ODE classification

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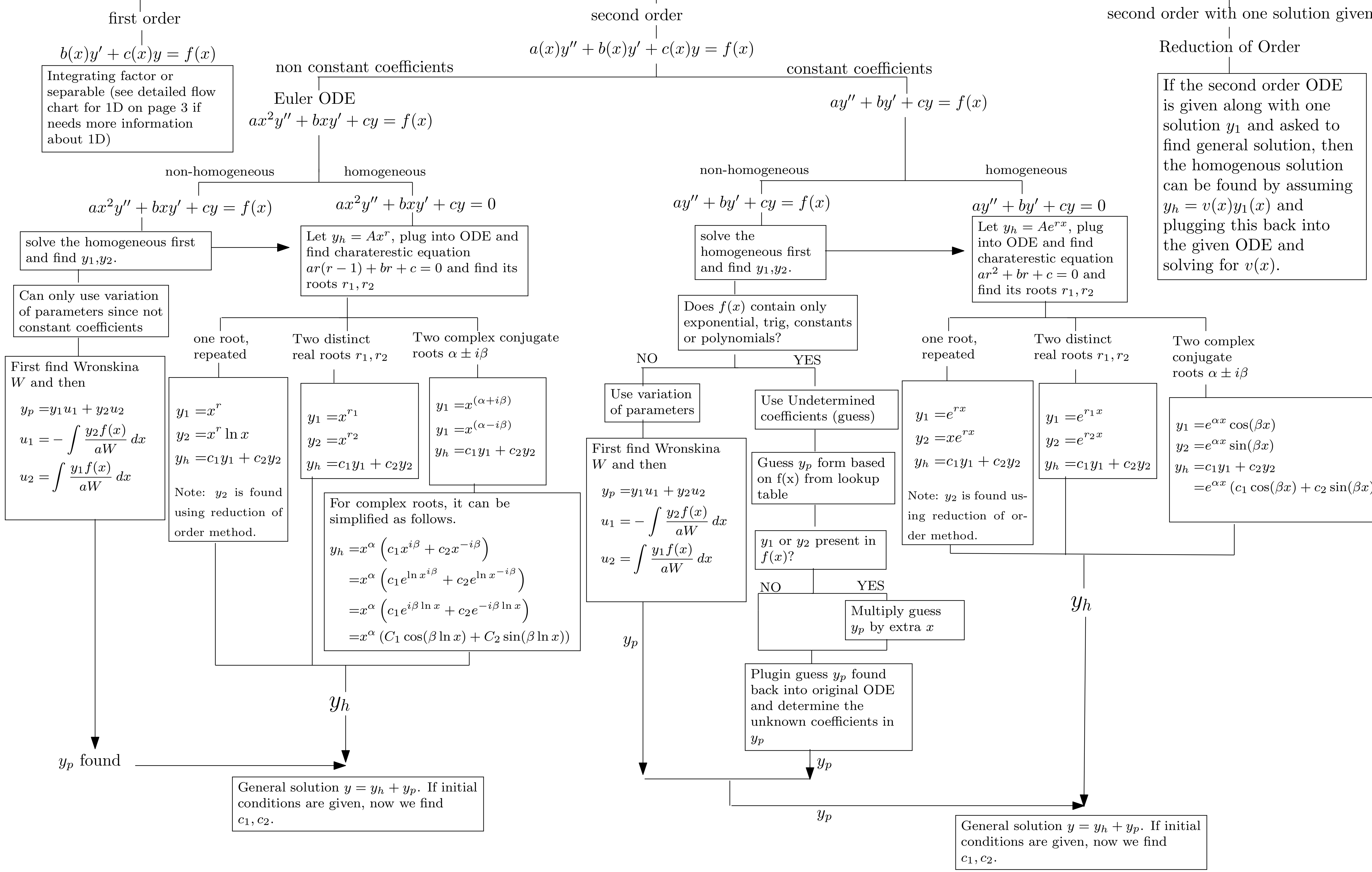
