HW 6 Mathematics 503, Mathematical Modeling, CSUF , June 24, 2007

Nasser M. Abbasi

Summer 2007 Compiled on October 14, 2025 at 5:10pm [public]

Contents

1 animations

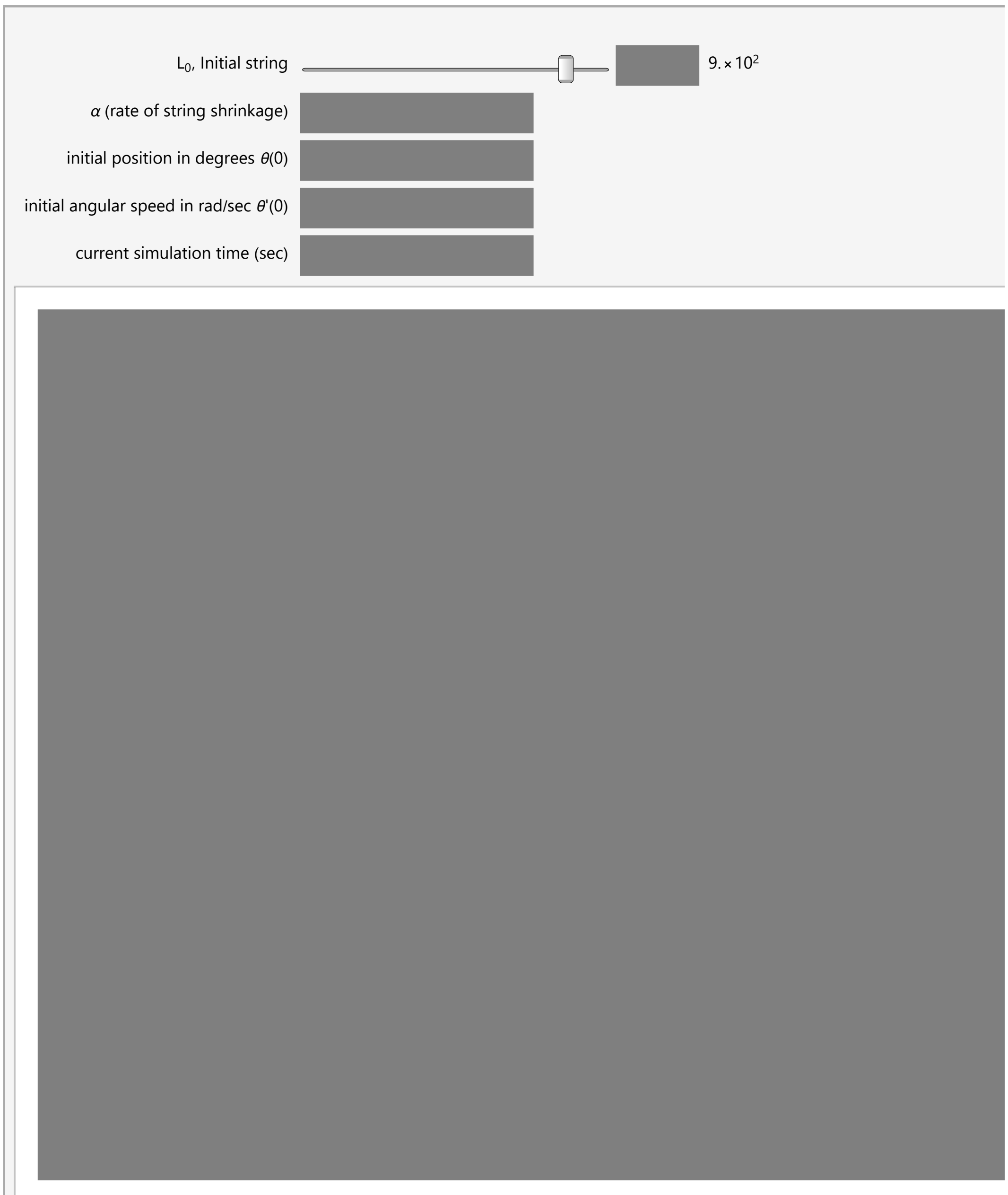
Click on shrink_pendulum3.swf (adobe flash file) (it will take 2-3 second for the simulation to start.)

