On stress measures in deformed solids

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Abstract

Different known stress measures used in continuum mechanics during deformation analysis are derived and geometrically illustrated. The deformed solid body is subjected to rigid body rotation tensor $\tilde{Q}$. Expressions formulated showing how the deformation geometrical tensors $\tilde{U}$, $\tilde{V}$ and $\tilde{R}$ are transformed under this rigid body motion.
In addition, each stress measure is also analyzed under this rigid rotation.
For each stress tensor, the appropriate strain tensor to use in the material stress-strain constitutive relation is derived analytically. The famous paper by Professor Atlrui [2] was used as the main framework and guide for all these derivations.

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1 Conclusion and results

1.1 Different stress tensors

Stress	Stress measure	Generally Symmetrical ?
$\tilde{\tau}$ Cauchy	$\mathbf{df} = (da\ \mathbf{n}) \cdot \tilde{\tau}$	Yes
$\tilde{\mathbf{t}}$ First Piola-Kirchhoff	$\tilde{\mathbf{t}} = J\ \tilde{\mathbf{F}}^{-1} \cdot \tilde{\tau}$	No
$\tilde{\mathbf{s}}_1$Second Piola-Kirchhoff	$\tilde{\mathbf{s}}_1 = J\ \tilde{\mathbf{F}}^{-1} \cdot \tilde{\tau} \cdot \tilde{\mathbf{F}}^{-T}$	Yes
$\tilde{\sigma}$ Kirchhoff	$\tilde{\sigma} = J\ \tilde{\tau}$	Yes
$\tilde{\mathbf{\Gamma}}$	$\tilde{\mathbf{\Gamma}} = \tilde{\mathbf{R}}^T \cdot \tilde{\tau} \cdot \tilde{\mathbf{R}}$	Yes
$\tilde{\mathbf{r}}^*$ Biot-Lure	$\tilde{\mathbf{r}}^* = J\ \tilde{\mathbf{F}}^{-1} \cdot \tilde{\tau} \cdot \tilde{\mathbf{R}}$	No
$\tilde{\mathbf{r}}$ Jaumann	$\tilde{\mathbf{r}} = \frac{\left(\tilde{\mathbf{r}}^* + \tilde{\mathbf{r}}^{*T}\right)}{2}$	Yes
$\tilde{\mathbf{T}}^*$	$J\ \tilde{\mathbf{V}}^{-1} \cdot \tilde{\tau}$	No
$\tilde{\mathbf{T}}$	$\tilde{\mathbf{T}} = \frac{\left(\tilde{\mathbf{T}}^* + \tilde{\mathbf{T}}^{*T}\right)}{2}$	Yes

1.2 Deformation gradient tensor under rigid body transformation $\tilde{\mathbf{Q}}$

Tensor	$\tilde{\mathbf{Q}}$ based transformation
$\tilde{\mathbf{F}}$ The deformation gradient	$\tilde{\mathbf{F}}_q = \tilde{\mathbf{Q}} \cdot \tilde{\mathbf{F}}$
$\tilde{\mathbf{U}}$ Stretch before rotation $\tilde{\mathbf{R}}$	$\tilde{\mathbf{U}}_q = \tilde{\mathbf{U}}$
$\tilde{\mathbf{V}}$ Stretch after rotation $\tilde{\mathbf{R}}$	$\tilde{\mathbf{V}}_q = \tilde{\mathbf{Q}} \cdot \tilde{\mathbf{V}} \cdot \tilde{\mathbf{Q}}^T$

1.3 Stress tensors under rigid body transformation $\tilde{\mathbf{Q}}$

Stress	$\tilde{\mathbf{Q}}$ based transformation	Transforms Similar to
$\tilde{\tau}$ Cauchy	$\tilde{\tau}_q = \tilde{\mathbf{Q}} \cdot \tilde{\tau} \cdot \tilde{\mathbf{Q}}^T$	$\tilde{\mathbf{V}}$
$\tilde{\mathbf{t}}$ First Piola-Kirchhoff	$\tilde{\mathbf{t}}_q = \tilde{\mathbf{Q}} \cdot \tilde{\mathbf{t}}$	$\tilde{\mathbf{F}}$
$\tilde{\mathbf{s}}_1$Second Piola-Kirchhoff	$\tilde{\mathbf{s}}_{1_q} = \tilde{\mathbf{s}}_1$	$\tilde{\mathbf{U}}$
$\tilde{\sigma}$ Kirchhoff	$\tilde{\sigma} = J\ \left(\tilde{\mathbf{Q}} \cdot \tilde{\tau} \cdot \tilde{\mathbf{Q}}^T\right)$	$\tilde{\mathbf{V}}$
$\tilde{\mathbf{\Gamma}}$	$\tilde{\mathbf{\Gamma}}_q = \tilde{\mathbf{\Gamma}}$	$\tilde{\mathbf{U}}$
$\tilde{\mathbf{r}}^*$ Biot-Lure	$\tilde{\mathbf{r}}^*_q = \tilde{\mathbf{r}}^*$	$\tilde{\mathbf{U}}$
$\tilde{\mathbf{r}}$ Jaumann	$\tilde{\mathbf{r}}_q = \tilde{\mathbf{r}}$	$\tilde{\mathbf{U}}$
$\tilde{\mathbf{T}}^*$	$\tilde{\mathbf{T}}^*_q = \tilde{\mathbf{T}}^*$	$\tilde{\mathbf{U}}$
$\tilde{\mathbf{T}}$	$\tilde{\mathbf{T}}_q = \tilde{\mathbf{T}}$	$\tilde{\mathbf{U}}$

1.4 Conjugate pairs (Stress tensor/Strain tensor)

If we refer to W as the current amount of energy stored in a unit volume as a result of the body undergoing deformation, then the time rate at which this energy changes will equal the stress tensor $\tilde{\mathbf{B}}$ times the strain rate $\frac{\partial \tilde{\mathbf{A}}}{\partial t}$. Hence we write

$$\boxed{\dot{W} = \tilde{\mathbf{B}} : \frac{\partial \tilde{\mathbf{A}}}{\partial t}}$$

The following table lists the stress tensor $\tilde{\mathbf{B}}$, the strain rate $\frac{\partial \tilde{\mathbf{A}}}{\partial t}$ and the strain $\tilde{\mathbf{A}}$.

Stress tensor $\tilde{\mathbf{B}}$	Strain tensor rate $\frac{\partial \tilde{\mathbf{A}}}{\partial t}$	Strain tensor $\tilde{\mathbf{A}}$
$\tilde{\tau}$ Cauchy	$\frac{1}{J}\frac{1}{2}\left(\dot{\mathbf{F}} \cdot \tilde{\mathbf{F}}^{-1} + \left(\dot{\mathbf{F}} \cdot \tilde{\mathbf{F}}^{-1}\right)^T\right)$	Almansi strain tensor $\tilde{\mu} = \frac{1}{J}\frac{1}{2}\left(\tilde{\mathbf{F}}^{-T} \cdot \tilde{\mathbf{F}}^{-1} - \tilde{\mathbf{I}}\right)$
$\tilde{\sigma}$ Kirchhoff	$\frac{1}{2}\left(\dot{\mathbf{F}} \cdot \tilde{\mathbf{F}}^{-1} + \left(\dot{\mathbf{F}} \cdot \tilde{\mathbf{F}}^{-1}\right)^T\right)$	$\frac{1}{2}\left(\tilde{\mathbf{F}}^{-T} \cdot \tilde{\mathbf{F}}^{-1} - \tilde{\mathbf{I}}\right)$
$\tilde{\mathbf{t}}$ 1^{st} Piola-Kirchhoff	$\frac{1}{J}\dot{\mathbf{F}}^T$	$\frac{1}{J}\tilde{\mathbf{F}}^T$
$\tilde{\mathbf{s}}_1$ 2^{nd} Piola-Kirchhoff	$\frac{1}{J}\dot{\gamma}$	Green-Lagrange strain tensor $\frac{1}{J}\tilde{\gamma} = \frac{1}{2J}\left(\tilde{\mathbf{F}}^{\mathbf{T}} \cdot \tilde{\mathbf{F}} - \tilde{\mathbf{I}}\right)$
$\tilde{\mathbf{r}}^*$ Biot-Lure	$\frac{1}{J}\dot{\mathbf{U}}$	$\frac{1}{J}\mathbf{U}$
$\tilde{\mathbf{r}}$ Jaumann	$\dot{\mathbf{U}}$	$\tilde{\mathbf{U}}$
$\tilde{\mathbf{\Gamma}}$	$\frac{1}{2}\left(\tilde{\mathbf{U}}^{-1} \cdot \dot{\mathbf{U}} + \dot{\mathbf{U}} \cdot \tilde{\mathbf{U}}^{-1}\right)$	$\ln\left(\tilde{\mathbf{U}}\right)$ (For isotropic material only)
$\tilde{\mathbf{T}}$	$\dot{\mathbf{V}}$ (for isotropic only)	$\tilde{\mathbf{V}}$ (For isotropic material only)

2 Overview of geometry and mathematical notations used

Position and deformation measurements is of central importance in Continuum mechanics.

There are two methods employed to accomplish this. The Lagrangian and the Eulerian methods.

In the Lagrangian method, a particle position and speed are measured in reference to a fixed stationary observer based coordinates systems. This is called the referential coordinates system where the observer is located. In other words, we use a coordinate system attached to a stationary observer. Hence in the lagrangian method, the particle movment is always measured from this original global frame of reference.

In Eulerian methods, we attach a frame of reference local to the area of interest where we wish to make the measurement, and we measure the particle movment relative to this local coordinates systems. Hence we measure the position and speed of the the particle as it moves relative to this local frame of reference.

In Continuum mechanics we use the lagrangian method and in Fluid mechanics the Eulerian method is used.

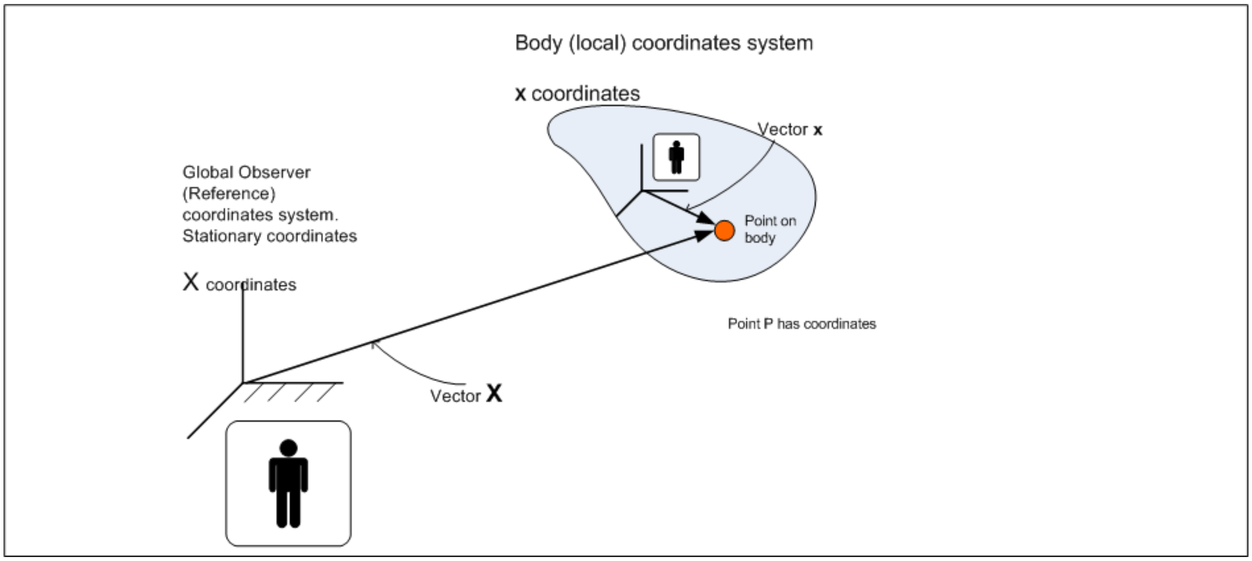
In Continuum mechanics we can attach a local frame of reference to the body itself. Hence this frame of reference will move along with the body as it deforms. We can make measurments relative to this local frame of reference, this convert these measurments back relative to the global frame of reference.

We can find a coordinate transformation that gives back the coordinates of a point on a body relative to the global reference frame, given the coordinates of the same point as measured in the local reference frame. This transformation is given by

$$\mathbf{X} = \mathbf{A}\mathbf{x} + \mathbf{d}$$

Where $\mathbf{X}$ is the coordinate vector relative the global frame of reference, $\mathbf{x}$ is the coordinate vector relative the local/body frame of reference and $\mathbf{A}$ is the $n \times n$ matrix (where $n = 3$ for normal 3D space) that represents pure rotation, and $\mathbf{d}$ is an n-dimensional vector that represents pure translation.

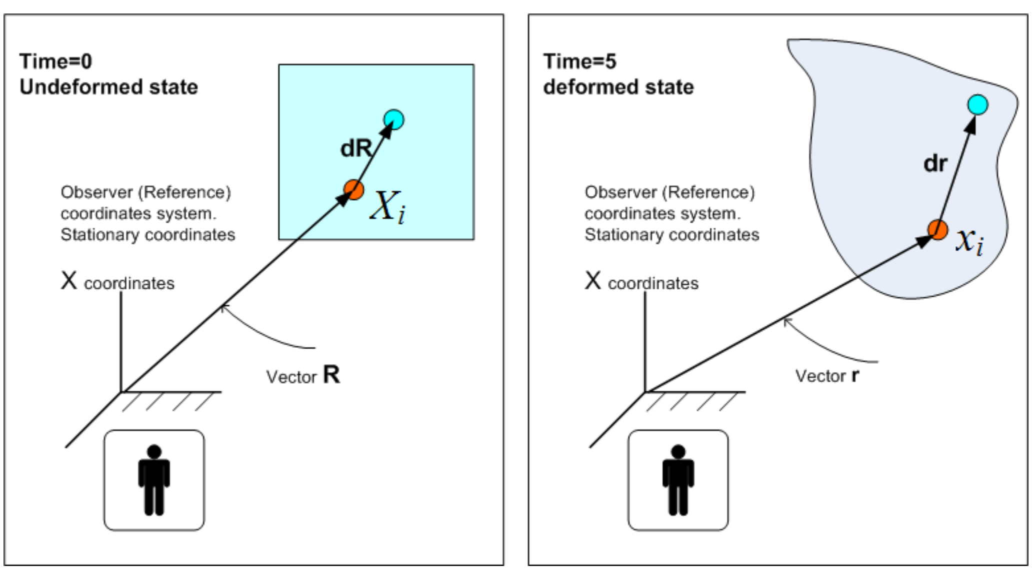
This diagram illustrates these basic differences.



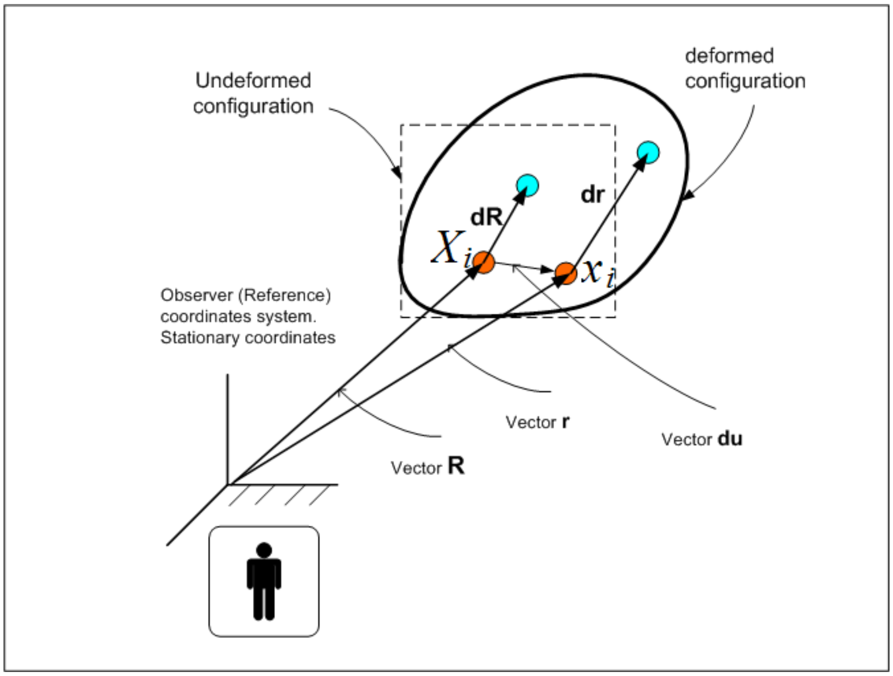
In general, we are usually interested in differential changes when a body deformed. In other words, we are interested in measuring how a differential vector that represents the orientation of one point relative to another changes as the body deformed.

Considering the Lagrangian method from now on. We now shift attention to what happens when the body starts to deform. We start by choosing the global reference frame. This is where all measurements are made against. We distinguish measurements made when the body is underformed from those measurements made when the body deforms. We use upper case X_i for the coordinates of a point on the body when the body is undeformed, and we use a lower case x_i for the coordinates of the same point when measured in reference to this same global coordinate system but after the body has deformed.

In this diagram below we assume we took a snap shot of the system after 5 units of time and we measure the deformation to illustrate the notation used.



Another way to represent the above is to use the same diagram to show both the undeformed and the deformed configuration on it as follows.



We call the undeformed configuration, henceforth will be referred to as the body *state*, as state B. By state we mean the set of independent variables needed to fully describe the forces and geometry of the body.

We assume the body, when in the undeformed state B is free of internal stresses and no traction forces act on it.

