

Computer Algebra Independent Integration Tests

Summer 2024

1-Algebraic-functions/1.1-Binomial/1.1.3-General-
binomial/1.1.3.4/57-1.1.3.4-d

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CHAPTER 1

INTRODUCTION

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This report gives the result of running the computer algebra independent integration test. The download section in on the main webpage contains links to download the problems in plain text format used for all CAS systems. The number of integrals in this report is [43]. This is test number [57].

1.1 Listing of CAS systems tested

The following are the CAS systems tested:

1. Mathematica 14 (January 9, 2024) on windows 10 pro.
2. Rubi 4.17.3 (Sept 25, 2023) on Mathematica 14 on windows 10m pro.
3. Maple 2024 (March 1, 2024) on windows 10 pro.
4. Maxima 5.47 (June 1, 2023) using Lisp SBCL 2.4.0 on Linux Manjaro 23.1.2 KDE via sagemath 10.3.
5. FriCAS 1.3.10 built with sbcl 2.3.11 (January 10, 2024) on Linux Manjaro 23.1.2 KDE via sagemath 10.3.
6. Giac/Xcas 1.9.0-99 on Linux via sagemath 10.3.
7. Sympy 1.12 using Python 3.11.6 (Nov 14 2023, 09:36:21) [GCC 13.2.1 20230801] on Linux Manjaro 23.1.2 KDE.
8. Mupad using Matlab 2021a with Symbolic Math Toolbox Version 8.7 on windows 10.
9. Reduce CSL rev 6687 (January 9, 2024) on Linux Manjaro 23.1.2 KDE.

Maxima and Fricas and Giac are called using Sagemath. This was done using Sagemath `integrate` command by changing the name of the algorithm to use the different CAS systems.

Sympy was run directly in Python not via sagemath.

Reduce was called directly.

1.2 Results

Important note: A number of problems in this test suite have no antiderivative in closed form. This means the antiderivative of these integrals can not be expressed in terms of elementary, special functions or Hypergeometric2F1 functions. RootSum and RootOf are not allowed. If a CAS returns the above integral unevaluated within the time limit, then the result is counted as passed and assigned an A grade.

However, if CAS times out, then it is assigned an F grade even if the integral is not integrable, as this implies CAS could not determine that the integral is not integrable in the time limit.

If a CAS returns an antiderivative to such an integral, it is assigned an A grade automatically and this special result is listed in the introduction section of each individual test report to make it easy to identify as this can be important result to investigate.

The results given in in the table below reflects the above.

System	% solved	% Failed
Rubi	100.00 (43)	0.00 (0)
Mathematica	100.00 (43)	0.00 (0)
Maple	100.00 (43)	0.00 (0)
Fricas	100.00 (43)	0.00 (0)
Maxima	100.00 (43)	0.00 (0)
Reduce	100.00 (43)	0.00 (0)
Giac	97.67 (42)	2.33 (1)
Mupad	86.05 (37)	13.95 (6)
Sympy	41.86 (18)	58.14 (25)

Table 1.1: Percentage solved for each CAS

The table below gives additional break down of the grading of quality of the antiderivatives generated by each CAS. The grading is given using the letters A,B,C and F with A being the best quality. The grading is accomplished by comparing the antiderivative generated with the optimal antiderivatives included in the test suite. The following table describes the meaning of these grades.

grade	description
A	Integral was solved and antiderivative is optimal in quality and leaf size.
B	Integral was solved and antiderivative is optimal in quality but leaf size is larger than twice the optimal antiderivatives leaf size.
C	Integral was solved and antiderivative is non-optimal in quality. This can be due to one or more of the following reasons <ol style="list-style-type: none"> 1. antiderivative contains a hypergeometric function and the optimal antiderivative does not. 2. antiderivative contains a special function and the optimal antiderivative does not. 3. antiderivative contains the imaginary unit and the optimal antiderivative does not.
F	Integral was not solved. Either the integral was returned unevaluated within the time limit, or it timed out, or CAS hanged or crashed or an exception was raised.

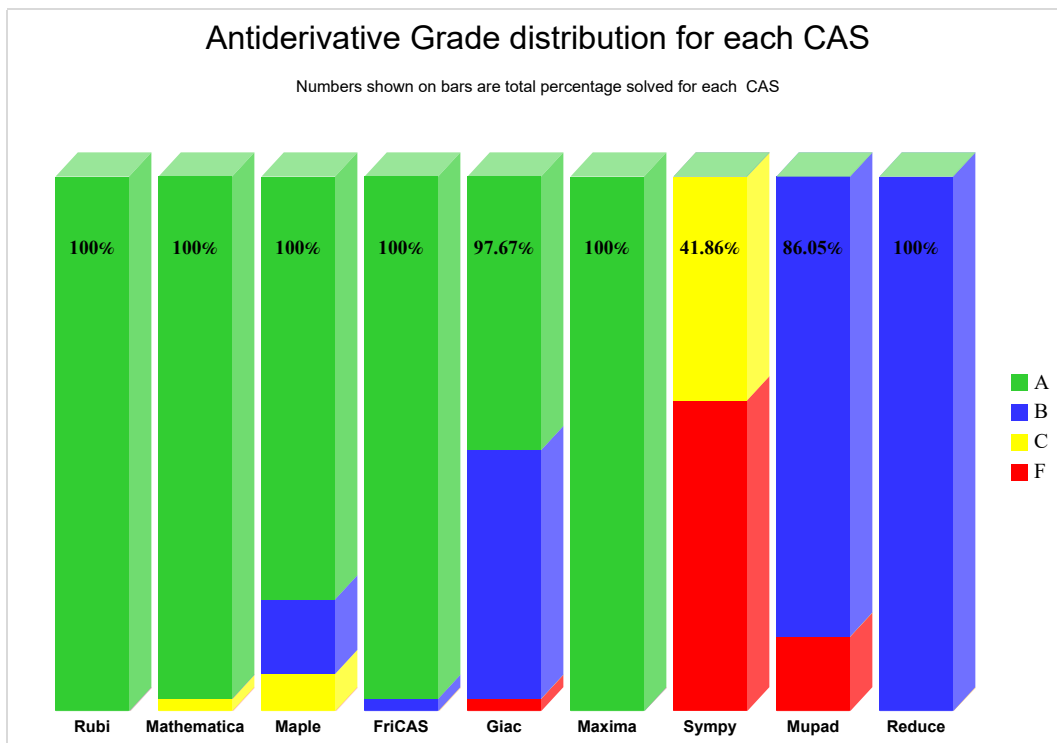
Table 1.2: Description of grading applied to integration result

Grading is implemented for all CAS systems. Based on the above, the following table summarizes the grading for this test suite.

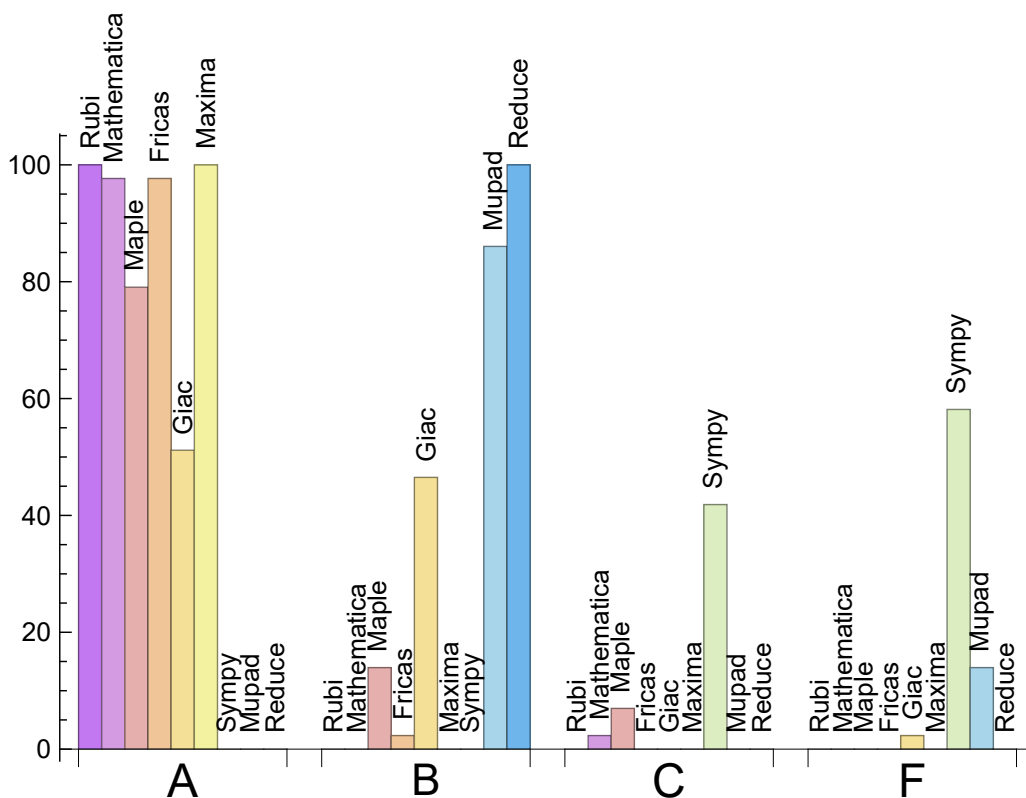
System	% A grade	% B grade	% C grade	% F grade
Rubi	100.000	0.000	0.000	0.000
Maxima	100.000	0.000	0.000	0.000
Mathematica	97.674	0.000	2.326	0.000
Fricas	97.674	2.326	0.000	0.000
Maple	79.070	13.953	6.977	0.000
Giac	51.163	46.512	0.000	2.326
Mupad	0.000	86.047	0.000	13.953
Reduce	0.000	100.000	0.000	0.000
Sympy	0.000	0.000	41.860	58.140

Table 1.3: Antiderivative Grade distribution of each CAS

The following is a Bar chart illustration of the data in the above table.



The figure below compares the grades of the CAS systems.



The following table shows the distribution of the different types of failures for each CAS. There are 3 types failures. The first is when CAS returns the input within the time limit, which means it could not solve it. This is the typical failure and given as **F**.

The second failure is due to time out. CAS could not solve the integral within the 3 minutes time limit which is assigned. This is assigned **F(-1)**.

The third is due to an exception generated, indicated as **F(-2)**. This most likely indicates an interface problem between sagemath and the CAS (applicable only to FriCAS, Maxima and Giac) or it could be an indication of an internal error in the CAS itself. This type of error requires more investigation to determine the cause.

System	Number failed	Percentage normal failure	Percentage time-out failure	Percentage exception failure
Rubi	0	0.00	0.00	0.00
Mathematica	0	0.00	0.00	0.00
Fricas	0	0.00	0.00	0.00
Maple	0	0.00	0.00	0.00
Maxima	0	0.00	0.00	0.00
Reduce	0	0.00	0.00	0.00
Giac	1	100.00	0.00	0.00
Mupad	6	0.00	100.00	0.00
Sympy	25	36.00	64.00	0.00

Table 1.4: Failure statistics for each CAS

1.3 Time and leaf size Performance

The table below summarizes the performance of each CAS system in terms of time used and leaf size of results.

Mean size is the average leaf size produced by the CAS (before any normalization). The Normalized mean is relative to the mean size of the optimal anti-derivative given in the input files.

For example, if CAS has **Normalized mean** of 3, then the mean size of its leaf size is 3 times as large as the mean size of the optimal leaf size.

Median size is value of leaf size where half the values are larger than this and half are smaller (before any normalization). i.e. The Middle value.

Similarly the **Normalized median** is relative to the median leaf size of the optimal.

For example, if a CAS has Normalized median of 1.2, then its median is 1.2 as large as the median leaf size of the optimal.

System	Mean time (sec)
Maxima	0.07
Fricas	0.10
Maple	0.14
Giac	0.16
Reduce	0.22
Rubi	0.37
Mathematica	0.40
Mupad	13.29
Sympy	47.45

Table 1.5: Time performance for each CAS

System	Mean size	Normalized mean	Median size	Normalized median
Mathematica	80.09	0.91	74.00	0.79
Fricas	87.65	0.97	80.00	0.89
Rubi	91.93	0.97	85.00	1.00
Maxima	101.63	1.04	89.00	1.02
Maple	114.60	1.26	108.00	1.18
Reduce	139.81	1.46	125.00	1.26
Giac	187.69	1.86	144.00	1.69
Sympy	276.44	4.55	175.00	2.77
Mupad	444.78	4.03	152.00	1.93

Table 1.6: Leaf size performance for each CAS

1.4 Performance based on number of rules Rubi used

This section shows how each CAS performed based on the number of rules Rubi needed to solve the same integral. One diagram is given for each CAS.

On the y axis is the percentage solved which Rubi itself needed the number of rules given the x axis. These plots show that as more rules are needed then most CAS system percentage of solving decreases which indicates the integral is becoming more complicated to solve.

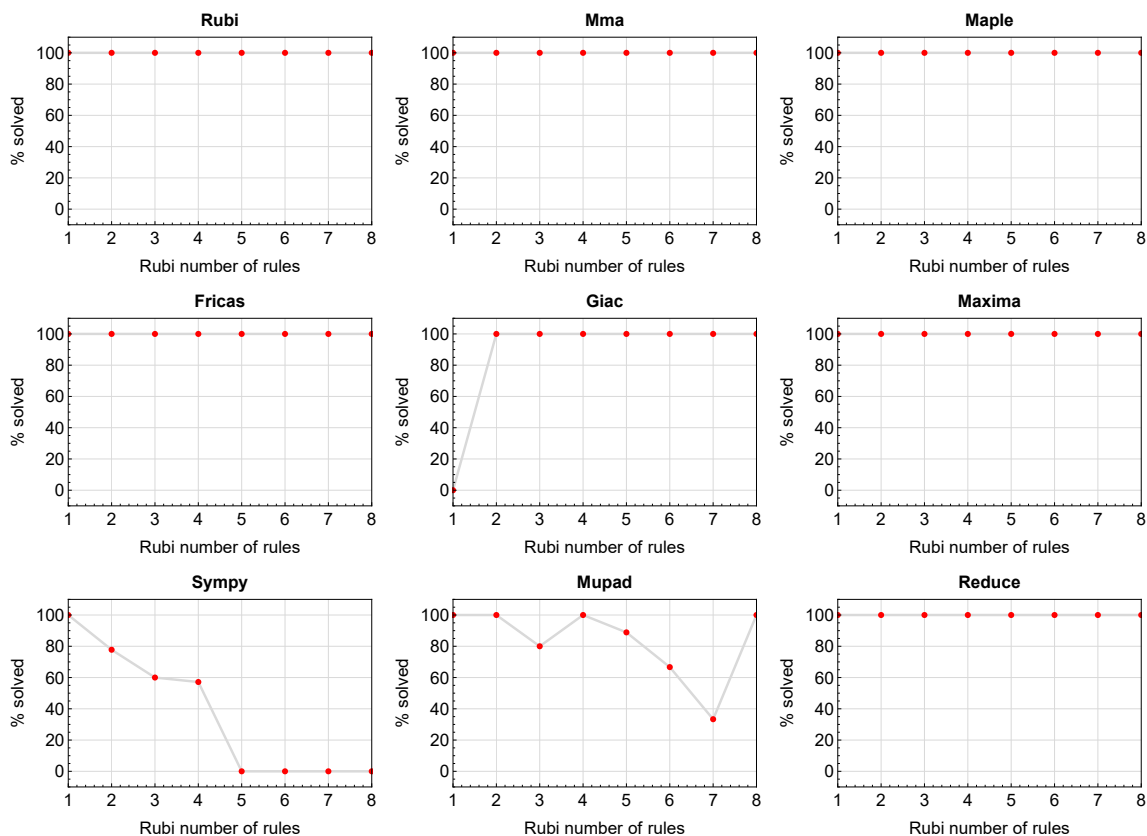


Figure 1.1: Solving statistics per number of Rubi rules used

1.5 Performance based on number of steps Rubi used

This section shows how each CAS performed based on the number of steps Rubi needed to solve the same integral. Note that the number of steps Rubi needed can be much higher than the number of rules, as the same rule could be used more than once.

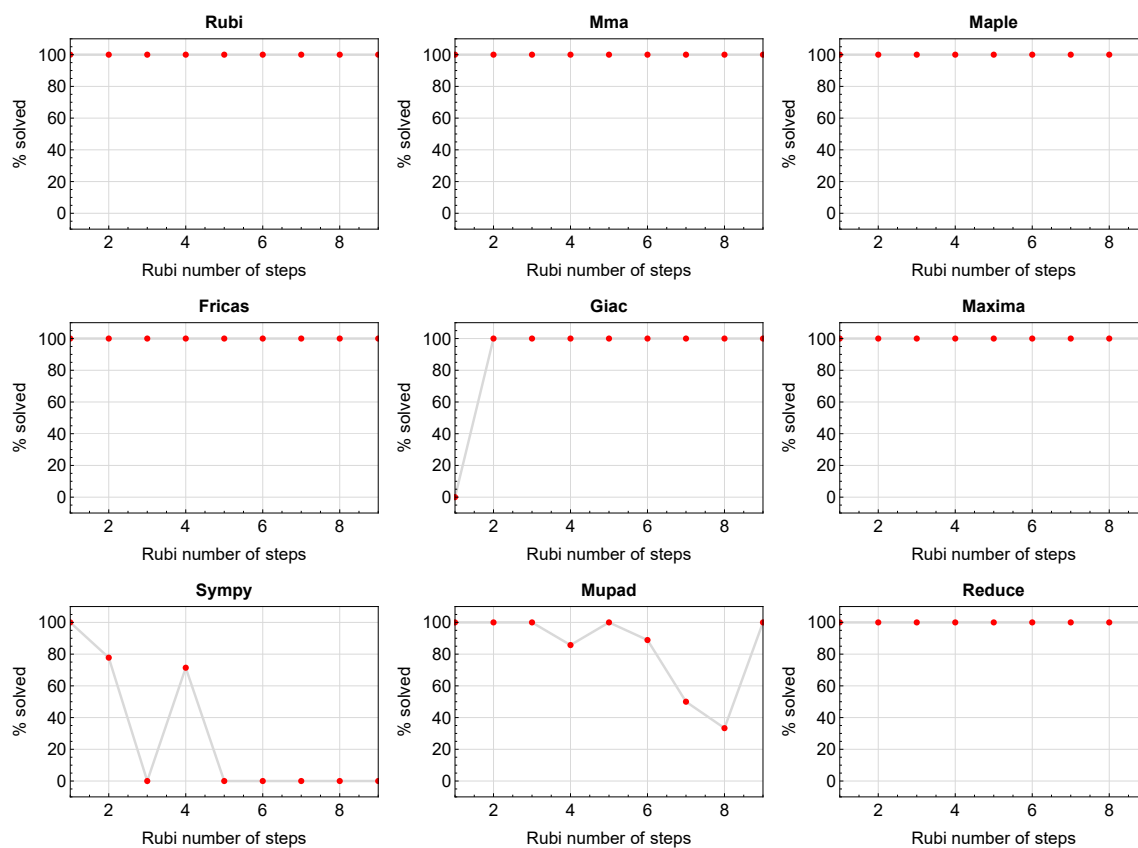


Figure 1.2: Solving statistics per number of Rubi steps used

The above diagram shows that the percentage of solved integrals decreases for most CAS systems as the number of steps increases. As expected, for integrals that required less steps by Rubi, CAS systems had more success which indicates the integral was not as hard to solve. As Rubi needed more steps to solve the integral, the solved percentage decreased for most CAS systems which indicates the integral is becoming harder to solve.

1.6 Solved integrals histogram based on leaf size of result

The following shows the distribution of solved integrals for each CAS system based on leaf size of the antiderivatives produced by each CAS. It shows that most integrals solved produced leaf size less than about 100 to 150. The bin size used is 40.

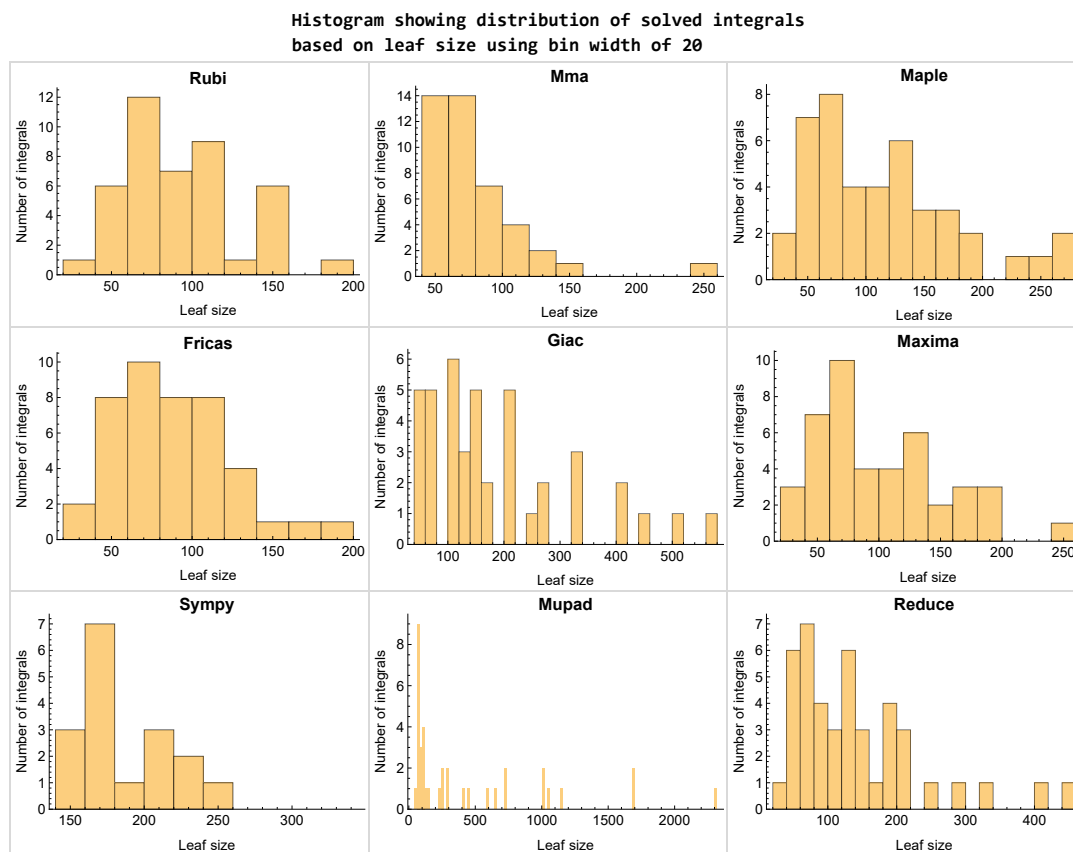


Figure 1.3: Solved integrals based on leaf size distribution

1.7 Solved integrals histogram based on CPU time used

The following shows the distribution of solved integrals for each CAS system based on CPU time used in seconds. The bin size used is 0.1 second.

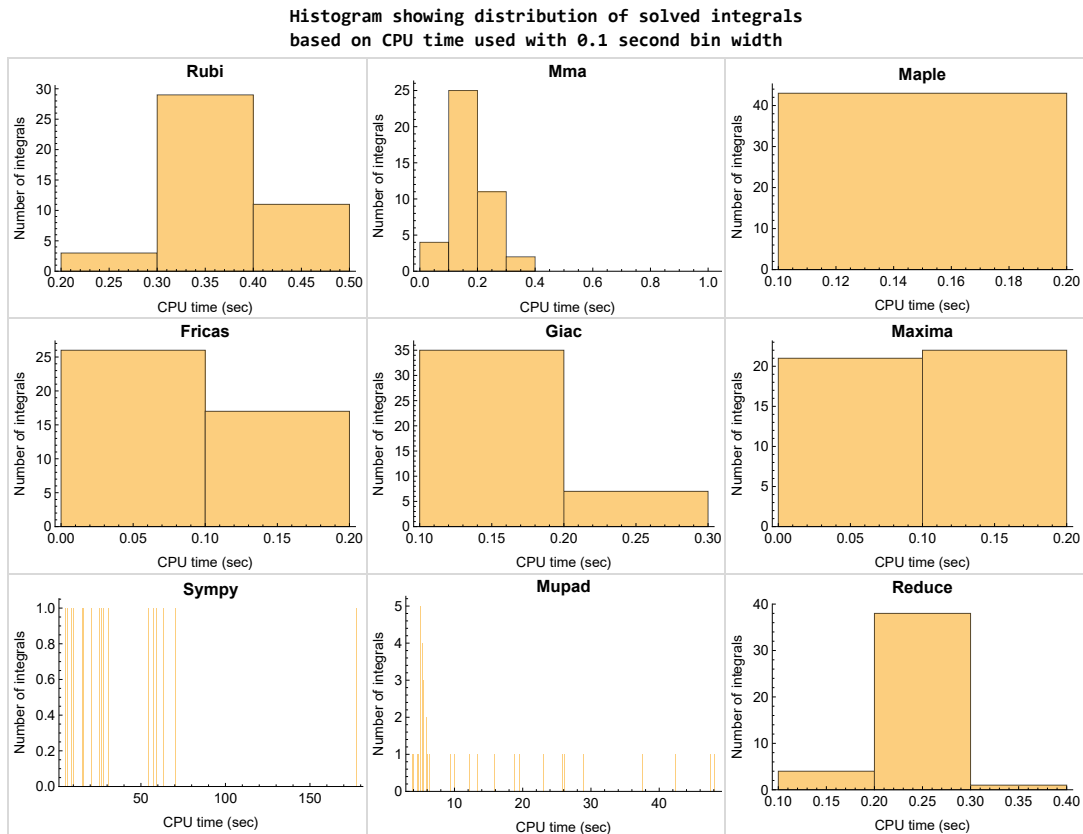


Figure 1.4: Solved integrals histogram based on CPU time used

1.8 Leaf size vs. CPU time used

The following shows the relation between the CPU time used to solve an integral and the leaf size of the antiderivative.

