HW 7

EMA 545
Mechanical Vibrations

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Contents

1 problem 1

Problem 1: (20 pts) Use FFT techniques to find the response of your vibration isolation system to the turbulent flight profile in "FlightAccel.mat" on the course website. (To do this, you will have to assume that the aircraft experiences this exact same flight profile over and over again.)

  a.) Provide at least one plot comparing the response of the aircraft to the (hopefully improved) response on the vibration isolator.

  b.) Report on the values of mass, stiffness and damping that you used and how they would be realized in practice (e.g. if you use a beam as a leaf spring, what would its dimensions be?).

1.1 Part(a)

Vibration isolation was based on reducing absolute acceleration of passenger under turbulent external forces. This was done by isolating the passenger from the base motion subjected to external absolute acceleration. Hence the model is based on the following diagram
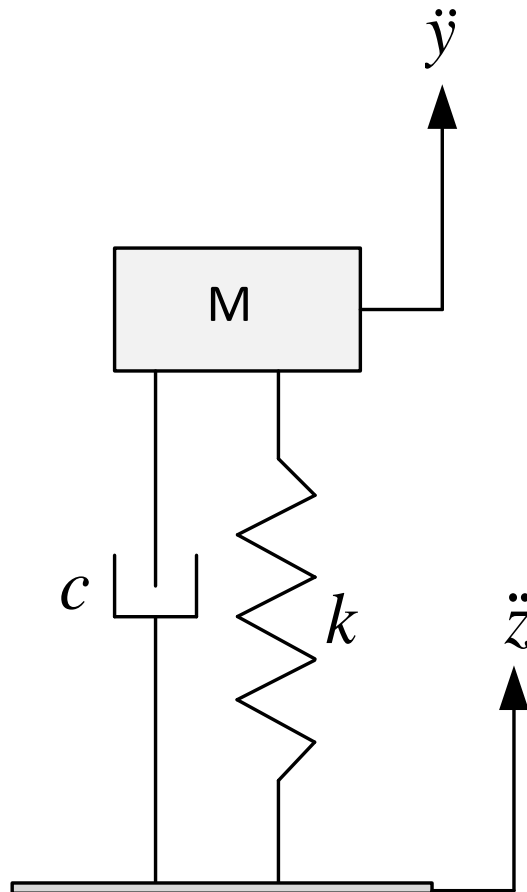
Hence EQM of motion is

$$my'' + c(y' - z') + k(y - z) = 0$$
$$my'' + cy' + ky = cz' + kz \tag{1}$$

We are given the time history of the turbulent acceleration. Hence in frequency domain we can write

$$z'' = \text{Re}\left\{Z_n^{acc} e^{i(\omega_1 n)t}\right\}$$

Where Z_n^{acc} is the complex amplitude of the nth harmonic component in the acceleration data. Let $\omega_1 n \equiv \varpi_n$ then using the above, In frequency domain Eq 1 becomes

$$\text{Re}\left\{\left(-m\varpi_n^2 + i\varpi_n c + k\right)Y_n e^{i\varpi_n t}\right\} = \text{Re}\left\{\left(c\frac{Z_n^{acc}}{i\varpi_n} + k\frac{Z_n^{acc}}{-\varpi_n^2}\right)e^{i\varpi_n t}\right\}$$

$$Y_n = \left(\frac{\frac{c}{i\varpi_n} - \frac{k}{\varpi_n^2}}{-m\varpi_n^2 + i\varpi_n c + k}\right)Z_n^{acc}$$

The above gives the transfer function between the displacement of the passenger and the external acceleration. In otherwords

$$y(t) = \text{Re}\left\{\left(\frac{\frac{c}{i\varpi_n} - \frac{k}{\varpi_n^2}}{-m\varpi_n^2 + i\varpi_n c + k}\right)Z_n^{acc} e^{i(\omega_1 n)t}\right\}$$

