# my mathematics cheat sheet

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## Contents



A place to keep quick notes about Math that I keep forgetting. This is meant to be a scratch notes to write math notes. Can contain errors and/or not complete. Use at your own risk.

# 1 general notes

1. Methods to find Green function are
   (a) Fredholm theory
   (b) methods of images
   (c) separation of variables
   (d) Laplace transform

   reference wikipedia I need to make one example and apply each of the above methods on it.

2. we can't just say $f(z)$ is Analytic and stop. Have to say $f(z)$ is analytic *at a point* $z_0$ or *in a region*. $f(z)$ is analytic at a point $z_0$ if the power series for $f(z)$ expanded around $z_0$ converges to $f(z)$ evaluated at $z_0$. (i.e. $f(z)$ has Taylor series that converges to $f(z)$ at the point).
   For a region, the above test will be true for every point in that region. For example, $f(z) = \sqrt{z}$ is not analytic at $z_0 = 0$. This is because $f'(z)$ when evaluated at $z_0 = 0$ is a singularity (a fancy name for saying it blows up at $z_0 = 0$). But $f(0) = 0$, so we see that $f(z)$ is not analytic at $z_0 = 0$ since its power series does not converge to the same value as the function *at* $z_0 = 0$
   But this is only for this point $z_0 = 0$. The function $f(z_0)$ is analytic at another point, say $z_0 = 1$.

3. In solving an ODE with constant coefficient just use the characteristic equation to solve the solution.

4. In solving an ODE with coefficients that are functions that depends on the independent variable, as in $y''(x) + q(x)\,y'(x) + p(x)\,y(x) = 0$, first classify the point $x_0$ type. This means to check how $p(x)$ and $q(x)$ behaves at $x_0$. We are talking about the ODE here, not the solution yet. There are 3 kinds of points. $x_0$ can be normal, or regular singular point, or irregular singular point. Normal $x_0$ means $p(x)$ and $q(x)$ have Taylor series expansion $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ that converges to $y(x)$ at $x_0$.
   Regular singular point $x_0$ means that the above test fails, but $\lim_{x\to x_0} (x - x_0)\,q(x)$ has a convergent Taylor series, and also that $\lim_{x\to x_0} (x - x_0)^2\,p(x)$ now has a convergent Taylor series at $x_0$. All this just means we can get rid of the singularity. i.e. $x_0$ is a removable singularity. If this is the case, then the solution at $x_0$ can be assumed to have a Frobenius series $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\alpha}$ where $a_0 \neq 0$ and $\alpha$ must be integer values.
   The third type of point, is the hard one. Called irregular singular point. We can't get rid of it using the above trick. So we also say the ODE has an essential singularity at $x_0$ (another fancy name for irregular singular point, I think). What this means is that we can't approximate the solution at $x_0$ using either Taylor nor Frobenius series.
   If the point is an irregular singular point, then use the methods of asymptotic. See *advanced mathematical methods for scientists and engineers* chapter 3. For normal point, use $y(x) = \sum_{n=0}^{\infty} a_n x^n$, for regular singular point use $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$. Remember, to solve for $r$ first. This should give two values. If you get one root, then use reduction of order to find second solution.

5. For Taylor series, leading behavior is $a_0$ no controlling factor? For Frobenius series, leading behavior term is $a_0 x^\alpha$ and controlling factor is $x^\alpha$. For asymptotic series, controlling factor is assumed to be $e^{S(x)}$ always. proposed by Carlini (1817)

6. Method to find the leading behavior of the solution $y(x)$ near irregular singular point using asymptotic is called the dominant balance method.

7. When solving $\epsilon y'' + p(x)\,y' + q(x)\,y = 0$ for very small $\epsilon$ then use WKB method, if there is no boundary layer between the boundary conditions. If the ODE non-linear, can't use WKB, has to use boundary layer (B.L.). Example $\epsilon y'' + yy' - y = 0$ with $y(0) = 0, y(1) = -2$ then use BL.

8. good exercise is to solve say $\epsilon y'' + (1+x)y' + y = 0$ with $y(0) = y(1)$ using both B.L. and WKB and compare the solutions, they should come out the same. $y \sim \frac{2}{1+x} - \exp\left(\frac{-x}{\epsilon} - \frac{x^2}{2\epsilon}\right) + O(\epsilon)$. with BL had to do the matching between the outer and the inner solutions. WKB is easier. But can't use it for non-linear ODE.

9. When there is rapid oscillation over the entire domain, WKB is better. Use WKB to solve Schrodinger equation where $\epsilon$ becomes function of $\hbar$ (Planck's constant, $6.62606957 \times 10^{-34}$ $\text{m}^2\text{kg/s}$)

10. In second order ODE with non constant coefficient, $y''(x) + p(x)\,y'(x) + q(x)\,y(x) = 0$, if we know one solution $y_1(x)$, then a method called the *reduction of order* can be used to find the second solution $y_2(x)$. Write $y_2(x) = u(x)\,y_1(x)$, plug this in the ODE, and solve for $u(x)$. The final solution will be $y(x) = c_1 y_1(x) + c_2 y_2(x)$. Now apply I.C.'s to find $c_1, c_2$.

11. To find particular solution to $y''(x) + p(x)\,y'(x) + q(x)\,y(x) = f(x)$, we can use a method called *undetermined coefficients*. But a better method is called *variation of parameters*, In this method, assume $y_p(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$ where $y_1(x), y_2(x)$ are the two linearly independent solutions of the homogeneous ODE and $u_1(x), u_2(x)$ are to be determined. This ends up with $u_1(x) = -\int \frac{y_2(x) f(x)}{W} dx$ and $u_2(x) = \int \frac{y_1(x) f(x)}{W} dx$. Remember to put the ODE in standard form first, so $a = 1$, i.e. $ay''(x) + \cdots$. In here, $W$ is the Wronskian $W = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$

12. Two solutions of $y''(x) + p(x)\,y'(x) + q(x)\,y(x) = 0$ are linearly independent if $W(x) \neq 0$, where $W$ is the Wronskian.

13. Green function takes the homogeneous solution and the forcing function and constructs a particular solution. For PDE's, we always want a symmetric Green's function.

14. To get a symmetric Green's function given an ODE, start by converting the ODE to a Sturm-Liouville form first. This way the Green's function comes out symmetric.

15. For numerical solutions of field problems, there are basically two different problems: Those with closed boundaries and those with open boundaries but with initial conditions. Closed boundaries are elliptical problems which can be cast in the form $Au = f$, and the other are either hyperbolic or parabolic.

16. For numerical solution of elliptical problems, the basic layout is something like this: Always start with trial solution $u(x)$ such that $u_{trial}(x) = \sum_{i=0}^{i=N} C_i \phi_i(x)$ where the $C_i$ are the unknowns to be determined and the $\phi_i$ are set of linearly independent functions (polynomials) in $x$. How to determine those $C_i$ comes next. Use either residual method (Galerkin) or variational methods (Ritz). For residual, we make a function based on the error $R = A - u_{trial} f$. It all

comes down to solving $\int f(R) = 0$ over the domain. This is a picture

```
|
+---------------+-----------------------------------+
|                                                   |
residual                          Variational (sub u_trial in I(u)
|                            where I(u) is functional to minimize.
|
+----------------+------------+----------+
|                |            |          |
Absolute error   collocation   subdomain orthogonality
....
+----------------------+------------+
|                      |            |
method of moments   Galerkin     least squares
```

17. Geometric probability distribution. Use when you want an answer to the question: What is the probability you have to do the experiment $N$ times to finally get the output you are looking for, given that a probability of $p$ showing up from doing one experiment.

    For example: What is the probability one has to flip a fair coin $N$ times to get a head? The answer is $P(X = N) = (1-p)^{k-1}p$. So for a fair coin, $p = \frac{1}{2}$ that a head will show up from one flip. So the probability we have to flip a coin 10 times to get a head is $P(X = 10) = (1-0.5)^9(0.5) = 0.00097$ which is very low as expected.

18. To generate random variable drawn from some distribution different from uniform distribution, by only using uniform distribution $U(0,1)$ do this: Lets say we want to generate random number from exponential distribution with mean $\mu$. This distribution has $pdf(X) = \frac{1}{\mu}e^{\frac{-x}{\mu}}$, the first step is to find the cdf of exponential distribution, which is known to be $F(x) = P(X <= x) = 1 - e^{\frac{-x}{\mu}}$. Now find the inverse of this, which is $F^{-1}(x) = -\mu\ln(1-x)$. Then generate a random number from the uniform distribution $U(0,1)$. Let this value be called $z$. Now plug this value into $F^{-1}(z)$, this gives a random number from exponential distribution, which will be $-\mu ln(1-z)$ (take the natural log of both side of $F(x)$).

    This method can be used to generate random variables from any other distribution by knowing on $U(0,1)$. But it requires knowing the CDF and the inverse of the CDF for the other distribution. This is called the **inverse CDF method**. Another method is called the **rejection method**

19. Given $u$, a r.v. from uniform distribution over [0,1], then to obtain $v$, a r.v. from uniform distribution over [A,B], then the relation is $v = A + (B-A)u$.

20. When solving using F.E.M. is best to do everything using isoparametric element (natural coordinates), then find the Jacobian of transformation between the natural and physical coordinates to evaluate the integrals needed. For the force function, using Gaussian quadrature method.

21. A solution to differential equation is a function that can be expressed as a convergent series. (Cauchy. Briot and Bouquet, Picard)

22. To solve a first order ODE using integrating factor.

$$x'(t) + p(t)x(t) = f(t)$$

then as long as it is linear and $p(t), f(t)$ are integrable functions in $t$, then follow these steps

(a) multiply the ODE by function $I(t)$, this is called the integrating factor.

$$I(t)x'(t) + I(t)p(t)x(t) = I(t)f(t)$$

(b) We solve for $I(t)$ such that the left side satisfies

$$\frac{d}{dt}(I(t)x(t)) = I(t)x'(t) + I(t)p(t)x(t)$$

(c) Solving the above for $I(t)$ gives

$$\begin{aligned} I'(t)x(t) + I(t)x'(t) &= I(t)x'(t) + I(t)p(t)x(t) \\ I'(t)x(t) &= I(t)p(t)x(t) \\ I'(t) &= I(t)p(t) \\ \frac{dI}{I} &= p(t)dt \end{aligned}$$

Integrating both sides gives

$$\begin{aligned} \ln(I) &= \int p(t)dt \\ I(t) &= e^{\int p(t)dt} \end{aligned}$$

(d) Now eq(1) can be written as

$$\frac{d}{dt}(I(t)x(t)) = I(t)f(t)$$

We now integrate the above to give

$$\begin{aligned} I(t)x(t) &= \int I(t)f(t)\,dt + C \\ x(t) &= \frac{\int I(t)f(t)\,dt + C}{I(t)} \end{aligned}$$

Where $I(t)$ is given by (2). Hence

$$x(t) = \frac{\int e^{\int p(t)dt} f(t)\,dt + C}{e^{\int p(t)dt}}$$

23. a polynomial is called ill-conditions if we make small change to one of its coefficients and this causes large change to one of the roots.

