EGME 511
HW1 SOLUTION

1.4)

Solution:

$$\ddot{x} - \dot{x} + x = 0 \quad x_0 = 1, \quad v_0 = 0 \tag{1}$$

Let $x(t) = Ce^{\lambda t}$

$$Ce^{\lambda t}(\lambda^2 - \lambda + 1) = 0$$

$$\lambda_{1,2} = \frac{1}{2}\left(1 \pm j\sqrt{3}\right)$$

$$x(t) = A_1' e^{1/2(1+j\sqrt{3})t} + A_2' e^{1/2(1-j\sqrt{3})t} \tag{2}$$

Using Euler's equation,

$$x(t) = e^{\frac{1}{2}t}\left(A_1 \cos\sqrt{3}t + A_2 \sin\sqrt{3}t\right)$$

Apply the initial conditions and obtain,

$$1 = A_1$$

$$0 = \frac{1}{2}A_1 + \sqrt{3}A_2, \quad A_2 = -\frac{1}{2\sqrt{3}}$$

To obtain the solution in the form of Eq.(3),

$$x(t) = A_3 e^{\frac{1}{2}t} \sin\left(\sqrt{3}t + \phi\right)$$

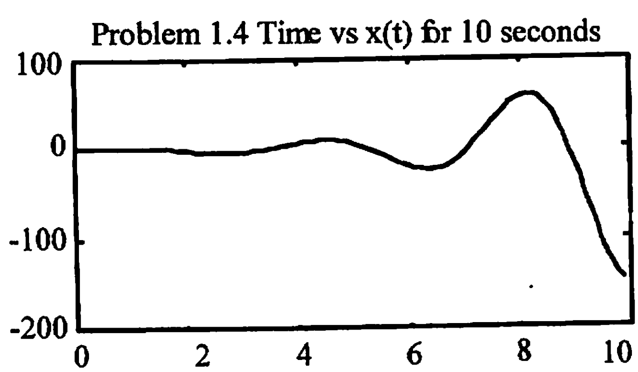
Using the trigonometric identities from problem (1) and get,

$$A_3 = 1.041$$

$$\phi = 0.281 \text{ rad.}$$

$$x(t) = 1.041e^{0.5t} \sin\left(\sqrt{3}t + 0.281)\right)$$

The graph of the response of the system is shown below.



1.9)

Solution: Using equation (1.22) directly yields:

$$\zeta = \frac{1}{\sqrt{2}}, M_p = \frac{1}{2\frac{1}{\sqrt{2}}\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2}} = 1$$

1.12)

Solutions: $m\ddot{x} + c\dot{x} + kx = F\sin\omega t,\ \zeta = 1.1$ and $\omega_n^2 = 4$ (1)

Assume $x_p(t) = X\sin(\omega t - \phi)$ and substitute into (1),

$$X = \frac{F/k}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

$$\phi = \tan^{-1}\left(\frac{2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2}\right)$$

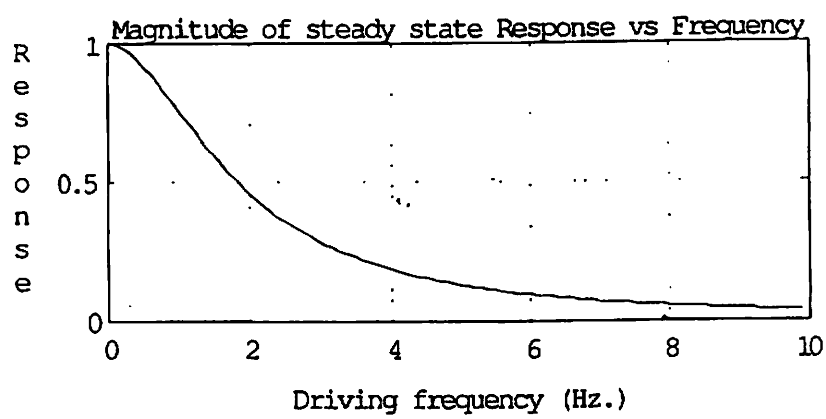
The particular solution is

$$x_p(t) = \frac{F/k}{\sqrt{(1-0.25\omega^2)^2 + 1.21\omega^2}}\sin(\omega t - \phi).$$

where

$$\phi = \tan^{-1}\left(\frac{1.1\omega}{1-0.25\omega^2}\right)$$

The response is plotted in the figure shown below, where the y-axis represents $\frac{Xk}{F}$.