Let

$$Y_n = \left(\frac{\frac{c}{i\varpi_n} - \frac{k}{\varpi_n^2}}{-m\varpi_n^2 + i\varpi_n c + k}\right)Z_n^{acc}$$

then the transfer function is

$$H(\varpi_n) = \frac{Y_n}{Z_n^{acc}} = \frac{\frac{-ic}{\varpi_n} - \frac{k}{\varpi_n^2}}{-m\varpi_n^2 + i\varpi_n c + k}$$

$$= -\frac{1}{\varpi_n^2}\frac{(k + ic\varpi_n)}{\left(k - m\varpi_n^2\right) + i\varpi_n c}$$

Hence phase is

$$\arg(H(\varpi_n)) = \tan^{-1}\left(\frac{c\varpi_n}{k}\right) - \tan^{-1}\left(\frac{\varpi_n c}{k - m\varpi_n^2}\right)$$

and magnitude is

$$|H(\varpi_n)| = \left|\frac{Y_n}{Z_n^{acc}}\right| = \frac{1}{\varpi_n}\frac{\sqrt{k^2 + c^2\varpi_n^2}}{\sqrt{\left(k - m\varpi_n^2\right)^2 + (\varpi_n c)^2}}$$

These can be written in terms of ζ and ω_{nat} as follows. From $H(\omega_n) = -\frac{1}{\omega_n^2} \frac{(k+ic\omega_n)}{(k-m\omega_n^2)+i\omega_n c}$, dividing numerator and denominator by $k = m\omega_{nat}^2$ and using $c = 2\zeta m\omega_{nat}$ then

$$H(\omega_n) = -\frac{1}{\omega_n^2} \frac{\left(1 + \frac{i2\zeta m\omega_{nat}\omega_n}{m\omega_{nat}^2}\right)}{\left(1 - \frac{m\omega_n^2}{m\omega_{nat}^2}\right) + \frac{i\omega_n 2\zeta m\omega_{nat}}{m\omega_{nat}^2}} = -\frac{1}{\omega_n^2} \frac{\left(1 + \frac{i2\zeta\omega_n}{\omega_{nat}}\right)}{\left(1 - \frac{\omega_n^2}{\omega_{nat}^2}\right) + \frac{i\omega_n 2\zeta}{\omega_{nat}}}$$

Let $r_n = \frac{\omega_n}{\omega_{nat}}$ then the above becomes

$$H(\omega_n) = -\frac{1}{\omega_n^2} \frac{(1 + i2\zeta r_n)}{\left(1 - r_n^2\right) + i2r_n\zeta}$$

Hence

$$|H(\omega_n)| = \frac{1}{\omega_n} \frac{\sqrt{1 + (2\zeta r_n)^2}}{\sqrt{\left(1 - r_n^2\right)^2 + (2r_n\zeta)^2}}$$

$$\arg(H(\omega_n)) = \tan^{-1}(2\zeta r_n) - \tan^{-1}\frac{2r_n\zeta}{1 - r_n^2}$$

The following is a plot showing the passenger absolute acceleration $y''(t)$ over the period of 80 seconds against the turbulent acceleration $z''(t)$. We now see that passenger absolute acceleration is close to the nominal acceleration. This was done using the following values for the vibration isolation

M	100000 kg
ζ	0.72
k	38924 N/m
c	57746 Ns/m

The plot on the right side is the absolute acceleration of the passenger during flight in the turbulent case.
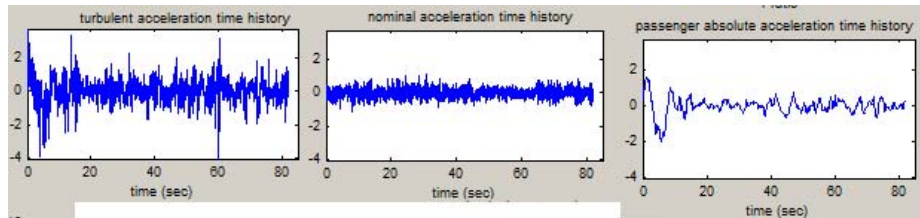
1.2 Part(b)

The length of first class cabinet was estimated to be $L = 15$ meters from looking at Boeing web page.