24. to find rank of matrix $A$ by hand, find the row echelon form, then count how many zero rows there are. subtract that from number of rows, i.e. $n$.

25. to find the basis of the column space of $A$, find the row echelon form and pick the columns with the pivots, there are the basis (the linearly independent columns of $A$).

26. For symmetric matrix $A$, its second norm is its spectral radius $\rho(A)$ which is the largest eigenvalue of $A$ (in absolute terms).

27. The eigenvalues of the inverse of matrix $A$ is the inverse of the eigenvalues of $A$.

28. If matrix $A$ of order $n \times n$, and it has $n$ distinct eigenvalues, then it can be diagonalized $A = V\Lambda V^{-1}$, where

$$\Lambda = \begin{pmatrix} e^{\lambda_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{\lambda n} \end{pmatrix}$$

and $V$ is matrix that has the $n$ eigenvectors as its columns.

29. $\lim_{k\to\infty} \int_{x_1}^{x_2} f_k(x)\,dx = \int_{x_1}^{x_2} \lim_{k\to\infty} f_k(x)\,dx$ only if $f_k(x)$ converges uniformly over $[x_1, x_2]$.

30. $A^3 = I$, has infinite number of $A$ solutions. Think of $A^3$ as 3 rotations, each of $120^0$, going back to where we started. Each rotation around a straight line. Hence infinite number of solutions.

31. How to integrate $I = \int \frac{\sqrt{x^3-1}}{x}\,dx$.

Let $u = x^3 + 1$, then $du = 3x^2 dx$ and the above becomes

$$I = \int \frac{\sqrt{u}}{3x^3}\,du = \frac{1}{3}\int \frac{\sqrt{u}}{u-1}\,du$$

Now let $u = \tan^2 v$ or $\sqrt{u} = \tan v$, hence $\frac{1}{2}\frac{1}{\sqrt{u}}du = \sec^2 v\,dv$ and the above becomes

$$\begin{aligned} I &= \frac{1}{3}\int \frac{\sqrt{u}}{\tan^2 v - 1}\left(2\sqrt{u}\sec^2 v\right)\,dv \\ &= \frac{2}{3}\int \frac{u}{\tan^2 v - 1}\sec^2 v\,dv \\ &= \frac{2}{3}\int \frac{\tan^2 v}{\tan^2 v - 1}\sec^2 v\,dv \end{aligned}$$

But $\tan^2 v - 1 = \sec^2 v$ hence

$$\begin{aligned} I &= \frac{2}{3}\int \tan^2 v\,dv \\ &= \frac{2}{3}(\tan v - v) \end{aligned}$$

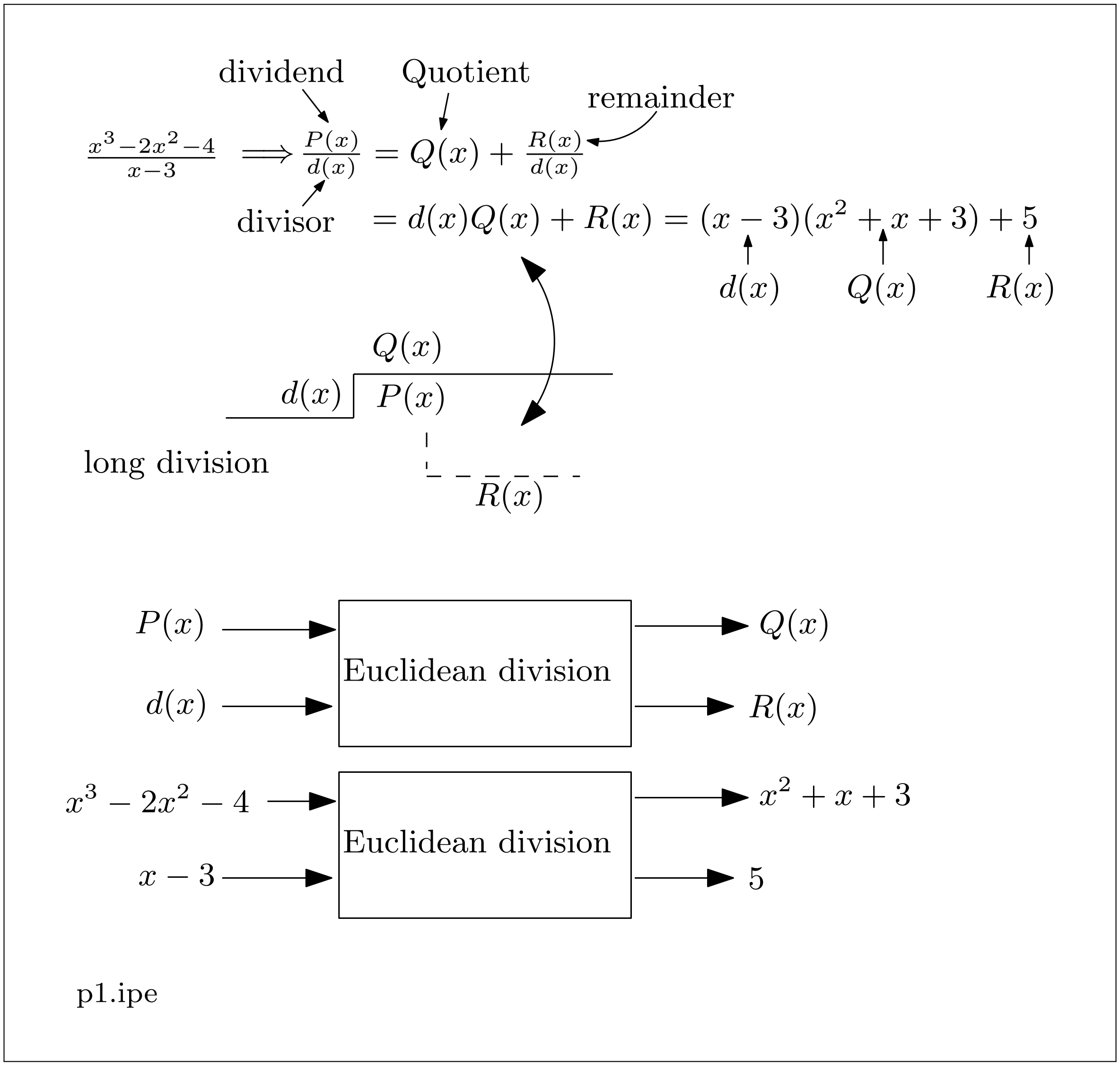
Substituting back

$$I = \frac{2}{3}\left(\sqrt{u} - \arctan\left(\sqrt{u}\right)\right)$$

Substituting back

$$I = \frac{2}{3}\left(\sqrt{x^3+1} - \arctan\left(\sqrt{x^3+1}\right)\right)$$

32. (added Nov. 4, 2015) Made small diagram to help me remember long division terms used.




33. If a linear ODE is equidimensional, as in $a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n01)} + \dots$ for example $x^2 y'' - 2y = 0$ then use ansatz $y = x^r$ this will give equation in $r$ only. Solve for $r$ and obtain $y_1 = x^{r_1}, y_2 = x^{r_2}$ and the solution will be

$$y = c_1 y_1 + c_2 y_2$$

For example, for the above ode, the solution is $c_1 x^2 + \frac{c_2}{x}$. This ansatz works only if ODE is equidimensional. So can't use it on $xy'' + y = 0$ for example.

If $r$ is multiple root, use $x^r, x^r \log(x), x^r (\log(x))^2 \dots$ as solutions.

34. for $x^i$, where $i = \sqrt{-1}$, write it as $x = e^{\log x}$ hence $x^i = e^{i \log x} = \cos(\log x) + i \sin(\log x)$

35. For Sturm Liouville, write it as $py'' + p'y' + (q + \lambda\sigma)\, y = 0$. Now it is easy to compare to. For example, $x'' + \lambda x = 0$, then we see that $p = 1, q = 0, \sigma = 1$. And for $xy'' + x' + \lambda xy = 0$ then $p = 1, q = 0, \sigma = x$. Do not divide by $x$ first to re-write $xy'' + x' + \lambda xy = 0$ as $y'' + \frac{1}{x} x' + \lambda y = 0$ and then compare, as this makes it hard to compare to standard form and will make error.

Books write Sturm Liouville as $\frac{d}{dx}(py') + qy + \lambda\sigma y = 0$ but I find this confusing to compare to this way. Better to expand it to $py'' + p'y' + (q + \lambda\sigma) y = 0$ first so now it is in a more familiar form and can read it out more directly.

36. Some integral tricks: $\int \sqrt{a^2 - x^2} dx$ use $x = a \sin\theta$. For $\int \sqrt{a^2 + x^2} dx$ use $x = a \tan\theta$ and for $\int \sqrt{x^2 - a^2} dx$ use $x = a \sec\theta$.

37. $y'' + x^n y = 0$ is called Emden-Fowler form.

38. For second order ODE, boundary value problem, with eigenvalue (Sturm-Liouville), remember that having two boundary conditions is not enough to fully solve it. One boundary condition is used to find first constant of integration, second boundary condition is used to find the eigenvalues. We still need another input to find the second constant of integration. This is normally done by giving the initial value. This problem happens as part of initial value, boundary value problem. The point is, with boundary value and eigenvalue also present, we need 3 inputs to fully solve it. Two boundary conditions is not enough.