The result for Fracas, Maxima and Giac is shifted more to the right than the other CAS system due to the use of sagemath to call them, which causes an initial slight delay in the timing to start the integration due to overhead of starting a new process each time. This should also be taken into account when looking at the timing of these three CAS systems. Direct calls not using sagemath would result in faster timings, but current implementation uses sagemath as this makes testing much easier to do.

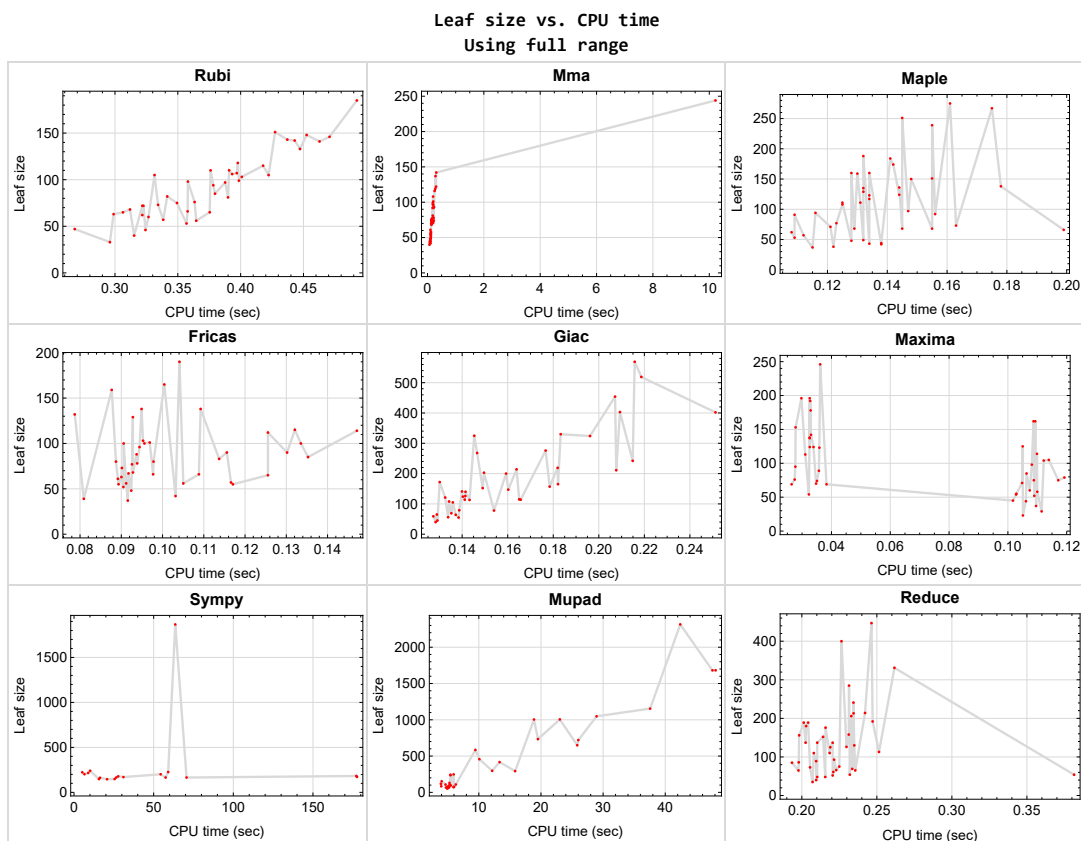


Figure 1.5: Leaf size vs. CPU time. Full range

1.9 list of integrals with no known antiderivative

{}

1.10 List of integrals solved by CAS but has no known antiderivative

Rubi {}

Mathematica {}

Maple {}

Maxima {}

Fricas {}

Sympy {}

Giac {}

Reduce {}

Mupad {}

1.11 list of integrals solved by CAS but failed verification

The following are integrals solved by CAS but the verification phase failed to verify the anti-derivative produced is correct. This does not necessarily mean that the anti-derivative is wrong as additional methods of verification might be needed, or more time is needed (3 minutes time limit was used). These integrals are listed here to make it possible to do further investigation to determine why the result could not be verified.

Rubi {}

Mathematica {12, 14, 16, 17, 18, 19, 21, 42}

Maple {}

Maxima Verification phase not currently implemented.

Fricas Verification phase not currently implemented.

Sympy Verification phase not currently implemented.

Giac Verification phase not currently implemented.

Reduce Verification phase not currently implemented.

Mupad Verification phase not currently implemented.

1.12 Timing

The command `AbsoluteTiming[]` was used in Mathematica to obtain the elapsed time for each integrate call. In Maple, the command `Usage` was used as in the following example

```
cpu_time := Usage(assign ('result_of_int',int(expr,x)),output='realtime')
```

For all other CAS systems, the elapsed time to complete each integral was found by taking the difference between the time after the call completed from the time before the call was made. This was done using Python's `time.time()` call.

All elapsed times shown are in seconds. A time limit of 3 CPU minutes was used for each integral. If the integrate command did not complete within this time limit, the integral was aborted and considered to have failed and assigned an F grade. The time used by failed integrals due to time out was not counted in the final statistics.

1.13 Verification

A verification phase was applied on the result of integration for **Rubi** and **Mathematica**.

Future version of this report will implement verification for the other CAS systems. For the integrals whose result was not run through a verification phase, it is assumed that the antiderivative was correct.

Verification phase also had 3 minutes time out. An integral whose result was not verified could still be correct, but further investigation is needed on those integrals. These integrals were marked in the summary table below and also in each integral separate section so they are easy to identify and locate.

1.14 Important notes about some of the results

Important note about Maxima results

Since tests were run in a batch mode, and using an automated script, then any integral where Maxima needed an interactive response from the user to answer a question during the evaluation of the integral will fail.

The exception raised is `ValueError`. Therefore Maxima results is lower than what would result if Maxima was run directly and each question was answered correctly.

The percentage of such failures were not counted for each test file, but for an example, for the `Timofeev` test file, there were about 14 such integrals out of total 705, or about 2 percent. This percentage can be higher or lower depending on the specific input test file.

Such integrals can be identified by looking at the output of the integration in each section for Maxima. The exception message will indicate the cause of error.

Maxima `integrate` was run using SageMath with the following settings set by default

```
'besselexpand : true'  
'display2d : false'  
'domain : complex'  
'keepfloat : true'  
'load(to_poly_solve)'  
'load(simplify_sum)'  
'load(abs_integrate)' 'load(diag)'
```

SageMath automatic loading of Maxima `abs_integrate` was found to cause some problems. So the following code was added to disable this effect.

```
from sage.interfaces.maxima_lib import maxima_lib  
maxima_lib.set('extra_definite_integration_methods', '[]')  
maxima_lib.set('extra_integration_methods', '[]')
```

See <https://ask.sagemath.org/question/43088/integrate-results-that-are-different-from-using-maxima/> for reference.

Important note about FriCAS result

There were few integrals which failed due to SageMath interface and not because FriCAS system could not do the integration.

These will fail With error `Exception raised: NotImplementedError`.

The number of such cases seems to be very small. About 1 or 2 percent of all integrals. These can be identified by looking at the exception message given in the result.

Important note about finding leaf size of antiderivative

For Mathematica, Rubi, and Maple, the builtin system function `LeafSize` was used to find the leaf size of each antiderivative.

The other CAS systems (SageMath and Sympy) do not have special builtin function for this purpose at this time. Therefore the leaf size for Fricas and Sympy antiderivative was determined using the following function, thanks to user `slelievre` at https://ask.sagemath.org/question/57123/could-we-have-a-leaf_count-function-in-base-sagemath/

```
def tree_size(expr):
    r"""
    Return the tree size of this expression.
    """
    if expr not in SR:
        # deal with lists, tuples, vectors
        return 1 + sum(tree_size(a) for a in expr)
    expr = SR(expr)
    x, aa = expr.operator(), expr.operands()
    if x is None:
        return 1
    else:
        return 1 + sum(tree_size(a) for a in aa)
```

For Sympy, which was called directly from Python, the following code was used to obtain the leafsize of its result

```
try:  
    # 1.7 is a fudge factor since it is low side from actual leaf count  
    leafCount = round(1.7*count_ops(anti))  
  
except Exception as ee:  
    leafCount = 1
```

Important note about Mupad results

Matlab's symbolic toolbox does not have a leaf count function to measure the size of the antiderivative. Maple was used to determine the leaf size of Mupad output by post processing Mupad result.

Currently no grading of the antiderivative for Mupad is implemented. If it can integrate the problem, it was assigned a B grade automatically as a placeholder. In the future, when grading function is implemented for Mupad, the tests will be rerun again.

The following is an example of using Matlab's symbolic toolbox (Mupad) to solve an integral

```
integrand = evalin(symengine, 'cos(x)*sin(x)')  
the_variable = evalin(symengine, 'x')  
anti = int(integrand,the_variable)
```

Which gives $\sin(x)^2/2$

1.15 Current tree layout of integration tests

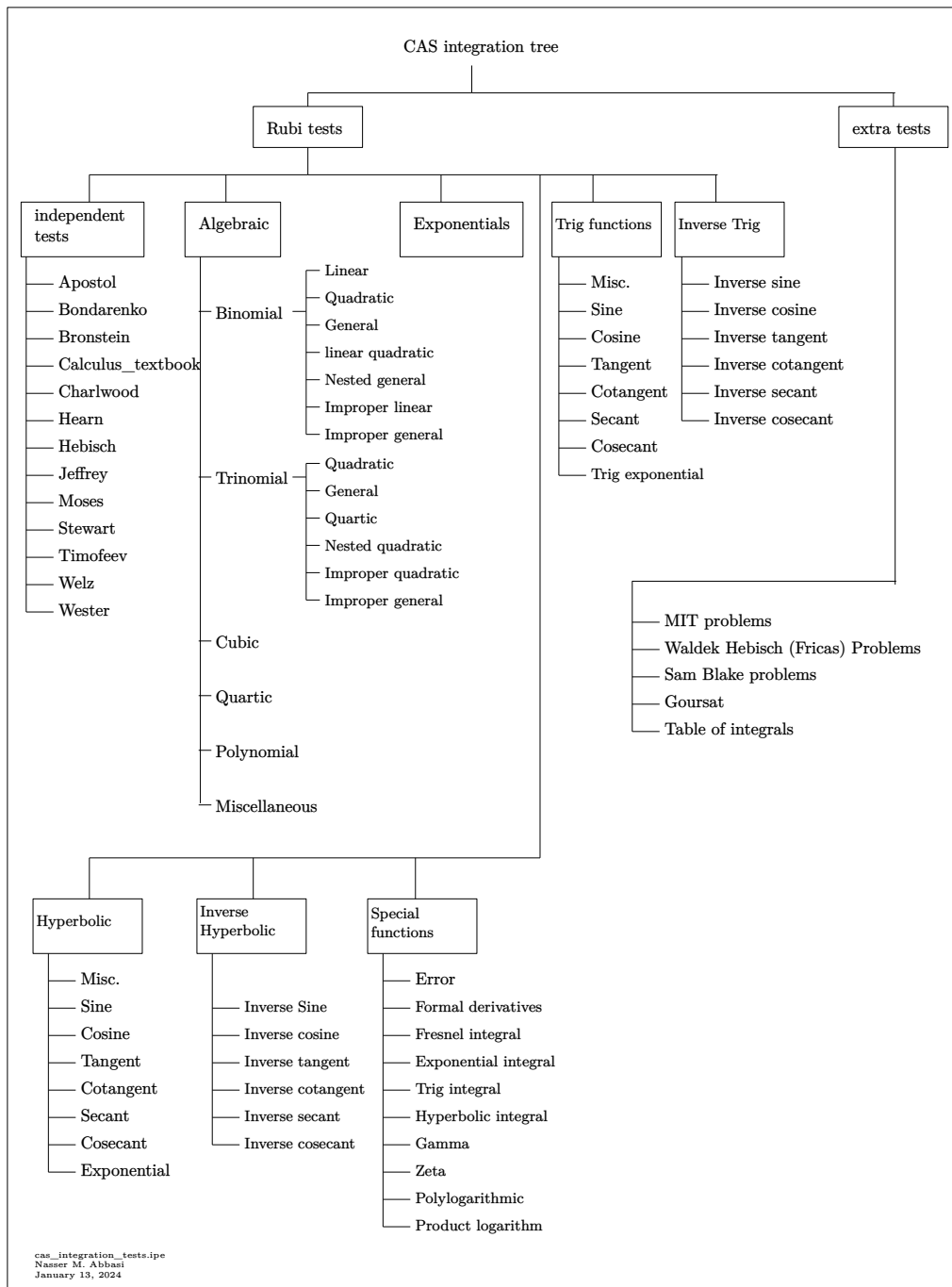
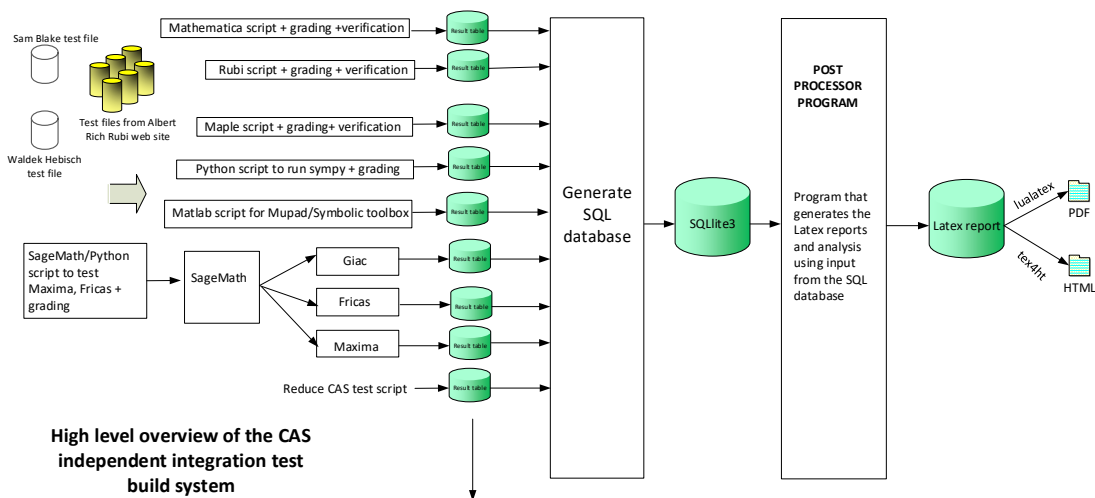


Figure 1.6: CAS integration tests tree

1.16 Design of the test system

The following diagram gives a high level view of the current test build system.



High level overview of the CAS independent integration test build system

One record (line) per one integral result. The line is CSV comma separated. This is description of each record

1. integer, the problem number.
2. integer. 0 for failed, 1 for passed, -1 for timeout, -2 for CAS specific exception. (this is not the grade field)
3. integer. Leaf size of result.
4. integer. Leaf size of the optimal antiderivative.
5. number. CPU time used to solve this integral. 0 if failed.
6. string. The integral in Latex format
7. string. The input used in CAS own syntax.
8. string. The result (antiderivative) produced by CAS in Latex format
9. string. The optimal antiderivative in Latex format.
10. integer. 0 or 1. Indicates if problem has known antiderivative or not
11. String. The result (antiderivative) in CAS own syntax.
12. String. The grade of the antiderivative. Can be "A", "B", "C", or "E"
13. String. Small string description of why the grade was given.
14. integer. 1 if result was verified or 0 if not verified. (For mma, rubi and maple only)

The following fields are present only in Rubi Table file

15. integer. Number of steps used.
16. integer. Number of rules used.
17. integer. Integrand leaf size.
18. real number. Ratio. Field 16 over field 17
19. String of form "{n,n,...}" which is list of the rules used by Rubi
20. String. The optimal antiderivative in Mathematica syntax

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Design note

CHAPTER 2

DETAILED SUMMARY TABLES OF RESULTS

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2.1 List of integrals sorted by grade for each CAS

Rubi	25
Mma	25
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Rubi

A grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43 }

B grade { }

C grade { }

F normal fail { }

F(-1) timedout fail { }

F(-2) exception fail { }

Mma

A grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42 }

B grade { }

C grade { 43 }

F normal fail { }

F(-1) timedout fail { }

F(-2) exception fail { }

Maple

A grade { 1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 19, 20, 21, 22, 23, 24, 25, 26, 29, 30, 31, 32, 33, 35, 38, 40, 41, 42, 43 }

B grade { 4, 16, 27, 34, 37, 39 }

C grade { 18, 28, 36 }

F normal fail { }

F(-1) timedout fail { }

F(-2) exception fail { }

Fricas

A grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 38, 39, 40, 41, 42, 43 }

B grade { 36 }

C grade { }

F normal fail { }

F(-1) timedout fail { }

F(-2) exception fail { }

Maxima

A grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43 }

B grade { }

C grade { }

F normal fail { }

F(-1) timedout fail { }

F(-2) exception fail { }

Giac

A grade { 4, 5, 10, 12, 13, 14, 15, 16, 17, 18, 22, 23, 24, 25, 26, 27, 28, 30, 32, 33, 34, 42 }

B grade { 1, 2, 3, 6, 7, 8, 9, 11, 19, 20, 21, 29, 31, 35, 36, 37, 38, 39, 40, 41 }

C grade { }

F normal fail { 43 }

F(-1) timedout fail { }

F(-2) exception fail { }

Mupad

A grade { }

B grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 33, 35, 38, 40, 42, 43 }

C grade { }

F normal fail { }

F(-1) timedout fail { 32, 34, 36, 37, 39, 41 }

F(-2) exception fail { }

Sympy

A grade { }

B grade { }

C grade { 13, 15, 17, 18, 20, 23, 25, 27, 28, 30, 33, 35, 36, 37, 38, 40, 42, 43 }

F normal fail { 1, 2, 3, 4, 5, 7, 8, 9, 10 }

F(-1) timedout fail { 6, 11, 12, 14, 16, 19, 21, 22, 24, 26, 29, 31, 32, 34, 39, 41 }

F(-2) exception fail { }

Reduce

A grade { }

B grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26,
27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43 }

C grade { }

F normal fail { }

F(-1) timedout fail { }

F(-2) exception fail { }

2.2 Detailed conclusion table per each integral for all CAS systems

Detailed conclusion table per each integral is given by the table below. The elapsed time is in seconds. For failed result it is given as **F(-1)** if the failure was due to timeout. It is given as **F(-2)** if the failure was due to an exception being raised, which could indicate a bug in the system. If the failure was due to integral not being evaluated within the time limit, then it is given as **F**.

In this table, the column **N.S.** means **normalized size** and is defined as $\frac{\text{antiderivative leaf size}}{\text{optimal antiderivative leaf size}}$. To make the table fit the page, the name **Mathematica** was abbreviated to **MMA**.

Problem 1	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	152	151	97	92	178	114	0	569	113	152
N.S.	1	0.99	0.64	0.61	1.17	0.75	0.00	3.74	0.74	1.00
time (sec)	N/A	0.427	0.188	0.156	0.033	0.147	0.000	0.216	0.251	4.023

Problem 2	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	109	110	75	68	124	90	0	454	89	118
N.S.	1	1.01	0.69	0.62	1.14	0.83	0.00	4.17	0.82	1.08
time (sec)	N/A	0.376	0.147	0.145	0.033	0.130	0.000	0.207	0.210	3.895

Problem 3	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	67	72	49	44	70	66	0	330	65	83
N.S.	1	1.07	0.73	0.66	1.04	0.99	0.00	4.93	0.97	1.24
time (sec)	N/A	0.322	0.113	0.138	0.035	0.098	0.000	0.183	0.198	3.957

Problem 4	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	B	A	A	F	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	80	81	75	174	52	80	0	78	130	248
N.S.	1	1.01	0.94	2.18	0.65	1.00	0.00	0.98	1.62	3.10
time (sec)	N/A	0.390	0.144	0.142	0.109	0.098	0.000	0.154	0.235	5.950

Problem 5	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	96	99	74	138	98	85	0	157	180	584
N.S.	1	1.03	0.77	1.44	1.02	0.89	0.00	1.64	1.88	6.08
time (sec)	N/A	0.399	0.223	0.178	0.108	0.135	0.000	0.178	0.203	9.439

Problem 6	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F(-1)	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	118	142	99	150	162	100	0	324	213	1004
N.S.	1	1.20	0.84	1.27	1.37	0.85	0.00	2.75	1.81	8.51
time (sec)	N/A	0.443	0.191	0.148	0.109	0.091	0.000	0.196	0.234	18.878

Problem 7	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	207	185	142	184	246	138	0	519	241	2314
N.S.	1	0.89	0.69	0.89	1.19	0.67	0.00	2.51	1.16	11.18
time (sec)	N/A	0.492	0.311	0.141	0.036	0.109	0.000	0.219	0.234	42.417

Problem 8	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	158	143	116	159	192	112	0	403	189	1681
N.S.	1	0.91	0.73	1.01	1.22	0.71	0.00	2.55	1.20	10.64
time (sec)	N/A	0.437	0.252	0.130	0.033	0.126	0.000	0.209	0.204	47.592

Problem 9	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	114	105	92	135	137	88	0	276	137	734
N.S.	1	0.92	0.81	1.18	1.20	0.77	0.00	2.42	1.20	6.44
time (sec)	N/A	0.331	0.183	0.132	0.033	0.094	0.000	0.177	0.210	19.515

Problem 10	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	104	97	74	117	105	83	0	114	125	243
N.S.	1	0.93	0.71	1.12	1.01	0.80	0.00	1.10	1.20	2.34
time (sec)	N/A	0.388	0.205	0.134	0.114	0.114	0.000	0.166	0.219	5.431

Problem 11	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F(-1)	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	84	85	81	124	75	100	0	165	126	236
N.S.	1	1.01	0.96	1.48	0.89	1.19	0.00	1.96	1.50	2.81
time (sec)	N/A	0.380	0.185	0.144	0.109	0.096	0.000	0.182	0.230	5.438

Problem 12	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F(-1)	A	B	B
verified	N/A	Yes	No	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	125	118	96	129	153	96	0	172	152	1154
N.S.	1	0.94	0.77	1.03	1.22	0.77	0.00	1.38	1.22	9.23
time (sec)	N/A	0.398	0.206	0.132	0.028	0.094	0.000	0.130	0.214	37.565

Problem 13	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	C	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	103	98	61	57	95	55	216	108	54	108
N.S.	1	0.95	0.59	0.55	0.92	0.53	2.10	1.05	0.52	1.05
time (sec)	N/A	0.358	0.105	0.112	0.028	0.089	8.901	0.134	0.232	4.641

Problem 14	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F(-1)	A	B	B
verified	N/A	Yes	No	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	87	82	77	111	113	77	0	121	110	720
N.S.	1	0.94	0.89	1.28	1.30	0.89	0.00	1.39	1.26	8.28
time (sec)	N/A	0.342	0.162	0.125	0.031	0.093	0.000	0.133	0.208	26.012

Problem 15	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	C	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	65	65	43	38	54	37	202	59	35	66
N.S.	1	1.00	0.66	0.58	0.83	0.57	3.11	0.91	0.54	1.02
time (sec)	N/A	0.306	0.082	0.122	0.032	0.092	6.542	0.127	0.207	4.708

Problem 16	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	B	A	A	F(-1)	A	B	B
verified	N/A	Yes	No	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	47	47	58	91	74	55	0	69	69	293
N.S.	1	1.00	1.23	1.94	1.57	1.17	0.00	1.47	1.47	6.23
time (sec)	N/A	0.268	0.120	0.109	0.035	0.117	0.000	0.135	0.234	15.841