External loads are now applied to the body resulting in a change of state. The new state can be a result of only a deformation in the body shape, or due to only a rigid body translation/rotation, or it could be a result of a combination of deformation and rigid body motion.

The deformation will take sometime t to complete. However, in this discussion we are only interested in the final deformed state, which we call state b. Hence no functions of time will be involved in this analysis.

We assume that the boundary conditions remain the same in state B and in state b. This means that if the solid body was in physical contact with some external non-moving supporting configuration, then after deformation, the body will remain in physical contact with these supports and at the same points of contact as before the deformation began. This means the body is free to deform everywhere, except it is constrained to deform at those specific points it is in contact with the support. For the rigid body rotation part, we assume the body along with its support will rotate together.

A very important operator in continuum mechanics is called the deformation tensor $\tilde{\mathbf{F}}$. (A tensor can be viewed as an operator which takes a vector and maps it to another vector). This tensor allows us to determine the deformed differential vector dr knowing the undeformed differential vector dR as follows.

$$\mathbf{dr} = \tilde{\mathbf{F}} \cdot \mathbf{dR}$$

This tensor if a field tensor in general. This means the actual value of $\tilde{\mathbf{F}}$ changes depending on the location of the body where $\tilde{\mathbf{F}}$ is evaluated. So it is a function of the body coordinates. See [4] for examples how to calculate $\tilde{\mathbf{F}}$ for very simple cases of deformations in 2D. See appendix for derivation of $\tilde{\mathbf{F}}$ in the specific case of normal cartesian coordinates.

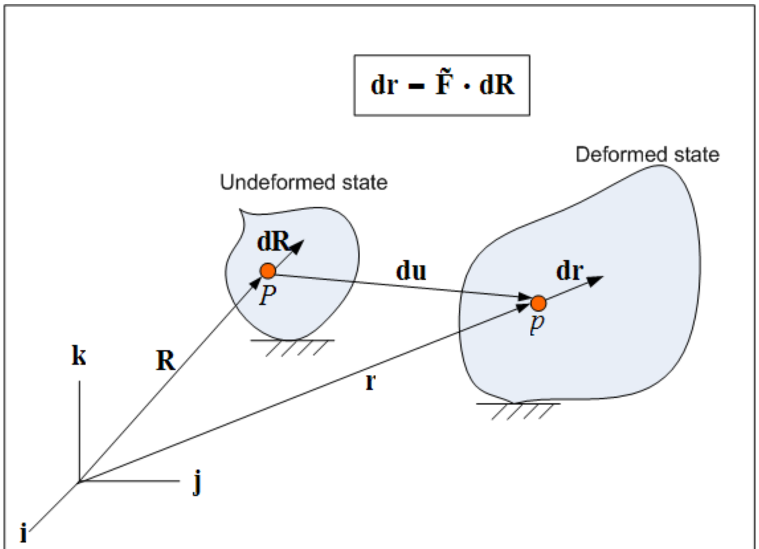
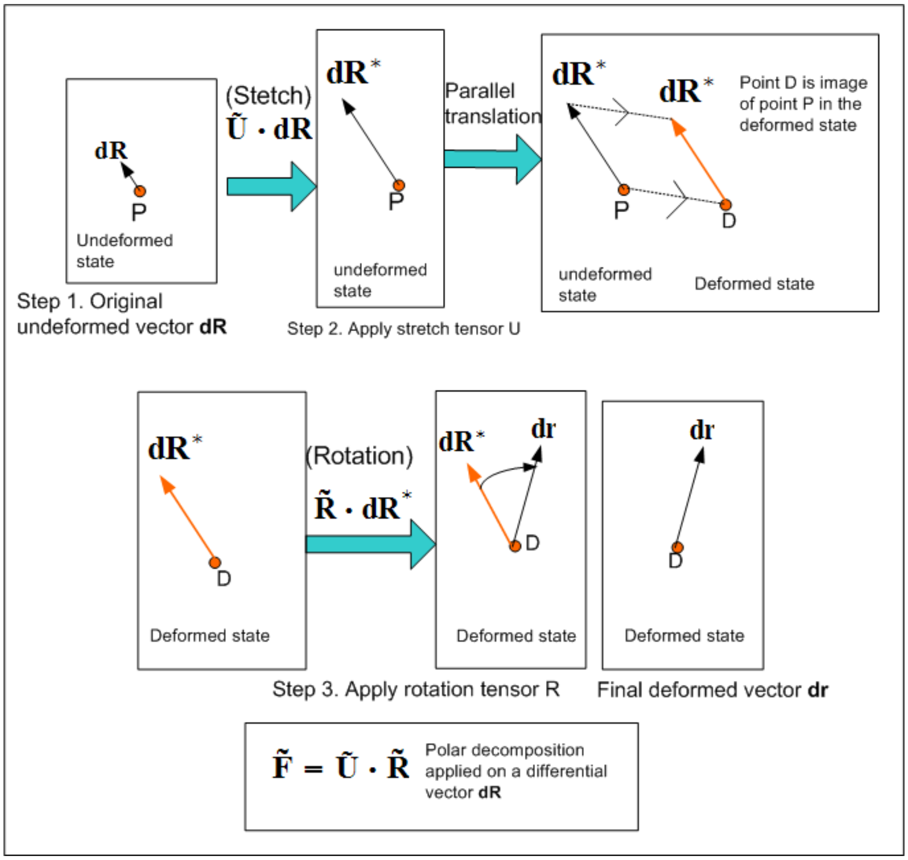


Illustration showing the main vectors involved in the deformation analysis.

2.1 Illustration of the polar decomposition of the deformation gradient tensor $\tilde{F}$

2.1.1 Polar decomposition applied to a vector



The effect of applying the deformation gradient tensor $\tilde{\mathbf{F}}$ on a vector $\mathbf{dR}$ can be considered to have the same result as the effect of first applying a stretch deforming tensor $\tilde{\mathbf{U}}$ (Also called the deformation tensor) on $\mathbf{dR}$, resulting in a vector $\mathbf{dR}^*$, then followed by applying a rotation deforming tensor $\tilde{\mathbf{R}}$ on this new vector $\mathbf{dR}^*$ to produce the final vector $\mathbf{dr}$.

Hence we write $\tilde{\mathbf{F}} = \tilde{\mathbf{U}} \cdot \tilde{\mathbf{R}}$ and hence

$$\mathbf{dr} = \tilde{\mathbf{F}} \cdot \mathbf{dR}$$

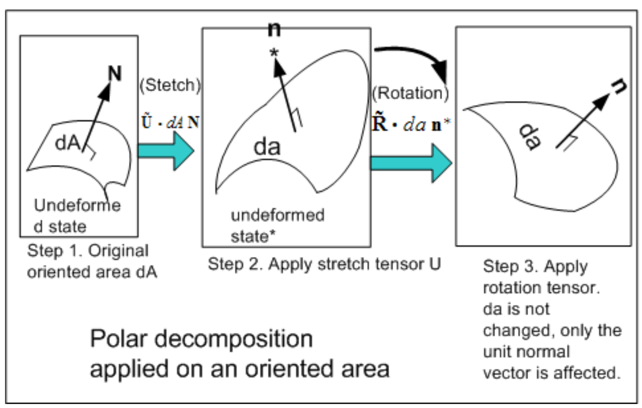
Using polar decomposition we write

$$\begin{aligned}\tilde{\mathbf{F}}\cdot\mathbf{dR} &= \left(\tilde{\mathbf{R}}\cdot\tilde{\mathbf{U}}\right)\cdot\mathbf{dR}\\ &= \tilde{\mathbf{R}}\cdot\overbrace{\left(\tilde{\mathbf{U}}\cdot\mathbf{dR}\right)}^{\mathbf{dR}^*}\\ &= \tilde{\mathbf{R}}\cdot\mathbf{dR}^*\\ &= \mathbf{dr}\end{aligned}$$

This is called polar decomposition of $\tilde{\mathbf{F}}$, and it always possible to find such decomposition. In addition, this decomposition is unique for each $\tilde{\mathbf{F}}$.

2.1.2 Polar decomposition applied to an oriented area

An oriented area in the undeformed state is $dA\ \mathbf{N}$ (Where $\mathbf{N}$ is a unit normal to dA). This area becomes $da\ \mathbf{n}^*$ after the application of the stretch tensor $\tilde{\mathbf{U}}$. It is clear that rotation will not have an effect on the area da itself, but it will rotate the unit vector $\mathbf{n}^*$ which is normal to da to become the unit vector $\mathbf{n}$. This is illustrated in the diagram below.



Now that we have given a brief description of the geometry and the important tensor $\tilde{\mathbf{F}}$ we are ready to start the discussion of the main topic of this paper.

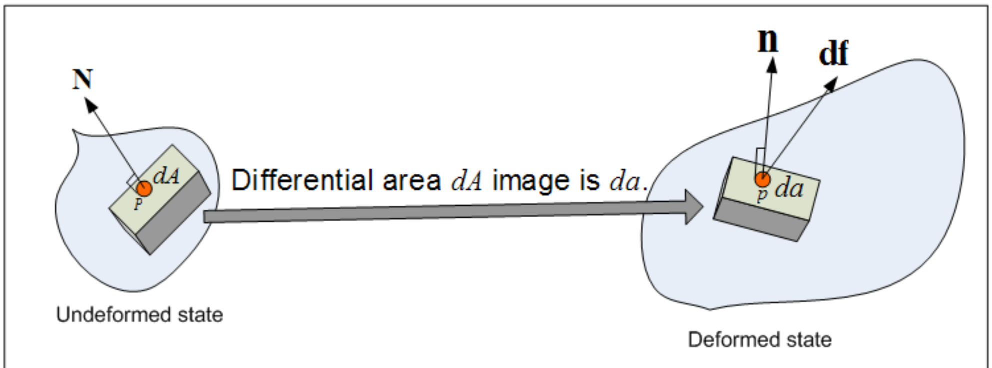
3 Stress Measures

Before outlining the different stress measures, we describe and illustrate the different entities involved.

Given the undeformed state B, we consider a point P in B and then follow its location in the deformed state b (Lagrangian description). Consider a differential area dA at P at the surface of B. Consider a unit vector $\mathbf{dN}$ normal to this area. After deformation, this differential area will be deformed to a new differential area da in the deformed state b. Let $\mathbf{dn}$ be the unit vector normal to da.

Let $\mathbf{df}$ be the differential force vector that represents the resultant of the total internal forces acting on da in the deformed state b.

The following diagram illustrates the above.



Differential area deformation and normal unit vectors used in stress measurment

3.1 Cauchy stress measure

The Cauchy stress measure $\tilde{\tau}$ is is a measure of

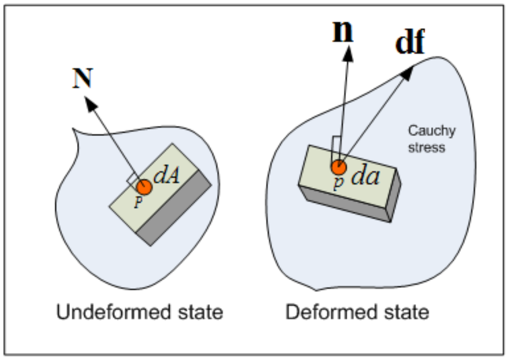
$$\boxed{\text{force per unit area in the deformed state}}$$

hence it is called the true measure of stress. From the above definition we see that

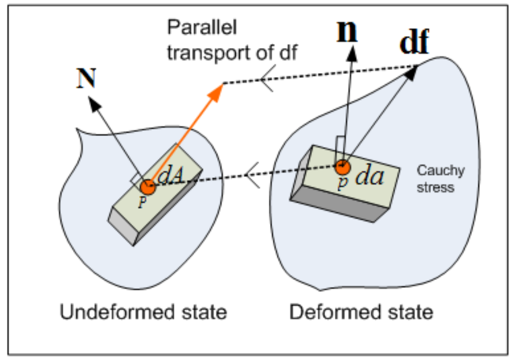
$$\boxed{\mathbf{df} = (da\ \mathbf{n}) \cdot \tilde{\tau}}$$

Cauchy stress tensor is in general (in absence of body couples) a symmetric tensor.

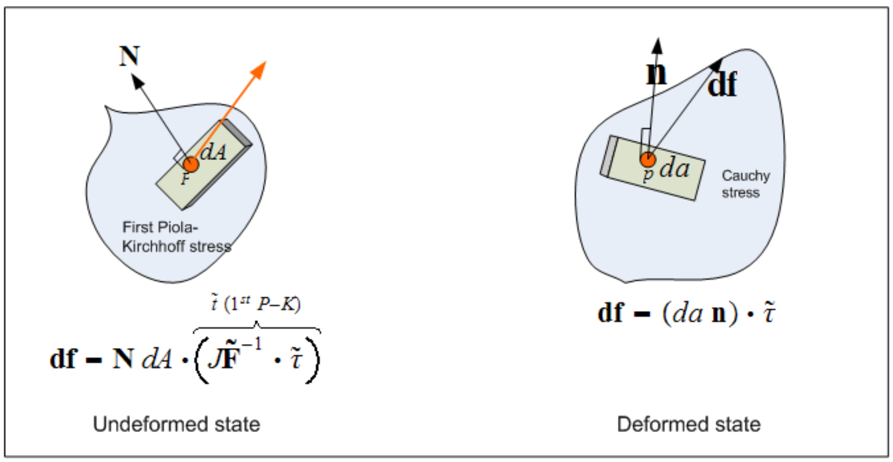
3.2 First Piola-Kirchhoff or Piola-Lagrange stress measure



Step 1. Determine Cauchy stress in deformed state



Step 2. Parallel transport forces from deformed state to undeformed state on area image.



Step 3. First piola-kirchhoff stress measure determined in the undeformed state.

From the above diagram we see that this stress $\tilde{\mathbf{t}}$ can be regarded as

$$\boxed{\text{The force in the deformed body per unit undeformed area}}.$$

It is derived as follows.

Start by moving the vector **df** (the result of internal forces in the deformed state) which acts on the deformed area da in parallel transport to the image of da in the undeformed state, which will be the differential area dA.

Hence we obtain in the undeformed state

$$\mathbf{df} = (dA\ \mathbf{N}) \cdot \tilde{\mathbf{t}} \tag{1}$$

Now given that

$$\mathbf{N}\ dA = \frac{1}{J} (da\ \mathbf{n}) \cdot \tilde{\mathbf{F}}$$

Which is a relationship derived from geometrical consideration [2]. Then from the above equation we obtain

$$da\ \mathbf{n} = J\ \ (\mathbf{N}\ dA) \cdot \tilde{\mathbf{F}}^{-1}$$

But since $\mathbf{df} = (da\ \mathbf{n}) \cdot \tilde{\tau}$, then using the above equation, we can write

$$\begin{aligned} \mathbf{df} &= \left(J\ \ (\mathbf{N}\ dA) \cdot \tilde{\mathbf{F}}^{-1}\right) \cdot \tilde{\tau} \\ &= \mathbf{N}\ dA \cdot \left(J\tilde{\mathbf{F}}^{-1} \cdot \tilde{\tau}\right) \end{aligned} \tag{2}$$

compare (1) to (2)

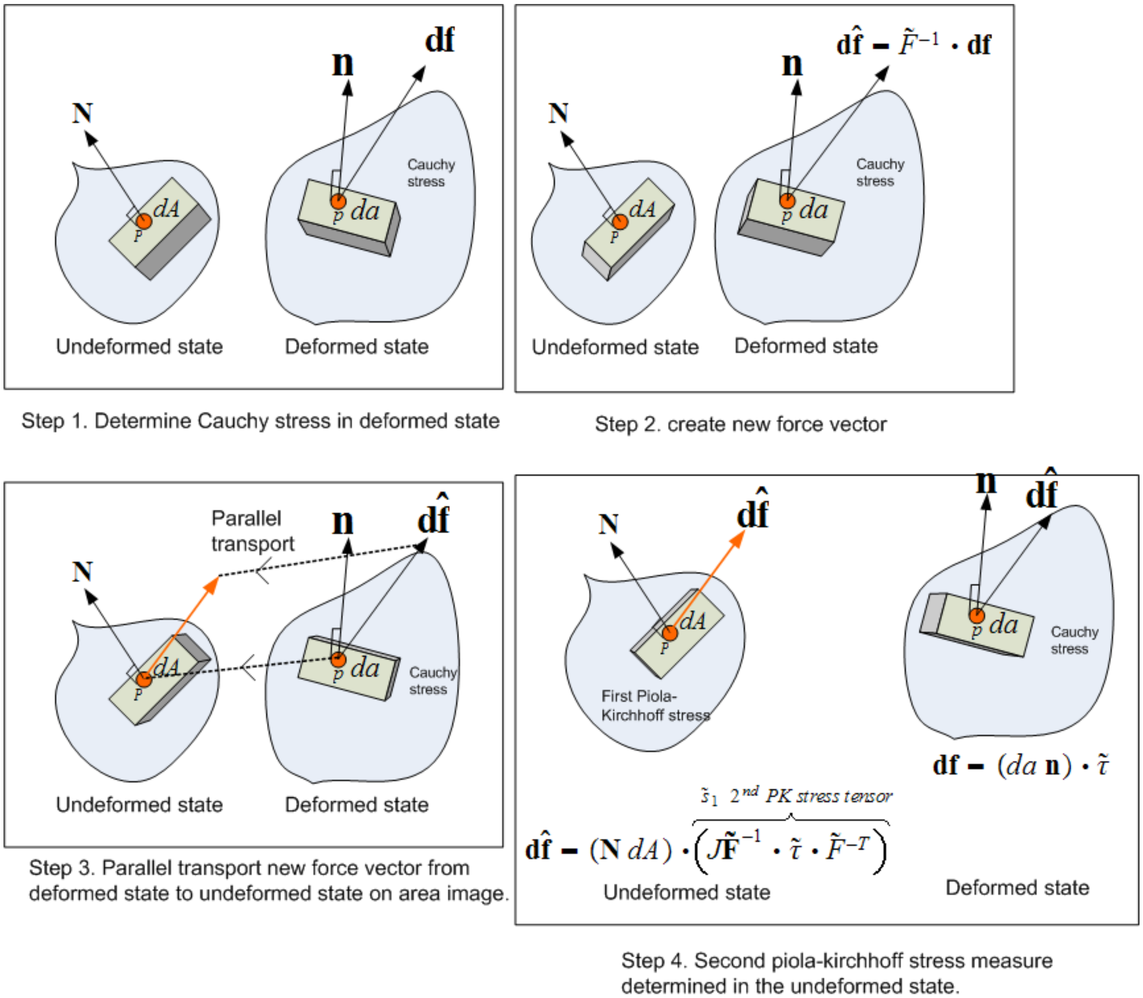
$$(dA\ \mathbf{N}) \cdot \tilde{\mathbf{t}}\ = (\mathbf{N}\ dA) \cdot J\tilde{\mathbf{F}}^{-1} \cdot \tilde{\tau}$$

Hence we determine that

$$\boxed{\tilde{\mathbf{t}} = J\ \tilde{\mathbf{F}}^{-1} \cdot \tilde{\tau}}$$

First Piola-Kirchhoff stress tensor in general is unsymmetrical.