Source code is

My first Manipulate program. Simulate of a pendulum which is being pulled up as it swings.

```
Remove["Global`*"];


maxLength = 1000;
u = Manipulate[Module[{g, r, simRate = 0.001},
        r = 1 - α t;
        eq = r θ''[t] - 2 α θ'[t] + g Sin[θ[t]];
        g = 9.8;
        maxTime = 1/α;
        s = θ /. First[NDSolve[{eq == 0, θ[0] == θ0 * Pi / 180, θ'[0] == v0}, θ, {t, 0,
                    maxTime - 0.001}, MaxSteps → 200000, MaxStepFraction → 0.0001]];


        getNormalizedAngle[θ_] := Module[{q = θ},
              {
                If[q >= 0, q = Mod[q, 2 Pi], {q = -q;
                     q = Mod[q, 2 Pi];
                     q = -q;
                     q = q + 2 Pi}];
                q * 180 / Pi
              }
           ];

        getQuadrent[a_] := Module[{q},
              {
                If[a ≥ 0 && a ≤ Pi / 2, q = 1];
                If[a > Pi / 2 && a ≤ Pi, q = 2];
                If[a > Pi && a ≤ 3/2 Pi, q = 3];
                If[a > 3/2 Pi && a ≤ 2 Pi, q = 4];
                q
              }
           ];
        coordinates[s_, t_, l_, α_] := Module[{θ, p = False, q},
              {θ = s[t];
                If[p, Print["t=", t, " θ=", θ * 180 / Pi]];
                If[θ >= 0, θ = Mod[θ, 2 Pi], {θ = -θ;
                     θ = Mod[θ, 2 Pi];
                     θ = -θ;
                     θ = θ + 2 Pi}];
                q = First[getQuadrent[θ]];
                If[p, Print["after mod t=", t, " θ=", θ * 180 / Pi]];
                If[p, Print["q=", q]];

                r = 1 - α t;
```

```
            If[p, Print["r=", r]];
            If[q == 1, {x = r Sin[θ], y = -r Cos[θ]}];
            If[q == 2, {θ = θ - Pi/2; x = r Cos[θ], y = r Sin[θ]}];
            If[q == 3, {θ = θ - Pi; x = -r Sin[θ], y = r Cos[θ]}];
            If[q == 4, {θ = θ - 3/2 Pi; x = -r Cos[θ], y = -r Sin[θ]}];
            If[p, Print["x=", x, " y=", y]];
            If[p, Print["******"]];
            {x, y}
          }
        ];
   d = coordinates[s, ts, l, α];
   Graphics[ { {Gray, Dashed , Line[{{-maxLength, 0}, {maxLength, 0}}]},
         {Gray, Dashed , Line[{{0, -maxLength}, {0, maxLength}}]},
         Text["θ= " <> ToString[First[getNormalizedAngle[ s[ts] ] ]], {.4 l, .9 l}],
         Text["current string length L(t)=L0-α t=" <> ToString[r], {.4 l, .8 l}],
         Text["current time t (sec)=" <> ToString[ts], {.4 l, .7 l}],
         Text["L[t] θ''[t]-2 α θ'[t]+g Sin[θ[t]]=0", {.4 l, .6 l}],
         Line[{ {0, 0}, First[d]} ],
         PointSize[.03], Red, Point[First[d]]},
      PlotRange → {{-l - .1 l, l + .1 l}, {-l - .1 l, l + .1 l}}, ImageSize → {450, 325}]
],
{{l, 0.9 maxLength, "L0, Initial string"}, 1, maxLength, 1, Appearance → "Labeled"},
{{α, 1, "α (rate of string shrinkage)"}, 0.1, 10, 0.1, Appearance → "Labeled"},
{{θ0, 45, "initial position in degrees θ(0)"}, 0, 360, 1, Appearance → "Labeled"},
{{v0, 0, "initial angular speed in rad/sec θ'(0)"},
   0, 100, 1, Appearance → "Labeled"},
{{ts, 0, "current simulation time (sec)"}, 0, 1/α - 0.001, (0.0001 1/α),
   Appearance → "Labeled"}, ContinuousAction → True, AutorunSequencing -> {{5, 60}},
FrameLabel → "A shrinking Pendulum. by Nasser Abbasi, July 2007"]
```



