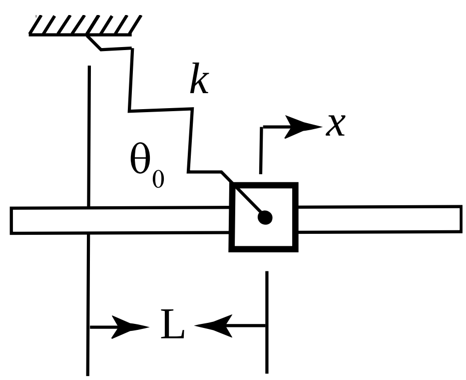
Homework #8
EMA 545, Spring 2013

Problem 1: (40 points)

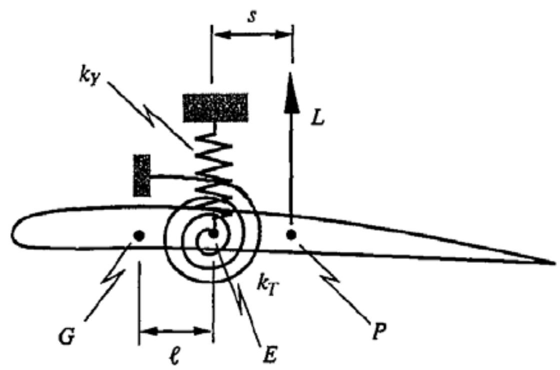
- **a.)** Find the nonlinear equation of motion for the system pictured below. The block has mass *m* and the guide can be approximated as frictionless. In the position shown the spring is unstretched and the angle between the spring and guide bar is θ_0.
- **b.)** Linearize your equation of motion for small deflections from the position shown (i.e. using a Taylor series expansion on $k(x)$ about x=0). Use a computer to plot k(x) versus the linear approximation for L=1 m, k=1000 N/m and θ_0 = 45 degrees for x ranging from -1 m to +1 m.
- **c.)** Find the equations of motion for the system using the stiff spring approximation and assuming small displacements from an equilibrium position defined by L=1 m, k=1000 N/m and θ_0 = 45 degrees. Compare your result with your linearized result from part (b).
- **d.)** Using m=1, find the response of the nonlinear system (in part a) using ode45 and plot the displacement of the mass over a few cycles when it is released from rest at x(0)=0.1 and also at x(0)=0.5 meters. Overlay both curves on the same set of axes. How does the period of the response compare with the linearized natural frequency in each case? In what other way(s) does the nonlinearity manifest itself in the response of the system when x(0)=0.5?



Problem 2: Exercise 1.27 from Ginsberg.

A standard model for a wing has a translational spring k_y and a torsional spring k_T representing the elastic rigidity. Point E represents the elastic center because static application of a vertical force at that point results in upward displacement without an associated rotation. The design of the wing is such that horizontal movement of point E is negligible. The lift force *L* acts at point P, which is called the center of pressure. The lift force may be treated as known. When the wing is in its static equilibrium position, points G, E and P form a horizontal line. Point G is the center of mass, and the radius of gyration of the wing about that point is r_G. Denote the mass of the wing *m*. Derive the equations of motion for the wing, <u>assuming small displacements (and small rotational displacements)</u>. Put the equations in matrix form and <u>check the units and sign of each</u>

term in your EOM. (Hint: use the displacement of the center of gravity and the rotation of the wing as generalized coordinates.)

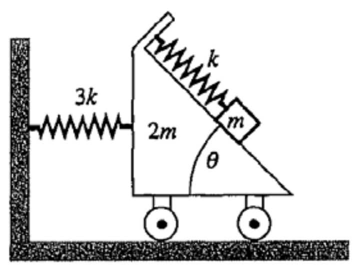


EXERCISE 1.27

Problem 3: Use the power balance method and the stiff spring approximation to find the equation of motion of the system pictured in Problem 1.16.

Problem 4: Exercise 1.33 from Ginsberg: (be very careful to write a correct expression for the acceleration of the small block.) Check the unit and sign of each term in your EOM.

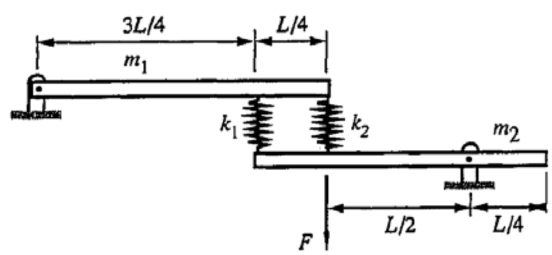
1.33 Determine the equations of motion governing a pair of generalized coordinates that locate the position of the cart and the sliding block. Friction is negligible.



EXERCISE 1.33

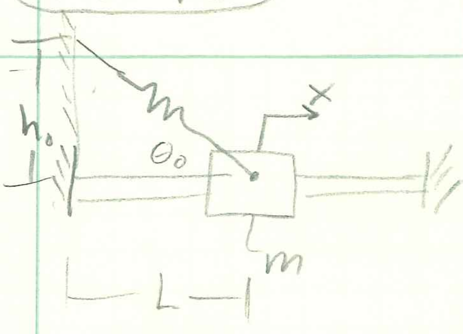
Problem 5: Exercise 1.30 from Ginsberg: Use the stiff spring approximation and assume small deflections of both bars. Check the units and sign of each term in your EOM. Gravity acts downward (same direction as the force, *F*).

