 0.3048 = 21.336$ meter
upward camber Δ	$2.5'' = 0.20833$ft	$2.5 \times 0.0254 = 0.0635$ meter

0.2.2 Car data

	imperial
mass of car	$\frac{1500}{32.2} = 46.584 \frac{lb.s^2}{ft}$
speed of car	30 mile/hr= 44.0 ft/sec
critical damping ratio is ζ	0.75
spring constant k	2400 lb/ft
natural frequency $\omega_n = \sqrt{\frac{k}{m}}$	$\sqrt{\frac{2400}{46.584}} = 7.1777$ rad/sec
natural frequency $f_n = \frac{\omega_n}{2\pi}$	$\frac{7.1777}{2\pi} = 1.1424$ Hz
natural period $T_n = \frac{1}{f_n}$	$\frac{1}{1.1424} = 0.87535$ sec
natural damped frequency $\omega_d = \omega_n\sqrt{1-\zeta^2}$	$7.1777\sqrt{1-0.75^2} = 4.7476$ rad/sec
natural damped frequency $f_d = \frac{\omega_d}{2\pi}$	$\frac{4.7476}{2\pi} = 0.7556$ Hz
T_p time to driver over one span $= \frac{\lambda}{v}$	1.591 sec
T_a time to cross the bridge (duration of loading)	$7 \times 1.591 = 11.137$ sec

0.2.3 Results

a_n values found for up to $n = 10$

a_n distribution

n	a[n]
0	24.0897
1	-16.0598
2	-3.21196
3	-1.37655
4	-0.764752
5	-0.48666
6	-0.336919
7	-0.247074
8	-0.188939
9	-0.149162
10	-0.12075

Peak relative displacement of the driver

Maximum relative displacement was 0.24 inch and it occurred during transient phase.

Peak total displacement of the driver

0.165 inch + 2.5 inch =2.665 inch and it occurred during steady state phase at multiples of half the period T_p while on the bridge.

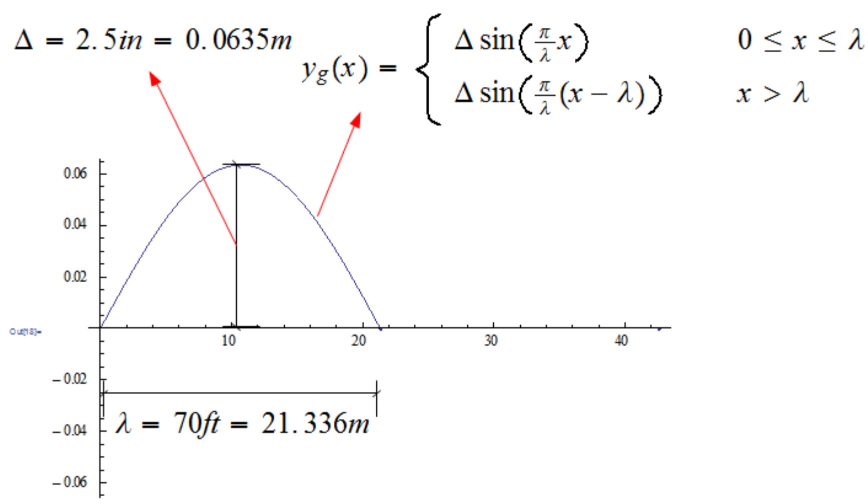
Number of a_n terms used

In addition to a_0 term, the next 5 terms are used for a total of 6 terms.

0.3 Analysis

0.3.1 Generate load equation

The equation of the ground is shown in this diagram



Description of one span and equation of road

Therefore, the equation of span is

$$y_g(x) = \Delta \sin\left(\frac{\pi}{\lambda}x\right) \qquad 0 \leq x \leq \lambda$$

Hence, we convert it to be a function of time using $x = vt$, hence

$$\begin{aligned} y_g(t) &= \Delta \sin\left(\frac{\pi}{\lambda}vt\right) \qquad 0 \leq t \leq \frac{\lambda}{v} \\ &= \Delta \sin(\varpi t) \qquad 0 \leq t \leq T_p \\ &= \Delta \sin\left(\frac{\pi}{T_p}t\right) \qquad 0 \leq t \leq T_p \end{aligned}$$