Problem 17	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	C	A	B	B
verified	N/A	Yes	No	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	46	46	45	62	29	48	162	45	61	77
N.S.	1	1.00	0.98	1.35	0.63	1.04	3.52	0.98	1.33	1.67
time (sec)	N/A	0.324	0.087	0.108	0.111	0.092	16.297	0.129	0.221	6.016

Problem 18	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	A	A	C	A	B	B
verified	N/A	Yes	No	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	33	33	48	77	44	56	148	56	52	61
N.S.	1	1.00	1.45	2.33	1.33	1.70	4.48	1.70	1.58	1.85
time (sec)	N/A	0.296	0.094	0.123	0.106	0.105	15.637	0.134	0.221	5.000

Problem 19	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F(-1)	B	B	B
verified	N/A	Yes	No	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	60	60	55	71	45	57	0	114	110	297
N.S.	1	1.00	0.92	1.18	0.75	0.95	0.00	1.90	1.83	4.95
time (sec)	N/A	0.327	0.112	0.121	0.102	0.117	0.000	0.141	0.219	12.153

Problem 20	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	C	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	62	62	42	37	54	52	146	105	75	53
N.S.	1	1.00	0.68	0.60	0.87	0.84	2.35	1.69	1.21	0.85
time (sec)	N/A	0.322	0.109	0.115	0.103	0.091	20.733	0.136	0.225	4.933

Problem 21	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F(-1)	B	B	B
verified	N/A	Yes	No	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	99	94	78	94	85	78	0	268	156	650
N.S.	1	0.95	0.79	0.95	0.86	0.79	0.00	2.71	1.58	6.57
time (sec)	N/A	0.378	0.218	0.116	0.106	0.094	0.000	0.147	0.198	25.835

Problem 22	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F(-1)	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	164	148	119	160	196	115	0	203	189	1682
N.S.	1	0.90	0.73	0.98	1.20	0.70	0.00	1.24	1.15	10.26
time (sec)	N/A	0.453	0.276	0.134	0.033	0.132	0.000	0.150	0.201	48.106

Problem 23	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	C	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	118	110	74	68	124	66	240	124	65	130
N.S.	1	0.93	0.63	0.58	1.05	0.56	2.03	1.05	0.55	1.10
time (sec)	N/A	0.391	0.135	0.129	0.034	0.109	10.110	0.140	0.236	5.283

Problem 24	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F(-1)	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	118	107	92	136	142	90	0	140	137	1048
N.S.	1	0.91	0.78	1.15	1.20	0.76	0.00	1.19	1.16	8.88
time (sec)	N/A	0.397	0.208	0.144	0.033	0.116	0.000	0.141	0.203	28.947

Problem 25	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	C	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	72	72	48	43	69	42	223	65	40	76
N.S.	1	1.00	0.67	0.60	0.96	0.58	3.10	0.90	0.56	1.06
time (sec)	N/A	0.323	0.105	0.134	0.038	0.103	5.170	0.129	0.210	4.982

Problem 26	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F(-1)	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	68	68	68	111	89	63	0	79	86	417
N.S.	1	1.00	1.00	1.63	1.31	0.93	0.00	1.16	1.26	6.13
time (sec)	N/A	0.312	0.141	0.131	0.036	0.090	0.000	0.139	0.198	13.340

Problem 27	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	B	A	A	C	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	56	56	54	108	37	61	178	55	85	108
N.S.	1	1.00	0.96	1.93	0.66	1.09	3.18	0.98	1.52	1.93
time (sec)	N/A	0.365	0.105	0.125	0.110	0.089	27.746	0.138	0.193	6.300

Problem 28	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	A	A	C	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	57	57	57	97	55	68	165	64	66	77
N.S.	1	1.00	1.00	1.70	0.96	1.19	2.89	1.12	1.16	1.35
time (sec)	N/A	0.338	0.122	0.147	0.103	0.093	26.508	0.137	0.223	5.204

Problem 29	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F(-1)	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	76	76	70	123	60	73	0	141	158	457
N.S.	1	1.00	0.92	1.62	0.79	0.96	0.00	1.86	2.08	6.01
time (sec)	N/A	0.363	0.132	0.134	0.107	0.090	0.000	0.140	0.231	10.091

Problem 30	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	C	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	75	75	54	49	75	67	170	126	93	79
N.S.	1	1.00	0.72	0.65	1.00	0.89	2.27	1.68	1.24	1.05
time (sec)	N/A	0.349	0.122	0.132	0.117	0.092	31.008	0.141	0.221	4.995

Problem 31	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F(-1)	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	123	115	101	151	114	100	0	325	214	1005
N.S.	1	0.93	0.82	1.23	0.93	0.81	0.00	2.64	1.74	8.17
time (sec)	N/A	0.418	0.205	0.155	0.110	0.134	0.000	0.145	0.242	23.034

Problem 32	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F(-1)	A	B	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	169	141	137	275	196	190	0	214	331	0
N.S.	1	0.83	0.81	1.63	1.16	1.12	0.00	1.27	1.96	0.00
time (sec)	N/A	0.463	0.284	0.161	0.030	0.104	0.000	0.164	0.262	0.000

Problem 33	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	C	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	123	103	72	68	123	80	226	200	73	90
N.S.	1	0.84	0.59	0.55	1.00	0.65	1.84	1.63	0.59	0.73
time (sec)	N/A	0.401	0.141	0.155	0.036	0.089	59.109	0.159	0.205	5.241

Problem 34	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	B	A	A	F(-1)	A	B	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	120	133	108	251	138	159	0	147	285	0
N.S.	1	1.11	0.90	2.09	1.15	1.32	0.00	1.22	2.38	0.00
time (sec)	N/A	0.447	0.199	0.145	0.033	0.088	0.000	0.160	0.231	0.000

Problem 35	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	C	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	76	73	45	42	69	56	201	152	48	67
N.S.	1	0.96	0.59	0.55	0.91	0.74	2.64	2.00	0.63	0.88
time (sec)	N/A	0.334	0.109	0.138	0.026	0.091	54.360	0.149	0.210	5.220

Problem 36	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	A	B	C	B	B	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	63	63	71	160	76	129	182	113	192	0
N.S.	1	1.00	1.13	2.54	1.21	2.05	2.89	1.79	3.05	0.00
time (sec)	N/A	0.299	0.201	0.128	0.027	0.093	177.218	0.143	0.247	0.000

Problem 37	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	B	A	A	C	B	B	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	65	65	70	188	58	101	172	115	206	0
N.S.	1	1.00	1.08	2.89	0.89	1.55	2.65	1.77	3.17	0.00
time (sec)	N/A	0.375	0.173	0.132	0.110	0.097	177.654	0.165	0.233	0.000

Problem 38	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	C	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	67	66	51	48	71	103	165	219	137	73
N.S.	1	0.99	0.76	0.72	1.06	1.54	2.46	3.27	2.04	1.09
time (sec)	N/A	0.358	0.119	0.128	0.105	0.095	57.483	0.182	0.221	4.955

Problem 39	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	B	A	A	F(-1)	B	B	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	117	105	93	239	104	138	0	211	400	0
N.S.	1	0.90	0.79	2.04	0.89	1.18	0.00	1.80	3.42	0.00
time (sec)	N/A	0.422	0.231	0.155	0.112	0.095	0.000	0.208	0.226	0.000

Problem 40	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	C	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	119	106	77	73	125	132	165	242	176	104
N.S.	1	0.89	0.65	0.61	1.05	1.11	1.39	2.03	1.48	0.87
time (sec)	N/A	0.393	0.166	0.163	0.105	0.079	70.645	0.215	0.216	5.311

Problem 41	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F(-1)	B	B	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	166	146	122	267	162	165	0	402	447	0
N.S.	1	0.88	0.73	1.61	0.98	0.99	0.00	2.42	2.69	0.00
time (sec)	N/A	0.471	0.304	0.175	0.109	0.100	0.000	0.251	0.246	0.000

Problem 42	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	C	A	B	B
verified	N/A	Yes	No	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	40	40	40	53	23	39	148	40	48	72
N.S.	1	1.00	1.00	1.32	0.58	0.98	3.70	1.00	1.20	1.80
time (sec)	N/A	0.315	0.071	0.109	0.105	0.081	25.457	0.128	0.216	5.916

Problem 43	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	C	A	A	A	C	F	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	53	53	244	66	79	65	1867	0	54	96
N.S.	1	1.00	4.60	1.25	1.49	1.23	35.23	0.00	1.02	1.81
time (sec)	N/A	0.357	10.220	0.199	0.119	0.126	63.551	0.000	0.381	5.484

2.3 Detailed conclusion table specific for Rubi results

The following table is specific to Rubi only. It gives additional statistics for each integral. the column **steps** is the number of steps used by Rubi to obtain the antiderivative. The **rules** column is the number of unique rules used. The **integrand size** column is the leaf size of the integrand. Finally the ratio $\frac{\text{number of rules}}{\text{integrand size}}$ is also given. The larger this ratio is, the harder the integral is to solve. In this test file, problem number [7] had the largest ratio of [.25806499999999989]

Table 2.1: Rubi specific breakdown of results for each integral

#	grade	number of steps used	number of unique rules	normalized antiderivative leaf size	integrand leaf size	$\frac{\text{number of rules}}{\text{integrand leaf size}}$
1	A	6	6	0.99	31	0.194
2	A	4	4	1.01	31	0.129
3	A	2	2	1.07	29	0.069
4	A	6	5	1.01	31	0.161
5	A	6	5	1.03	31	0.161
6	A	6	5	1.20	31	0.161
7	A	9	8	0.89	31	0.258
8	A	7	6	0.91	31	0.194
9	A	5	4	0.92	28	0.143
10	A	5	4	0.93	31	0.129
11	A	6	5	1.01	31	0.161
12	A	5	5	0.94	29	0.172
13	A	4	4	0.95	29	0.138
14	A	3	3	0.94	29	0.103
15	A	2	2	1.00	27	0.074
16	A	2	2	1.00	26	0.077
17	A	4	3	1.00	29	0.103
18	A	2	2	1.00	29	0.069
19	A	4	3	1.00	29	0.103
20	A	2	2	1.00	29	0.069
21	A	6	5	0.95	29	0.172

Continued on next page

Table 2.1 – continued from previous page

#	grade	number of steps used	number of unique rules	normalized antiderivative leaf size	integrand leaf size	$\frac{\text{number of rules}}{\text{integrand leaf size}}$
22	A	8	7	0.90	31	0.226
23	A	4	4	0.93	31	0.129
24	A	6	5	0.91	31	0.161
25	A	2	2	1.00	29	0.069
26	A	4	3	1.00	28	0.107
27	A	4	3	1.00	31	0.097
28	A	4	3	1.00	31	0.097
29	A	4	3	1.00	31	0.097
30	A	2	2	1.00	31	0.065
31	A	6	5	0.93	31	0.161
32	A	8	7	0.83	31	0.226
33	A	4	4	0.84	31	0.129
34	A	7	6	1.11	31	0.194
35	A	2	2	0.96	29	0.069
36	A	4	3	1.00	28	0.107
37	A	4	3	1.00	31	0.097
38	A	2	2	0.99	31	0.065
39	A	6	5	0.90	31	0.161
40	A	4	4	0.89	31	0.129
41	A	8	7	0.88	31	0.226
42	A	4	3	1.00	31	0.097
43	A	1	1	1.00	57	0.018

CHAPTER 3

LISTING OF INTEGRALS

3.1	$\int x^5 \sqrt{-c+dx} \sqrt{c+dx} (a+bx^2) dx$	44
3.2	$\int x^3 \sqrt{-c+dx} \sqrt{c+dx} (a+bx^2) dx$	52
3.3	$\int x \sqrt{-c+dx} \sqrt{c+dx} (a+bx^2) dx$	59
3.4	$\int \frac{\sqrt{-c+dx} \sqrt{c+dx} (a+bx^2)}{x} dx$	65
3.5	$\int \frac{\sqrt{-c+dx} \sqrt{c+dx} (a+bx^2)}{x^3} dx$	72
3.6	$\int \frac{\sqrt{-c+dx} \sqrt{c+dx} (a+bx^2)}{x^5} dx$	79
3.7	$\int x^4 \sqrt{-c+dx} \sqrt{c+dx} (a+bx^2) dx$	87
3.8	$\int x^2 \sqrt{-c+dx} \sqrt{c+dx} (a+bx^2) dx$	97
3.9	$\int \sqrt{-c+dx} \sqrt{c+dx} (a+bx^2) dx$	105
3.10	$\int \frac{\sqrt{-c+dx} \sqrt{c+dx} (a+bx^2)}{x^2} dx$	112
3.11	$\int \frac{\sqrt{-c+dx} \sqrt{c+dx} (a+bx^2)}{x^4} dx$	119
3.12	$\int \frac{x^4 (a+bx^2)}{\sqrt{-1+cx} \sqrt{1+cx}} dx$	126
3.13	$\int \frac{x^3 (a+bx^2)}{\sqrt{-1+cx} \sqrt{1+cx}} dx$	133
3.14	$\int \frac{x^2 (a+bx^2)}{\sqrt{-1+cx} \sqrt{1+cx}} dx$	140
3.15	$\int \frac{x (a+bx^2)}{\sqrt{-1+cx} \sqrt{1+cx}} dx$	147
3.16	$\int \frac{a+bx^2}{\sqrt{-1+cx} \sqrt{1+cx}} dx$	153
3.17	$\int \frac{a+bx^2}{x \sqrt{-1+cx} \sqrt{1+cx}} dx$	159
3.18	$\int \frac{a+bx^2}{x^2 \sqrt{-1+cx} \sqrt{1+cx}} dx$	165
3.19	$\int \frac{a+bx^2}{x^3 \sqrt{-1+cx} \sqrt{1+cx}} dx$	171
3.20	$\int \frac{a+bx^2}{x^4 \sqrt{-1+cx} \sqrt{1+cx}} dx$	177
3.21	$\int \frac{a+bx^2}{x^5 \sqrt{-1+cx} \sqrt{1+cx}} dx$	183
3.22	$\int \frac{x^4 (a+bx^2)}{\sqrt{-c+dx} \sqrt{c+dx}} dx$	190
3.23	$\int \frac{x^3 (a+bx^2)}{\sqrt{-c+dx} \sqrt{c+dx}} dx$	199

3.24	$\int \frac{x^2(a+bx^2)}{\sqrt{-c+dx}\sqrt{c+dx}} dx$	206
3.25	$\int \frac{x(a+bx^2)}{\sqrt{-c+dx}\sqrt{c+dx}} dx$	213
3.26	$\int \frac{a+bx^2}{\sqrt{-c+dx}\sqrt{c+dx}} dx$	219
3.27	$\int \frac{a+bx^2}{x\sqrt{-c+dx}\sqrt{c+dx}} dx$	225
3.28	$\int \frac{a+bx^2}{x^2\sqrt{-c+dx}\sqrt{c+dx}} dx$	231
3.29	$\int \frac{a+bx^2}{x^3\sqrt{-c+dx}\sqrt{c+dx}} dx$	237
3.30	$\int \frac{a+bx^2}{x^4\sqrt{-c+dx}\sqrt{c+dx}} dx$	244
3.31	$\int \frac{a+bx^2}{x^5\sqrt{-c+dx}\sqrt{c+dx}} dx$	250
3.32	$\int \frac{x^4(a+bx^2)}{(-c+dx)^{3/2}(c+dx)^{3/2}} dx$	258
3.33	$\int \frac{x^3(a+bx^2)}{(-c+dx)^{3/2}(c+dx)^{3/2}} dx$	266
3.34	$\int \frac{x^2(a+bx^2)}{(-c+dx)^{3/2}(c+dx)^{3/2}} dx$	273
3.35	$\int \frac{x(a+bx^2)}{(-c+dx)^{3/2}(c+dx)^{3/2}} dx$	280
3.36	$\int \frac{a+bx^2}{(-c+dx)^{3/2}(c+dx)^{3/2}} dx$	286
3.37	$\int \frac{a+bx^2}{x(-c+dx)^{3/2}(c+dx)^{3/2}} dx$	292
3.38	$\int \frac{a+bx^2}{x^2(-c+dx)^{3/2}(c+dx)^{3/2}} dx$	298
3.39	$\int \frac{a+bx^2}{x^3(-c+dx)^{3/2}(c+dx)^{3/2}} dx$	304
3.40	$\int \frac{a+bx^2}{x^4(-c+dx)^{3/2}(c+dx)^{3/2}} dx$	311
3.41	$\int \frac{a+bx^2}{x^5(-c+dx)^{3/2}(c+dx)^{3/2}} dx$	318
3.42	$\int \frac{1+c^2x^2}{x\sqrt{-1+cx}\sqrt{1+cx}} dx$	326
3.43	$\int \frac{x \frac{-2b^2c+a^2d}{b^2c+a^2d} (c+dx^2)}{\sqrt{-a+bx}\sqrt{a+bx}} dx$	332

3.1 $\int x^5 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx$

Optimal result	44
Mathematica [A] (verified)	45
Rubi [A] (verified)	45
Maple [A] (verified)	48
Fricas [A] (verification not implemented)	48
Sympy [F]	49
Maxima [A] (verification not implemented)	49
Giac [B] (verification not implemented)	50
Mupad [B] (verification not implemented)	51
Reduce [B] (verification not implemented)	51

Optimal result

Integrand size = 31, antiderivative size = 152

$$\int x^5 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx = \frac{c^4(bc^2 + ad^2) (-c + dx)^{3/2}(c + dx)^{3/2}}{3d^8} + \frac{c^2(3bc^2 + 2ad^2) (-c + dx)^{5/2}(c + dx)^{5/2}}{5d^8} + \frac{(3bc^2 + ad^2) (-c + dx)^{7/2}(c + dx)^{7/2}}{7d^8} + \frac{b(-c + dx)^{9/2}(c + dx)^{9/2}}{9d^8}$$

output

```
1/3*c^4*(a*d^2+b*c^2)*(d*x-c)^(3/2)*(d*x+c)^(3/2)/d^8+1/5*c^2*(2*a*d^2+3*b*c^2)*(d*x-c)^(5/2)*(d*x+c)^(5/2)/d^8+1/7*(a*d^2+3*b*c^2)*(d*x-c)^(7/2)*(d*x+c)^(7/2)/d^8+1/9*b*(d*x-c)^(9/2)*(d*x+c)^(9/2)/d^8
```

Mathematica [A] (verified)

Time = 0.19 (sec) , antiderivative size = 97, normalized size of antiderivative = 0.64

$$\int x^5 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx$$

$$= \frac{(-c + dx)^{3/2} (c + dx)^{3/2} (3ad^2(8c^4 + 12c^2d^2x^2 + 15d^4x^4) + b(16c^6 + 24c^4d^2x^2 + 30c^2d^4x^4 + 35d^6x^6))}{315d^8}$$

input

```
Integrate[x^5*Sqrt[-c + d*x]*Sqrt[c + d*x]*(a + b*x^2),x]
```

output

```
((-c + d*x)^(3/2)*(c + d*x)^(3/2)*(3*a*d^2*(8*c^4 + 12*c^2*d^2*x^2 + 15*d^4*x^4) + b*(16*c^6 + 24*c^4*d^2*x^2 + 30*c^2*d^4*x^4 + 35*d^6*x^6)))/(315*d^8)
```

Rubi [A] (verified)

Time = 0.43 (sec) , antiderivative size = 151, normalized size of antiderivative = 0.99, number of steps used = 6, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}} = 0.194$, Rules used = {960, 111, 27, 111, 27, 83}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x^5 (a + bx^2) \sqrt{dx - c} \sqrt{c + dx} dx$$

$$\downarrow 960$$

$$\frac{1}{3} \left(3a + \frac{2bc^2}{d^2} \right) \int x^5 \sqrt{dx - c} \sqrt{c + dx} dx + \frac{bx^6 (dx - c)^{3/2} (c + dx)^{3/2}}{9d^2}$$

$$\downarrow 111$$

$$\frac{1}{3} \left(3a + \frac{2bc^2}{d^2} \right) \left(\frac{\int 4c^2 x^3 \sqrt{dx - c} \sqrt{c + dx} dx}{7d^2} + \frac{x^4 (dx - c)^{3/2} (c + dx)^{3/2}}{7d^2} \right) + \frac{bx^6 (dx - c)^{3/2} (c + dx)^{3/2}}{9d^2}$$

$$\downarrow 27$$

$$\frac{1}{3} \left(3a + \frac{2bc^2}{d^2} \right) \left(\frac{4c^2 \int x^3 \sqrt{dx-c} \sqrt{c+dx} dx}{7d^2} + \frac{x^4(dx-c)^{3/2}(c+dx)^{3/2}}{7d^2} \right) + \frac{bx^6(dx-c)^{3/2}(c+dx)^{3/2}}{9d^2}$$

↓ 111

$$\frac{1}{3} \left(3a + \frac{2bc^2}{d^2} \right) \left(\frac{4c^2 \left(\frac{\int 2c^2 x \sqrt{dx-c} \sqrt{c+dx} dx}{5d^2} + \frac{x^2(dx-c)^{3/2}(c+dx)^{3/2}}{5d^2} \right)}{7d^2} + \frac{x^4(dx-c)^{3/2}(c+dx)^{3/2}}{7d^2} \right) + \frac{bx^6(dx-c)^{3/2}(c+dx)^{3/2}}{9d^2}$$

↓ 27

$$\frac{1}{3} \left(3a + \frac{2bc^2}{d^2} \right) \left(\frac{4c^2 \left(\frac{2c^2 \int x \sqrt{dx-c} \sqrt{c+dx} dx}{5d^2} + \frac{x^2(dx-c)^{3/2}(c+dx)^{3/2}}{5d^2} \right)}{7d^2} + \frac{x^4(dx-c)^{3/2}(c+dx)^{3/2}}{7d^2} \right) + \frac{bx^6(dx-c)^{3/2}(c+dx)^{3/2}}{9d^2}$$

↓ 83

$$\frac{1}{3} \left(\frac{4c^2 \left(\frac{2c^2(dx-c)^{3/2}(c+dx)^{3/2}}{15d^4} + \frac{x^2(dx-c)^{3/2}(c+dx)^{3/2}}{5d^2} \right)}{7d^2} + \frac{x^4(dx-c)^{3/2}(c+dx)^{3/2}}{7d^2} \right) \left(3a + \frac{2bc^2}{d^2} \right) + \frac{bx^6(dx-c)^{3/2}(c+dx)^{3/2}}{9d^2}$$

input `Int[x^5*sqrt[-c + d*x]*sqrt[c + d*x]*(a + b*x^2),x]`

output

```
(b*x^6*(-c + d*x)^(3/2)*(c + d*x)^(3/2))/(9*d^2) + ((3*a + (2*b*c^2)/d^2)*
((x^4*(-c + d*x)^(3/2)*(c + d*x)^(3/2))/(7*d^2) + (4*c^2*((2*c^2*(-c + d*x)
)^(3/2)*(c + d*x)^(3/2))/(15*d^4) + (x^2*(-c + d*x)^(3/2)*(c + d*x)^(3/2))
/(5*d^2)))/(7*d^2))/3
```

