

1.

$T = \text{kinetic energy at time zero} = \frac{1}{2} m (\dot{x}_0)^2$

Let $x_2 = \text{maximum displacement on right side.}$

$V = \text{potential energy in spring at displacement}$
 $x_2 = \frac{1}{2} k_2 x_2^2$ ($\dot{x}$ is zero at x_2)

Since $T = V$, $x_2 = \sqrt{\frac{m(\dot{x}_0)^2}{k_2}} = \sqrt{\frac{m}{k_2}} \dot{x}_0$

Let $x_1 = \text{maximum displacement to left side.}$ $V = \frac{1}{2} k_1 x_1^2$

$T = V$ gives $x_1 = \sqrt{\frac{m}{k_1}} (\dot{x}_0) = \sqrt{\frac{m}{k_1}} \dot{x}_0$

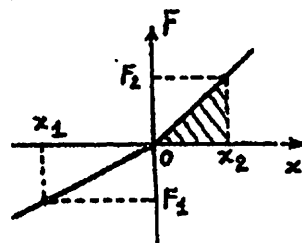
(a) Since $k_1 < k_2$, maximum deflection $= x_1 = \sqrt{\frac{m}{k_1}} \dot{x}_0$

(b) Period of vibration for a spring-mass system is $\tau_n = 2\pi \sqrt{\frac{m}{k}}$

In the present case, $\tau_n = (\text{time for } m \text{ to go to } x = x_1 \text{ from } x = 0 \text{ and return to } x = 0) +$

$(\text{time for } m \text{ to go to } x = x_2 \text{ from } x = 0 \text{ and return to } x = 0)$

$$\therefore \tau_n = \pi \left(\sqrt{\frac{m}{k_1}} + \sqrt{\frac{m}{k_2}} \right)$$



PI

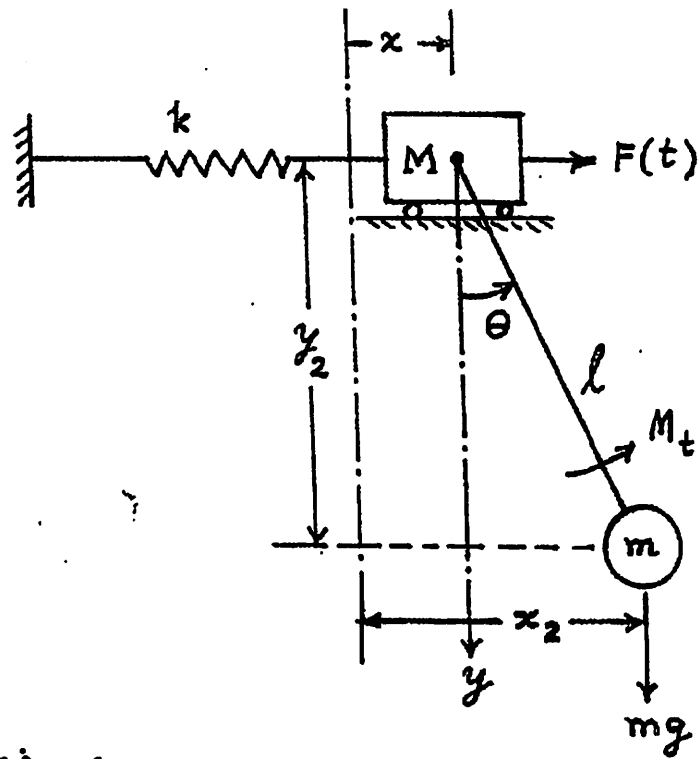
Key solution

HW # 3

5-11

4/27/09

2.



