Collection of PDE animations

Nasser M. Abbasi

May 8, 2017 compiled on — Monday May 08, 2017 at 06:17 PM

Contents

These are collection of PDE problems solved analytically and animated. Most of the animations where done in Mathematica and some in Matlab. This note shows only the problem description and final solution and the animation. The detailed solution is in my course notes elsewhere on my site in different places. Will add links to detailed analytical solutions soon. (Too many HW's, and too little time).

1 Summary table

Heat PDE $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ in $1D$ (in a rod)

Left side	Right side	initial condition	$\lambda = 0$	$\lambda > 0$	analyt
$u(0) = 0$	$u(L) = 0$	$u(x,0) = \begin{cases} x & 0 < x < \frac{L}{2} \\ L - x & \frac{L}{2} < x < L \end{cases}$	No	$\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1,2,3,\cdots$ $X_n = B_n \sin\left(\sqrt{\lambda_n}x\right)$	$\sum_{n=1}^{\infty}$
$u(0) = 0$	$u(L) = 0$	$u(x,0) = 100$	No	$\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1,2,3,\cdots$ $X_n = B_n \sin\left(\sqrt{\lambda_n}x\right)$	$\sum_{n=1}^{\infty}$
$u(0) = T_0$	$u(L) = 0$	$u(x,0) = x$	No	$\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1,2,3,\cdots$ $X_n = B_n \sin\left(\sqrt{\lambda_n}x\right)$	$T_0 -$
$\frac{\partial u(0)}{\partial x} = 0$	$\frac{\partial u(L)}{\partial x} = 0$	$u(x,0) = x$	$\lambda_0 = 0$ $X_0 = A_0$	$\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1,2,3,\cdots$ $X_n = A_n \cos\left(\sqrt{\lambda_n}x\right)$	$A_0 +$
$\frac{\partial u(0)}{\partial x} = 0$	$u(L) = T_0$	$u(x,0) = 0$	No	$\lambda_n = \left(\frac{n\pi}{2L}\right)^2, n = 1,3,5,\cdots$ $X_n = A_n \cos\left(\sqrt{\lambda_n}x\right)$	$T_0 +$
$\frac{\partial u(0)}{\partial x} = 0$	$u(L) = 0$	$u(x,0) = f(x)$	No	$\lambda_n = \left(\frac{n\pi}{2L}\right)^2, n = 1,3,5,\cdots$ $X_n = A_n \cos\left(\sqrt{\lambda_n}x\right)$	$\sum_{n=1}^{\infty}$
$u(0) = 0$	$\frac{\partial u(L)}{\partial x} = 0$	$u(x,0) = f(x)$	No	$\lambda_n = \left(\frac{n\pi}{2L}\right)^2, n = 1,3,5,\cdots$ $X_n = B_n \sin\left(\sqrt{\lambda_n}x\right)$	$\sum_{n=1}^{\infty}$
$u(0) = 0$	$u(L) + \frac{\partial u(L)}{\partial x} = 0$	$u(x,0) = f(x)$	$\lambda_0 = 0$ $X_0 = A_0$	$\tan\left(\sqrt{\lambda_n}L\right) = -\lambda_n$ $X_\lambda = B_\lambda \sin\left(\sqrt{\lambda_n}x\right)$	$A_0 +$

Heat PDE $\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2} - \beta u$ in $1D$ (in a rod) with $\alpha, \beta > 0$ for $0 < x < \pi$

Left side	Right side	initial condition	$\lambda = 0$	$\lambda > 0$	analytical solution $u(x,t)$
$\frac{\partial u(0,t)}{\partial x} = 0$	$\frac{\partial u(\pi,t)}{\partial x} = 0$	$u(x,0) = x$	$\lambda_0 = 0$ $X_0 = A_0$	$\lambda_n = n^2, n = 1,2,3,\cdots$ $X(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nx)$	$\frac{\pi}{2} + c_0\left(e^{-\beta t} - 1\right) + \frac{2}{\pi}\sum_{n=}^{\infty}$

(TO DO) Heat PDE for periodic conditions $u(-L) = u(L)$ and $\frac{\partial u(-L)}{\partial x} = \frac{\partial u(L)}{\partial x}$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1,2,3,\cdots$$

$$u(x,t) = \overbrace{a_0}^{\lambda=0} + \overbrace{\sum_{n=1}^{\infty} A_n \cos\left(\sqrt{\lambda_n}x\right) e^{-k\lambda_n t} + \sum_{n=1}^{\infty} B_n \sin\left(\sqrt{\lambda_n}x\right) e^{-k\lambda_n t}}^{\lambda>0}$$

2 Homogeneous Heat PDE in 1D

2.1 Boundary conditions both Dirichlet and Homogeneous

2.1.1 Example 1

Solve

$$\frac{1}{k}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L, t > 0$$

Boundary conditions

$$u(0,t) = 0^0$$
$$u(L,t) = 0^0$$

Initial conditions

$$u(x,0)=\begin{cases} x & 0<x<\frac{L}{2} \\ L-x & \frac{L}{2}<x<L \end{cases}$$

Solution. $\lambda_n=\frac{n\pi}{L}$

$$u(x,t)=\frac{4L}{\pi^2}\sum_{n=\text{odd}}^{\infty}\left(\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\right)\exp\left(-k\lambda_n^2 t\right)\sin(\lambda_n x)$$

2.1.2 Example 2

A bar length $L=10$ cm with insulated sides is initially at 100 degrees. starting at $t=0$, the ends are held at zero degree. Find the temperature distribution in the bar at time t

$$\frac{1}{\alpha^2}\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}\qquad 0<x<L,t>0$$

Boundary conditions

$$u(0,t)=0^0$$
$$u(L,t)=0^0$$

Initial conditions

$$u(x,0)=T_0=100^0$$

<u>Solution</u>

Let

$$u=X(x)T(t)$$

Substituting this into the PDE gives

$$\frac{1}{\alpha^2}T'X=X''T$$

Dividing by XT

$$\frac{1}{\alpha^2}\frac{T'}{T}=\frac{X''}{X}$$

Therefore

$$\frac{1}{\alpha^2}\frac{T'}{T}=\frac{X''}{X}=-\lambda$$

Where λ is some constant. Hence we obtain two ODE's to solve. They are

$$X''+\lambda X=0$$
$$X(0)=0$$
$$X(L)=0$$

And

$$T'+\lambda\alpha^2 T=0$$

<u>case $\lambda=0$</u>

$$X''=0$$
$$X=Ax+B$$

From first boundary conditions, $X(0)=0$, the above becomes

$$0=B$$

Hence $X=Ax$. From second boundary conditions $0=AL$, or $A=0$, therefore trivial solution and $\lambda=0$ is not an eigenvalue.

<u>case $\lambda>0$</u>

$$X''+\lambda X=0$$

The solution is

$$X = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$

From first B.C. $0 = A$ and the solution becomes $X = B\sin\left(\sqrt{\lambda}x\right)$. From second B.C.

$$0 = B\sin\left(\sqrt{\lambda}L\right)$$

For non-trivial solution, we want $\sin\left(\sqrt{\lambda}L\right) = 0$ or

$$\begin{aligned}\sqrt{\lambda}L &= n\pi \\ \lambda &= \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, 3, \cdots\end{aligned}$$

Therefore

$$X_n(x) = b_n \sin\left(\frac{n\pi}{L}x\right)$$

Now we solve the time ODE $T' + \lambda_n \alpha^2 T = 0$. This has solution

$$T_n = a_n e^{-\lambda_n \alpha^2 t}$$

Therefore the solution becomes

$$\begin{aligned}u &= \sum_{n=1}^{\infty} X_n(x)\, T_n(t) \\ &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) a_n e^{-\left(\frac{n\pi}{L}\right)^2 \alpha^2 t} \\ &= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\alpha\right)^2 t}\end{aligned}$$

To find B_n, we use orthoginality. At $t = 0$, we are given $u(x,0)$. Hence at $t = 0$ the solution becomes

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

Multiplying both sides by $\sin\left(\frac{m\pi}{L}x\right)$ and integrating

$$\begin{aligned}\int_0^L u(x,0)\sin\left(\frac{m\pi}{L}x\right)dx &= \int_0^L \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx \\ &= \sum_{n=1}^{\infty} B_n \int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx \\ &= B_m \frac{L}{2}\end{aligned}$$

Therefore

$$\begin{aligned}B_m &= \frac{2}{L}\int_0^L u(x,0)\sin\left(\frac{m\pi}{L}x\right)dx \\ &= \frac{2}{L}\int_0^L T_0 \sin\left(\frac{m\pi}{L}x\right)dx \\ &= \frac{2}{L}T_0\left(\frac{-L}{m\pi}\cos\left(\frac{m\pi}{L}x\right)_0^L\right) \\ &= \frac{-2}{m\pi}T_0\left(\cos(m\pi) - \cos(0)\right) \\ &= \frac{-2}{m\pi}T_0\left((-1)^m - 1\right)\end{aligned}$$

Therefore, the solution is

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\alpha\right)^2 t} \\
&= \frac{-2T_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n}\left((-1)^n - 1\right) \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\alpha\right)^2 t} \\
&= \frac{4T_0}{\pi} \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\alpha\right)^2 t}
\end{aligned}
$$

Here is an animation.