A shrinking Pendulum. by Nasser Abbasi, July 2007

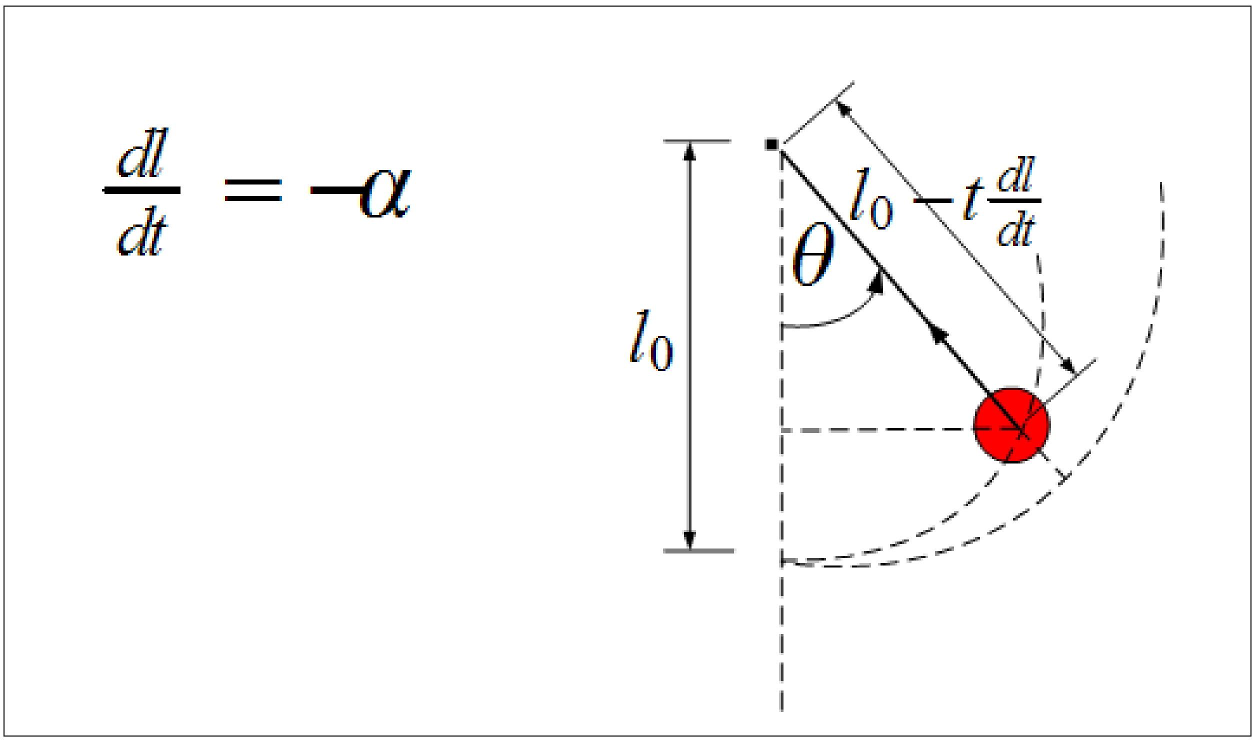
```
(*Export["shrink_pendulum3.swf",u];*)
```

2 Problem 1 (section 3.5,#9, page 197)

problem:

Consider a simple plane pendulum with a bob of mass m attached to a string of length l. After the pendulum is set in motion the string is shortened by a constant rate $\frac{dl}{dt} = -\alpha$. Formulate Hamilton's principle and determine the equation of motion. Compare the Hamiltonian to the total energy. Is energy conserved?

Solution:



Assume initial string length is l, and assume $t(0) = 0$, then at time t we have

$$r(t) = l - \alpha t$$

K.E. First note that

$$\begin{aligned}\dot{x} &= \frac{d}{dt}(r(t)\sin\theta(t)) \\ &= \dot{r}\sin\theta + r\cos\theta\dot{\theta}\end{aligned}$$

and

$$\begin{aligned}\dot{y} &= \frac{d}{dt}(r(t)\cos\theta(t)) \\ &= \dot{r}\cos\theta - r\sin\theta\dot{\theta}\end{aligned}$$

Now

$$\begin{aligned}T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2}m\Big(\big(\dot{r}\sin\theta + r\cos\theta\dot{\theta}\big)^2 + \big(\dot{r}\cos\theta - r\sin\theta\dot{\theta}\big)^2\Big) \\ &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \\ &= \frac{1}{2}m(\alpha^2 + r^2\dot{\theta}^2)\end{aligned}$$