$$x_2 = x + \ell \sin \theta ; \dot{x}_2 = \dot{x} + \ell \dot{\theta} \cos \theta$$

$$y_2 = \ell \cos \theta ; \dot{y}_2 = -\ell \dot{\theta} \sin \theta$$

$$\begin{aligned} T &= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m (\dot{x}_2^2 + \dot{y}_2^2) \\ &= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m [(\dot{x} + \ell \dot{\theta} \cos \theta)^2 + (-\ell \dot{\theta} \sin \theta)^2] \\ &= \frac{1}{2} (M + m) \dot{x}^2 + \frac{1}{2} m \ell^2 \dot{\theta}^2 + m \ell \dot{x} \dot{\theta} \cos \theta \end{aligned}$$

$$V = \frac{1}{2} k x^2 + m g \ell (1 - \cos \theta)$$

$$Q_x = F(t) ; Q_\theta = M_t(t)$$

Equations of motion:

$$(M + m) \ddot{x} + m \ell \ddot{\theta} \cos \theta - m \ell \dot{\theta}^2 \sin \theta + k x = F(t) \quad (1)$$

$$m \ell^2 \ddot{\theta} + m \ell \ddot{x} \cos \theta - m \ell \dot{x} \dot{\theta} \sin \theta + m g \ell \sin \theta = M_t(t) \quad (2)$$

Using the approximations

$$\cos \theta \approx 1 - \frac{\theta^2}{2} ; \sin \theta \approx \theta - \frac{\theta^3}{6}$$

Eqs. (1) and (2) can be expressed as

$$(M + m) \ddot{x} + m \ell \ddot{\theta} - \frac{1}{2} m \ell \theta^2 \ddot{\theta} - m \ell \theta \dot{\theta}^2 + \frac{1}{6} m \ell \theta^3 \dot{\theta}^2 + k x = F(t) \quad (3)$$

$$m \ell^2 \ddot{\theta} + m \ell \ddot{x} - \frac{1}{2} m \ell \theta^2 \ddot{x} - m \ell \theta \dot{\theta} \dot{x} + \frac{1}{6} m \ell \theta^3 \dot{\theta} \dot{x} + m g \ell \theta - \frac{1}{6} m g \ell \theta^3 = M_t(t) \quad (4)$$

By neglecting the nonlinear terms, the linearized equations of motion can be written as

$$(M + m) \ddot{x} + m \ell \ddot{\theta} + k x = F(t) \quad (5)$$

$$m \ell^2 \ddot{\theta} + m \ell \ddot{x} + m g \ell \theta = M_t(t) \quad (6)$$

3.

$$\ddot{\theta} + \omega_0^2 \left(\theta - \frac{1}{6} \theta^3 \right) = 0 \quad (E_1)$$

This equation is similar to Eq. (13.9) with

$$x = \theta, \quad \omega = \omega_0, \quad F(x) = F(\theta) = \theta - \frac{1}{6} \theta^3.$$

Eq. (E₁) can be rewritten as

$$\frac{d}{d\theta} (\dot{\theta}^2) + 2\omega_0^2 \left(\theta - \frac{1}{6} \theta^3 \right) = 0 \quad (E_2)$$

which upon integration gives

$$\dot{\theta}^2 = 2\omega_0^2 \int_{\theta}^{\theta_0} F(\eta) \cdot d\eta = 2\omega_0^2 \int_{\theta}^{\theta_0} \left(\eta - \frac{1}{6} \eta^3 \right) \cdot d\eta$$

$$= 2\omega_0^2 \left(\frac{1}{2} \eta^2 - \frac{1}{24} \eta^4 \right)_{\theta}^{\theta_0} = \omega_0^2 \left(\theta_0^2 - \frac{1}{12} \theta_0^4 - \theta^2 + \frac{1}{12} \theta^4 \right) \quad (E_3)$$

$$= \omega_0^2 (\theta_0^2 - \theta^2) \left\{ 1 - \frac{1}{12} (\theta_0^2 + \theta^2) \right\} \quad (E_4)$$