2.2 Boundary conditions both Dirichlet, one homogeneous other end nonHomogeneous

A bar length L

$$
\frac{1}{\alpha^2}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L, t > 0
$$

Boundary conditions

$$
\begin{aligned}
u(0,t) &= T_0 = 100^0 \\
u(L,t) &= 0
\end{aligned}
$$

Initial conditions

$$
u(x,0) = x
$$

Solution, $\lambda_n = \frac{n\pi}{L}$

$$
u(x,t) = T_0 - \frac{T_0}{L}x - \frac{4T_0}{\pi}\sum_{n=\text{even}}^{\infty} \frac{1}{n}\exp\left(-\alpha^2\lambda_n^2 t\right)\sin(\lambda_n x)
$$

2.3 Boundary conditions both Neumann and homogeneous

A bar length L

$$
\frac{1}{\alpha^2}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L, t > 0
$$

Boundary conditions, insulated

$$
\begin{aligned}
\left.\frac{\partial u}{\partial x}\right|_{x=0} &= 0 \\
\left.\frac{\partial u}{\partial x}\right|_{x=L} &= 0
\end{aligned}
$$

Initial conditions

$$
u(x,0) = x
$$

Solution, $\lambda_n = \frac{n\pi}{L}$

$$
u(x,t) = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=\text{odd}}^{\infty} \frac{1}{n^2}\exp\left(-\alpha^2\lambda_n^2 t\right)\cos(\lambda_n x)
$$

2.4 Boundary conditions one end Neumann, the other nonHomogeneous Dirichlet

A bar length $L=2$

$$\frac{1}{\alpha^2}\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} \qquad 0<x<L, t>0$$

Boundary conditions, left end only insulated

$$\left.\frac{\partial u}{\partial x}\right|_{x=0}=0$$
$$u(L,t)=T_0=100^0$$

Initial conditions

$$u(x,0)=0^0$$

Solution, $\lambda_n=\frac{2n+1}{2}\frac{\pi}{L}$

$$u(x,t)=T_0-\frac{4T_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}\exp\left(-\alpha^2\lambda_n^2 t\right)\cos(\lambda_n x)$$

2.5 Boundary conditions at both ends homogeneous Neumann (insulated)

See my solution for exam 1, Math 322. UW, Fall 2016, problem 3.

Solve, $0<x<\pi$

Solve

$$\frac{\partial u}{\partial t}=\alpha\frac{\partial^2 u}{\partial x^2}-\beta u \qquad 0<x<L, t>0$$

over $0<x<\pi$ with, $a>0,\beta>0$

$$\frac{\partial u(0,t)}{\partial x}=0$$
$$\frac{\partial u(\pi,t)}{\partial x}=0$$

And initial conditions

$$u(x,0)=x$$

Solution

$$u(x,t)=\frac{\pi}{2}+c_0\left(e^{-\beta t}-1\right)+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1^n-1)}{n^2}\cos(nx)\,e^{-\left(n^2\alpha+\beta\right)t}$$

Solution is not unique, since there is unknown c_0. To find c_0, the solvability condition for $\nabla^2 u=0$ with Neumann B.C. is used, which says that total flux must be zero at steady state, which is the case here, since flux is zero as given (insulated). Since solvability condition is satisfied, then energy is conserved. Solution at $t=\infty$ from above is

$$u(x,\infty)=\frac{1}{2}\pi-c_0$$

Since energy is conserved then comparing it to energy at initial conditions

$$\int_0^\pi \rho c u(x,0)\,dx=\int_0^\pi \rho c u(x,\infty)\,dx$$

But $u(x,0)=x$, hence

$$\int_0^\pi x dx=\int_0^\pi\left(\frac{\pi}{2}-c_0\right)dx$$
$$c_0=0$$

Hence the final solution is

$$u(x,t) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1^n - 1)}{n^2} \cos(nx)\, e^{-(n^2\alpha+\beta)t}$$

Using $\alpha = 2, \beta = 1$, here is animation for 3.5 seconds

2.6 Boundary conditions at one end non-homogeneous Neumannm other end homogeneous Dirichlet

This 1D heat PDE has one end with boundary condition that is time dependent.

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L; t > 0 \\ u_x(0,t) &= e^t \\ u(L,t) &= 0 \\ u(x,0) &= 0 \end{aligned}$$

Now let

$$\begin{aligned} r(x,t) &= e^t x - Le^t \\ &= e^t (x - L) \end{aligned}$$

Be some reference temperature distribution that only needs to satisfy the boundary conditions given. i.e. $\frac{\partial r}{\partial x}(0,t) = e^t, r(L,t) = 0$. Then the difference temperature distribution is

$$v(x,t) = u(x,t) - r(x,t)$$

Since u, r satisfy the nonhomogeneous B.C's, then v satisfies the homogeneous B.C which is $\frac{\partial v}{\partial x}(0,t) = 0, v(L,t) = 0$. With the initial conditions

$$\begin{aligned} v(x,0) &= u(x,0) - r(x,0) \\ &= 0 - (x - L) \\ &= -x + L \end{aligned}$$

Since $v(x,t)$ has homogeneous B.C. it can be easily solved. The solution can be found by separation of variables to be

$$v(x,t) = \sum_{n=1,3,5,\cdots}^{\infty} A_n \cos\left(\sqrt{\lambda_n} x\right) e^{-k\lambda_n t}$$

Where eigenvalue $\lambda_n = \left(\frac{n\pi}{2L}\right)^2$. A_n is now found from initial conditions.

$$v(x,0) = \sum_{n=1,3,5,\cdots}^{\infty} A_n \cos\left(\frac{n\pi}{2L} x\right)$$

Multiplying both sides by $\cos\left(\frac{m\pi}{2L}x\right)$, integrating and changing the order of integration and summation on RHS gives

$$\int_0^L (-x + L)\cos\left(\frac{m\pi}{2L}x\right) dx = \sum_{n=1,3,5,\cdots}^{\infty} A_n \int_0^L \cos\left(\frac{m\pi}{2L}x\right)\cos\left(\frac{n\pi}{2L}x\right) dx$$

By orthogonality of cos the above reduces to one term

$$\int_0^L (-x+L)\cos\left(\frac{m\pi}{2L}x\right)dx = A_m\frac{L}{2}$$

$$A_n = \frac{2}{L}\int_0^L (-x+L)\cos\left(\frac{n\pi}{2L}x\right)dx$$

Hence the solution is

$$v(x,t) = \sum_{n=1,3,5,\cdots}^{\infty} A_n \cos\left(\frac{n\pi}{2L}x\right)e^{-k\left(\frac{n\pi}{2L}\right)^2 t}$$

But $u = v + r$, therefore the final solution is

$$u(x,t) = e^t(x-L) + \sum_{n=1,3,5,\cdots}^{\infty} A_n \cos\left(\frac{n\pi}{2L}x\right)e^{-k\left(\frac{n\pi}{2L}\right)^2 t}$$

Here is animation using Mathematica, for $L = 1, k = 0.01$ for 2 seconds.

3 Nonhomogeneous heat PDE in 1D

3.1 Nonhomogeneous PDE and Nonhomogeneous boundary conditions

3.1.1 example 1

problem Solve

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \sin(5x)e^{-2t} \qquad 0 < x < \pi, t > 0\\
u(0,t) &= 1\\
u(\pi,t) &= 0\\
u(x,0) &= 0
\end{aligned}$$

Note: the boundary and initial conditions that are inconsistent at left end (but this is how the problem is, from the textbook, so please do not blame me).

Solution

This problem has nonhomogeneous B.C. and non-homogenous in the PDE itself (source present). First step is to use reference function to remove the nonhomogeneous B.C. then use the method of eigenfunction expansion on the resulting problem.

Let

$$r(x) = c_1 x + c_2$$

At $x = 0, r(x) = 1$, hence $1 = c_2$ and at $x = \pi, r(x) = 0$, hence $0 = c_1\pi + 1$ or $c_1 = -\frac{1}{\pi}$, hence

$$\boxed{r(x) = 1 - \frac{x}{\pi}}$$

Therefore

$$u(x,t) = v(x,t) + r(x)$$

Where $v(x,t)$ solution for the given PDE but with homogeneous B.C., therefore

$$\begin{aligned}
\frac{\partial v(x,t)}{\partial t} &= \frac{\partial^2 v(x,t)}{\partial x^2} + e^{-2t}\sin 5x \qquad (1)\\
v(0,t) &= 0\\
v(\pi,t) &= 0\\
v(x,0) &= u(x,0) - r(x) = 0 - \left(1 - \frac{x}{\pi}\right) = \frac{x}{\pi} - 1
\end{aligned}$$

We now solve (1). This is homogeneous in the PDE itself. To solve, we first solve the nonhomogeneous PDE in order to find the eigenfunctions. Hence we need to solve

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2}$$

This has solution

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\phi_n(x) \tag{2}$$

With

$$\begin{aligned} \phi_n(x) &= \sin\left(\sqrt{\lambda_n}x\right) \qquad n = 1,2,3\cdots \\ \lambda_n &= n^2 \qquad n = 1,2,3\cdots \end{aligned}$$

Plug-in (2) back into (1) gives

$$\begin{aligned} \sum_{n=1}^{\infty} a_n'(t)\,\phi_n(x) &= \sum_{n=1}^{\infty} a_n(t)\,\phi_n(x) + e^{-2t}\sin 5x \\ &= \sum_{n=1}^{\infty} a_n(t)\,\frac{\partial^2}{\partial x^2}\phi_n(x) + e^{-2t}\sin 5x \end{aligned}$$

