Comparing total error against truncation error at each grid point resulting from the numerical solution of $u''(x) = \exp(x)$ using finite difference discretization

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Introduction

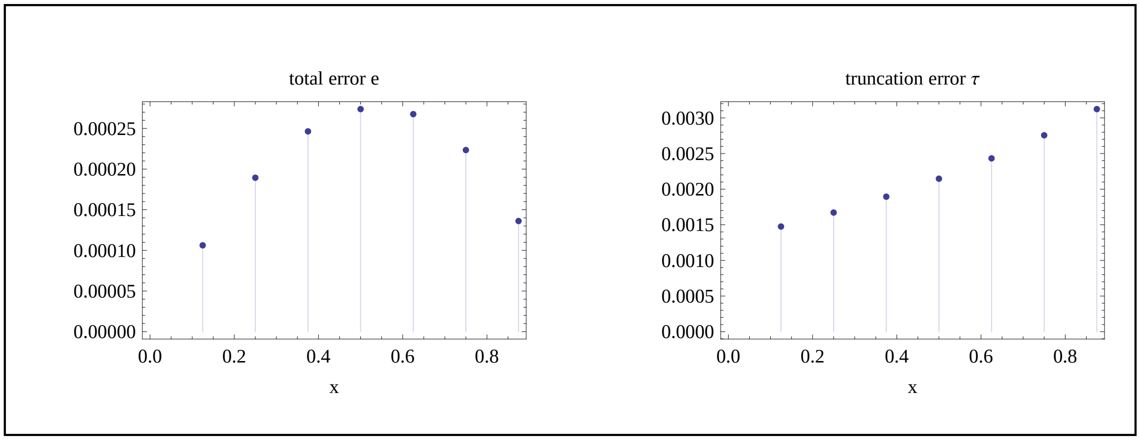
The differential equation $u''[x] == e^x$ with Dirichlet boundary conditions $u[0] = 0$, $u[1] = 0$ is solved numerically by approximating $u''[x]$ by $\frac{u_{i-1} - 2 u_i + u_{i+1}}{h^2}$ where h is the spacing between grid points. The truncation error τ at each x_i is given by $\frac{h^2}{12} u^{(4)}(x_i)$ while the total error e at the same point is given by $U_i - u_i$ where U_i is the numerical solution found at that point and u_i is the exact solution $u(x)$ evaluated at the same point.

A small program is written to show the distribution of e and τ along the length of the domain [0, 1] as h is made smaller and smaller.

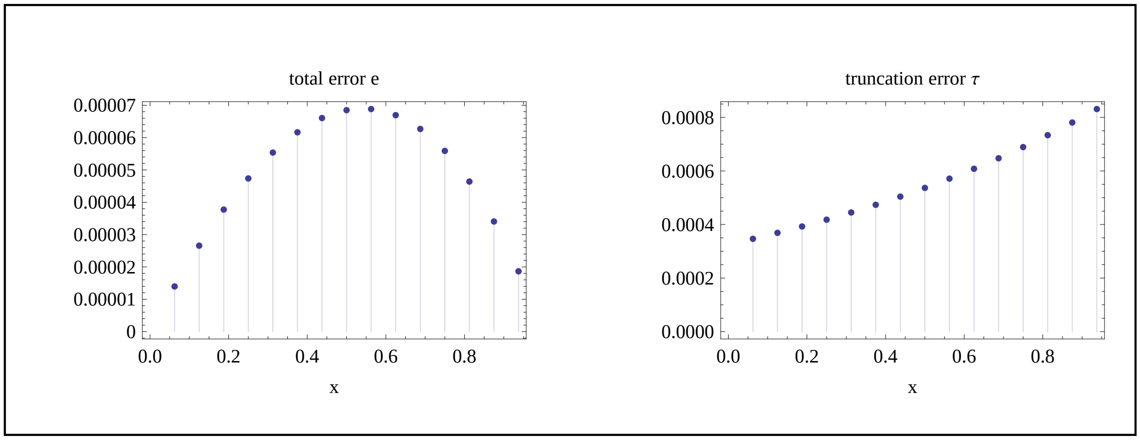
This was run for few iterations, where h was divided by half each time, the system was solved for U and the result is shown in a series of plots.