39. If given ODE $y''(x) + p(x) y'(x) + q(x) y(x) = 0$ and we are asked to classify singular at $x = \infty$, then let $x = \frac{1}{t}$ and check what happens at $t = 0$. The $\frac{d^2}{dx^2}$ operator becomes $\left(2t^3 \frac{d}{dt} + t^4 \frac{d^2}{dt^2}\right)$ and $\frac{d}{dx}$ operator becomes $-t^2 \frac{d}{dt}$. And write the ode now where $t$ is the independent variable, and follow standard operating procedures. i.e. look at $\lim_{t\to 0} xp(t)$ and $\lim_{t\to 0} x^2 q(t)$ and see if these are finite or not. To see how the operator are mapped, always start with $x = \frac{1}{t}$ then write $\frac{d}{dx} = \frac{d}{dt}\frac{dt}{dx}$ and write $\frac{d^2}{dx^2} = \left(\frac{d}{dx}\right)\left(\frac{d}{dx}\right)$. For example, $\frac{d}{dx} = -t^2 \frac{d}{dt}$ and

$$\begin{aligned}
\frac{d^2}{dx^2} &= \left(-t^2 \frac{d}{dt}\right)\left(-t^2 \frac{d}{dt}\right) \\
&= -t^2 \left(-2t \frac{d}{dt} - t^2 \frac{d^2}{dt^2}\right) \\
&= \left(2t^3 \frac{d}{dt} + t^4 \frac{d^2}{dt^2}\right)
\end{aligned}$$

Then the new ODE becomes

$$\begin{aligned}
\left(2t^3 \frac{d}{dt} + t^4 \frac{d^2}{dt^2}\right) y(t) + p(t)\left(-t^2 \frac{d}{dt} y(t)\right) + q(t) y(t) &= 0 \\
t^4 \frac{d^2}{dt^2} y + \left(-t^2 p(t) + 2t^3\right) \frac{d}{dt} y + q(t) y &= 0 \\
\frac{d^2}{dt^2} y + \frac{(-p(t) + 2t)}{t^2} \frac{d}{dt} y + \frac{q(t)}{t^4} y &= 0
\end{aligned}$$

The above is how the ODE will always become after the transformation. Remember to change $p(x)$ to $p(t)$ using $x = \frac{1}{t}$ and same for $q(x)$. Now the new $p$ is $\frac{(-p(t)+2t)}{t^2}$ and the new $q$ is $\frac{q(t)}{t^4}$. Then do $\lim_{t\to 0} t \frac{(p(t)+2t^3)}{t^4}$ and $\lim_{t\to 0} t^2 \frac{q(t)}{t^4}$ as before.

40. If the ODE $a(x) y'' + b(x) y' + c(x) y = 0$, and say $0 \leq x \leq 1$, is either non-linear or linear, and there is essential singularity at either end, then use boundary layer or WKB. But Boundary layer method works on non-linear ODE's (and also on linear ODE) and only if the boundary layer is at end of the domain, ie. at $x = 0$ or $x = 1$. WKB method on the other hand, works only on linear ODE, but the singularity can be any where (i.e. inside the domain). As rule of

thumb, if the ODE is linear, use WKB. If the ODE is non-linear, we must use boundary layer. Another difference, is that with boundary layer, we need to do matching phase at the interface between the boundary layer and the outer layer in order to find the constants of integrations. This can be tricky and is the hardest part of solving using boundary layer. Using WKB, no matching phase is needed. We apply the boundary conditions to the whole solution obtained. See my HWs for NE 548 for problems solved from Bender and Orszag text book.

## 2 things to find out

1. find out why I get different answers here. branch cut handling?

   Mathematica 9.01

```
Clear[x, y, k1, k2, k3];
sol = Solve[Cos[x - y] == k1 + k2*Cos[y] + k3*Cos[x], y]
sol /. {x -> 1, k1 -> 2, k2 -> 3, k3 -> 4}
N[%]
```

   gives

```
{{y -> -2.81197 - 1.04751 I},
{y -> 2.81197 + 1.04751 I},
{y -> -2.81197 + 1.04751 I},
{y -> 2.81197 - 1.04751 I}}
```

   Maple 18

```
restart;
eq:= cos(x-y)=k1+k2*cos(y)+k3*cos(x):
sol:=solve(eq,y):
evalf(subs({x=1,k1=2,k2=3,k3=4},{sol}));
```

   gives

```
{2.811969896 - 1.047512037 I,
2.811969896 + 1.047512037 I}
```

## 3 Solving exact first order ODE

(Added Sept. 30, 2016). When solving an exact ODE $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ setup the following two ODE's are setup

$$\frac{\partial \Psi}{\partial x} = M(x,y) \tag{1}$$

$$\frac{\partial \Psi}{\partial y} = N(x,y) \tag{2}$$

Next, the first ODE is integrated w.r.t. $x$, leading to

$$\Psi = g(x,y) + f(y) \tag{3}$$

Where $f(y)$ replaces the "integration constant". Now the above is differentiated w.r.t. $y$ and the resulting equation is compared to (2) to solve for $f'(y)$. Next $f(y)$ is found by integrating. Then now that $f(y)$ is found, then $\Psi$ is found from (3). And since $\Psi$ is some constant, then an implicit solution for $y(x)$ is thus obtained.

# 4 Direct solving of some simple PDE's

Some simple PDE's can be solved by direct integration, here are few examples.

Example 1

$$\frac{\partial z(x,y)}{\partial x} = 0$$

Integrate w.r.t. $x$., and remember that now constant of integration will be function of $y$, hence

$$z(x,y) = f(y)$$

Example 2

$$\frac{\partial^2 z(x,y)}{\partial x^2} = x$$

Integrating once w.r.t. $x$ gives

$$\frac{\partial z(x,y)}{\partial x} = \frac{x^2}{2} + f(y)$$

Integrating again gives

$$z(x,y) = \frac{x^3}{6} + xf(y) + g(y)$$

Example 3

$$\frac{\partial^2 z(x,y)}{\partial y^2} = y$$

Integrating once w.r.t. $y$ gives

$$\frac{\partial z(x,y)}{\partial y} = \frac{y^2}{2} + f(x)$$

Integrating again gives

$$z(x,y) = \frac{y^3}{6} + yf(x) + g(x)$$

Example 4

$$\frac{\partial^2 z(x,y)}{\partial x \partial y} = 0$$

Integrating once w.r.t $x$ gives

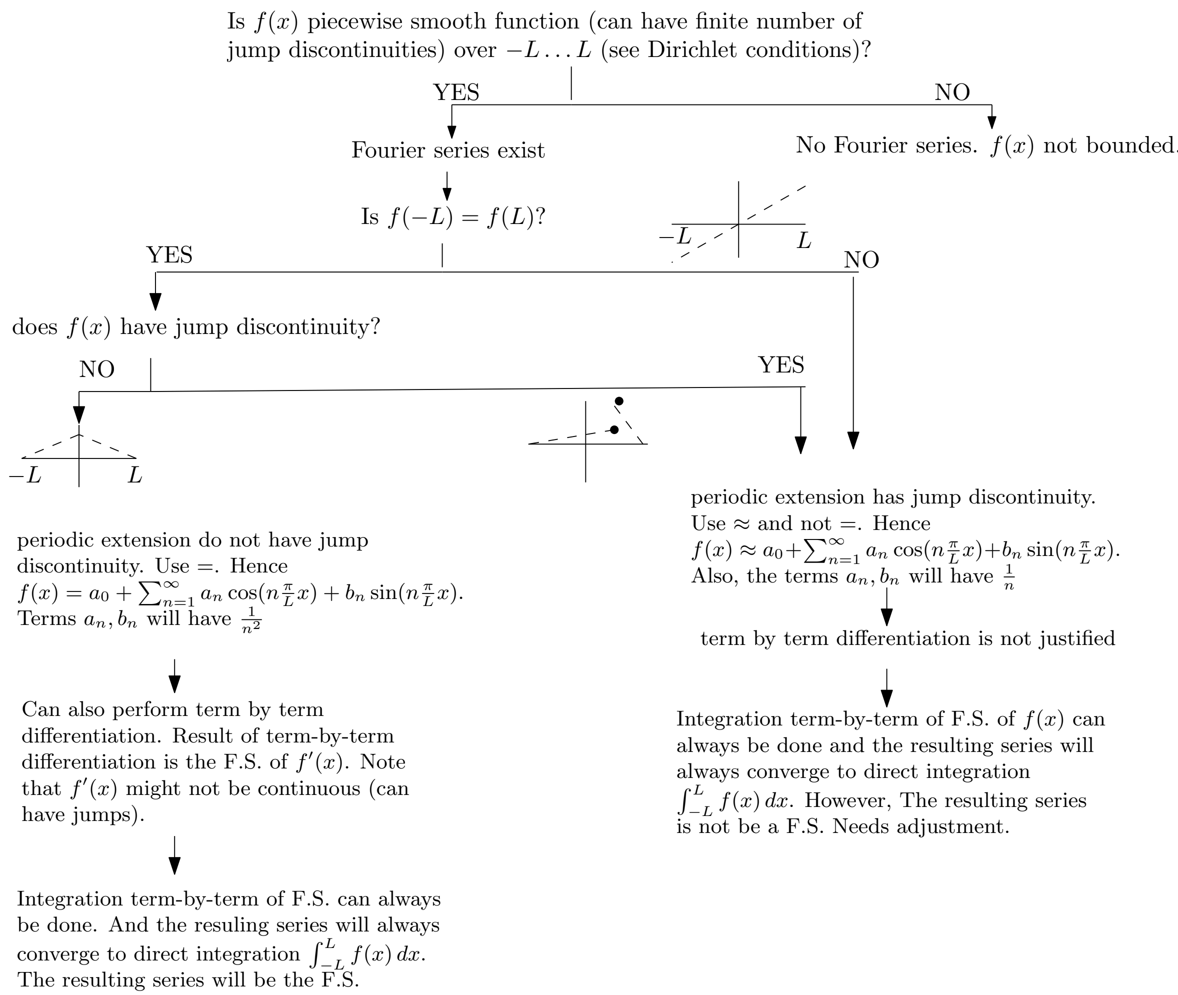
$$\frac{\partial z(x,y)}{\partial y} = f(y)$$

Integrating again w.r.t. $y$ gives

$$z(x,y) = \int f(y)\,dy + g(x)$$

# 5 Fourier series flow chart

(added Oct. 20, 2016).