But $\frac{\partial^2}{\partial x^2}\phi_n(x) = -\lambda_n\phi_n = -n\phi_n$, hence the above becomes

$$\begin{aligned} \sum_{n=1}^{\infty} a_n'(t)\,\phi_n(x) + n^2 a_n(t)\,\phi_n(x) &= e^{-2t}\sin 5x \\ \sum_{n=1}^{\infty} \left(a_n'(t) + n^2 a_n(t)\right)\sin(nx) &= e^{-2t}\sin 5x \end{aligned}$$

Therefore, since Fourier series expansion is unique, we can compare coefficients and obtain

$$a_n'(t) + n^2 a_n(t) = \begin{cases} e^{-2t} & n = 5 \\ 0 & n \neq 5 \end{cases}$$

<u>For the case $n = 5$</u>

$$\begin{aligned} a_5'(t) + 25a_5(t) &= e^{-2t} \\ \frac{d}{dt}\left(a_5(t)\,e^{25t}\right) &= e^{23t} \\ a_5(t)\,e^{25t} &= \int e^{23t}dt + c \\ &= \frac{e^{23t}}{23} + c \end{aligned}$$

Hence

$$a_5(t) = \frac{e^{-2t}}{23} + ce^{-25t}$$

At $t = 0$, $a_5(0) = \frac{1}{23} + c$, hence

$$c = a_5(0) - \frac{1}{23}$$

And the solution becomes

$$a_5(t) = \frac{1}{23}e^{-2t} + \left(a_5(0) - \frac{1}{23}\right)e^{-25t}$$

For the case $n \neq 5$

$$\begin{aligned} a_n'(t) + n^2 a_n(t) &= 0 \\ \frac{d}{dt}\left(a_n(t)\, e^{n^2 t}\right) &= 0 \\ a_n(t)\, e^{n^2 t} &= c \\ a_n(t) &= c e^{-n^2 t} \end{aligned}$$

At $t = 0$, $a_n(0) = c$, hence

$$a_n(t) = a_n(0)\, e^{-nt}$$

Therefore

$$a_n(t) = \begin{cases} \frac{1}{23} e^{-2t} + \left(a_5(0) - \frac{1}{23}\right) e^{-25t} & n = 5 \\ a_n(0)\, e^{-n^2 t} & n \neq 5 \end{cases}$$

To find $a_n(0)$ we use orthogonality. Since $u(x,t) = v(x,t) + r(x)$, then

$$u(x,t) = \left(\sum_{n=1}^{\infty} a_n(t) \sin(nx)\right) + \left(1 - \frac{x}{\pi}\right)$$

And at $t = 0$ the above becomes

$$0 = \left(\sum_{n=1}^{\infty} a_n(0) \sin(nx)\right) + \left(1 - \frac{x}{\pi}\right)$$

Or

$$\frac{x}{\pi} - 1 = \sum_{n=1}^{\infty} a_n(0) \sin(nx)$$

Applying orthogonality

$$\int_0^{\pi} \left(\frac{x}{\pi} - 1\right) \sin(n'x)\, dx = a_{n'}(0) \int_0^{\pi} \sin^2(n'x)\, dx$$

Therefore

$$\begin{aligned} a_n(0) &= \frac{\int_0^{\pi} \left(\frac{x}{\pi} - 1\right) \sin(nx)\, dx}{\frac{\pi}{2}} \\ &= \frac{2}{\pi} \int_0^{\pi} \left(\frac{x}{\pi} - 1\right) \sin(nx)\, dx \\ &= \frac{2}{\pi}\left[-\int_0^{\pi} \sin(nx)\, dx + \frac{1}{\pi}\int_0^{\pi} x \sin(nx)\, dx\right] \\ &= \frac{2}{\pi}\left[-\left(\frac{-\cos(nx)}{n}\right)_0^{\pi} + \frac{1}{\pi}\left(\frac{\sin(nx)}{n^2} - \frac{x\cos(nx)}{n}\right)_0^{\pi}\right] \\ &= \frac{2}{\pi}\left[\left(\frac{\cos(n\pi)}{n} - \frac{1}{n}\right) + \frac{1}{\pi}\left(\left(\frac{\sin(n\pi)}{n^2} - \frac{\pi\cos(n\pi)}{n}\right) - \left(\frac{\sin(0)}{n^2} - \frac{0\cos(0)}{n}\right)\right)\right] \\ &= \frac{2}{\pi}\left[\left(\frac{-1^n}{n} - \frac{1}{n}\right) + \frac{1}{\pi}\left(0 - \frac{\pi(-1)^n}{n}\right)\right] \\ &= \frac{2}{\pi}\left[\frac{(-1)^n}{n} - \frac{1}{n} - \frac{(-1)^n}{n}\right] \\ &= \frac{-2}{n\pi} \end{aligned}$$

Therefore $a_5(0) = \frac{-2}{5\pi}$. Hence

$$a_n(t) = \begin{cases} \frac{1}{23} e^{-2t} + \left(\frac{-2}{5\pi} - \frac{1}{23}\right) e^{-25t} & n = 5 \\ \frac{-2}{n\pi} e^{-n^2 t} & n \neq 5 \end{cases}$$

Where

$$
\begin{aligned}
u(x,t) &= v(x,t) + r(x) \\
&= \left(\sum_{n=1}^{\infty} a_n(t)\sin(nx)\right) + \left(1 - \frac{x}{\pi}\right)
\end{aligned}
$$

Animation for 3 seconds

4 Homogeneous Wave PDE in 1D (plugged string)

4.1 Boundary conditions both homogeneous. One end Neumann, other end Dirichlet

Solve, $0 < x < L$

$$
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L, t > 0
$$

Boundary conditions, $t > 0$

$$
\begin{aligned}
u(0,t) &= 0 \\
\left.\frac{\partial u}{\partial x}\right|_{x=L} &= 0
\end{aligned}
$$

Initial conditions, $t = 0$

$$
\begin{aligned}
\frac{\partial u(x,0)}{\partial t} &= 0 \\
u(x,0) = f(x) &= \begin{cases} \frac{3h}{L}x & 0 < x < \frac{L}{3} \\ h & \frac{L}{3} < x < L \end{cases}
\end{aligned}
$$

Use $c = 4, h = 0.1, L = 1$.
Solution, $C_n = \frac{24h}{((2n-1)\pi)^2}\sin\left(\frac{(2n-1)}{6}\pi\right), \lambda_n = \frac{(2n-1)\pi}{2L}$

$$
u(x,t) = \sum_{n=1}^{\infty} C_n \cos(\lambda_n c t)\sin(\lambda_n x)
$$

4.2 Boundary conditions both homogeneous. One end Neumann, other Dirichlet. Damping present.

Solve, $0 < x < L$

$$
\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L, t > 0
$$

Boundary conditions, $t > 0$

$$
\begin{aligned}
u(0,t) &= 0 \\
\left.\frac{\partial u}{\partial x}\right|_{x=L} &= 0
\end{aligned}
$$

Initial conditions, $t = 0$

$$
\begin{aligned}
\frac{\partial u(x,0)}{\partial t} &= 0 \\
u(x,0) = f(x) &= \begin{cases} \frac{3h}{L}x & 0 < x < \frac{L}{3} \\ h & \frac{L}{3} < x < L \end{cases}
\end{aligned}
$$

Use $c = 4, h = 0.1, L = 1$. Consider three cases for damping, $b = 0.5\frac{\pi c}{L}, b = \frac{\pi c}{L}, b = 2\frac{\pi c}{L}$
Solution case (underdamped) $b = 0.5\frac{\pi c}{L}$.

$$
\begin{aligned}
\lambda_n &= \frac{(2n-1)\pi}{2L} \\
\omega_n &= \lambda_n c \\
\xi_n &= \frac{b}{2\omega_n} \\
C_n &= \frac{24h}{((2n-1)\pi)^2} \sin\left(\frac{(2n-1)}{6}\pi\right) \\
\beta_n &= \omega_n \sqrt{1-\xi_n^2}
\end{aligned}
$$

$$
u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\frac{b}{2}t} \left(\cos(\beta_n t) + \frac{b}{2\beta_n}\sin(\beta_n t)\right) \sin(\lambda_n x)
$$

Solution case (critical damped)
$b = \frac{\pi c}{L}$.

$$
u(x,t) = C_1 \left(e^{\frac{-b}{2}t} + \frac{b}{2} t e^{\frac{-b}{2}t}\right) \sin(\lambda_1 x) + \sum_{n=2}^{\infty} C_n e^{-\frac{b}{2}t} \left(\cos(\beta_n t) + \frac{b}{2\beta_n}\sin(\beta_n t)\right) \sin(\lambda_n x)
$$

Solution case (overdamped damped)
$b = 2\frac{\pi c}{L}$. First mode only overdamped, rest underdamped.