Since the maximum value of θ is θ_0 , we assume

$$\theta(t) = \theta_0 \sin \beta \quad (E_5)$$

$$\text{Thus } \theta_0^2 - \theta^2 = \theta_0^2 - \theta_0^2 \sin^2 \beta = \theta_0^2 \cos^2 \beta \quad (E_6)$$

$$\theta_0^2 + \theta^2 = \theta_0^2 (1 + \sin^2 \beta) \quad (E_7)$$

$$\text{and } \dot{\theta} = A_0 \cos \beta \frac{d\beta}{dt} \quad (E_8)$$

Substitution of Eqs. (E₆) to (E₈) into (E₄) gives

$$\theta_0^2 \cos^2 \beta \left(\frac{d\beta}{dt} \right)^2 = \omega_0^2 \theta_0^2 \cos^2 \beta \left\{ 1 - \frac{1}{12} \theta_0^2 (1 + \sin^2 \beta) \right\}$$

i.e.,

$$\left(\frac{d\beta}{dt} \right)^2 = \omega_0^2 \left(1 - \frac{1}{12} \theta_0^2 \right) \left\{ 1 - \frac{\theta_0^2 \sin^2 \beta}{12 \left(1 - \frac{1}{12} \theta_0^2 \right)} \right\} \quad (E_9)$$

Defining

$$a^2 = \frac{\theta_0^2}{12 \left(1 - \frac{1}{12} \theta_0^2 \right)} \quad (E_{10})$$

Eq. (E₉) can be used to express (taking positive root):

$$\frac{d\beta}{dt} = \omega_0 \left(1 - \frac{1}{12} \theta_0^2 \right)^{\frac{1}{2}} \left(1 - a^2 \sin^2 \beta \right)^{\frac{1}{2}} \quad (E_{11})$$

i.e.,

$$\omega_0 \left(1 - \frac{1}{12} \theta_0^2 \right)^{\frac{1}{2}} dt = \int \frac{d\beta}{\sqrt{1 - a^2 \sin^2 \beta}} \quad (E_{12})$$

Integration of (E₁₂) yields

$$\omega_0 \left(1 - \frac{1}{12} \theta_0^2 \right)^{\frac{1}{2}} (t - t_0) = \int_{\beta_0}^{\beta} \frac{d\beta}{\sqrt{1 - a^2 \sin^2 \beta}} \quad (E_{13})$$

Using the initial conditions $\beta_0 = 0$ at $t_0 = 0$, Eq. (E₁₃) can be reduced to

$$\omega_0 \left(1 - \frac{1}{12} \theta_0^2\right)^{\frac{1}{2}} \cdot t = \int_0^{\beta} \frac{d\beta}{\sqrt{1 - a^2 \sin^2 \beta}} = F(a, \beta) \quad (E_{14})$$

where $F(a, \beta)$ is an incomplete elliptic integral of the first kind. Using $\beta = \frac{\pi}{2}$ when $\theta = \theta_0$ and $\beta = 0$ when $\theta = 0$, we get for one-quarter period,

$$\frac{\tau}{4} = t = \frac{1}{\omega_0 \left(1 - \frac{1}{12} \theta_0^2\right)^{\frac{1}{2}}} \cdot F\left(a, \frac{\pi}{2}\right) \quad (E_{15})$$

Thus the time period of the pendulum is given by

$$\tau = \frac{4}{\omega_0 \left(1 - \frac{1}{12} \theta_0^2\right)^{\frac{1}{2}}} \cdot F\left(a, \frac{\pi}{2}\right) \quad (E_{16})$$

4
o

$$\ddot{x} + 0.1(x^2 - 1)\dot{x} + x = 0 \quad \text{or} \quad \ddot{x} = -[0.1(x^2 - 1)\dot{x} + x]$$

$$\text{Let } x = x_1, \quad \dot{x}_1 = x_2 = f_1(x_1, x_2)$$

$$\dot{x}_2 = -[0.1(x_1^2 - 1)x_2 + x_1] = f_2(x_1, x_2)$$

For equilibrium,

$$f_1 = 0 \Rightarrow x_2 = 0; \quad f_2 = 0 \Rightarrow x_1 = 0$$

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

where

$$a_{11} = \left. \frac{\partial f_1}{\partial x_1} \right|_{(0,0)} = 0, \quad a_{12} = \left. \frac{\partial f_1}{\partial x_2} \right|_{(0,0)} = 1,$$

$$a_{21} = \left. \frac{\partial f_2}{\partial x_1} \right|_{(0,0)} = -(0.2x_1x_2 + 1) \Big|_{(0,0)} = -1,$$