For cos and sin Fourier series, we need the following conditions to be able to use $f(x) = \dots$ instead of $f(x) \approx \dots$
For *cos* series, the conditions are less strict than for *sin* series. For *cos* all what is needed is that $f(x)$ defined over $0 \cdots L$ not have jump discontinuity. Since the even extension will automatically make $f(-L) = f(L)$. For *sin* series, not only we need $f(x)$ not have jump discontinuity over $0 \cdots L$, but also we need $f(-L) = f(L) = 0$ and we also need $f(0) = 0$. If any of the above conditions fail, then we must use $f(x) \approx \dots$

Nasser M. Abbasi (fs.ipe). Oct 2016

# 6 Linear combination of two solution is solution to ODE

If $y_1, y_2$ are two solutions to $ay'' + by' + cy = 0$ then to show that $c_1 y_1 + c_2 y_2$ is also solution:

$$\begin{aligned} ay_1'' + by_1' + cy_1 &= 0 \\ ay_2'' + by_2' + cy_2 &= 0 \end{aligned}$$

Multiply the first ODE by $c_1$ and second ODE by $c_2$

$$\begin{aligned} a\left(c_1 y_1\right)'' + b\left(c_1 y_1\right)' + c\left(c_1 y_1\right) &= 0 \\ a\left(c_2 y_2\right)'' + b\left(c_2 y_2\right)' + c\left(c_2 y_2\right) &= 0 \end{aligned}$$

Add the above two equations, using linearity of differentials

$$a\left(c_1 y_1 + c_2 y_2\right)'' + b\left(c_1 y_1 + c_2 y_2\right)' + c\left(c_1 y_1 + c_2 y_2\right) = 0$$

Therefore $c_1 y_1 + c_2 y_2$ satisfies the original ODE. Hence solution.

# 7 To find the Wronskian ODE

Since

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

Where $y_1, y_2$ are two solutions to $ay'' + by' + cy = 0$. Write

$$\begin{aligned} ay_1'' + py_1' + cy_1 &= 0 \\ ay_2'' + py_2' + cy_2 &= 0 \end{aligned}$$

Multiply the first ODE above by $y_2$ and the second by $y_1$

$$\begin{aligned} ay_2 y_1'' + py_2 y_1' + cy_2 y_1 &= 0 \\ ay_1 y_2'' + py_1 y_2' + cy_1 y_2 &= 0 \end{aligned}$$

Subtract the second from the first

$$a\left(y_2 y_1'' - y_1 y_2''\right) + p\left(y_2 y_1' - y_1 y_2'\right) = 0 \tag{1}$$

But

$$p\left(y_2 y_1' - y_1 y_2'\right) = -pW \tag{2}$$

And

$$\begin{aligned} \frac{dW}{dx} &= \frac{d}{dx}\left(y_1 y_2' - y_2 y_1'\right) \\ &= y_1' y_2' + y_1 y_2'' - y_2' y_1' - y_2 y_1'' \\ &= y_1 y_2'' - y_2 y_1'' \end{aligned} \tag{3}$$

Substituting (2,3) into (1) gives the Wronskian differential equation

$$\begin{aligned} -a\left(\frac{dW}{dx}\right) - pW &= 0 \\ aW' + pW &= 0 \end{aligned}$$

Whose solution is

$$W(x) = Ce^{-\int \frac{p}{a} dx}$$

Where $C$ is constant of integration.

Remember: $W(x_0) = 0$ does not mean the two functions are linearly dependent. The functions can still be Linearly independent on other interval, It just means $x_0$ can't be in the domain of the solution for two functions to be solutions. However, if the two functions are linearly dependent, then this implies $W = 0$ everywhere. So to check if two functions are L.D., need to show that $W = 0$ everywhere.

# 8 Sturm-Liouville Notes

## 8.1 Definitions

Regular Sturm-Liouville ODE is an eigenvalue boundary value ODE. This means, it the ODE has an eigenvalue in it $\lambda$, where solution exists only for specific values of eigenvalues. The ODE is

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)\,y(x) + \lambda\sigma(x)\,y(x) = 0 \qquad a < x < b$$

Or

$$py'' + p'y' + (q + \lambda\sigma)\,y = 0$$

With the restrictions that $p(x), q(x), \sigma(x)$ are real functions that are continuous everywhere over $a \leq x \leq b$ and also we need $p(x) > 0, \sigma(x) > 0$. The $\sigma(x)$ is called the weight function. But this is not all. The boundary conditions must be linear homogeneous, of this form

$$\beta_1 y(a) + \beta_2 y'(a) = 0$$
$$\beta_3 y(b) + \beta_4 y'(b) = 0$$

Where $\beta_i$ are just real constants. Some of them can be zero but not all. For example, $\beta_1 y(a) = 0, \beta_4 y'(b) = 0$ is OK. So boundary conditions do not have to mixed, but they can be in general. But they must be homogeneous. Notice that periodic boundary conditions are not allowed. Well, they are allowed, but then the problem is no longer called Sturm-Liouville. The above is just the definition of the equation and its boundary conditions. Below is list of the important properties of this ODE. Each one of these properties have a proof.

1. All eigenvalues $\lambda_n$ are real. No complex $\lambda_n$.

2. Each eigenvalue $\lambda_n$, will have one and only one real eigenfunction $\phi_n(x)$ associated with it. (in higher dimensions, the eigenvalue problem can have more than one eigenfunction associated with one eigenvalue. For example, heat PDE in rectangle). But we can use Gram-schmidt to make these eigenfunctions linearly independent if we need to). But for exam, just worry about 1D for now.

3. There is smallest $\lambda$, called $\lambda_1$ and there are infinite number of eigenvalues. Hence eigenvalues are ordered $\lambda_1 < \lambda_2 < \lambda_3 \cdots$

4. Each eigenfunction $\phi_n(x)$ will have $n-1$ zeros in $a < x < b$. Note that this does not include the end points. This means, $\phi_3(x)$ will have two zeros inside the domain. i.e. there are two $x$ locations where $\phi_3(x) = 0$ in $a < x < b$.

5. Eigenfunctions $\phi_n(x)$ make a complete set of basis. This means any piecewise continuous function $f(x)$ can be represented as $f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x)$. This is called the generalized Fourier series.

6. This follows from (5). Each eigenfunction is orthogonal to another eigenfunction, but with the weight in there. This means $\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx = 0$ if $n \neq m$.

7. Rayleigh quotient relates an eigenvalue to its eigenfunction. Starting with the SL ODE $(p\phi')' + q\phi = -\lambda\sigma\phi$, then multiplying by $\phi$ and integrating, we obtain $\int_a^b \phi(p\phi')' + q\phi^2 dx = -\int_a^b \lambda\sigma\phi^2 dx$ and solving for $\lambda$, gives

$$\lambda = \frac{\int_a^b \phi(p\phi')' + q\phi^2 dx}{\int_a^b \phi^2\sigma(x)\,dx}$$

Carrying integration by parts on first part of the integral in numerator, it becomes

$$\lambda = \frac{(-p\phi\phi')_b^a + \int_a^b p(\phi')^2 - q\phi^2 dx}{\int_a^b \phi^2\sigma(x)\,dx}$$

But this becomes much simpler when we plug-in the boundary conditions that we must use, making the above

$$\lambda = \frac{\int_a^b p(\phi')^2 - q\phi^2 dx}{\int_a^b \phi^2\sigma(x)\,dx}$$

And if $q = 0, p = 1$ and $\sigma(x) = 1$, then it becomes $\lambda = \frac{\int_a^b (\phi')^2 dx}{\int_a^b \phi^2 dx}$. Rayleigh quotient is useful to show that $\lambda$ can be positive without solving for $\phi$ and also used to estimate the value of the minimum $\lambda$ by replacing $\phi$ by a trial function and actually solving for $\lambda$ using $\lambda = \frac{\int_a^b \phi(p\phi')' + q\phi^2 dx}{\int_a^b \phi^2\sigma(x)dx}$ to obtain a numerical estimate of $\lambda_1$.

8. There is symmetry relation. In operator form, let $L \equiv \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)$, then we have $\int_a^b uL[v] - vL[u]\,dx = 0$ where $u, v$ are any two different eigenfunctions.

9. There is also what is called Lagrange identity, which says $uL[v] - vL[u] = \frac{d}{dx}(p(uv' - vu'))$. This comes from simply expanding $L[v], L[u]$ and simplifying things. Just calculus.

10. Green formula, follows from Lagrange identity, which just gives the integral form $\int_a^b uL[v] - vL[u]\,dx = [p(uv' - vu')]_a^b$. But form Sturm-Liouville, we know that $\int_a^b uL[v] - vL[u]\,dx = 0$ (from 8). So this really just says that $[p(uv' - vu')]_a^b = 0$. But we know this already from boundary conditions. So I am not sure why this is useful for Sturm-Liouville now. Since it is just saying the same thing again.

11. If $p(x) = 0$ at the left or right end (boundaries), then the problem is now called singular Sturm-Liouville. This is actually the important case. In regular S.L., $p(x)$ must be positive everywhere. We only consider $p(x) = 0$ at the ends, not in the middle or any other place. When this happens, then solutions to the ODE generate special functions, such as Bessel, Legendre, Chebyshev. These special functions are solution to the Singular S.L. ODE, not the regular S.L. ODE. In addition, at the end where $p(x) = 0$, the boundary conditions must be bounded. Not the regular boundary conditions above. For example, if $p(a) = 0$ where $x = a$

is say the left end, the the boundary conditions at the left end must be $y(a) < \infty, y'(a) < \infty$. Notice that singular can be at left end, right end or at both at same time. At the end where $p(x) = 0$, the boundary conditions must be bounded type.

12. Symmetry relation
$$\int_a^b uL[v] - vL[u]\,dx = 0$$
where $u, v$ are are two eigenfunction. But remember, this does NOT mean the integrand is identically zero, or $uL[v] - vL[u] = 0$. Only when $u, v$ happened to have same eigenvalue, only then we can say that $uL[v] - vL[u] = 0$. But for unique eigenfunctions only the integral form is zero $\int_a^b uL[v] - vL[u]\,dx = 0$. This is important difference, used in the proof below, that different eigenfunctions have different eigenvalues (for the scalar SL case). For higher dimensions, we can have more than one eigenfunction for same eigenvalue.