$$
\begin{aligned}
\omega_1 &= 2\frac{\pi c}{L} \\
C_1 &= \frac{1}{1 - \frac{-\frac{b}{2}+\omega_1\sqrt{\zeta^2-1}}{-\frac{b}{2}-\omega_1\sqrt{\zeta^2-1}}} \frac{24h}{\pi^2} \sin\left(\frac{\pi}{6}\right) \\
C_n &= \frac{24h}{((2n-1)\pi)^2} \sin\left(\frac{(2n-1)}{6}\pi\right) \qquad n > 1
\end{aligned}
$$

$$
\begin{aligned}
u(x,t) = C_1 &\left(e^{\left(-\frac{b}{2}+\omega_1\sqrt{\zeta^2-1}\right)t} - \frac{-\frac{b}{2}+\omega_1\sqrt{\zeta^2-1}}{-\frac{b}{2}-\omega_1\sqrt{\zeta^2-1}} e^{\left(-\frac{b}{2}-\omega_1\sqrt{\zeta^2-1}\right)t}\right) \sin(\lambda_1 x) \\
&+ \sum_{n=2}^{\infty} C_n e^{-\frac{b}{2}t} \left(\cos(\beta_n t) + \frac{b}{2\beta_n}\sin(\beta_n t)\right) \sin(\lambda_n x)
\end{aligned}
$$

5 Homogeneous Wave PDE in 1D, semi-infinite domain

5.1 Neumann boundary conditions at $x = 0$

(10) This is animation of solution of $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ for $x \geq 0, t \geq 0$ with initial conditions $u(x,0) = \begin{cases} 1 & 4 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$ and $\frac{\partial u(x,0)}{\partial t} = 0$ and boundary condition $\frac{\partial u(0,t)}{\partial x} = 0$.

5.2 Dirichlet boundary conditions at $x = 0$

This is the same problem as above, but with $u(0,t) = 0$

6 FitzHugh-Nagumo in 2D

6.1 Example 1

The equations to solve are the following on the unit square in 2D.

$$\begin{aligned}
\frac{\partial v(x,y,t)}{\partial t} &= D\nabla^2 v + (a-v)(v-1)v - w + I \\
\frac{\partial w(x,y,t)}{\partial t} &= \epsilon(v - \gamma w)
\end{aligned}$$

Using $a = 0.1, \gamma = 2, \epsilon = 0.005, D = 5 \times 10^{-5}, I = 0$, hence the PDE's are

$$\begin{aligned}
\frac{\partial v(x,y,t)}{\partial t} &= \left(5 \times 10^{-5}\right)\nabla^2 v + (0.1-v)(v-1)v - w \\
\frac{\partial w(x,y,t)}{\partial t} &= 0.005(v - 2w)
\end{aligned}$$

Initial conditions, $t = 0$

$$\begin{aligned}
v(x,y,0) &= \exp\left(-100\left(x^2 + y^2\right)\right) \\
w(x,y,0) &= 0
\end{aligned}$$

Boundary conditions are homogeneous Neumann for v. (I solved this numerically, fractional step method. ADI for the diffusion solve).

6.2 Example 2

The equations to solve are the following on the unit square in 2D.

$$\begin{aligned}
\frac{\partial v(x,y,t)}{\partial t} &= D\nabla^2 v + (a-v)(v-1)v - w + I \\
\frac{\partial w(x,y,t)}{\partial t} &= \epsilon(v - \gamma w)
\end{aligned}$$

Using $a = 0.1, \gamma = 2, \epsilon = 0.005, D = 5 \times 10^{-5}, I = 0$, hence the PDE's are

$$\begin{aligned}
\frac{\partial v(x,y,t)}{\partial t} &= \left(5 \times 10^{-5}\right)\nabla^2 v + (0.1-v)(v-1)v - w \\
\frac{\partial w(x,y,t)}{\partial t} &= 0.005(v - 2w)
\end{aligned}$$

Initial conditions, $t = 0$

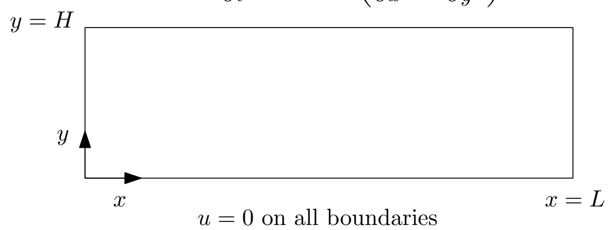
$$\begin{aligned}
v(x,y,0) &= 1 - 2x \\
w(x,y,0) &= 0.05y
\end{aligned}$$

Boundary conditions are homogeneous Neumann for v. (I solved this numerically, fractional step method. ADI for the diffusion solve).

7 Homogeneous Wave PDE in 2D

7.1 Rectangular membrane

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$



Using Cartesian coordinates. Wave displacement is $u \equiv u(x,y,t)$ (out of page).

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$
$$0 < x < L$$
$$0 < y < H$$

Boundary conditions on x

$$u(0,y,t) = 0$$
$$u(L,y,t) = 0$$

And boundary conditions on y

$$u(x,0,t) = 0$$
$$u(x,H,t) = 0$$

Initial conditions

$$u(x,y,0) = f(x,y)$$
$$\frac{\partial u}{\partial t}(x,y,0) = g(x,y)$$

Let $u = X(x)\,Y(y)\,T(t)$. Plug in the PDE

$$\frac{1}{c^2}T''XY = X''YT + Y''XT$$
$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y}$$

Hence

$$\frac{1}{c^2}\frac{T''}{T} = -\lambda$$
$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda$$

Hence time ODE becomes

$$T'' + c^2\lambda T = 0$$

And the space ODE is

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda$$

Separating this again

$$\frac{X''}{X} = -\lambda - \frac{Y''}{Y}$$

Using new separation variable μ, we obtain two new ODE's

$$\begin{aligned}\frac{X''}{X} &= -\mu \\ -\lambda - \frac{Y''}{Y} &= -\mu\end{aligned}$$

Or

$$\begin{aligned}X'' + \mu X &= 0 \\ Y'' + Y\left(\lambda - \mu\right) &= 0\end{aligned}$$

Solving for X ODE, and knowing that $\mu > 0$ from earlier, we obtain

$$X = A\cos\left(\sqrt{\mu}x\right) + B\sin\left(\sqrt{\mu}x\right)$$

Applying B.C. at $x = 0$

$$0 = A$$

Hence $X\left(x\right) = B\sin\left(\sqrt{\mu}x\right)$. Applying B.C. at $x = L$

$$0 = B\sin\left(\sqrt{\mu}L\right)$$

Hence

$$\begin{aligned}\sqrt{\mu}L &= n\pi \\ \mu &= \left(\frac{n\pi}{L}\right)^2 \qquad n = 1,2,3,\cdots\end{aligned}$$

Therefore the X solution is

$$X_n\left(x\right) = B_n \sin\left(\left(\frac{n\pi}{L}\right)x\right) \qquad n = 1,2,3,\cdots$$

Solving the Y ODE

$$Y'' + Y\left(\lambda - \left(\frac{n\pi}{L}\right)^2\right) = 0$$

The solution is

$$Y = A\cos\left(\sqrt{\lambda - \left(\frac{n\pi}{L}\right)^2}y\right) + B\sin\left(\sqrt{\lambda - \left(\frac{n\pi}{L}\right)^2}y\right)$$

Applying first B.C.

$$0 = A$$

Hence

$$Y = B\sin\left(\sqrt{\lambda - \left(\frac{n\pi}{L}\right)^2}y\right)$$

Applying second B.C.

$$0 = B\sin\left(\sqrt{\lambda - \left(\frac{n\pi}{L}\right)^2}H\right)$$

Hence

$$\begin{aligned}\sqrt{\lambda - \left(\frac{n\pi}{L}\right)^2}H &= m\pi \qquad m = 1,2,3,\cdots \\ \lambda_{nm} - \left(\frac{n\pi}{L}\right)^2 &= \left(\frac{m\pi}{H}\right)^2 \\ \lambda_{nm} &= \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2 \qquad n = 1,2,3,\cdots, m = 1,2,3,\cdots\end{aligned}$$

Hence the Y solution is

$$Y_{nm} = B_{nm}\sin\left(\frac{m\pi}{H}y\right) \qquad n = 1,2,3,\cdots, m = 1,2,3,\cdots$$

Now we solve the time T ode

$$\begin{aligned}
T''_{nm} + c^2\lambda_{nm}T_{nm} &= 0 \\
T_{nm}\left(t\right) &= A_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right) + B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)
\end{aligned}$$

Combining all solution , and merging all constants into two, we find

$$\begin{aligned}
u_{nm}\left(x,y,t\right) &= X_n\left(x\right)Y_{nm}\left(y\right)T_{nm}\left(t\right) \\
u\left(x,y,t\right) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} X_m\left(x\right)Y_{mn}\left(y\right)T_{mn}\left(t\right) \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\sin\left(\left(\frac{n\pi}{L}\right)x\right)\sin\left(\frac{m\pi}{H}y\right)\cos\left(c\sqrt{\lambda_{nm}}t\right) \\
&+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\sin\left(\left(\frac{n\pi}{L}\right)x\right)\sin\left(\frac{m\pi}{H}y\right)\sin\left(c\sqrt{\lambda_{nm}}t\right) \qquad (1)
\end{aligned}$$

We now use initial conditions to find A_{nm}, B_{nm}. At $t=0$

$$\begin{aligned}
u\left(x,y,0\right) &= f\left(x,y\right) \\
\frac{\partial u}{\partial t}\left(x,y,0\right) &= g\left(x,y\right)
\end{aligned}$$