Where in the above $\varpi = \frac{\pi}{T_p}$ is the fundamental frequency of the ground motion. Hence $y_g'(t) = \Delta \frac{\pi}{T_p} \cos\left(\frac{\pi}{T_p}t\right)$ and

$$y_g''(t) = -\Delta \left(\frac{\pi}{T_p}\right)^2 \sin\left(\frac{\pi}{T_p}t\right)$$

And

$$\beta = \frac{T_n}{T_p} = \frac{0.87535}{1.591} = 0.55019$$

Then load in one span $0 < t < T_p$ is

$$P_a(t) = m\Delta \left(\frac{\pi}{T_p}\right)^2 \sin\left(\frac{\pi}{T_p}t\right)$$

Let

$$
\begin{aligned}
P_o &= m\Delta\left(\frac{\pi}{T_p}\right)^2 = (46.584)\,(0.208\,33)\left(\frac{\pi}{1.591}\right)^2 \\
&= 37.840 \text{ lb}
\end{aligned}
$$

Then the load becomes

$$
P_a(t) = P_o \sin\left(\frac{\pi}{T_p}t\right) \tag{1}
$$

0.3.2 Convert load to Fourier series

Now we need to convert Eq 1 to Fourier series[1]. Let $\tilde{P}_a(t)$ be the Fourier series approximation to $P_a(t)$, hence

$$
\begin{aligned}
\tilde{P}_a(t) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{2\pi}{T_p}t\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\frac{2\pi}{T_p}t\right) \\
a_0 &= \frac{1}{T_p}\int_0^{T_p} P_a(t)dt \\
a_n &= \frac{2}{T_p}\int_0^{T_p} P_a(t)\cos\left(n\frac{2\pi}{T_p}t\right)dt \\
b_n &= \frac{2}{T_p}\int_0^{T_p} P_a(t)\sin\left(n\frac{2\pi}{T_p}t\right)dt
\end{aligned}
$$

Hence

$$
\begin{aligned}
a_0 &= \frac{1}{T_p}\int_0^{T_p} P_a(t)dt = \frac{P_o}{T_p}\int_0^{T_p} \sin\left(\frac{\pi}{T_p}t\right)dt = \frac{P_o}{T_p}\left(\frac{-\cos\left(\frac{\pi}{T_p}t\right)}{\frac{\pi}{T_p}}\right)_0^{T_p} = -\frac{P_o}{\pi}\left(\cos(\pi) - 1\right) \\
&= \frac{2P_o}{\pi} = \frac{2\,(37.840)}{\pi} \\
&= 24.090 \text{ lb}
\end{aligned}
$$

And

$$
\begin{aligned}
a_n &= \frac{2}{T_p}\int_0^{T_p} P_a(t)\cos\left(n\frac{2\pi}{T_p}t\right)dt \\
&= \frac{2P_o}{T_p}\int_0^{T_p} \sin\left(\frac{\pi}{T_p}t\right)\cos\left(n\frac{2\pi}{T_p}t\right)dt \\
&= \frac{4P_o}{\pi - 4n^2\pi}\cos(n\pi)^2
\end{aligned}
$$

But $\cos(n\pi)^2 = 1$ Hence

$$
a_n = \frac{4P_o}{\pi - 4n^2\pi}
$$

[1] The Fourier series can also be found using complex form. This was done in the appendix.

and

$$\begin{aligned} b_n &= \frac{1}{T_p}\int_0^{T_p} P_a(t)\sin\left(2\pi n\frac{t}{T_p}\right)dt \\ &= \frac{2P_o}{T_p}\int_0^{T_p}\sin\left(\frac{\pi}{T_p}t\right)\cos\left(n\frac{2\pi}{T_p}t\right)dt \\ &= \frac{2P_o}{\pi-4n^2\pi}\sin(2n\pi) \end{aligned}$$