P.E.

$$\begin{aligned}V &= mgl - mg(r\cos\theta) \\ &= mg(l - r\cos\theta)\end{aligned}$$

Hence

$$\begin{aligned}J(\theta) &= \int_0^T (T - V)\,dt \\ &= \int_0^T \frac{1}{2}m(\alpha^2 + r^2\dot{\theta}^2) - mg(l - r\cos\theta)\ \ dt\end{aligned}$$

Hence

$$L(t,\theta(t),\dot{\theta}(t)) = \frac{1}{2}m(\alpha^2 + r^2\dot{\theta}^2) - mg(l - r\cos\theta) \tag{1}$$

Hence the Euler-Lagrange equations are

$$L_\theta - \frac{d}{dt}(L_{\dot{\theta}}) = 0 \tag{2}$$

But

$$\boxed{L_\theta = -mgr\sin\theta}$$

and

$$\boxed{L_{\dot{\theta}} = mr^2\dot{\theta}}$$

and

$$\frac{d}{dt}(L_{\dot{\theta}}) = m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta})$$

But $\dot{r} = -\alpha$, the above becomes

$$\boxed{\tfrac{d}{dt}(L_{\dot{\theta}}) = m(r^2\ddot{\theta} - 2r\alpha\dot{\theta})}$$

Hence $L_\theta - \frac{d}{dt}(L_{\dot{\theta}}) = 0$ becomes

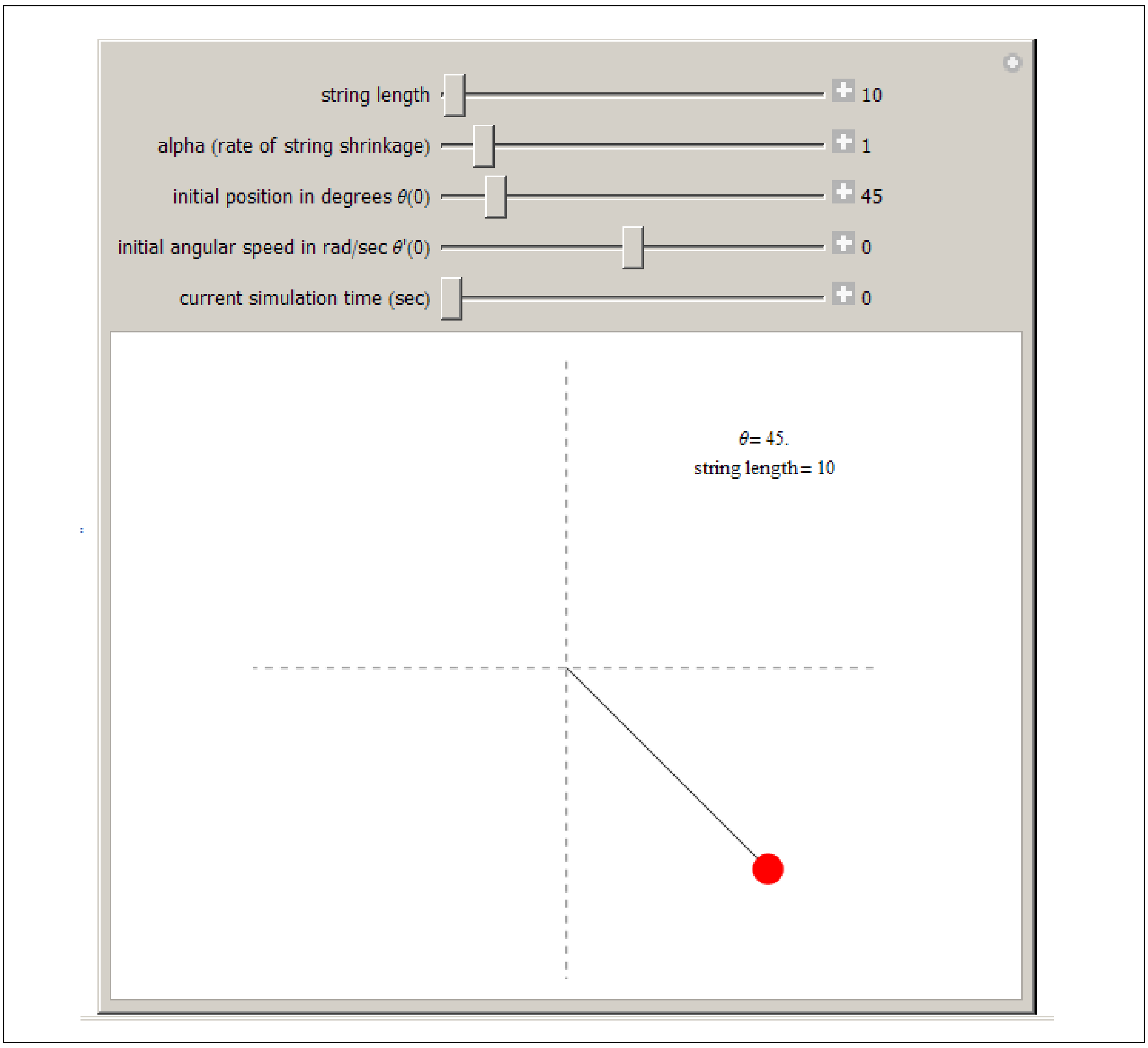
$$\begin{aligned} -mgr\sin\theta - m(r^2\ddot{\theta} - 2\alpha\dot{\theta}r) &= 0 \\ r^2\ddot{\theta} - 2\alpha\dot{\theta}r + gr\sin\theta &= 0 \end{aligned}$$