Applying first initial condition to (1) gives

$$f\left(x,y\right) = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} A_{nm}\sin\left(\frac{m\pi}{H}y\right)\right)\sin\left(\left(\frac{n\pi}{L}\right)x\right)$$

Applying 2D orthogonality gives

$$\begin{aligned}
\int_0^L\int_0^H f\left(x,y\right)\sin\left(\left(\frac{n\pi}{L}\right)x\right)\sin\left(\frac{m\pi}{H}y\right)\ dxdy &= A_{nm}\left(\frac{L}{2}\right)\left(\frac{H}{2}\right) \\
A_{nm} &= \frac{4}{LH}\int_0^L\int_0^H f\left(x,y\right)\sin\left(\left(\frac{n\pi}{L}\right)x\right)\sin\left(\frac{m\pi}{H}y\right)\ dxdy
\end{aligned}$$

Taking time derivative of (1)

$$\begin{aligned}
\frac{\partial u}{\partial t}\left(x,y,t\right) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} -c\sqrt{\lambda_{nm}}A_{nm}\sin\left(\left(\frac{n\pi}{L}\right)x\right)\sin\left(\frac{m\pi}{H}y\right)\sin\left(c\sqrt{\lambda_{nm}}t\right) \\
&+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c\sqrt{\lambda_{nm}}B_{nm}\sin\left(\left(\frac{n\pi}{L}\right)x\right)\sin\left(\frac{m\pi}{H}y\right)\cos\left(c\sqrt{\lambda_{nm}}t\right)
\end{aligned}$$

AT $t=0$ the above becomes

$$g\left(x,y\right) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c\sqrt{\lambda_{nm}}B_{nm}\sin\left(\left(\frac{n\pi}{L}\right)x\right)\sin\left(\frac{m\pi}{H}y\right)$$

Applying 2D orthogonality gives

$$\begin{aligned}
\int_0^L\int_0^H g\left(x,y\right)\sin\left(\left(\frac{n\pi}{L}\right)x\right)\sin\left(\frac{m\pi}{H}y\right)\ dxdy &= B_{nm}\left(\frac{L}{2}\right)\left(\frac{H}{2}\right) \\
B_{nm} &= \frac{4}{LH}\int_0^L\int_0^H g\left(x,y\right)\sin\left(\left(\frac{n\pi}{L}\right)x\right)\sin\left(\frac{m\pi}{H}y\right)\ dxdy
\end{aligned}$$

Summary of solution

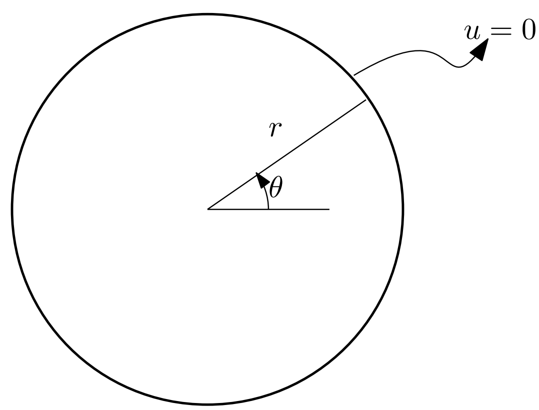
$$
\begin{aligned}
u(x,y,t) &= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} A_{nm}\sin\left(\frac{m\pi}{H}y\right)\cos\left(c\sqrt{\lambda_{nm}}t\right)\right)\sin\left(\left(\frac{n\pi}{L}\right)x\right)\\
&+\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\sin\left(\frac{m\pi}{H}y\right)\sin\left(c\sqrt{\lambda_{nm}}t\right)\right)\sin\left(\left(\frac{n\pi}{L}\right)x\right)\\
A_{nm} &= \frac{4}{LH}\int_0^L\int_0^H f(x,y)\sin\left(\left(\frac{n\pi}{L}\right)x\right)\sin\left(\frac{m\pi}{H}y\right)\,dxdy\\
B_{nm} &= \frac{4}{LH}\int_0^L\int_0^H g(x,y)\sin\left(\left(\frac{n\pi}{L}\right)x\right)\sin\left(\frac{m\pi}{H}y\right)\,dxdy\\
\lambda_{nm} &= \left(\frac{m\pi}{H}\right)^2+\left(\frac{n\pi}{L}\right)^2
\end{aligned}
$$

Here are few animations, all using $L=1, H=2, c=0.1$. For different modes. The mode n,m is given below each animation. These run for 100 seconds. Initial conditions are $u(x,y,0)=x\cos(y)$ and $\frac{\partial}{\partial t}u(x,y,0)=0$. Boundary conditions zero on all 4 edges.

7.2 circular membrane

7.2.1 not circularly symmetric

$$\frac{\partial^2 u(r,\theta,t)}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)$$



Using polar coordinates. Disk has radius a. Wave displacement is $u\equiv u(r,\theta,t)$ (out of page).

$$
\begin{aligned}
\frac{\partial^2 u(r,\theta,t)}{\partial t^2} &= c^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)\\
0 &< r < a\\
-\pi &< \theta < \pi
\end{aligned}
$$

Boundary conditions on r

$$
\begin{aligned}
|u(0,\theta,t)| &< \infty\\
u(a,\theta,t) &= 0
\end{aligned}
$$

And boundary conditions on θ

$$
\begin{aligned}
u(r,-\pi,t) &= u(r,\pi,t)\\
\frac{\partial u(r,-\pi,t)}{\partial \theta} &= \frac{\partial u(r,\pi,t)}{\partial \theta}
\end{aligned}
$$

Initial conditions

$$
\begin{aligned}
u(r,\theta,0) &= f(r,\theta) \\
\frac{\partial u}{\partial t}(r,\theta,0) &= g(r,\theta)
\end{aligned}
$$

Let $u = T(t)\,R(r)\,\Theta(\theta)$. Plug in the PDE

$$
\frac{1}{c^2}T''R\Theta = R''T\Theta + \frac{1}{r}R'T\Theta + \frac{1}{r^2}\Theta''RT
$$

Dividing by $RT\Theta$

$$
\frac{1}{c^2}\frac{T''}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}
$$

Hence

$$
\begin{aligned}
\frac{1}{c^2}\frac{T''}{T} &= -\lambda \\
\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} &= -\lambda
\end{aligned}
$$

The time ODE is

$$
T'' + c^2\lambda T = 0
$$

Now we separate again the space ODE's (remember to move the λ with the R and not the Θ)

$$
\begin{aligned}
\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \lambda &= -\frac{1}{r^2}\frac{\Theta''}{\Theta} \\
r^2\frac{R''}{R} + r\frac{R'}{R} + r^2\lambda &= -\frac{\Theta''}{\Theta}
\end{aligned}
$$

Let separation constant be μ, therefore

$$
\begin{aligned}
-\frac{\Theta''}{\Theta} &= \mu \\
\Theta'' + \mu\Theta &= 0
\end{aligned}
$$

With periodic boundary conditions and

$$
\begin{aligned}
r^2\frac{R''}{R} + r\frac{R'}{R} + r^2\lambda &= \mu \\
r^2R'' + rR' + \lambda r^2R - \mu R &= 0 \\
rR'' + R' - \frac{\mu}{r}R &= -\lambda rR
\end{aligned}
$$

Now it is in SL form, where $p = r, q = -\frac{\mu}{r}, \sigma = r$. This is singular SL. Can be written as

$$
R'' + \frac{1}{r}R' + \left(\lambda - \frac{\mu}{r^2}\right)R = 0
$$

Before we solve the above R ODE, we solve the $\Theta'' + \mu\Theta = 0$ to find μ Eigenvalues. The solution is

$$
\Theta = A\cos\left(\sqrt{\mu}\theta\right) + B\sin\left(\sqrt{\mu}\theta\right)
$$

With B.C $\Theta(-\pi) = \Theta(\pi)$ and $\Theta'(-\pi) = \Theta'(\pi)$. From first B.C. we obtain

$$
\begin{aligned}
A\cos\left(\sqrt{\mu}\pi\right) - B\sin\left(\sqrt{\mu}\pi\right) &= A\cos\left(\sqrt{\mu}\pi\right) + B\sin\left(\sqrt{\mu}\pi\right) \\
2B\sin\left(\sqrt{\mu}\pi\right) &= 0 \qquad (1)
\end{aligned}
$$

Looking at second B.C. $\Theta'(-\pi) = \Theta'(\pi)$

$$
\Theta'(\theta) = -A\sqrt{\mu}\sin\left(\sqrt{\mu}\theta\right) + \sqrt{\mu}B\cos\left(\sqrt{\mu}\theta\right)
$$

Hence

$$\begin{aligned} A\sqrt{\mu}\sin\left(\sqrt{\mu}\pi\right)+\sqrt{\mu}B\cos\left(\sqrt{\mu}\pi\right) &= -A\sqrt{\mu}\sin\left(\sqrt{\mu}\pi\right)+\sqrt{\mu}B\cos\left(\sqrt{\mu}\pi\right)\\ A\sqrt{\mu}\sin\left(\sqrt{\mu}\pi\right) &= -A\sqrt{\mu}\sin\left(\sqrt{\mu}\pi\right)\\ 2A\sin\left(\sqrt{\mu}\pi\right) &= 0 \end{aligned} \tag{2}$$

From (1,2), we see that both are satisfied if

$$\begin{aligned} \sqrt{\mu}\pi &= n\pi \qquad n=1,2,3,\dots\\ \mu &= n^2 \end{aligned}$$

Hence

$$\Theta_n = A_n\cos\left(n\theta\right)+B_n\sin\left(n\theta\right)$$

There is another solution for $\mu=0$ which is constant (that is why one of the sums below starts from $n=0$). We can combine the zero eigenvalue with the above and write

$$\Theta_n = A_n\cos\left(n\theta\right)+B_n\sin\left(n\theta\right) \qquad n=0,1,2,3,\dots$$

Since at $n=0$ the above reduces to constant A_0.