1.30 Both bars in the linkage are horizontal, as shown, when the system is in static equilibrium. Determine the linearized equations of motion for this system.



EXERCISE 1.30

Problem 6: Exercise 4.1 in Ginsberg. Solve the eigenvalue problem by hand to get the natural frequencies and mode shapes. You may check your answers with Matlab.

a) 

$$|\vec{F}_{spr}| = k\Delta$$

$$\sum \vec{F}\cdot\hat{\imath} = -|\vec{F}_{spr}|\cos\theta = m\ddot{x}$$

$$h = L\tan\theta_0$$

Length $= L_{spr}$ $\quad L_{spr}^2 = h^2 + (L+x)^2$

$$\cos\theta = \frac{L+x}{\sqrt{h^2+(L+x)^2}}$$

$$\Delta = (L_{curr} - L_0) = \sqrt{h^2+(L+x)^2} - \sqrt{h^2+L^2}$$

So, everything is known, and eom becomes

$$-k\left(\sqrt{h^2+(L+x)^2} - \sqrt{h^2+L^2}\right)\left(\frac{L+x}{\sqrt{h^2+(L+x)^2}}\right) = m\ddot{x}$$

or: $\boxed{m\ddot{x} + K(x) = 0,\ K(x) = k\left(\sqrt{h^2+(L+x)^2} - \sqrt{h^2+L^2}\right)\left(\frac{L+x}{\sqrt{h^2+(L+x)^2}}\right)}$

b) Linearize: expand $K(x)$ in a taylor series

$$K(x) = K(0) + \frac{\partial K}{\partial x}\bigg|_{x=0}\Delta x + \text{H.O.T.}$$

$$K(x) \approx 0 + +k\Big[(h^2+(L+x)^2)^{-1/2}(\cancel{\tfrac{1}{2}})(\cancel{2}(L+x))\left(\frac{L+x}{(h+(L+x)^2)^{1/2}}\right)$$

$$+ \underbrace{\left(\sqrt{h^2+(L+x)^2} - \sqrt{h^2+L^2}\right)}_{0\ @\ x=0}\frac{L}{\sqrt{h^2+(L+x)^2}} +$$

$$+ \underbrace{\left(\sqrt{h^2+(L+x)^2} - \sqrt{h^2+L^2}\right)}_{0\ @\ x=0}(L+x)(h+(L+x)^2)^{-3/2}(-\tfrac{1}{2})(\cancel{2}(L+x))\Big]\Delta x$$

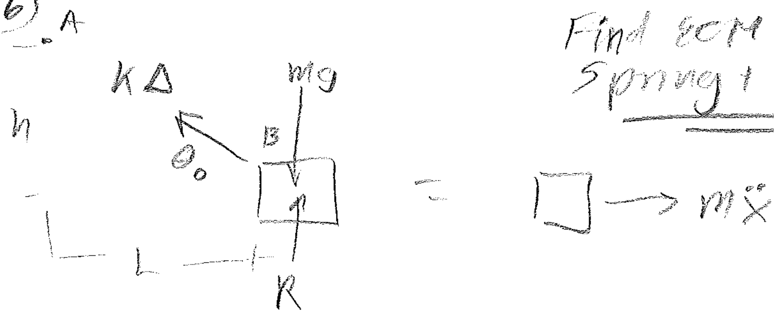
eval at $x=0$

$$K(x) \approx +k\left[\frac{(L+x)^2}{h^2+(L+x)^2}\right]\bigg|_{x=0}\Delta x = \boxed{+k\left(\frac{L^2}{h^2+L^2}\right)\Delta x \approx K(x)}$$

HW#2, EMA545 | Spring 2011

(P6)

Find EOM Assuming stiff Spring + small deformations!



$$\Delta = u_B - \cancel{u_A}^{\,0} \quad (e_{B/A})$$
$$= (x\hat{i})\cdot(\cos\theta\,\hat{i} - \sin\theta\,\hat{j}) = x\cos\theta$$

$$\Sigma F\cdot\hat{i} = -(Kx\cos\theta)\cos\theta = m\ddot{x}$$

$$\boxed{m\ddot{x} + K\cos^2\theta\, x = 0}$$

Compare with HW#1 →

$$\cos\theta = \frac{L}{\sqrt{h^2+L^2}} \rightarrow \quad m\ddot{x} + \frac{KL^2}{h^2+L^2}x = 0 \quad ✓$$