3.3 Second Piola-Kirchhoff stress tensor



From the above diagram we see that this stress $\tilde{s}_1$ can be regarded as

Modified version of forces in the deformed body per unit undeformed area.

This stress measure $\tilde{s}_1$ is similar to the first Piola-Kirchhoff stress measure, except that instead of parallel transporting the force **df** from the deformed state to the undeformed state, we first create a force vector $\hat{\mathbf{df}}$ derived from **df** and then parallel transport this new vector. Everything else remains the same. The purpose of this is that the second piola-Kirchhoff stress tensor will now be a symmetric tensor while the first Piola-Kirchhoff stress tensor was unsymmetric

$$\hat{\mathbf{df}} = \tilde{\mathbf{F}}^{-1} \cdot \mathbf{df} \tag{1}$$

Hence in the undeformed state (after we parallel transport $\hat{\mathbf{df}}$ to dA) we have the relationship

$$\mathbf{d\hat{f}} = (dA\ \mathbf{N}) \cdot \tilde{\mathbf{s}}_1 \tag{2}$$

And in the deformed state we have the relation

$$\mathbf{df} = (da\ \mathbf{n}) \cdot \tilde{\tau} \tag{3}$$

As before, we now proceed to find an expression for $\tilde{\mathbf{s}}_1$ in terms of the Cauchy stress tensor $\tilde{\tau}$. Given that

$$da\ \mathbf{n} = J\ \ (\mathbf{N}\ dA) \cdot \tilde{\mathbf{F}}^{-1}$$

Substitute the above in eq (3) we obtain

$$\mathbf{df} = \Big(J\ \ (\mathbf{N}\ dA) \cdot \tilde{\mathbf{F}}^{-1}\Big) \cdot \tilde{\tau}$$

But from (1) $\mathbf{df} = \tilde{\mathbf{F}}^T \cdot \mathbf{d\hat{f}}$, hence the above equation becomes

$$\tilde{\mathbf{F}}^T \cdot \mathbf{d\hat{f}} = \Big(J\ \ (\mathbf{N}\ dA) \cdot \tilde{\mathbf{F}}^{-1}\Big) \cdot \tilde{\tau}$$

Hence

$$\begin{aligned} \mathbf{d\hat{f}} &= \Big(J\ \ (\mathbf{N}\ dA) \cdot \tilde{\mathbf{F}}^{-1}\Big) \cdot \tilde{\tau} \cdot \tilde{\mathbf{F}}^{-T} \\ &= (\mathbf{N}\ dA) \cdot \Big(J\tilde{\mathbf{F}}^{-1} \cdot \tilde{\tau} \cdot \tilde{\mathbf{F}}^{-T}\Big) \end{aligned} \tag{4}$$

Compare (4) with (2)

$$\begin{aligned} \mathbf{d\hat{f}} &= (\mathbf{N}\ dA) \cdot \overbrace{\Big(J\tilde{\mathbf{F}}^{-1} \cdot \tilde{\tau} \cdot \tilde{\mathbf{F}}^{-T}\Big)}^{\tilde{s}_1 \quad 2^{nd}\ PK\ stress\ tensor} \\ &= (dA\ \mathbf{N}) \cdot \tilde{\mathbf{s}}_1 \end{aligned}$$

Hence the second Piola-Kirchhoff stress tensor is

$$\boxed{\tilde{\mathbf{s}}_1 = J\ \tilde{\mathbf{F}}^{-1} \cdot \tilde{\tau} \cdot \tilde{\mathbf{F}}^{-T}}$$

The second Piola-Kirchhoff stress tensor is in general symmetric.

3.4 Kirchhoff stress tensor

Kirchhoff stress tensor $\tilde{\sigma}$ is a scalar multiple of the true stress tensor $\tilde{\tau}$. The scale factor is the determinant of $\tilde{\mathbf{F}}$, the deformation gradient tensor.

Hence

$$\boxed{\tilde{\sigma} = J\ \tilde{\tau}}$$

$\tilde{\sigma}$ is symmetric when $\tilde{\tau}$ is symmetric which is in general the case.

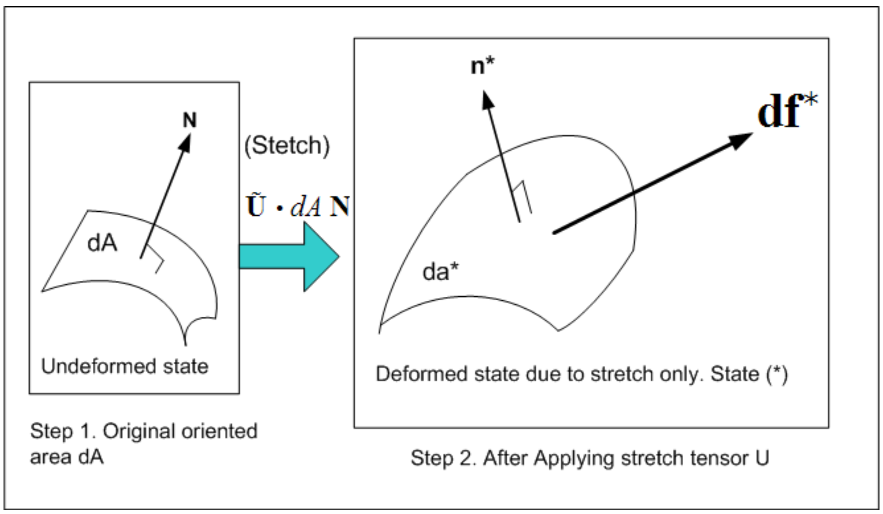
3.5 $\tilde{\mathbf{\Gamma}}$ stress tensor

The $\tilde{\mathbf{\Gamma}}$ stress tensor is a result of internal forces generated due to the application of the stretch tensor only. Hence this stress acts on the area deformed due to stretch only. Hence this stress represents

$$\boxed{\text{forces due to stretch only in the stretched body per unit stretched area}}.$$

Assume these are called $\mathbf{df}^*$, then by this definition we have

$$\mathbf{df}^* = (da\ \mathbf{n}^*) \cdot \tilde{\mathbf{\Gamma}} \tag{1}$$



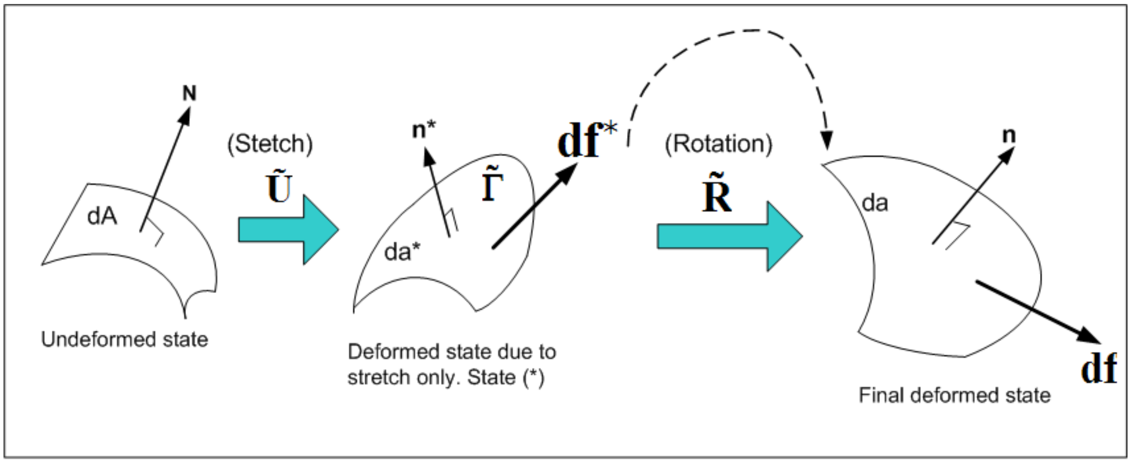
While in the final deformed state we have as before the following relation

$$\mathbf{df} = (da\ \mathbf{n}) \cdot \tilde{\tau} \tag{2}$$

What the above means is that we can consider the stretched state as a partial deformed state, and the final deformed state as the result of applying the rotation tensor on the stretched state. In the final deformed state the result of the internal forces is **df** while in the stretched state (which we designate all the variables in that state with a star *) the internal forces are termed $\mathbf{df}^*$

Hence we can write

$$\mathbf{df} = \tilde{\mathbf{R}} \cdot \mathbf{df}^* \tag{3}$$



$\tilde{\mathbf{\Gamma}}$ Stress tensor exist in the stretched state only

eq (3) can be written as $\mathbf{df}^* = \mathbf{df} \cdot \tilde{\mathbf{R}}$, then substitute this into (1) we get

$$\overbrace{\mathbf{df}}^{(da\ \mathbf{n})\cdot\tilde{\tau}} \cdot\tilde{\mathbf{R}} = (da\ \mathbf{n}^*) \cdot \tilde{\mathbf{\Gamma}} \tag{4}$$

Now substitute for $\mathbf{df}$ in the above equation the expression for $\mathbf{df}$ in eq (2) we obtain

$$(da\ \mathbf{n}) \cdot \tilde{\tau} \cdot \tilde{\mathbf{R}} = (da\ \mathbf{n}^*) \cdot \tilde{\mathbf{\Gamma}} \tag{5}$$

But $da\ \mathbf{n} = \tilde{\mathbf{R}} \cdot (da\ \mathbf{n}^*)$ hence the above equation becomes

$$\begin{aligned} \tilde{\mathbf{R}} \cdot (da\ \mathbf{n}^*) \cdot \tilde{\tau} \cdot \tilde{\mathbf{R}} &= (da\ \mathbf{n}^*) \cdot \tilde{\mathbf{\Gamma}} \\ (da\ \mathbf{n}^*) \cdot \tilde{\mathbf{R}}^T \cdot \tilde{\tau} \cdot \tilde{\mathbf{R}} &= (da\ \mathbf{n}^*) \cdot \tilde{\mathbf{\Gamma}} \\ (da\ \mathbf{n}^*) \cdot \left(\tilde{\mathbf{R}}^T \cdot \tilde{\tau} \cdot \tilde{\mathbf{R}}\right) &= (da\ \mathbf{n}^*) \cdot \tilde{\mathbf{\Gamma}} \end{aligned}$$

Hence

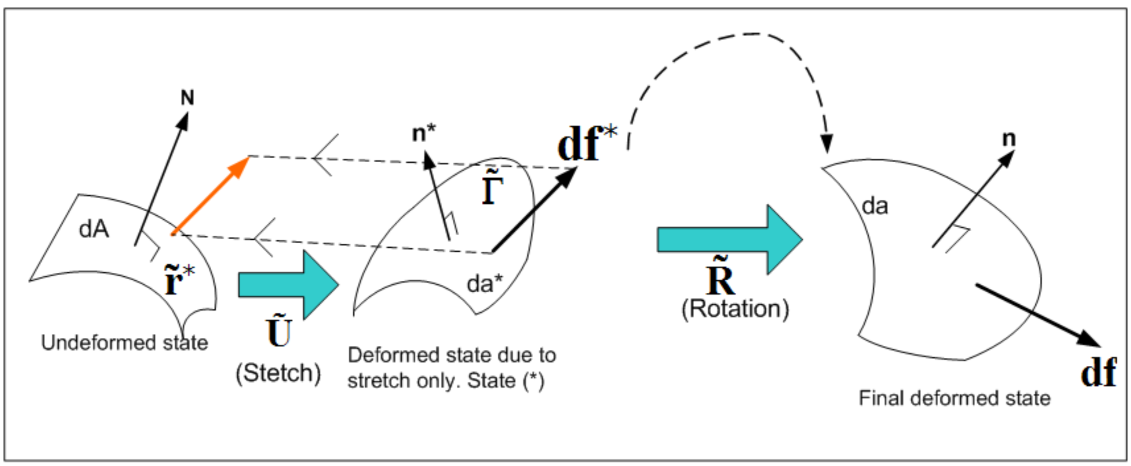
$$\boxed{\tilde{\mathbf{\Gamma}} = \tilde{\mathbf{R}}^T \cdot \tilde{\tau} \cdot \tilde{\mathbf{R}}}$$

3.6 Biot-Lure $\tilde{\mathbf{r}}^*$ stress tensor

This stress measure exists in the undeformed state as a result of parallel translation of the $\mathbf{df}^*$ forces generated in the stretched state back to the undeformed state and applying this force into the image of the stretched area in the undeformed state. Hence this stress can be considered as

forces due to stretch only applied in the undeformed body per unit undeformed area

Hence in a sense, it is one step more involved than the $\tilde{\mathbf{\Gamma}}$ stress tensor described earlier. The following diagram illustrates the above.



Biot-Lure $\tilde{\mathbf{r}}^*$ Stress tensor exist in the undeformed state only

From the above diagram we can now write an expression for the Biot-Lure stress tensor

$$\mathbf{df}^* = (dA\ \mathbf{N}) \cdot \tilde{\mathbf{r}}^*$$

And now we proceed to find an expression for $\tilde{\mathbf{r}}^*$.

Since $\mathbf{df}^* = \mathbf{df} \cdot \tilde{\mathbf{R}}$, then the above equation becomes

$$\mathbf{df} \cdot \tilde{\mathbf{R}} = (dA\ \mathbf{N}) \cdot \tilde{\mathbf{r}}^*$$

But $\mathbf{df} = da\ \mathbf{n} \cdot \tilde{\tau}$, hence the above equation becomes

$$(da\ \mathbf{n} \cdot \tilde{\tau}) \cdot \tilde{\mathbf{R}} = (dA\ \mathbf{N}) \cdot \tilde{\mathbf{r}}^*$$

But $da\ \mathbf{n} = J\ \ (dA\ \mathbf{N}) \cdot \tilde{\mathbf{F}}^{-1}$ hence the above equation becomes

$$\left(J\ \ (dA\ \mathbf{N})\cdot\tilde{\mathbf{F}}^{-1}\cdot\tilde{\tau}\right)\cdot\tilde{\mathbf{R}}=(dA\ \mathbf{N})\cdot\tilde{\mathbf{r}}^{*}$$
$$(dA\ \mathbf{N})\cdot\left(J\ \tilde{\mathbf{F}}^{-1}\cdot\tilde{\tau}\cdot\tilde{\mathbf{R}}\right)=(dA\ \mathbf{N})\cdot\tilde{\mathbf{r}}^{*}$$

Hence by comparison we see that

$$\boxed{\tilde{\mathbf{r}}^{*}=J\ \tilde{\mathbf{F}}^{-1}\cdot\tilde{\tau}\cdot\tilde{\mathbf{R}}}$$

The stress tensor $\tilde{\mathbf{r}}^{*}$ is unsymmetric when $\tilde{\tau}$ is symmetric which is in the general is the case.

3.7 Jaumann stress tensor $\tilde{\mathbf{r}}$

This stress tensor is introduced to create a symmetric stress tensor from the Biot-Lure stress tensor as follows

$$\boxed{\tilde{\mathbf{r}}=\frac{\left(\tilde{\mathbf{r}}^{*}+\tilde{\mathbf{r}}^{*T}\right)}{2}}$$

No physical interpretation of this stress tensor can be made similar to the Biot-Lure stress tensor.