Proofs for these properties follows

## 8.2 Proof symmetry of operator

Given regular Sturm Liouville (RSL) ODE

$$\begin{aligned}(py')' + qy &= -\lambda\sigma y \qquad (1)\\ py'' + p'y' + qy &= -\lambda\sigma y\end{aligned}$$

In operator form

$$L \equiv \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)$$

And (1) becomes

$$L[y] = -\lambda\sigma y$$

When solving RSL ode, since it is an eigenvalue ODE with associated boundary conditions, we will get infinite number of non-negative eigenvalues. For each eigenvalue, there is associated with it one eigenfunction (for 1D case). Looking at any two different eigenfunctions, say $u, v$, then the symmetry relation says the following

$$\int_a^b uL[v]\,dx = \int_a^b vL[u]\,dx$$

Now we will show the above is true. This requires integration by parts two times. We start from the LHS expression and at the end we should end up with the integral on the RHS. Let $I = \int_a^b uL[v]\,dx$, then

$$\begin{aligned}I &= \int_a^b uL[v]\,dx \qquad (1)\\ &= \int_a^b u\left(\frac{d}{dx}\left(p\frac{d}{dx}v\right) + qv\right)dx\end{aligned}$$

Where we used $L[v] = \left(\frac{d}{dx}\left(p\frac{d}{dx}\right) + q\right)v = \left(\frac{d}{dx}\left(p\frac{d}{dx}v\right) + qv\right)$ in the above. Hence

$$I = \int_a^b u\overbrace{\frac{d}{dx}\left(p\frac{dv}{dx}\right)}^{dV}dx + \int_a^b qvu\,dx \qquad (2)$$

Now we will do integration by parts on the first integral. Let $I_2 = \int_a^b u \frac{d}{dx}\left(p\frac{dv}{dx}\right)dx$. Using $\int U dV = UV - \int V dU$, and if we let $U = u, dV = \frac{d}{dx}\left(p\frac{dv}{dx}\right)$, then $dU = \frac{du}{dx}, V = p\frac{dv}{dx}$. Hence

$$I_2 = \left(up\frac{dv}{dx}\right)_a^b - \int_a^b \overbrace{\frac{dv}{dx}}^{dV} \overbrace{p\frac{du}{dx}}^{U} dx$$

We now apply integration by parts again to the second integral above. But now let $U = p\frac{du}{dx}$ and $dV = \frac{dv}{dx}$, hence $dU = \frac{d}{dx}\left(p\frac{du}{dx}\right)$ and $V = v$, therefore the above becomes

$$I_2 = \left(up\frac{dv}{dx}\right)_a^b - \left[\left(p\frac{du}{dx}v\right)_a^b - \int_a^b v\frac{d}{dx}\left(p\frac{du}{dx}\right)dx\right]$$

Substituting the above into (2) gives

$$I = \left(up\frac{dv}{dx}\right)_a^b - \left[\left(p\frac{du}{dx}v\right)_a^b - \int_a^b v\frac{d}{dx}\left(p\frac{du}{dx}\right)dx\right] + \int_a^b qvudx \tag{3}$$

Now comes the part where the boundary conditions are important. In RSL, the boundary conditions are such that all the terms $\left(up\frac{dv}{dx}\right)_a^b, \left(p\frac{du}{dx}v\right)_a^b$ vanish. This is because the boundary conditions are

$$\begin{aligned}\beta_1 u(a) + \beta_2 u'(a) &= 0\\ \beta_3 u(b) + \beta_4 u'(b) &= 0\end{aligned}$$

And

$$\begin{aligned}\beta_1 v(a) + \beta_2 v'(a) &= 0\\ \beta_3 v(b) + \beta_4 v'(b) &= 0\end{aligned}$$

So now (3) becomes

$$\begin{aligned}I &= -\left[-\int_a^b v\frac{d}{dx}\left(p\frac{du}{dx}\right)dx\right] + \int_a^b qvudx\\ &= \int_a^b v\frac{d}{dx}\left(p\frac{du}{dx}\right)dx + \int_a^b qvudx\\ &= \int_a^b v\left(\frac{d}{dx}\left(p\frac{du}{dx}\right) + qu\right)dx\\ &= \int_a^b vL[u]\,dx\end{aligned}$$

Therefore, we showed that $\int_a^b uL[v]\,dx = \int_a^b vL[u]\,dx$. The only thing to watch for here, is which term to make $U$ and which to make $dV$ when making integration by parts. To remember, always start with $uL[v]$ and make $dV$ the term with $\frac{dv}{dx}$ in them during integration parts.

## 8.3 Proof eigenvalues are real

Assume $\lambda$ is complex. The corresponding eigenfunctions are complex also. Let $\phi$ be corresponding eigenfunction

$$L[\phi] = -\lambda\sigma\phi \tag{1}$$

Taking complex conjugate of both sides, and since $\sigma$ is real, we obtain

$$\overline{L[\phi]} = -\overline{\lambda}\sigma\overline{\phi}$$

But $\overline{L[\phi]} = L\left[\overline{\phi}\right]$ since all the coefficients of the ODE are real. The above becomes

$$L\left[\overline{\phi}\right] = -\overline{\lambda}\sigma\overline{\phi} \tag{2}$$

But by symmetry, we know that

$$\int_a^b \phi L\left[\overline{\phi}\right] - \overline{\phi}L[\phi]\,dx = 0$$

Substituting (1),(2) into the above gives

$$\begin{aligned}
\int_a^b \phi\left(-\overline{\lambda}\sigma\overline{\phi}\right) - \overline{\phi}\left(-\lambda\sigma\phi\right)dx &= 0\\
\int_a^b -\overline{\lambda}\sigma\phi\overline{\phi} + \lambda\sigma\overline{\phi}\phi dx &= 0\\
\int_a^b \left(\lambda-\overline{\lambda}\right)\left(\sigma\overline{\phi}\phi\right)dx &= 0\\
\left(\lambda-\overline{\lambda}\right)\int_a^b \sigma\overline{\phi}\phi dx &= 0
\end{aligned}$$

But $\overline{\phi}\phi = |\phi|$ which is positive. Also the weight $\sigma(x)$ is positive by definition. Hence for the above to zero, it must be that $\lambda = \overline{\lambda}$. Which means $\lambda$ is real. QED.

So the main tools to use in this proof: Definition of $L[\phi] = -\lambda\sigma\phi$ , and symmetry relation and that $\overline{L[\phi]} = L\left[\overline{\phi}\right]$. This might come up in the exam.

## 8.4 Proof eigenfunctions are unique

Now will show that there is one eigenfunction associated with each eigenvalue (Again, this is for 1D, it is possible to get more than one eigenfunction for same eigenvalue for 2D, as mentioned earlier). By contradiction, assume that $\lambda$ has two eigenfunctions $\phi_1,\phi_2$ associated with it. Hence

$$\begin{aligned}
L[\phi_1] &= -\lambda\sigma\phi_1\\
L[\phi_2] &= -\lambda\sigma\phi_2
\end{aligned}$$

From the first equation, $\lambda = \frac{-L[\phi_1]}{\sigma\phi_1}$, substituting this into the second equation gives

$$\begin{aligned}
L[\phi_2] &= \frac{L[\phi_1]}{\phi_1}\phi_2\\
\phi_1 L[\phi_2] - \phi_2 L[\phi_1] &= 0
\end{aligned}$$

By Lagrange identity, $\phi_1 L[\phi_2] - \phi_2 L[\phi_1] = \frac{d}{dx}\left(p\left(\phi_1\phi_2' - \phi_2\phi_1'\right)\right)$, hence this means that

$$\begin{aligned}
\frac{d}{dx}\left(p\left(\phi_1\phi_2' - \phi_2\phi_1'\right)\right) &= 0\\
p\left(\phi_1\phi_2' - \phi_2\phi_1'\right) &= c_1
\end{aligned}$$

Where $c_1$ is some constant. This is the main difference between the above argument, and between Largrange identity. This can be confusing. So let me talk more about this. In Lagrange identity, we write

$$\phi_1 L[\phi_2] - \phi_2 L[\phi_1] = \frac{d}{dx}\left(p\left(\phi_1\phi_2' - \phi_2\phi_1'\right)\right)$$

And when $\phi_2, \phi_1$ also satisfy the SL boundary condition, only then we say that

$$\int_a^b \phi_1 L[\phi_2] - \phi_2 L[\phi_1]\, dx = 0$$
$$p\left(\phi_1\phi_2' - \phi_2\phi_1'\right)\Big|_a^b = 0$$

But the above is not the same as saying $\phi_1 L[\phi_2] - \phi_2 L[\phi_1] = 0$. This is important to keep in mind. Only the integral form is zero for any two functions with the SL B.C..Now we continue. We showed that $p\left(\phi_1\phi_2' - \phi_2\phi_1'\right) = c$. In SL, this constant is zero due to B.C. Hence

$$p\left(\phi_1\phi_2' - \phi_2\phi_1'\right) = 0$$

But $p > 0$ by definition. Hence $\phi_1\phi_2' - \phi_2\phi_1' = 0$ or

$$\frac{d}{dx}\left(\frac{\phi_2}{\phi_1}\right) = 0$$
$$\frac{\phi_2}{\phi_1} = c_2$$

or $\phi_2 = c_2\phi_1$. So the eigenfunctions are linearly dependent. One is just scaled version of the other. But eigenfunction must be linearly independent. Hence assumption is not valid, and there can not be two linearly independent eigenfunctions for same eigenvalue. Notice also that $\phi_1\phi_2' - \phi_2\phi_1' = 0$ is the just the Wronskian. When it is zero, we know the functions are linearly dependent. The important part in the above proof, is that $\phi_1\phi_2' - \phi_2\phi_1' = 0$ only when $\phi_1, \phi_2$ happened to have same eigenvalue.