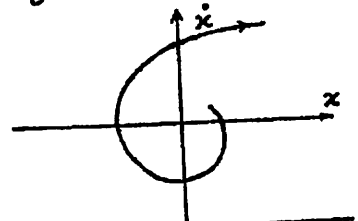
$$a_{22} = \left. \frac{\partial f_2}{\partial x_2} \right|_{(0,0)} = -[0.1(x_1^2 - 1)] \Big|_{(0,0)} = 0.1$$

$$\text{we find } p = 0.1, \quad q = 1$$

$$\lambda_1, \lambda_2 = \frac{1}{2}(0.1 \pm \sqrt{0.01 - 4}) = \text{complex with positive real parts}$$

Since $p > 0$, the system is unstable at the equilibrium point $(x, \dot{x}) = (0, 0)$.

Hence the phase-plane trajectory in the neighborhood of the equilibrium position appears as shown in the figure.



[EQ 4A]

[EQ 4B]

[EQ 4C]

5

$$\text{Equation of motion: } \ddot{x} + f \frac{x}{|\dot{x}|} + \omega_n^2 x = 0 \quad (E_1)$$

$$\text{i.e. } \ddot{x} + \omega_n^2 (x+a) = 0 \quad \text{for } \dot{x} > 0 \quad (E_2)$$

$$\text{and } \ddot{x} + \omega_n^2 (x-a) = 0 \quad \text{for } \dot{x} < 0 \quad (E_3)$$

$$\text{where } a = f/\omega_n^2 \quad (E_4)$$

Multiplying by $2\dot{x}$ and integrating, (E_2) and (E_3) yield

$$\dot{x}^2 + \omega_n^2 (x+a)^2 = R_j^2 \quad \text{for } \dot{x} > 0 \quad (E_5)$$

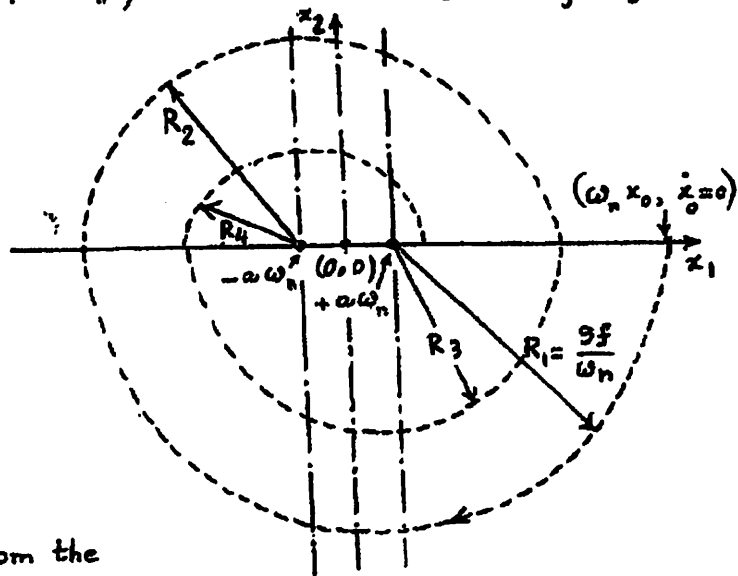
$$\dot{x}^2 + \omega_n^2 (x-a)^2 = R_{j+1}^2 \quad \text{for } \dot{x} < 0 \quad (E_6)$$

where R_j^2 and R_{j+1}^2 are integration constants which are to be computed at each switching of sign of $\dot{x}$.

We can plot the trajectories of a representative point whose coordinates are

$$x_1 = \omega_n x, \quad x_2 = \dot{x} \quad (E_7)$$

Eqs. (E₅) and (E₆) show that the trajectory is made of semicircles whose centers are located at $x = -a$ (or $x_1 = -a\omega_n$) and $x = +a$ (or $x_1 = +a\omega_n$) as shown in the following figure.