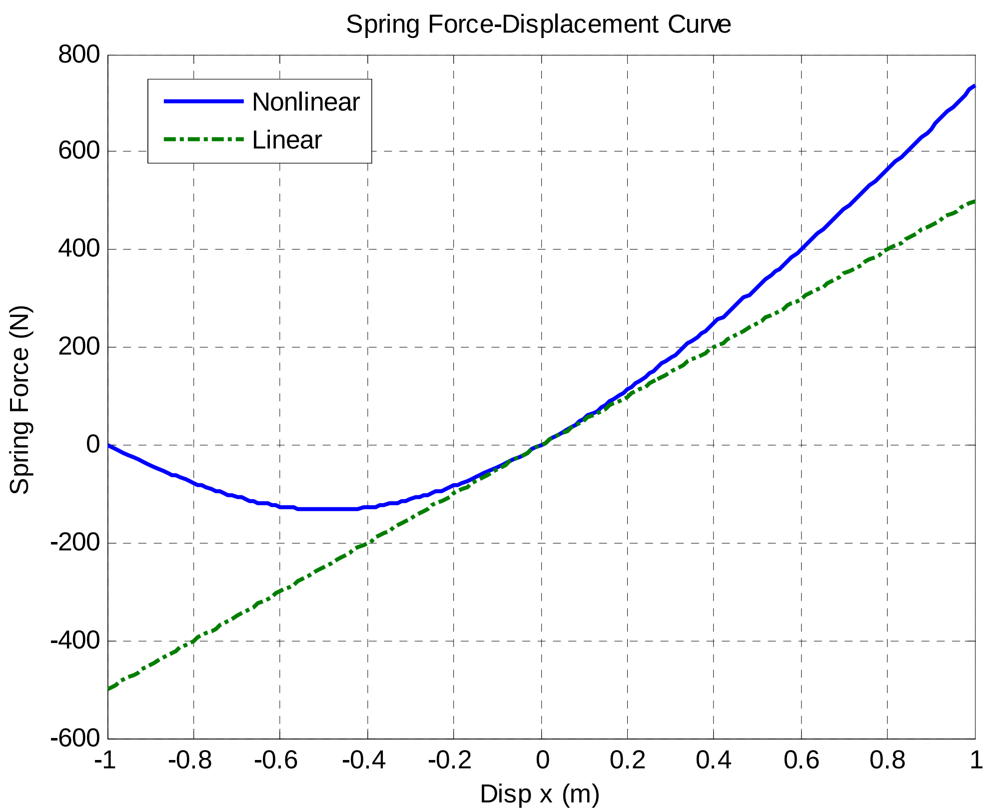
```
% Part (b)
% Plot nonlinear k(x) for large deformations of spring-mass system.
% M.S. Allen, Spring 2011, EMA 545

k=1000; %N/m
L = 1; %m
theta = 45*pi/180; % rad
xs = [-1:0.01:1]; % m
h = L*tan(theta);

kx = k*((h^2+(L+xs).^2).^(1/2)-
sqrt(h^2+L^2)).*((L+xs)./(h^2+(L+xs).^2).^(1/2));

klin = k*(L^2/(h^2+L^2));

figure(1)
plot(xs,kx,xs,klin*xs,'-.'); set(get(gca,'Children'),'LineWidth',2);
grid on;
xlabel('Disp x (m)'); ylabel('Spring Force (N)');
title('Spring Force-Displacement Curve');
legend('Nonlinear','Linear');
```

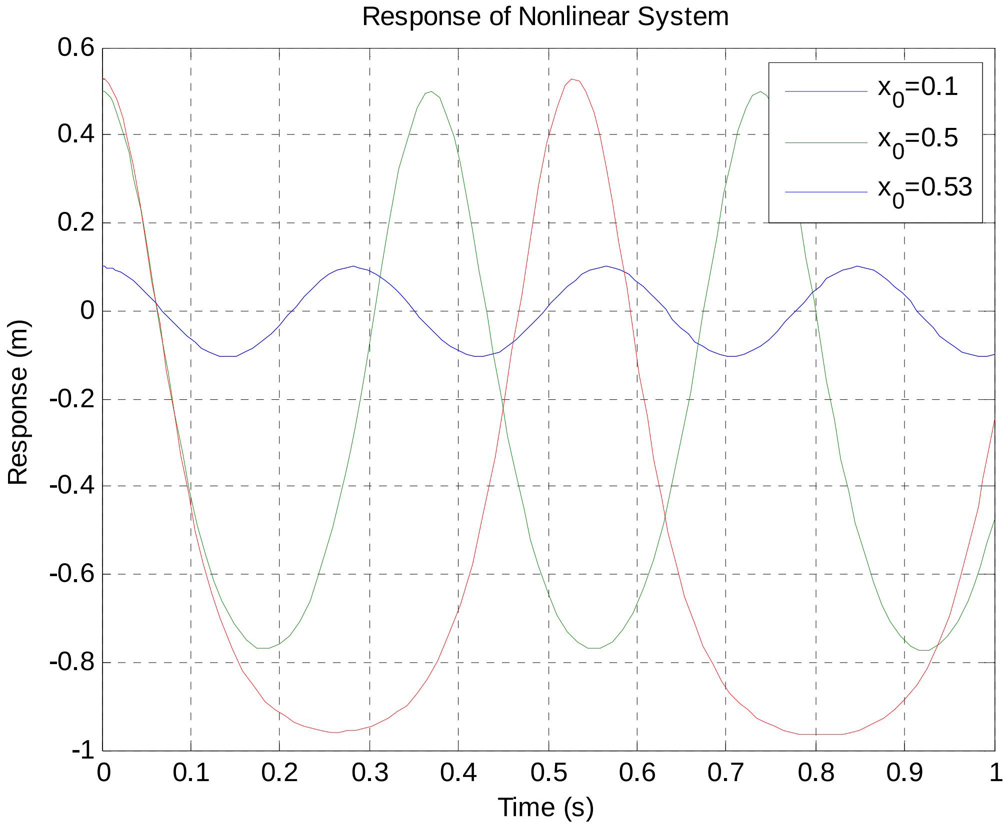



```
% Part (d)
% Find response to a small disturbance.
m=1;
wn_lin = sqrt(klin/m)
```

```
eom = @(t,x) [x(2)+0*t; -(1/m)*(k*((h^2+(L+x(1)).^2).^(1/2)-
sqrt(h^2+L^2)).*((L+x(1))./(h^2+(L+x(1)).^2).^(1/2)))]

[ts1,y1]=ode45(eom,[0,1],[0.1; 0]);
[ts2,y2]=ode45(eom,[0,1],[0.5; 0]);

figure(2)
plot(ts1,y1(:,1),ts2,y2(:,1)); hold on; grid on;
xlabel('Time (s)'); ylabel('Response (m)');
title('Response of Nonlinear System');
```