## 8.5 Proof eigenfunctions are real

The idea of this proof is to assume the eigenfunction is complex, then show that its real part and its complex part both satisfy the ODE and the boundary conditions. But since they are both use the same eigenvalue, then the real part and the complex part must be linearly dependent. This implies the eigenfunction must be real. (think of Argand diagram)

Assume that $\phi = U + iV$ is complex eigenfunction with real part $U$ and complex part $V$. Then since

$$L[\phi] = -\lambda\sigma\phi$$

The above is just writing the Sturm-Liouville ODE in operator form, where $L$ is the operator as above. Now we have

$$L[U + iV] = -\lambda\sigma(U + iV)$$

By linearity of operator $L$

$$L[U] + iL[V] = -\lambda\sigma U - i\lambda\sigma V$$

Which implies

$$L[U] = -\lambda\sigma U$$
$$L[V] = -\lambda\sigma V$$

So we showed that the real and complex part satisfy S.L. Now we need to show they are satisfy S.L. boundary conditions. Since

$$\begin{aligned}\beta_1\phi(a)+\beta_2\phi'(a)&=0\\ \beta_3\phi(b)+\beta_4\phi'(b)&=0\end{aligned}$$

Where $x=a, x=b$ are the left and right ends of the domain. Then

$$\begin{aligned}\beta_1(U(a)+iV(a))+\beta_2\left(U'(a)+iV'(a)\right)&=0\\ \beta_3(U(b)+iV(b))+\beta_4\left(U'(b)+iV'(b)\right)&=0\end{aligned}$$

Hence

$$\begin{aligned}\beta_1U(a)+\beta_2U'(a)&=0\\ \beta_1V(a)+\beta_2V'(a)&=0\\ \beta_3U(b)+\beta_4U'(b)&=0\\ \beta_3V(b)+\beta_4V'(b)&=0\end{aligned}$$

So the above means both $U$ and $V$ satisfy the boundary conditions of S.L. But since both $U, V$ have the same eigenvalue, then they must be linearly dependent, since we know with S.L. each eigenfunction (or one linearly dependent to it) have only one eigenvalue. This means

$$V=cU$$

Where $c$ is some constant. In other words,

$$\begin{aligned}\phi&=U+iV\\ &=U+icU\\ &=U(1+ic)\\ &=c_0U\end{aligned}$$

Where $c_0$ is new constant. (OK it happens to be complex constant, but it is OK to do so, we always do this trick in other places, if it will make me feel better, I could take the magnitude of the constant). So all what the above says, is that we assumed $\phi$ to be complex, and found that it is real. So it can't be complex.

## 8.6 Proof eigenfunctions are orthogonal with weight

Given two different eigenfunctions $\phi_1, \phi_2$. Hence

$$L[\phi_1]=-\lambda_1\sigma\phi_1 \tag{1}$$
$$L[\phi_2]=-\lambda_2\sigma\phi_2 \tag{2}$$

From symmetry integral relation, since these eigenfunctions also satisfy S.L. boundary conditions, we can write

$$\int_a^b \phi_1L[\phi_2]-\phi_2L[\phi_1]\,dx=0$$

Replacing (1,2) into the above

$$\int_a^b \phi_1 \left(-\lambda_2 \sigma \phi_2\right) - \phi_2 \left(-\lambda_1 \sigma \phi_1\right) dx = 0$$
$$\int_a^b -\lambda_2 \sigma \phi_1 \phi_2 + \lambda_1 \sigma \phi_2 \phi_1 dx = 0$$
$$\int_a^b \left(\lambda_1 - \lambda_2\right) \left(\sigma \phi_1 \phi_2\right) dx = 0$$
$$\left(\lambda_1 - \lambda_2\right) \int_a^b \sigma \phi_1 \phi_2 dx = 0$$

But $\left(\lambda_1 - \lambda_2\right) \neq 0$ since there are different eigenvalues for different eigenfunctions. Hence

$$\int_a^b \sigma \phi_1 (x) \phi_2 (x) dx = 0$$

Which means $\phi_{1,}\phi_2$ are orthogonal to each others with weight $\sigma(x)$

## 8.7 Special functions generated from solving singular Sturm-Liouville

When S.L. is singular, meaning $p = 0$ at one or both ends, we end with important class of ODE's, whose solutions are special functions (not sin or cos) as the case with the regular S.L. Recall that S.L. is

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x) y(x) + \lambda \sigma(x) y(x) = 0 \tag{1}$$
$$py'' + p'y' + qy = -\lambda \sigma y$$

With the regular S.L., we say that $p(x) > 0$ over the whole domain, including end points. But with singular, this is not the case. Here are three important S.L. ODE's that are singular.

Bessel equation:

$$xy'' + x'y' = -\lambda \sigma y \qquad 0 < x < 1$$

Or in standard form

$$\left(xy'\right)' = -\lambda \sigma y$$

So we see that $p(x) = x, q(x) = 0, \sigma(x) = x$. We see that at $x = 0$, then $p(x) = 0$, which what makes it singular. (also the weight happens to be zero also), but we only care about $p$ being zero or not, at one of the ends. So to check if S.L. is singular or not, just need to check if $p(x)$ is zero or not at one of the ends. As mentioned before, when $p = 0$ at one of the ends, we can't use the standard B.C. for the regular S.L. instead, at the end where $p = 0$, we must use what is called bounded boundary conditions, which is in this case $y(0) < \infty, y'(0) < \infty$. The solution to this ODE will be in terms of Bessel functions. Notice that this happened to be singular, due to the domain starting at $x = 0$. If the domain happened to be from $x = 0.5$ to say $x = 1$, then it is no longer singular S.L. but regular one.

Legendre equation

$$\left(\left(1 - x^2\right) y'\right)' = -\lambda y \qquad 0 < x < 1$$

Or it can be written as

$$\left(1 - x^2\right) y'' - 2xy' = -\lambda y$$

We see that $p = 1 - x^2$. And now it happens to be that $p = 0$ at $x = 1$. So it is singular at the other end compared to Bessel. In this case $q = 0, \sigma = 1$. Again, the boundary conditions at $x = 1$ must now be bounded. i.e. $y(1) < \infty, y'(1) < \infty$. On the other end, where $p$ is not zero, we still use the standard boundary conditions $\beta_1 y(0) + \beta_2 y'(0) = 0$.

Chebyshev equation

$$\left(\sqrt{1-x^2}y'\right)' = -\lambda\sqrt{1-x^2}y \qquad -1 < x < 1$$

Or

$$\sqrt{1-x^2}y'' - \frac{1}{2}\frac{2x}{\sqrt{1-x^2}}y' = -\lambda\sqrt{1-x^2}y$$

So we see that $p = \sqrt{1-x^2}, q = 0, \sigma = \sqrt{1-x^2}$. So where is $p = 0$ here? At $x = -1$ then $p = \sqrt{1-1} = 0$ and at $x = -1$ then $p = 0$ also. So this is singular at both ends. So we need to use bounded boundary conditions at both ends now to solve this.

$$\begin{aligned} y(-1) &< \infty \\ y'(-1) &< \infty \\ y(+1) &< \infty \\ y'(+1) &< \infty \end{aligned}$$

The solution to this singular S.L. is given in terms of special function Chebyshev.

### 8.8 Some notes

1. Solution to $xy'' + y' = -\lambda xy$. Set in S-L form, we get $(py')' + qy = -\lambda\sigma y$, then $p = x, q = 0, \sigma = x$. For $a < x < b$ with $y(b) = 0, y(a) < \infty$. This is singular S-L. Using Frobenius series, the solution comes out to be $y(x) = \sum_{n=1}^{\infty} c_n J_0\left(\sqrt{\lambda_n}x\right)$, where $J_0$ is Bessel function of first kind. $\sqrt{\lambda_n}$ are roots of $0 = J_0\left(\sqrt{\lambda}b\right)$. (*check* if this should be $0 = J_0\left(\sqrt{\lambda}(b-a)\right)$. $c_n$ is constant, which still needs to be found.

2. We can find approximate solution to S-L ODE for large eigenvalue, without solving the ODE using series method, but by using WKB. Given $(py')' + qy = -\lambda\sigma y$ which is standard form S-L. Using physical optics correction, we obtain the solution as

$$y(x) \sim (\sigma(x)p(x))^{\frac{-1}{4}}\left(c_1\cos\left(\sqrt{\lambda}\int_0^x\sqrt{\frac{\sigma(t)}{p(t)}}dt\right) + c_1\sin\left(\sqrt{\lambda}\int_0^x\sqrt{\frac{\sigma(t)}{p(t)}}dt\right)\right)$$

   Where $c_1, c_2$ are found from boundary conditions now. The above is valid for "large" $\lambda$, and is found by first letting $\varepsilon^2 = \frac{1}{\lambda}$ and then assuming $y(x) = \exp\left(\frac{1}{\varepsilon}\sum_{n=0}^{\infty}\varepsilon^n S_n(x)\right)$. And working though the WKB method. Remember, WKB only works for linear homogeneous ODE and is used to estimate solution for large $\lambda$ (or small $\varepsilon$).

3. in 1D, S-L is $(py')' + qy = -\lambda\sigma y$ with $0 < x < L$, with B.C. given by $\beta_1 y(0) + \beta_2 y'(0) = 0$ and $\beta_1 y(L) + \beta_2 y'(L) = 0$, and in higher dimensions, this problem becomes $\nabla\cdot(p\nabla\phi) + q\phi = -\lambda\sigma\phi$ with boundary conditions now written as $\beta_1\phi + \beta_2(\nabla\phi\cdot\hat{n}) = 0$ on boundary $\Gamma$. In higher dimensions, $\phi \equiv \phi(\bar{x})$.

4. Lagrange's identity in 1D is

$$uL(v) - vL(u) = \frac{d}{dx}\left(p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right)$$

where $L$ above is the S-L operator, as in $L \equiv \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$ so that we write $L(u) = -\lambda\sigma y$.

5. When it says eigenfunctions are normalized, it should mean that $\int_a^b \phi_n^2(x)\,\sigma(x)\ dx = 1$ where $\sigma(x)$ is the weight function (comes from SL equation) and $a < x < b$ is the domain. This is for 1D SL. Remember for example, that $\int_0^L \cos^2\left(\sqrt{\lambda_n}x\right) dx = \frac{L}{2}$. where $\sqrt{\lambda_n} = \frac{n\pi}{L}$. Here, it is not normalized since the result is not one. The weight here is just one and here $\phi_n(x) = \cos\left(\frac{n\pi}{L}x\right)$

6. Heat PDE in cylinder is $\frac{\partial T(r,z,\theta,t)}{\partial t} = k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}\right)$. For steady state, it becomes, say cylinder has length $L$

$$\begin{aligned} 0 = \nabla^2 T(r,z,\theta) &= \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} \\ |T(0,z,\theta)| &< \infty \\ |T'(0,z,\theta)| &< \infty \\ T(r,0) &= f(r) \\ R(r,L) &= g(r) \end{aligned}$$

And for axis symmetric (no $\theta$ dependency), it becomes

$$0 = \nabla^2 T(r,z) = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}$$

The solution is found by separation of variables. For the above is in terms of Bessel function order zero

$$T(r,z) = \sum_{n=1}^{\infty} J_0\left(\sqrt{\lambda_0}r\right)\left(A_n e^{\sqrt{\lambda_n}z} + B_n e^{\sqrt{\lambda_n}z}\right)$$

which is found using series method $T = \sum_{n=0}^{\infty} a_n r^{n+\alpha}$. The eigenvalues $\lambda_n$ are zeros of $0 = J_0\left(\sqrt{\lambda}r_0\right)$ where $r_0$ is disk radius.