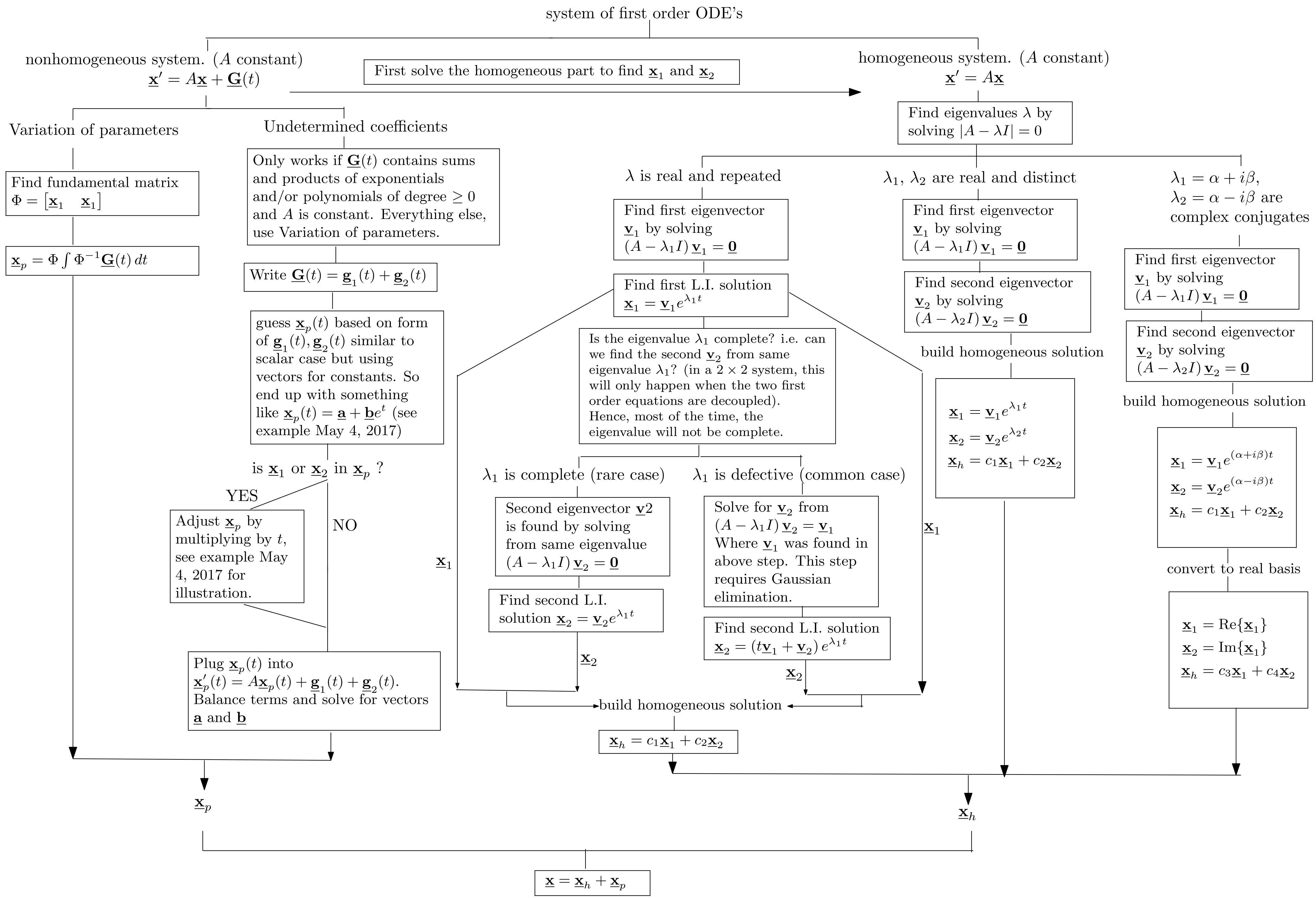
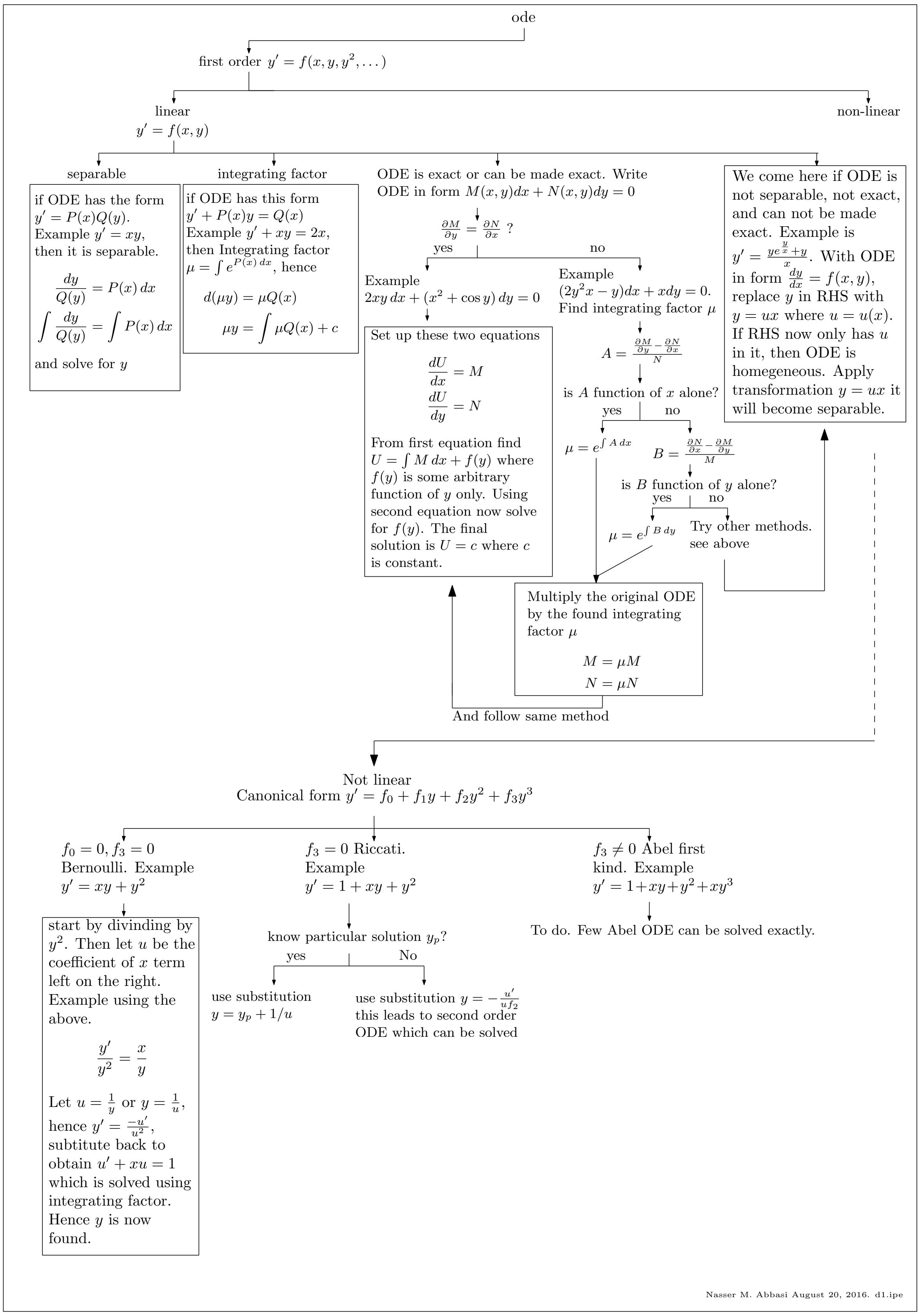