Now that we know $\mu_n=n^2$, from solving the θ part, we go and solve the r ODE. For each n, the solution to the r (Bessel) ode

$$R''+\frac{1}{r}R'+\left(\lambda-\frac{n^2}{r^2}\right)R=0$$

The solution turns out to be

$$R_{nm}\left(r\right)=J_n\left(\sqrt{\lambda_{nm}}r\right) \qquad m=1,2,3,\cdots$$

Where λ_{nm} is found from roots of $0=J_n\left(\sqrt{\lambda_{nm}}a\right)$ giving the eigenvalues. Now the time ODE is solved

$$\begin{aligned} T_{nm}''+c^2\lambda_{nm}T_{nm} &= 0\\ T_{nm} &= C_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)+D_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right) \qquad n=0,1,2,3,\dots,m=1,2,3,\cdots \end{aligned}$$

Hence the solution is

$$\begin{aligned} u\left(r,\theta,t\right) &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}T_{nm}R_{nm}\Theta_n\\ &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left(C_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)+D_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\right)J_n\left(\sqrt{\lambda_{nm}}r\right)\left(A_n\cos\left(n\theta\right)+B_n\sin\left(n\theta\right)\right) \end{aligned}$$

We now break this sum as follows

$$\begin{aligned} u\left(r,\theta,t\right) = &\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left(C_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)+D_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\right)J_n\left(\sqrt{\lambda_{nm}}r\right)A_n\cos\left(n\theta\right)\\ &+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(C_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)+D_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\right)J_n\left(\sqrt{\lambda_{nm}}r\right)B_n\sin\left(n\theta\right) \end{aligned}$$

Or

$$\begin{aligned} u\left(r,\theta,t\right) = &\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}C_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)J_n\left(\sqrt{\lambda_{nm}}r\right)A_n\cos\left(n\theta\right)+D_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)J_n\left(\sqrt{\lambda_{nm}}r\right)A_n\cos\left(n\theta\right)\\ &+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)J_n\left(\sqrt{\lambda_{nm}}r\right)B_n\sin\left(n\theta\right)+D_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)J_n\left(\sqrt{\lambda_{nm}}r\right)B_n\sin\left(n\theta\right) \end{aligned}$$

Then we break the above into 4 sums

$$\begin{aligned}
u(r,\theta,t) &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} C_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right) J_n\left(\sqrt{\lambda_{nm}}r\right) A_n\cos(n\theta)\\
&+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} D_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right) J_n\left(\sqrt{\lambda_{nm}}r\right) A_n\cos(n\theta)\\
&+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right) J_n\left(\sqrt{\lambda_{nm}}r\right) B_n\sin(n\theta)\\
&+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} D_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right) J_n\left(\sqrt{\lambda_{nm}}r\right) B_n\sin(n\theta)
\end{aligned}$$

Finally, we merge constants in the above as follows

$$\begin{aligned}
A_nC_{nm} &\equiv A_{nm}\\
A_nD_{nm} &\equiv B_{nm}\\
B_nC_{nm} &\equiv C_{nm}\\
B_nD_{nm} &\equiv D_{nm}
\end{aligned}$$

Hence the final solution now becomes

$$\begin{aligned}
u(r,\theta,t) &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right) J_n\left(\sqrt{\lambda_{nm}}r\right)\cos(n\theta)\\
&+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right) J_n\left(\sqrt{\lambda_{nm}}r\right)\cos(n\theta)\\
&+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right) J_n\left(\sqrt{\lambda_{nm}}r\right)\sin(n\theta)\\
&+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} D_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right) J_n\left(\sqrt{\lambda_{nm}}r\right)\sin(n\theta) \qquad (3)
\end{aligned}$$

Now initial conditions $u(r,\theta,0)=f(r,\theta)$ is used to find A_{nm},C_{nm} using orthogonality. At $t=0$ the solution simplifies to (all terms with $\sin\left(c\sqrt{\lambda_{nm}}t\right)$ vanish giving

$$\begin{aligned}
u(r,\theta,t) &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}J_n\left(\sqrt{\lambda_{nm}}r\right)\cos(n\theta)\\
&+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}J_n\left(\sqrt{\lambda_{nm}}r\right)\sin(n\theta)
\end{aligned}$$

Hence

$$f(r,\theta)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}J_n\left(\sqrt{\lambda_{nm}}r\right)\cos(n\theta)+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}J_n\left(\sqrt{\lambda_{nm}}r\right)\sin(n\theta) \qquad (4)$$

When iterating over m index, the terms $\cos(n\theta)$ and $\sin(n\theta)$ will be constant. So for each n, we have $\sum_{m=1}^{\infty} A_{nm}J_n\left(\sqrt{\lambda_{nm}}r\right)$ and $\sum_{m=1}^{\infty} C_{nm}J_n\left(\sqrt{\lambda_{nm}}r\right)$. So orthogonality is carried out on the m index on the Bessel functions, as in $\int_0^a J_n\left(\sqrt{\lambda_{nm}}r\right)J_n\left(\sqrt{\lambda_{nm'}}r\right)\sigma dr$. But $\sigma=r$ in this problem (from the Bessel ODE above). Multiplying (4) by $J_n\left(\sqrt{\lambda_{nm'}}r\right)$ and integrating

$$\begin{aligned}
\int_0^a f(r,\theta)J_n\left(\sqrt{\lambda_{nm'}}r\right) rdr &= \sum_{n=0}^{\infty}\left(\int_0^a\sum_{m=1}^{\infty} A_{nm}J_n\left(\sqrt{\lambda_{nm}}r\right)J_n\left(\sqrt{\lambda_{nm'}}r\right) rdr\right)\cos(n\theta)\\
&+\sum_{n=1}^{\infty}\left(\int_0^a\sum_{m=1}^{\infty} C_{nm}J_n\left(\sqrt{\lambda_{nm}}r\right)J_n\left(\sqrt{\lambda_{nm'}}r\right) rdr\right)\sin(n\theta)
\end{aligned}$$

Or

$$\int_0^a f(r,\theta) J_n\left(\sqrt{\lambda_{nm'}}r\right) rdr = \sum_{n=0}^{\infty} A_{nm'} \left(\int_0^a J_n^2\left(\sqrt{\lambda_{nm'}}r\right) rdr\right)\cos(n\theta) + \sum_{n=1}^{\infty} C_{nm'} \left(\int_0^a J_n^2\left(\sqrt{\lambda_{nm'}}r\right) rdr\right)\sin(n\theta)$$

Replacing m' back with m, the above becomes

$$\int_0^a f(r,\theta) J_n\left(\sqrt{\lambda_{nm}}r\right) rdr = \sum_{n=0}^{\infty} A_{nm} \left(\int_0^a J_n^2\left(\sqrt{\lambda_{nm}}r\right) rdr\right)\cos(n\theta) + \sum_{n=1}^{\infty} C_{nm} \left(\int_0^a J_n^2\left(\sqrt{\lambda_{nm}}r\right) rdr\right)\sin(n\theta) \tag{5}$$

We now apply orthogonality on m using the cos. Hence

$$\int_{-\pi}^{\pi}\left(\int_0^a f(r,\theta) J_n\left(\sqrt{\lambda_{nm}}r\right) rdr\right)\cos(n'\theta)\,d\theta = \int_{-\pi}^{\pi}\sum_{n=0}^{\infty} A_{nm} \left(\int_0^a J_n^2\left(\sqrt{\lambda_{nm}}r\right) rdr\right)\cos(n\theta)\cos(n'\theta)\,d\theta + \int_{-\pi}^{\pi}\sum_{n=1}^{\infty} C_{nm} \left(\int_0^a J_n^2\left(\sqrt{\lambda_{nm}}r\right) rdr\right)\sin(n\theta)\cos(n'\theta)\,d\theta$$

Or

$$\int_{-\pi}^{\pi}\int_0^a f(r,\theta) J_n\left(\sqrt{\lambda_{nm}}r\right)\cos(n'\theta)\,r\ d\theta dr = \int_{-\pi}^{\pi} A_{n'm} \left(\int_0^a J_n^2\left(\sqrt{\lambda_{nm}}r\right) rdr\right)\cos^2(n'\theta)\,d\theta + \int_{-\pi}^{\pi} C_{n'm} \left(\int_0^a J_n^2\left(\sqrt{\lambda_{nm}}r\right) rdr\right)\sin(n'\theta)\cos(n'\theta)\,d\theta$$

The $\int\sin(n'\theta)\cos(n'\theta)\,\sigma d\theta$ term goes to zero, and we are left with (also, changed n' back to n since we are doing with summation)

$$\int_{-\pi}^{\pi}\int_0^a f(r,\theta) J_n\left(\sqrt{\lambda_{nm}}r\right)\cos(n\theta)\,r\ d\theta dr = A_{nm}\int_{-\pi}^{\pi}\int_0^a J_n^2\left(\sqrt{\lambda_{nm}}r\right)\cos^2(n\theta)\,rd\theta dr$$