7. Heat PDE in 2D polar is $\frac{\partial T(r,\theta,t)}{\partial t} = k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}\right)$ and for steady state it becomes $0 = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}$. After separating, the $r$ ODE becomes Euler ODE. Use guess $R(r) = r^p$ for solution. The solution will be

$$T(r,z) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta)\, r^n + \sum_{n=1}^{\infty} B_n \sin(n\theta)\, r^n$$

8. To change $p(x)\,y'' + q(x)\,y' + r(x)\,y(x) = 0$ to S.L. form, multiply the ODE by $\mu = \frac{1}{p(x)}\exp\left(\int \frac{q(x)}{p(x)} dx\right)$

9. In using method of eigenfunction expansion to solve nonhomogeneous B.C. PDE, the eigenfunction used $\phi_n(x)$ are the ones from solving the PDE with homogeneous B.C. For example, if given $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ with nonhomogeneous B.C. , say $u(0,t) = A, u(L,t) = B$, and we want

to use $u(x,t) \sim \sum_{n=1}^{\infty} b_n(t)\,\phi_n(x)$ on the nonhomogeneous B.C. PDE, then those $\phi_n(x)$ are from the corresponding homogeneous B.C. PDE. They should be something like $\sin\left(\frac{n\pi}{L}x\right)$ and so on. See the table at the top. That is why we write $\sim$ above and not $=$ and remember not to do term-by-term differentiation in $x$ when this is the case. We can do term-by-term differentiation only of the PDE itself also has homogeneous B.C. , even if it had a source term there also. The point is, $\phi_n(x)$ always come from solving the nonhomogeneous B.C. PDE. (This normally means solving a Sturm-Liouville ODE).

10. I found new relation for eigenvalue $\lambda$, but it is probably not useful as the one the book has. Here how to derive it
Given S-L

$$\frac{d}{dx}\left(p\frac{dy}{dx}\right) + qy = -\lambda\sigma y \tag{1}$$

Let $\phi_n(x)$ be the eigenfunction associated with eigenvalue $\lambda_n$. Since eigenfunctions satisfy the ODE itself, then we can write, for any arbitrary eigenfunction (subscript removed for clarity in what follows)

$$\frac{d}{dx}\left(p\phi'\right) + q\phi = -\lambda\sigma\phi$$
$$p'\phi' + p\phi'' + q\phi = -\lambda\sigma\phi$$

Integrating both sides

$$\int_a^b p'\phi' dx + \int_a^b p\phi'' dx + \int_a^b q\phi dx = -\lambda\int_a^b \sigma\phi dx \tag{2}$$

Looking at $\int_a^b p\phi'' dx$. Integrating by part. Let $u = p, dv = \phi'' \rightarrow du = p', v = \phi'$, hence

$$\begin{aligned}\int_a^b p\phi'' dx &= [uv]_a^b - \int_a^b v du \\ &= \left[p\phi'\right]_a^b - \int_a^b p'\phi' dx \end{aligned} \tag{3}$$

Substituting (3) into (2) gives

$$\begin{aligned}\int_a^b p'\phi' dx + \left[p\phi'\right]_0^L - \int_a^b p'\phi' dx + \int_a^b q\phi dx &= -\lambda\int_a^b \sigma\phi dx \\ \left[p\phi'\right]_0^L + \int_a^b q\phi dx &= -\lambda\int_a^b \sigma\phi dx\end{aligned}$$

Hence

$$\lambda = -\frac{[p\phi']_a^b + \int_a^b q\phi dx}{\int_a^b \sigma\phi dx}$$

compare to the one in the book $\lambda = \frac{(-p\phi\phi')_b^a + \int_a^b p(\phi')^2 - q\phi^2 dx}{\int_a^b \phi^2\sigma(x) dx}$.

## 8.9 Methods of solutions to some problems

Problem 1 If the problem gives S-L equations, and asks to find estimate on the smallest eigenvalue, then use Rayleigh quotient for $\lambda$. And write $\lambda_{\min} = \lambda_1 \leq \frac{(-p\phi\phi')_b^a + \int_a^b p(\phi')^2 - q\phi^2 dx}{\int_a^b \phi^2 \sigma(x) dx}$. Now we do not need to solve the SL to find solution $\phi$, this is the whole point. Everything in RHS is given, except for, of course the solution $\phi$. Here comes the main idea: Come up with any trial $\phi_{trial}$ and use it in place of $\phi$. This trial function just needs to satisfy the boundary conditions, which is also given. Then all what we need to do is just evaluate the integral. Pick the simplest $\phi_{trial}$ function of $x$ which satisfies boundary conditions. All other terms $p, q, \sigma$ we can read from the given problem. At the end, we should get a numerical value for the integral. This is the upper limit of the lowest eigenvalue $\lambda_1$

Problem 2 We are given SL problem with boundary conditions, and asked to show that $\lambda \geq 0$ without solving the ODE to find $\phi$: Use Rayleigh quotient and argue that the denominator can't be zero (else eigenfunction is zero, which is not possible) and it also can't be negative. Argue about the numerator not being negative. Do not solve the ODE! this is the whole point of using Rayleigh quotient.

Problem We are given an SL problem with boundary conditions and asked to estimate large $\lambda$ and corresponding eigenfunction. This is different from being asked to estimate the smallest eigenvalue, where we use Rayleigh quotient and trial function. In this one, we instead use WKB. For 1D, just use what is called the physical optics correction method, given by $\phi(x) \approx (\sigma p)^{\frac{-1}{4}} \left( c_1 \cos\left(\sqrt{\lambda} \int_0^x \sqrt{\frac{\sigma(t)}{p(t)}} dt\right) + c_2 \sin\left(\sqrt{\lambda} \int_0^x \sqrt{\frac{\sigma(t)}{p(t)}} dt\right)\right)$. Where $\sigma, p$ in this are all known functions (from the problem itself). Notice that $q$ is not there. Now use the boundary conditions and solve for one of the constants, should come to be zero. Use the second boundary condition and (remember, the boundary conditions are homogenous), and we get one equation in $\lambda$ and for non-trivial solution, solve for allowed values of $\lambda_n$. This gives the large eigenvalue estimate. i.e. for $\lambda_n$ when $n$ is very large. Depending on problem, $n$ does not have be too large to get good accurate estimate compared with exact solution. See HW7, problem 5.9.2 for example.

Problem We are given 1D heat ODE $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$. If B.C. are homogenous, we are done. We know the solution. Separation of variables. If the B.C. is not homogenous, then we have a small problem. We can't do separation of variables. If you can find equilibrium solution, $u_r(x)$, where this solution only needs to satisfy the non-homogenous B.C. then write solution as $u(x,t) = v(x,t) + u_r(x)$, and plug this in the PDE. This will give $\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}$ but this now satisfies the homogenous B.C. This is the case if the original non-homogenous B.C were Dirichlet. If they were Neumann, then we will get $\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + extra$ where $extra$ is extra term that do not vanish because $u_r(x)$ do not vanish in this case after double differentiating. We solve for $v(x,t)$ now, since it has homogenous B.C., but it has extra term there, which we treat as new source. We apply initial conditions now also to final all the eigenfunction expansion coefficients. We are done. Now we know $u(x,t)$.

But if we can find the reference function $u_r$ or we do not want to use this method, then we can use another method, called eigenfunction expansion. We assume $u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x)$ and plug this in the PDE. But now we can't do term by term differentiation, since $\phi_n$ are the eigenfunctions that satisfies the homogenous B.C., while $u(x,t)$ has non-homogenous B.C. So the trick is to use Green formula, to re-writes $\int_0^L k\frac{\partial^2 u}{\partial x^2}\phi_n(x)\,dx$ as $\int_0^L ku\frac{\partial^2 \phi_n(x)}{\partial x^2}dx$ plus the contribution from the boundary terms. (this is like doing integration by parts twice, but much easier). Now rewrite $\frac{\partial^2 \phi_n(x)}{\partial x^2} = -\lambda_n \phi_n(x)$. See page 355-356, Haberman.

### 8.10 Examples converting second order linear ODE to Sturm-Liouville

Any linear second order ODE can be converted to S-L form. The S-L form is the following

$$\left(p(x)\frac{dy}{dx}\right)' + (q(x) + \lambda r(x))\, y(x) = 0 \tag{1}$$

Sometimes there is a minus sign $-\left(p(x)\frac{dy}{dx}\right)'$ there. I really never understood why some book put a minus sign and some do not. May be I'll find out one day. But for now, (1) is used as the S-L form. The goal now, is given any general form eigenvalue ODE, we want to convert it (rewrite it) in the above form. The second order linear ODE will have this form

$$a(x)\, y'' + b(x)\, y' + (c(x) + \lambda)\, y = 0 \tag{2}$$

The parameter $\lambda$ in the S-L form, is the eigenvalue. We really only use S-L form for eigenvalue problems. First, we will show how to convert (2) to (1), and then show few examples. We are given (2), and want to convert it to (1). The first step is to convert (2) to standard form

$$y'' + \frac{b(x)}{a(x)}y' + \frac{(c(x) + \lambda)}{a(x)}y = 0$$

Then multiply (2) by some unknown $\mu(x)$ which is assumed positive.

$$\mu(x)\, y'' + \mu(x)\frac{b(x)}{a(x)}y' + \mu(x)\frac{(c(x) + \lambda)}{a(x)}y = 0$$

Now, rewrite $\mu(x)\, y''$ as $(\mu(x)\, y')' - \mu'(x)\, y'$. These are the same thing. Now we replace this in the above and obtain

$$\begin{aligned}\left((\mu(x)\, y')' - \mu'(x)\, y'\right) + \mu(x)\frac{b(x)}{a(x)}y' + \mu(x)\frac{(c(x) + \lambda)}{a(x)}y &= 0\\ (\mu(x)\, y')' + y'\left(\mu(x)\frac{b(x)}{a(x)} - \mu'(x)\right) + \mu(x)\frac{(c(x) + \lambda)}{a(x)}y &= 0 \end{aligned}\tag{3}$$

Here comes the main trick in all of this. We want to force $\mu(x)\frac{b(x)}{a(x)} - \mu'(x)$ to be zero. This implies $\mu(x) = e^{\int \frac{b(x)}{a(x)}dx}$. (this comes from just solving the ODE $\mu(x)\frac{b(x)}{a(x)} - \mu'(x) = 0$. Therefore, if $\mu(x) = e^{\int \frac{b(x)}{a(x)}dx}$, then (3) becomes

$$\begin{aligned}(\mu(x)\, y')' + \left(\mu(x)\frac{(c(x) + \lambda)}{a(x)}\right)y &= 0\\ (\mu(x)\, y')' + \left(\mu(x)\frac{c(x)}{a(x)} + \frac{\mu(x)}{a(x)}\lambda\right)y &= 0\end{aligned}$$

We are done. Hence

$$p(x) = \mu(x) = e^{\int \frac{b(x)}{a(x)}dx}$$

And

$$q(x) = \mu(x) \frac{c(x)}{a(x)}$$
$$r(x) = \frac{\mu(x)}{a(x)}$$

Let see how this works on some examples

### 8.10.1 Example 1

Convert $y'' + \lambda y = 0$ to S-L. We see the general form here is $a(x) y'' + b(x) y' + (c(x) + \lambda) y = 0$, with $a = 1, b = 0, c = 0$. Hence $\mu(x) = e^{\int \frac{b(x)}{a(x)} dx} = 1$, therefore the SL form is

$$\left(p(x) \frac{dy}{dx}\right)' + (q(x) + \lambda r(x)) y(x) = 0 \tag{1}$$

Where

$$p(x) = 1$$
$$q(x) = \mu(x) \frac{c(x)}{a(x)} = 0$$
$$r(x) = \frac{\mu(x)}{a(x)} = 1$$

Hence (1) becomes

$$(y')' + \lambda y(x) = 0$$

This was easy, since the ODE given was already in SL form.