Resonance does not occur because the system is overdamped.

1.18)

Solution: Gathering up terms, the equation of motion can be written as

$$\ddot{x}(t) + 4x(t) = 0.5\sin 2t$$

In this form it is clear that the natural frequency and the driving frequency are both 2 rad/s, hence the system is in resonance. The homogeneous form is undamped, hence stable. However, with a bounded input of 0.5sin2*t*, the response becomes unbounded and, hence the forced response is unstable. The solution can be computed as (see Inman 2001, page 96):

$$x(t) = \frac{v_o}{2}\sin 2t + x_0\cos 2t + 0.125t\sin 2t$$

which clearly grows without bound as it oscillates.

1.20)

Solution:

The equation of motion with controller is:

$$2\ddot{x} + (0.8 + Kg_1)\dot{x} + (8 + Kg_2)x = Ku(t) \qquad (1)$$

$$m = 2,\ c = 0.8 + Kg_1,\ k = 8 + Kg_2 \qquad (2)$$

The design expressions for overshoot and settling time are only valid for underdamped systems, if $1 - \zeta^2 > 0$. Substitution for the value of ζ yields that

$$1 - \frac{c^2}{4km} > 0 \Rightarrow 4km > c^2 \qquad (3)$$

Substituting (2) into (3) and simplifying,

$$\underline{64 + 8Kg_2 > (0.8 + Kg_1)^2}$$

To insure BIBO stability of the closed loop system, the equivalent open loop system must be asymptotically stable. This requires the coefficients to be positive: $\underline{Kg_2 > -8 \text{ and } Kg_1 \geq -0.8}$. Note that in general, negative feedback is used so that Kg_1 and Kg_2 are usually positive. However, in order to obtain a specified settling time and overshoot, it may be that the gains g_i could be negative, hence stability must be checked.

1.21)

Solution: Since the open loop system is already stable, only a damping term needs to be added by the controller. Choosing, $K = 1$, and $g_2 = 0$ yields:

$$4\ddot{x}(t) + g_1\dot{x}(t) + 16x(t) = f(t)$$

This yields that $\omega_n = 2$ and

$$2\zeta\omega_n = \frac{g_1}{4} \Rightarrow g_1 = 16\zeta$$

However from equation (1.34) the settling time is

$$t_s = \frac{3.2}{\omega_n\zeta} \Rightarrow \zeta = \frac{3.2}{2t_s}$$

Combining these last two expression yields

$$g_1 = 16\frac{3.2}{2t_s} = \underline{25.6}$$

1.22) Compute the equilibrium positions of the pendulum equation: $m\ell^2\ddot{\theta}(t) + mg\ell\sin\theta(t) = 0$.

Solution: First put the system into first order form by defining the two states of position and velocity: $x_1 = \theta$, $x_2 = \dot{\theta} = \dot{x}_1$, and writing the equations of motion in state space, or first order form (diving through by the leading coefficient):

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -\frac{g}{\ell}\sin(x_1(t))$$

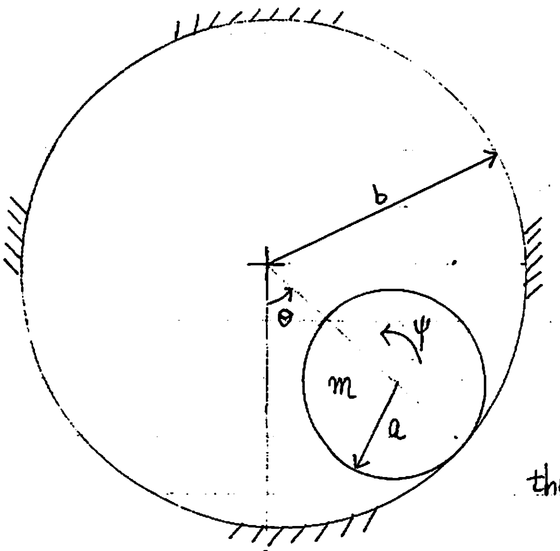
$$\Rightarrow \mathbf{F}(t) = \begin{bmatrix} x_2 \\ \sin x_1 \end{bmatrix}$$