Definitions of rubi rules used

- rule 27 $\text{Int}[(a_*)(Fx_), x_Symbol] \rightarrow \text{Simp}[a \text{ Int}[Fx, x], x] /; \text{FreeQ}[a, x] \ \&\& \ !\text{MatchQ}[Fx, (b_*)(Gx_)] /; \text{FreeQ}[b, x]$
- rule 83 $\text{Int}[(a_.) + (b_.)*(x_)*((c_.) + (d_.)*(x_))^{(n_.)}*((e_.) + (f_.)*(x_))^{(p_.)}, x_] \rightarrow \text{Simp}[b*(c + d*x)^{(n + 1)}*((e + f*x)^{(p + 1)}/(d*f*(n + p + 2))), x] /; \text{FreeQ}[\{a, b, c, d, e, f, n, p\}, x] \ \&\& \ \text{NeQ}[n + p + 2, 0] \ \&\& \ \text{EqQ}[a*d*f*(n + p + 2) - b*(d*e*(n + 1) + c*f*(p + 1)), 0]$
- rule 111 $\text{Int}[(a_.) + (b_.)*(x_))^{(m_.)}*((c_.) + (d_.)*(x_))^{(n_.)}*((e_.) + (f_.)*(x_))^{(p_.)}, x_] \rightarrow \text{Simp}[b*(a + b*x)^{(m - 1)}*(c + d*x)^{(n + 1)}*((e + f*x)^{(p + 1)}/(d*f*(m + n + p + 1))), x] + \text{Simp}[1/(d*f*(m + n + p + 1)) \text{ Int}[(a + b*x)^{(m - 2)}*(c + d*x)^n*(e + f*x)^p*\text{Simp}[a^2*d*f*(m + n + p + 1) - b*(b*c*e*(m - 1) + a*(d*e*(n + 1) + c*f*(p + 1))) + b*(a*d*f*(2*m + n + p) - b*(d*e*(m + n) + c*f*(m + p)))*x, x], x] /; \text{FreeQ}[\{a, b, c, d, e, f, n, p\}, x] \ \&\& \ \text{GtQ}[m, 1] \ \&\& \ \text{NeQ}[m + n + p + 1, 0] \ \&\& \ \text{IntegerQ}[m]$
- rule 960 $\text{Int}[(e_.)*(x_))^{(m_.)}*((a1_.) + (b1_.)*(x_)^{(non2_.)})^{(p_.)}*((a2_.) + (b2_.)*(x_)^{(non2_.)})^{(p_.)}*((c_.) + (d_.)*(x_))^{(n_.)}, x_Symbol] \rightarrow \text{Simp}[d*(e*x)^{(m + 1)}*(a1 + b1*x^{(n/2)})^{(p + 1)}*((a2 + b2*x^{(n/2)})^{(p + 1)}/(b1*b2*e*(m + n*(p + 1) + 1))), x] - \text{Simp}[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(b1*b2*(m + n*(p + 1) + 1)) \text{ Int}[(e*x)^m*(a1 + b1*x^{(n/2)})^p*(a2 + b2*x^{(n/2)})^p, x], x] /; \text{FreeQ}[\{a1, b1, a2, b2, c, d, e, m, n, p\}, x] \ \&\& \ \text{EqQ}[\text{non2}, n/2] \ \&\& \ \text{EqQ}[a2*b1 + a1*b2, 0] \ \&\& \ \text{NeQ}[m + n*(p + 1) + 1, 0]$

Maple [A] (verified)

Time = 0.16 (sec) , antiderivative size = 92, normalized size of antiderivative = 0.61

method	result	size
gospers	$\frac{(dx-c)^{\frac{3}{2}}(dx+c)^{\frac{3}{2}}(35bx^6d^6+45ad^6x^4+30bc^2d^4x^4+36ac^2d^4x^2+24bc^4d^2x^2+24ac^4d^2+16bc^6)}{315d^8}$	92
orering	$-\frac{(dx+c)^{\frac{3}{2}}(-dx+c)(35bx^6d^6+45ad^6x^4+30bc^2d^4x^4+36ac^2d^4x^2+24bc^4d^2x^2+24ac^4d^2+16bc^6)\sqrt{dx-c}}{315d^8}$	98
default	$-\frac{\sqrt{dx-c}\sqrt{dx+c}(-d^2x^2+c^2)(35bx^6d^6+45ad^6x^4+30bc^2d^4x^4+36ac^2d^4x^2+24bc^4d^2x^2+24ac^4d^2+16bc^6)}{315d^8}$	104
risch	$\frac{\sqrt{dx+c}(-35bx^8d^8-45ad^8x^6+5bc^2d^6x^6+9ac^2d^6x^4+6bc^4d^4x^4+12ac^4d^4x^2+8bc^6d^2x^2+24ac^6d^2+16bc^8)(-dx+c)}{315\sqrt{dx-c}d^8}$	122

input `int(x^5*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x,method=_RETURNVERBOSE)`

output `1/315/d^8*(d*x-c)^(3/2)*(d*x+c)^(3/2)*(35*b*d^6*x^6+45*a*d^6*x^4+30*b*c^2*d^4*x^4+36*a*c^2*d^4*x^2+24*b*c^4*d^2*x^2+24*a*c^4*d^2+16*b*c^6)`

Fricas [A] (verification not implemented)

Time = 0.15 (sec) , antiderivative size = 114, normalized size of antiderivative = 0.75

$$\int x^5 \sqrt{-c+dx} \sqrt{c+dx} (a+bx^2) dx$$

$$= \frac{(35bd^8x^8 - 16bc^8 - 24ac^6d^2 - 5(bc^2d^6 - 9ad^8)x^6 - 3(2bc^4d^4 + 3ac^2d^6)x^4 - 4(2bc^6d^2 + 3ac^4d^4)x^2)\sqrt{dx+c}\sqrt{dx-c}}{315d^8}$$

input `integrate(x^5*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x, algorithm="fricas")`

output `1/315*(35*b*d^8*x^8 - 16*b*c^8 - 24*a*c^6*d^2 - 5*(b*c^2*d^6 - 9*a*d^8)*x^6 - 3*(2*b*c^4*d^4 + 3*a*c^2*d^6)*x^4 - 4*(2*b*c^6*d^2 + 3*a*c^4*d^4)*x^2)*sqrt(d*x + c)*sqrt(d*x - c)/d^8`

Sympy [F]

$$\int x^5 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx = \int x^5 (a + bx^2) \sqrt{-c + dx} \sqrt{c + dx} dx$$

input `integrate(x**5*(d*x-c)**(1/2)*(d*x+c)**(1/2)*(b*x**2+a), x)`

output `Integral(x**5*(a + b*x**2)*sqrt(-c + d*x)*sqrt(c + d*x), x)`

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 178, normalized size of antiderivative = 1.17

$$\begin{aligned} \int x^5 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx = & \frac{(d^2 x^2 - c^2)^{\frac{3}{2}} b x^6}{9 d^2} + \frac{2 (d^2 x^2 - c^2)^{\frac{3}{2}} b c^2 x^4}{21 d^4} \\ & + \frac{(d^2 x^2 - c^2)^{\frac{3}{2}} a x^4}{7 d^2} + \frac{8 (d^2 x^2 - c^2)^{\frac{3}{2}} b c^4 x^2}{105 d^6} \\ & + \frac{4 (d^2 x^2 - c^2)^{\frac{3}{2}} a c^2 x^2}{35 d^4} \\ & + \frac{16 (d^2 x^2 - c^2)^{\frac{3}{2}} b c^6}{315 d^8} + \frac{8 (d^2 x^2 - c^2)^{\frac{3}{2}} a c^4}{105 d^6} \end{aligned}$$

input `integrate(x^5*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x, algorithm="maxima")`

output `1/9*(d^2*x^2 - c^2)^(3/2)*b*x^6/d^2 + 2/21*(d^2*x^2 - c^2)^(3/2)*b*c^2*x^4/d^4 + 1/7*(d^2*x^2 - c^2)^(3/2)*a*x^4/d^2 + 8/105*(d^2*x^2 - c^2)^(3/2)*b*c^4*x^2/d^6 + 4/35*(d^2*x^2 - c^2)^(3/2)*a*c^2*x^2/d^4 + 16/315*(d^2*x^2 - c^2)^(3/2)*b*c^6/d^8 + 8/105*(d^2*x^2 - c^2)^(3/2)*a*c^4/d^6`

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 569 vs. $2(128) = 256$.

Time = 0.22 (sec) , antiderivative size = 569, normalized size of antiderivative = 3.74

$$\int x^5 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx = \text{Too large to display}$$

input `integrate(x^5*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x, algorithm="giac")`

output

```
1/40320*(168*(((2*((d*x + c))*(4*(d*x + c))*(5*(d*x + c)/d^5 - 31*c/d^5) + 3
21*c^2/d^5) - 451*c^3/d^5)*(d*x + c) + 745*c^4/d^5)*(d*x + c) - 405*c^5/d^
5)*sqrt(d*x + c)*sqrt(d*x - c) - 150*c^6*log(abs(-sqrt(d*x + c) + sqrt(d*x
- c)))/d^5)*a*c + 3*(((2*((4*(5*(d*x + c))*(6*(d*x + c))*(7*(d*x + c)/d^7 -
57*c/d^7) + 1219*c^2/d^7) - 12463*c^3/d^7)*(d*x + c) + 64233*c^4/d^7)*(d*
x + c) - 53963*c^5/d^7)*(d*x + c) + 59465*c^6/d^7)*(d*x + c) - 23205*c^7/d
^7)*sqrt(d*x + c)*sqrt(d*x - c) - 7350*c^8*log(abs(-sqrt(d*x + c) + sqrt(d
*x - c)))/d^7)*b*c + 24*(1050*c^7*log(abs(-sqrt(d*x + c) + sqrt(d*x - c)))
+ (2835*c^6 - (6335*c^5 - 2*(4781*c^4 - (4551*c^3 - 4*(5*(6*d*x - 37*c)*(
d*x + c) + 661*c^2)*(d*x + c))*(d*x + c))*(d*x + c))*(d*x + c))*sqrt(d*x +
c)*sqrt(d*x - c))*a/d^5 + (22050*c^9*log(abs(-sqrt(d*x + c) + sqrt(d*x -
c))) + (69615*c^8 - (205275*c^7 - 2*(216993*c^6 - (310203*c^5 - 4*(75293*c
^4 - 5*(9833*c^3 - 2*(7*(8*d*x - 65*c)*(d*x + c) + 2073*c^2)*(d*x + c))*(d
*x + c))*(d*x + c))*(d*x + c))*(d*x + c))*sqrt(d*x + c)*sqrt(d*x
- c))*b/d^7)/d
```

Mupad [B] (verification not implemented)

Time = 4.02 (sec) , antiderivative size = 152, normalized size of antiderivative = 1.00

$$\int x^5 \sqrt{-c+dx} \sqrt{c+dx} (a+bx^2) dx = -\sqrt{dx-c} \left(\frac{(16bc^8 + 24ac^6d^2) \sqrt{c+dx}}{315d^8} - \frac{bx^8 \sqrt{c+dx}}{9} + \frac{x^4 (6bc^4d^4 + 9ac^2d^6) \sqrt{c+dx}}{315d^8} + \frac{x^2 (8bc^6d^2 + 12ac^4d^4) \sqrt{c+dx}}{315d^8} - \frac{x^6 (45ad^8 - 5b^2c^2d^6) \sqrt{c+dx}}{315d^8} \right)$$

input `int(x^5*(a + b*x^2)*(c + d*x)^(1/2)*(d*x - c)^(1/2),x)`output `-(d*x - c)^(1/2)*(((16*b*c^8 + 24*a*c^6*d^2)*(c + d*x)^(1/2))/(315*d^8) - (b*x^8*(c + d*x)^(1/2))/9 + (x^4*(9*a*c^2*d^6 + 6*b*c^4*d^4)*(c + d*x)^(1/2))/(315*d^8) + (x^2*(12*a*c^4*d^4 + 8*b*c^6*d^2)*(c + d*x)^(1/2))/(315*d^8) - (x^6*(45*a*d^8 - 5*b*c^2*d^6)*(c + d*x)^(1/2))/(315*d^8))`**Reduce [B] (verification not implemented)**

Time = 0.25 (sec) , antiderivative size = 113, normalized size of antiderivative = 0.74

$$\int x^5 \sqrt{-c+dx} \sqrt{c+dx} (a+bx^2) dx = \frac{\sqrt{dx+c} \sqrt{dx-c} (35bd^8x^8 + 45ad^8x^6 - 5b^2c^2d^6x^6 - 9ac^2d^6x^4 - 6b^2c^4d^4x^4 - 12ac^4d^4x^2 - 8b^2c^6d^2x^2 - 2b^2c^8)}{315d^8}$$

input `int(x^5*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x)`output `(sqrt(c + d*x)*sqrt(-c + d*x)*(-24*a*c**6*d**2 - 12*a*c**4*d**4*x**2 - 9*a*c**2*d**6*x**4 + 45*a*d**8*x**6 - 16*b*c**8 - 8*b*c**6*d**2*x**2 - 6*b*c**4*d**4*x**4 - 5*b*c**2*d**6*x**6 + 35*b*d**8*x**8))/(315*d**8)`

3.2 $\int x^3 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx$

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Optimal result

Integrand size = 31, antiderivative size = 109

$$\int x^3 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx = \frac{c^2(bc^2 + ad^2)(-c + dx)^{3/2}(c + dx)^{3/2}}{3d^6} + \frac{(2bc^2 + ad^2)(-c + dx)^{5/2}(c + dx)^{5/2}}{5d^6} + \frac{b(-c + dx)^{7/2}(c + dx)^{7/2}}{7d^6}$$

```
output 1/3*c^2*(a*d^2+b*c^2)*(d*x-c)^(3/2)*(d*x+c)^(3/2)/d^6+1/5*(a*d^2+2*b*c^2)*(d*x-c)^(5/2)*(d*x+c)^(5/2)/d^6+1/7*b*(d*x-c)^(7/2)*(d*x+c)^(7/2)/d^6
```

Mathematica [A] (verified)

Time = 0.15 (sec) , antiderivative size = 75, normalized size of antiderivative = 0.69

$$\int x^3 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx = \frac{(-c + dx)^{3/2}(c + dx)^{3/2} (7ad^2(2c^2 + 3d^2x^2) + b(8c^4 + 12c^2d^2x^2 + 15d^4x^4))}{105d^6}$$

```
input Integrate[x^3*Sqrt[-c + d*x]*Sqrt[c + d*x]*(a + b*x^2),x]
```

output

$$\frac{((-c + d*x)^{(3/2)}*(c + d*x)^{(3/2)}*(7*a*d^2*(2*c^2 + 3*d^2*x^2) + b*(8*c^4 + 12*c^2*d^2*x^2 + 15*d^4*x^4))}{(105*d^6)}$$

Rubi [A] (verified)

Time = 0.38 (sec) , antiderivative size = 110, normalized size of antiderivative = 1.01, number of steps used = 4, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}} = 0.129$, Rules used = {960, 111, 27, 83}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x^3(a + bx^2) \sqrt{dx - c} \sqrt{c + dx} dx$$

$$\downarrow 960$$

$$\frac{1}{7} \left(7a + \frac{4bc^2}{d^2} \right) \int x^3 \sqrt{dx - c} \sqrt{c + dx} dx + \frac{bx^4(dx - c)^{3/2}(c + dx)^{3/2}}{7d^2}$$

$$\downarrow 111$$

$$\frac{1}{7} \left(7a + \frac{4bc^2}{d^2} \right) \left(\frac{\int 2c^2 x \sqrt{dx - c} \sqrt{c + dx} dx}{5d^2} + \frac{x^2(dx - c)^{3/2}(c + dx)^{3/2}}{5d^2} \right) + \frac{bx^4(dx - c)^{3/2}(c + dx)^{3/2}}{7d^2}$$

$$\downarrow 27$$

$$\frac{1}{7} \left(7a + \frac{4bc^2}{d^2} \right) \left(\frac{2c^2 \int x \sqrt{dx - c} \sqrt{c + dx} dx}{5d^2} + \frac{x^2(dx - c)^{3/2}(c + dx)^{3/2}}{5d^2} \right) + \frac{bx^4(dx - c)^{3/2}(c + dx)^{3/2}}{7d^2}$$

$$\downarrow 83$$

$$\frac{1}{7} \left(\frac{2c^2(dx - c)^{3/2}(c + dx)^{3/2}}{15d^4} + \frac{x^2(dx - c)^{3/2}(c + dx)^{3/2}}{5d^2} \right) \left(7a + \frac{4bc^2}{d^2} \right) + \frac{bx^4(dx - c)^{3/2}(c + dx)^{3/2}}{7d^2}$$

input

$$\text{Int}[x^3 \sqrt{-c + d*x} * \sqrt{c + d*x} * (a + b*x^2), x]$$

output

$$\frac{(b^2x^4(-c + dx)^{3/2}(c + dx)^{3/2})/(7d^2) + ((7a + (4b^2c^2)/d^2) * ((2c^2(-c + dx)^{3/2}(c + dx)^{3/2})/(15d^4) + (x^2(-c + dx)^{3/2} * (c + dx)^{3/2})/(5d^2)))/7$$

Defintions of rubi rules used

rule 27

$$\text{Int}[(a_*)(Fx_), x_Symbol] \rightarrow \text{Simp}[a \text{ Int}[Fx, x], x] /; \text{FreeQ}[a, x] \ \&\& \ !\text{MatchQ}[Fx, (b_*)(Gx_)] /; \text{FreeQ}[b, x]$$

rule 83

$$\text{Int}[(a_.) + (b_.) * (x_)] * ((c_.) + (d_.) * (x_))^{(n_.)} * ((e_.) + (f_.) * (x_))^{(p_.)}, x_] \rightarrow \text{Simp}[b * (c + d * x)^{(n + 1)} * ((e + f * x)^{(p + 1)} / (d * f * (n + p + 2))), x] /; \text{FreeQ}[\{a, b, c, d, e, f, n, p\}, x] \ \&\& \ \text{NeQ}[n + p + 2, 0] \ \&\& \ \text{EqQ}[a * d * f * (n + p + 2) - b * (d * e * (n + 1) + c * f * (p + 1)), 0]$$

rule 111

$$\text{Int}[(a_.) + (b_.) * (x_)]^{(m_.)} * ((c_.) + (d_.) * (x_))^{(n_.)} * ((e_.) + (f_.) * (x_))^{(p_.)}, x_] \rightarrow \text{Simp}[b * (a + b * x)^{(m - 1)} * (c + d * x)^{(n + 1)} * ((e + f * x)^{(p + 1)} / (d * f * (m + n + p + 1))), x] + \text{Simp}[1 / (d * f * (m + n + p + 1)) \text{ Int}[(a + b * x)^{(m - 2)} * (c + d * x)^n * (e + f * x)^p * \text{Simp}[a^2 * d * f * (m + n + p + 1) - b * (b * c * e * (m - 1) + a * (d * e * (n + 1) + c * f * (p + 1))) + b * (a * d * f * (2 * m + n + p) - b * (d * e * (m + n) + c * f * (m + p))) * x, x], x], x] /; \text{FreeQ}[\{a, b, c, d, e, f, n, p\}, x] \ \&\& \ \text{GtQ}[m, 1] \ \&\& \ \text{NeQ}[m + n + p + 1, 0] \ \&\& \ \text{IntegerQ}[m]$$

rule 960

$$\text{Int}[(e_.) * (x_)]^{(m_.)} * ((a1_.) + (b1_.) * (x_))^{(non2_.)} * ((a2_.) + (b2_.) * (x_))^{(non2_.)} * ((c_.) + (d_.) * (x_))^{(n_.)}, x_Symbol] \rightarrow \text{Simp}[d * (e * x)^{(m + 1)} * (a1 + b1 * x^{(n/2)})^{(p + 1)} * ((a2 + b2 * x^{(n/2)})^{(p + 1)} / (b1 * b2 * e * (m + n * (p + 1) + 1))), x] - \text{Simp}[(a1 * a2 * d * (m + 1) - b1 * b2 * c * (m + n * (p + 1) + 1)) / (b1 * b2 * (m + n * (p + 1) + 1)) \text{ Int}[(e * x)^m * (a1 + b1 * x^{(n/2)})^p * (a2 + b2 * x^{(n/2)})^p, x], x] /; \text{FreeQ}[\{a1, b1, a2, b2, c, d, e, m, n, p\}, x] \ \&\& \ \text{EqQ}[non2, n/2] \ \&\& \ \text{EqQ}[a2 * b1 + a1 * b2, 0] \ \&\& \ \text{NeQ}[m + n * (p + 1) + 1, 0]$$

Maple [A] (verified)

Time = 0.14 (sec) , antiderivative size = 68, normalized size of antiderivative = 0.62

method	result	size
gospers	$\frac{(dx-c)^{\frac{3}{2}}(dx+c)^{\frac{3}{2}}(15bx^4d^4+21ad^4x^2+12b^2c^2d^2x^2+14ac^2d^2+8bc^4)}{105d^6}$	68
orering	$-\frac{(dx+c)^{\frac{3}{2}}(-dx+c)(15bx^4d^4+21ad^4x^2+12b^2c^2d^2x^2+14ac^2d^2+8bc^4)\sqrt{dx-c}}{105d^6}$	74
default	$-\frac{\sqrt{dx-c}\sqrt{dx+c}(-d^2x^2+c^2)(15bx^4d^4+21ad^4x^2+12b^2c^2d^2x^2+14ac^2d^2+8bc^4)}{105d^6}$	80
risch	$\frac{\sqrt{dx+c}(-15bx^6d^6-21ad^6x^4+3bc^2d^4x^4+7ac^2d^4x^2+4bc^4d^2x^2+14ac^4d^2+8bc^6)(-dx+c)}{105\sqrt{dx-c}d^6}$	98

input `int(x^3*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x,method=_RETURNVERBOSE)`

output
$$\frac{1}{105d^6}(d*x-c)^{\frac{3}{2}}(d*x+c)^{\frac{3}{2}}(15*b*d^4*x^4+21*a*d^4*x^2+12*b*c^2*d^2*x^2+14*a*c^2*d^2+8*b*c^4)$$

Fricas [A] (verification not implemented)

Time = 0.13 (sec) , antiderivative size = 90, normalized size of antiderivative = 0.83

$$\int x^3\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2) dx$$

$$= \frac{(15bd^6x^6 - 8bc^6 - 14ac^4d^2 - 3(bc^2d^4 - 7ad^6)x^4 - (4bc^4d^2 + 7ac^2d^4)x^2)\sqrt{dx+c}\sqrt{dx-c}}{105d^6}$$

input `integrate(x^3*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x, algorithm="fricas")`

output
$$\frac{1}{105}(15*b*d^6*x^6 - 8*b*c^6 - 14*a*c^4*d^2 - 3*(b*c^2*d^4 - 7*a*d^6)*x^4 - (4*b*c^4*d^2 + 7*a*c^2*d^4)*x^2)*\text{sqrt}(d*x + c)*\text{sqrt}(d*x - c)/d^6$$

Sympy [F]

$$\int x^3 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx = \int x^3 (a + bx^2) \sqrt{-c + dx} \sqrt{c + dx} dx$$

input `integrate(x**3*(d*x-c)**(1/2)*(d*x+c)**(1/2)*(b*x**2+a), x)`

output `Integral(x**3*(a + b*x**2)*sqrt(-c + d*x)*sqrt(c + d*x), x)`

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 124, normalized size of antiderivative = 1.14

$$\begin{aligned} \int x^3 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx = & \frac{(d^2 x^2 - c^2)^{\frac{3}{2}} b x^4}{7 d^2} + \frac{4 (d^2 x^2 - c^2)^{\frac{3}{2}} b c^2 x^2}{35 d^4} \\ & + \frac{(d^2 x^2 - c^2)^{\frac{3}{2}} a x^2}{5 d^2} + \frac{8 (d^2 x^2 - c^2)^{\frac{3}{2}} b c^4}{105 d^6} \\ & + \frac{2 (d^2 x^2 - c^2)^{\frac{3}{2}} a c^2}{15 d^4} \end{aligned}$$

input `integrate(x^3*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a), x, algorithm="maxima")`

output `1/7*(d^2*x^2 - c^2)^(3/2)*b*x^4/d^2 + 4/35*(d^2*x^2 - c^2)^(3/2)*b*c^2*x^2/d^4 + 1/5*(d^2*x^2 - c^2)^(3/2)*a*x^2/d^2 + 8/105*(d^2*x^2 - c^2)^(3/2)*b*c^4/d^6 + 2/15*(d^2*x^2 - c^2)^(3/2)*a*c^2/d^4`

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 454 vs. $2(91) = 182$.