Using <u>Steel, Structural ASTM-A36</u> I beam as a cantilever beam for the implementation, then using $k = \frac{3EI}{L^3}$ results in

$$38924 = \frac{3\left(200 \times 10^9\right)I}{15^3}$$
$$I = 2.1895 \times 10^{-4} \text{ m}^4$$

Using rectangle cross section $I = \frac{bh^3}{12}$. Letting $h = 20$ cm, then $b = \frac{\left(2.1895 \times 10^{-4}\right)12}{0.2^3} = 0.32843$ meter or 32 cm.

2 Problem 2

3.60 A one-degree-of-freedom system is subjected to a pulse excitation in the form of a parabola,

$$Q = 2000\frac{t(T - t)}{T^2}[h(t) - h(t - T)] \quad \text{N}$$

The system mass is 0.5 kg, and the natural frequency is 100 Hz. The pulse duration T equals the undamped period of free vibration period. The system is at rest in the equilibrium position at $t = 0$.

(a) Use FFT techniques to evaluate the response when $\zeta = 0.20$.

(b) Use FFT techniques to evaluate the response when $\zeta = 0.002$.

(c) Use superposition and Appendix B to derive the analytical solution for this pulse. Compare the analytical and FFT results for $\zeta = 0.2$ and $\zeta = 0.002$.

2.1 part(a)

$$Q = 2000t\frac{(T - t)}{T^2}[h(t) - h(t - T)]$$

$$m = 0.5 \text{ kg}$$
$$\omega_n = 2\pi f_n$$
$$f_n = 100 \text{ Hz}$$

Hence pulse duration is $\frac{1}{f} = 0.01$ sec.

$$my'' + cy' + ky = Q(t)$$

In the frequency domain assuming that the force $Q(t)$ can be represented in its Fourier series as

$$Q(t) = \text{Re}\left(\sum_n Q_n e^{i\omega_1 nt}\right)$$

where ω_1 is the fundamental frequency for $Q(t)$ which depends on the period we choose to select to sample over. In this example, I selected $3T$ as the overall period to sample over so that it covers the pulse duration and an additional time to show the free vibration part as well and to compare to the analytical solution. Hence the EQM becomes

$$Y_n = \frac{Q_n}{-m(n\omega_1)^2 + ic(n\omega_1) + k}$$

$k = \omega_n^2 m$ hence dividing the numerator and denominator by k we obtain

$$Y_n = \frac{\frac{Q_n}{K}}{\left(1 - \frac{m(n\omega_1)^2}{\omega_n^2 m}\right) + \frac{ic(n\omega_1)}{\omega_n^2 m}}$$

$$= \frac{1}{k} \frac{1}{\left(1 - r_n^2\right) + i2\zeta r_n} Q_n$$

where $r_n = \frac{n\omega_1}{\omega_n}$. Hence response is

$$y(t) = \text{Re}\left(\sum_n Y_n e^{i\omega_1 nt}\right)$$

$$= \text{Re}\left(\sum_n \frac{1}{k} \frac{1}{\left(1 - r_n^2\right) + i2\zeta r_n} Q_n e^{i\omega_1 nt}\right)$$

$y(t)$ is found by taking the IFFT of $\sum_n \frac{1}{k} \frac{1}{\left(1 - r_n^2\right) + i2\zeta r_n} Q_n$.

Q_n values are found by taking the FFT of $Q(t)$. We start by sampling $Q(t)$. To obtain the solution for say $t = 0 \cdots 3T$, then we have to assume that the period of the signal is actually $3T$ and sample over this whole time from $0 \cdots 3T - delt$. Then we use FFT on the result. Then find the response by doing IFFT. Using $N = 128$ over $t = 0 \cdots 0.03$ seconds, the following solution was obtained