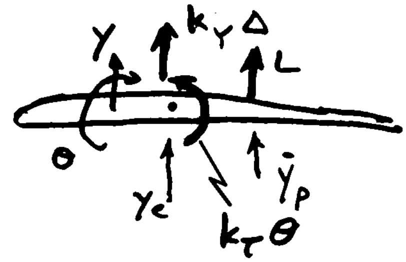


The period of the nonlinear response in each case is given below (found using ginput on the plot). The linearized natural frequency is 22.36 rad/s and the corresponding period is 0.281 seconds.

Initial Displacement	Period (s)
x_0=0.1	0.2823
x_0=0.5	0.3698
x_0=0.53	0.5311

The behavior of the system is quite peculiar. The period becomes longer (frequency lower) as the system approaches the region where the stiffness vanishes. As shown, with a slightly larger initial displacement of 0.53, the mass almost comes to rest as the mass approaches x=-1, which is the other equilibrium position. Incidentally, the body panels of a hypersonic aircraft, which I am studying as part of an Air Force grant, can behave very similarly. They buckle due to thermal expansion and then as they vibrate they may jump between two equilibria.

Exercise 1.27



$q_1 = Y$ of G , $q_2 = \theta$ (cw)

$\Delta = -Y_E$

$Y_E = Y - l\theta, \ \dot{Y}_P = \dot{Y} - (l+s)\dot{\theta}$

$T = \frac{1}{2} m \dot{y}^2 + \frac{1}{2} I_G \dot{\theta}^2$

$I_G = m r_G^2$

$T = \frac{1}{2} m \dot{y}^2 + \frac{1}{2}(m r_G^2)\dot{\theta}^2 = \frac{1}{2}[M_{11}\dot{y}^2 + M_{22}\dot{\theta}^2 + 2M_{12}\dot{y}\dot{\theta}]$

$M_{11} = m, \ M_{22} = m r_G^2, \ M_{12} = 0$

$V = \frac{1}{2} k_Y \Delta^2 + \frac{1}{2} k_T \theta^2 = \frac{1}{2} k_Y (y - l\theta)^2 + \frac{1}{2} k_T \theta^2$

$= \frac{1}{2}[k_Y y^2 + (k_Y l^2 + k_T)\theta^2 - 2 k_Y l Y \theta]$

$= \frac{1}{2}[K_{11} y^2 + K_{22}\theta^2 + 2K_{12} Y\theta]$

$K_{11} = K_Y, \ K_{22} = k_Y l^2 + K_T, \ K_{12} = -K_Y l$

$P_{dis} = 0, \ P_{in} = L\dot{Y}_P = L[\dot{y} - (l+s)\dot{\theta}] = Q_1\dot{y} + Q_2\dot{\theta}$

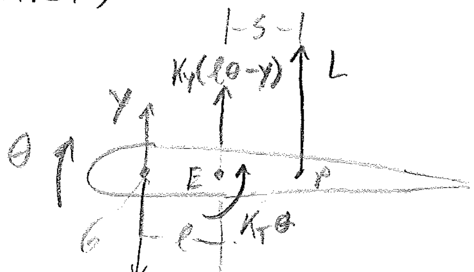
$Q_1 = L, \ Q_2 = -L(l+s)$

$[M]\{\ddot{q}\} + [K]\{q\} = \{Q\}$

$\Rightarrow$ where $\{q\} = [Y \ \ \theta]^T, \ [M] = \begin{bmatrix} m & 0 \\ 0 & m r_G^2 \end{bmatrix}$

$\Rightarrow \ [K] = \begin{bmatrix} k_Y & -k_Y l \\ -k_Y l & (k_Y l^2 + k_T) \end{bmatrix}, \ \{Q\} = \begin{Bmatrix} L \\ -L(l+s) \end{Bmatrix}$

P1 (1.27)



$\cancel{mg}$, leave off - write about static eq.