Time = 0.21 (sec) , antiderivative size = 454, normalized size of antiderivative = 4.17

$$\int x^3 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx$$

$$= \frac{70 \left(\left((dx + c) \left(2(dx + c) \left(\frac{3(dx+c)}{d^3} - \frac{13c}{d^3} \right) + \frac{43c^2}{d^3} \right) - \frac{39c^3}{d^3} \right) \sqrt{dx + c} \sqrt{dx - c} - \frac{18c^4 \log(|-\sqrt{dx+c} + \sqrt{dx-c}|)}{d^3} \right) a}{1}$$

input `integrate(x^3*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x, algorithm="giac")`

output

$$\frac{1}{1680} \cdot (70 \cdot ((d \cdot x + c) \cdot (2 \cdot (d \cdot x + c) \cdot (3 \cdot (d \cdot x + c) / d^3 - 13 \cdot c / d^3) + 43 \cdot c^2 / d^3) - 39 \cdot c^3 / d^3) \cdot \sqrt{d \cdot x + c} \cdot \sqrt{d \cdot x - c} - 18 \cdot c^4 \cdot \log(\text{abs}(-\sqrt{d \cdot x + c} + \sqrt{d \cdot x - c}))) / d^3) \cdot a \cdot c + 7 \cdot ((2 \cdot ((d \cdot x + c) \cdot (4 \cdot (d \cdot x + c) \cdot (5 \cdot (d \cdot x + c) / d^5 - 31 \cdot c / d^5) + 321 \cdot c^2 / d^5) - 451 \cdot c^3 / d^5) \cdot (d \cdot x + c) + 745 \cdot c^4 / d^5) \cdot (d \cdot x + c) - 405 \cdot c^5 / d^5) \cdot \sqrt{d \cdot x + c} \cdot \sqrt{d \cdot x - c} - 150 \cdot c^6 \cdot \log(\text{abs}(-\sqrt{d \cdot x + c} + \sqrt{d \cdot x - c}))) / d^5) \cdot b \cdot c + 14 \cdot (90 \cdot c^5 \cdot \log(\text{abs}(-\sqrt{d \cdot x + c} + \sqrt{d \cdot x - c}))) + (195 \cdot c^4 - (295 \cdot c^3 - 2 \cdot (3 \cdot (4 \cdot d \cdot x - 17 \cdot c) \cdot (d \cdot x + c) + 133 \cdot c^2) \cdot (d \cdot x + c)) \cdot (d \cdot x + c)) \cdot \sqrt{d \cdot x + c} \cdot \sqrt{d \cdot x - c}) \cdot a / d^3 + (1050 \cdot c^7 \cdot \log(\text{abs}(-\sqrt{d \cdot x + c} + \sqrt{d \cdot x - c}))) + (2835 \cdot c^6 - (6335 \cdot c^5 - 2 \cdot (4781 \cdot c^4 - (4551 \cdot c^3 - 4 \cdot (5 \cdot (6 \cdot d \cdot x - 37 \cdot c) \cdot (d \cdot x + c) + 661 \cdot c^2) \cdot (d \cdot x + c))) \cdot (d \cdot x + c)) \cdot (d \cdot x + c)) \cdot \sqrt{d \cdot x + c} \cdot \sqrt{d \cdot x - c}) \cdot b / d^5) / d$$
Mupad [B] (verification not implemented)

Time = 3.89 (sec) , antiderivative size = 118, normalized size of antiderivative = 1.08

$$\int x^3 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx = -\sqrt{dx - c} \left(\frac{(8bc^6 + 14ac^4d^2) \sqrt{c + dx}}{105d^6} - \frac{bx^6 \sqrt{c + dx}}{7} + \frac{x^2(4bc^4d^2 + 7ac^2d^4) \sqrt{c + dx}}{105d^6} - \frac{x^4(21ad^6 - 3bc^2d^4) \sqrt{c + dx}}{105d^6} \right)$$

input `int(x^3*(a + b*x^2)*(c + d*x)^(1/2)*(d*x - c)^(1/2),x)`

output `-(d*x - c)^(1/2)*(((8*b*c^6 + 14*a*c^4*d^2)*(c + d*x)^(1/2))/(105*d^6) - (b*x^6*(c + d*x)^(1/2))/7 + (x^2*(7*a*c^2*d^4 + 4*b*c^4*d^2)*(c + d*x)^(1/2)))/(105*d^6) - (x^4*(21*a*d^6 - 3*b*c^2*d^4)*(c + d*x)^(1/2))/(105*d^6))`

Reduce [B] (verification not implemented)

Time = 0.21 (sec) , antiderivative size = 89, normalized size of antiderivative = 0.82

$$\int x^3 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx$$

$$= \frac{\sqrt{dx + c} \sqrt{dx - c} (15b d^6 x^6 + 21a d^6 x^4 - 3b c^2 d^4 x^4 - 7a c^2 d^4 x^2 - 4b c^4 d^2 x^2 - 14a c^4 d^2 - 8b c^6)}{105d^6}$$

input `int(x^3*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x)`

output `(sqrt(c + d*x)*sqrt(-c + d*x)*(-14*a*c**4*d**2 - 7*a*c**2*d**4*x**2 + 21*a*d**6*x**4 - 8*b*c**6 - 4*b*c**4*d**2*x**2 - 3*b*c**2*d**4*x**4 + 15*b*d**6*x**6))/(105*d**6)`

3.3 $\int x\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2) dx$

Optimal result	59
Mathematica [A] (verified)	59
Rubi [A] (verified)	60
Maple [A] (verified)	61
Fricas [A] (verification not implemented)	61
Sympy [F]	62
Maxima [A] (verification not implemented)	62
Giac [B] (verification not implemented)	63
Mupad [B] (verification not implemented)	63
Reduce [B] (verification not implemented)	64

Optimal result

Integrand size = 29, antiderivative size = 67

$$\int x\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2) dx = \frac{(bc^2 + ad^2)(-c+dx)^{3/2}(c+dx)^{3/2}}{3d^4} + \frac{b(-c+dx)^{5/2}(c+dx)^{5/2}}{5d^4}$$

output

```
1/3*(a*d^2+b*c^2)*(d*x-c)^(3/2)*(d*x+c)^(3/2)/d^4+1/5*b*(d*x-c)^(5/2)*(d*x+c)^(5/2)/d^4
```

Mathematica [A] (verified)

Time = 0.11 (sec) , antiderivative size = 49, normalized size of antiderivative = 0.73

$$\int x\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2) dx = \frac{(-c+dx)^{3/2}(c+dx)^{3/2}(2bc^2 + 5ad^2 + 3bd^2x^2)}{15d^4}$$

input

```
Integrate[x*Sqrt[-c + d*x]*Sqrt[c + d*x]*(a + b*x^2),x]
```

output

```
((-c + d*x)^(3/2)*(c + d*x)^(3/2)*(2*b*c^2 + 5*a*d^2 + 3*b*d^2*x^2))/(15*d^4)
```

Rubi [A] (verified)

Time = 0.32 (sec) , antiderivative size = 72, normalized size of antiderivative = 1.07, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}} = 0.069$, Rules used = {960, 83}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x(a + bx^2) \sqrt{dx - c} \sqrt{c + dx} dx$$

$$\downarrow 960$$

$$\frac{1}{5} \left(5a + \frac{2bc^2}{d^2} \right) \int x \sqrt{dx - c} \sqrt{c + dx} dx + \frac{bx^2(dx - c)^{3/2}(c + dx)^{3/2}}{5d^2}$$

$$\downarrow 83$$

$$\frac{(dx - c)^{3/2}(c + dx)^{3/2} \left(5a + \frac{2bc^2}{d^2} \right)}{15d^2} + \frac{bx^2(dx - c)^{3/2}(c + dx)^{3/2}}{5d^2}$$

input `Int[x*Sqrt[-c + d*x]*Sqrt[c + d*x]*(a + b*x^2),x]`

output `((5*a + (2*b*c^2)/d^2)*(-c + d*x)^(3/2)*(c + d*x)^(3/2))/(15*d^2) + (b*x^2*(-c + d*x)^(3/2)*(c + d*x)^(3/2))/(5*d^2)`

Defintions of rubi rules used

rule 83 `Int[((a_.) + (b_.)*(x_))*((c_.) + (d_.)*(x_))^(n_.)*((e_.) + (f_.)*(x_))^(p_.), x_] :> Simp[b*(c + d*x)^(n + 1)*((e + f*x)^(p + 1)/(d*f*(n + p + 2))), x] /; FreeQ[{a, b, c, d, e, f, n, p}, x] && NeQ[n + p + 2, 0] && EqQ[a*d*f*(n + p + 2) - b*(d*e*(n + 1) + c*f*(p + 1)), 0]`

rule 960

```
Int[((e_.)*(x_))^(m_.)*((a1_) + (b1_.)*(x_)^(non2_.))^(p_.)*((a2_) + (b2_.)
*(x_)^(non2_.))^(p_.)*((c_) + (d_.)*(x_)^(n_)), x_Symbol] := Simp[d*(e*x)^(
m + 1)*(a1 + b1*x^(n/2))^(p + 1)*((a2 + b2*x^(n/2))^(p + 1)/(b1*b2*e*(m + n
*(p + 1) + 1))), x] - Simp[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/
(b1*b2*(m + n*(p + 1) + 1)) Int[(e*x)^m*(a1 + b1*x^(n/2))^p*(a2 + b2*x^(n
/2))^p, x], x] /; FreeQ[{a1, b1, a2, b2, c, d, e, m, n, p}, x] && EqQ[non2,
n/2] && EqQ[a2*b1 + a1*b2, 0] && NeQ[m + n*(p + 1) + 1, 0]
```

Maple [A] (verified)

Time = 0.14 (sec) , antiderivative size = 44, normalized size of antiderivative = 0.66

method	result	size
gospers	$\frac{(dx-c)^{\frac{3}{2}}(dx+c)^{\frac{3}{2}}(3bx^2d^2+5ad^2+2bc^2)}{15d^4}$	44
orering	$-\frac{(dx+c)^{\frac{3}{2}}(-dx+c)(3bx^2d^2+5ad^2+2bc^2)\sqrt{dx-c}}{15d^4}$	50
default	$-\frac{\sqrt{dx-c}\sqrt{dx+c}(-d^2x^2+c^2)(3bx^2d^2+5ad^2+2bc^2)}{15d^4}$	56
risch	$\frac{\sqrt{dx+c}(-3bx^4d^4-5ad^4x^2+bc^2d^2x^2+5ac^2d^2+2bc^4)(-dx+c)}{15\sqrt{dx-c}d^4}$	73

input

```
int(x*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x,method=_RETURNVERBOSE)
```

output

```
1/15/d^4*(d*x-c)^(3/2)*(d*x+c)^(3/2)*(3*b*d^2*x^2+5*a*d^2+2*b*c^2)
```

Fricas [A] (verification not implemented)

Time = 0.10 (sec) , antiderivative size = 66, normalized size of antiderivative = 0.99

$$\int x\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2) dx$$

$$= \frac{(3bd^4x^4 - 2bc^4 - 5ac^2d^2 - (bc^2d^2 - 5ad^4)x^2)\sqrt{dx+c}\sqrt{dx-c}}{15d^4}$$

input

```
integrate(x*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x, algorithm="fricas")
```

output $1/15*(3*b*d^4*x^4 - 2*b*c^4 - 5*a*c^2*d^2 - (b*c^2*d^2 - 5*a*d^4)*x^2)*\sqrt{d*x + c}*\sqrt{d*x - c}/d^4$

Sympy [F]

$$\int x\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2) dx = \int x(a+bx^2)\sqrt{-c+dx}\sqrt{c+dx} dx$$

input `integrate(x*(d*x-c)**(1/2)*(d*x+c)**(1/2)*(b*x**2+a), x)`

output `Integral(x*(a + b*x**2)*sqrt(-c + d*x)*sqrt(c + d*x), x)`

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 70, normalized size of antiderivative = 1.04

$$\int x\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2) dx = \frac{(d^2x^2 - c^2)^{\frac{3}{2}}bx^2}{5d^2} + \frac{2(d^2x^2 - c^2)^{\frac{3}{2}}bc^2}{15d^4} + \frac{(d^2x^2 - c^2)^{\frac{3}{2}}a}{3d^2}$$

input `integrate(x*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a), x, algorithm="maxima")`

output $1/5*(d^2*x^2 - c^2)^{(3/2)}*b*x^2/d^2 + 2/15*(d^2*x^2 - c^2)^{(3/2)}*b*c^2/d^4 + 1/3*(d^2*x^2 - c^2)^{(3/2)}*a/d^2$

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 330 vs. $2(55) = 110$.

Time = 0.18 (sec) , antiderivative size = 330, normalized size of antiderivative = 4.93

$$\int x\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2) dx$$

$$= \frac{5 \left(\left((dx+c) \left(2(dx+c) \left(\frac{3(dx+c)}{d^3} - \frac{13c}{d^3} \right) + \frac{43c^2}{d^3} \right) - \frac{39c^3}{d^3} \right) \sqrt{dx+c}\sqrt{dx-c} - \frac{18c^4 \log(|-\sqrt{dx+c}+\sqrt{dx-c}|)}{d^3} \right) bc}{1}$$

input `integrate(x*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x, algorithm="giac")`

output `1/120*(5*(((d*x + c)*(2*(d*x + c)*(3*(d*x + c)/d^3 - 13*c/d^3) + 43*c^2/d^3) - 39*c^3/d^3)*sqrt(d*x + c)*sqrt(d*x - c) - 18*c^4*log(abs(-sqrt(d*x + c) + sqrt(d*x - c)))/d^3)*b*c - 60*(2*c^2*log(abs(-sqrt(d*x + c) + sqrt(d*x - c))) - sqrt(d*x + c)*sqrt(d*x - c)*(d*x - 2*c))*a*c/d + 20*(6*c^3*log(abs(-sqrt(d*x + c) + sqrt(d*x - c))) + ((2*d*x - 5*c)*(d*x + c) + 9*c^2)*sqrt(d*x + c)*sqrt(d*x - c))*a/d + (90*c^5*log(abs(-sqrt(d*x + c) + sqrt(d*x - c))) + (195*c^4 - (295*c^3 - 2*(3*(4*d*x - 17*c)*(d*x + c) + 133*c^2)*(d*x + c))*(d*x + c))*sqrt(d*x + c)*sqrt(d*x - c))*b/d^3)/d`

Mupad [B] (verification not implemented)

Time = 3.96 (sec) , antiderivative size = 83, normalized size of antiderivative = 1.24

$$\int x\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2) dx = \sqrt{dx-c} \left(\frac{bx^4\sqrt{c+dx}}{5} - \frac{(2bc^4 + 5a^2d^2)\sqrt{c+dx}}{15d^4} + \frac{x^2(5ad^4 - bc^2d^2)\sqrt{c+dx}}{15d^4} \right)$$

input `int(x*(a + b*x^2)*(c + d*x)^(1/2)*(d*x - c)^(1/2),x)`

output

```
(d*x - c)^(1/2)*((b*x^4*(c + d*x)^(1/2))/5 - ((2*b*c^4 + 5*a*c^2*d^2)*(c +
d*x)^(1/2))/(15*d^4) + (x^2*(5*a*d^4 - b*c^2*d^2)*(c + d*x)^(1/2))/(15*d^
4))
```

Reduce [B] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 65, normalized size of antiderivative = 0.97

$$\int x\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2) dx$$

$$= \frac{\sqrt{dx+c}\sqrt{dx-c}(3bd^4x^4 + 5ad^4x^2 - bc^2d^2x^2 - 5ac^2d^2 - 2bc^4)}{15d^4}$$

input

```
int(x*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x)
```

output

```
(sqrt(c + d*x)*sqrt(-c + d*x)*(-5*a*c**2*d**2 + 5*a*d**4*x**2 - 2*b*c*
*4 - b*c**2*d**2*x**2 + 3*b*d**4*x**4))/(15*d**4)
```

3.4 $\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x} dx$

Optimal result	65
Mathematica [A] (verified)	65
Rubi [A] (verified)	66
Maple [B] (verified)	68
Fricas [A] (verification not implemented)	68
Sympy [F]	69
Maxima [A] (verification not implemented)	69
Giac [A] (verification not implemented)	69
Mupad [B] (verification not implemented)	70
Reduce [B] (verification not implemented)	70

Optimal result

Integrand size = 31, antiderivative size = 80

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x} dx = a\sqrt{-c+dx}\sqrt{c+dx} + \frac{b(-c+dx)^{3/2}(c+dx)^{3/2}}{3d^2} - ac \arctan\left(\frac{\sqrt{-c+dx}\sqrt{c+dx}}{c}\right)$$

output

```
a*(d*x-c)^(1/2)*(d*x+c)^(1/2)+1/3*b*(d*x-c)^(3/2)*(d*x+c)^(3/2)/d^2-a*c*arctan((d*x-c)^(1/2)*(d*x+c)^(1/2)/c)
```

Mathematica [A] (verified)

Time = 0.14 (sec) , antiderivative size = 75, normalized size of antiderivative = 0.94

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x} dx = \frac{\sqrt{-c+dx}\sqrt{c+dx}(-bc^2+3ad^2+bd^2x^2)}{3d^2} - 2ac \arctan\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)$$

input

```
Integrate[(Sqrt[-c + d*x]*Sqrt[c + d*x]*(a + b*x^2))/x,x]
```

output

```
(Sqrt[-c + d*x]*Sqrt[c + d*x]*(-(b*c^2) + 3*a*d^2 + b*d^2*x^2))/(3*d^2) -
2*a*c*ArcTan[Sqrt[-c + d*x]/Sqrt[c + d*x]]
```

Rubi [A] (verified)

Time = 0.39 (sec) , antiderivative size = 81, normalized size of antiderivative = 1.01, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.161$, Rules used = {960, 112, 27, 103, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{(a + bx^2) \sqrt{dx - c} \sqrt{c + dx}}{x} dx \\
 & \quad \downarrow \text{960} \\
 & a \int \frac{\sqrt{dx - c} \sqrt{c + dx}}{x} dx + \frac{b(dx - c)^{3/2}(c + dx)^{3/2}}{3d^2} \\
 & \quad \downarrow \text{112} \\
 & a \left(\sqrt{dx - c} \sqrt{c + dx} - \int \frac{c^2}{x \sqrt{dx - c} \sqrt{c + dx}} dx \right) + \frac{b(dx - c)^{3/2}(c + dx)^{3/2}}{3d^2} \\
 & \quad \downarrow \text{27} \\
 & a \left(\sqrt{dx - c} \sqrt{c + dx} - c^2 \int \frac{1}{x \sqrt{dx - c} \sqrt{c + dx}} dx \right) + \frac{b(dx - c)^{3/2}(c + dx)^{3/2}}{3d^2} \\
 & \quad \downarrow \text{103} \\
 & a \left(\sqrt{dx - c} \sqrt{c + dx} - c^2 d \int \frac{1}{dc^2 + d(dx - c)(c + dx)} d(\sqrt{dx - c} \sqrt{c + dx}) \right) + \\
 & \quad \frac{b(dx - c)^{3/2}(c + dx)^{3/2}}{3d^2} \\
 & \quad \downarrow \text{218} \\
 & a \left(\sqrt{dx - c} \sqrt{c + dx} - c \arctan \left(\frac{\sqrt{dx - c} \sqrt{c + dx}}{c} \right) \right) + \frac{b(dx - c)^{3/2}(c + dx)^{3/2}}{3d^2}
 \end{aligned}$$

input

```
Int[(Sqrt[-c + d*x]*Sqrt[c + d*x]*(a + b*x^2))/x,x]
```

output $(b*(-c + d*x)^{(3/2)}*(c + d*x)^{(3/2)})/(3*d^2) + a*(\text{Sqrt}[-c + d*x]*\text{Sqrt}[c + d*x] - c*\text{ArcTan}[(\text{Sqrt}[-c + d*x]*\text{Sqrt}[c + d*x])/c])$

Defintions of rubi rules used

rule 27 $\text{Int}[(a_*)(Fx_), x_Symbol] \rightarrow \text{Simp}[a \quad \text{Int}[Fx, x], x] \;/; \text{FreeQ}[a, x] \ \&\& \ !\text{MatchQ}[Fx, (b_)*(Gx_)] \;/; \text{FreeQ}[b, x]$

rule 103 $\text{Int}[1/(\text{Sqrt}[(a_.) + (b_.)*(x_)]*\text{Sqrt}[(c_.) + (d_.)*(x_)]*((e_.) + (f_.)*(x_))), x_] \rightarrow \text{Simp}[b*f \quad \text{Subst}[\text{Int}[1/(d*(b*e - a*f)^2 + b*f^2*x^2), x], x, \text{Sqrt}[a + b*x]*\text{Sqrt}[c + d*x]], x] \;/; \text{FreeQ}[\{a, b, c, d, e, f\}, x] \ \&\& \ \text{EqQ}[2*b*d*e - f*(b*c + a*d), 0]$

rule 112 $\text{Int}(((a_.) + (b_.)*(x_))^{(m_.)}*((c_.) + (d_.)*(x_))^{(n_.)}*((e_.) + (f_.)*(x_))^{(p_.)}, x_] \rightarrow \text{Simp}[(a + b*x)^m*(c + d*x)^n*(e + f*x)^{p+1}/(f*(m + n + p + 1)), x] - \text{Simp}[1/(f*(m + n + p + 1)) \quad \text{Int}[(a + b*x)^{m-1}*(c + d*x)^{n-1}*(e + f*x)^p*\text{Simp}[c*m*(b*e - a*f) + a*n*(d*e - c*f) + (d*m*(b*e - a*f) + b*n*(d*e - c*f))*x, x], x], x] \;/; \text{FreeQ}[\{a, b, c, d, e, f, p\}, x] \ \&\& \ \text{GtQ}[m, 0] \ \&\& \ \text{GtQ}[n, 0] \ \&\& \ \text{NeQ}[m + n + p + 1, 0] \ \&\& \ (\text{IntegersQ}[2*m, 2*n, 2*p] \ || \ (\text{IntegersQ}[m, n + p] \ || \ \text{IntegersQ}[p, m + n]))$

rule 218 $\text{Int}(((a_) + (b_.)*(x_)^2)^{-1}, x_Symbol] \rightarrow \text{Simp}[(\text{Rt}[a/b, 2]/a)*\text{ArcTan}[x/\text{Rt}[a/b, 2]], x] \;/; \text{FreeQ}[\{a, b\}, x] \ \&\& \ \text{PosQ}[a/b]$

rule 960 $\text{Int}(((e_.)*(x_))^{(m_.)}*((a1_.) + (b1_.)*(x_)^{(non2_.)})^{(p_.)}*((a2_.) + (b2_.)*(x_)^{(non2_.)})^{(p_.)}*((c_.) + (d_.)*(x_)^{(n_.)}), x_Symbol] \rightarrow \text{Simp}[d*(e*x)^{(m+1)}*(a1 + b1*x^{(n/2)})^{(p+1)}*((a2 + b2*x^{(n/2)})^{(p+1)})/(b1*b2*e*(m + n*(p + 1) + 1)), x] - \text{Simp}[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(b1*b2*(m + n*(p + 1) + 1)) \quad \text{Int}[(e*x)^m*(a1 + b1*x^{(n/2)})^p*(a2 + b2*x^{(n/2)})^p, x], x] \;/; \text{FreeQ}[\{a1, b1, a2, b2, c, d, e, m, n, p\}, x] \ \&\& \ \text{EqQ}[non2, n/2] \ \&\& \ \text{EqQ}[a2*b1 + a1*b2, 0] \ \&\& \ \text{NeQ}[m + n*(p + 1) + 1, 0]$

Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 173 vs. $2(66) = 132$.