force Q(t) and its reponse. 16 samples, delT=0.000236

reponse at zeta=0.200000
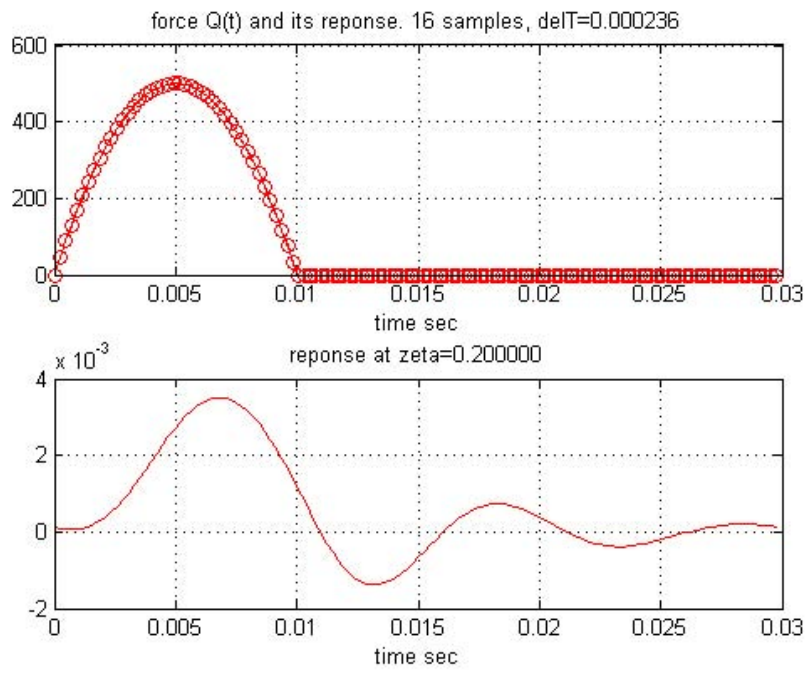
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%by Nasser M. Abbasi, HW 7, EMA 545
close all;
T     = 0.01; %sec
duration = 3*T;   %duration to find solution over
N     = 128;
delT = duration/(N-1);
w1    = 2*pi/duration; %fundamental freq rad/sec
t     = linspace(0,(duration-delT),N);
Qt    = @(t) (2000*t.*(T-t))/T^2.*(t<=T)+0*(t>T)

subplot(2,1,1)
plot(t,Qt(t),'r-o');
hold on;
plot(0:delT:duration,Qt(0:delT:duration),'r');

title(sprintf('force Q(t) and its reponse. 16 samples, delT=%f',delT));
xlabel('time sec');
grid;

m       = 0.5; %mass kg
wn      = 2*pi*100; %natural freq
k       = wn^2*m;   %stiffness N/meter
[Q,ws] = fft_easy(Qt(t),delT);

zeta    = 0.002;
I       = sqrt(-1);
y       = ifft_easy( (Q/k)./( (1-(ws/wn).^2) + 2*I*zeta*ws/wn),ws);