But $\sin(2n\pi)=0$ for all integer n, hence $b_n=0$.Therefore

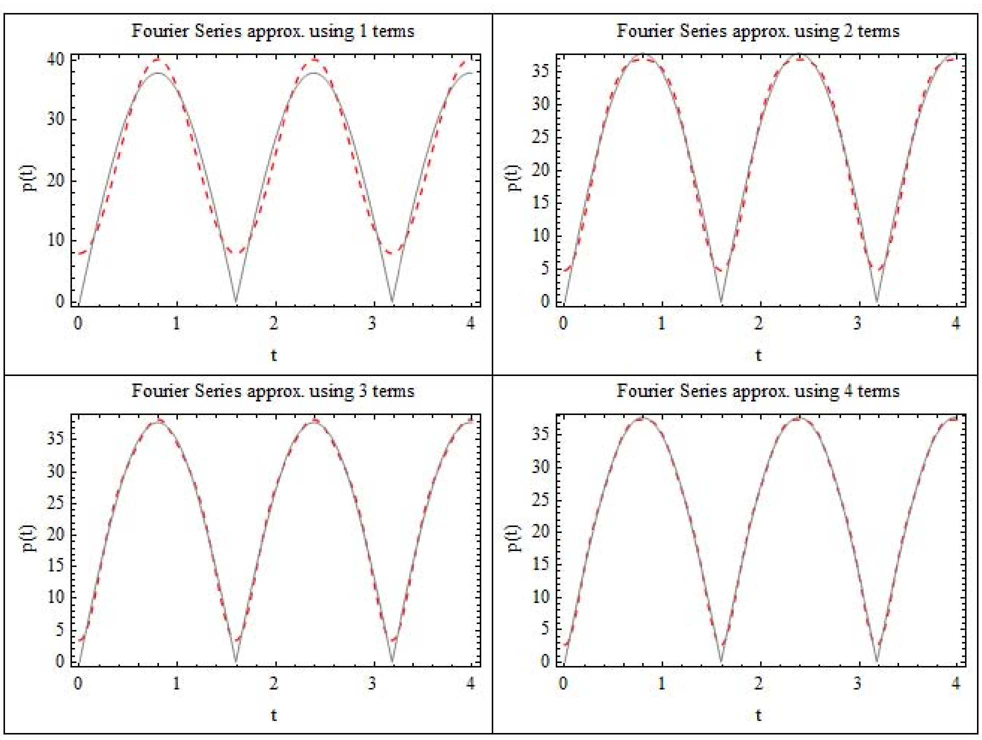
$$\begin{aligned} \tilde{P}_a(t) &= a_0+\sum_{n=1}^{\infty}a_n\cos\left(2\pi n\frac{t}{T_p}\right) \\ &= a_0+\sum_{n=1}^{\infty}\frac{4P_o}{\pi-4n^2\pi}\cos\left(2\pi n\frac{t}{T_p}\right) \end{aligned}$$

Using the numerical values found, we obtain

$$\tilde{P}_a(t) = 24.0897+\sum_{n=1}^{\infty}\frac{4\,(37.840)}{\pi-4n^2\pi}\cos\left(2\pi n\frac{t}{1.591}\right)$$

0.3.3 Plot of load and its Fourier series approximation

This plot below shows $P_a(t)$ and its Fourier series approximation $\tilde{P}_a(t)$ as more terms are added. This was plotted for $t=0\cdots 5$ sec. This was done to verify that the Fourier series approximation is correct before going to the next stage of the analysis. The actual calculations used the first 6 terms of a_n.



0.3.4 Finding the steady state response

The equation of motion of the car is

$$my'' + cy' + ky = cy_g' + ky_g$$

Let $y - y_g = u$ which is the distance between m and the ground. Hence the equation of motion now becomes

$$\begin{aligned} m\left(u'' + y_g''\right) + c\left(u' + y_g'\right) + k\left(u + y_g\right) &= cy_g' + ky_g \\ mu'' + cu' + ku &= -my_g'' \end{aligned} \tag{2}$$

Hence Eq 2 becomes

$$\begin{aligned} mu'' + cu' + ku &= m\Delta\left(\frac{\pi}{T_p}\right)^2 \sin\left(\frac{\pi}{T_p}t\right) \\ &= P_a(t) \\ &= \sum_{n=0}^{\infty} a_n \cos\left(2\pi n\frac{t}{T_p}\right) \\ &= \operatorname{Re}\left\{\sum_{n=0}^{\infty} a_n e^{in\varpi t}\right\} \end{aligned}$$

