Analytical solution to specific Stockes first problem PDE

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Solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{1}$$
$$0 < x < L$$
$$t > 0$$

Initial conditions

$$u(0,x) = 0$$

Boundary conditions

$$u(0,t) = \sin(t)$$
$$u(t,L) = 0$$

Let

$$u = v + u_E \tag{2}$$

where $u_E(x,t)$ is steady state solution that only needs to satisfy boundary conditions and $v(x,t)$ satisfies the PDE itself but with homogenous B.C. At steady state, the PDE becomes

$$0 = k\frac{d^2 u_E}{dx^2}$$
$$u_E(0) = \sin(t)$$
$$u_E(L) = 0$$

The solution is $u_E(t) = \left(\frac{L-x}{L}\right)\sin(t)$. Hence (2) becomes

$$u(x,t) = v(x,t) + \left(\frac{L-x}{L}\right)\sin(t)$$

Substituting the above in (1) gives

$$\frac{\partial v}{\partial t} + \left(\frac{L-x}{L}\right)\cos(t) = k\frac{\partial^2 v}{\partial x^2}$$
$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + \left(\frac{x-L}{L}\right)\cos(t)$$
$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + Q(x,t) \tag{3}$$

With boundary conditions $u_0(0,t) = 0, u(L,t) = 0$. This is now in standard form and separation of variables can be used to solve it.

$$Q(x,t) = \left(\frac{x-L}{L}\right)\cos(t)$$

Now acts as a source term. The eigenfunctions are known to be $\Phi_n(x) = \sin\left(\sqrt{\lambda_n}x\right)$ where $\lambda_n = \left(\frac{n\pi}{L}\right)^2$. Hence by eigenfunction expansion, the solution to (3) is

$$v(x,t) = \sum_{n=1}^{\infty} B_n(t)\,\Phi_n(x) \tag{3A}$$

Substituting this into (3) gives

$$\sum_{n=1}^{\infty} \frac{dB_n(t)}{dt}\Phi_n(x) = k\sum_{n=1}^{\infty} B_n(t)\,\Phi_n''(x) + Q(x,t) \tag{4}$$

Expanding $Q(x,t)$ using same basis (eigenfunctions) gives

$$Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\,\Phi_n(x)$$

Applying orthogonality

$$\begin{aligned}\int_0^L Q(x,t)\,\Phi_m(x)\,dx &= \int_0^L \sum_{n=1}^{\infty} q_n(t)\,\Phi_n(x)\,\Phi_m(x)\,dx\\ &= \sum_{n=1}^{\infty} q_n(t)\int_0^L \Phi_n(x)\,\Phi_m(x)\,dx\end{aligned}$$

But $\sum_{n=1}^{\infty}\int_0^L \Phi_n(x)\,\Phi_m(x)\,dx = \int_0^L \Phi_m^2(x)\,dx = \frac{L}{2}$ since $\Phi_n(x) = \sin\left(\frac{n\pi}{L}x\right)$ and the above simplifies to

$$\begin{aligned}\int_0^L Q(x,t)\,\Phi_n(x)\,dx &= \frac{L}{2}q_n(t)\\ q_n(t) &= \frac{2}{L}\int_0^L Q(x,t)\sin\left(\frac{n\pi}{L}x\right)dx\end{aligned}$$

But $Q(x,t) = \left(\frac{x-L}{L}\right)\cos(t)$, hence

$$\begin{aligned}q_n(t) &= \frac{2}{L}\int_0^L \left(\frac{x-L}{L}\right)\cos(t)\sin\left(\frac{n\pi}{L}x\right)dx\\ &= \frac{-2}{n\pi}\cos(t)\end{aligned}$$

Therefore $Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\,\Phi_n(x) = \sum_{n=1}^{\infty}\frac{-2}{n\pi}\cos(t)\sin\left(\frac{n\pi}{L}x\right)$ and (4) becomes

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{dB_n(t)}{dt}\Phi_n(x) &= k\sum_{n=1}^{\infty}B_n(t)\,\Phi_n''(x) - \sum_{n=1}^{\infty}\frac{2}{n\pi}\cos(t)\sin\left(\frac{n\pi}{L}x\right)\\ \frac{dB_n(t)}{dt}\sin\left(\frac{n\pi}{L}x\right) &= kB_n(t)\left(-\frac{n^2\pi^2}{L^2}\sin\left(\frac{n\pi}{L}x\right)\right) - \frac{2}{n\pi}\cos(t)\sin\left(\frac{n\pi}{L}x\right)\\ \frac{dB_n(t)}{dt} + B_n(t)\,k\frac{n^2\pi^2}{L^2} &= -\frac{2}{n\pi}\cos(t)\end{aligned}$$