subplot(2,1,2);
plot(t,y,'r');
title(sprintf('reponse at zeta=%f',zeta));
xlabel('time sec');
grid;
```
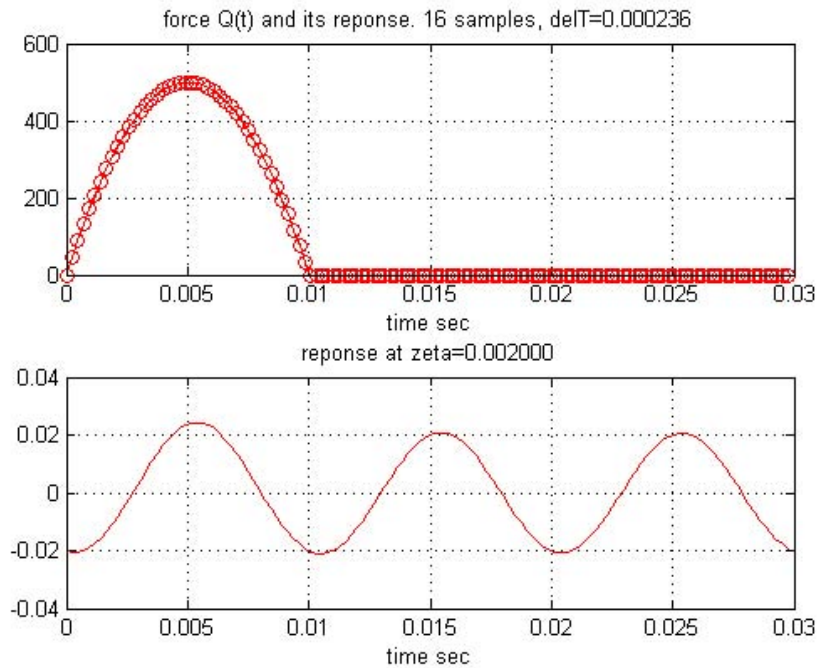
2.2 Part(b)

For $\zeta = 0.002$ the above Matlab script was modified and the following solution resulted.

force Q(t) and its reponse. 16 samples, delT=0.000236

reponse at zeta=0.002000

Now we compare the above with the analytical solution.

2.3 Part(c)

The pulse can be written as

$$F = Q(t)[h(t) - h(t - T)]$$
$$= Q(t)h(t) - Q(t)h(t - T)$$

Let $t' = t - T$, hence $t = t' + T$, therefore the above becomes

$$F = Q(t)h(t) - Q(t' + T)h(t')$$

But $Q(t) = \frac{2000t(T-t)}{T^2}$. Let $\frac{2000}{T^2} = \beta$ since it is a constant. Hence $Q(t) = \beta t(T - t)$. Now we write the above F as

$$
\begin{aligned}
F &= \beta t(T - t)h(t) - \beta(t' + T)(T - (t' + T))h(t') \\
&= \left(\beta T t - \beta t^2\right)h(t) - \beta(t' + T)(-t')h(t') \\
&= \left(\beta T t - \beta t^2\right)h(t) + \beta\left((t')^2 + Tt'\right)h(t') \\
&= \beta T t h(t) - \beta t^2 h(t) + \beta T(t')^2 + \beta T t' h(t') \quad (2)
\end{aligned}
$$

So we see that the response to F will be the response to a unit impulse $h(t)$ with forcing basis functions that are $1, t, t^2$. Now we can use the solution from back of the book appendix B to sum the responses in order to find the final response and compare to the FFT method.

From appendix B, the response to unit ramp $th(t)$ is

$$r(th(t)) = \frac{1}{m\omega_n^3}\left(\omega_n t - 2\zeta + e^{-\zeta\omega_n t}\left[2\zeta\cos\omega_d t - \left(1 - 2\zeta^2\right)\frac{\omega_n}{\omega_d}\sin\omega_d t\right]\right)h(t)$$

and the response to quadratic $t^2 h(t)$ is

$$s\left(t^2 h(t)\right) = \frac{1}{m\omega_n^4}\left((\omega_n t)^2 - 4\zeta\omega_n t - 2\left(1 - 4\zeta^2\right) + e^{-\zeta\omega_n t}\left[2\left(1 - 4\zeta^2\right)\cos\omega_d t + \left(6\zeta - 8\zeta^3\right)\frac{\omega_n}{\omega_d}\sin\omega_d t\right]\right)h(t)$$