first order
$b(x)y' + c(x)y = f(x)$
Integrating factor or separable (see detailed flow chart for 1D on page 3 if needs more information about 1D)
second order
$a(x)y'' + b(x)y' + c(x)y = f(x)$
non constant coefficients
Euler ODE
$ax^2y'' + bxy' + cy = f(x)$
non-homogeneous
homogeneous
$ax^2y'' + bxy' + cy = f(x)$
solve the homogeneous first and find y_1,y_2.
Can only use variation of parameters since not constant coefficients
First find Wronskina W and then
$y_p = y_1 u_1 + y_2 u_2$
$u_1 = -\int \frac{y_2 f(x)}{aW}\,dx$
$u_2 = \int \frac{y_1 f(x)}{aW}\,dx$
y_p found
$ax^2y'' + bxy' + cy = 0$
Let $y_h = Ax^r$, plug into ODE and find charaterestic equation $ar(r-1) + br + c = 0$ and find its roots r_1, r_2
one root, repeated
$y_1 = x^r$
$y_2 = x^r \ln x$
$y_h = c_1 y_1 + c_2 y_2$
Note: y_2 is found using reduction of order method.
Two distinct real roots r_1, r_2
$y_1 = x^{r_1}$
$y_2 = x^{r_2}$
$y_h = c_1 y_1 + c_2 y_2$
Two complex conjugate roots $\alpha \pm i\beta$
$y_1 = x^{(\alpha + i\beta)}$
$y_1 = x^{(\alpha - i\beta)}$
$y_h = c_1 y_1 + c_2 y_2$
For complex roots, it can be simplified as follows.
$y_h = x^\alpha \left(c_1 x^{i\beta} + c_2 x^{-i\beta}\right)$
$= x^\alpha \left(c_1 e^{\ln x^{i\beta}} + c_2 e^{\ln x^{-i\beta}}\right)$
$= x^\alpha \left(c_1 e^{i\beta \ln x} + c_2 e^{-i\beta \ln x}\right)$
$= x^\alpha \left(C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)\right)$
y_h
General solution $y = y_h + y_p$. If initial conditions are given, now we find c_1, c_2.
constant coefficients
$ay'' + by' + cy = f(x)$
non-homogeneous
homogeneous
$ay'' + by' + cy = f(x)$
solve the homogeneous first and find y_1,y_2.
Does $f(x)$ contain only exponential, trig, constants or polynomials?
NO
YES
Use variation of parameters
First find Wronskina W and then
$y_p = y_1 u_1 + y_2 u_2$
$u_1 = -\int \frac{y_2 f(x)}{aW}\,dx$
$u_2 = \int \frac{y_1 f(x)}{aW}\,dx$
y_p
Use Undetermined coefficients (guess)
Guess y_p form based on f(x) from lookup table
y_1 or y_2 present in $f(x)$?
NO
YES
Multiply guess y_p by extra x
Plugin guess y_p found back into original ODE and determine the unknown coefficients in y_p
y_p
y_p
$ay'' + by' + cy = 0$
Let $y_h = Ae^{rx}$, plug into ODE and find charaterestic equation $ar^2 + br + c = 0$ and find its roots r_1, r_2
one root, repeated
$y_1 = e^{rx}$
$y_2 = xe^{rx}$
$y_h = c_1 y_1 + c_2 y_2$
Note: y_2 is found using reduction of order method.
Two distinct real roots r_1, r_2
$y_1 = e^{r_1 x}$
$y_2 = e^{r_2 x}$
$y_h = c_1 y_1 + c_2 y_2$
Two complex conjugate roots $\alpha \pm i\beta$
$y_1 = e^{\alpha x}\cos(\beta x)$
$y_2 = e^{\alpha x}\sin(\beta x)$
$y_h = c_1 y_1 + c_2 y_2$
$= e^{\alpha x}\left(c_1 \cos(\beta x) + c_2 \sin(\beta x)\right)$
y_h
General solution $y = y_h + y_p$. If initial conditions are given, now we find c_1, c_2.
second order with one solution given
Reduction of Order
If the second order ODE is given along with one solution y_1 and asked to find general solution, then the homogenous solution can be found by assuming $y_h = v(x)y_1(x)$ and plugging this back into the given ODE and solving for $v(x)$.