Hence

$$A_{nm} = \frac{\int_{-\pi}^{\pi}\int_0^a f(r,\theta) J_n\left(\sqrt{\lambda_{nm}}r\right)\cos(n\theta)\,r\ d\theta dr}{\int_{-\pi}^{\pi}\int_0^a J_n^2\left(\sqrt{\lambda_{nm}}r\right)\cos^2(n\theta)\,r\ d\theta dr}$$

We now need to apply orthogonality on n using the sin. to find C_{nm}. Using (5)

$$\int_{-\pi}^{\pi}\left(\int_0^a f(r,\theta) J_n\left(\sqrt{\lambda_{nm}}r\right) rdr\right)\sin(n'\theta)\,rd\theta = \int_{-\pi}^{\pi}\sum_{n=0}^{\infty} A_{nm} \left(\int_0^a J_n^2\left(\sqrt{\lambda_{nm}}r\right) rdr\right)\cos(n\theta)\sin(n'\theta)\,d\theta + \int_{-\pi}^{\pi}\sum_{n=1}^{\infty} C_{nm} \left(\int_0^a J_n^2\left(\sqrt{\lambda_{nm}}r\right) rdr\right)\sin(n\theta)\sin(n'\theta)\,d\theta$$

Hence

$$\int_{-\pi}^{\pi}\left(\int_0^a f(r,\theta) J_n\left(\sqrt{\lambda_{nm}}r\right) rdr\right)\sin(n\theta)\,d\theta = C_{nm}\int_{-\pi}^{\pi}\int_0^a J_n^2\left(\sqrt{\lambda_{nm}}r\right)\sin^2(n\theta)\,r\ d\theta dr$$

Therefore

$$C_{nm} = \frac{\int_{-\pi}^{\pi}\int_0^a f(r,\theta) J_n\left(\sqrt{\lambda_{nm}}r\right)\sin(n\theta)\,r\ d\theta dr}{\int_{-\pi}^{\pi}\int_0^a J_n^2\left(\sqrt{\lambda_{nm}}r\right)\sin^2(n\theta)\,r\ d\theta dr}$$

Now we will look at the second initial conditions $\frac{\partial u}{\partial t}(r,\theta,0) = g(r,\theta)$. Taking derivative w.r.t. time t of the solution in (3) gives

$$\begin{aligned}\frac{\partial u}{\partial t}(r,\theta,t) &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} -c\sqrt{\lambda_{nm}}A_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)J_n\left(\sqrt{\lambda_{nm}}r\right)\cos(n\theta)\\ &+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c\sqrt{\lambda_{nm}}B_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)J_n\left(\sqrt{\lambda_{nm}}r\right)\cos(n\theta)\\ &+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} -c\sqrt{\lambda_{nm}}C_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)J_n\left(\sqrt{\lambda_{nm}}r\right)\sin(n\theta)\\ &+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c\sqrt{\lambda_{nm}}D_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)J_n\left(\sqrt{\lambda_{nm}}r\right)\sin(n\theta)\end{aligned}$$

At time $t=0$ the above becomes (all terms with $\sin\left(c\sqrt{\lambda_{nm}}t\right)$ vanish).

$$\begin{aligned}g(r,\theta) &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c\sqrt{\lambda_{nm}}B_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)J_n\left(\sqrt{\lambda_{nm}}r\right)\cos(n\theta)\\ &+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c\sqrt{\lambda_{nm}}D_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)J_n\left(\sqrt{\lambda_{nm}}r\right)\sin(n\theta)\end{aligned}$$

Now orthogonality is used. At $t=0$ the above becomes

$$\begin{aligned}g(r,\theta) &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c\sqrt{\lambda_{nm}}B_{nm}J_n\left(\sqrt{\lambda_{nm}}r\right)\cos(n\theta)\\ &+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c\sqrt{\lambda_{nm}}D_{nm}J_n\left(\sqrt{\lambda_{nm}}r\right)\sin(n\theta)\end{aligned}$$

Similarly to the above we now find B_{nm} and D_{nm}. The only difference, is that now we have extra $c\sqrt{\lambda_{nm}}$ terms that show up. The final result will be

$$B_{nm} = \frac{\int_{-\pi}^{\pi}\int_0^a g(r,\theta)J_n\left(\sqrt{\lambda_{nm}}r\right)\cos(n\theta)\,r\ d\theta dr}{c\sqrt{\lambda_{nm}}\int_{-\pi}^{\pi}\int_0^a J_n^2\left(\sqrt{\lambda_{nm}}r\right)\cos^2(n\theta)\,r\ d\theta dr}$$

And

$$D_{nm} = \frac{\int_{-\pi}^{\pi}\int_0^a g(r,\theta)J_n\left(\sqrt{\lambda_{nm}}r\right)\sin(n\theta)\,r\ d\theta dr}{c\sqrt{\lambda_{nm}}\int_{-\pi}^{\pi}\int_0^a J_n^2\left(\sqrt{\lambda_{nm}}r\right)\sin^2(n\theta)\,r\ d\theta dr}$$

Summary of solution

$$\begin{aligned}u(r,\theta,t) &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)J_n\left(\sqrt{\lambda_{nm}}r\right)\cos(n\theta)\\ &+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)J_n\left(\sqrt{\lambda_{nm}}r\right)\cos(n\theta)\\ &+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)J_n\left(\sqrt{\lambda_{nm}}r\right)\sin(n\theta)\\ &+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} D_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)J_n\left(\sqrt{\lambda_{nm}}r\right)\sin(n\theta)\end{aligned}$$

$$A_{nm} = \frac{\int_{-\pi}^{\pi}\int_0^a f(r,\theta)J_n\left(\sqrt{\lambda_{nm}}r\right)\cos(n\theta)\,r\ d\theta dr}{\int_{-\pi}^{\pi}\int_0^a J_n^2\left(\sqrt{\lambda_{nm}}r\right)\cos^2(n\theta)\,r\ d\theta dr}$$

$$C_{nm} = \frac{\int_{-\pi}^{\pi}\int_0^a f(r,\theta)J_n\left(\sqrt{\lambda_{nm}}r\right)\sin(n\theta)\,r\ d\theta dr}{\int_{-\pi}^{\pi}\int_0^a J_n^2\left(\sqrt{\lambda_{nm}}r\right)\sin^2(n\theta)\,r\ d\theta dr}$$

$$B_{nm} = \frac{\int_{-\pi}^{\pi}\int_0^a g(r,\theta)\, J_n\left(\sqrt{\lambda_{nm}}r\right)\cos(n\theta)\, r\ d\theta dr}{c\sqrt{\lambda_{nm}}\int_{-\pi}^{\pi}\int_0^a J_n^2\left(\sqrt{\lambda_{nm}}r\right)\cos^2(n\theta)\, r\ d\theta dr}$$

$$D_{nm} = \frac{\int_{-\pi}^{\pi}\int_0^a g(r,\theta)\, J_n\left(\sqrt{\lambda_{nm}}r\right)\sin(n\theta)\, r\ d\theta dr}{c\sqrt{\lambda_{nm}}\int_{-\pi}^{\pi}\int_0^a J_n^2\left(\sqrt{\lambda_{nm}}r\right)\sin^2(n\theta)\, r\ d\theta dr}$$

With λ_{nm} being the solutions for $0 = J_n\left(\sqrt{\lambda_{nm}}a\right)$. For each n, we find $\lambda_{n,1}, \lambda_{n,2}, \lambda_{n,3}, \cdots$, which are the zeros of the Bessel $J_n(x)$ function. So for each n, we have infinite number of zeros. This generates all the needed λ_{nm}. Hence $\sqrt{\lambda_{nm}}a = BesselJZero(n,m)$, therefore $\sqrt{\lambda_{nm}} = \frac{a}{BesselJZero(n,m)}$

The following animations run for 80 seconds. They are for different n,m modes. All have zero for initial velocity, which means $g(r,\theta) = 0$. Radius used is $a = 1$ and $c = 0.2$, initial position used is $f(r,\theta) = r\theta$.

