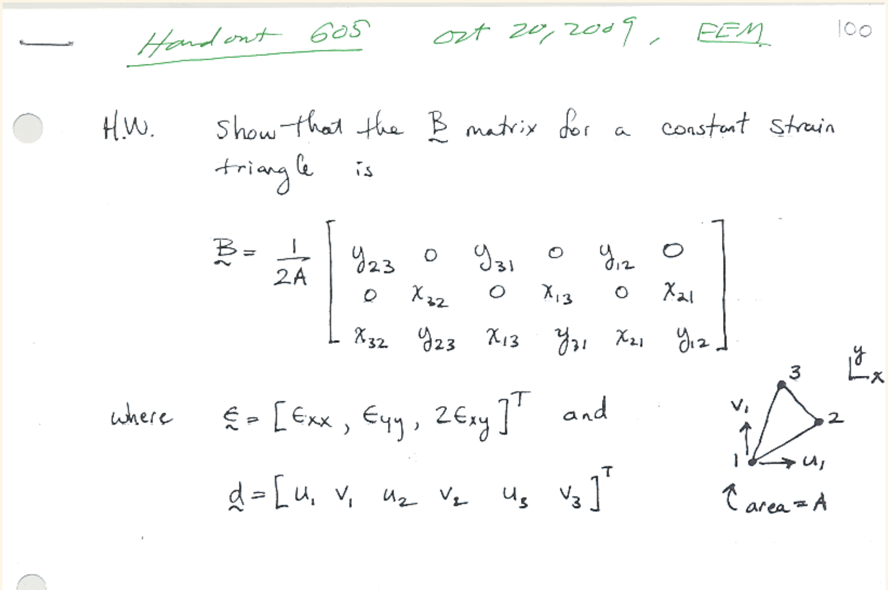
The problem to solve

by Nasser M. Abbasi (oct 2009)



Handout 60S oct 20, 2009, FEM 100

H.W. Show that the $\underset{\sim}{B}$ matrix for a constant strain triangle is

$$\underset{\sim}{B} = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

where $\underset{\sim}{\epsilon} = [\epsilon_{xx}, \epsilon_{yy}, 2\epsilon_{xy}]^T$ and

$\underset{\sim}{d} = [u_1 \; v_1 \; u_2 \; v_2 \; u_3 \; v_3]^T$

In this solution, I start directly by solving for the vector field $\{u, v\}$ and starting from the general degrees of freedom, and from it by matrix inversion, find the shape function matrix N (in terms of nodal degrees of freedom). This involves inversting a 6 by 6 matrix. But Ok, I am using a computer. By hand, I would use the method I showed in the analytical note part of this assignment which involves inverting only a 3 by 3 matrix.

In[1]:=
```
Needs["Notation`"]
```

In[2]:=
```
nNodes = 3;
nDegreeOfFreedom = 6;
Symbolize[ u1 ]; Symbolize[ u2 ]; Symbolize[ u3 ]; Symbolize[ v1 ]; Symbolize[ v2 ]
Symbolize[ v3 ]; Symbolize[ a1 ]; Symbolize[ a2 ]; Symbolize[ a3 ]; Symbolize[ a4 ]
Symbolize[ a5 ]; Symbolize[ a6 ]; Symbolize[ x1 ]; Symbolize[ x2 ]; Symbolize[ x3 ]
Symbolize[ y1 ]; Symbolize[ y2 ]; Symbolize[ y3 ]
```

Start by defining the *u* and *v* trial functions (linear polynomials in x and y)

In[8]:=
```
nodalDegreesOfFreedom = {u1, v1, u2, v2, u3, v3};
generalDegreesOfFreedom = {a1, a2, a3, a4, a5, a6};
u = a1 + a2 x + a3 y;
v = a4 + a5 x + a6 y;
```

set up the $u = X\, a$ equation

In[12]:=
```
{b, xMat} = Normal[CoefficientArrays[{u, v}, generalDegreesOfFreedom]];
Print[ToString[MatrixForm[{{"u"}, {"v"}}], FormatType → TraditionalForm] <> "=",
 ToString[MatrixForm[xMat], FormatType → StandardForm] <>
  ToString[MatrixForm[generalDegreesOfFreedom], FormatType → StandardForm]]
```

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix}$$

Now find the shape functions. Start by expression nodal unknowns in terms of nodal coordinates

In[14]:=
```
u1 = u /. {x → x1, y → y1}
v1 = v /. {x → x1, y → y1}
u2 = u /. {x → x2, y → y2}
v2 = v /. {x → x2, y → y2}
u3 = u /. {x → x3, y → y3}
v3 = v /. {x → x3, y → y3}
```