system of first order ODE's
nonhomogeneous system. (A constant)
$\underline{\mathbf{x}}' = A\underline{\mathbf{x}} + \underline{\mathbf{G}}(t)$
First solve the homogeneous part to find $\underline{\mathbf{x}}_1$ and $\underline{\mathbf{x}}_2$
homogeneous system. (A constant)
$\underline{\mathbf{x}}' = A\underline{\mathbf{x}}$
Find eigenvalues λ by solving $|A - \lambda I| = 0$
Variation of parameters
Undetermined coefficients
Find fundamental matrix $\Phi = [\underline{\mathbf{x}}_1 \quad \underline{\mathbf{x}}_1]$
$\underline{\mathbf{x}}_p = \Phi \int \Phi^{-1} \underline{\mathbf{G}}(t)\, dt$
Only works if $\underline{\mathbf{G}}(t)$ contains sums and products of exponentials and/or polynomials of degree ≥ 0 and A is constant. Everything else, use Variation of parameters.
Write $\underline{\mathbf{G}}(t) = \underline{\mathbf{g}}_1(t) + \underline{\mathbf{g}}_2(t)$
guess $\underline{\mathbf{x}}_p(t)$ based on form of $\underline{\mathbf{g}}_1(t), \underline{\mathbf{g}}_2(t)$ similar to scalar case but using vectors for constants. So end up with something like $\underline{\mathbf{x}}_p(t) = \underline{\mathbf{a}} + \underline{\mathbf{b}}e^t$ (see example May 4, 2017)
is $\underline{\mathbf{x}}_1$ or $\underline{\mathbf{x}}_2$ in $\underline{\mathbf{x}}_p$?
YES
Adjust $\underline{\mathbf{x}}_p$ by multiplying by t, see example May 4, 2017 for illustration.
NO
Plug $\underline{\mathbf{x}}_p(t)$ into $\underline{\mathbf{x}}_p'(t) = A\underline{\mathbf{x}}_p(t) + \underline{\mathbf{g}}_1(t) + \underline{\mathbf{g}}_2(t)$. Balance terms and solve for vectors $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$
$\underline{\mathbf{x}}_p$
λ is real and repeated
Find first eigenvector $\underline{\mathbf{v}}_1$ by solving $(A - \lambda_1 I)\,\underline{\mathbf{v}}_1 = \underline{\mathbf{0}}$
Find first L.I. solution $\underline{\mathbf{x}}_1 = \underline{\mathbf{v}}_1 e^{\lambda_1 t}$
Is the eigenvalue λ_1 complete? i.e. can we find the second $\underline{\mathbf{v}}_2$ from same eigenvalue λ_1? (in a 2×2 system, this will only happen when the two first order equations are decoupled). Hence, most of the time, the eigenvalue will not be complete.
λ_1 is complete (rare case)
λ_1 is defective (common case)
Second eigenvector $\underline{\mathbf{v}}2$ is found by solving from same eigenvalue $(A - \lambda_1 I)\,\underline{\mathbf{v}}_2 = \underline{\mathbf{0}}$
Find second L.I. solution $\underline{\mathbf{x}}_2 = \underline{\mathbf{v}}_2 e^{\lambda_1 t}$
Solve for $\underline{\mathbf{v}}_2$ from $(A - \lambda_1 I)\,\underline{\mathbf{v}}_2 = \underline{\mathbf{v}}_1$ Where $\underline{\mathbf{v}}_1$ was found in above step. This step requires Gaussian elimination.
Find second L.I. solution $\underline{\mathbf{x}}_2 = (t\underline{\mathbf{v}}_1 + \underline{\mathbf{v}}_2)\, e^{\lambda_1 t}$
$\underline{\mathbf{x}}_1$
$\underline{\mathbf{x}}_2$
$\underline{\mathbf{x}}_2$
$\underline{\mathbf{x}}_1$
build homogeneous solution
$\underline{\mathbf{x}}_h = c_1\underline{\mathbf{x}}_1 + c_2\underline{\mathbf{x}}_2$
λ_1, λ_2 are real and distinct
Find first eigenvector $\underline{\mathbf{v}}_1$ by solving $(A - \lambda_1 I)\,\underline{\mathbf{v}}_1 = \underline{\mathbf{0}}$
Find second eigenvector $\underline{\mathbf{v}}_2$ by solving $(A - \lambda_2 I)\,\underline{\mathbf{v}}_2 = \underline{\mathbf{0}}$
build homogeneous solution
$\underline{\mathbf{x}}_1 = \underline{\mathbf{v}}_1 e^{\lambda_1 t}$
$\underline{\mathbf{x}}_2 = \underline{\mathbf{v}}_2 e^{\lambda_2 t}$
$\underline{\mathbf{x}}_h = c_1\underline{\mathbf{x}}_1 + c_2\underline{\mathbf{x}}_2$
$\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$ are complex conjugates
Find first eigenvector $\underline{\mathbf{v}}_1$ by solving $(A - \lambda_1 I)\,\underline{\mathbf{v}}_1 = \underline{\mathbf{0}}$
Find second eigenvector $\underline{\mathbf{v}}_2$ by solving $(A - \lambda_2 I)\,\underline{\mathbf{v}}_2 = \underline{\mathbf{0}}$
build homogeneous solution
$\underline{\mathbf{x}}_1 = \underline{\mathbf{v}}_1 e^{(\alpha+i\beta)t}$
$\underline{\mathbf{x}}_2 = \underline{\mathbf{v}}_2 e^{(\alpha-i\beta)t}$
$\underline{\mathbf{x}}_h = c_1\underline{\mathbf{x}}_1 + c_2\underline{\mathbf{x}}_2$
convert to real basis
$\underline{\mathbf{x}}_1 = \mathrm{Re}\{\underline{\mathbf{x}}_1\}$
$\underline{\mathbf{x}}_2 = \mathrm{Im}\{\underline{\mathbf{x}}_1\}$
$\underline{\mathbf{x}}_h = c_3\underline{\mathbf{x}}_1 + c_4\underline{\mathbf{x}}_2$
$\underline{\mathbf{x}}_h$
$\underline{\mathbf{x}} = \underline{\mathbf{x}}_h + \underline{\mathbf{x}}_p$