R_1 can be obtained from the initial conditions using Eq. (E₆) as:

$$R_1^2 = 0^2 + \omega_n^2 \left(\frac{10f}{\omega_n^2} - \frac{f}{\omega_n^2} \right)^2 = \left(\frac{9f}{\omega_n} \right)^2 ; \quad R_1 = \frac{9f}{\omega_n} \quad (E_8)$$

Notice that the radii of the circles $R_1, R_2, \dots$ decrease according to the relation

$$R_j = R_{j-1} - 2a\omega_n ; \quad j = 1, 2, \dots$$

and the system will stop when

$$R_k \leq 2a\omega_n$$

$$\text{Here } R_1 = \frac{9f}{\omega_n}, \quad R_2 = R_1 - \frac{2f}{\omega_n} = \frac{7f}{\omega_n}, \quad R_3 = R_2 - \frac{2f}{\omega_n} = \frac{5f}{\omega_n},$$

$$R_4 = R_3 - \frac{2f}{\omega_n} = \frac{3f}{\omega_n}, \quad R_5 = R_4 - \frac{2f}{\omega_n} = \frac{f}{\omega_n},$$

and the motion stops at this point (after five half-cycles) since $R_6 < 2a\omega_n = \frac{2f}{\omega_n}$.

6

$$\ddot{\theta} + c\dot{\theta} + \sin \theta = 0 \quad \text{or} \quad \ddot{\theta} = -c\dot{\theta} - \sin \theta$$

$$\text{Let } x = \theta \text{ and } y = \frac{dx}{dt} = \dot{\theta}$$

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -cy - \sin x \quad (E_1)$$

Equilibrium or critical point (where $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$) of this system is $(x=0, y=0)$. Linearization of Eqs. (E_1) about the equilibrium point (origin) leads to

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -cy - x$$

$$\text{or} \quad \begin{Bmatrix} dx/dt \\ dy/dt \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} \quad (E_2)$$

The eigenvalues of this system are given by

$$\left| \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\text{i.e.} \quad \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda - c \end{vmatrix} = 0 \quad \text{i.e.,} \quad \lambda^2 + \lambda c + 1 \equiv \lambda^2 + p\lambda + q = 0$$

$$\text{i.e.,} \quad \lambda_{1,2} = -\frac{c}{2} \pm \sqrt{\left(\frac{c}{2}\right)^2 - 1} \quad (E_3)$$

$$\text{If } c=0: \quad p=0; q=1; \lambda_{1,2} = \pm \sqrt{-1}$$

The origin will be a center.

If $0 < c < 2$: $p > 0$; $q > 0$; $\lambda_{1,2} = \text{complex conjugates}$
The origin will be a stable focal point (spiral point).

If $c=2$: $p > 0$; $q > 0$; $\lambda_{1,2} = \text{negative and equal}$.
The origin will be a stable nodal point.

If $c > 2$: $p > 0$; $q > 0$; If $\lambda_{1,2} = \text{negative real}$, the origin will be a stable nodal point.

If $-2 < c < 0$: $p < 0$; $q > 0$; $\lambda_{1,2} = \text{complex conjugates}$.
The origin will be an unstable focal point (spiral point).

7

$$\text{Equation of motion:} \quad \ddot{\theta} + 0.5\dot{\theta} + \sin \theta = 0.8 \quad (E_1)$$

$$\text{Let } x = \theta \text{ and } y = dx/dt$$

$$\therefore \frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\sin x - 0.5y + 0.8$$

p9 (E2)

$$\frac{dy}{dx} = \frac{-\sin x - 0.5y + 0.8}{y} \quad (E_3)$$

At $(x = \sin^{-1} 0.8, y = 0)$, $\frac{dy}{dx} = \frac{0}{0}$ and hence it will be an equilibrium point. To investigate the nature of singularity, we rewrite Eqs. (E₂) in linearized form as

$$\left. \begin{aligned} \frac{dx}{dt} &= (0)x + (1)y \\ \frac{dy}{dt} &= (0)x - 0.5y \end{aligned} \right\} \quad (E_4)$$