Out[14]= $a_1 + a_2 x_1 + a_3 y_1$

Out[15]= $a_4 + a_5 x_1 + a_6 y_1$

Out[16]= $a_1 + a_2 x_2 + a_3 y_2$

Out[17]= $a_4 + a_5 x_2 + a_6 y_2$

Out[18]= $a_1 + a_2 x_3 + a_3 y_3$

Out[19]= $a_4 + a_5 x_3 + a_6 y_3$

Write the $u = A\,a$ equation

In[20]:=
```
{b, aMat} = Normal[CoefficientArrays[{u1, v1, u2, v2, u3, v3}, generalDegreesOfFreedom]];

Print[ToString[MatrixForm[Transpose[{{"u1", "v1", "u2", "v2", "u3", "v3"}}]],
   FormatType → TraditionalForm] <> "=",
 ToString[MatrixForm[aMat], FormatType → StandardForm] <>
  ToString[MatrixForm[generalDegreesOfFreedom], FormatType → StandardForm]]
```

$$\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix}$$

Find $a = A^{-1}\,u$ from the above by matrix inversion

```
shapeFunctions = xMat.Inverse[aMat];
A = {{D[#1, x] &, 0 &}, {0 &, D[#1, y] &}, {D[#1, y] &, D[#1, x] &}};
bMat = Inner[#1[#2] &, A, shapeFunctions, Plus];

(bMat = Simplify[Assuming[Element[{y1, y2, y3, x1, x2, x3}, Reals], bMat]]) // MatrixForm
```

Out[29]//MatrixForm=

$$\begin{pmatrix} \frac{-y_2+y_3}{x_3(-y_1+y_2)+x_2(y_1-y_3)+x_1(-y_2+y_3)} & 0 & \frac{y_1-y_3}{x_2 y_1-x_3 y_1-x_1 y_2+x_3 y_2+x_1 y_3-x_2 y_3} & 0 \\ 0 & \frac{x_2-x_3}{x_2 y_1-x_3 y_1-x_1 y_2+x_3 y_2+x_1 y_3-x_2 y_3} & 0 & \frac{x_1-x_3}{x_3(y_1-y_2)+x_1(y_2-y_3)+x_2(-y_1+y} \\ \frac{x_2-x_3}{x_2 y_1-x_3 y_1-x_1 y_2+x_3 y_2+x_1 y_3-x_2 y_3} & \frac{-y_2+y_3}{x_3(-y_1+y_2)+x_2(y_1-y_3)+x_1(-y_2+y_3)} & \frac{x_1-x_3}{x_3(y_1-y_2)+x_1(y_2-y_3)+x_2(-y_1+y_3)} & \frac{y_1-y_3}{x_2 y_1-x_3 y_1-x_1 y_2+x_3 y_2+x_1 y_3-x_2} \end{pmatrix}$$

Factor the determinant term from the above to the outside.

```
den = Denominator[bMat[[1, 1]]] ;
bMat = bMat * den;
Print[ToString[1 / den, FormatType → StandardForm] <>
  ToString[MatrixForm[Simplify[bMat]], FormatType → StandardForm]]
```

$$\frac{1}{x_3(-y_1+y_2)+x_2(y_1-y_3)+x_1(-y_2+y_3)}\begin{pmatrix} -y_2+y_3 & 0 & y_1-y_3 & 0 & -y_1+y_2 & 0 \\ 0 & x_2-x_3 & 0 & -x_1+x_3 & 0 & x_1-x_2 \\ x_2-x_3 & -y_2+y_3 & -x_1+x_3 & y_1-y_3 & x_1-x_2 & -y_1+y_2 \end{pmatrix}$$

But area of triangle is

In[33]:=

```
area = (1 / 2) Cross[{x2 - x1, y2 - y1, 0}, {x3 - x1, y3 - y1, 0}][[3]]
```

Out[33]= $\frac{1}{2}(-x_2 y_1 + x_3 y_1 + x_1 y_2 - x_3 y_2 - x_1 y_3 + x_2 y_3)$

Hence B matrix becomes

In[34]:=

```
Panel[Style[ToString[1 / "2 area", FormatType → StandardForm] <>
   ToString[MatrixForm[Simplify[bMat]], FormatType → StandardForm], 18]]
```

Out[34]=

$$\frac{1}{2\ \text{area}}\begin{pmatrix} -y_2+y_3 & 0 & y_1-y_3 & 0 & -y_1+y_2 & 0 \\ 0 & x_2-x_3 & 0 & -x_1+x_3 & 0 & x_1-x_2 \\ x_2-x_3 & -y_2+y_3 & -x_1+x_3 & y_1-y_3 & x_1-x_2 & -y_1+y_2 \end{pmatrix}$$

```
finalB = 1/(2 area) bMat;
```

Out[87]//MatrixForm=

$$\begin{pmatrix} \frac{y_2-y_3}{-x_3 y_2+x_2 y_3+y_2 x_1-y_3 x_1-x_2 y_1+x_3 y_1} & 0 & \frac{y_3-y_1}{-x_3 y_2+x_2 y_3+y_2 x_1-y_3 x_1-x_2 y_1+x_3 y_1} & 0 \\ 0 & \frac{-x_2+x_3}{-x_3 y_2+x_2 y_3+y_2 x_1-y_3 x_1-x_2 y_1+x_3 y_1} & 0 & \frac{-x_3+x_1}{-x_3 y_2+x_2 y_3+y_2 x_1-y_3 x_1-x_2 y_1+x} \\ \frac{-x_2+x_3}{-x_3 y_2+x_2 y_3+y_2 x_1-y_3 x_1-x_2 y_1+x_3 y_1} & \frac{y_2-y_3}{-x_3 y_2+x_2 y_3+y_2 x_1-y_3 x_1-x_2 y_1+x_3 y_1} & \frac{-x_3+x_1}{-x_3 y_2+x_2 y_3+y_2 x_1-y_3 x_1-x_2 y_1+x_3 y_1} & \frac{y_3-y_1}{-x_3 y_2+x_2 y_3+y_2 x_1-y_3 x_1-x_2 y_1+x} \end{pmatrix}$$