ode
first order $y' = f(x, y, y^2, \dots)$
linear
$y' = f(x, y)$
non-linear
separable
if ODE has the form $y' = P(x)Q(y)$. Example $y' = xy$, then it is separable.
$\frac{dy}{Q(y)} = P(x)\,dx$
$\int \frac{dy}{Q(y)} = \int P(x)\,dx$
and solve for y
integrating factor
if ODE has this form $y' + P(x)y = Q(x)$ Example $y' + xy = 2x$, then Integrating factor $\mu = \int e^{P(x)\,dx}$, hence
$d(\mu y) = \mu Q(x)$
$\mu y = \int \mu Q(x) + c$
ODE is exact or can be made exact. Write ODE in form $M(x,y)dx + N(x,y)dy = 0$
$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$?
yes
no
Example $2xy\,dx + (x^2 + \cos y)\,dy = 0$
Set up these two equations
$\frac{dU}{dx} = M$
$\frac{dU}{dy} = N$
From first equation find $U = \int M\,dx + f(y)$ where $f(y)$ is some arbitrary function of y only. Using second equation now solve for $f(y)$. The final solution is $U = c$ where c is constant.
Example $(2y^2 x - y)dx + xdy = 0$. Find integrating factor μ
$A = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$
is A function of x alone?
yes
no
$\mu = e^{\int A\,dx}$
$B = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$
is B function of y alone?
yes
no
$\mu = e^{\int B\,dy}$
Try other methods. see above
Multiply the original ODE by the found integrating factor μ
$M = \mu M$
$N = \mu N$
And follow same method
We come here if ODE is not separable, not exact, and can not be made exact. Example is $y' = \frac{ye^{\frac{y}{x}} + y}{x}$. With ODE in form $\frac{dy}{dx} = f(x,y)$, replace y in RHS with $y = ux$ where $u = u(x)$. If RHS now only has u in it, then ODE is homegeneous. Apply transformation $y = ux$ it will become separable.
Not linear
Canonical form $y' = f_0 + f_1 y + f_2 y^2 + f_3 y^3$
$f_0 = 0, f_3 = 0$ Bernoulli. Example $y' = xy + y^2$
start by dividing by y^2. Then let u be the coefficient of x term left on the right. Example using the above.
$\frac{y'}{y^2} = \frac{x}{y}$
Let $u = \frac{1}{y}$ or $y = \frac{1}{u}$, hence $y' = \frac{-u'}{u^2}$, subtitute back to obtain $u' + xu = 1$ which is solved using integrating factor. Hence y is now found.
$f_3 = 0$ Riccati. Example $y' = 1 + xy + y^2$
know particular solution y_p?
yes
No
use substitution $y = y_p + 1/u$
use substitution $y = -\frac{u'}{uf_2}$ this leads to second order ODE which can be solved
$f_3 \neq 0$ Abel first kind. Example $y' = 1 + xy + y^2 + xy^3$
To do. Few Abel ODE can be solved exactly.
Nasser M. Abbasi August 20, 2016. d1.ipe