Thus the eigenvalues of the system are given by

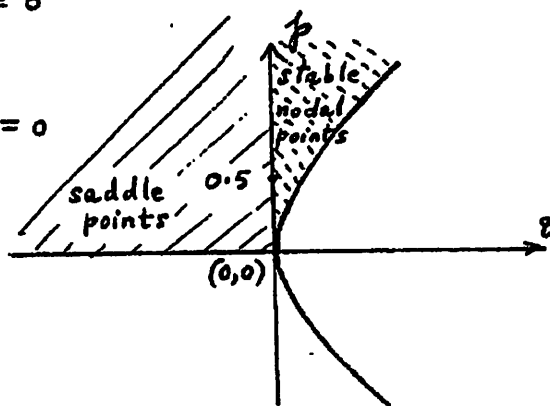
$$\left| \begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0 \quad \text{or} \quad \begin{vmatrix} -\lambda & 1 \\ 0 & -0.5 - \lambda \end{vmatrix} = 0$$

$$\text{i.e., } \lambda^2 + 0.5\lambda \equiv \lambda^2 + p\lambda + q = 0$$

$$\therefore \lambda_1 = 0, \lambda_2 = \text{negative}$$

Here $p = \text{positive}$, $q = 0$, $\lambda_1 = 0$ and $\lambda_2 = \text{negative}$.

Thus the equilibrium point falls on the border of saddle points and stable nodal points as shown in the adjacent figure.



$$\frac{dx}{dt} = (0)x + (1)y \quad (E_1)$$

$$\frac{dy}{dt} = -1.x - c.y + (0.1)x^3 \quad (E_2)$$

Eqs. (E₁) and (E₂) are zero at $(x=0, y=0)$. Hence the origin $(0,0)$ will be equilibrium point (singularity). The eigenvalues are given by

$$\left| \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} -\lambda & 1 \\ -1 & -c-\lambda \end{vmatrix} = 0$$

$$\text{i.e., } \lambda^2 + \lambda c + 1 \equiv \lambda^2 + p\lambda + q = 0$$

$$\text{i.e., } \lambda_{1,2} = \left\{ \frac{-c \pm \sqrt{c^2 - 4}}{2} \right\}$$

For $c > 0$ and $c < 2$:

$p > 0$, $q > 0$ and $\lambda_{1,2} = \text{complex conjugates}$.

Hence the origin will be a stable focus (or spiral point).

For $c \geq 2$:

$p > 0$, $q > 0$; $\lambda_{1,2} = \text{negative real}$.

Hence the origin will be a stable nodal point.

9

Van der Pol's equation: $\ddot{x} - \alpha(1-x^2)\dot{x} + x = 0, \quad \alpha > 0$ (E₁)

Assume $x(t) = x_0(t) + \alpha x_1(t) + \alpha^2 x_2(t)$ (E₂)

$\omega_0^2 = 1 = \omega^2 - \alpha \omega_1 - \alpha^2 \omega_2$ (E₃)

where $\omega_0^2 = 1 = \text{coefficient of } x \text{ in } E_1.$

Substitution of (E₂) and (E₃) into (E₁) gives

$$\begin{aligned} \alpha^0 [\ddot{x}_0 + \omega^2 x_0] + \alpha^1 [\ddot{x}_1 - \dot{x}_0 + \dot{x}_0 x_0^2 - \omega_1 x_0 + \omega^2 x_1] \\ + \alpha^2 [\ddot{x}_2 - \dot{x}_1 + \dot{x}_1 x_0^2 + 2 x_0 \dot{x}_0 x_1 - \omega_2 x_0 - \omega_1 x_1 + \omega^2 x_2] \\ + \alpha^3 [\dots] + \dots = 0 \end{aligned} \quad (E_4)$$

Setting coefficient of α^0 in (E₄) to zero, we obtain

$$\ddot{x}_0 + \omega^2 x_0 = 0, \quad \text{i.e.,} \quad x_0(t) = A_1 \cos \omega t + A_2 \sin \omega t \quad (E_5)$$