Time = 0.14 (sec) , antiderivative size = 174, normalized size of antiderivative = 2.18

method	result
default	$\frac{\sqrt{dx-c}\sqrt{dx+c} \left(b d^2 x^2 \sqrt{-c^2} \sqrt{d^2 x^2 - c^2} + 3 \ln \left(-\frac{2(c^2 - \sqrt{-c^2} \sqrt{d^2 x^2 - c^2})}{x} \right) a c^2 d^2 + 3 \sqrt{-c^2} \sqrt{d^2 x^2 - c^2} a d^2 - b c^2 \sqrt{-c^2} \sqrt{d^2 x^2 - c^2} \right)}{3 \sqrt{d^2 x^2 - c^2} \sqrt{-c^2} d^2}$

input `int((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a)/x,x,method=_RETURNVERBOSE)`

output
$$\frac{1}{3} * (d*x-c)^{(1/2)} * (d*x+c)^{(1/2)} * (b*d^2*x^2 * (-c^2)^{(1/2)} * (d^2*x^2-c^2)^{(1/2)} + 3 * \ln(-2 * (c^2 - (-c^2)^{(1/2)} * (d^2*x^2-c^2)^{(1/2)}) / x) * a * c^2 * d^2 + 3 * (-c^2)^{(1/2)} * (d^2*x^2-c^2)^{(1/2)} * a * d^2 - b * c^2 * (-c^2)^{(1/2)} * (d^2*x^2-c^2)^{(1/2)}) / (d^2 * x^2 - c^2)^{(1/2)} / (-c^2)^{(1/2)} / d^2$$

Fricas [A] (verification not implemented)

Time = 0.10 (sec) , antiderivative size = 80, normalized size of antiderivative = 1.00

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x} dx$$

$$= -\frac{6acd^2 \arctan\left(-\frac{dx-\sqrt{dx+c}\sqrt{dx-c}}{c}\right) - (bd^2x^2 - bc^2 + 3ad^2)\sqrt{dx+c}\sqrt{dx-c}}{3d^2}$$

input `integrate((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a)/x,x, algorithm="fricas")`

output
$$-1/3 * (6 * a * c * d^2 * \arctan(-(d*x - \sqrt{d*x + c}) * \sqrt{d*x - c}) / c) - (b * d^2 * x^2 - b * c^2 + 3 * a * d^2) * \sqrt{d*x + c} * \sqrt{d*x - c}) / d^2$$

Sympy [F]

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x} dx = \int \frac{(a+bx^2)\sqrt{-c+dx}\sqrt{c+dx}}{x} dx$$

input `integrate((d*x-c)**(1/2)*(d*x+c)**(1/2)*(b*x**2+a)/x,x)`

output `Integral((a + b*x**2)*sqrt(-c + d*x)*sqrt(c + d*x)/x, x)`

Maxima [A] (verification not implemented)

Time = 0.11 (sec) , antiderivative size = 52, normalized size of antiderivative = 0.65

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x} dx = ac \arcsin\left(\frac{c}{d|x|}\right) + \sqrt{d^2x^2 - c^2}a + \frac{(d^2x^2 - c^2)^{\frac{3}{2}}b}{3d^2}$$

input `integrate((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a)/x,x, algorithm="maxima")`

output `a*c*arcsin(c/(d*abs(x))) + sqrt(d^2*x^2 - c^2)*a + 1/3*(d^2*x^2 - c^2)^(3/2)*b/d^2`

Giac [A] (verification not implemented)

Time = 0.15 (sec) , antiderivative size = 78, normalized size of antiderivative = 0.98

$$\begin{aligned} & \int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x} dx \\ &= 2ac \arctan\left(\frac{(\sqrt{dx+c}-\sqrt{dx-c})^2}{2c}\right) \\ & \quad + \frac{1}{3}\sqrt{dx+c}\sqrt{dx-c}\left((dx+c)\left(\frac{(dx+c)b}{d^2} - \frac{2bc}{d^2}\right) + 3a\right) \end{aligned}$$

input `integrate((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a)/x,x, algorithm="giac")`

output

```
2*a*c*arctan(1/2*(sqrt(d*x + c) - sqrt(d*x - c))^2/c) + 1/3*sqrt(d*x + c)*
sqrt(d*x - c)*((d*x + c)*((d*x + c)*b/d^2 - 2*b*c/d^2) + 3*a)
```

Mupad [B] (verification not implemented)

Time = 5.95 (sec) , antiderivative size = 248, normalized size of antiderivative = 3.10

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x} dx$$

$$= a\sqrt{-c}\sqrt{c}\ln\left(\frac{(\sqrt{c+dx}-\sqrt{c})^2}{(\sqrt{-c}-\sqrt{dx-c})^2}+1\right)$$

$$- a\sqrt{-c}\sqrt{c}\ln\left(\frac{\sqrt{c+dx}-\sqrt{c}}{\sqrt{-c}-\sqrt{dx-c}}\right) - \frac{b(c^2-d^2x^2)\sqrt{c+dx}\sqrt{dx-c}}{3d^2}$$

$$- \frac{8a\sqrt{-c}\sqrt{c}(\sqrt{c+dx}-\sqrt{c})^2}{(\sqrt{-c}-\sqrt{dx-c})^2\left(\frac{(\sqrt{c+dx}-\sqrt{c})^4}{(\sqrt{-c}-\sqrt{dx-c})^4} - \frac{2(\sqrt{c+dx}-\sqrt{c})^2}{(\sqrt{-c}-\sqrt{dx-c})^2} + 1\right)}$$

input

```
int(((a + b*x^2)*(c + d*x)^(1/2)*(d*x - c)^(1/2))/x,x)
```

output

```
a*(-c)^(1/2)*c^(1/2)*log(((c + d*x)^(1/2) - c^(1/2))^2/((-c)^(1/2) - (d*x
- c)^(1/2))^2 + 1) - a*(-c)^(1/2)*c^(1/2)*log(((c + d*x)^(1/2) - c^(1/2))/
((-c)^(1/2) - (d*x - c)^(1/2))) - (b*(c^2 - d^2*x^2)*(c + d*x)^(1/2)*(d*x
- c)^(1/2))/(3*d^2) - (8*a*(-c)^(1/2)*c^(1/2)*((c + d*x)^(1/2) - c^(1/2))^
2)/(((c + d*x)^(1/2) - c^(1/2))^2*((c + d*x)^(1/2) - c^(1/2))^4/((-c)^(
1/2) - (d*x - c)^(1/2))^4 - (2*((c + d*x)^(1/2) - c^(1/2))^2)/((-c)^(1/2)
- (d*x - c)^(1/2))^2 + 1))
```

Reduce [B] (verification not implemented)

Time = 0.23 (sec) , antiderivative size = 130, normalized size of antiderivative = 1.62

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x} dx$$

$$= \frac{-6atan\left(\frac{\sqrt{dx-c}+\sqrt{dx+c}-\sqrt{c}}{\sqrt{c}}\right)acd^2 + 6atan\left(\frac{\sqrt{dx-c}+\sqrt{dx+c}+\sqrt{c}}{\sqrt{c}}\right)acd^2 + 3\sqrt{dx+c}\sqrt{dx-c}ad^2 - \sqrt{dx+c}\sqrt{dx-c}ad^2}{3d^2}$$

input `int((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a)/x,x)`

output `(- 6*atan((sqrt(- c + d*x) + sqrt(c + d*x) - sqrt(c))/sqrt(c))*a*c*d**2 + 6*atan((sqrt(- c + d*x) + sqrt(c + d*x) + sqrt(c))/sqrt(c))*a*c*d**2 + 3*sqrt(c + d*x)*sqrt(- c + d*x)*a*d**2 - sqrt(c + d*x)*sqrt(- c + d*x)*b*c**2 + sqrt(c + d*x)*sqrt(- c + d*x)*b*d**2*x**2)/(3*d**2)`

3.5 $\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^3} dx$

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Optimal result

Integrand size = 31, antiderivative size = 96

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^3} dx = b\sqrt{-c+dx}\sqrt{c+dx} - \frac{a\sqrt{-c+dx}\sqrt{c+dx}}{2x^2} - \frac{(2bc^2 - ad^2) \arctan\left(\frac{\sqrt{-c+dx}\sqrt{c+dx}}{c}\right)}{2c}$$

output

```
b*(d*x-c)^(1/2)*(d*x+c)^(1/2)-1/2*a*(d*x-c)^(1/2)*(d*x+c)^(1/2)/x^2-1/2*(-a*d^2+2*b*c^2)*arctan((d*x-c)^(1/2)*(d*x+c)^(1/2)/c)/c
```

Mathematica [A] (verified)

Time = 0.22 (sec) , antiderivative size = 74, normalized size of antiderivative = 0.77

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^3} dx = \frac{\sqrt{-c+dx}\sqrt{c+dx}(-a+2bx^2)}{2x^2} + \left(-2bc + \frac{ad^2}{c}\right) \arctan\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)$$

input

```
Integrate[(Sqrt[-c + d*x]*Sqrt[c + d*x]*(a + b*x^2))/x^3,x]
```

output

```
(Sqrt[-c + d*x]*Sqrt[c + d*x]*(-a + 2*b*x^2))/(2*x^2) + (-2*b*c + (a*d^2)/c)*ArcTan[Sqrt[-c + d*x]/Sqrt[c + d*x]]
```

Rubi [A] (verified)

Time = 0.40 (sec) , antiderivative size = 99, normalized size of antiderivative = 1.03, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.161$, Rules used = {956, 112, 27, 103, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(a + bx^2) \sqrt{dx - c} \sqrt{c + dx}}{x^3} dx$$

$$\downarrow 956$$

$$\frac{1}{2} \left(2b - \frac{ad^2}{c^2} \right) \int \frac{\sqrt{dx - c} \sqrt{c + dx}}{x} dx + \frac{a(dx - c)^{3/2} (c + dx)^{3/2}}{2c^2 x^2}$$

$$\downarrow 112$$

$$\frac{1}{2} \left(2b - \frac{ad^2}{c^2} \right) \left(\sqrt{dx - c} \sqrt{c + dx} - \int \frac{c^2}{x \sqrt{dx - c} \sqrt{c + dx}} dx \right) + \frac{a(dx - c)^{3/2} (c + dx)^{3/2}}{2c^2 x^2}$$

$$\downarrow 27$$

$$\frac{1}{2} \left(2b - \frac{ad^2}{c^2} \right) \left(\sqrt{dx - c} \sqrt{c + dx} - c^2 \int \frac{1}{x \sqrt{dx - c} \sqrt{c + dx}} dx \right) + \frac{a(dx - c)^{3/2} (c + dx)^{3/2}}{2c^2 x^2}$$

$$\downarrow 103$$

$$\frac{1}{2} \left(2b - \frac{ad^2}{c^2} \right) \left(\sqrt{dx - c} \sqrt{c + dx} - c^2 d \int \frac{1}{dc^2 + d(dx - c)(c + dx)} d(\sqrt{dx - c} \sqrt{c + dx}) \right) + \frac{a(dx - c)^{3/2} (c + dx)^{3/2}}{2c^2 x^2}$$

$$\downarrow 218$$

$$\frac{1}{2} \left(2b - \frac{ad^2}{c^2} \right) \left(\sqrt{dx - c} \sqrt{c + dx} - c \arctan \left(\frac{\sqrt{dx - c} \sqrt{c + dx}}{c} \right) \right) + \frac{a(dx - c)^{3/2} (c + dx)^{3/2}}{2c^2 x^2}$$

input

```
Int[(Sqrt[-c + d*x]*Sqrt[c + d*x]*(a + b*x^2))/x^3,x]
```

output

$$\frac{(a*(-c + dx)^{3/2}*(c + dx)^{3/2})/(2*c^2*x^2) + ((2*b - (a*d^2)/c^2)*(Sqrt[-c + dx]*Sqrt[c + dx] - c*ArcTan[(Sqrt[-c + dx]*Sqrt[c + dx])/c]))}{2}$$

Defintions of rubi rules used

rule 27

$$\text{Int}[(a_*)(Fx_), x_Symbol] \rightarrow \text{Simp}[a \text{ Int}[Fx, x], x] \text{ ; FreeQ}[a, x] \ \&\& \ !\text{MatchQ}[Fx, (b_)*(Gx_)] \text{ ; FreeQ}[b, x]$$

rule 103

$$\text{Int}[1/(\text{Sqrt}[(a_.) + (b_.)*(x_)]*\text{Sqrt}[(c_.) + (d_.)*(x_)]*((e_.) + (f_.)*(x_))), x_] \rightarrow \text{Simp}[b*f \text{ Subst}[\text{Int}[1/(d*(b*e - a*f)^2 + b*f^2*x^2), x], x, \text{Sqrt}[a + b*x]*\text{Sqrt}[c + d*x]], x] \text{ ; FreeQ}[\{a, b, c, d, e, f\}, x] \ \&\& \ \text{EqQ}[2*b*d*e - f*(b*c + a*d), 0]$$

rule 112

$$\text{Int}[((a_.) + (b_.)*(x_))^{(m_)}*((c_.) + (d_.)*(x_))^{(n_)}*((e_.) + (f_.)*(x_))^{(p_)}, x_] \rightarrow \text{Simp}[(a + b*x)^m*(c + d*x)^n*(e + f*x)^{p+1}/(f*(m + n + p + 1)), x] - \text{Simp}[1/(f*(m + n + p + 1)) \text{ Int}[(a + b*x)^{m-1}*(c + d*x)^{n-1}*(e + f*x)^p*\text{Simp}[c*m*(b*e - a*f) + a*n*(d*e - c*f) + (d*m*(b*e - a*f) + b*n*(d*e - c*f))*x, x], x] \text{ ; FreeQ}[\{a, b, c, d, e, f, p\}, x] \ \&\& \ \text{GtQ}[m, 0] \ \&\& \ \text{GtQ}[n, 0] \ \&\& \ \text{NeQ}[m + n + p + 1, 0] \ \&\& \ (\text{IntegersQ}[2*m, 2*n, 2*p] \ || \ (\text{IntegersQ}[m, n + p] \ || \ \text{IntegersQ}[p, m + n]))$$

rule 218

$$\text{Int}[(a_.) + (b_.)*(x_)^2)^{-1}, x_Symbol] \rightarrow \text{Simp}[(\text{Rt}[a/b, 2]/a)*\text{ArcTan}[x/\text{Rt}[a/b, 2]], x] \text{ ; FreeQ}[\{a, b\}, x] \ \&\& \ \text{PosQ}[a/b]$$

rule 956

$$\text{Int}[(e_.)*(x_))^{(m_)}*((a1_.) + (b1_.)*(x_)^{(non2_.)})^{(p_)}*((a2_.) + (b2_.)*(x_)^{(non2_.)})^{(p_)}*((c_.) + (d_.)*(x_)^{(n_)}), x_Symbol] \rightarrow \text{Simp}[c*(e*x)^{(m+1)}*(a1 + b1*x^{(n/2)})^{(p+1)}*((a2 + b2*x^{(n/2)})^{(p+1)})/(a1*a2*e^{(m+1)}), x] + \text{Simp}[(a1*a2*d*(m+1) - b1*b2*c*(m + n*(p+1) + 1))/(a1*a2*e^{n*(m+1)}) \text{ Int}[(e*x)^{(m+n)}*(a1 + b1*x^{(n/2)})^p*(a2 + b2*x^{(n/2)})^p, x], x] \text{ ; FreeQ}[\{a1, b1, a2, b2, c, d, e, p\}, x] \ \&\& \ \text{EqQ}[\text{non2}, n/2] \ \&\& \ \text{EqQ}[a2*b1 + a1*b2, 0] \ \&\& \ (\text{IntegerQ}[n] \ || \ \text{GtQ}[e, 0]) \ \&\& \ ((\text{GtQ}[n, 0] \ \&\& \ \text{LtQ}[m, -1]) \ || \ (\text{LtQ}[n, 0] \ \&\& \ \text{GtQ}[m + n, -1])) \ \&\& \ !\text{ILtQ}[p, -1]$$

Maple [A] (verified)

Time = 0.18 (sec) , antiderivative size = 138, normalized size of antiderivative = 1.44

method	result
risch	$\frac{a(-dx+c)\sqrt{dx+c}}{2x^2\sqrt{dx-c}} - \frac{\left(\frac{(ad^2-2bc^2)\ln\left(\frac{-2c^2+2\sqrt{-c^2}\sqrt{d^2x^2-c^2}}{x}\right) - b\sqrt{(dx-c)(dx+c)}}{2\sqrt{-c^2}} \right) \sqrt{(dx-c)(dx+c)}}{\sqrt{dx-c}\sqrt{dx+c}}$
default	$-\frac{\sqrt{dx-c}\sqrt{dx+c}\left(\ln\left(-\frac{2(c^2-\sqrt{-c^2}\sqrt{d^2x^2-c^2})}{x}\right)\right)ad^2x^2-2\ln\left(-\frac{2(c^2-\sqrt{-c^2}\sqrt{d^2x^2-c^2})}{x}\right)bc^2x^2-2bx^2\sqrt{-c^2}\sqrt{d^2x^2-c^2}+\sqrt{-c^2}}{2\sqrt{d^2x^2-c^2}x^2\sqrt{-c^2}}$

input `int((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a)/x^3,x,method=_RETURNVERBOSE)`

output
$$\frac{1}{2}aa(-dx+c)(d*x+c)^{(1/2)}/x^2/(d*x-c)^{(1/2)} - \frac{1}{2}(ad^2-2bc^2)/(-c^2)^{(1/2)}*\ln\left(\frac{-2c^2+2(-c^2)^{(1/2)}*(d^2*x^2-c^2)^{(1/2)}}{x}\right) - b*((d*x-c)*(d*x+c))^{(1/2)}*((d*x-c)*(d*x+c))^{(1/2)}/(d*x-c)^{(1/2)}/(d*x+c)^{(1/2)}$$

Fricas [A] (verification not implemented)

Time = 0.14 (sec) , antiderivative size = 85, normalized size of antiderivative = 0.89

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^3} dx$$

$$= -\frac{2(2bc^2-ad^2)x^2 \arctan\left(-\frac{dx-\sqrt{dx+c}\sqrt{dx-c}}{c}\right) - (2bcx^2-ac)\sqrt{dx+c}\sqrt{dx-c}}{2cx^2}$$

input `integrate((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a)/x^3,x, algorithm="fricas")`

output
$$-1/2*(2*(2*b*c^2 - a*d^2)*x^2*\arctan(-(d*x - \text{sqrt}(d*x + c))*\text{sqrt}(d*x - c))/c) - (2*b*c*x^2 - a*c)*\text{sqrt}(d*x + c)*\text{sqrt}(d*x - c)/(c*x^2)$$

Sympy [F]

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^3} dx = \int \frac{(a+bx^2)\sqrt{-c+dx}\sqrt{c+dx}}{x^3} dx$$

input `integrate((d*x-c)**(1/2)*(d*x+c)**(1/2)*(b*x**2+a)/x**3,x)`

output `Integral((a + b*x**2)*sqrt(-c + d*x)*sqrt(c + d*x)/x**3, x)`

Maxima [A] (verification not implemented)

Time = 0.11 (sec) , antiderivative size = 98, normalized size of antiderivative = 1.02

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^3} dx = bc \arcsin\left(\frac{c}{d|x|}\right) - \frac{ad^2 \arcsin\left(\frac{c}{d|x|}\right)}{2c} + \sqrt{d^2x^2 - c^2}b - \frac{\sqrt{d^2x^2 - c^2}ad^2}{2c^2} + \frac{(d^2x^2 - c^2)^{\frac{3}{2}}a}{2c^2x^2}$$

input `integrate((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a)/x^3,x, algorithm="maxima")`

output `b*c*arcsin(c/(d*abs(x))) - 1/2*a*d^2*arcsin(c/(d*abs(x)))/c + sqrt(d^2*x^2 - c^2)*b - 1/2*sqrt(d^2*x^2 - c^2)*a*d^2/c^2 + 1/2*(d^2*x^2 - c^2)^(3/2)*a/(c^2*x^2)`

Giac [A] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 157, normalized size of antiderivative = 1.64

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^3} dx = \frac{\sqrt{dx+c}\sqrt{dx-c}bd + \frac{(2bc^2d-ad^3) \arctan\left(\frac{(\sqrt{dx+c}-\sqrt{dx-c})^2}{2c}\right)}{c} + \frac{2(ad^3(\sqrt{dx+c}-\sqrt{dx-c})^6 - 4ac^2d^3(\sqrt{dx+c}-\sqrt{dx-c})^2)}{((\sqrt{dx+c}-\sqrt{dx-c})^4 + 4c^2)^2}}{d}$$

input `integrate((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a)/x^3,x, algorithm="giac")`

output $(\sqrt{d*x + c}*\sqrt{d*x - c}*b*d + (2*b*c^2*d - a*d^3)*\arctan(1/2*(\sqrt{d*x + c} - \sqrt{d*x - c}))^2/c)/c + 2*(a*d^3*(\sqrt{d*x + c} - \sqrt{d*x - c}))^6 - 4*a*c^2*d^3*(\sqrt{d*x + c} - \sqrt{d*x - c})^2)/((\sqrt{d*x + c} - \sqrt{d*x - c})^4 + 4*c^2)^2/d$

Mupad [B] (verification not implemented)

Time = 9.44 (sec) , antiderivative size = 584, normalized size of antiderivative = 6.08

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^3} dx = b\sqrt{-c}\sqrt{c}\ln\left(\frac{(\sqrt{c+dx}-\sqrt{c})^2}{(\sqrt{-c}-\sqrt{dx-c})^2}+1\right) - \frac{\frac{a\sqrt{-c}d^2}{32c^{3/2}} + \frac{a\sqrt{-c}d^2(\sqrt{c+dx}-\sqrt{c})^2}{16c^{3/2}(\sqrt{-c}-\sqrt{dx-c})^2} - \frac{15a\sqrt{-c}d^2(\sqrt{c+dx}-\sqrt{c})^4}{32c^{3/2}(\sqrt{-c}-\sqrt{dx-c})^4}}{\frac{(\sqrt{c+dx}-\sqrt{c})^2}{(\sqrt{-c}-\sqrt{dx-c})^2} + \frac{2(\sqrt{c+dx}-\sqrt{c})^4}{(\sqrt{-c}-\sqrt{dx-c})^4} + \frac{(\sqrt{c+dx}-\sqrt{c})^6}{(\sqrt{-c}-\sqrt{dx-c})^6}} - b\sqrt{-c}\sqrt{c}\ln\left(\frac{\sqrt{c+dx}-\sqrt{c}}{\sqrt{-c}-\sqrt{dx-c}}\right) + \frac{a\sqrt{-c}d^2\ln\left(\frac{\sqrt{c+dx}-\sqrt{c}}{\sqrt{-c}-\sqrt{dx-c}}\right)}{2c^{3/2}} - \frac{a\sqrt{-c}d^2\ln\left(\frac{(\sqrt{c+dx}-\sqrt{c})^2}{(\sqrt{-c}-\sqrt{dx-c})^2}+1\right)}{2c^{3/2}} - \dots$$

input `int(((a + b*x^2)*(c + d*x)^(1/2)*(d*x - c)^(1/2))/x^3,x)`

output

```

b*(-c)^(1/2)*c^(1/2)*log(((c + d*x)^(1/2) - c^(1/2))^2/((-c)^(1/2) - (d*x
- c)^(1/2))^2 + 1) - ((a*(-c)^(1/2)*d^2)/(32*c^(3/2)) + (a*(-c)^(1/2)*d^2*
((c + d*x)^(1/2) - c^(1/2))^2)/(16*c^(3/2)*((-c)^(1/2) - (d*x - c)^(1/2))^
2) - (15*a*(-c)^(1/2)*d^2*((c + d*x)^(1/2) - c^(1/2))^4)/(32*c^(3/2)*((-c)
^(1/2) - (d*x - c)^(1/2))^4)/(((c + d*x)^(1/2) - c^(1/2))^2/((-c)^(1/2) -
(d*x - c)^(1/2))^2 + (2*((c + d*x)^(1/2) - c^(1/2))^4)/((-c)^(1/2) - (d*x
- c)^(1/2))^4 + ((c + d*x)^(1/2) - c^(1/2))^6/((-c)^(1/2) - (d*x - c)^(1/
2))^6) - b*(-c)^(1/2)*c^(1/2)*log(((c + d*x)^(1/2) - c^(1/2))/((-c)^(1/2)
- (d*x - c)^(1/2))) + (a*(-c)^(1/2)*d^2*log(((c + d*x)^(1/2) - c^(1/2))/((-
c)^(1/2) - (d*x - c)^(1/2))))/(2*c^(3/2)) - (a*(-c)^(1/2)*d^2*log(((c + d
*x)^(1/2) - c^(1/2))^2/((-c)^(1/2) - (d*x - c)^(1/2))^2 + 1))/(2*c^(3/2))
- (a*(-c)^(1/2)*d^2*((c + d*x)^(1/2) - c^(1/2))^2)/(32*c^(3/2)*((-c)^(1/2)
- (d*x - c)^(1/2))^2) - (8*b*(-c)^(1/2)*c^(1/2)*((c + d*x)^(1/2) - c^(1/2
))^2)/((-c)^(1/2) - (d*x - c)^(1/2))^2*((c + d*x)^(1/2) - c^(1/2))^4/((-
c)^(1/2) - (d*x - c)^(1/2))^4 - (2*((c + d*x)^(1/2) - c^(1/2))^2)/((-c)^(1
/2) - (d*x - c)^(1/2))^2 + 1)

```

Reduce [B] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 180, normalized size of antiderivative = 1.88

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^3} dx$$

$$= \frac{2 \operatorname{atan}\left(\frac{\sqrt{dx-c}+\sqrt{dx+c}-\sqrt{c}}{\sqrt{c}}\right) a d^2 x^2 - 4 \operatorname{atan}\left(\frac{\sqrt{dx-c}+\sqrt{dx+c}-\sqrt{c}}{\sqrt{c}}\right) b c^2 x^2 - 2 \operatorname{atan}\left(\frac{\sqrt{dx-c}+\sqrt{dx+c}+\sqrt{c}}{\sqrt{c}}\right) a d^2 x^2 + 4 \operatorname{atan}\left(\frac{\sqrt{dx-c}+\sqrt{dx+c}+\sqrt{c}}{\sqrt{c}}\right) b c^2 x^2}{2c x^2}$$

input

```
int((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a)/x^3,x)
```

output

```

(2*atan((sqrt(-c + d*x) + sqrt(c + d*x) - sqrt(c))/sqrt(c))*a*d**2*x**2
- 4*atan((sqrt(-c + d*x) + sqrt(c + d*x) - sqrt(c))/sqrt(c))*b*c**2*x**2
- 2*atan((sqrt(-c + d*x) + sqrt(c + d*x) + sqrt(c))/sqrt(c))*a*d**2*x**
2 + 4*atan((sqrt(-c + d*x) + sqrt(c + d*x) + sqrt(c))/sqrt(c))*b*c**2*x*
*2 - sqrt(c + d*x)*sqrt(-c + d*x)*a*c + 2*sqrt(c + d*x)*sqrt(-c + d*x)
*b*c*x**2)/(2*c*x**2)