Results

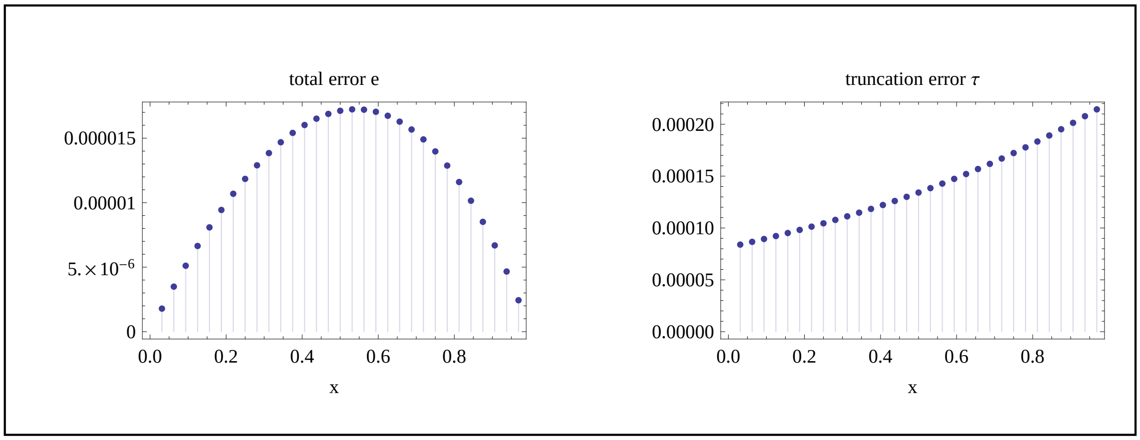
number of grid points = 9, spacing h=0.125



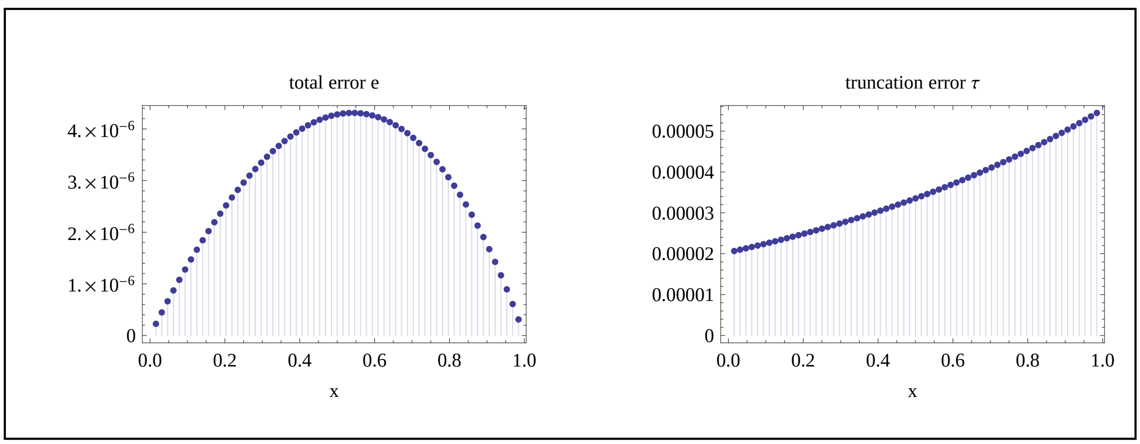
number of grid points = 17, spacing h=0.0625



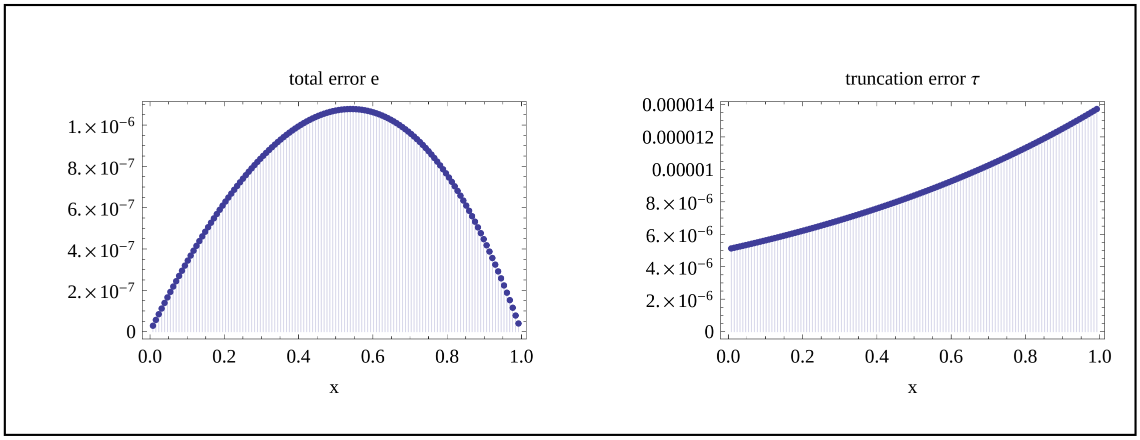
number of grid points = 33, spacing h=0.03125