Hence the ODE becomes, after dividing by common factor r

$$\boxed{r\ddot{\theta} - 2\alpha\dot{\theta} + g\sin\theta = 0}$$

This is a second order nonlinear differential equation. Notice that when $l = \alpha t$ it will mean that the string has been pulled all the way back to the pivot and $r(t) = 0$. So when running the solution it needs to run from $t = 0$ up to $t = \frac{l}{\alpha}$.

A small simulation was done for the above solution which can be run for different parameters to see the effect more easily. Here is a screen shot.



Now we need to determine the Hamiltonian of the system.

$$H = -L(t, \theta, \phi(t, \theta, p)) + \phi(t, \theta, p)\, p \tag{3}$$

Where we define a new variable p called the canonical momentum by

$$\begin{aligned} p &\equiv L_{\dot\theta}(t, \theta, \dot\theta) \\ &= mr^2\dot\theta \end{aligned}$$

Hence

$$\dot\theta = \frac{p}{mr^2}$$

This implies that

$$\phi(t, \theta, p) = \frac{p}{mr^2}$$

Then from (1) and (3), we now calculate H

$$\begin{aligned} H &= -L(t, \theta, \phi(t, \theta, p)) + \phi(t, \theta, p)\, p \\ &= -\left\{\frac{1}{2}m\left(\alpha^2 + r^2\left(\frac{p}{mr^2}\right)^2\right) - mg(l - r\cos\theta)\right\} + \left(\frac{p}{mr^2}\right)p \\ &= -\frac{1}{2}m\left(\alpha^2 + \frac{p^2}{m^2r^2}\right) + mg(l - r\cos\theta) + \frac{p^2}{mr^2} \\ &= -\frac{1}{2}m\alpha^2 - \frac{1}{2}\frac{p^2}{mr^2} + mg(l - r\cos\theta) + \frac{p^2}{mr^2} \end{aligned}$$

Hence the Hamiltonian is

$$H = \frac{1}{2}\frac{p^2}{mr^2} + mg(l - r\cos\theta) - \frac{1}{2}m\alpha^2 \tag{5}$$

Now we are asked to compare H to the total energy. The total instantaneous energy of the system is $(T+V)$, hence we need to determine if $H = T+V$ or not.

$$T + V = \frac{1}{2}m(\alpha^2 + r^2\dot{\theta}^2) + mg(l - r\cos\theta) \tag{6}$$

To make comparing (5) and (6) easier, I need to either replace p by $mr^2\dot{\theta}$ in (5) or replace $\dot{\theta}$ by $\frac{p}{mr^2}$. Lets do the later, hence (6) becomes

$$\begin{aligned} T + V &= \frac{1}{2}m\left(\alpha^2 + r^2\left(\frac{p}{mr^2}\right)^2\right) + mg(l - r\cos\theta) \\ &= \frac{1}{2}m\left(\alpha^2 + \frac{p^2}{m^2r^2}\right) + mg(l - r\cos\theta) \\ &= \frac{1}{2}m\alpha^2 + \frac{1}{2}\frac{p^2}{mr^2} + mg(l - r\cos\theta) \end{aligned} \tag{7}$$