$$\Sigma M_g = -I\ddot{\theta}$$

$$K_T\theta + lK_y(l\theta - y) + L(s+l) = -I_g\ddot{\theta}$$

$$\boxed{m r_g^2\ddot{\theta} + K_T\theta + K_y l^2\theta - lK_y y = -(s+l)L}$$

$$\Sigma F_y = K_y(l\theta - y) + L = m\ddot{y}$$

$$\boxed{m\ddot{y} + K_y y - lK_y\theta = L}$$

✓ +L causes $\ddot{\theta} < 0$, $\ddot{y} > 0$

✓ Spring stiffnesses + on diagonal, neg off diagonal

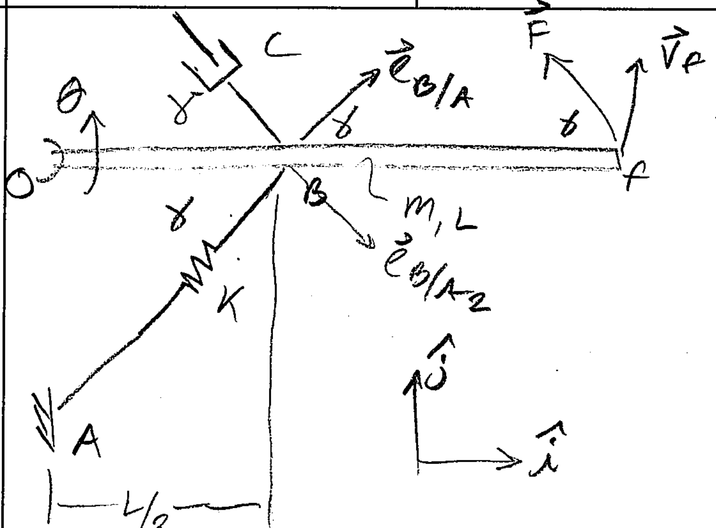
Grader, −1 or −2 pts if no check of EOM!

* Note, bend and twist are coupled (as in a real wing) since the elastic center ≠ center of mass.

$$\boxed{\begin{bmatrix} I_g & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix}\ddot{\theta} \\ \ddot{y}\end{Bmatrix} + \begin{bmatrix} (K_T + K_y l^2) & -lK_y \\ -lK_y & K_y \end{bmatrix}\begin{Bmatrix}\theta \\ y\end{Bmatrix} = \begin{Bmatrix} -(s+l)L \\ L \end{Bmatrix}}$$

Units - Kg-m²·rad/s², N-m/rad ✓, N/m·m·m ✓

1.16



Find EOM using P.B.

$T = \frac{1}{2} I_0 \dot{\theta}^2 \rightarrow$

$m_{11} = I_0 = \frac{1}{3} m L^2$

Use stiff Spring approx.

$\Delta \approx u_B - u_A \quad , \dot{u}_B = \vec{V}_B \cdot \vec{e}_{B/A}$

$\vec{V}_B = \frac{L}{3} \dot{\theta} \hat{j}$

$\vec{e}_{B/A} = \cos\gamma \hat{i} + \sin\gamma \hat{j}$

$\dot{u}_B = \frac{L}{3} \dot{\theta} \sin\gamma$

$\Delta = \left(\frac{L}{3} \theta \sin\gamma - \overset{u_A = 0}{0}\right)$

$V = \frac{1}{2} K \Delta^2 = \frac{1}{2} K \frac{L^2}{9} \sin^2\gamma \, \theta^2 \Rightarrow K_{11} = \frac{KL^2}{9} \sin^2\gamma$

$P_{dis} = c \dot{\Delta}^2 \qquad \dot{\Delta} = \vec{V}_B \cdot (\vec{e}_{B/A})_2 = \frac{L}{3} \dot{\theta} \hat{j} \cdot (\cos\gamma \hat{i} - \sin\gamma \hat{j})$

$= c\left(-\frac{L}{3} \dot{\theta} \sin\gamma\right)^2 = \underbrace{\frac{cL^2}{9} \sin^2\gamma}_{c_{11}} \dot{\theta}^2 \Rightarrow c_{11} = \frac{cL^2}{9} \sin^2\gamma$

$P_{in} = F \cdot \vec{V}_f = F \sin\gamma L \dot{\theta}$

$P_{in} = FL \sin\gamma \dot{\theta} = Q_1 \dot{q}_1 \rightarrow Q_1 = FL \sin\gamma$

EOM:

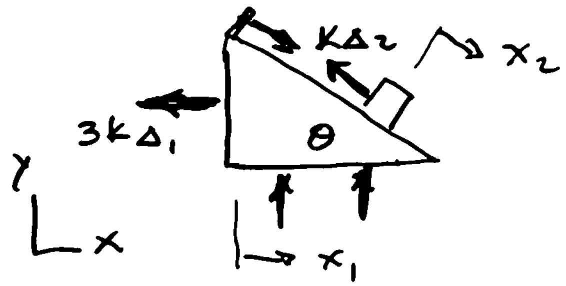
$$\boxed{\frac{1}{3} m L^2 \ddot{\theta} + \frac{cL^2}{9} \sin^2\gamma \, \dot{\theta} + \frac{KL^2}{9} \sin^2\gamma = FL \sin\gamma}$$

✓ signs ~ +F causes pos $\ddot{\theta}$ or pos. θ, etc...

