Euler's solution to the Basel problem

Nasser M. Abbasi

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1 Introduction

This gives step by step the solution Euler came up to show that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^2} &= 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\\ &= \frac{\pi^2}{6}\end{aligned}$$

2 Solution steps

Starting with the Taylor series of $\sin x$

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$$

We want only non-zero roots. But $\sin x$ has root at $x=0$. Hence dividing both sides by x gives

$$\frac{\sin x}{x} = 1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots \tag{A}$$

The left side is the $\operatorname{sinc}(x)$ which has no zero root. All its roots are at integer multiple of π. Using Fundamental theory of algebra which says for finite polynomial of degree n it can be written as product of its n roots, then the above becomes

$$\frac{\sin x}{x} = \cdots\left(1-\frac{x}{-3\pi}\right)\left(1-\frac{x}{-2\pi}\right)\left(1-\frac{x}{-\pi}\right)\left(1-\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1-\frac{x}{3\pi}\right)\cdots \tag{1}$$

Euler assumed this applies to infinite polynomial as it does for finite polynomial. It took 100 years for Karl Weierstrass to shows this is true with his Weierstrass factorization theorem. (1) can be rewritten as

$$
\begin{aligned}
\frac{\sin x}{x} &= \overbrace{\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{\pi}\right)}\overbrace{\left(1+\frac{x}{2\pi}\right)\left(1-\frac{x}{2\pi}\right)}\overbrace{\left(1+\frac{x}{3\pi}\right)\left(1-\frac{x}{3\pi}\right)}\cdots && (2)\\
&= \left(1-\frac{1}{\pi^2}x^2\right)\left(1-\frac{1}{4\pi^2}x^2\right)\left(1-\frac{1}{9\pi^2}x^2\right)\cdots && (3)
\end{aligned}
$$

Where each term in (3) is the product of each two terms under brackets in (2). Hence (3) becomes (adding two more terms to the end)

$$
\frac{\sin x}{x} = \left(1-\frac{1}{\pi^2}x^2\right)\left(1-\frac{1}{4\pi^2}x^2\right)\left(1-\frac{1}{9\pi^2}x^2\right)\left(1-\frac{1}{16\pi^2}x^2\right)\left(1-\frac{1}{25\pi^2}x^2\right)\cdots
$$

Multiplying throughout gives

$$
\begin{aligned}
\frac{\sin x}{x} &= \overbrace{\left(1-\frac{1}{\pi^2}x^2\right)\left(1-\frac{1}{4\pi^2}x^2\right)}\left(1-\frac{1}{9\pi^2}x^2\right)\left(1-\frac{1}{16\pi^2}x^2\right)\left(1-\frac{1}{25\pi^2}x^2\right)\cdots\\
&= \overbrace{\left(1-\frac{1}{4\pi^2}x^2-\frac{1}{\pi^2}x^2+\frac{x^4}{4\pi^4}\right)\left(1-\frac{1}{9\pi^2}x^2\right)}\left(1-\frac{1}{16\pi^2}x^2\right)\left(1-\frac{1}{25\pi^2}x^2\right)\cdots\\
&= \overbrace{\left(1-\frac{1}{9\pi^2}x^2+\frac{1}{36\pi^4}x^4-\frac{1}{4\pi^2}x^2+\frac{1}{9\pi^4}x^4-\frac{1}{\pi^2}x^2+\frac{1}{4\pi^4}x^4-\frac{1}{36\pi^6}x^6\right)}\left(1-\frac{1}{25\pi^2}x^2\right)\cdots
\end{aligned}
$$

And so on. Collecting only the terms with x^2 gives

$$
\frac{\sin x}{x} = \left[x^2\left(-\frac{1}{\pi^2}-\frac{1}{4\pi^2}-\frac{1}{9\pi^2}-\frac{1}{16\pi^2}-\cdots\right)\right]+x^4(\cdots)+x^6(\cdots)+\cdots \tag{4}
$$

But from (A) we had

$$
\frac{\sin x}{x} = 1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots
$$

Hence comparing coefficients of x^2 shows that

$$
-\frac{1}{\pi^2}-\frac{1}{4\pi^2}-\frac{1}{9\pi^2}-\frac{1}{16\pi^2}-\cdots = -\frac{1}{3!}
$$

Hence

$$
\begin{aligned}
\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\frac{1}{16\pi^2}+\cdots &= \frac{1}{6}\\
1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots &= \frac{\pi^2}{6}
\end{aligned}
$$

Or

$$
\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}
$$

The above derivation made Euler very famous at the time as this was not known result. The same process can be used to find $\sum_{n=1}^{\infty}\frac{1}{n^4}$ and for all other even powers. Currently there is no known result for odd powers. But it was proved that for $\sum_{n=1}^{\infty}\frac{1}{n^3}$ the sum must be irrational number.

3 References

1. https://en.wikipedia.org/wiki/Basel_problem wikipedia article on Basel problem.

2. https://www.math.cmu.edu/~bwsulliv/basel-problem.pdf Basel problem different proofs by Sullivan.

3. https://web.williams.edu/Mathematics/sjmiller/public_html/hudson/Emmell,%20Amber_Euler%20&%20The%20Basel%20Problem.pdf Emmell article on Basel problem.