If H is the total energy, then (7)-(6) should come out to be zero, lets find out

$$\begin{aligned} H - (T+V) &= \left(\frac{1}{2}\frac{p^2}{mr^2} + mg(l - r\cos\theta) - \frac{1}{2}m\alpha^2\right) - \left(\frac{1}{2}m\alpha^2 + \frac{1}{2}\frac{p^2}{mr^2} + mg(l - r\cos\theta)\right) \\ &= -m\alpha^2 \end{aligned}$$

Hence we see that

$$\boxed{H - (T+V) \neq 0}$$

Hence H does not represents the total energy, and the $\boxed{\text{energy of the system is not conserved}}$.[1]

[1] Reading a reference on Noether's theorem, total energy is written as $-(T+V)$ not $(T+V)$, this would not make a difference in showing that $H \neq$ total energy, just different calculations results as shown below, but the same conclusion

$$-(T+V) = -\left(\frac{1}{2}m(\alpha^2 + r^2\dot{\theta}^2) + mg(l - r\cos\theta)\right) \tag{6}$$

To make comparing (5) and (6) easier, I need to either replace p by $mr^2\dot{\theta}$ in (5) or replace $\dot{\theta}$ by $\frac{p}{mr^2}$, let do the later, hence (6) becomes

$$\begin{aligned} -(T+V) &= -\left(\frac{1}{2}m\left(\alpha^2 + r^2\left(\frac{p}{mr^2}\right)^2\right) + mg(l - r\cos\theta)\right) \\ &= -\left(\frac{1}{2}m\left(\alpha^2 + \frac{p^2}{m^2r^2}\right) + mg(l - r\cos\theta)\right) \\ &= -\left(\frac{1}{2}m\alpha^2 + \frac{1}{2}\frac{p^2}{mr^2} + mg(l - r\cos\theta)\right) \end{aligned} \tag{7}$$

If H is the total energy, then (7)-(6) should come out to be zero, lets find out

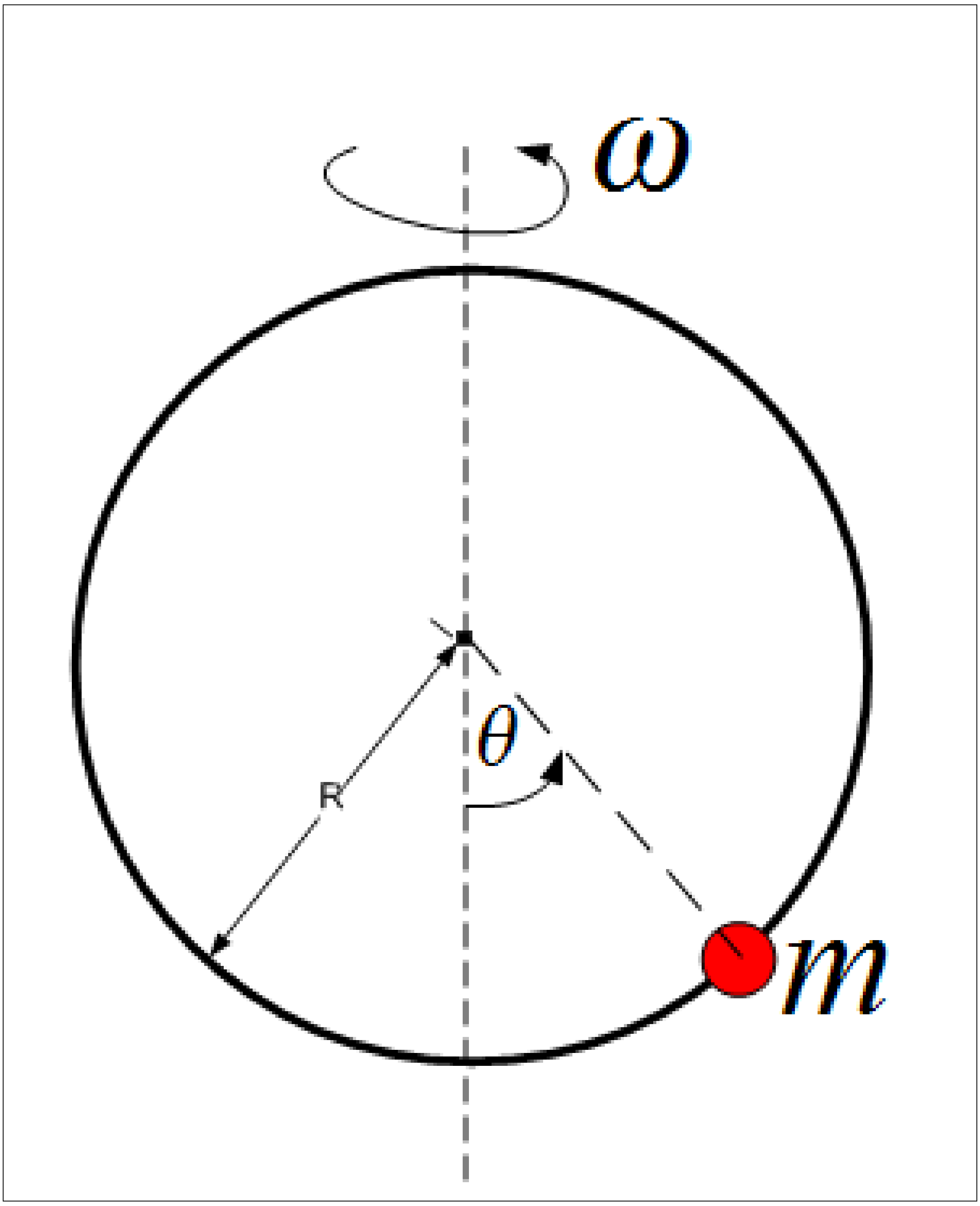
$$\begin{aligned} H - (-(T+V)) &= \left(\frac{1}{2}\frac{p^2}{mr^2} + mg(l - r\cos\theta) - \frac{1}{2}m\alpha^2\right) + \left(\frac{1}{2}m\alpha^2 + \frac{1}{2}\frac{p^2}{mr^2} + mg(l - r\cos\theta)\right) \\ &= \frac{p^2}{mr^2} + 2mg(l - r\cos\theta) \end{aligned}$$