number of grid points = 65, spacing h=0.015625



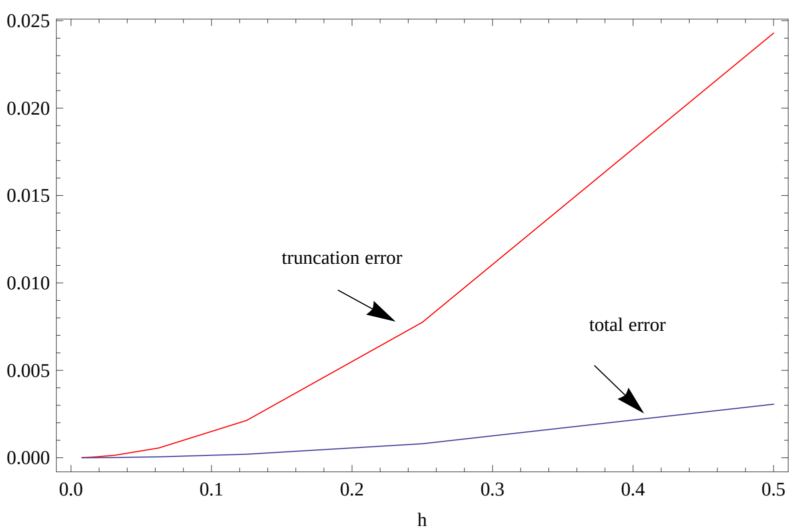
number of grid points = 129, spacing h=0.0078125

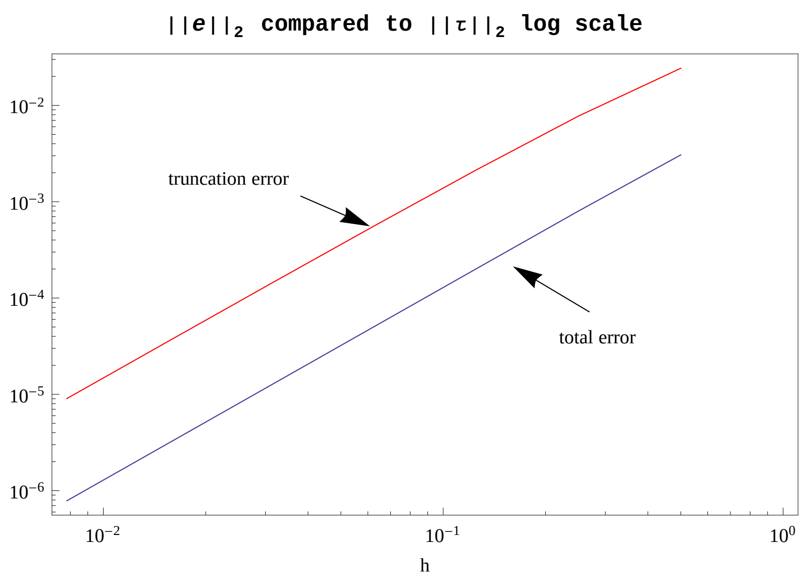


- **Observations on the above plots**

The total error e was largest in the middle of the grid, and smallest at the edges, while the truncation error was smallest at the left edge and largest at the right edge.
Comparing e to τ on each grid point is useful to understand the distribution of errors along the domain, but for verification that the numerical solution converges to the exact solution as h becomes smaller, the grid norm of e is the one examined to verify that it is less than or equal to the grid norm of τ.
Therefore, the following plots are generated which shows how $\|e\|_2$ compares to $\|\tau\|_2$ as h becomes smaller. These norms are grid norms and not the standard norms.

$||e||_2$ compared to $||\tau||_2$ linear scale





- **Observations on the above plots**

The total error has smaller grid norm. This shows convergence, since this implies that $\| e \| \le \| A^{-1} \| \ \| \tau \|$, therefore showing that $\| A^{-1} \| =\ O(1)$ which is the condition for convergence. The above results confirms this for this problem.