```

3.6 $\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^5} dx$

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Optimal result

Integrand size = 31, antiderivative size = 118

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^5} dx = -\frac{\left(4b + \frac{ad^2}{c^2}\right) \sqrt{-c+dx}\sqrt{c+dx}}{8x^2} + \frac{a(-c+dx)^{3/2}(c+dx)^{3/2}}{4c^2x^4} + \frac{d^2(4bc^2 + ad^2) \arctan\left(\frac{\sqrt{-c+dx}\sqrt{c+dx}}{c}\right)}{8c^3}$$

output

```
-1/8*(4*b+a*d^2/c^2)*(d*x-c)^(1/2)*(d*x+c)^(1/2)/x^2+1/4*a*(d*x-c)^(3/2)*(d*x+c)^(3/2)/c^2/x^4+1/8*d^2*(a*d^2+4*b*c^2)*arctan((d*x-c)^(1/2)*(d*x+c)^(1/2)/c)/c^3
```


Mathematica [A] (verified)

Time = 0.19 (sec) , antiderivative size = 99, normalized size of antiderivative = 0.84

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^5} dx$$

$$= \frac{c\sqrt{-c+dx}\sqrt{c+dx}(-2ac^2 - 4bc^2x^2 + ad^2x^2) + 2d^2(4bc^2 + ad^2)x^4 \arctan\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)}{8c^3x^4}$$

input

```
Integrate[(Sqrt[-c + d*x]*Sqrt[c + d*x]*(a + b*x^2))/x^5,x]
```

output

```
(c*Sqrt[-c + d*x]*Sqrt[c + d*x]*(-2*a*c^2 - 4*b*c^2*x^2 + a*d^2*x^2) + 2*d^2*(4*b*c^2 + a*d^2)*x^4*ArcTan[Sqrt[-c + d*x]/Sqrt[c + d*x]])/(8*c^3*x^4)
```

Rubi [A] (verified)

Time = 0.44 (sec) , antiderivative size = 142, normalized size of antiderivative = 1.20, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.161$, Rules used = {956, 105, 105, 103, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(a+bx^2)\sqrt{dx-c}\sqrt{c+dx}}{x^5} dx$$

$$\downarrow 956$$

$$\frac{1}{4}\left(\frac{ad^2}{c^2} + 4b\right) \int \frac{\sqrt{dx-c}\sqrt{c+dx}}{x^3} dx + \frac{a(dx-c)^{3/2}(c+dx)^{3/2}}{4c^2x^4}$$

$$\downarrow 105$$

$$\frac{1}{4}\left(\frac{ad^2}{c^2} + 4b\right) \left(\frac{1}{2}d \int \frac{\sqrt{c+dx}}{x^2\sqrt{dx-c}} dx - \frac{\sqrt{dx-c}(c+dx)^{3/2}}{2cx^2} \right) + \frac{a(dx-c)^{3/2}(c+dx)^{3/2}}{4c^2x^4}$$

$$\downarrow 105$$

$$\frac{1}{4} \left(\frac{ad^2}{c^2} + 4b \right) \left(\frac{1}{2} d \left(d \int \frac{1}{x\sqrt{dx-c}\sqrt{c+dx}} dx + \frac{\sqrt{dx-c}\sqrt{c+dx}}{cx} \right) - \frac{\sqrt{dx-c}(c+dx)^{3/2}}{2cx^2} \right) + \frac{a(dx-c)^{3/2}(c+dx)^{3/2}}{4c^2x^4}$$

↓ 103

$$\frac{1}{4} \left(\frac{ad^2}{c^2} + 4b \right) \left(\frac{1}{2} d \left(d^2 \int \frac{1}{dc^2 + d(dx-c)(c+dx)} d(\sqrt{dx-c}\sqrt{c+dx}) + \frac{\sqrt{dx-c}\sqrt{c+dx}}{cx} \right) - \frac{\sqrt{dx-c}(c+dx)^{3/2}}{2cx^2} \right) + \frac{a(dx-c)^{3/2}(c+dx)^{3/2}}{4c^2x^4}$$

↓ 218

$$\frac{1}{4} \left(\frac{ad^2}{c^2} + 4b \right) \left(\frac{1}{2} d \left(\frac{d \arctan \left(\frac{\sqrt{dx-c}\sqrt{c+dx}}{c} \right)}{c} + \frac{\sqrt{dx-c}\sqrt{c+dx}}{cx} \right) - \frac{\sqrt{dx-c}(c+dx)^{3/2}}{2cx^2} \right) + \frac{a(dx-c)^{3/2}(c+dx)^{3/2}}{4c^2x^4}$$

input `Int[(Sqrt[-c + d*x]*Sqrt[c + d*x]*(a + b*x^2))/x^5,x]`

output `(a*(-c + d*x)^(3/2)*(c + d*x)^(3/2))/(4*c^2*x^4) + ((4*b + (a*d^2)/c^2)*(-1/2*(Sqrt[-c + d*x]*(c + d*x)^(3/2))/(c*x^2) + (d*((Sqrt[-c + d*x]*Sqrt[c + d*x])/(c*x) + (d*ArcTan[(Sqrt[-c + d*x]*Sqrt[c + d*x])/c])/c))/2))/4`

Defintions of rubi rules used

rule 103 `Int[1/(Sqrt[(a_.) + (b_.)*(x_)]*Sqrt[(c_.) + (d_.)*(x_)]*((e_.) + (f_.)*(x_))), x_] := Simp[b*f Subst[Int[1/(d*(b*e - a*f)^2 + b*f^2*x^2), x], x, Sqrt[a + b*x]*Sqrt[c + d*x]], x] /; FreeQ[{a, b, c, d, e, f}, x] && EqQ[2*b*d*e - f*(b*c + a*d), 0]`

rule 105 `Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_)*((e_.) + (f_.)*(x_))^(p_), x_] := Simp[(a + b*x)^(m + 1)*(c + d*x)^n*(e + f*x)^(p + 1)/((m + 1)*(b*e - a*f)), x] - Simp[n*((d*e - c*f)/((m + 1)*(b*e - a*f)) Int[(a + b*x)^(m + 1)*(c + d*x)^(n - 1)*(e + f*x)^p, x], x] /; FreeQ[{a, b, c, d, e, f, m, p}, x] && EqQ[m + n + p + 2, 0] && GtQ[n, 0] && (SumSimplerQ[m, 1] || !SumSimplerQ[p, 1]) && NeQ[m, -1]`

rule 218 `Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(Rt[a/b, 2]/a)*ArcTan[x/Rt[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]`

rule 956 `Int[((e_.)*(x_))^(m_.)*((a1_.) + (b1_.)*(x_)^(non2_.))^(p_.)*((a2_.) + (b2_.)*(x_)^(non2_.))^(p_.)*((c_.) + (d_.)*(x_)^(n_)), x_Symbol] := Simp[c*(e*x)^(m + 1)*(a1 + b1*x^(n/2))^(p + 1)*((a2 + b2*x^(n/2))^(p + 1)/(a1*a2*e^(m + 1))), x] + Simp[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(a1*a2*e^n*(m + 1)) Int[(e*x)^(m + n)*(a1 + b1*x^(n/2))^p*(a2 + b2*x^(n/2))^p, x], x] /; FreeQ[{a1, b1, a2, b2, c, d, e, p}, x] && EqQ[non2, n/2] && EqQ[a2*b1 + a1*b2, 0] && (IntegerQ[n] || GtQ[e, 0]) && ((GtQ[n, 0] && LtQ[m, -1]) || (LtQ[n, 0] && GtQ[m + n, -1])) && !ILtQ[p, -1]`

Maple [A] (verified)

Time = 0.15 (sec) , antiderivative size = 150, normalized size of antiderivative = 1.27

method	result
risch	$\frac{\sqrt{dx+c}(-dx+c)(-ad^2x^2+4bc^2x^2+2ac^2)}{8x^4c^2\sqrt{dx-c}} - \frac{d^2(ad^2+4bc^2)\ln\left(\frac{-2c^2+2\sqrt{-c^2}\sqrt{d^2x^2-c^2}}{x}\right)\sqrt{(dx-c)(dx+c)}}{8c^2\sqrt{-c^2}\sqrt{dx-c}\sqrt{dx+c}}$
default	$-\frac{\sqrt{dx-c}\sqrt{dx+c}\left(\ln\left(-\frac{2(c^2-\sqrt{-c^2}\sqrt{d^2x^2-c^2})}{x}\right)\right)ad^4x^4+4\ln\left(-\frac{2(c^2-\sqrt{-c^2}\sqrt{d^2x^2-c^2})}{x}\right)bc^2d^2x^4-\sqrt{-c^2}\sqrt{d^2x^2-c^2}ad^2x^2+4}{8c^2\sqrt{d^2x^2-c^2}x^4\sqrt{-c^2}}$

input `int((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a)/x^5,x,method=_RETURNVERBOSE)`

output

```
1/8*(d*x+c)^(1/2)*(-d*x+c)*(-a*d^2*x^2+4*b*c^2*x^2+2*a*c^2)/x^4/c^2/(d*x-c)^(1/2)-1/8*d^2*(a*d^2+4*b*c^2)/c^2/(-c^2)^(1/2)*ln((-2*c^2+2*(-c^2)^(1/2)*(d^2*x^2-c^2)^(1/2))/x)*((d*x-c)*(d*x+c))^(1/2)/(d*x-c)^(1/2)/(d*x+c)^(1/2)
```

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 100, normalized size of antiderivative = 0.85

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^5} dx$$

$$= \frac{2(4bc^2d^2+ad^4)x^4 \arctan\left(-\frac{dx-\sqrt{dx+c}\sqrt{dx-c}}{c}\right) - (2ac^3+(4bc^3-acd^2)x^2)\sqrt{dx+c}\sqrt{dx-c}}{8c^3x^4}$$

input

```
integrate((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a)/x^5,x, algorithm="fricas")
```

output

```
1/8*(2*(4*b*c^2*d^2+a*d^4)*x^4*arctan(-(d*x-sqrt(d*x+c))*sqrt(d*x-c))/c)-(2*a*c^3+(4*b*c^3-a*c*d^2)*x^2)*sqrt(d*x+c)*sqrt(d*x-c)/(c^3*x^4)
```

Sympy [F(-1)]

Timed out.

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^5} dx = \text{Timed out}$$

input

```
integrate((d*x-c)**(1/2)*(d*x+c)**(1/2)*(b*x**2+a)/x**5,x)
```

output

Timed out

Maxima [A] (verification not implemented)

Time = 0.11 (sec) , antiderivative size = 162, normalized size of antiderivative = 1.37

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^5} dx = -\frac{bd^2 \arcsin\left(\frac{c}{d|x|}\right)}{2c} - \frac{ad^4 \arcsin\left(\frac{c}{d|x|}\right)}{8c^3} - \frac{\sqrt{d^2x^2-c^2}bd^2}{2c^2} - \frac{\sqrt{d^2x^2-c^2}ad^4}{8c^4} + \frac{(d^2x^2-c^2)^{\frac{3}{2}}b}{2c^2x^2} + \frac{(d^2x^2-c^2)^{\frac{3}{2}}ad^2}{8c^4x^2} + \frac{(d^2x^2-c^2)^{\frac{3}{2}}a}{4c^2x^4}$$

input `integrate((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a)/x^5,x, algorithm="maxima")`

output `-1/2*b*d^2*arcsin(c/(d*abs(x)))/c - 1/8*a*d^4*arcsin(c/(d*abs(x)))/c^3 - 1/2*sqrt(d^2*x^2 - c^2)*b*d^2/c^2 - 1/8*sqrt(d^2*x^2 - c^2)*a*d^4/c^4 + 1/2*(d^2*x^2 - c^2)^(3/2)*b/(c^2*x^2) + 1/8*(d^2*x^2 - c^2)^(3/2)*a*d^2/(c^4*x^2) + 1/4*(d^2*x^2 - c^2)^(3/2)*a/(c^2*x^4)`

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 324 vs. 2(100) = 200.

Time = 0.20 (sec) , antiderivative size = 324, normalized size of antiderivative = 2.75

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^5} dx = \frac{(4bc^2d^3+ad^5) \arctan\left(\frac{(\sqrt{dx+c}-\sqrt{dx-c})^2}{2c}\right)}{c^3} - \frac{2(4bc^2d^3(\sqrt{dx+c}-\sqrt{dx-c})^{14}-ad^5(\sqrt{dx+c}-\sqrt{dx-c})^{14}+16bc^4d^3(\sqrt{dx+c}-\sqrt{dx-c})^{10}+20bc^3d^3(\sqrt{dx+c}-\sqrt{dx-c})^8+16bc^2d^3(\sqrt{dx+c}-\sqrt{dx-c})^6+4bc^2d^3(\sqrt{dx+c}-\sqrt{dx-c})^4+4bc^2d^3(\sqrt{dx+c}-\sqrt{dx-c})^2+4bc^2d^3)}{c^3}$$

input `integrate((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a)/x^5,x, algorithm="giac")`

output

```
-1/4*((4*b*c^2*d^3 + a*d^5)*arctan(1/2*(sqrt(d*x + c) - sqrt(d*x - c))^2/c
)/c^3 - 2*(4*b*c^2*d^3*(sqrt(d*x + c) - sqrt(d*x - c))^14 - a*d^5*(sqrt(d*
x + c) - sqrt(d*x - c))^14 + 16*b*c^4*d^3*(sqrt(d*x + c) - sqrt(d*x - c))^
10 + 28*a*c^2*d^5*(sqrt(d*x + c) - sqrt(d*x - c))^10 - 64*b*c^6*d^3*(sqrt(
d*x + c) - sqrt(d*x - c))^6 - 112*a*c^4*d^5*(sqrt(d*x + c) - sqrt(d*x - c)
)^6 - 256*b*c^8*d^3*(sqrt(d*x + c) - sqrt(d*x - c))^2 + 64*a*c^6*d^5*(sqrt
(d*x + c) - sqrt(d*x - c))^2)/(((sqrt(d*x + c) - sqrt(d*x - c))^4 + 4*c^2)
^4*c^2))/d
```

Mupad [B] (verification not implemented)

Time = 18.88 (sec) , antiderivative size = 1004, normalized size of antiderivative = 8.51

$$\int \frac{\sqrt{-c + dx}\sqrt{c + dx}(a + bx^2)}{x^5} dx = \text{Too large to display}$$

input

```
int(((a + b*x^2)*(c + d*x)^(1/2)*(d*x - c)^(1/2))/x^5,x)
```

output

```
((a*(-c)^(1/2)*d^4)/(1024*c^(7/2)) + (a*(-c)^(1/2)*d^4*((c + d*x)^(1/2) -
c^(1/2))^2)/(128*c^(7/2)*((-c)^(1/2) - (d*x - c)^(1/2))^2) + (11*a*(-c)^(1
/2)*d^4*((c + d*x)^(1/2) - c^(1/2))^4)/(512*c^(7/2)*((-c)^(1/2) - (d*x - c
)^(1/2))^4) + (7*a*(-c)^(1/2)*d^4*((c + d*x)^(1/2) - c^(1/2))^6)/(256*c^(7
/2)*((-c)^(1/2) - (d*x - c)^(1/2))^6) - (239*a*(-c)^(1/2)*d^4*((c + d*x)^(
1/2) - c^(1/2))^8)/(1024*c^(7/2)*((-c)^(1/2) - (d*x - c)^(1/2))^8) + (a*(-
c)^(1/2)*d^4*((c + d*x)^(1/2) - c^(1/2))^10)/(256*c^(7/2)*((-c)^(1/2) - (d
*x - c)^(1/2))^10)/(((c + d*x)^(1/2) - c^(1/2))^4/((-c)^(1/2) - (d*x - c)
^(1/2))^4 + (4*((c + d*x)^(1/2) - c^(1/2))^6)/((-c)^(1/2) - (d*x - c)^(1/2)
)^6 + (6*((c + d*x)^(1/2) - c^(1/2))^8)/((-c)^(1/2) - (d*x - c)^(1/2))^8
+ (4*((c + d*x)^(1/2) - c^(1/2))^10)/((-c)^(1/2) - (d*x - c)^(1/2))^10 + (
(c + d*x)^(1/2) - c^(1/2))^12/((-c)^(1/2) - (d*x - c)^(1/2))^12) - ((b*(-c
)^(1/2)*d^2)/(32*c^(3/2)) + (b*(-c)^(1/2)*d^2*((c + d*x)^(1/2) - c^(1/2))^
2)/(16*c^(3/2)*((-c)^(1/2) - (d*x - c)^(1/2))^2) - (15*b*(-c)^(1/2)*d^2*((
c + d*x)^(1/2) - c^(1/2))^4)/(32*c^(3/2)*((-c)^(1/2) - (d*x - c)^(1/2))^4)
)/(((c + d*x)^(1/2) - c^(1/2))^2/((-c)^(1/2) - (d*x - c)^(1/2))^2 + (2*((c
+ d*x)^(1/2) - c^(1/2))^4)/((-c)^(1/2) - (d*x - c)^(1/2))^4 + ((c + d*x)^(
1/2) - c^(1/2))^6/((-c)^(1/2) - (d*x - c)^(1/2))^6) + (a*(-c)^(1/2)*d^4*1
og(((c + d*x)^(1/2) - c^(1/2))/((-c)^(1/2) - (d*x - c)^(1/2))))/(8*c^(7/2)
) + (b*(-c)^(1/2)*d^2*log(((c + d*x)^(1/2) - c^(1/2))/((-c)^(1/2) - (d*...
```

Reduce [B] (verification not implemented)

Time = 0.23 (sec) , antiderivative size = 213, normalized size of antiderivative = 1.81

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^5} dx$$

$$= \frac{2\operatorname{atan}\left(\frac{\sqrt{dx-c}+\sqrt{dx+c}-\sqrt{c}}{\sqrt{c}}\right) a d^4 x^4 + 8\operatorname{atan}\left(\frac{\sqrt{dx-c}+\sqrt{dx+c}-\sqrt{c}}{\sqrt{c}}\right) b c^2 d^2 x^4 - 2\operatorname{atan}\left(\frac{\sqrt{dx-c}+\sqrt{dx+c}+\sqrt{c}}{\sqrt{c}}\right) a d^4 x^4 - 8\operatorname{atan}\left(\frac{\sqrt{dx-c}+\sqrt{dx+c}+\sqrt{c}}{\sqrt{c}}\right) b c^2 d^2 x^4}{8c^3 x^4}$$

input

```
int((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a)/x^5,x)
```

output

```
(2*atan((sqrt(-c+d*x)+sqrt(c+d*x)-sqrt(c))/sqrt(c))*a*d**4*x**4
+ 8*atan((sqrt(-c+d*x)+sqrt(c+d*x)-sqrt(c))/sqrt(c))*b*c**2*d**2
*x**4 - 2*atan((sqrt(-c+d*x)+sqrt(c+d*x)+sqrt(c))/sqrt(c))*a*d**
4*x**4 - 8*atan((sqrt(-c+d*x)+sqrt(c+d*x)+sqrt(c))/sqrt(c))*b*c*
*2*d**2*x**4 - 2*sqrt(c+d*x)*sqrt(-c+d*x)*a*c**3 + sqrt(c+d*x)*sqr
t(-c+d*x)*a*c*d**2*x**2 - 4*sqrt(c+d*x)*sqrt(-c+d*x)*b*c**3*x**2
)/(8*c**3*x**4)
```

3.7 $\int x^4 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx$

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Optimal result

Integrand size = 31, antiderivative size = 207

$$\int x^4 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx = -\frac{c^4(5bc^2 + 8ad^2) x \sqrt{-c + dx} \sqrt{c + dx}}{128d^6} - \frac{c^2(5bc^2 + 8ad^2) x^3 \sqrt{-c + dx} \sqrt{c + dx}}{192d^4} + \frac{1}{48} \left(8a + \frac{5bc^2}{d^2} \right) x^5 \sqrt{-c + dx} \sqrt{c + dx} + \frac{bx^5(-c + dx)^{3/2}(c + dx)^{3/2}}{8d^2} - \frac{c^6(5bc^2 + 8ad^2) \operatorname{arctanh}\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)}{64d^7}$$

output

```
-1/128*c^4*(8*a*d^2+5*b*c^2)*x*(d*x-c)^(1/2)*(d*x+c)^(1/2)/d^6-1/192*c^2*(8*a*d^2+5*b*c^2)*x^3*(d*x-c)^(1/2)*(d*x+c)^(1/2)/d^4+1/48*(8*a+5*b*c^2/d^2)*x^5*(d*x-c)^(1/2)*(d*x+c)^(1/2)+1/8*b*x^5*(d*x-c)^(3/2)*(d*x+c)^(3/2)/d^2-1/64*c^6*(8*a*d^2+5*b*c^2)*arctanh((d*x-c)^(1/2)/(d*x+c)^(1/2))/d^7
```


Mathematica [A] (verified)

Time = 0.31 (sec) , antiderivative size = 142, normalized size of antiderivative = 0.69

$$\int x^4 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx$$

$$= \frac{dx \sqrt{-c + dx} \sqrt{c + dx} (8ad^2(-3c^4 - 2c^2d^2x^2 + 8d^4x^4) - b(15c^6 + 10c^4d^2x^2 + 8c^2d^4x^4 - 48d^6x^6)) - 6c^6(5b^2c^2 + 8ad^2) \operatorname{ArcTanh}\left[\frac{\sqrt{-c + dx}}{\sqrt{c + dx}}\right]}{384d^7}$$

input

```
Integrate[x^4*Sqrt[-c + d*x]*Sqrt[c + d*x]*(a + b*x^2),x]
```

output

```
(d*x*Sqrt[-c + d*x]*Sqrt[c + d*x]*(8*a*d^2*(-3*c^4 - 2*c^2*d^2*x^2 + 8*d^4*x^4) - b*(15*c^6 + 10*c^4*d^2*x^2 + 8*c^2*d^4*x^4 - 48*d^6*x^6)) - 6*c^6*(5*b*c^2 + 8*a*d^2)*ArcTanh[Sqrt[-c + d*x]/Sqrt[c + d*x]])/(384*d^7)
```