Now that we have the basis solutions, we can apply them to EQ 2

$$F = \beta T(r(t) + r(t')) - \beta T(s(t) - s(t'))$$
$$= \beta(r(t) + r(t - T)) - \beta T(s(t) - s(t - T))$$
$$= (\beta T)\frac{1}{m\omega_n^3}\left(\omega_n t - 2\zeta + e^{-\zeta\omega_n t}\left[2\zeta\cos\omega_d t - \left(1 - 2\zeta^2\right)\frac{\omega_n}{\omega_d}\sin\omega_d t\right]\right)h(t)$$
$$+ (\beta T)\frac{1}{m\omega_n^3}\left(\omega_n t' - 2\zeta + e^{-\zeta\omega_n t'}\left[2\zeta\cos\omega_d t' - \left(1 - 2\zeta^2\right)\frac{\omega_n}{\omega_d}\sin\omega_d t'\right]\right)h(t')$$
$$- (\beta)\frac{1}{m\omega_n^4}\left((\omega_n t)^2 - 4\zeta\omega_n t - 2\left(1 - 4\zeta^2\right) + e^{-\zeta\omega_n t}\left[2\left(1 - 4\zeta^2\right)\cos\omega_d t + \left(6\zeta - 8\zeta^3\right)\frac{\omega_n}{\omega_d}\sin\omega_d t\right]\right)h(t)$$
$$+ (\beta)\frac{1}{m\omega_n^4}\left((\omega_n t')^2 - 4\zeta\omega_n t' - 2\left(1 - 4\zeta^2\right) + e^{-\zeta\omega_n t'}\left[2\left(1 - 4\zeta^2\right)\cos\omega_d t' + \left(6\zeta - 8\zeta^3\right)\frac{\omega_n}{\omega_d}\sin\omega_d t'\right]\right)h(t')$$

In the above, $\omega_d = \omega_n\sqrt{1 - \zeta^2}$. To plot this solution, the following small script was used and was run for both $\zeta = 0.2$ and $\zeta = 0.002$

For $\zeta = 0.2$

For $\zeta = 0.002$
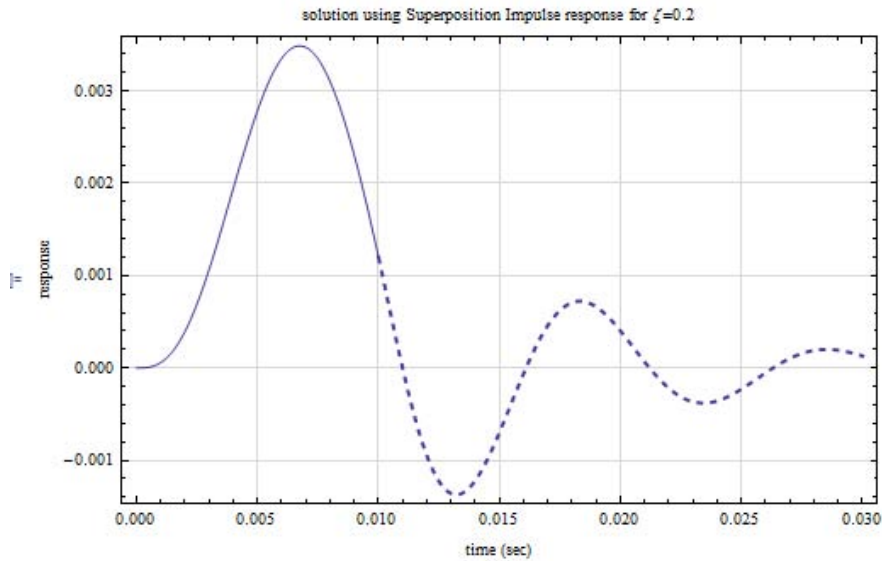
solution using Superposition Impulse response for ζ=0.002
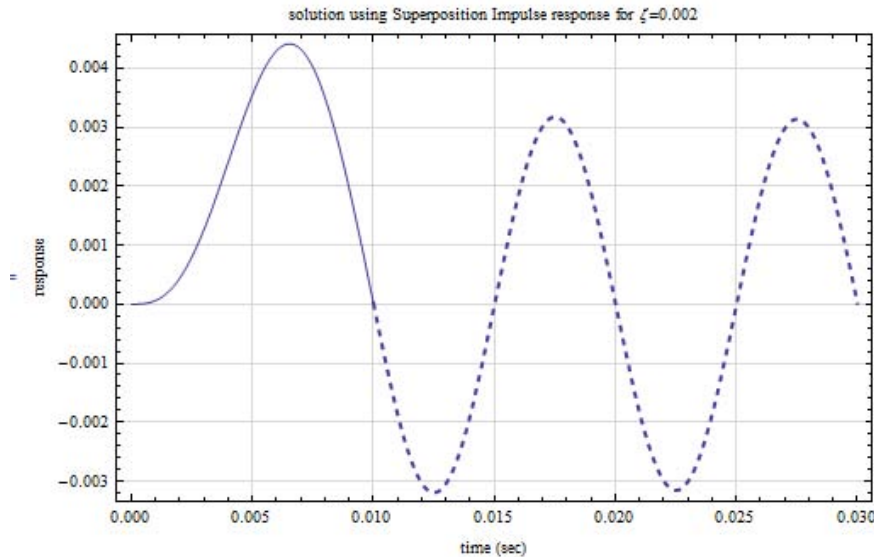
2.4 Conclusions

The analytical solution, using superposition agreed with the FFT solution for $\zeta = 0.2$. However, for some reason which I am not able to determine why yet, the FFT solution when $\zeta = 0.002$ did not agree with the analytical solution. The analytical solution was verified to be correct using another numerical ODE solver. So the FFT method for some reason is not giving accurate result for $\zeta = 0.002$. The same Matlab script was used for both cases. I tried increasing the sampling rate but that did not change the result. Please see Appendix for verification and the code used to plot the analytical solutions.