3.8 The stress tensor $\tilde{\mathbf{T}}^*$

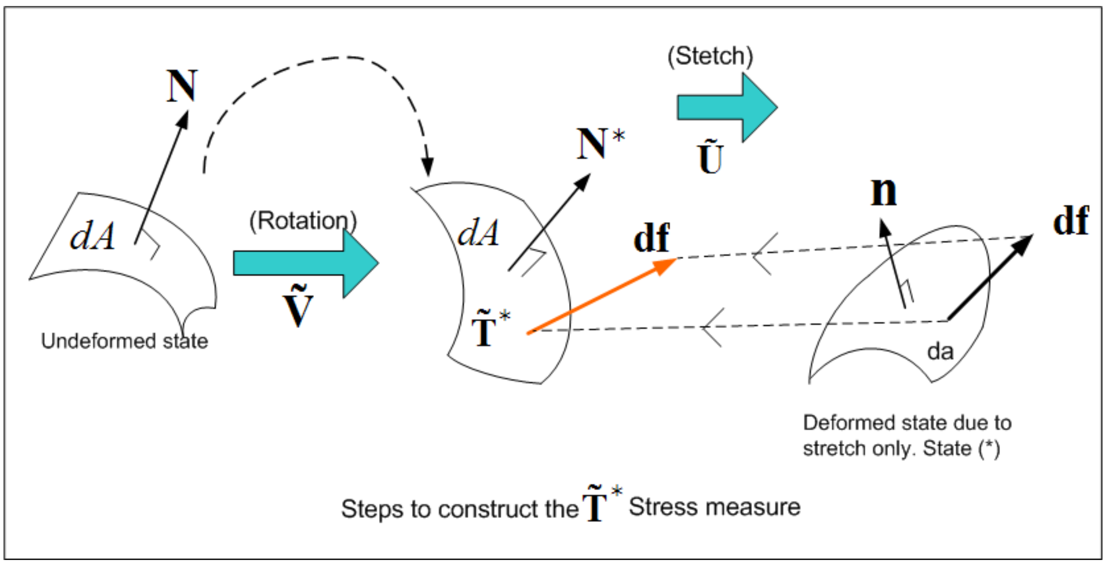
This stress tensor is defined in the rotated state without any stretch being applied before hand. The forces that act on the rotated area has been parallel transported from the forces that was generated in the final deformed state. Hence this stress can be considered as

forces due to final deformation applied in the rotated body per unit undeformed area

The following diagram illustrates this. Notice that since rotation has been applied before stretch, then the polar decomposition of $\tilde{\mathbf{F}}$ is now written as

$$\tilde{\mathbf{F}} = \tilde{\mathbf{U}} \cdot \tilde{\mathbf{V}}$$

Where $\tilde{\mathbf{V}}$ is the rotation tensor (which was called $\tilde{\mathbf{R}}$ when it is applied after stretch), and $\tilde{\mathbf{U}}$ is the stretch tensor.



From the above diagram we see that

$$\mathbf{df} = (dA\ \mathbf{N}^*) \cdot \tilde{\mathbf{T}}^*$$

But $\mathbf{df} = da\ \mathbf{n} \cdot \tilde{\tau}$, the above equation becomes

$$da\ \mathbf{n} \cdot \tilde{\tau} = (dA\ \mathbf{N}^*) \cdot \tilde{\mathbf{T}}^*$$

But $da\ \mathbf{n} = J\left(dA\ \mathbf{N}^* \cdot \tilde{\mathbf{V}}^{-1}\right)$ hence the above equation becomes

$$J\left(dA\ \mathbf{N}^{*}\cdot\tilde{\mathbf{V}}^{-1}\right)\cdot\tilde{\tau}=(dA\ \mathbf{N}^{*})\cdot\tilde{\mathbf{T}}^{*}$$
$$(dA\ \mathbf{N}^{*})\cdot J\ \tilde{\mathbf{V}}^{-1}\cdot\ \tilde{\tau}=(dA\ \mathbf{N}^{*})\cdot\tilde{\mathbf{T}}^{*}$$

Hence

$$\boxed{\tilde{\mathbf{T}}^{*}=J\ \tilde{\mathbf{V}}^{-1}\cdot\ \tilde{\tau}}$$

$\tilde{\mathbf{T}}^{*}$ is unsymmetric when $\tilde{\tau}$ is symmetric.

3.9 The stress tensor $\tilde{\mathbf{T}}$

This stress tensor is introduced to create a symmetric stress tensor from the $\tilde{\mathbf{T}}^{*}$ stress tensor as follows

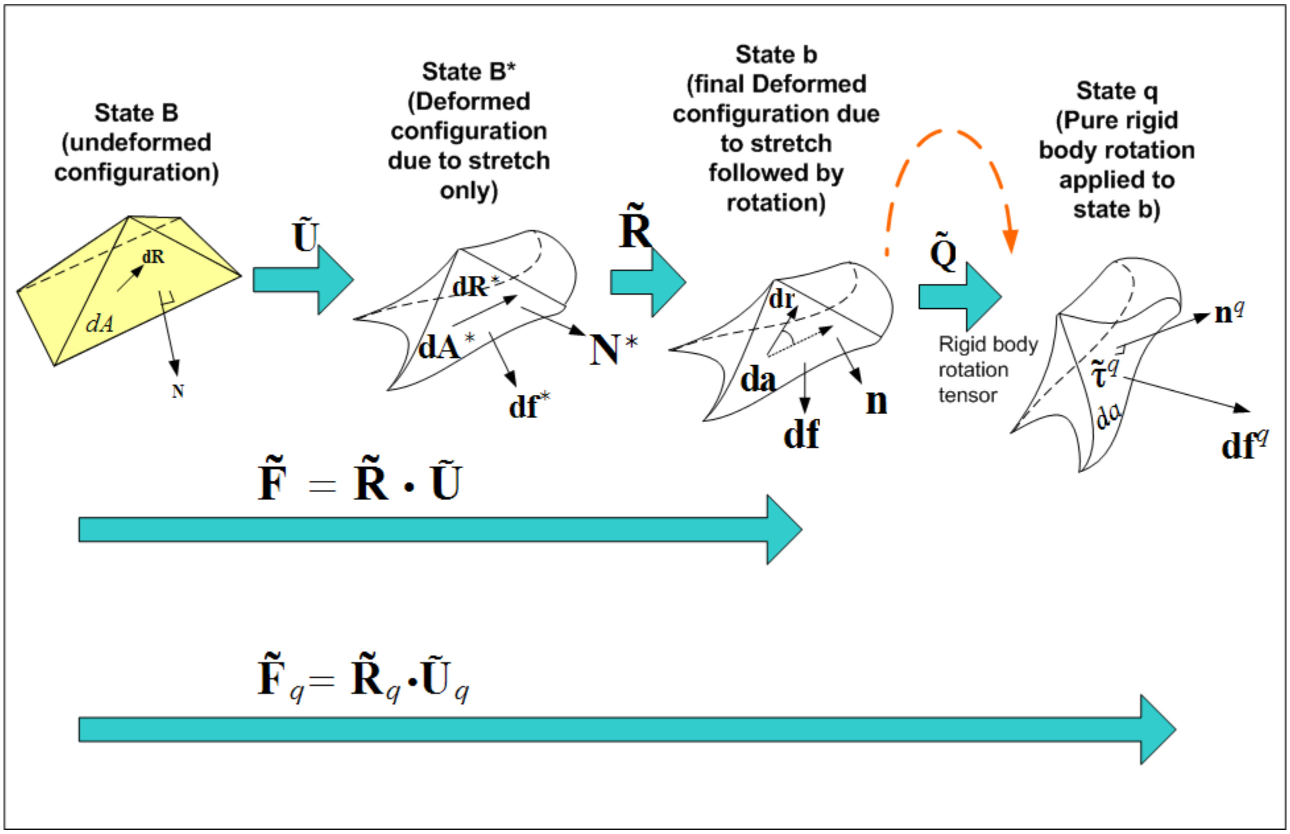
$$\boxed{\tilde{\mathbf{T}}=\frac{\left(\tilde{\mathbf{T}}^{*}+\tilde{\mathbf{T}}^{*T}\right)}{2}}$$

No physical interpretation of this stress tensor can be made similar to the $\tilde{\mathbf{T}}^{*}$ stress tensor.

4 Geometry and stress tensors transformation due to rigid body rotation

We now consider what happens to the geometrical deforming tensors $\tilde{\mathbf{F}}, \tilde{\mathbf{U}}$ and $\tilde{\mathbf{V}}$ when the body when in its final deformed state, is then subjected to a pure rigid body rotation $\tilde{\mathbf{Q}}$.

Next we consider what happens to the various stress tensors we derived above under the same $\tilde{\mathbf{Q}}$.



The polar decomposition of $\tilde{\mathbf{F}}$ is given by

$$\boxed{\tilde{\mathbf{F}} = \tilde{\mathbf{R}} \cdot \tilde{\mathbf{U}}}$$

where $\tilde{\mathbf{F}}$ is the deformation gradient tensor, $\tilde{\mathbf{U}}$ is the stretch before rotation $\tilde{\mathbf{R}}$ tensor, and $\tilde{\mathbf{R}}$ is the rotation tensor. The polar decomposition of $\tilde{\mathbf{F}}$ can also be written as

$$\boxed{\tilde{\mathbf{F}} = \tilde{\mathbf{V}} \cdot \tilde{\mathbf{R}}}$$

where $\tilde{\mathbf{V}}$ is the stretch after rotation $\tilde{\mathbf{R}}$ tensor.

We need to determine the effect of applying pure rigid body rotation $\tilde{\mathbf{Q}}$ on $\tilde{\mathbf{F}}, \tilde{\mathbf{U}}$ and $\tilde{\mathbf{V}}$.

In each of the following derivations we have the following setting in place: We have a body that is originally in the undeformed state B. Loads are applied and the body. The body will undergo deformation governed by the deformation gradient tensor $\tilde{\mathbf{F}}$ resulting in the body being in the final deformed state state b with a stress tensor $\tilde{\tau}$ at point p. If we consider the body under the effect of $\tilde{\mathbf{U}}$ first (stretch), then the new state will be called B^*, then after applying the effect of $\tilde{\mathbf{R}}$ (point to point rotation tensor), then the state will be called b (which is the final deformation state).

If however we apply $\tilde{\mathbf{R}}$ first (rotation), then the new state will also be called B^* and then when we apply the stretch $\tilde{\mathbf{V}}$ the state will becomes b (which is the final deformation state).

Now, from state b, which is the final deformation state, we apply a pure rigid body rotation tensor $\tilde{\mathbf{Q}}$ to the whole body (with its fixed supports if any), hence there will be no changes in the body shape, and we let the new state be called q.

We can also consider the change of state from state B all the way to state q to be the result of a new deformation gradient tensor which we call called $\tilde{\mathbf{F}}_q$. The polar decomposition of $\tilde{\mathbf{F}}_q$ can also be written as $\boxed{\tilde{\mathbf{F}}_q = \tilde{\mathbf{R}}_q \cdot \tilde{\mathbf{U}}_q}$ or as $\boxed{\tilde{\mathbf{F}}_q = \tilde{\mathbf{V}}_q \cdot \tilde{\mathbf{R}}_q}$.

We seek to compare $\tilde{\mathbf{F}}$ to $\tilde{\mathbf{F}}_q$, $\tilde{\mathbf{U}}$ to $\tilde{\mathbf{U}}_q$ and $\tilde{\mathbf{V}}$ to $\tilde{\mathbf{V}}_q$ to see the effect of the rigid body rotation on these tensors.

4.1 Deformation tensors transformation ($\tilde{\mathbf{F}}, \tilde{\mathbf{U}}$ and $\tilde{\mathbf{V}}$) due to rigid body rotation $\tilde{\mathbf{Q}}$

4.1.1 Transformation of $\tilde{\mathbf{F}}$ (the deformation gradient tensor)

From the above diagram we see that $\boxed{\tilde{\mathbf{F}}_q = \tilde{\mathbf{Q}} \cdot \tilde{\mathbf{F}}}$

4.1.2 Transformation of $\tilde{\mathbf{U}}$ (the stretch before rotation $\tilde{\mathbf{R}}$ tensor)

We know that

$$\tilde{\mathbf{U}} = \left(\tilde{\mathbf{F}}^T \cdot \tilde{\mathbf{F}}\right)^{\frac{1}{2}} \tag{1}$$

Similarly,

$$\tilde{\mathbf{U}}_q = \left(\tilde{\mathbf{F}}_q^T \cdot \tilde{\mathbf{F}}_q\right)^{\frac{1}{2}}$$

But $\tilde{\mathbf{F}}_q = \tilde{\mathbf{Q}} \cdot \tilde{\mathbf{F}}$, hence the above becomes

$$\tilde{\mathbf{U}}_q = \left(\left(\tilde{\mathbf{Q}} \cdot \tilde{\mathbf{F}}\right)^T \cdot \left(\tilde{\mathbf{Q}} \cdot \tilde{\mathbf{F}}\right)\right)^{\frac{1}{2}}$$

But from linear algebra we can write $\left(\tilde{\mathbf{Q}} \cdot \tilde{\mathbf{F}}\right)^T = \tilde{\mathbf{F}}^T \cdot \tilde{\mathbf{Q}}^T$ hence the above becomes

$$\tilde{\mathbf{U}}_q = \left(\left(\tilde{\mathbf{F}}^T \cdot \tilde{\mathbf{Q}}^T\right) \cdot \left(\tilde{\mathbf{Q}} \cdot \tilde{\mathbf{F}}\right)\right)^{\frac{1}{2}}$$

But $\tilde{\mathbf{Q}}^T \cdot \tilde{\mathbf{Q}} = \tilde{\mathbf{I}}$ since $\tilde{\mathbf{Q}}$ is orthogonal, hence the above becomes

$$\tilde{\mathbf{U}}_q = \left(\tilde{\mathbf{F}}^T \cdot \tilde{\mathbf{F}}\right)^{\frac{1}{2}} \tag{2}$$

Now compare (1) and (2) we see they are the same. Hence

$$\boxed{\tilde{\mathbf{U}}_q = \tilde{\mathbf{U}}}$$

This means that

$$\boxed{\tilde{\mathbf{U}} \text{ does not change under pure rigid rotation}}.$$

4.1.3 Transformation of $\tilde{\mathbf{V}}$ (The stretch after rotation $\tilde{\mathbf{R}}$ tensor)

Since

$$\tilde{\mathbf{F}}_q = \tilde{\mathbf{Q}} \cdot \tilde{\mathbf{F}} \tag{1}$$

and $\tilde{\mathbf{F}} = \tilde{\mathbf{R}} \cdot \tilde{\mathbf{U}}$ by polar decomposition on $\tilde{\mathbf{F}}$, hence the above can be written as

$$\tilde{\mathbf{F}}_q = \tilde{\mathbf{Q}} \cdot \left(\tilde{\mathbf{R}} \cdot \tilde{\mathbf{U}} \right)$$

Apply polar decomposition on $\tilde{\mathbf{F}}_q$ to obtain $\tilde{\mathbf{F}}_q = \tilde{\mathbf{R}}_q \cdot \tilde{\mathbf{U}}_q$ hence the above becomes

$$\tilde{\mathbf{R}}_q \cdot \tilde{\mathbf{U}}_q = \tilde{\mathbf{Q}} \cdot \tilde{\mathbf{R}} \cdot \tilde{\mathbf{U}}$$

But we found earlier that $\tilde{\mathbf{U}}_q = \tilde{\mathbf{U}}$ hence the above becomes

$$\begin{aligned} \tilde{\mathbf{R}}_q \cdot \tilde{\mathbf{U}} &= \tilde{\mathbf{Q}} \cdot \tilde{\mathbf{R}} \cdot \tilde{\mathbf{U}} \\ \tilde{\mathbf{R}}_q &= \tilde{\mathbf{Q}} \cdot \tilde{\mathbf{R}} \end{aligned} \tag{2}$$

Now we utilize the second form of polar decomposition on $\tilde{\mathbf{F}}_q$ and write

$$\tilde{\mathbf{F}}_q = \tilde{\mathbf{V}}_q \cdot \tilde{\mathbf{R}}_q$$

Substitute (2) into the above equation, it becomes

$$\tilde{\mathbf{F}}_q = \tilde{\mathbf{V}}_q \cdot \left(\tilde{\mathbf{Q}} \cdot \tilde{\mathbf{R}} \right)$$

Substitute (1) into the above we obtain

$$\tilde{\mathbf{Q}} \cdot \tilde{\mathbf{F}} = \tilde{\mathbf{V}}_q \cdot \tilde{\mathbf{Q}} \cdot \tilde{\mathbf{R}}$$

Since $\tilde{\mathbf{Q}} \cdot \tilde{\mathbf{R}}$ is invertible (check), we can write the above as

$$\tilde{\mathbf{V}}_q = \tilde{\mathbf{Q}} \cdot \tilde{\mathbf{F}} \cdot \left(\tilde{\mathbf{Q}} \cdot \tilde{\mathbf{R}} \right)^{-1}$$

But $\left(\tilde{\mathbf{Q}} \cdot \tilde{\mathbf{R}} \right)^{-1} = \tilde{\mathbf{R}}^T \cdot \tilde{\mathbf{Q}}^T$ (Since $\tilde{\mathbf{Q}} \cdot \tilde{\mathbf{R}}$ is an orthogonal matrix, check). Hence the above becomes

$$\tilde{\mathbf{V}}_q = \tilde{\mathbf{Q}} \cdot \tilde{\mathbf{F}} \cdot \tilde{\mathbf{R}}^T \cdot \tilde{\mathbf{Q}}^T \tag{3}$$

But from polar decomposition we know that $\tilde{\mathbf{F}} = \tilde{\mathbf{V}} \cdot \tilde{\mathbf{R}}$, hence $\tilde{\mathbf{V}} = \tilde{\mathbf{F}} \cdot \tilde{\mathbf{R}}^{-1}$, but $\tilde{\mathbf{R}}^{-1} = \tilde{\mathbf{R}}^T$ since it is an orthogonal matrix, hence

$$\tilde{\mathbf{V}} = \tilde{\mathbf{F}} \cdot \tilde{\mathbf{R}}^T \tag{4}$$

Substitute (4) in (3) we obtain

$$\boxed{\tilde{\mathbf{V}}_q = \tilde{\mathbf{Q}} \cdot \tilde{\mathbf{V}} \cdot \tilde{\mathbf{Q}}^T}$$

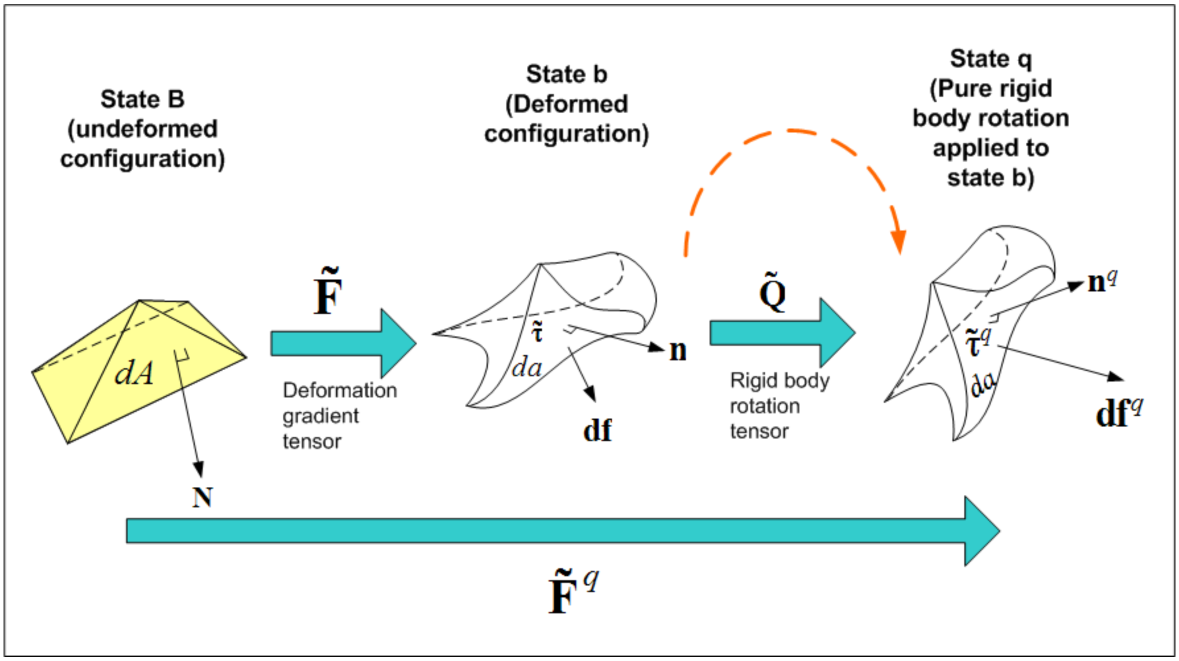
This above is how $\tilde{\mathbf{V}}$ transforms due to rigid rotation $\tilde{\mathbf{Q}}$.