$\sin\gamma \sim \gamma \rightarrow 0$., no contribution ✓

Units ✓ all moments

Exercise 1.33



$q_1 = x_1, \quad q_2 = x_2$

$\bar{v}_{cart} = \dot{x}_1 \bar{i}$

$\bar{v}_{bl} = \bar{v}_{cart} + \bar{v}_{bl/cart}$

$\quad = \dot{x}_1 \bar{i} + \dot{x}_2 (\cos\theta\, \bar{i} - \sin\theta\, \bar{j})$

$T = \frac{1}{2}(2m)\dot{x}_1^2 + \frac{1}{2} m\, \bar{v}_{bl} \cdot \bar{v}_{bl}$

$$T = \tfrac{1}{2}(2m)\dot{x}_1^2 + \tfrac{1}{2} m\left[(\dot{x}_1 + \dot{x}_2\cos\theta)^2 + (\dot{x}_2 \sin\theta)^2\right]$$

$$= \tfrac{1}{2}\left[3m\,\dot{x}_1^2 + m\,\dot{x}_2^2 + 2m\,\dot{x}_1\dot{x}_2\cos\theta\right]$$

$$= \tfrac{1}{2}\left[M_{11}\dot{x}_1^2 + M_{22}\dot{x}_2^2 + 2M_{12}\dot{x}_1\dot{x}_2\right]$$

$M_{11} = 3m, \quad M_{22} = M, \quad M_{12} = m\cos\theta$

$V = \frac{1}{2}(3k)\Delta_1^2 + \frac{1}{2}k\,\Delta_2^2$ $\quad$ {gravity only affects equil position of block)

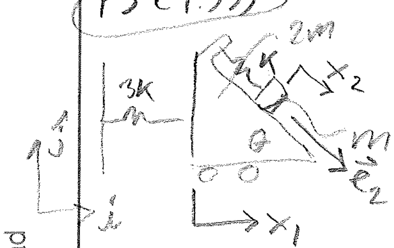
$\Delta_1 = x_1, \quad \Delta_2 = x_2$

$$V = \tfrac{1}{2}\left[K_{11}x_1^2 + K_{22}x_2^2 + 2K_{12}x_1x_2\right]$$

$K_{11} = 3k, \quad K_{22} = k, \quad K_{12} = 0$

$$[M] = m\begin{bmatrix} 3 & \cos\theta \\ \cos\theta & 1 \end{bmatrix}, \quad [K] = k\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

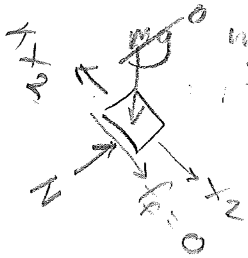
P3 (1.33)



- let x_2 denote disp. of block relative to the large cart relative to static equilibrium.

$\vec{a}_2 = \vec{a}\big|_{relative} + \vec{a}\big|_{cart}$

$= \ddot{x}_2 \vec{e}_2 + \ddot{x}_1 \hat{i}$

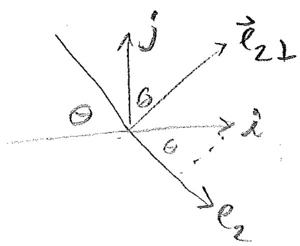


$= m\vec{a}_2 \qquad \Sigma F \cdot \vec{e}_2 = -Kx_2 = m(\ddot{x}_2 + \ddot{x}_1(\vec{e}_2 \cdot \hat{i}))$

$e_2 = \cos\theta\,\hat{i} - \sin\theta\,\hat{j}$

$-Kx_2 = m\ddot{x}_2 + m\cos\theta\,\ddot{x}_1$

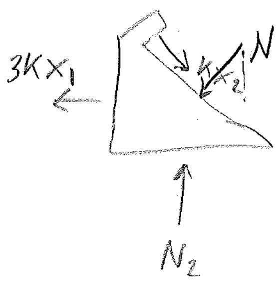
$\boxed{m\ddot{x}_2 + m\cos\theta\,\ddot{x}_1 + Kx_2 = 0}$



if $x_2 = 0$, $\ddot{x}_2 = -\cos\theta\,\ddot{x}_1$ ✓ units ✓

$\Sigma F \cdot \vec{e}_{2\perp} = N = m(\ddot{x}_2 \vec{e}_2 + \ddot{x}_1 \hat{i}) \cdot \vec{e}_{2\perp} = m\ddot{x}_1 \vec{e}_{2\perp}$

$N = m\ddot{x}_1 (\cos\theta\,\hat{j} + \sin\theta\,\hat{i}) \cdot \hat{i} = m\ddot{x}_1 \sin\theta = N$



$= 2m\ddot{x}_1 \rightarrow$

$\hat{e}_2$

$\Sigma F \cdot \hat{i} = -3Kx_1 + Kx_2(\cos\theta\,\hat{i} - \sin\theta\,\hat{j}) \cdot \hat{i} - N\sin\theta$

$= 2m\ddot{x}_1$

$2m\ddot{x}_1 + (m\ddot{x}_1 \sin\theta)\sin\theta - 1Kx_2\cos\theta + 3Kx_1 = 0$

$$\begin{bmatrix} 2m + m\sin^2\theta & 0 \\ m\cos\theta & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 3K & -K\cos\theta \\ 0 & K \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Note, EOM are non-symmetric. Can happen sometimes when using this Newton-Euler method!