3   Problem 3

Problem 3: Exercise 1.11 from Ginsberg. (For the proof described in (b), set m_2=0 and see what your equation of motion reduces to.)

1.11 When the system in the sketch is at its static equilibrium position, there is no axial force in each spring.

(a) Derive equations of motion for the horizontal displacements x_1 and x_2 measured from the equilibrium position.

(b) Prove that if $m_2 = 0$, the equation of motion for x_1 is the same as that obtained by replacing the four springs by a single equivalent spring. Identify the way in which the springs are connected—for example, series or parallel—according to the equivalent spring stiffness.
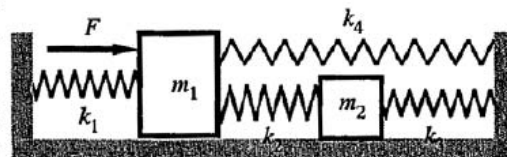
EXERCISE 1.11

3.1 Part(a)

Let T be the kinetic energy and V be the potential energy. Then equation of motion for a generalized coordinate q_i is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

Where L is the Lagrangian $L = T - V$ and Q_i is the generalized force in the q_i direction.

Assuming $x_2 > x_1$ and masses are moving to the right. For x_1 we obtain

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_1\dot{x}_2^2$$
$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2 + \frac{1}{2}k_4x_1^2 + \frac{1}{2}k_3x_2^2$$
$$Q_1 = F$$
$$Q_2 = 0$$

Hence

$$L = T - V$$
$$= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_1\dot{x}_2^2 - \left(\frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2 + \frac{1}{2}k_4x_1^2 + \frac{1}{2}k_3x_2^2\right)$$

$$\frac{\partial L}{\partial \dot{x}_1} = m_1\dot{x}_1$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1$$
$$\frac{\partial L}{\partial x_1} = -k_1x_1 - k_2(x_2 - x_1)(-1) - k_4x_1$$