Now that we have obtained the transformation of $\tilde{\mathbf{F}}, \tilde{\mathbf{U}}$ and $\tilde{\mathbf{V}}$ we proceed to find how each one of the stress tensors derived earlier transforms due to $\tilde{\mathbf{Q}}$.

4.2 Stress tensors transformation due to rigid body rotation $\tilde{\mathbf{Q}}$

4.2.1 Transformation of stress tensor $\tilde{\tau}$ (Cauchy stress tensor)

We now calculate the stress $\tilde{\tau}_q$ (Cauchy stress in state q) at the point b_q. Since this is a rigid body rotation, then the area da will not change, only the unit normal vector $\mathbf{n}$ will change to $\mathbf{n}_q$.



The tensor $\tilde{\mathbf{Q}}$ maps the vector $\mathbf{df}$ to the vector $\mathbf{df}_q$

$$\mathbf{df}_q = \tilde{\mathbf{Q}} \cdot \mathbf{df} \tag{1}$$

But in state b (the deformed state), the Cauchy stress tensor is given by

$$\mathbf{df} = (da\ \mathbf{n}) \cdot \tilde{\tau} \tag{2}$$

Substitute (2) into (1)

$$\mathbf{df}_q = \tilde{\mathbf{Q}} \cdot (da\ \mathbf{n}) \cdot \tilde{\tau}$$

Exchange the order of $\tilde{\tau}$ and $da\ \mathbf{n}$, hence use the transpose of $\tilde{\tau}$

$$\mathbf{df}_q = \tilde{\mathbf{Q}} \cdot \tilde{\tau}^T \cdot (da\ \mathbf{n}) \tag{3}$$

However since the tensor $\tilde{\mathbf{Q}}$ maps the oriented area $(da\ \mathbf{n})$ to the oriented area $(da\ \mathbf{n}_q)$ then we write

$$(da\ \mathbf{n}_q) = \tilde{\mathbf{Q}} \cdot (da\ \mathbf{n})$$

Or in other words

$$\tilde{\mathbf{Q}}^T \cdot (da\ \mathbf{n}_q) = da\ \mathbf{n} \tag{4}$$

Substitute (4) into (3)

$$\begin{aligned} \mathbf{df}_q &= \tilde{\mathbf{Q}} \cdot \tilde{\tau}^T \cdot \tilde{\mathbf{Q}}^T \cdot (da\ \mathbf{n}_q) \\ &= (da\ \mathbf{n}_q) \cdot \tilde{\mathbf{Q}} \cdot \tilde{\tau} \cdot \tilde{\mathbf{Q}}^T \end{aligned} \tag{5}$$

However, the stress $\tilde{\tau}_q$ in state q is given by $\mathbf{df}_q = (da\ \mathbf{n}_q) \cdot \tilde{\tau}_q$, hence the above equation becomes

$$(da\ \mathbf{n}_q) \cdot \tilde{\tau}_q = (da\ \mathbf{n}_q) \cdot \tilde{\mathbf{Q}} \cdot \tilde{\tau} \cdot \tilde{\mathbf{Q}}^T$$

In other words

$$\boxed{\tilde{\tau}_q = \tilde{\mathbf{Q}} \cdot \tilde{\tau} \cdot \tilde{\mathbf{Q}}^T}$$

This implies

the true stress tensor has changed in the deformed body subjected to pure rigid rotation.

By comparing the above transformation result with the deformation tensors transformation, we see that the

true Cauchy stress $\tilde{\tau}$ transforms similarly to the tensor $\tilde{\mathbf{V}}$

4.2.2 Transformation of first Piola-Kirchhoff stress tensor

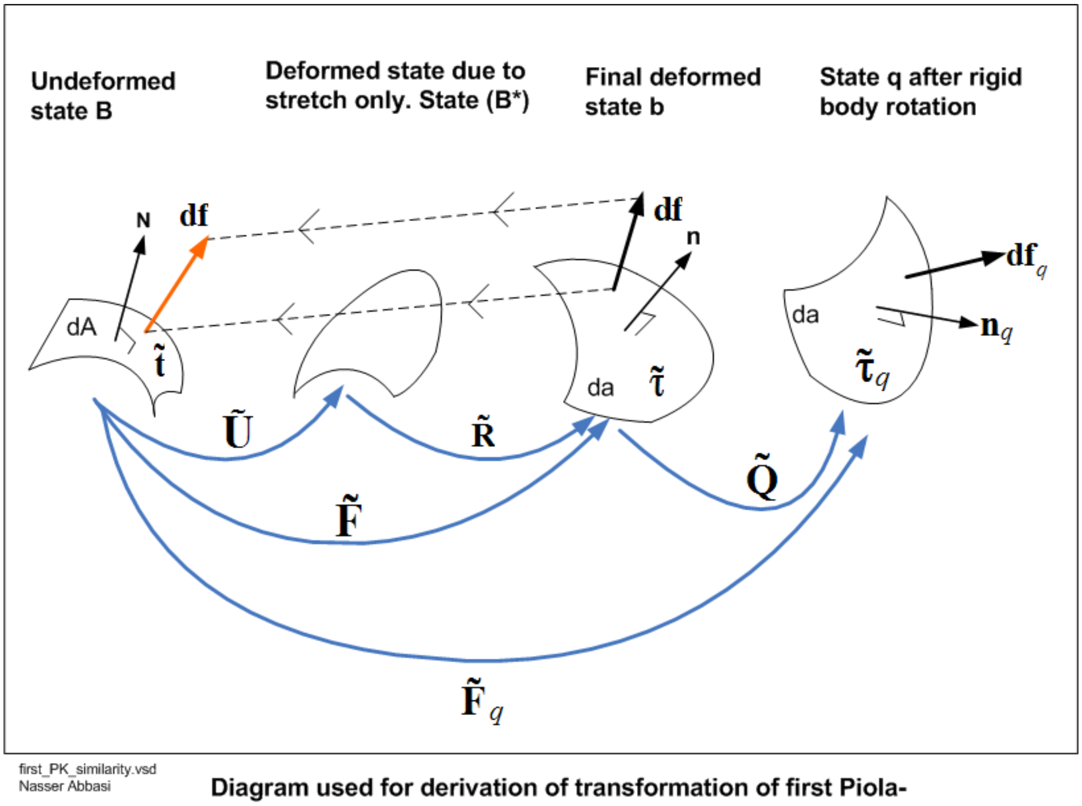


Diagram used for derivation of transformation of first Piola-Kirchhoff stress tensor under rigid body rotation

The transformation of the first Piola-Kirchhoff stress tensor $\tilde{\mathbf{t}}$ is shown below.

Earlier we have shown that

$$\tilde{\mathbf{t}} = J\tilde{\mathbf{F}}^{-1}\cdot\tilde{\tau}$$

This implies

$$\tilde{\mathbf{t}}_q = J\ \tilde{\mathbf{F}}_q^{-1}\cdot\tilde{\tau}_q$$

However, we have found earlier that $\tilde{\tau}_q = \tilde{\mathbf{Q}}\cdot\tilde{\tau}\cdot\tilde{\mathbf{Q}}^T$ then the above becomes

$$\tilde{\mathbf{t}}_q = J\ \tilde{\mathbf{F}}_q^{-1}\cdot\tilde{\mathbf{Q}}\cdot\tilde{\tau}\cdot\tilde{\mathbf{Q}}^T \tag{1}$$

Now we need to find an expression for $\tilde{\mathbf{F}}_q^{-1}$.

We know that $\tilde{\mathbf{F}}_q = \tilde{\mathbf{Q}}\cdot\tilde{\mathbf{F}}$, hence $\tilde{\mathbf{F}}_q^{-1} = \left(\tilde{\mathbf{Q}}\cdot\tilde{\mathbf{F}}\right)^{-1}$, hence $\tilde{\mathbf{F}}_q^{-1} = \tilde{\mathbf{F}}^{-1}\cdot\tilde{\mathbf{Q}}^{-1}$ but since $\tilde{\mathbf{Q}}$ is orthogonal, then $\tilde{\mathbf{Q}}^{-1} = \tilde{\mathbf{Q}}^T$, hence

$$\tilde{\mathbf{F}}_q^{-1} = \tilde{\mathbf{F}}^{-1}\cdot\tilde{\mathbf{Q}}^T$$

Now substitute the above in eq (1), we obtain

$$\begin{aligned}\tilde{\mathbf{t}}_q &= J\ \tilde{\mathbf{F}}^{-1}\cdot\tilde{\mathbf{Q}}^T\cdot\tilde{\mathbf{Q}}\cdot\tilde{\tau}\cdot\tilde{\mathbf{Q}}^T\\ &= J\ \tilde{\mathbf{F}}^{-1}\cdot\tilde{\tau}\cdot\tilde{\mathbf{Q}}^T\end{aligned}$$

However, since $\tilde{\mathbf{t}} = J\tilde{\mathbf{F}}^{-1}\cdot\tilde{\tau}$ then the above simplifies to

$$\tilde{\mathbf{t}}_q = \tilde{\mathbf{t}}\cdot\tilde{\mathbf{Q}}^T$$

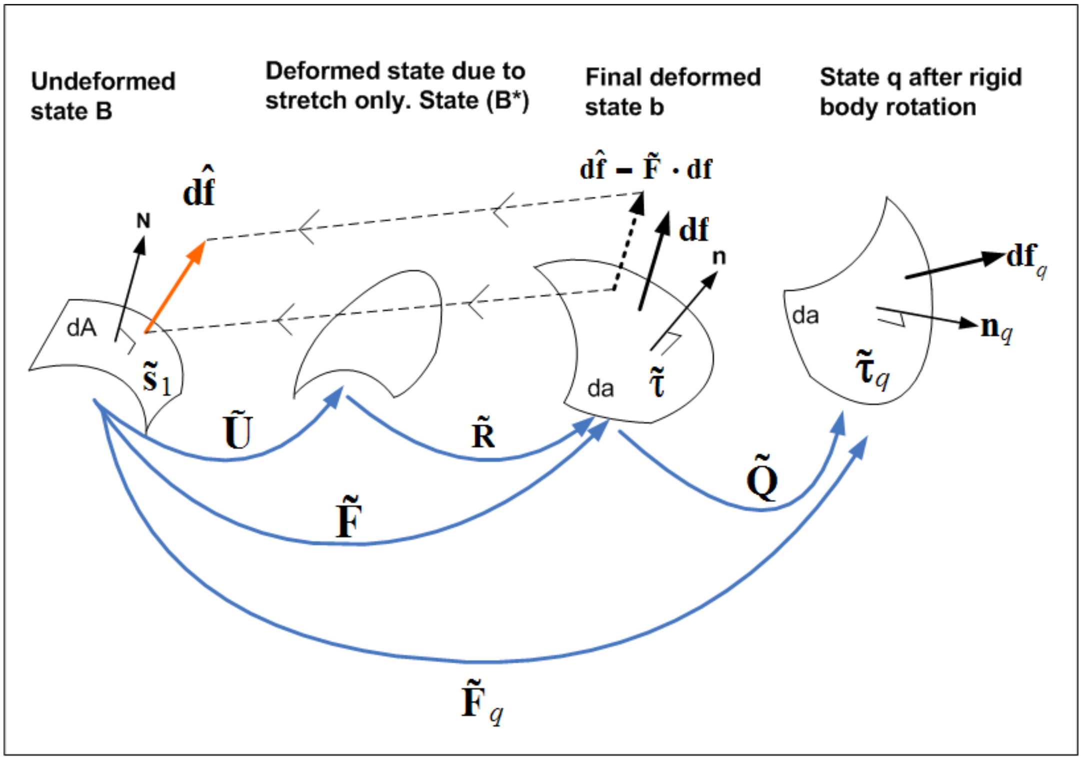
Hence

$$\tilde{\mathbf{t}}_q = \tilde{\mathbf{Q}}\cdot\tilde{\mathbf{t}}$$

Now, by examining how the geometrical tensors transform, we saw before that $\tilde{\mathbf{F}}_q = \tilde{\mathbf{Q}}\cdot\tilde{\mathbf{F}}$ hence

$$\boxed{\tilde{\mathbf{t}}\text{ transforms similarly to }\tilde{\mathbf{F}}}.$$

4.2.3 Transformation of second Piola-Kirchhoff stress tensor



second_PK_similarity.vsd
Nasser Abbasi

Diagram used for derivation of transformation of second Piola-Kirchhoff stress tensor under rigid body rotation

The transformation of the second Piola-Kirchhoff stress tensor $\tilde{\mathbf{s}}_1$ is shown below.

Earlier we have shown that

$$\tilde{\mathbf{s}}_1 = J\ \tilde{\mathbf{F}}^{-1} \cdot \tilde{\tau} \cdot \tilde{\mathbf{F}}^{-T}$$

Hence

$$\tilde{\mathbf{s}}_{1_q} = J\ \tilde{\mathbf{F}}_q^{-1} \cdot \tilde{\tau}_q \cdot \tilde{\mathbf{F}}_q^{-T}$$

We need to find an expression for $\tilde{\mathbf{F}}_q^{-1}$. We know that $\tilde{\mathbf{F}}_q = \tilde{\mathbf{Q}} \cdot \tilde{\mathbf{F}}$, hence $\tilde{\mathbf{F}}_q^{-1} = \left(\tilde{\mathbf{Q}} \cdot \tilde{\mathbf{F}}\right)^{-1}$, hence $\tilde{\mathbf{F}}_q^{-1} = \tilde{\mathbf{F}}^{-1} \cdot \tilde{\mathbf{Q}}^{-1}$ but since $\tilde{\mathbf{Q}}$ is orthogonal, then $\tilde{\mathbf{Q}}^{-1} = \tilde{\mathbf{Q}}^T$, hence $\boxed{\tilde{\mathbf{F}}_q^{-1} = \tilde{\mathbf{F}}^{-1} \cdot \tilde{\mathbf{Q}}^T}$. So the above equation becomes

$$\tilde{\mathbf{s}}_{1_q} = J\ \left(\tilde{\mathbf{F}}^{-1} \cdot \tilde{\mathbf{Q}}^T\right) \cdot \tilde{\tau}_q \cdot \tilde{\mathbf{F}}_q^{-T}$$

$\tilde{\mathbf{F}}_q^{-T} = \left(\tilde{\mathbf{F}}_q^{-1}\right)^T$ hence $\tilde{\mathbf{F}}_q^{-T} = \left(\tilde{\mathbf{F}}^{-1} \cdot \tilde{\mathbf{Q}}^T\right)^T$ then $\boxed{\tilde{\mathbf{F}}_q^{-T} = \tilde{\mathbf{Q}} \cdot \tilde{\mathbf{F}}^{-T}}$ hence the above equation becomes

$$\tilde{\mathbf{s}}_{1_q} = J \left(\tilde{\mathbf{F}}^{-1} \cdot \tilde{\mathbf{Q}}^T\right) \cdot \tilde{\tau}_q \cdot \left(\tilde{\mathbf{Q}} \cdot \tilde{\mathbf{F}}^{-T}\right)$$

But we found earlier that $\tilde{\tau}_q = \tilde{\mathbf{Q}} \cdot \tilde{\tau} \cdot \tilde{\mathbf{Q}}^T$ hence the above equation becomes

$$\begin{aligned}\tilde{\mathbf{s}}_{1_q} &= J \left(\tilde{\mathbf{F}}^{-1} \cdot \tilde{\mathbf{Q}}^T\right) \cdot \left(\tilde{\mathbf{Q}} \cdot \tilde{\tau} \cdot \tilde{\mathbf{Q}}^T\right) \cdot \left(\tilde{\mathbf{Q}} \cdot \tilde{\mathbf{F}}^{-T}\right) \\ &= J \, \tilde{\mathbf{F}}^{-1} \cdot \tilde{\tau} \cdot \tilde{\mathbf{F}}^{-T}\end{aligned}$$

So

$$\boxed{\tilde{\mathbf{s}}_{1_q} = J \, \tilde{\mathbf{F}}^{-1} \cdot \tilde{\tau} \cdot \tilde{\mathbf{F}}^{-T}}$$

But this is the same as $\tilde{\mathbf{s}}_1$ hence

$$\boxed{\tilde{\mathbf{s}}_{1_q} = \tilde{\mathbf{s}}_1}$$

Now since we found earlier that $\tilde{\mathbf{U}}_q = \tilde{\mathbf{U}}$ hence we conclude that

$$\boxed{\tilde{\mathbf{s}}_1 \text{ transforms similarly to } \tilde{\mathbf{U}}.}$$

4.2.4 Transformation of Kirchhoff stress tensor $\tilde{\sigma}$

Since $\tilde{\sigma}$ is a scalar multiple of $\tilde{\tau}$ and we have found that $\tilde{\tau}$ is a conjugate pair with $\tilde{\mathbf{V}}$ hence we conclude that

$$\boxed{\tilde{\sigma} \text{ transforms similarly to } \tilde{\mathbf{V}}}$$

4.2.5 Transformation of $\tilde{\mathbf{\Gamma}}$ stress tensor

The transformation of the second $\tilde{\mathbf{\Gamma}}$ stress tensor is shown below.