Now we ask, can the above be zero? Since $\frac{p^2}{mr^2}$ is always ≥ 0, and since $(l - r\cos\theta)$ represents the remaining length of the string, hence it is a positive quantity (until such time the string has been pulled all the way in), Therefore RHS above > 0. Hence H does not represent the total energy of the system. Hence the energy is not conserved.

3 Problem 1 (section 3.5,#9, page 197)

problem: A bead of mass m slides down the rim of a circular hoop of radius R. The hoop stands vertically and rotates about its diameter with angular velocity ω. Determine the equation of motion of the bead.

Answer:



Kinetic energy T is made up of 2 components, one due to motion of the bead along the hoop itself with speed $R\dot{\theta}$, and another due to motion with angular speed ω which has speed given by $R\sin\theta\omega$

Hence

$$
\begin{aligned}
T &= \frac{1}{2}m\left(\left(R\dot{\theta}\right)^2 + \left(R\sin\theta\omega\right)^2\right) \\
&= \frac{1}{2}mR^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right)
\end{aligned}
$$

P.E. V is due to the bead movement up and down the hoop, which is the standard V for a pendulum given by

$$V = mgR(1 - \cos\theta)$$

Hence

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2} mR^2 \left(\dot{\theta}^2 + \omega^2 \sin^2\theta\right) - mgR(1-\cos\theta) \end{aligned}$$

Hence

$$L_\theta = mR^2 \left(\omega^2 \sin\theta \cos\theta\right) - mgR\sin\theta$$

and

$$L_{\dot{\theta}} = mR^2 \dot{\theta}$$

Hence

$$\frac{d}{dt} L_{\dot{\theta}} = mR^2 \ddot{\theta}$$

Hence

$$\begin{aligned} L_\theta - \frac{d}{dt} L_{\dot{\theta}} &= 0 \\ mR^2 \left(\omega^2 \sin\theta\cos\theta\right) - mgR\sin\theta - mR^2\ddot{\theta} &= 0 \\ \omega^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta - \ddot{\theta} &= 0 \end{aligned}$$

Hence the ODE is

$$\boxed{\ddot{\theta} + \sin\theta\left(\tfrac{g}{R} - \omega^2\cos\theta\right) = 0}$$

With initial conditions $\theta(0) = \theta_0, \dot{\theta}(0) = \dot{\theta}_0$