Where $\varpi = \frac{2\pi}{T_p}$ is the fundamental loading harmonic. Let $u_{ss}(n) = \operatorname{Re}\left\{U_n e^{in\varpi t}\right\}$ be the response due to the n term in the loading function. Hence the equation of motion now becomes

$$\begin{aligned} m\operatorname{Re}\left\{\sum_{n=0}^{\infty} -n^2\varpi^2 U_n e^{in\varpi t}\right\} + c\operatorname{Re}\left\{\sum_{n=0}^{\infty} i\varpi n U_n e^{in\varpi t}\right\} + k\operatorname{Re}\left\{\sum_{n=0}^{\infty} U_n e^{in\varpi t}\right\} &= \operatorname{Re}\left\{\sum_{n=0}^{\infty} a_n e^{in\varpi t}\right\} \\ \left(-n^2\varpi^2 m + cin\varpi + k\right) U_n &= a_n \\ U_n &= \frac{a_n}{-n^2\varpi^2 m + cin\varpi + k} \\ &= \frac{a_n}{k}\frac{1}{\left(1 - n^2r^2\right) + 2i\zeta nr} \end{aligned}$$

Hence the transfer function is

$$\begin{aligned} \left(-n^2\varpi^2 m + cin\varpi + k\right) U_n &= a_n \\ U_n &= \frac{a_n}{-n^2\varpi^2 m + cin\varpi + k} \\ &= \frac{a_n}{k}\frac{1}{\left(1 - n^2r^2\right) + 2i\zeta nr} \end{aligned}$$

Therefore, steady state response is

$$\begin{aligned} y_{ss}(t) &= \operatorname{Re}\left\{\sum_{n=0}^{\infty} U_n e^{in\varpi t}\right\} \\ &= \operatorname{Re}\left\{\sum_{n=0}^{\infty} \frac{a_n}{k}\frac{1}{\left(1 - n^2r^2\right) + 2i\zeta nr} e^{in\varpi t}\right\} \\ &= \operatorname{Re}\left\{\sum_{n=0}^{\infty} \overbrace{\frac{a_n}{k} D(\zeta, r, n)}^{U_n} e^{in\varpi t}\right\} \end{aligned} \tag{3}$$

Where $D(\zeta, r, n)$ is the n^{th} harmonic dynamic magnification factor

$$|D(\zeta, r, n)| = \sqrt{\frac{1}{\left(1 - n^2 r^2\right)^2 + (2\zeta n r)^2}}$$

and

$$\arg D(\zeta, r, n) = -\tan^{-1}\left(\frac{2\zeta n r}{1 - n^2 r^2}\right)$$

In the above,

$$r = \frac{\varpi}{\omega_{nat}} = \frac{\frac{2\pi}{T_p}}{\omega_{nat}} = \frac{\frac{2\pi}{1.591}}{7.1777} = \boxed{0.5502}$$

This is a list of the magnitude of U_n for different n value to examine the contribution of each harmonic to the steady state response.

Distribution of U_{ss} harmonics

n	U_n	$\|U_n\|$	phase(U_n) degree
0	0.0100374	0.0100374	0.
1	-0.00170524 + 0.00366837 i	0.00404534	-114.931
2	0.0000312072 + 0.000443912 i	0.000445007	-85.9787
3	0.0000425907 + 0.000111137 i	0.000119018	-69.0317
4	0.000024122 + 0.0000376552 i	0.000044719	-57.3563
5	0.0000134002 + 0.0000153014 i	0.0000203396	-48.7897
6	$7.7639\times10^{-6} + 7.05943\times10^{-6}$ i	0.0000104935	-42.2791
7	$4.72164\times10^{-6} + 3.58384\times10^{-6}$ i	5.92772×10^{-6}	-37.1994
8	$3.00344\times10^{-6} + 1.96149\times10^{-6}$ i	3.58722×10^{-6}	-33.1478
9	$1.98761\times10^{-6} + 1.14081\times10^{-6}$ i	2.29173×10^{-6}	-29.8542
10	$1.36131\times10^{-6} + 6.97572\times10^{-7}$ i	1.52963×10^{-6}	-27.1318

0.3.5 Find the transient solution

From the steady state solution $u_{ss}(t)$ we found above, we now find $u_{ss}(0)$ and $u'_{ss}(0)$ these are the initial conditions, but in opposite sign, that the transient solution have to satisfy.