Earlier we have shown that

$$\tilde{\mathbf{\Gamma}} = \tilde{\mathbf{R}}^T \cdot \tilde{\tau} \cdot \tilde{\mathbf{R}}$$

Hence

$$\tilde{\mathbf{\Gamma}}_q = \tilde{\mathbf{R}}_q^T \cdot \tilde{\tau}_q \cdot \tilde{\mathbf{R}}_q$$

Since $\tilde{\tau}_q = \tilde{\mathbf{Q}} \cdot \tilde{\tau} \cdot \tilde{\mathbf{Q}}^T$ then the above becomes

$$\tilde{\mathbf{\Gamma}}_q = \tilde{\mathbf{R}}_q^T \cdot \left(\tilde{\mathbf{Q}} \cdot \tilde{\tau} \cdot \tilde{\mathbf{Q}}^T\right) \cdot \tilde{\mathbf{R}}_q \tag{1}$$

But $\tilde{\mathbf{F}}_q = \tilde{\mathbf{Q}} \cdot \tilde{\mathbf{F}}$ and also by polar decomposition we can write $\tilde{\mathbf{F}}_q = \tilde{\mathbf{R}}_q \cdot \tilde{\mathbf{U}}_q$ hence

$$\begin{aligned} \tilde{\mathbf{Q}} \cdot \tilde{\mathbf{F}} &= \tilde{\mathbf{R}}_q \cdot \tilde{\mathbf{U}}_q \\ \tilde{\mathbf{Q}} &= \tilde{\mathbf{R}}_q \cdot \tilde{\mathbf{U}}_q \cdot \tilde{\mathbf{F}}^{-1} \end{aligned}$$

But $\tilde{\mathbf{F}} = \tilde{\mathbf{R}} \cdot \tilde{\mathbf{U}}$ hence $\tilde{\mathbf{F}}^{-1} = \left(\tilde{\mathbf{R}} \cdot \tilde{\mathbf{U}}\right)^{-1} = \tilde{\mathbf{U}}^{-1} \cdot \tilde{\mathbf{R}}^{-1}$, the above becomes

$$\tilde{\mathbf{Q}} = \tilde{\mathbf{R}}_q \cdot \tilde{\mathbf{U}}_q \cdot \left(\tilde{\mathbf{U}}^{-1} \cdot \tilde{\mathbf{R}}^{-1}\right)$$

But since $\tilde{\mathbf{U}}_q = \tilde{\mathbf{U}}$ then the above becomes

$$\begin{aligned} \tilde{\mathbf{Q}} &= \tilde{\mathbf{R}}_q \cdot \overbrace{\tilde{\mathbf{U}} \cdot \tilde{\mathbf{U}}^{-1}} \cdot \tilde{\mathbf{R}}^{-1} \\ &= \tilde{\mathbf{R}}_q \cdot \tilde{\mathbf{R}}^{-1} \end{aligned}$$

But $\tilde{\mathbf{R}}^{-1} = \tilde{\mathbf{R}}^T$ hence the above becomes

$$\tilde{\mathbf{Q}} = \tilde{\mathbf{R}}_q \cdot \tilde{\mathbf{R}}^T \tag{2}$$

And hence

$$\begin{aligned} \tilde{\mathbf{Q}}^T &= \left(\tilde{\mathbf{R}}_q \cdot \tilde{\mathbf{R}}^T\right)^T \\ &= \tilde{\mathbf{R}} \cdot \tilde{\mathbf{R}}_q^T \end{aligned} \tag{3}$$

Substitute (2) and (3) into (1) we obtain

$$\tilde{\mathbf{\Gamma}}_q = \tilde{\mathbf{R}}_q^T \cdot \left(\tilde{\mathbf{R}}_q \cdot \tilde{\mathbf{R}}^T\right) \cdot \tilde{\tau} \cdot \left(\tilde{\mathbf{R}} \cdot \tilde{\mathbf{R}}_q^T\right) \cdot \tilde{\mathbf{R}}_q$$

Since $\tilde{\mathbf{R}}$ and $\tilde{\mathbf{R}}_q^T$ are orthogonal, then the above reduces to

$$\tilde{\mathbf{\Gamma}}_q = \tilde{\mathbf{R}}^T \cdot \tilde{\tau} \cdot \tilde{\mathbf{R}}$$

But $\tilde{\mathbf{\Gamma}} = \tilde{\mathbf{R}}^T \cdot \tilde{\tau} \cdot \tilde{\mathbf{R}}$ hence we conclude that

$$\boxed{\tilde{\mathbf{\Gamma}}_q = \tilde{\mathbf{\Gamma}}}$$

$$\boxed{\tilde{\mathbf{\Gamma}} \text{ transforms similarly to } \tilde{\mathbf{U}}}$$

4.2.6 Transformation of Biot-Lure stress tensor $\tilde{\mathbf{r}}^*$

The transformation of the Biot-Lure stress tensor $\tilde{\mathbf{r}}^*$ is shown below.

Earlier we have shown that

$$\tilde{\mathbf{r}}^* = J\ \tilde{\mathbf{F}}^{-1} \cdot \tilde{\tau} \cdot \tilde{\mathbf{R}} \tag{1}$$

Hence

$$\tilde{\mathbf{r}}_q^* = J\ \tilde{\mathbf{F}}_q^{-1} \cdot \tilde{\tau}_q \cdot \tilde{\mathbf{R}}_q$$

But $\tilde{\tau}_q = \tilde{\mathbf{Q}} \cdot \tilde{\tau} \cdot \tilde{\mathbf{Q}}^T$ hence the above becomes

$$\tilde{\mathbf{r}}_q^* = J\ \tilde{\mathbf{F}}_q^{-1} \cdot \tilde{\mathbf{Q}} \cdot \tilde{\tau} \cdot \tilde{\mathbf{Q}}^T \cdot \tilde{\mathbf{R}}_q \tag{2}$$

Since $\tilde{\mathbf{F}}_q = \tilde{\mathbf{Q}} \cdot \tilde{\mathbf{F}}$ then $\tilde{\mathbf{F}}_q^{-1} = \left(\tilde{\mathbf{Q}} \cdot \tilde{\mathbf{F}}\right)^{-1} = \tilde{\mathbf{F}}^{-1} \cdot \tilde{\mathbf{Q}}^{-1}$ but $\tilde{\mathbf{Q}}$ is orthogonal, hence $\tilde{\mathbf{Q}}^{-1} = \tilde{\mathbf{Q}}^T$, hence $\tilde{\mathbf{F}}_q^{-1} = \tilde{\mathbf{F}}^{-1} \cdot \tilde{\mathbf{Q}}^T$ hence eq (2) can be written as

$$\tilde{\mathbf{r}}_q^* = J\ \left(\tilde{\mathbf{F}}^{-1} \cdot \tilde{\mathbf{Q}}^T\right) \cdot \tilde{\mathbf{Q}} \cdot \tilde{\tau} \cdot \tilde{\mathbf{Q}}^T \cdot \tilde{\mathbf{R}}_q \tag{3}$$

Now we seek to resolve $\tilde{\mathbf{R}}_q$.

We know that $\tilde{\mathbf{F}}_q = \tilde{\mathbf{Q}} \cdot \tilde{\mathbf{F}}$ and also by polar decomposition $\tilde{\mathbf{F}}_q = \tilde{\mathbf{R}}_q \cdot \tilde{\mathbf{U}}_q$ hence we can write

$$\begin{aligned} \tilde{\mathbf{R}}_q \cdot \tilde{\mathbf{U}}_q &= \tilde{\mathbf{Q}} \cdot \tilde{\mathbf{F}} \\ \tilde{\mathbf{R}}_q &= \tilde{\mathbf{Q}} \cdot \tilde{\mathbf{F}} \cdot \tilde{\mathbf{U}}_q^{-1} \end{aligned} \tag{4}$$

Substitute (4) into (3) we obtain

$$\begin{aligned} \tilde{\mathbf{r}}_q^* &= J\ \left(\tilde{\mathbf{F}}^{-1} \cdot \tilde{\mathbf{Q}}^T\right) \cdot \tilde{\mathbf{Q}} \cdot \tilde{\tau} \cdot \tilde{\mathbf{Q}}^T \cdot \left(\tilde{\mathbf{Q}} \cdot \tilde{\mathbf{F}} \cdot \tilde{\mathbf{U}}_q^{-1}\right) \\ &= J\ \tilde{\mathbf{F}}^{-1} \cdot \overbrace{\tilde{\mathbf{Q}}^T \cdot \tilde{\mathbf{Q}}} \cdot \tilde{\tau} \cdot \overbrace{\tilde{\mathbf{Q}}^T \cdot \tilde{\mathbf{Q}}} \cdot \tilde{\mathbf{F}} \cdot \tilde{\mathbf{U}}_q^{-1} \\ &= J\ \tilde{\mathbf{F}}^{-1} \cdot \tilde{\tau} \cdot \tilde{\mathbf{F}} \cdot \tilde{\mathbf{U}}_q^{-1} \end{aligned}$$

But since $\tilde{\mathbf{F}} = \tilde{\mathbf{R}} \cdot \tilde{\mathbf{U}}$ then the above becomes

$$
\begin{aligned}
\tilde{\mathbf{r}}_q^* &= J\ \tilde{\mathbf{F}}^{-1}\cdot\tilde{\tau}\cdot \overbrace{\tilde{\mathbf{F}}}^{\tilde{\mathbf{R}}\cdot\tilde{\mathbf{U}}} \cdot\tilde{\mathbf{U}}_q^{-1} \\
&= J\ \tilde{\mathbf{F}}^{-1}\cdot\tilde{\tau}\cdot\tilde{\mathbf{R}}\cdot\tilde{\mathbf{U}}\cdot\tilde{\mathbf{U}}_q^{-1}
\end{aligned}
$$

But we found earlier that $\tilde{\mathbf{U}} = \tilde{\mathbf{U}}_q$ hence the above becomes

$$
\begin{aligned}
\tilde{\mathbf{r}}_q^* &= J\ \tilde{\mathbf{F}}^{-1}\cdot\tilde{\tau}\cdot\tilde{\mathbf{R}}\cdot\overbrace{\tilde{\mathbf{U}}\cdot\tilde{\mathbf{U}}^{-1}} \\
&= J\ \tilde{\mathbf{F}}^{-1}\cdot\tilde{\tau}\cdot\tilde{\mathbf{R}}
\end{aligned}
$$

But from eq (1) we see that

$$\boxed{\tilde{\mathbf{r}}_q^* = \tilde{\mathbf{r}}^*}$$

Hence since $\tilde{\mathbf{U}}_q = \tilde{\mathbf{U}}$, we conclude that

$$\boxed{\tilde{\mathbf{r}}^* \text{ transforms similarly to } \tilde{\mathbf{U}}}$$

4.2.7 Transformation of Juamann stress tensor $\tilde{\mathbf{r}}$

The transformation of the Juamann stress tensor $\tilde{\mathbf{r}}$ is shown below.

Earlier we have shown that

$$\tilde{\mathbf{r}} = \frac{\left(\tilde{\mathbf{r}}^{*} + \tilde{\mathbf{r}}^{*T}\right)}{2} \tag{1}$$

From (1) we see that

$$\tilde{\mathbf{r}}_q = \frac{\left(\tilde{\mathbf{r}}_q^{*} + \tilde{\mathbf{r}}_q^{*T}\right)}{2}$$

But since we found that $\tilde{\mathbf{r}}_q^{*} = \tilde{\mathbf{r}}^{*}$ then the above becomes

$$\tilde{\mathbf{r}}_q = \frac{\left(\tilde{\mathbf{r}}^{*} + \tilde{\mathbf{r}}^{*T}\right)}{2}$$

Hence

$$\boxed{\tilde{\mathbf{r}}_q = \tilde{\mathbf{r}}}$$

Hence since $\tilde{\mathbf{U}}_q = \tilde{\mathbf{U}}$, we conclude that

$$\boxed{\tilde{\mathbf{r}} \text{ transforms similarly to } \tilde{\mathbf{U}}}$$

4.2.8 Transformation of $\tilde{\mathbf{T}}^*$ stress tensor

We found earlier that

$$\tilde{\mathbf{T}}^* = J\ \tilde{\mathbf{V}}^{-1} \cdot\ \tilde{\tau}$$

Hence

$$\tilde{\mathbf{T}}_q^* = J\ \tilde{\mathbf{V}}_q^{-1} \cdot\ \tilde{\tau}_q$$

But $\tilde{\tau}_q = \tilde{\mathbf{Q}} \cdot \tilde{\tau} \cdot \tilde{\mathbf{Q}}^T$ and $\tilde{\mathbf{V}}_q = \tilde{\mathbf{Q}} \cdot \tilde{\mathbf{V}} \cdot \tilde{\mathbf{Q}}^T$ then the above becomes

$$\begin{aligned}
\tilde{\mathbf{T}}_q^* &= J\ \left(\tilde{\mathbf{Q}} \cdot \tilde{\mathbf{V}} \cdot \tilde{\mathbf{Q}}^T\right)^{-1} \cdot\ \left(\tilde{\mathbf{Q}} \cdot \tilde{\tau} \cdot \tilde{\mathbf{Q}}^T\right) \\
&= J\ \tilde{\mathbf{Q}}^{-T} \cdot \left(\tilde{\mathbf{Q}} \cdot \tilde{\mathbf{V}}\right)^{-1} \cdot \tilde{\mathbf{Q}} \cdot \tilde{\tau} \cdot \tilde{\mathbf{Q}}^T \\
&= J\ \tilde{\mathbf{Q}}^{-T} \cdot \tilde{\mathbf{V}}^{-1} \cdot \overbrace{\tilde{\mathbf{Q}}^{-1} \cdot\ \tilde{\mathbf{Q}}} \cdot \tilde{\tau} \cdot \tilde{\mathbf{Q}}^T \\
&= J\ \tilde{\mathbf{Q}}^{-T} \cdot \tilde{\mathbf{V}}^{-1} \cdot \tilde{\tau} \cdot \tilde{\mathbf{Q}}^T \\
&= J\ \tilde{\mathbf{V}}^{-1} \cdot\ \tilde{\tau} \\
&= \tilde{\mathbf{T}}^*
\end{aligned}$$

Hence

$$\boxed{\tilde{\mathbf{T}}_q^* = \tilde{\mathbf{T}}^*}$$

Hence

$$\boxed{\tilde{\mathbf{T}}^* \text{ is conjugate pair with } \tilde{\mathbf{U}}}$$

4.2.9 Transformation of $\tilde{\mathbf{T}}$ stress tensor

Since $\tilde{\mathbf{T}} = \frac{\left(\tilde{\mathbf{T}}^* + \tilde{\mathbf{T}}^{*T}\right)}{2}$ and $\tilde{\mathbf{T}}^*$ is conjugate pair with $\tilde{\mathbf{U}}$ then

$$\boxed{\tilde{\mathbf{T}}^* \text{ is conjugate pair with } \tilde{\mathbf{U}}}$$

5 Constitutive Equations using conjugate pairs for non-linear elastic materials with large deformations: Hyper-elasticity

In formulating the constitutive relation for a material we seek a formula that relates the stress measure to the strain measure. Hence, using a specific stress measure, we need to use the correct strain measure.

Hence the problem at hand is the following: Given a stress tensor, one of the many stress tensors we discussed earlier, how can we determine the correct strain tensor to use with it?

To make the discussion general, we designate the stress tensor by $\tilde{\mathbf{B}}$ and its conjugate pair, the strain tensor, by $\tilde{\mathbf{A}}$.

Hence $\tilde{\mathbf{B}}$ could be any of the stress measures discussed earlier, such as the Cauchy stress tensor $\tilde{\tau}$, the second piola-kirchhoff stress tensor $\tilde{\mathbf{s}}_1$, etc..., we will now determine the strain tensor to use. We call $\left(\tilde{\mathbf{B}}, \tilde{\mathbf{A}}\right)$ the conjugate pair tensors.