$\cos\theta \cdot (2) + (1) \longrightarrow$ replace w/ (1)

$$3m = \begin{bmatrix} 2m + m(\sin^2\theta + \cos^2\theta) + m\cos\theta \\ m\cos\theta \quad m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 3K & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

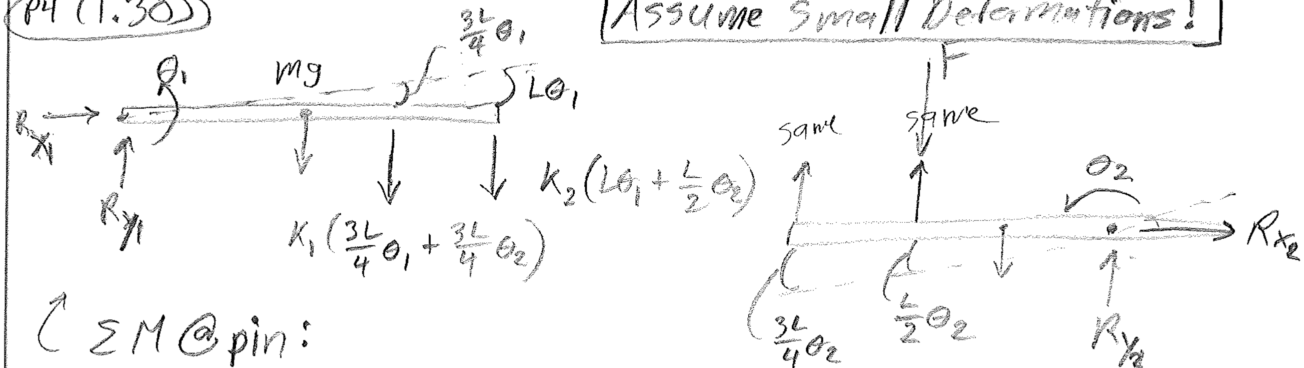
$$\begin{bmatrix} 3m & m\cos\theta \\ m\cos\theta & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 3K & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

✓ springs are uncoupled ~ x_2 doesn't affect spring (3K)

✓ Mass coupled, checked above, $\ddot{x}_2 = 0 \longrightarrow 3m\ddot{x}_1 + 3Kx_1 = 0$ ✓

P4 (1.30)

$\boxed{\text{Assume Small Deformations!}}$



$\circlearrowleft$ ΣM @ pin:

$$-\left(\frac{3L}{4}\right)K_1\left(\frac{3L}{4}\theta_1+\frac{3L}{4}\theta_2\right)-LK_2\left(L\theta_1+\frac{L}{2}\theta_2\right)-mg\frac{L}{2}\cos\theta_1=\frac{1}{3}m_1L^2\ddot{\theta}_1$$

$$\boxed{\frac{1}{3}m_1L^2\ddot{\theta}_1+\left(\frac{9L^2}{16}K_1+L^2K_2\right)\theta_1+\left(\frac{9L^2}{16}K_1+\frac{L^2}{2}K_2\right)\theta_2=\cancel{-mg\frac{L}{2}\cos\theta_1}^{\text{static}}} \quad (1)$$

*Last term falls out → it is a constant moment for small θ_1 and must be zero in static equilibrium.

Bar 2: ΣM @ pin

$$-K_1\frac{3L}{4}(\theta_1+\theta_2)\left(\frac{3L}{4}\right)-K_2\left(L\theta_1+\frac{L}{2}\theta_2\right)\left(\frac{L}{2}\right)+\left(F\frac{L}{2}\right)+\cancel{mg\frac{L}{4}\cos\theta_2}=I_2\ddot{\theta}_2$$

Static, no stiffness effect for small θ_2

$$I_2=\frac{1}{12}mL^2+m\left(\frac{L}{4}\right)^2=\left(\frac{1}{12}+\frac{1}{16}\right)mL^2\ddot{\theta}_2$$

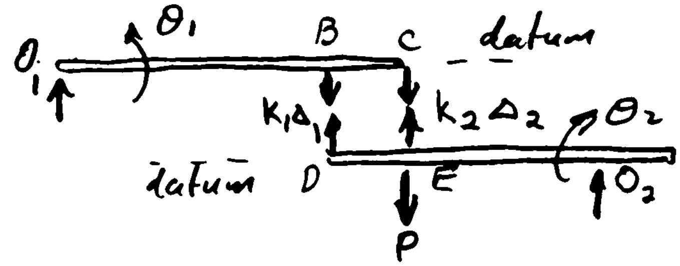
$$\boxed{\left(\frac{1}{12}+\frac{1}{16}\right)mL^2\ddot{\theta}_2+\left(\frac{9L^2}{16}K_1+\frac{L^2}{2}K_2\right)\theta_1+\left(\frac{9L^2}{16}K_1+\frac{L^2}{4}K_2\right)\theta_2=\frac{FL}{2}} \quad (2)$$

✓ units kg·m² rad/s² ✓, N/m·m²·rad ✓

✓ $\theta_1=0$, $\theta_2>0$, $\ddot{\theta}_1<0$ ✓ (in EOM 1)

✓ $\theta_2=0$, $\theta_1>0$, $\ddot{\theta}_2<0$ ✓ (in EOM 2) $F>0\rightarrow\ddot{\theta}_2>0$ ✓

Exercise 1.30



Given equil position is horizontal

Find eqs of motion

Solution: $q_1=\theta_1$, $q_2=\theta_2$

Both bars rotate about fixed points $\Rightarrow \omega_1=\dot\theta_1$, $\omega_2=\dot\theta_2$

$$T=\frac{1}{2}I_{O1}\dot\theta_1^2+\frac{1}{2}I_{O2}\dot\theta_2^2$$

where $I_{O1}=\frac{1}{3}m_1L^2$, $I_{O2}=\frac{1}{12}m_2L^2+m\left(\frac{L}{4}\right)^2=\frac{7}{48}m_2L^2$

Match to standard form: $T=\frac{1}{2}\left[(M_{11}\dot\theta_1^2+M_{22}\dot\theta_2^2)+2M_{12}\dot\theta_1\dot\theta_2\right]$