From above, we found the steady state solution to be

$$y_{ss}(t) = \operatorname{Re}\left\{\sum_{n=0}^{\infty} U_n e^{in\varpi t}\right\}$$

Hence

$$y'_{ss}(t) = \operatorname{Re}\left\{\sum_{n=0}^{\infty} in\varpi U_n e^{in\varpi t}\right\}$$

At time $t = 0$ the above becomes

$$y_{ss}(0) = \operatorname{Re}\left\{\sum_{n=0}^{\infty} U_n\right\}$$

$$y'_{ss}(0) = \operatorname{Re}\left\{\sum_{n=0}^{\infty} in\varpi U_n\right\}$$

Now we need to decide on how many harmonics to use in order to determine $y_{ss}(0)$ and $y'_{ss}(0)$. From above we see that after $\boxed{n = 5}$ then a_n became very small. Hence we will use up to $n = 5$ to find the initial conditions from the above 2 equations.

$$y_{ss}(0) = \operatorname{Re}\left\{\sum_{n=0}^{5} U_n\right\} = \operatorname{Re}\left\{\sum_{n=0}^{5} \frac{a_n}{k} \frac{1}{\left(1 - n^2 r^2\right) + 2i\zeta n r}\right\}$$
$$= 0.0084435 \text{ ft} = 0.101322 \text{ inch}$$

and for the initial velocity we obtain

$$\begin{aligned} y'_{ss}(0) &= \operatorname{Re}\left\{\sum_{n=0}^{\infty} in\varpi U_n\right\} \\ &= \operatorname{Re}\left\{\sum_{n=0}^{\infty} in\varpi \frac{a_n}{k}\frac{1}{\left(1-n^2r^2\right)+2i\zeta nr}\right\} \\ &= -0.020207 \text{ ft/sec} = -0.242484 \text{ inch/sec} \end{aligned}$$

Now the transient solution for damped system is given by

$$u_{tr}(t) = e^{-\zeta\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right)$$

with

$$\begin{aligned} u_{tr}(0) &= -0.0084435 \\ u'_{tr}(0) &= +0.020207 \end{aligned}$$

Hence

$$A = u_{tr}(0) = \boxed{-0.0084435}$$

Taking derivative of $u_{tr}(t)$ gives

$$u'_{tr}(t) = \zeta\omega_n e^{-\zeta\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right) + e^{-\zeta\omega_n t}\left(-A\omega_d\sin\omega_d t + B\omega_d\cos\omega_d t\right)$$

Hence at $t = 0$ we obtain

$$\begin{aligned} u'_{tr}(0) &= \zeta\omega_n A + B\omega_d \\ B &= \frac{u'_{tr}(0) - \zeta\omega_n A}{\omega_d} \end{aligned}$$

But $u'_{tr}(0) = +0.020207$ ft/sec, $A = -0.0084435$ ft, $\zeta = 0.75$, $\omega_d = 4.7476$ rad/sec, $\omega_n = 7.1777$ rad/sec, hence