### 8.10.2 Example 2

Convert

$$x^2 y'' + xy' + (\lambda^2 x^2 - n^2) y = 0$$

Rewrite as

$$y'' + \frac{1}{x} y' + \left(\lambda - \frac{n^2}{x^2}\right) y = 0$$

We see the general form here is $a(x) y'' + b(x) y' + (c(x) + \lambda) y = 0$, with $a = 1, b = \frac{1}{x}, c = -\frac{n^2}{x^2}$. Hence

$$\mu(x) = e^{\int \frac{b(x)}{a(x)} dx} = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

Therefore the SL form is

$$\left(p(x) \frac{dy}{dx}\right)' + (q(x) + \lambda r(x)) y(x) = 0 \tag{1}$$

Where

$$p(x) = \mu(x) = x$$
$$q(x) = \mu(x)\frac{c(x)}{a(x)} = x\left(\frac{-n^2}{x^2}\right) = -\frac{n^2}{x}$$
$$r(x) = \frac{\mu(x)}{a(x)} = x$$

Hence (1) becomes

$$\left(x\frac{dy}{dx}\right)' + \left(-\frac{n^2}{x} + \lambda x\right) y(x) = 0$$

### 8.10.3 Example 3

Convert

$$y'' - \frac{2x}{1-x^2}y' + \frac{\lambda}{1-x^2}y = 0$$

Rewrite as

$$\left(1-x^2\right)y'' - 2xy' + \lambda y = 0$$

We see the general form here is $a(x)y'' + b(x)y' + (c(x)+\lambda)y = 0$, with $a = \left(1-x^2\right), b = -2x, c = 0$. Hence

$$\mu(x) = e^{\int \frac{b}{a}dx} = e^{-\int \frac{2x}{1-x^2}dx} = e^{-(-\ln(x-1)-\ln(x+1))} = e^{\ln(x-1)}e^{\ln(x+1)} = (x-1)(x+1) = x^2-1$$

Therefore the SL form is

$$\left(p(x)\frac{dy}{dx}\right)' + (q(x) + \lambda r(x))\, y(x) = 0 \qquad (1)$$

Where

$$p(x) = \mu(x) = x^2-1$$
$$q(x) = \mu(x)\frac{c(x)}{a(x)} = 0$$
$$r(x) = \frac{\mu(x)}{a(x)} = \frac{x^2-1}{x^2-1} = 1$$

Hence (1) becomes

$$\left(\left(x^2-1\right)\frac{dy}{dx}\right)' + \lambda y(x) = 0$$

### 8.10.4 Example 4

Convert

$$\left(1-x^2\right)y''-xy'+\lambda y=0$$

We see the general form here is $a\left(x\right)y''+b\left(x\right)y'+\left(c\left(x\right)+\lambda\right)y=0$, with $a=\left(1-x^2\right),b=-x,c=0$. Hence

$$\mu\left(x\right)=e^{\int\frac{b}{a}dx}=e^{-\int\frac{x}{1-x^2}dx}=e^{-\left(-\frac{1}{2}\ln(x-1)-\frac{1}{2}\ln(x+1)\right)}=e^{\frac{1}{2}\ln(x-1)}e^{\frac{1}{2}\ln(x+1)}=\sqrt{(x-1)}\sqrt{(x+1)}=\sqrt{x^2-1}$$

Therefore the SL form is

$$\left(p\left(x\right)\frac{dy}{dx}\right)'+\left(q\left(x\right)+\lambda r\left(x\right)\right)y\left(x\right)=0 \tag{1}$$

Where

$$\begin{aligned}p\left(x\right)&=\mu\left(x\right)=\sqrt{x^2-1}\\ q\left(x\right)&=\mu\left(x\right)\frac{c\left(x\right)}{a\left(x\right)}=0\\ r\left(x\right)&=\frac{\mu\left(x\right)}{a\left(x\right)}=\frac{\sqrt{x^2-1}}{\left(1-x^2\right)}=\frac{\sqrt{x^2-1}}{-\left(x^2-1\right)}=-\frac{1}{\sqrt{x^2-1}}\end{aligned}$$

Hence (1) becomes

$$\left(\sqrt{x^2-1}\frac{dy}{dx}\right)'-\frac{\lambda}{\sqrt{x^2-1}}y\left(x\right)=0$$

I seem to have a sign mistake there. I just do not see it now. This should come out to be $p\left(x\right)=\sqrt{1-x^2},r\left(x\right)=\frac{1}{\sqrt{1-x^2}}$

# 9 Green functions notes

1. Green function is what is called impulse response in control. But it is more general, and can be used for solving PDE also. Given a differential equation with some forcing function on the right side. To solve this, we replace the forcing function with an impulse. The solution of the DE now is called the impulse response, which is the Green's function of the differential equation. Now to find the solution to the original problem with the original forcing function, we just convolve the Green function with the original forcing function. Here is an example. Suppose we want to solve $L\left[y\left(t\right)\right]=f\left(t\right)$ with zero initial conditions. Then we solve $L\left[g\left(t\right)\right]=\delta\left(t\right)$. The solution is $g\left(t\right)$. Now $y\left(t\right)=g\left(t\right)\circledast f\left(t\right)$. This is for initial value problem. For example. $y'\left(t\right)+kx=e^{at}$, with $y\left(0\right)=0$. Then we solve $g'\left(t\right)+kg=\delta\left(t\right)$. The solution is $g\left(t\right)=\begin{cases}e^{-kt} & t>0\\ 0 & t<0\end{cases}$, this is for causal system. Hence $y\left(t\right)=g\left(t\right)\circledast f\left(t\right)$. The nice thing here, is that once we find $g\left(t\right)$, we can solve $y'\left(t\right)+kx=f\left(t\right)$ for any $f\left(t\right)$ by just convolving the Green function (impulse response) with the new $f\left(t\right)$.

2. We can think of Green function as an inverse operator. Given $L\left[y\left(t\right)\right]=f\left(t\right)$, we want to find solution $y\left(t\right)=\int_{-\infty}^{\infty}G\left(t;\tau\right)f\left(\tau\right)d\tau$. So in a sense, $G\left(t;\tau\right)$ is like $L^{-1}\left[y\left(t\right)\right]$.

3. Need to add notes for Green function for Sturm-Liouville boundary value ODE. Need to be clear on what boundary conditions to use. What is B.C. is not homogeneous?

4. Green function properties:

   (a) $G(t;\tau)$ is continuous at $t=\tau$. This is where the impulse is located.

   (b) The derivative $G'(t)$ just before $t=\tau$ is not the same as $G'(t)$ just after $t=\tau$. i.e. $G'(t;t-\varepsilon)-G'(t;t+\varepsilon)\neq 0$. This means there is discontinuity in derivative.

   (c) $G(t;\tau)$ should satisfy same boundary conditions as original PDE or ODE (this is for Sturm-Liouville or boundary value problems).

   (d) $L[G(t;\tau)]=0$ for $t\neq\tau$

   (e) $G(x;\tau)$ is symmetric. i.e. $G(x;\tau)=G(\tau;x)$.

5. When solving for $G(t;\tau)$, in context of 1D, hence two boundary conditions, one at each end, and second order ODE (Sturm-Liouville), we now get two solutions, one for $t<\tau$ and one for $t>\tau$. So we have 4 constants of integrations to find (this is for second order ODE) not just two constants as normally one would get , since now we have 2 different solutions. Two of these constants from the two boundary conditions, and two more come from property of Green function as mentioned above. $G(t;\tau)=\begin{cases} A_1y_1+A_2y_2 & t<0 \\ A_3y_1+A_4y_2 & t>0 \end{cases}$

## 10 Laplace transform notes

1. Remember that $u_c(t)f(t-c)\iff e^{-cs}F(s)$ and $u_c(t)f(t)\iff e^{-cs}\mathcal{L}\{f(t+c)\}$. For example, if we are given $u_2(t)t$, then $\mathcal{L}(u_2(t)t)=e^{-2s}\mathcal{L}\{t+2\}=e^{-2s}\left(\frac{1}{s^2}+\frac{2}{s}\right)=e^{-2s}\left(\frac{1+2s}{s^2}\right)$. Do not do $u_c(t)f(t)\iff e^{-cs}\mathcal{L}\{f(t)\}$ ! That will be a big error. We use this allot when asked to write a piecewise function using Heaviside functions.

## 11 References

Too many references used, but will try to remember to start recording books used from now on. Here is current list

1. Applied partial differential equation, by Haberman

2. Advanced Mathematical Methods for Scientists and Engineers, Bender and Orszag, Springer.

3. Boundary value problems in physics and engineering, Frank Chorlton, Van Norstrand, 1969

4. Class notes. Math 322. University Wisconsin, Madison. Fall 2016. By Professor Smith. Math dept.

5. various pages Wikipedia.

6. good note on Sturm-Liouville `http://ramanujan.math.trinity.edu/rdaileda/teach/s12/m3357/lectures/lecture_4_10_short.pdf`