Thus $M_{11}=\frac{1}{3}m_1L^2$, $M_{22}=\frac{7}{48}m_2L^2$, $M_{12}=0$

$$V=V_{sp}+V_{gr}$$

Points on bar 1 displace upward by $r\theta_1$ } r = distance from pivot

Point on bar 2 displace upward by $r\theta_2$

$\Delta_1=\frac{3L}{4}\theta_1-\frac{3L}{4}\theta_2$, $\Delta_2=L\theta_1-\frac{L}{2}\theta_2$

$$V_{sp}=\frac{1}{2}k_1\Delta_1^2+\frac{1}{2}k_2\Delta_2^2$$
$$=\frac{1}{2}k_1\left(\frac{3L}{4}\theta_1-\frac{3L}{4}\theta_2\right)^2+\frac{1}{2}k_2\left(L\theta_1-\frac{L}{2}\theta_2\right)^2$$
$$=\frac{1}{2}\left[\left(\frac{9}{16}k_1L^2+k_2L^2\right)\theta_1^2+\left(\frac{9}{16}k_1L^2+\frac{1}{4}k_2L^2\right)\theta_2^2-2\left(\frac{9}{16}k_1L^2+\frac{1}{2}k_2L^2\right)\theta_1\theta_2\right]$$

Standard: $V=\frac{1}{2}\left[(K_{11}\theta_1^2+K_{22}\theta_2^2)+2K_{12}\theta_1\theta_2\right]$

Thus $(K_{11})_{sp}=\frac{9}{16}k_1L^2+k_2L^2$, $(K_{22})_{sp}=\frac{9}{16}k_1L^2+\frac{1}{4}k_2L^2$

$(K_{12})_{sp}=-\left(\frac{9}{16}k_1L^2+\frac{1}{2}k_2L^2\right)$

$V_{gr}=m_1gh_1+m_2gh_2 \Rightarrow$ datum for each bar at its pin

so $h_1=\frac{L}{2}\sin\theta_1$, $h_2=\frac{L}{4}\sin\theta_2 \Rightarrow V_{gr}=m_1g\frac{L}{2}\sin\theta_1+m_2g\frac{L}{4}\sin\theta_2$

$\frac{\partial^2 V_{gr}}{\partial\theta_1^2}=-m_1g\frac{L}{2}\sin\theta_1$, $\frac{\partial^2 V_{gr}}{\partial\theta_2^2}=-m_2g\frac{L}{4}\sin\theta_2$, $\frac{\partial^2 V_{gr}}{\partial\theta_1\partial\theta_2}=0$

Evaluate at $\theta_1=\theta_2=0 \Rightarrow (K_{11})_{gr}=(K_{22})_{gr}=(K_{12})_{gr}=0$

Thus $K_{11}=(K_{11})_{sp}$, $K_{22}=(K_{22})_{sp}$, $K_{12}=(K_{12})_{sp}$

No dashpots $\Rightarrow [C]=[0]$

$\mathcal{P}_{in}=Pv_E$, but $v_E=\frac{L}{2}\dot\theta_2$ upward $\Rightarrow \mathcal{P}_{in}=-P\frac{L}{2}\dot\theta_2$

Standard $\mathcal{P}_{in}=Q_1\dot\theta_1+Q_2\dot\theta_2$

Thus $Q_1 = 0$, $Q_2 = -P\frac{L}{2}$

Standard eq of motion: $[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \{Q\}$

$$\begin{bmatrix} \frac{1}{3}m_1L^2 & 0 \\ 0 & \frac{7}{48}m_2L^2 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} (\frac{9}{16}k_1L^2 + k_2L^2) & -(\frac{9}{16}k_1L^2 + \frac{1}{2}k_2L^2) \\ -(\frac{9}{16}k_1L^2 + \frac{1}{2}k_2L^2) & (\frac{9}{16}k_1L^2 + \frac{1}{4}k_2L^2) \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ -P\frac{L}{2} \end{Bmatrix} \quad \Longleftarrow$$

Exercise 4.1

$$\left[[K]-\omega^2[M]\right]\{\phi\} = \begin{bmatrix} 200-4\omega^2 & 200 \\ 200 & 800-2\omega^2 \end{bmatrix}\{\phi\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left|[K]-\omega^2[M]\right| = (200-4\omega^2)(800-2\omega^2) - (200)^2 = 0$$

$$8\omega^4 - 3600\omega^2 + 120000 = 0 \Rightarrow$$

$$\omega_1^2 = 36.25, \quad \omega_2^2 = 413.75$$

Then $\begin{bmatrix} 200-4\omega_j^2 & 200 \\ x & x \end{bmatrix}\begin{Bmatrix} 1 \\ \phi_{2j} \end{Bmatrix} = \begin{Bmatrix} 0 \\ x \end{Bmatrix}$

$$\phi_{2j} = -\frac{200-4\omega_j^2}{200}$$

First mode $\omega_1 = 6.021$ rad/s, $\{\phi_1\} = \begin{Bmatrix} 1 \\ -0.275 \end{Bmatrix}$ $\Leftarrow$

Second mode $\omega_2 = 20.341$ rad/s, $\{\phi_2\} = \begin{Bmatrix} 1 \\ 7.275 \end{Bmatrix}$