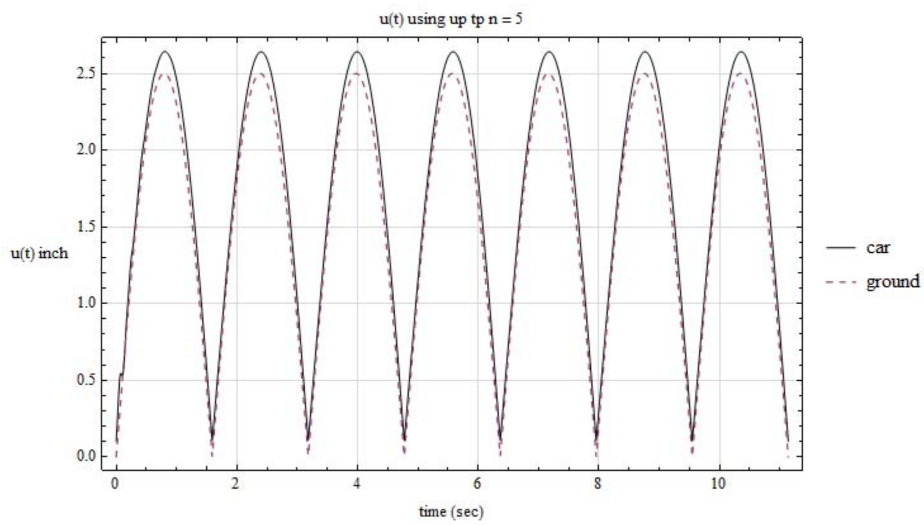
$$\begin{aligned} B &= \frac{0.020207 - 0.75\times 7.1777\times(-0.0084435)}{4.7476} \\ &= 0.01383 \end{aligned}$$

Therefore

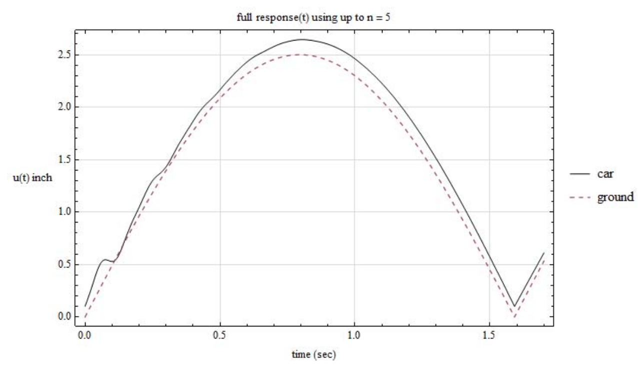
$$\begin{aligned} u_{tr}(t) &= e^{-\zeta\omega_n t}\left(-0.0084435\cos\omega_d t + 0.01383\sin\omega_d t\right) \\ &= e^{-0.75(7.1777)t}\left(-0.0084435\cos(4.7476t) + 0.01383\sin 4.7476t\right) \end{aligned}$$

This solution is now added to the steady state solution.

0.3.6 Plot of the absolute total displacement with the bridge for both steady state and transient combined



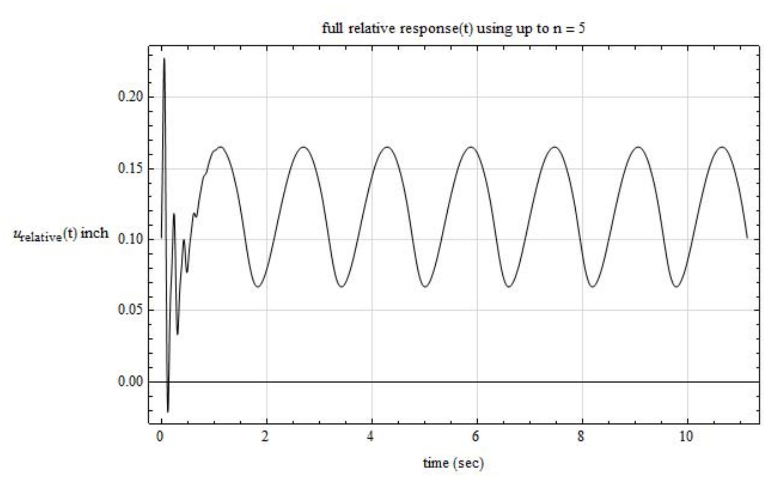
Zooming on the first 1.8 seconds shows more clearly the effect of transient solution