Setting $\mathbf{F} = \mathbf{0}$ yields the equilibrium points:

$$\mathbf{x}_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ 0 \end{bmatrix}, \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}, \begin{bmatrix} 3\pi \\ 0 \end{bmatrix}, \dots \begin{bmatrix} n\pi \\ 0 \end{bmatrix}, \dots$$

Example of Lagrangian Dynamics

Gear Problem

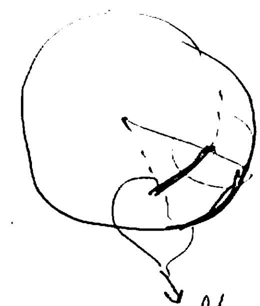


Assumptions

1) Outer gear is fixed
2) Inner gear rolls due to gravity force
3) Radius of gyration of small gear $= K$
4) One degree of freedom system $[\theta]$

$$a\dot{\psi} = (b-a)\dot{\theta} \quad ✓$$

thus $V = (b-a)\dot{\theta}$



then 2 must be equal for no slippage

Kinetic Energy : $T = \frac{1}{2}mv^2 + \frac{1}{2}I\dot{\psi}^2$

where v = linear velocity of center of mass $= (b-a)\dot{\theta}$

I = rotational inertia of center of mass $= mk^2$

thus $T = \frac{1}{2}m[\underbrace{(b-a)\dot{\theta}}_{V}]^2 + \frac{1}{2}(\underbrace{mk^2}_{I})\left[\frac{b-a}{a}\dot{\theta}\right]^2$

Potential Energy

$$V = mg(b-a)(1-\cos\theta) \quad ✓$$

$$-\frac{\partial V}{\partial \theta} = -mg(b-a)\sin\theta$$

From Lagrange's Equation for conservative system,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = -\frac{\partial V}{\partial \theta}$$

$$\frac{\partial T}{\partial \dot{\theta}} = \frac{1}{2}m(b-a)^2(2)\dot{\theta} + \frac{1}{2}mk^2\left(\frac{b-a}{a}\right)^2 2\dot{\theta}$$

$$\frac{\partial T}{\partial \dot{\theta}} = m(b-a)^2\dot{\theta} + mk^2\left(\frac{b-a}{a}\right)^2\dot{\theta}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = m(b-a)^2\ddot{\theta} + mk^2\left(\frac{b-a}{a}\right)^2\ddot{\theta}$$

$$\frac{\partial T}{\partial \theta} = 0 \qquad \text{and} \qquad -\frac{\partial V}{\partial \theta} = -mg(b-a)\sin\theta$$

Hence $\frac{d}{dt}\frac{\partial T}{\partial \dot\theta} - \frac{\partial T}{\partial \theta} = -\frac{\partial V}{\partial \theta}$ becomes

$$m(b-a)^2\ddot\theta + mk^2\left(\frac{b-a}{a}\right)^2\ddot\theta + mg(b-a)\sin\theta = 0$$

Also by dividing by $(b-a)$

$$(b-a)\ddot\theta + \frac{k^2(b-a)}{a^2}\ddot\theta + g\sin\theta = 0$$

$$(b-a)\ddot\theta\left[1 + \frac{k^2}{a^2}\right] + g\sin\theta = 0$$

$$(b-a)\left[\frac{a^2+k^2}{a^2}\right]\ddot\theta + g\sin\theta = 0$$

$$\ddot\theta + \frac{g}{(b-a)\left[\frac{a^2+k^2}{a^2}\right]}\sin\theta = 0$$