Cases for $n = 0$

Cases for $n = 1$

Cases for $n = 2$

Cases for $n = 3$

7.2.2 case circularly symmetric

In this case, there is no θ dependency in boundary conditions or in initial conditions. Using polar coordinates. Disk has radius a. Wave displacement is $u \equiv u(r,t)$ (out of page).

$$\frac{\partial^2 u(r,\theta,t)}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$
$$0 < r < a$$

Boundary conditions on r

$$|u(0,t)| < \infty$$
$$u(a,t) = 0$$

Initial conditions

$$u(r,0) = f(r)$$
$$\frac{\partial u}{\partial t}(r,0) = g(r)$$

Let $u = T(t)\,R(r)$. Plug in the PDE

$$\frac{1}{c^2}T''R = R''T + \frac{1}{r}R'T$$

Dividing by RT

$$\frac{1}{c^2}\frac{T''}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R}$$

Hence

$$\frac{1}{c^2}\frac{T''}{T} = -\lambda$$
$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\lambda$$

The time ODE is

$$T'' + c^2\lambda T = 0$$

And the r ODE is (Sturm-Liouville)

$$rR'' + R' + \lambda rR = 0$$

Where $p=r,q=0,\sigma=r$. This is singular SL. The solution turns out to be

$$R_n(r)=A_nJ_0\left(\sqrt{\lambda_n}r\right)\qquad n=1,2,3,\cdots$$

Where λ_n is found from roots of $0=J_n\left(\sqrt{\lambda_n}a\right)$ giving the eigenvalues. Now the time ODE is solved

$$T_n''+c^2\lambda_nT_n=0$$
$$T_n=B_n\cos\left(c\sqrt{\lambda_n}t\right)+C_n\sin\left(c\sqrt{\lambda_n}t\right)\qquad n=1,2,3,\dots,$$

Hence the solution is

$$\begin{aligned}u(r,t)&=\sum_{n=1}^{\infty}T_nR_n\\&=\sum_{n=1}^{\infty}A_n\cos\left(c\sqrt{\lambda_n}t\right)J_0\left(\sqrt{\lambda_n}r\right)+B_n\sin\left(c\sqrt{\lambda_n}t\right)J_0\left(\sqrt{\lambda_n}r\right)\end{aligned}\tag{1}$$

Now initial conditions $u(r,0)=f(r)$ is used to find A_n using orthogonality. At $t=0$ the solution simplifies to

$$u(r,0)=\sum_{n=1}^{\infty}A_nJ_0\left(\sqrt{\lambda_n}r\right)$$

Hence

$$f(r)=\sum_{n=1}^{\infty}A_nJ_0\left(\sqrt{\lambda_n}r\right)$$
$$\int_0^a f(r)J_0\left(\sqrt{\lambda_n}r\right)rdr=A_n\int_0^a J_0^2\left(\sqrt{\lambda_n}r\right)rdr$$
$$A_n=\frac{\int_0^a f(r)J_0\left(\sqrt{\lambda_n}r\right)rdr}{\int_0^a J_0^2\left(\sqrt{\lambda_n}r\right)rdr}$$

Now we will look at the second initial conditions $\frac{\partial u}{\partial t}(r,0)=g(r)$. Taking derivative w.r.t. time t of the solution in (1) gives

$$\frac{\partial u}{\partial t}(r,t)=\sum_{n=1}^{\infty}-c\sqrt{\lambda_n}A_n\sin\left(c\sqrt{\lambda_n}t\right)J_0\left(\sqrt{\lambda_n}r\right)+B_nc\sqrt{\lambda_n}\cos\left(c\sqrt{\lambda_n}t\right)J_0\left(\sqrt{\lambda_n}r\right)$$

At time $t=0$ the above becomes

$$g(r)=\sum_{n=1}^{\infty}B_nc\sqrt{\lambda_n}J_0\left(\sqrt{\lambda_n}r\right)$$

Now orthogonality is used. The above becomes

$$B_n=\frac{\int_0^a g(r)J_0\left(\sqrt{\lambda_n}r\right)rdr}{c\sqrt{\lambda_n}\int_0^a J_0^2\left(\sqrt{\lambda_n}r\right)rdr}$$

<u>Summary of solution</u>

$$u(r,t)=\sum_{n=1}^{\infty}A_n\cos\left(c\sqrt{\lambda_n}t\right)J_0\left(\sqrt{\lambda_n}r\right)+B_n\sin\left(c\sqrt{\lambda_n}t\right)J_0\left(\sqrt{\lambda_n}r\right)$$
$$A_n=\frac{\int_0^a f(r)J_0\left(\sqrt{\lambda_n}r\right)rdr}{\int_0^a J_0^2\left(\sqrt{\lambda_n}r\right)rdr}$$
$$B_n=\frac{\int_0^a g(r)J_0\left(\sqrt{\lambda_n}r\right)rdr}{c\sqrt{\lambda_n}\int_0^a J_0^2\left(\sqrt{\lambda_n}r\right)rdr}$$

With λ_n being the solutions for $0=J_0\left(\sqrt{\lambda_n}a\right)$. We have infinite number of zeros. This generates all the needed λ_n. Hence $\sqrt{\lambda_n}a=BesselJZero(0,n)$, therefore $\sqrt{\lambda_n}=\frac{a}{BesselJZero(0,n)}$

This animation runs for 40 seconds. Using radius $a=1$ and zero initial velocity. Initial position is $u(r,0)=f(r)=r$. And $c=0.2$. Number of terms in the sum used was 30. Which means $u(r,t)=\sum_{n=1}^{30}A_n\cos\left(c\sqrt{\lambda_n}t\right)J_0\left(\sqrt{\lambda_n}r\right)$. The B_n terms are all zero since initial velocity $g(r)$ was zero.

8 Solitons wave animation (non-linear wave pde)

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

Assuming special solution $u = f(\xi)$ where $\xi = x - c\ \not{t}$, this PDE is transformed to non-linear first order ODE

$$-c\frac{f^2}{2} + f^3 + \frac{1}{2}\left(\frac{df}{d\xi}\right)^2 = 0$$

The above is solved analytically (Krvskal, Zabrsky 1965) and the solution is

$$f(\xi) = \left(\frac{1}{2}c\right)\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct)\right)$$

Tall waves move fast but have smaller period, short wave move slow. Tall wave pass through short wave and leave as they enter. Here are two animations and the above solution. This first animation has one tall wave passing though short wave

This animation shows three waves

Code used for the above is

by Nasser M. Abbasi, April 2017

In[4]:=
```
SetDirectory[NotebookDirectory[]];
```

In[5]:=
```
<< MaTeX`
SetOptions[MaTeX, "Preamble" →
   {"\\usepackage{color,txfonts}
\\usepackage{amsmath}
\\DeclareMathOperator{\\sech}{sech}
"}];
```

Manipulate to test before exporting

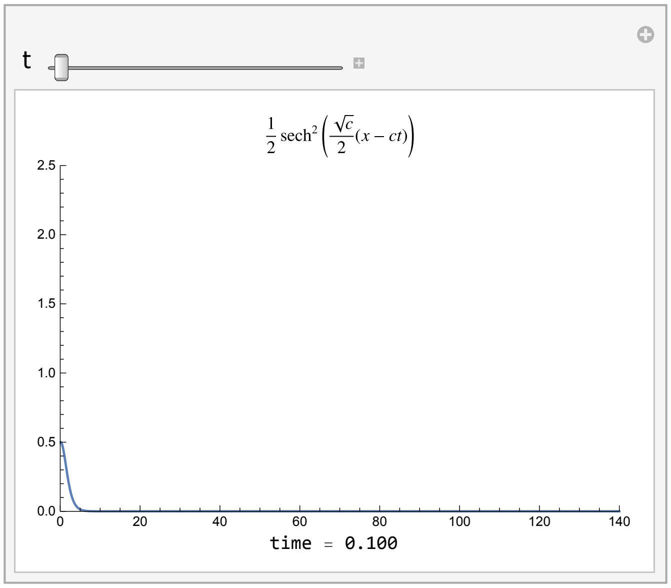
In[7]:= $f[x\_, c\_, t\_] := \frac{1}{2} c \, \text{Sech}\left[\frac{\sqrt{c}}{2} (x - c\,t)\right]^2;$

```
title = MaTeX["\\frac{1}{2} \\sech^2\\left( \\frac{\\sqrt c}{2} (x-c t)  \\right)"]

Manipulate[
 Grid[{
   {
    Plot[
     If[t > 40, f[x, 2.9, t - 40] + f[x, 1.5, t - 20] + f[x, 1, t],
      If[t > 20, f[x, 1.5, t - 20] + f[x, 1, t],
       f[x, 1, t]
      ]
     ], {x, 0, 140},
     PlotRange → {{0, 140}, {0, 2.5}},
     ImageSize → 400,
     Frame → False,
     PlotLabel → title]
   },
   {Row[{"time = ", NumberForm[t, {5, 3}]}]}
  }
 ],
 {t, 0.1, 100, .1}
]
```

Out[8]= $\frac{1}{2} \operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct)\right)$

Out[9]=



First animation

```
data = Table[Grid[{
      {
       Plot[
        If[t > 20, f[x, 2, t - 20] + f[x, 1, t], f[x, 1, t]], {x, 0, 80},
        PlotRange → {{0, 80}, {0, 2}},
        ImageSize → 400,
        Frame → False,
        PlotLabel → title]
      },
      {Row[{"time = ", NumberForm[t, {5, 3}]}]}
     }
    ], {t, 0, 80, .3}
   ];

Export["solitons.gif", data, "AnimationRepetitions" → Infinity]
```

solitons.gif

Second animation

```
data = Table[Grid[{
     {
      Plot[
       If[t > 40, f[x, 2.9, t - 40] + f[x, 1.5, t - 20] + f[x, 1, t],
        If[t > 20, f[x, 1.5, t - 20] + f[x, 1, t],
         f[x, 1, t]
        ]
       ], {x, 0, 140},
       PlotRange → {{0, 140}, {0, 2.5}},
       ImageSize → 400,
       Frame → False,
       PlotLabel → title]
     },
     {Row[{"time = ", NumberForm[t, {5, 3}]}]}
    }
   ],
   {t, 0.1, 100, .3}];
```

```
Export["solitons_3.gif", data, "AnimationRepetitions" → Infinity]
```

```
solitons_3.gif
```