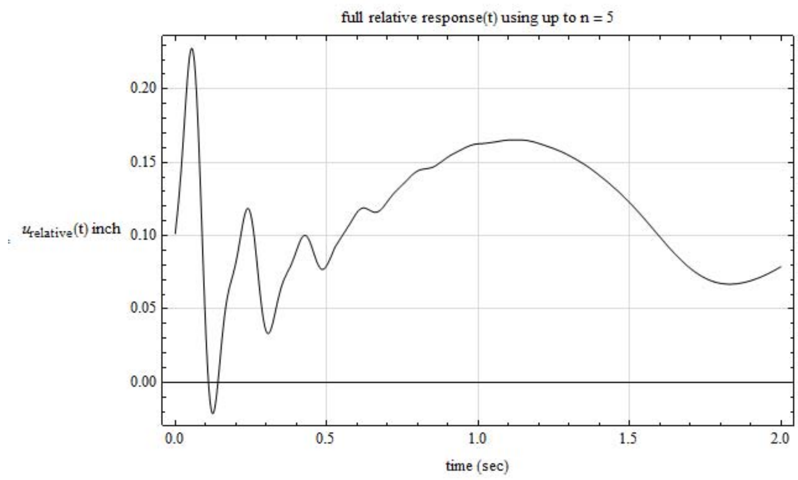


The transient solution effect vanishes after about 1.5 second.

0.3.7 Plotting the full relative solution

To better see the solution obtained, we plot the relative displacement. This is the displacement felt by the passenger. First the solution is shown for the whole time to cross the bridge, then we zoom to the first 2 seconds to better see the transient solution





From the above we see that the maximum relative displacement is about 0.24 inch and it occurs during transient phase. During steady state, the maximum relative displacement is about 0.165 inch

0.4 Appendix

0.4.1 Finding Fourier series approximation using complex form

The Fourier series approximation can also be found using the complex representation. This is the derivation using this method which gives the same result as was found earlier.

$$\tilde{P}_a(t) = \frac{1}{2}Y_0 + \text{Re}\left(\sum_{n=1}^{\infty} Y_n e^{in\varpi t}\right)$$

Where

$$\begin{aligned} Y_n &= \frac{2}{T_p}\int_0^{T_p} P_a(t) e^{-in\varpi t}\, dt \\ &= \frac{2P_o}{T_p}\int_0^{T_p} \sin\left(\frac{\pi}{T_p}t\right) e^{-in\varpi t}\, dt \end{aligned} \tag{4}$$

Integration by parts, $\int u dv = uv - \int v du$, let $u = \sin\left(\frac{\pi}{T_p}t\right)$, hence $du = \frac{\pi}{T_p}\cos\left(\frac{\pi}{T_p}t\right)$ and $v = \frac{e^{-in\varpi t}}{-in\varpi}$, therefore the above becomes

$$\begin{aligned} Y_n &= \frac{2P_o}{T_p}\left(\left[\sin\left(\frac{\pi}{T_p}t\right)\frac{e^{-in\varpi t}}{-in\varpi}\right]_0^{T_p} - \int_0^{T_p} \frac{\pi}{T_p}\cos\left(\frac{\pi}{T_p}t\right)\frac{e^{-in\varpi t}}{-in\varpi}dt\right) \\ &= \frac{2P_o}{T_p}\left(\sin\left(\frac{\pi}{T_p}T_p\right)\frac{ie^{-in\varpi T_p}}{n\varpi} - \frac{i}{2n}\int_0^{T_p}\cos\left(\frac{\pi}{T_p}t\right)e^{-in\varpi t}dt\right) \\ &= -\frac{iP_o}{nT_p}\int_0^{T_p}\cos\left(\frac{\pi}{T_p}t\right)e^{-in\varpi t}dt \end{aligned} \tag{5}$$

Now integrate by parts again where now $\int u dv = uv - \int v du$, let $u = \cos\left(\frac{\pi}{T_p}t\right)$, hence $du = -\frac{\pi}{T_p}\sin\left(\frac{\pi}{T_p}t\right)$ and $v = \frac{e^{-in\varpi t}}{-in\varpi}$, therefore Eq 5 becomes