This is an ODE in $B_n(t)$ whose solution is

$$B_n(t) = C_n e^{-k\left(\frac{n^2\pi^2}{L^2}\right)t} - \frac{2L^2\left(kn^2\pi^2\cos t + L^2\sin t\right)}{n\pi\left(L^4 + k^2n^4\pi^4\right)}$$

From (3A) $v(x,t)$ now becomes

$$v(x,t) = \sum_{n=1}^{\infty} C_n e^{-k\left(\frac{n^2\pi^2}{L^2}\right)t}\sin\left(\frac{n\pi}{L}x\right) - \frac{2L^2\left(kn^2\pi^2\cos t + L^2\sin t\right)}{n\pi\left(L^4 + k^2n^4\pi^4\right)}\sin\left(\frac{n\pi}{L}x\right) \tag{5}$$

To find C_n, from initial conditions, at $t = 0$ the above becomes

$$0 = \sum_{n=1}^{\infty} C_n\sin\left(\frac{n\pi}{L}x\right) - \frac{2L^2\left(kn^2\pi^2\right)}{n\pi\left(L^4 + k^2n^4\pi^4\right)}\sin\left(\frac{n\pi}{L}x\right)$$

Hence

$$C_n = \frac{2L^2\left(kn^2\pi^2\right)}{n\pi\left(L^4 + k^2n^4\pi^4\right)}$$

Therefore (5) becomes

$$v(x,t) = \sum_{n=1}^{\infty}\left(\frac{2L^2\left(kn^2\pi^2\right)}{n\pi\left(L^4 + k^2n^4\pi^4\right)}e^{-k\left(\frac{n^2\pi^2}{L^2}\right)t} - \frac{2L^2\left(kn^2\pi^2\cos t + L^2\sin t\right)}{n\pi\left(L^4 + k^2n^4\pi^4\right)}\right)\sin\left(\frac{n\pi}{L}x\right)$$

And since $u = v + u_E$ then the solution is

$$u(x,t) = \left(\sum_{n=1}^{\infty}\left(\frac{2L^2\left(kn^2\pi^2\right)}{n\pi\left(L^4 + k^2n^4\pi^4\right)}e^{-k\left(\frac{n^2\pi^2}{L^2}\right)t} - \frac{2L^2\left(kn^2\pi^2\cos t + L^2\sin t\right)}{n\pi\left(L^4 + k^2n^4\pi^4\right)}\right)\sin\left(\frac{n\pi}{L}x\right)\right) + \left(\frac{L-x}{L}\right)\sin(t)$$

To simulate

```
ClearAll[t, x, n]
k = 1; L0 = 5; max = 400;
u[x_, t_] =
Sum[(((2*L0^2*(k*n^2*Pi^2))/(n*Pi*(L0^4 + k^2*n^4*Pi^4)))*
Exp[(-k)*((n^2*Pi^2)/L0^2)*t] -
(2*L0^2*(k*n^2*Pi^2*Cos[t] + L0^2*Sin[t]))/(n*
Pi*(L0^4 + k^2*Pi^4*n^4)))*Sin[((n*Pi)/L0)*x],
{n, 1, max}] + ((L0 - x)/L0)*Sin[t];
Manipulate[Grid[{{"Analytical solution"},
{Plot[Evaluate[u[x,t]],{x,0,5},PlotRange->{{0,5},{-1.1,1.1}},
ImageSize->400]}}],
{{t,0,"t"},0,100,.01}
]
```

Here is the animation from the above

Here is the numerical solution to compare with

```
ClearAll["Global`*"];
pdeset = {Derivative[1, 0][U][t, x] == Derivative[0, 2][U][t, x]}
ics = {U[0, x] == 0};
bcs = {U[t, 0] == Sin[t], U[t, 5] == 0};
ibcAll = {ics, bcs};
numericalSol = NDSolve[{pdeset, ibcAll}, U, {t, 0, 100}, {x, 0, 5}];
Manipulate[Grid[{{"Numerical solution"},
{Plot[Evaluate[U[t, x] /. numericalSol], {x, 0, 5},
PlotRange -> {{0, 5}, {-1, 1}}, ImageSize -> 400]}}],
{{t, 0, "t"}, 0, 100, .01}
]
```

Here is the animation from the above

Reference: stokes second problem question and answer