1 examples

1.1 second order, constant coeff.

1.1.1 second order, constant coeff. homogeneous

second order, constant coeff. homogeneous, one root repeated

$$y'' - 2y' + 1 = 0$$

Let $y = Ae^{rx}$ and plug into the above and simplify, we obtain the charaterstic equation

$$\begin{aligned} r^2 - 2r + 1 &= 0 \\ (r-1)^2 &= 0 \\ r &= 1 \end{aligned}$$

Repated root. Hence the two L.I. basis solutions are

$$\begin{aligned} y_1 &= e^x \\ y_2 &= xe^x \end{aligned}$$

And the homogeneous solution is

$$\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 e^x + c_2 e^x \end{aligned}$$

second order, constant coeff. homogeneous, two real distinct roots

$$y'' + y' - 2y = 0$$

Let $y = Ae^{rx}$ and plug into the above and simplify, we obtain the charaterstic equation

$$\begin{aligned} r^2 + r - 2 &= 0 \\ (r-1)(r+2) &= 0 \\ r_1 &= 1 \\ r_2 &= -2 \end{aligned}$$

Hence the L.I. basis solutions are

$$\begin{aligned} y_1 &= e^x \\ y_2 &= e^{-2x} \end{aligned}$$

And the homogeneous solution is

$$\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 e^x + c_2 e^{-2x} \end{aligned}$$

second order, constant coeff. homogeneous, two complex conjugate roots

$$y'' - 6y' + 13y = 0$$

Let $y = Ae^{rx}$ and plug into the above and simplify, we obtain the charaterstic equation

$$r^2 - 6r + 13 = 0$$

Whose roots are

$$\begin{aligned} r_1 &= 3 + 2i \\ r_2 &= 3 - 2i \end{aligned}$$

Hence the L.I. basis solutions are

$$
\begin{aligned}
y_1 &= e^{(3+2i)x} \\
y_2 &= e^{(3-2i)x}
\end{aligned}
$$

the homogeneous solution is

$$
\begin{aligned}
y_h &= c_1 y_1 + c_2 y_2 \\
&= c_1 e^{(3+2i)x} + c_2 e^{(3-2i)x}
\end{aligned}
$$

This can be converted to real basis using Euler relation which results in

$$
\begin{aligned}
y_h &= C_1 e^{3x} \cos 2x + C_2 e^{3} \sin 2x \\
&= e^{3x} \left(C_1 \cos 2x + C_2 \sin 2x\right)
\end{aligned}
$$

1.1.2