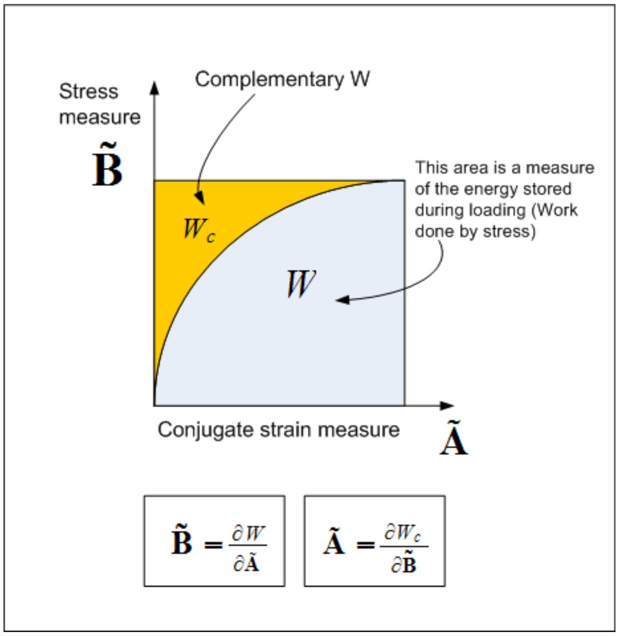
To guide us in finding $\tilde{\mathbf{A}}$ for each specific $\tilde{\mathbf{B}}$ we will be guided by physics.

If we refer to W as the current amount of energy stored in a unit volume as a result of the body undergoing deformation, then the time rate at which this energy changes will be equal to the stress times the strain rate. Hence we write

$$\boxed{\dot{W} = \tilde{\mathbf{B}} : \frac{\partial \tilde{\mathbf{A}}}{\partial t}}$$

Where : is the trace matrix operator. This is the rule by which we will use to determine $\tilde{\mathbf{A}}$.

On a stress-strain diagram we draw the following.



The strain measure $\tilde{\mathbf{A}}$ (the conjugate pair for the stress measure $\tilde{\mathbf{B}}$) must satisfy the relation

$$\frac{\partial W}{\partial t} = \frac{\partial W}{\partial \tilde{\mathbf{A}}} : \frac{\partial \tilde{\mathbf{A}}}{\partial t}$$
$$\dot{W} = \tilde{\mathbf{B}} : \dot{\mathbf{A}}$$

For each stress/strain conjugate pair, we now derive the terms $\frac{\partial W}{\partial t}, \frac{\partial \tilde{\mathbf{A}}}{\partial t}, \tilde{\mathbf{A}}$

5.1 Conjugate pair for Cauchy stress tensor

In the deformed state, the stress tensor is the true stress tensor, which is the cauchy stress $\tilde{\tau}$, and the strain rate in this state is known to be [2]

$$\frac{1}{2}\left(\tilde{\mathbf{e}} + \tilde{\mathbf{e}}^T\right)$$

where $\tilde{\mathbf{e}}$ is the velocity gradient tensor. It is shown in [2] that

$$\boxed{\tilde{\mathbf{e}} = \dot{\mathbf{F}} \cdot \tilde{\mathbf{F}}^{-1}}$$

Hence in the deformed state we can write

$$\dot{W} = \tilde{\tau} \colon \frac{1}{2}\left(\dot{\mathbf{F}} \cdot \tilde{\mathbf{F}}^{-1} + \tilde{\mathbf{F}}^{-T} \cdot \dot{\mathbf{F}}^{-1}\right)$$

In other words, the conjugate strain for the cauchy stress tensor is given by $\tilde{\mathbf{A}}$ such that

$$\frac{\partial \tilde{\mathbf{A}}}{\partial t} = \frac{1}{2}\left(\dot{\mathbf{F}} \cdot \tilde{\mathbf{F}}^{-1} + \tilde{\mathbf{F}}^{-T} \cdot \dot{\mathbf{F}}^{-1}\right)$$

$\tilde{\mathbf{A}}$ should come out to be the Almansi strain tensor, which is

$$\boxed{\tilde{\mathbf{A}} = \tfrac{1}{2}\left(\tilde{\mathbf{F}}^{-T} \cdot \tilde{\mathbf{F}}^{-1} - \tilde{\mathbf{I}}\right)}$$

(check)

5.2 Conjugate pair for second Piola-kirchhoff stress tensor $\tilde{\mathrm{s}}_1$

$$\dot{W} = \tilde{\mathbf{B}} \colon \frac{\partial \tilde{\mathbf{A}}}{\partial t}$$
$$= \tilde{\tau} \colon \tilde{\mathbf{e}}$$

Pre dot multiply $\tilde{\mathbf{e}}$ by $\tilde{\mathbf{I}} = \left(\tilde{\mathbf{F}}^{-T} \cdot \tilde{\mathbf{F}}^{T}\right)$ and post dot multiply it with $\tilde{\mathbf{I}} = \left(\tilde{\mathbf{F}} \cdot \tilde{\mathbf{F}}^{-1}\right)$ which will make no change in the value, hence we obtain

$$\dot{W} = \tilde{\tau} \colon \left(\tilde{\mathbf{F}}^{-\mathbf{T}} \cdot \tilde{\mathbf{F}}^{\mathbf{T}}\right) \cdot \tilde{\mathbf{e}} \cdot \left(\tilde{\mathbf{F}} \cdot \tilde{\mathbf{F}}^{-1}\right)$$

Using the properties of $:$ we can write the above as

$$\dot{W} = \left(\tilde{\mathbf{F}}^{-1} \cdot \tilde{\tau} \cdot \tilde{\mathbf{F}}^{-T}\right) \ : \ \left(\tilde{\mathbf{F}}^{\mathbf{T}} \cdot \tilde{\mathbf{e}} \cdot \tilde{\mathbf{F}}\right)$$

We have determined earlier that $\tilde{\mathbf{s}}_1 = J \ \tilde{\mathbf{F}}^{-1} \cdot \tilde{\tau} \cdot \tilde{\mathbf{F}}^{-T}$ hence $\tilde{\mathbf{F}}^{-1} \cdot \tilde{\tau} \cdot \tilde{\mathbf{F}}^{-\mathbf{T}} = \frac{\tilde{\mathbf{s}}_1}{J}$ hence the above equation becomes

$$\dot{W} = \frac{\tilde{\mathbf{s}}_1}{J} \ : \ \left(\tilde{\mathbf{F}}^{T} \cdot \tilde{\mathbf{e}} \cdot \tilde{\mathbf{F}}\right)$$

But $\tilde{\mathbf{e}} = \dot{\mathbf{F}} \cdot \tilde{\mathbf{F}}^{-1}$ hence the above becomes

$$\dot{W} = \frac{\tilde{\mathbf{s}}_1}{J} \ : \ \left(\tilde{\mathbf{F}}^{\mathbf{T}} \cdot \dot{\mathbf{F}} \cdot \tilde{\mathbf{F}}^{-\mathbf{1}} \cdot \tilde{\mathbf{F}}\right)$$
$$= \tilde{\mathbf{s}}_1 \ : \ \frac{1}{J} \left(\tilde{\mathbf{F}}^{\mathbf{T}} \cdot \dot{\mathbf{F}} \cdot \tilde{\mathbf{F}}^{-\mathbf{1}} \cdot \tilde{\mathbf{F}}\right)$$

Hence

$$\frac{\partial \tilde{\mathbf{A}}}{\partial t} = \frac{1}{J} \left(\tilde{\mathbf{F}}^{T} \cdot \dot{\mathbf{F}} \cdot \tilde{\mathbf{F}}^{-\mathbf{1}} \cdot \tilde{\mathbf{F}}\right)$$

we see that if $\tilde{\mathbf{A}} = \frac{1}{2}\left(\tilde{\mathbf{F}}^{\mathbf{T}} \cdot \tilde{\mathbf{F}} - \tilde{\mathbf{I}}\right)$ then $\frac{\partial \tilde{\mathbf{A}}}{\partial t} = \dot{\mathbf{F}}^{T} \cdot \tilde{\mathbf{F}} + \tilde{\mathbf{F}}^{T} \cdot \dot{\mathbf{F}}$

Hence

$$\boxed{\tilde{\mathbf{A}} = \frac{1}{2J}\left(\tilde{\mathbf{F}}^{\mathbf{T}} \cdot \tilde{\mathbf{F}} - \tilde{\mathbf{I}}\right)}$$

The advantage in using the second Piola Kirchhoff stress tensor instead of the Cauchy or the first Piola Kirchhoff stress tensor, is that with the second Piola Kirchhoff stress tensor, calculations are performed the reference configuration (undeformed state) where the state measurements are known instead of using the deformed configuration where state measurements are not known.

5.3 Conjugate pair for first Piola-kirchhoff stress tensor $\tilde{\mathbf{t}}$

$$\begin{aligned}\dot{W} &= \tilde{\mathbf{B}} \colon \frac{\partial \tilde{\mathbf{A}}}{\partial t} \\ &= \tilde{\tau} \colon \tilde{\mathbf{e}}\end{aligned}$$

But $\tilde{\mathbf{e}} = \dot{\mathbf{F}} \cdot \tilde{\mathbf{F}}^{-1}$ hence the above becomes

$$\dot{W} = \tilde{\tau} \colon \dot{\mathbf{F}} \cdot \tilde{\mathbf{F}}^{-1}$$

Using the property of $:$ we can rewrite $A \colon B \cdot C$ as $A \cdot C^T \colon B$ hence applying this property to the above expression we obtain

$$\dot{W} = \tilde{\tau} \cdot \tilde{\mathbf{F}}^{-T} \colon \dot{\mathbf{F}}$$

Now applying the property that $A \cdot C^T \colon B \to C \cdot A \colon B^T$ to the above results in

$$\dot{W} = \tilde{\mathbf{F}}^{-1} \cdot \tilde{\tau} \colon \dot{\mathbf{F}}^T$$

we found earlier that $\tilde{\mathbf{t}} = J\, \tilde{\mathbf{F}}^{-1} \cdot \tilde{\tau}$ hence replace this into the above we obtain

$$\dot{W} = \frac{1}{J} \tilde{\mathbf{t}} \colon \dot{\mathbf{F}}^T$$

Hence we see that $\frac{\partial \tilde{\mathbf{A}}}{\partial t} = \frac{1}{J} \dot{\mathbf{F}}^T$ hence

$$\boxed{\tilde{\mathbf{A}} = \frac{1}{J} \mathbf{F}^T}$$

5.4 Conjugate pair for $\tilde{\sigma}$ Kirchhoff stress tensor

Since $\tilde{\sigma}$ is just a scaled version of $\tilde{\tau}$ where

$$\tilde{\sigma} = J\ \tilde{\tau}$$

and we found that the strain tensor associated with $\tilde{\tau}$ is $\frac{1}{2J}\left(\tilde{\mathbf{F}}^{\mathbf{T}}\cdot\tilde{\mathbf{F}}-\tilde{\mathbf{I}}\right)$ hence the strain tensor associated with $\tilde{\sigma}$ is $\frac{1}{2}\left(\tilde{\mathbf{F}}^{\mathbf{T}}\cdot\tilde{\mathbf{F}}-\tilde{\mathbf{I}}\right)$

Hence

$$\boxed{\tilde{\mathbf{A}} = \frac{1}{2}\left(\tilde{\mathbf{F}}^{\mathbf{T}}\cdot\tilde{\mathbf{F}}-\tilde{\mathbf{I}}\right)}$$

5.5 Conjugate pair for $\tilde{\mathbf{r}}^{*}$ Biot-Lure stress tensor

$$\begin{aligned}\dot{W} &= \tilde{\mathbf{B}}\colon \frac{\partial\tilde{\mathbf{A}}}{\partial t}\\ &= \tilde{\tau}\colon \tilde{\mathbf{e}}\end{aligned}$$

But $\tilde{\mathbf{e}} = \dot{\mathbf{F}}\cdot\tilde{\mathbf{F}}^{-1}$ hence the above becomes

$$\begin{aligned}\dot{W} &= \tilde{\tau}\colon \dot{\mathbf{F}}\cdot\tilde{\mathbf{F}}^{-1}\\ &= \tilde{\tau}\colon \frac{1}{2}\left(\dot{\mathbf{F}}+\dot{\mathbf{F}}\right)\cdot\tilde{\mathbf{F}}^{-1}\\ &= \tilde{\tau}\colon \frac{1}{2}\left(\dot{\mathbf{F}}\cdot\tilde{\mathbf{I}}+\tilde{\mathbf{I}}\cdot\dot{\mathbf{F}}\right)\cdot\tilde{\mathbf{F}}^{-1}\end{aligned} \tag{1}$$

But $\left(\tilde{\mathbf{F}}^{-1}\cdot\tilde{\mathbf{F}}\right) = \tilde{\mathbf{I}}$, use this to replace the first $\tilde{\mathbf{I}}$ in equation (1) above.

Also $\left(\tilde{\mathbf{F}}\cdot\tilde{\mathbf{F}}^{-1}\right) = \tilde{\mathbf{I}}$, use this to replace the second $\tilde{\mathbf{I}}$ in equation (1) above.

Hence eq (1) becomes

$$\begin{aligned}\dot{W} &= \tilde{\tau}\colon \frac{1}{2}\left(\dot{\mathbf{F}}\cdot\left(\tilde{\mathbf{F}}^{-1}\cdot\tilde{\mathbf{F}}\right)+\left(\tilde{\mathbf{F}}\cdot\tilde{\mathbf{F}}^{-1}\right)\cdot\dot{\mathbf{F}}\right)\cdot\tilde{\mathbf{F}}^{-1}\\ &= \tilde{\tau}\colon \frac{1}{2}\left(\overbrace{\dot{\mathbf{F}}\cdot\tilde{\mathbf{F}}^{-1}}\cdot\tilde{\mathbf{F}}+\overbrace{\tilde{\mathbf{F}}\cdot\tilde{\mathbf{F}}^{-1}}\cdot\dot{\mathbf{F}}\right)\cdot\tilde{\mathbf{F}}^{-1}\end{aligned}$$

Switch the order of terms selected above by transposing them we get

$$\dot{W} = \tilde{\tau} : \frac{1}{2}\left(\overbrace{\tilde{\mathbf{F}}^{-T} \cdot \dot{\mathbf{F}}^{T}} \cdot \tilde{\mathbf{F}} + \overbrace{\tilde{\mathbf{F}}^{-T} \cdot \tilde{\mathbf{F}}^{T}} \cdot \dot{\mathbf{F}}\right) \cdot \tilde{\mathbf{F}}^{-1}$$

Take $\tilde{\mathbf{F}}^{-\mathbf{T}}$ as common factor we get

$$\dot{W} = \tilde{\tau} : \frac{1}{2}\left\{\tilde{\mathbf{F}}^{-T} \cdot \left(\dot{\mathbf{F}}^{T} \cdot \tilde{\mathbf{F}} + \tilde{\mathbf{F}}^{T} \cdot \dot{\mathbf{F}}\right)\right\} \cdot \tilde{\mathbf{F}}^{-1} \tag{2}$$

But

$$\dot{\mathbf{F}}^{T} \cdot \tilde{\mathbf{F}} + \tilde{\mathbf{F}}^{T} \cdot \dot{\mathbf{F}} = \frac{d}{dt}\left(\tilde{\mathbf{F}}^{T} \cdot \mathbf{F}\right)$$

Hence (2) becomes

$$\dot{W} = \tilde{\tau} : \frac{1}{2}\left\{\tilde{\mathbf{F}}^{-T} \cdot \frac{d}{dt}\left(\tilde{\mathbf{F}}^{T} \cdot \mathbf{F}\right)\right\} \cdot \tilde{\mathbf{F}}^{-1} \tag{3}$$

But $\frac{d}{dt}\left(\tilde{\mathbf{F}}^{T} \cdot \mathbf{F}\right) = \frac{d}{dt}\left(\tilde{\mathbf{U}}^{2}\right)$ since $\tilde{\mathbf{F}}^{T} \cdot \mathbf{F} = \tilde{\mathbf{U}}^{2}$

Hence (3) becomes

$$\dot{W} = \tilde{\tau} : \frac{1}{2}\left\{\tilde{\mathbf{F}}^{-T} \cdot \frac{d}{dt}\left(\tilde{\mathbf{U}}^{2}\right)\right\} \cdot \tilde{\mathbf{F}}^{-1} \tag{4}$$

But

$$\frac{d}{dt}\left(\tilde{\mathbf{U}}^{2}\right) = 2\left(\tilde{\mathbf{U}} \cdot \dot{\mathbf{U}}\right)$$

Hence (4) becomes

$$\dot{W} = \tilde{\tau} : \frac{1}{2}\left\{\tilde{\mathbf{F}}^{-T} \cdot 2\left(\tilde{\mathbf{U}} \cdot \dot{\mathbf{U}}\right)\right\} \cdot \tilde{\mathbf{F}}^{-1} \tag{5}$$

But

$$\tilde{\mathbf{U}} \cdot \dot{\mathbf{U}} = \dot{\mathbf{U}} \cdot \mathbf{U}$$

From symmetry of $\mathbf{U}$

Hence

$$2\left(\tilde{\mathbf{U}} \cdot \dot{\mathbf{U}}\right) = \tilde{\mathbf{U}} \cdot \dot{\mathbf{U}} + \dot{\mathbf{U}} \cdot \mathbf{U}$$

Hence (5) becomes

$$\dot{W} = \tilde{\tau}\colon \frac{1}{2}\left(\tilde{\mathbf{F}}^{-T}\cdot\left(\tilde{\mathbf{U}}\cdot\dot{\mathbf{U}}+\dot{\mathbf{U}}\cdot\mathbf{U}\right)\cdot\tilde{\mathbf{F}}^{-1}\right)$$