Assuming the initial conditions $x(0) = A$ and $\dot{x}(0) = 0$, we get

$$A_1 = A \text{ and } A_2 = 0. \text{ Thus } (E_5) \text{ reduces to} \quad (E_6)$$

$$x_0(t) = A \cos \omega t$$

Setting coefficient of α^1 to zero, in E₄, (E₄),

$$\begin{aligned} \ddot{x}_1 + \omega^2 x_1 &= \dot{x}_0 - \dot{x}_0 x_0^2 + \omega_1 x_0 \\ &= -A\omega \sin \omega t + A^3 \omega \sin \omega t \cos^2 \omega t + \omega_1 A \cos \omega t \\ &= \left(-A\omega + \frac{1}{4} A^3 \omega\right) \sin \omega t + \omega_1 A \cos \omega t + \frac{A^3 \omega}{4} \sin 3\omega t \leftarrow \end{aligned} \quad (E_7)$$

The coefficients of $\sin \omega t$ and $\cos \omega t$ must be zero in E₇. (E₇) to avoid secular terms. This gives

$$A = \pm 2, \quad \omega_1 = 0 \quad (E_8)$$

Thus the particular solution of (E₇) can be expressed as

$$x_1(t) = A_3 \sin 3\omega t + A_4 \cos 3\omega t \quad (E_9)$$

Substitution of (E₈) and (E₉) into E₇ gives

$$A_3 = \frac{1}{32} \frac{A^3}{\omega} \text{ and } A_4 = 0 \quad (E_{10})$$

$$\text{Thus } x_1(t) = \frac{1}{32} \frac{A^3}{\omega} \sin 3\omega t \quad (E_{11})$$

Finally, setting coefficient of α^2 in (E₄) to zero, we get

$$\ddot{x}_2 + \omega^2 x_2 = \dot{x}_1 - \dot{x}_1 x_0^2 - 2 x_0 \dot{x}_0 x_1 + \omega_2 x_0 + \omega_1 x_1 \quad (E_{12})$$

Substitution of (E₁₁), (E₆) and (E₈) into (E₁₂) leads to

$$\begin{aligned} \ddot{x}_2 + \omega^2 x_2 &= \frac{3}{32} A^3 \cos 3\omega t - \left(\frac{3}{32} A^3 \cos 3\omega t\right) A^2 \cos^2 \omega t \\ &\quad - 2(A \cos \omega t)(-A\omega \sin \omega t) \left(\frac{A^3}{32\omega} \sin 3\omega t\right) + \omega_2 A \cos \omega t \\ &= \left(-\frac{3}{128} A^5 + \frac{1}{64} A^5 + A\omega_2\right) \cos \omega t + \left(\frac{3}{32} A^3 - \frac{3}{64} A^5\right) \cos 3\omega t \\ &\quad + \left(-\frac{3}{128} A^5 - \frac{1}{64} A^5\right) \cos 5\omega t \end{aligned} \quad (E_{13})$$

P12

To avoid secular terms, the coefficient of $\cos \omega t$ in (E_{13}) must be zero. This gives $\omega_2 = \frac{1}{128} A^4$ (E₁₄)

With this, and using $A=2$, Eq. (E_{13}) reduces to

$$\ddot{x}_2 + \omega^2 x_2 = -\frac{3}{4} \cos 3\omega t - \frac{5}{4} \cos 5\omega t \quad (E_{15})$$

Assuming $x_2(t) = A_5 \cos 3\omega t + A_6 \cos 5\omega t$ (E₁₆)

we find, from Eq. (E_{15}) ,

$$A_5 = \frac{3}{32} \cdot \frac{1}{\omega^2}, \quad A_6 = \frac{5}{96} \cdot \frac{1}{\omega^2} \quad (E_{17})$$

$$\therefore x_2(t) = \frac{3}{32\omega^2} \cos 3\omega t + \frac{5}{96\omega^2} \cos 5\omega t \quad (E_{18})$$

Thus the complete solution, Eq. (E_2) and (E_3) , become

$$x(t) = 2 \cos \omega t + \frac{\alpha}{4\omega} \sin 3\omega t + \frac{3\alpha^2}{32\omega^2} \cos 3\omega t + \frac{5\alpha^2}{96\omega^2} \cos 5\omega t$$

and $\omega^2 = 1 + \frac{\alpha^2}{8}.$

P13