$$\begin{aligned} Y_n &= -\frac{iP_o}{nT_p}\left[\left(\cos\left(\frac{\pi}{T_p}t\right)\frac{e^{-in\varpi t}}{-in\varpi}\right)_0^{T_p} - \int_0^{T_p} -\frac{\pi}{T_p}\sin\left(\frac{\pi}{T_p}t\right)\frac{e^{-in\varpi t}}{-in\varpi}dt\right] \\ &= -\frac{iP_o}{nT_p}\left[\left(\cos\left(\frac{\pi}{T_p}T_p\right)\frac{e^{-in\frac{2\pi}{T_p}T_p}}{-in\varpi} - \frac{1}{-in\varpi}\right) + \frac{i}{n2}\int_0^{T_p}\sin\left(\frac{\pi}{T_p}t\right)e^{-in\varpi t}dt\right] \\ &= -\frac{iP_o}{nT_p}\left[\left(-i\frac{e^{-in2\pi}}{n\varpi} - \frac{i}{n\varpi}\right) + \frac{i}{n2}\int_0^{T_p}\sin\left(\frac{\pi}{T_p}t\right)e^{-in\varpi t}dt\right] \\ &= -\frac{P_o}{nT_p}\left(\frac{e^{-in2\pi}+1}{n\varpi}\right) + \frac{\Delta}{2n^2T_p}\int_0^{T_p}\sin\left(\frac{\pi}{T_p}t\right)e^{-in\varpi t}dt \end{aligned} \tag{6}$$

Now we see that the term $\int_0^{T_p}\sin\left(\frac{\pi}{T_p}t\right)e^{-in\varpi t}dt$ has repeated again. This term is the same as what we started with in Eq 4, therefore, we write

$$\int_0^{T_p}\sin\left(\frac{\pi}{T_p}t\right)e^{-in\varpi t}dt = \frac{T_p}{2\Delta}Y_n$$

and replace this term back into Eq 6, hence it becomes

$$\begin{aligned}
Y_n &= -\frac{P_o}{nT_p}\left(\frac{e^{-in2\pi}+1}{n\varpi}\right)+\frac{\Delta}{2n^2T_p}\frac{T_p}{2\Delta}Y_n \\
&= -\frac{P_o}{nT_p}\left(\frac{e^{-in2\pi}+1}{n\varpi}\right)+\frac{1}{2^2n^2}Y_n \\
Y_n-\frac{1}{2^2n^2}Y_n &= -\frac{P_o}{nT_p}\left(\frac{e^{-in2\pi}+1}{n\varpi}\right) \\
Y_n\left(1-\frac{1}{(2n)^2}\right) &= -\frac{P_o}{nT_p}\left(\frac{e^{-in2\pi}+1}{n\varpi}\right) \\
Y_n &= -\frac{2P_o\left(e^{-in2\pi}+1\right)}{\pi\left((2n)^2-1\right)}=\frac{2P_o\left(e^{-in2\pi}+1\right)}{\pi-\pi\,(2n)^2} \\
&= \frac{4P_o}{\pi\left(1-4n^2\right)}
\end{aligned}$$

And

$$\begin{aligned}
Y_0 &= \frac{2}{T_p}\int_0^{T_p}P_o\sin\left(\frac{\pi}{T_p}t\right)dt=\frac{2P_o}{T_p}\int_0^{T_p}\sin\left(\frac{\pi}{T_p}t\right)dt=\frac{2P_o}{T_p}\left(\frac{-\cos\left(\frac{\pi}{T_p}t\right)}{\frac{\pi}{T_p}}\right)_0^{T_p}=-\frac{2P_o}{\pi}\left(\cos\left(\frac{\pi}{T_p}T_p\right)-1\right) \\
&= -\frac{2P_o}{\pi}(-1-1) \\
&= \frac{4P_o}{\pi}
\end{aligned}$$

Therefore, the Fourier series approximation for ground motion is now

$$\begin{aligned}
\tilde{P}_a(t) &= \frac{1}{2}Y_0+\mathrm{Re}\left(\sum_{n=1}^{\infty}Y_ne^{in\varpi t}\right) \\
&= \frac{4P_o}{2\pi}+\mathrm{Re}\left(\sum_{n=1}^{\infty}\frac{4P_o}{\pi\left(1-4n^2\right)}e^{in\varpi t}\right) \\
&= \frac{2P_o}{\pi}+\mathrm{Re}\left(\sum_{n=1}^{\infty}\frac{4P_o}{\pi\left(1-4n^2\right)}e^{in\varpi t}\right)
\end{aligned}$$

We see that we obtained the same result using the classical Fourier series form.