From property of : we can rewrite the above as

$$\dot{W} = \tilde{\mathbf{F}}^{-1}\cdot\tilde{\tau}\cdot\tilde{\mathbf{F}}^{-T}\colon \frac{1}{2}\left(\tilde{\mathbf{U}}\cdot\dot{\mathbf{U}}+\dot{\mathbf{U}}\cdot\mathbf{U}\right)$$

But as we said above, $\frac{1}{2}\left(\tilde{\mathbf{U}}\cdot\dot{\mathbf{U}}+\dot{\mathbf{U}}\cdot\mathbf{U}\right)=\tilde{\mathbf{U}}\cdot\dot{\mathbf{U}}$ hence the above becomes

$$\dot{W} = \tilde{\mathbf{F}}^{-1}\cdot\tilde{\tau}\cdot\tilde{\mathbf{F}}^{-T}\colon\tilde{\mathbf{U}}\cdot\dot{\mathbf{U}}$$

Using property of : we can move $\tilde{\mathbf{U}}$ to the left of : to get

$$\dot{W} = \tilde{\mathbf{F}}^{-1}\cdot\tilde{\tau}\cdot\tilde{\mathbf{F}}^{-T}\cdot\tilde{\mathbf{U}}\colon\dot{\mathbf{U}}$$

But $\tilde{\mathbf{F}}^{-T}\cdot\tilde{\mathbf{U}}=\tilde{\mathbf{R}}$ hence the above becomes

$$\dot{W} = \overbrace{\tilde{\mathbf{F}}^{-1}\cdot\tilde{\tau}\cdot\tilde{\mathbf{R}}}\colon\dot{\mathbf{U}}$$

But we found earlier that $\tilde{\mathbf{r}}^{*}=J\ \tilde{\mathbf{F}}^{-1}\cdot\tilde{\tau}\cdot\tilde{\mathbf{R}}$

Hence $\dot{W}=\frac{1}{J}\tilde{\mathbf{r}}^{*}\colon\dot{\mathbf{U}}$ Hence

$$\dot{W} = \tilde{\mathbf{r}}^{*}\colon\frac{1}{J}\dot{\mathbf{U}}$$

Hence we see that $\frac{\partial\tilde{\mathbf{A}}}{\partial t}=\frac{1}{J}\dot{\mathbf{U}}$ then

$$\boxed{\tilde{\mathbf{A}}=\frac{1}{J}\mathbf{U}}$$

5.6 Conjugate pair for $\tilde{\mathrm{r}}$ Jaumann stress tensor

Since we found that $\tilde{\mathbf{r}} = \frac{\left(\tilde{\mathbf{r}}^{*}+\tilde{\mathbf{r}}^{*T}\right)}{2}$ then the conjugate pair for $\tilde{\mathbf{r}}$ is $\frac{\left(\frac{1}{J}\mathbf{U}+\frac{1}{J}\mathbf{U}^{T}\right)}{2}$

But $\tilde{\mathbf{U}}$ is symmetrical, hence

conjugate pair for $\tilde{\mathbf{r}}$ is $\frac{1}{J}\mathbf{U}$

i.e.

$$\tilde{\mathbf{A}} = \frac{1}{J}\mathbf{U}$$

The same as strain tensor associated with the Biot-Lure stress.

6 Stress-Strain relations using conjugate pairs based on complementary strain energy

TODO for future work.

7 Appendix

7.1 Derivation of the deformation gradient tensor $\tilde{\mathbf{F}}$ in normal Cartesian coordinates system

In what follows we derive the expression for the deformation gradient tensor $\tilde{\mathbf{F}}$. This tensor transform one vector into another vector.

For simplicity we will assume that the deformed and the undeformed states are described using the same coordinates system. In addition, we assume that this coordinates system is the normal Cartesian system with basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Later these expression will be written in the more general case where the coordinate systems are different and uses curvilinear coordinate. Other than using different notation, the derivation is the same in both cases.

Consider a point P in the undeformed state. This point will have coordinates (X_1, X_2, X_3). When the body undergoes deformation, this point will be displaced to a new location. We call the image of this point in the deformed state as point p. We refer to the coordinates of the the point p as (x_1, x_2, x_3).

The coordinates x_i are function of the coordinates X_j. These functions constitute the mapping between the undeformed shape and the deformed shape. These functions can be written in general as follows

$$
\begin{aligned}
x_1 &= f_1 \left(X_1, X_2, X_3\right) \\
x_2 &= f_2 \left(X_1, X_2, X_3\right) \\
x_3 &= f3 \left(X_1, X_2, X_3\right)
\end{aligned}
$$

Hence by knowing the functions f_i we can locate the position of any point in the deformed state if we know its position in the undeformed state. It is more customary to write the function f_i using the name of the coordinate itself. For example we can write $x_1 = x_1 \left(X_1, X_2, X_3\right)$ instead of $x_1 = f_1 \left(X_1, X_2, X_3\right)$ as we did above. However this can be a little confusing since it uses the letter x_i as function when on the RHS and a variable on the LHS. Hence we selected to use a new name for the mapping function.

From the above we can determine the expression for a differential change in each of the 3 coordinates using the differentiation chain rule as follows

$$
\begin{aligned}
dx_1 &= \frac{\partial f_1}{\partial X_1} dX_1 + \frac{\partial f_1}{\partial X_2} dX_2 + \frac{\partial f_1}{\partial X_3} dX_3 \\
dx_2 &= \frac{\partial f_2}{\partial X_1} dX_1 + \frac{\partial f_2}{\partial X_2} dX_2 + \frac{\partial f_2}{\partial X_3} dX_3 \\
dx_3 &= \frac{\partial f_3}{\partial X_1} dX_1 + \frac{\partial f_3}{\partial X_2} dX_2 + \frac{\partial f_3}{\partial X_3} dX_3
\end{aligned} \tag{1}
$$

Consider now a differential vector element $\mathbf{dr}$ in the deformed state. Hence this vector can be written as

$$\mathbf{dr} = \mathbf{i}\ dx_1 + \mathbf{j}\ dx_2 + \mathbf{k}\ dx_3 \tag{2}$$

Combining equations (1) and (2) we obtain

$$\begin{aligned}\mathbf{dr} = \mathbf{i}\ &\left(\frac{\partial f_1}{\partial X_1}dX_1 + \frac{\partial f_1}{\partial X_2}dX_2 + \frac{\partial f_1}{\partial X_3}dX_3\right)\\ +\mathbf{j}\ &\left(\frac{\partial f_2}{\partial X_1}dX_1 + \frac{\partial f_2}{\partial X_2}dX_2 + \frac{\partial f_2}{\partial X_3}dX_3\right)\\ +\mathbf{k}\ &\left(\frac{\partial f_3}{\partial X_1}dX_1 + \frac{\partial f_3}{\partial X_2}dX_2 + \frac{\partial f_3}{\partial X_3}dX_3\right)\end{aligned}$$

The above equation can be written in matrix form as follows

$$\begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \frac{\partial f_1}{\partial X_2} & \frac{\partial f_1}{\partial X_3} \\ \frac{\partial f_2}{\partial X_1} & \frac{\partial f_2}{\partial X_2} & \frac{\partial f_2}{\partial X_3} \\ \frac{\partial f_3}{\partial X_1} & \frac{\partial f_3}{\partial X_2} & \frac{\partial f_3}{\partial X_3} \end{pmatrix} \begin{pmatrix} dX_1 \\ dX_2 \\ dX_3 \end{pmatrix} \tag{3}$$

Hence we see that the components of $\mathbf{dr}$ can be obtained from the components $\mathbf{dR}$ by pre-multiplying the components of $\mathbf{dR}$ by the above 3×3 matrix. Hence this matrix acts as a transformation rule which maps one vector to another, it is a second order tensor, which is called the deformation gradient tensor $\tilde{\mathbf{F}}$

$$\mathbf{dr} = \tilde{\mathbf{F}} \cdot \mathbf{dR} \tag{4}$$

This relation can be written also in dyadic form as follows

$$\begin{aligned}&\mathbf{i}\ dx_1 + \mathbf{j}\ dx_2 + \mathbf{k}\ dx_3 =\\ &\left(\mathbf{ii}\frac{\partial f_1}{\partial X_1} + \mathbf{ij}\frac{\partial f_1}{\partial X_2} + \mathbf{ik}\frac{\partial f_1}{\partial X_3} + \mathbf{ji}\frac{\partial f_2}{\partial X_1} + \mathbf{jj}\frac{\partial f_2}{\partial X_2} + \mathbf{jk}\frac{\partial f_2}{\partial X_3} + \mathbf{ki}\frac{\partial f_3}{\partial X_1} + \mathbf{kj}\frac{\partial f_3}{\partial X_2} + \mathbf{kk}\frac{\partial f_3}{\partial X_3}\right)\\ &\cdot (\mathbf{i}\ dX_1 + \mathbf{j}\ dX_2 + \mathbf{k}\ dX_3)\end{aligned} \tag{5}$$

To carry the multiplication on the RHS in the above equation, we follow the normal dot product convention, using the following rules.

$$
\begin{aligned}
\mathbf{ii}\cdot\mathbf{i} &= \mathbf{i}\left(\mathbf{i}\cdot\mathbf{i}\right) = 1\\
\mathbf{ij}\cdot\mathbf{i} &= \mathbf{i}\left(\mathbf{j}\cdot\mathbf{i}\right) = 0\\
\mathbf{ik}\cdot\mathbf{i} &= \mathbf{i}\left(\mathbf{k}\cdot\mathbf{i}\right) = 0\\
\mathbf{ji}\cdot\mathbf{i} &= \mathbf{j}\left(\mathbf{i}\cdot\mathbf{i}\right) = 1\\
& etc\cdots
\end{aligned}
$$

Hence if we perform the multiplication we obtain

$$
\begin{aligned}
&\mathbf{i}\ dx_1 + \mathbf{j}\ dx_2 + \mathbf{k}\ dx_3 = \\
&\left(\mathbf{ii}\frac{\partial f_1}{\partial X_1} + \mathbf{ij}\frac{\partial f_1}{\partial X_2} + \mathbf{ik}\frac{\partial f_1}{\partial X_3} + \mathbf{ji}\frac{\partial f_2}{\partial X_1} + \mathbf{jj}\frac{\partial f_2}{\partial X_2} + \mathbf{jk}\frac{\partial f_2}{\partial X_3} + \mathbf{ki}\frac{\partial f_3}{\partial X_1} + \mathbf{kj}\frac{\partial f_3}{\partial X_2} + \mathbf{kk}\frac{\partial f_3}{\partial X_3}\right)\cdot\mathbf{i}\ dX_1\\
&+\left(\mathbf{ii}\frac{\partial f_1}{\partial X_1} + \mathbf{ij}\frac{\partial f_1}{\partial X_2} + \mathbf{ik}\frac{\partial f_1}{\partial X_3} + \mathbf{ji}\frac{\partial f_2}{\partial X_1} + \mathbf{jj}\frac{\partial f_2}{\partial X_2} + \mathbf{jk}\frac{\partial f_2}{\partial X_3} + \mathbf{ki}\frac{\partial f_3}{\partial X_1} + \mathbf{kj}\frac{\partial f_3}{\partial X_2} + \mathbf{kk}\frac{\partial f_3}{\partial X_3}\right)\cdot\mathbf{j}\ dX_2\\
&+\left(\mathbf{ii}\frac{\partial f_1}{\partial X_1} + \mathbf{ij}\frac{\partial f_1}{\partial X_2} + \mathbf{ik}\frac{\partial f_1}{\partial X_3} + \mathbf{ji}\frac{\partial f_2}{\partial X_1} + \mathbf{jj}\frac{\partial f_2}{\partial X_2} + \mathbf{jk}\frac{\partial f_2}{\partial X_3} + \mathbf{ki}\frac{\partial f_3}{\partial X_1} + \mathbf{kj}\frac{\partial f_3}{\partial X_2} + \mathbf{kk}\frac{\partial f_3}{\partial X_3}\right)\cdot\mathbf{k}\ dX_3
\end{aligned}
$$

Now we simplify the dot multiplication using the above mentioned rules to obtain

$$
\begin{aligned}
&\mathbf{i}\ dx_1 + \mathbf{j}\ dx_2 + \mathbf{k}\ dx_3 = \\
&\left(\mathbf{i}\frac{\partial f_1}{\partial X_1}dX_1 + \mathbf{0} + \mathbf{0} + \mathbf{j}\frac{\partial f_2}{\partial X_1}dX_1 + \mathbf{0} + \mathbf{0} + \mathbf{k}\frac{\partial f_3}{\partial X_1}dX_1 + \mathbf{0} + \mathbf{0}\right)\\
&+\left(\mathbf{0} + \mathbf{ij}\frac{\partial f_1}{\partial X_2}dX_2 + \mathbf{0} + \mathbf{0} + \mathbf{jj}\frac{\partial f_2}{\partial X_2}dX_2 + \mathbf{0} + \mathbf{0} + \mathbf{kj}\frac{\partial f_3}{\partial X_2}dX_2 + \mathbf{0}\right)\\
&+\left(\mathbf{0} + \mathbf{0} + \mathbf{ik}\frac{\partial f_1}{\partial X_3}dX_3 + \mathbf{0} + \mathbf{0} + \mathbf{jk}\frac{\partial f_2}{\partial X_3}dX_3 + \mathbf{0} + \mathbf{0} + \mathbf{kk}\frac{\partial f_3}{\partial X_3}dX_3\right)
\end{aligned}
$$

Simplifying we obtain

$$
\begin{aligned}
&\mathbf{i}\ dx_1 + \mathbf{j}\ dx_2 + \mathbf{k}\ dx_3 = \\
&\left(\mathbf{i}\frac{\partial f_1}{\partial X_1}dX_1 + \mathbf{j}\frac{\partial f_2}{\partial X_1}dX_1 + \mathbf{k}\frac{\partial f_3}{\partial X_1}dX_1\right) + \left(\mathbf{i}\frac{\partial f_1}{\partial X_2}dX_2 + \mathbf{j}\frac{\partial f_2}{\partial X_2}dX_2 + \mathbf{k}\frac{\partial f_3}{\partial X_2}dX_2\right)\\
&+\left(\mathbf{i}\frac{\partial f_1}{\partial X_3}dX_3 + \mathbf{j}\frac{\partial f_2}{\partial X_3}dX_3 + \mathbf{k}\frac{\partial f_3}{\partial X_3}dX_3\right)
\end{aligned}
$$

Collect similar terms on the RHS we obtain

$$\mathbf{i}\ dx_1 + \mathbf{j}\ dx_2 + \mathbf{k}\ dx_3 = \\ \mathbf{i}\left(\frac{\partial f_1}{\partial X_1}dX_1 + \frac{\partial f_1}{\partial X_2}dX_2 + \frac{\partial f_1}{\partial X_3}dX_3\right) + \mathbf{j}\left(\frac{\partial f_2}{\partial X_1}dX_1 + \frac{\partial f_2}{\partial X_2}dX_2 + \frac{\partial f_2}{\partial X_3}dX_3\right) \\ + \mathbf{k}\left(\frac{\partial f_3}{\partial X_1}dX_1 + \frac{\partial f_3}{\partial X_2}dX_2 + \frac{\partial f_3}{\partial X_3}dX_3\right)$$

comparing the components of the vector on the LHS with those component of the vector on the RHS we obtain equation (1) as expected.

In addition to the matrix form and the dyadic form, we can express the transformation from **dR** to **dr** using indices notation as follows

$$dx_i = \frac{\partial f_i}{\partial X_j} dX_j$$

7.2 Useful identities and formulas

A matrix **A** is orthogonal if $AA^T = I$ where I is the identity matrix.

If a matrix/tensor **A** is orthogonal then $\mathbf{A}^{-1} = \mathbf{A}^T$. In component form, $a_{ij}^{-1} = a_{ji}$

$\left(\tilde{\mathbf{A}} \cdot \tilde{\mathbf{B}}\right)^{-1} = \tilde{\mathbf{B}}^{-1} \cdot \tilde{\mathbf{A}}^{-1}$

$\left(\tilde{\mathbf{A}} \cdot \tilde{\mathbf{B}}\right)^{T} = \tilde{\mathbf{B}}^{T} \cdot \tilde{\mathbf{A}}^{T}$

$\left(\tilde{\mathbf{A}} \cdot \tilde{\mathbf{B}}\right)^{-T} = \left(\left(\tilde{\mathbf{A}} \cdot \tilde{\mathbf{B}}\right)^{T}\right)^{-1} = \left(\tilde{\mathbf{B}}^{T} \cdot \tilde{\mathbf{A}}^{T}\right)^{-1}$

$\tilde{\mathbf{F}} = \tilde{\mathbf{R}} \cdot \tilde{\mathbf{U}}$

$\tilde{\mathbf{F}} = \tilde{\mathbf{V}} \cdot \tilde{\mathbf{R}}$

$\tilde{\mathbf{U}} = \tilde{\mathbf{R}}^{T} \cdot \tilde{\mathbf{V}} \cdot \tilde{\mathbf{R}}$

$\tilde{\mathbf{V}} = \tilde{\mathbf{R}} \cdot \tilde{\mathbf{U}} \cdot \tilde{\mathbf{R}}^{T}$

$\tilde{\mathbf{U}} = \left(\tilde{\mathbf{F}}^{T} \cdot \tilde{\mathbf{F}}\right)^{\frac{1}{2}}$

$\tilde{\mathbf{F}}_q = \tilde{\mathbf{Q}} \cdot \tilde{\mathbf{F}}$

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