and

$$\frac{\partial L}{\partial \dot{x}_2} = m_1\dot{x}_2$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m_1\ddot{x}_2$$
$$\frac{\partial L}{\partial x_2} = -k_2(x_2 - x_1)(1) - k_3x_2$$

Hence the 2 EOM are for x_1

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = F$$

$$m_1 \ddot{x}_1 - (-k_1 x_1 + k_2(x_2 - x_1) - k_4 x_1) = F$$

$$m_1 \ddot{x}_1 + k_1 x_1 - k_2(x_2 - x_1) + k_4 x_1 = F$$

Therefore EOM 1

$$m_1 \ddot{x}_1 + (k_1 + k_2 + k_4)x_1 - k_2 x_2 = F$$

and for x_2

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

$$m_1 \ddot{x}_2 - (-k_2(x_2 - x_1) - k_3 x_2) = 0$$

$$m_1 \ddot{x}_2 + k_2(x_2 - x_1) + k_3 x_2 = 0$$

Hence EOM 2

$$m_1 \ddot{x}_2 + (k_2 + k_3)x_2 - k_2 x_1 = 0$$

Hence in Matrix form EOM are

$$MX'' + KX = Q$$

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} + \begin{pmatrix} (k_1 + k_2 + k_4) & -k_2 \\ -k_2 & (k_2 + k_3) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

3.2 Part(b)

If m_2 do not exist, then this means the springs k_2 and k_3 do not have a mass between them and so these need to be replaced by single spring, say k_5 found by finding equivalent spring in series
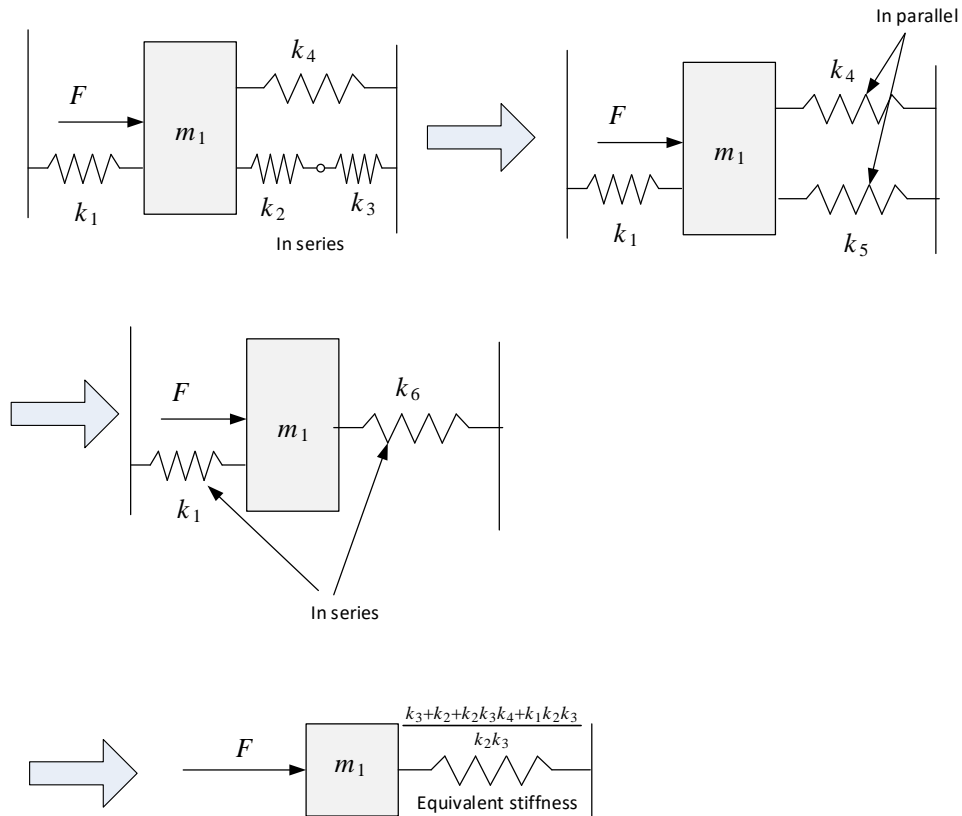
$$\frac{1}{k_5} = \frac{1}{k_2} + \frac{1}{k_3}$$

$$k_5 = \frac{k_3 + k_2}{k_2 k_3}$$

From above, EQM for m_1 becomes

$$m_1 \ddot{x}_1 + \left(k_1 + \overbrace{\left(\frac{k_3 + k_2}{k_2 k_3} \right)}^{k_5} + k_4 \right) x_1 = F$$

So now k_4 and k_4 are in parallel, hence we replace $k_5 + k_4$ by k_6 found from

$$k_6 = k_5 + k_4$$

$$= \left(\frac{k_3 + k_2}{k_2 k_3} \right) + k_4$$

$$k_6 = \frac{k_3 + k_2 + k_2 k_3 k_4}{k_2 k_3}$$

Hence EQM for m_1 now becomes

$$m_1 \ddot{x}_1 + \left(k_1 + \frac{\overbrace{k_3 + k_2 + k_2 k_3 k_4}^{k_6}}{k_2 k_3} \right) x_1 = F$$

and finally

$$m_1 \ddot{x}_1 + \frac{k_3 + k_2 + k_2 k_3 k_4 + k_1 k_2 k_3}{k_2 k_3} x_1 = F$$