Rubi [A] (verified)Time = 0.49 (sec) , antiderivative size = 185, normalized size of antiderivative = 0.89, number of steps used = 9, number of rules used = 8, $\frac{\text{number of rules}}{\text{integrand size}} = 0.258$, Rules used = {960, 111, 27, 101, 27, 40, 45, 221}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x^4 (a + bx^2) \sqrt{dx - c} \sqrt{c + dx} dx$$

$$\downarrow 960$$

$$\frac{1}{8} \left(8a + \frac{5bc^2}{d^2} \right) \int x^4 \sqrt{dx - c} \sqrt{c + dx} dx + \frac{bx^5 (dx - c)^{3/2} (c + dx)^{3/2}}{8d^2}$$

$$\downarrow 111$$

$$\frac{1}{8} \left(8a + \frac{5bc^2}{d^2} \right) \left(\frac{\int 3c^2 x^2 \sqrt{dx - c} \sqrt{c + dx} dx}{6d^2} + \frac{x^3 (dx - c)^{3/2} (c + dx)^{3/2}}{6d^2} \right) + \frac{bx^5 (dx - c)^{3/2} (c + dx)^{3/2}}{8d^2}$$

$$\downarrow 27$$

$$\begin{aligned}
& \frac{1}{8} \left(8a + \frac{5bc^2}{d^2} \right) \left(\frac{c^2 \int x^2 \sqrt{dx-c} \sqrt{c+dx} dx}{2d^2} + \frac{x^3(dx-c)^{3/2}(c+dx)^{3/2}}{6d^2} \right) + \\
& \quad \frac{bx^5(dx-c)^{3/2}(c+dx)^{3/2}}{8d^2} \\
& \quad \downarrow 101 \\
& \frac{1}{8} \left(8a + \frac{5bc^2}{d^2} \right) \left(\frac{c^2 \left(\frac{\int c^2 \sqrt{dx-c} \sqrt{c+dx} dx}{4d^2} + \frac{x(dx-c)^{3/2}(c+dx)^{3/2}}{4d^2} \right)}{2d^2} + \frac{x^3(dx-c)^{3/2}(c+dx)^{3/2}}{6d^2} \right) + \\
& \quad \frac{bx^5(dx-c)^{3/2}(c+dx)^{3/2}}{8d^2} \\
& \quad \downarrow 27 \\
& \frac{1}{8} \left(8a + \frac{5bc^2}{d^2} \right) \left(\frac{c^2 \left(\frac{c^2 \int \sqrt{dx-c} \sqrt{c+dx} dx}{4d^2} + \frac{x(dx-c)^{3/2}(c+dx)^{3/2}}{4d^2} \right)}{2d^2} + \frac{x^3(dx-c)^{3/2}(c+dx)^{3/2}}{6d^2} \right) + \\
& \quad \frac{bx^5(dx-c)^{3/2}(c+dx)^{3/2}}{8d^2} \\
& \quad \downarrow 40 \\
& \frac{1}{8} \left(8a + \frac{5bc^2}{d^2} \right) \left(\frac{c^2 \left(\frac{c^2 \left(\frac{1}{2} x \sqrt{dx-c} \sqrt{c+dx} - \frac{1}{2} c^2 \int \frac{1}{\sqrt{dx-c} \sqrt{c+dx}} dx \right)}{4d^2} + \frac{x(dx-c)^{3/2}(c+dx)^{3/2}}{4d^2} \right)}{2d^2} + \frac{x^3(dx-c)^{3/2}(c+dx)^{3/2}}{6d^2} \right) + \\
& \quad \frac{bx^5(dx-c)^{3/2}(c+dx)^{3/2}}{8d^2} \\
& \quad \downarrow 45 \\
& \frac{1}{8} \left(8a + \frac{5bc^2}{d^2} \right) \left(\frac{c^2 \left(\frac{c^2 \left(\frac{1}{2} x \sqrt{dx-c} \sqrt{c+dx} - c^2 \int \frac{1}{d - \frac{d(dx-c)}{c+dx}} d \frac{\sqrt{dx-c}}{\sqrt{c+dx}} \right)}{4d^2} + \frac{x(dx-c)^{3/2}(c+dx)^{3/2}}{4d^2} \right)}{2d^2} + \frac{x^3(dx-c)^{3/2}(c+dx)^{3/2}}{6d^2} \right) + \\
& \quad \frac{bx^5(dx-c)^{3/2}(c+dx)^{3/2}}{8d^2} \\
& \quad \downarrow 221
\end{aligned}$$

$$\frac{1}{8} \left(8a + \frac{5bc^2}{d^2} \right) \left(\frac{c^2 \left(\frac{\frac{1}{2}x\sqrt{dx-c}\sqrt{c+dx} - \frac{c^2 \operatorname{arctanh}\left(\frac{\sqrt{dx-c}}{\sqrt{c+dx}}\right)}{d}}{4d^2} + \frac{x(dx-c)^{3/2}(c+dx)^{3/2}}{4d^2} \right)}{2d^2} + \frac{x^3(dx-c)^{3/2}(c+dx)^{3/2}}{6d^2} \right) + \frac{bx^5(dx-c)^{3/2}(c+dx)^{3/2}}{8d^2}$$

input `Int[x^4*Sqrt[-c + d*x]*Sqrt[c + d*x]*(a + b*x^2),x]`

output `(b*x^5*(-c + d*x)^(3/2)*(c + d*x)^(3/2))/(8*d^2) + ((8*a + (5*b*c^2)/d^2)*((x^3*(-c + d*x)^(3/2)*(c + d*x)^(3/2))/(6*d^2) + (c^2*((x*(-c + d*x)^(3/2)*(c + d*x)^(3/2))/(4*d^2) + (c^2*((x*Sqrt[-c + d*x]*Sqrt[c + d*x])/2 - (c^2*ArcTanh[Sqrt[-c + d*x]/Sqrt[c + d*x]])/d))/(4*d^2)))/(2*d^2))/8`

Defintions of rubi rules used

rule 27 `Int[(a_)*(F_x_), x_Symbol] := Simp[a Int[F_x, x], x] /; FreeQ[a, x] && !MatchQ[F_x, (b_)*(G_x_)] /; FreeQ[b, x]`

rule 40 `Int[((a_) + (b_.)*(x_))^(m_)*((c_) + (d_.)*(x_))^(m_), x_Symbol] := Simp[x*(a + b*x)^m*((c + d*x)^m/(2*m + 1)), x] + Simp[2*a*c*(m/(2*m + 1)) Int[(a + b*x)^(m - 1)*(c + d*x)^(m - 1), x], x] /; FreeQ[{a, b, c, d}, x] && EqQ[b*c + a*d, 0] && IGtQ[m + 1/2, 0]`

rule 45 `Int[1/(Sqrt[(a_) + (b_.)*(x_)]*Sqrt[(c_) + (d_.)*(x_)]), x_Symbol] := Simp[2 Subst[Int[1/(b - d*x^2), x], x, Sqrt[a + b*x]/Sqrt[c + d*x]], x] /; FreeQ[{a, b, c, d}, x] && EqQ[b*c + a*d, 0] && !GtQ[c, 0]`

rule 101 $\text{Int}[(a_.) + (b_.)(x_)^2((c_.) + (d_.)(x_)^{(n_.)}((e_.) + (f_.)(x_)^{(p_.)}, x_] := \text{Simp}[b*(a + b*x)*(c + d*x)^{(n + 1)}*((e + f*x)^{(p + 1)}/(d*f*(n + p + 3))), x] + \text{Simp}[1/(d*f*(n + p + 3)) \text{Int}[(c + d*x)^n*(e + f*x)^p*\text{Simp}[a^2*d*f*(n + p + 3) - b*(b*c*e + a*(d*e*(n + 1) + c*f*(p + 1))) + b*(a*d*f*(n + p + 4) - b*(d*e*(n + 2) + c*f*(p + 2)))*x, x], x] /; \text{FreeQ}\{a, b, c, d, e, f, n, p\}, x] \&\& \text{NeQ}[n + p + 3, 0]$

rule 111 $\text{Int}[(a_.) + (b_.)(x_)^{(m_.)}((c_.) + (d_.)(x_)^{(n_.)}((e_.) + (f_.)(x_)^{(p_.)}, x_] := \text{Simp}[b*(a + b*x)^{(m - 1)}*(c + d*x)^{(n + 1)}*((e + f*x)^{(p + 1)}/(d*f*(m + n + p + 1))), x] + \text{Simp}[1/(d*f*(m + n + p + 1)) \text{Int}[(a + b*x)^{(m - 2)}*(c + d*x)^n*(e + f*x)^p*\text{Simp}[a^2*d*f*(m + n + p + 1) - b*(b*c*e*(m - 1) + a*(d*e*(n + 1) + c*f*(p + 1))) + b*(a*d*f*(2*m + n + p) - b*(d*e*(m + n) + c*f*(m + p)))*x, x], x] /; \text{FreeQ}\{a, b, c, d, e, f, n, p\}, x] \&\& \text{GtQ}[m, 1] \&\& \text{NeQ}[m + n + p + 1, 0] \&\& \text{IntegerQ}[m]$

rule 221 $\text{Int}[(a_.) + (b_.)(x_)^2)^{-1}, x_Symbol] := \text{Simp}[(\text{Rt}[-a/b, 2]/a)*\text{ArcTanh}[x/\text{Rt}[-a/b, 2]], x] /; \text{FreeQ}\{a, b\}, x] \&\& \text{NegQ}[a/b]$

rule 960 $\text{Int}[(e_.)(x_)^{(m_.)}((a1_.) + (b1_.)(x_)^{(non2_.)})^{(p_.)}((a2_.) + (b2_.)(x_)^{(non2_.)})^{(p_.)}((c_.) + (d_.)(x_)^{(n_.)}, x_Symbol] := \text{Simp}[d*(e*x)^{(m + 1)}*(a1 + b1*x^{(n/2)})^{(p + 1)}*((a2 + b2*x^{(n/2)})^{(p + 1)}/(b1*b2*e*(m + n*(p + 1) + 1))), x] - \text{Simp}[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(b1*b2*(m + n*(p + 1) + 1)) \text{Int}[(e*x)^m*(a1 + b1*x^{(n/2)})^p*(a2 + b2*x^{(n/2)})^p, x], x] /; \text{FreeQ}\{a1, b1, a2, b2, c, d, e, m, n, p\}, x] \&\& \text{EqQ}[non2, n/2] \&\& \text{EqQ}[a2*b1 + a1*b2, 0] \&\& \text{NeQ}[m + n*(p + 1) + 1, 0]$

Maple [A] (verified)

Time = 0.14 (sec) , antiderivative size = 184, normalized size of antiderivative = 0.89

method	result
risch	$\frac{x(-48b^6d^6 - 64ad^6x^4 + 8b^2d^4x^4 + 16a^2d^4x^2 + 10b^4d^2x^2 + 24a^4d^2 + 15b^6c^6)(-dx+c)\sqrt{dx+c}}{384d^6\sqrt{dx-c}} - \frac{c^6(8ad^2 + 5b^2c^2)\ln\left(\frac{d^2x}{\sqrt{d^2}} + \sqrt{d^2}\right)}{128d^6\sqrt{d^2}\sqrt{d}}$
default	$-\frac{\sqrt{dx-c}\sqrt{dx+c}\left(-48\text{csgn}(d)b^6d^7x^7\sqrt{d^2x^2-c^2}-64\text{csgn}(d)a^2d^7x^5\sqrt{d^2x^2-c^2}+8\text{csgn}(d)b^2c^2d^5x^5\sqrt{d^2x^2-c^2}+16\text{csgn}(d)a^2c^2d^5x^3\sqrt{d^2x^2-c^2}\right)}{384d^6\sqrt{dx-c}}$

input `int(x^4*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x,method=_RETURNVERBOSE)`

output
$$\frac{1}{384}x^4(-48bd^6x^6 - 64a^2d^6x^4 + 8b^2c^2d^4x^4 + 16a^2c^2d^4x^2 + 10b^2c^4d^2x^2 + 24a^2c^4d^2 + 15b^2c^6)(-dx+c)(d*x+c)^{1/2}/d^6/(d*x-c)^{1/2} - 1/128c^6(8a^2d^2+5b^2c^2)/d^6*\ln(d^2*x/(d^2)^{1/2}+(d^2*x^2-c^2)^{1/2})/(d^2)^{1/2}*((d*x-c)*(d*x+c))^{1/2}/(d*x-c)^{1/2}/(d*x+c)^{1/2}$$

Fricas [A] (verification not implemented)

Time = 0.11 (sec) , antiderivative size = 138, normalized size of antiderivative = 0.67

$$\int x^4 \sqrt{-c+dx} \sqrt{c+dx} (a+bx^2) dx = \frac{(48bd^7x^7 - 8(bc^2d^5 - 8ad^7)x^5 - 2(5bc^4d^3 + 8ac^2d^5)x^3 - 3(5bc^6d + 8ac^4d^3)x)\sqrt{dx+c}\sqrt{dx-c} + 3(5b^2c^8 + 8a^2c^6d^2)\log(-dx + \sqrt{dx+c}\sqrt{dx-c})}{384d^7}$$

input `integrate(x^4*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x, algorithm="fricas")`

output
$$\frac{1}{384}((48bd^7x^7 - 8(b^2c^2d^5 - 8ad^7)x^5 - 2(5b^2c^4d^3 + 8a^2c^2d^5)x^3 - 3(5b^2c^6d + 8a^2c^4d^3)x)\sqrt{dx+c}\sqrt{dx-c} + 3(5b^2c^8 + 8a^2c^6d^2)\log(-dx + \sqrt{dx+c}\sqrt{dx-c}))/d^7$$

Sympy [F]

$$\int x^4 \sqrt{-c+dx} \sqrt{c+dx} (a+bx^2) dx = \int x^4 (a+bx^2) \sqrt{-c+dx} \sqrt{c+dx} dx$$

input `integrate(x**4*(d*x-c)**(1/2)*(d*x+c)**(1/2)*(b*x**2+a),x)`

output `Integral(x**4*(a + b*x**2)*sqrt(-c + d*x)*sqrt(c + d*x), x)`

Maxima [A] (verification not implemented)

Time = 0.04 (sec) , antiderivative size = 246, normalized size of antiderivative = 1.19

$$\int x^4 \sqrt{-c+dx} \sqrt{c+dx} (a+bx^2) dx = \frac{(d^2x^2 - c^2)^{\frac{3}{2}} bx^5}{8 d^2} + \frac{5 (d^2x^2 - c^2)^{\frac{3}{2}} bc^2 x^3}{48 d^4}$$

$$+ \frac{(d^2x^2 - c^2)^{\frac{3}{2}} ax^3}{6 d^2}$$

$$- \frac{5 bc^8 \log(2 d^2 x + 2 \sqrt{d^2 x^2 - c^2} d)}{128 d^7}$$

$$- \frac{ac^6 \log(2 d^2 x + 2 \sqrt{d^2 x^2 - c^2} d)}{16 d^5}$$

$$+ \frac{5 \sqrt{d^2 x^2 - c^2} bc^6 x}{128 d^6} + \frac{\sqrt{d^2 x^2 - c^2} ac^4 x}{16 d^4}$$

$$+ \frac{5 (d^2 x^2 - c^2)^{\frac{3}{2}} bc^4 x}{64 d^6} + \frac{(d^2 x^2 - c^2)^{\frac{3}{2}} ac^2 x}{8 d^4}$$

input `integrate(x^4*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x, algorithm="maxima")`

output `1/8*(d^2*x^2 - c^2)^(3/2)*b*x^5/d^2 + 5/48*(d^2*x^2 - c^2)^(3/2)*b*c^2*x^3/d^4 + 1/6*(d^2*x^2 - c^2)^(3/2)*a*x^3/d^2 - 5/128*b*c^8*log(2*d^2*x + 2*sqrt(d^2*x^2 - c^2)*d)/d^7 - 1/16*a*c^6*log(2*d^2*x + 2*sqrt(d^2*x^2 - c^2)*d)/d^5 + 5/128*sqrt(d^2*x^2 - c^2)*b*c^6*x/d^6 + 1/16*sqrt(d^2*x^2 - c^2)*a*c^4*x/d^4 + 5/64*(d^2*x^2 - c^2)^(3/2)*b*c^4*x/d^6 + 1/8*(d^2*x^2 - c^2)^(3/2)*a*c^2*x/d^4`

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 519 vs. 2(177) = 354.

Time = 0.22 (sec) , antiderivative size = 519, normalized size of antiderivative = 2.51

$$\int x^4 \sqrt{-c+dx} \sqrt{c+dx} (a+bx^2) dx = \text{Too large to display}$$

input `integrate(x^4*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x, algorithm="giac")`

output

```

1/13440*(56*((2*((d*x + c)*(4*(d*x + c)*(5*(d*x + c)/d^5 - 31*c/d^5) + 32
1*c^2/d^5) - 451*c^3/d^5)*(d*x + c) + 745*c^4/d^5)*(d*x + c) - 405*c^5/d^5
)*sqrt(d*x + c)*sqrt(d*x - c) - 150*c^6*log(abs(-sqrt(d*x + c) + sqrt(d*x
- c)))/d^5)*a*d + (((2*((4*(5*(d*x + c)*(6*(d*x + c)*(7*(d*x + c)/d^7 - 57
*c/d^7) + 1219*c^2/d^7) - 12463*c^3/d^7)*(d*x + c) + 64233*c^4/d^7)*(d*x +
c) - 53963*c^5/d^7)*(d*x + c) + 59465*c^6/d^7)*(d*x + c) - 23205*c^7/d^7)
*sqrt(d*x + c)*sqrt(d*x - c) - 7350*c^8*log(abs(-sqrt(d*x + c) + sqrt(d*x
- c)))/d^7)*b*d + 112*(90*c^5*log(abs(-sqrt(d*x + c) + sqrt(d*x - c))) + (
195*c^4 - (295*c^3 - 2*(3*(4*d*x - 17*c)*(d*x + c) + 133*c^2)*(d*x + c))*
(d*x + c))*sqrt(d*x + c)*sqrt(d*x - c))*a*c/d^4 + 8*(1050*c^7*log(abs(-sqrt
(d*x + c) + sqrt(d*x - c))) + (2835*c^6 - (6335*c^5 - 2*(4781*c^4 - (4551*
c^3 - 4*(5*(6*d*x - 37*c)*(d*x + c) + 661*c^2)*(d*x + c))*(d*x + c))*
(d*x + c))*sqrt(d*x + c)*sqrt(d*x - c))*b*c/d^6)/d

```

Mupad [B] (verification not implemented)

Time = 42.42 (sec) , antiderivative size = 2314, normalized size of antiderivative = 11.18

$$\int x^4 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx = \text{Too large to display}$$

input

```
int(x^4*(a + b*x^2)*(c + d*x)^(1/2)*(d*x - c)^(1/2),x)
```

output

```

((35*a*c^6*((c + d*x)^(1/2) - c^(1/2))^3)/(12*((-c)^(1/2) - (d*x - c)^(1/2))^3) - (a*c^6*((c + d*x)^(1/2) - c^(1/2)))/(4*((-c)^(1/2) - (d*x - c)^(1/2))) + (757*a*c^6*((c + d*x)^(1/2) - c^(1/2))^5)/(4*((-c)^(1/2) - (d*x - c)^(1/2))^5) + (7339*a*c^6*((c + d*x)^(1/2) - c^(1/2))^7)/(4*((-c)^(1/2) - (d*x - c)^(1/2))^7) + (41929*a*c^6*((c + d*x)^(1/2) - c^(1/2))^9)/(6*((-c)^(1/2) - (d*x - c)^(1/2))^9) + (25661*a*c^6*((c + d*x)^(1/2) - c^(1/2))^11)/(2*((-c)^(1/2) - (d*x - c)^(1/2))^11) + (25661*a*c^6*((c + d*x)^(1/2) - c^(1/2))^13)/(2*((-c)^(1/2) - (d*x - c)^(1/2))^13) + (41929*a*c^6*((c + d*x)^(1/2) - c^(1/2))^15)/(6*((-c)^(1/2) - (d*x - c)^(1/2))^15) + (7339*a*c^6*((c + d*x)^(1/2) - c^(1/2))^17)/(4*((-c)^(1/2) - (d*x - c)^(1/2))^17) + (757*a*c^6*((c + d*x)^(1/2) - c^(1/2))^19)/(4*((-c)^(1/2) - (d*x - c)^(1/2))^19) + (35*a*c^6*((c + d*x)^(1/2) - c^(1/2))^21)/(12*((-c)^(1/2) - (d*x - c)^(1/2))^21) - (a*c^6*((c + d*x)^(1/2) - c^(1/2))^23)/(4*((-c)^(1/2) - (d*x - c)^(1/2))^23))/(d^5 - (12*d^5*((c + d*x)^(1/2) - c^(1/2))^2)/((-c)^(1/2) - (d*x - c)^(1/2))^2 + (66*d^5*((c + d*x)^(1/2) - c^(1/2))^4)/((-c)^(1/2) - (d*x - c)^(1/2))^4 - (220*d^5*((c + d*x)^(1/2) - c^(1/2))^6)/((-c)^(1/2) - (d*x - c)^(1/2))^6 + (495*d^5*((c + d*x)^(1/2) - c^(1/2))^8)/((-c)^(1/2) - (d*x - c)^(1/2))^8 - (792*d^5*((c + d*x)^(1/2) - c^(1/2))^10)/((-c)^(1/2) - (d*x - c)^(1/2))^10 + (924*d^5*((c + d*x)^(1/2) - c^(1/2))^12)/((-c)^(1/2) - (d*x - c)^(1/2))^12 - (792*d^5*((c + d*x)^(1/2) - c^(1/2)...

```

Reduce [B] (verification not implemented)

Time = 0.23 (sec) , antiderivative size = 241, normalized size of antiderivative = 1.16

$$\int x^4 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx$$

$$= \frac{-24\sqrt{dx + c} \sqrt{dx - c} a^4 d^3 x - 16\sqrt{dx + c} \sqrt{dx - c} a^2 d^5 x^3 + 64\sqrt{dx + c} \sqrt{dx - c} a d^7 x^5 - 15\sqrt{dx + c} \sqrt{dx - c} a^3 d^3 x^3 + 15\sqrt{dx + c} \sqrt{dx - c} a^2 d^5 x^5 - 15\sqrt{dx + c} \sqrt{dx - c} a d^7 x^7}{d^5}$$

input

```
int(x^4*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x)
```


output

```
( - 24*sqrt(c + d*x)*sqrt( - c + d*x)*a*c**4*d**3*x - 16*sqrt(c + d*x)*sqrt( - c + d*x)*a*c**2*d**5*x**3 + 64*sqrt(c + d*x)*sqrt( - c + d*x)*a*d**7*x**5 - 15*sqrt(c + d*x)*sqrt( - c + d*x)*b*c**6*d*x - 10*sqrt(c + d*x)*sqrt( - c + d*x)*b*c**4*d**3*x**3 - 8*sqrt(c + d*x)*sqrt( - c + d*x)*b*c**2*d**5*x**5 + 48*sqrt(c + d*x)*sqrt( - c + d*x)*b*d**7*x**7 - 48*log((sqrt( - c + d*x) + sqrt(c + d*x))/(sqrt(c)*sqrt(2)))*a*c**6*d**2 - 30*log((sqrt( - c + d*x) + sqrt(c + d*x))/(sqrt(c)*sqrt(2)))*b*c**8)/(384*d**7)
```

3.8 $\int x^2 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx$

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Optimal result

Integrand size = 31, antiderivative size = 158

$$\int x^2 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx = -\frac{c^2(bc^2 + 2ad^2) x \sqrt{-c + dx} \sqrt{c + dx}}{16d^4} + \frac{1}{8} \left(2a + \frac{bc^2}{d^2}\right) x^3 \sqrt{-c + dx} \sqrt{c + dx} + \frac{bx^3(-c + dx)^{3/2}(c + dx)^{3/2}}{6d^2} - \frac{c^4(bc^2 + 2ad^2) \operatorname{arctanh}\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)}{8d^5}$$

output

```
-1/16*c^2*(2*a*d^2+b*c^2)*x*(d*x-c)^(1/2)*(d*x+c)^(1/2)/d^4+1/8*(2*a+b*c^2/d^2)*x^3*(d*x-c)^(1/2)*(d*x+c)^(1/2)+1/6*b*x^3*(d*x-c)^(3/2)*(d*x+c)^(3/2)/d^2-1/8*c^4*(2*a*d^2+b*c^2)*arctanh((d*x-c)^(1/2)/(d*x+c)^(1/2))/d^5
```

Mathematica [A] (verified)

Time = 0.25 (sec) , antiderivative size = 116, normalized size of antiderivative = 0.73

$$\int x^2 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx$$

$$= \frac{dx \sqrt{-c + dx} \sqrt{c + dx} (-6ad^2(c^2 - 2d^2x^2) + b(-3c^4 - 2c^2d^2x^2 + 8d^4x^4)) - 6c^4(bc^2 + 2ad^2) \operatorname{arctanh}\left(\frac{\sqrt{-c + dx}}{\sqrt{c + dx}}\right)}{48d^5}$$

input

```
Integrate[x^2*Sqrt[-c + d*x]*Sqrt[c + d*x]*(a + b*x^2),x]
```

output

```
(d*x*Sqrt[-c + d*x]*Sqrt[c + d*x]*(-6*a*d^2*(c^2 - 2*d^2*x^2) + b*(-3*c^4 - 2*c^2*d^2*x^2 + 8*d^4*x^4)) - 6*c^4*(b*c^2 + 2*a*d^2)*ArcTanh[Sqrt[-c + d*x]/Sqrt[c + d*x]]/(48*d^5)
```

Rubi [A] (verified)Time = 0.44 (sec) , antiderivative size = 143, normalized size of antiderivative = 0.91, number of steps used = 7, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}} = 0.194$, Rules used = {960, 101, 27, 40, 45, 221}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x^2 (a + bx^2) \sqrt{dx - c} \sqrt{c + dx} dx$$

$$\downarrow 960$$

$$\frac{1}{2} \left(2a + \frac{bc^2}{d^2} \right) \int x^2 \sqrt{dx - c} \sqrt{c + dx} dx + \frac{bx^3 (dx - c)^{3/2} (c + dx)^{3/2}}{6d^2}$$

$$\downarrow 101$$

$$\frac{1}{2} \left(2a + \frac{bc^2}{d^2} \right) \left(\frac{\int c^2 \sqrt{dx - c} \sqrt{c + dx} dx}{4d^2} + \frac{x(dx - c)^{3/2} (c + dx)^{3/2}}{4d^2} \right) + \frac{bx^3 (dx - c)^{3/2} (c + dx)^{3/2}}{6d^2}$$

$$\downarrow 27$$

$$\frac{1}{2} \left(2a + \frac{bc^2}{d^2} \right) \left(\frac{c^2 \int \sqrt{dx-c} \sqrt{c+dx} dx}{4d^2} + \frac{x(dx-c)^{3/2}(c+dx)^{3/2}}{4d^2} \right) + \frac{bx^3(dx-c)^{3/2}(c+dx)^{3/2}}{6d^2}$$

↓ 40

$$\frac{1}{2} \left(2a + \frac{bc^2}{d^2} \right) \left(\frac{c^2 \left(\frac{1}{2} x \sqrt{dx-c} \sqrt{c+dx} - \frac{1}{2} c^2 \int \frac{1}{\sqrt{dx-c} \sqrt{c+dx}} dx \right)}{4d^2} + \frac{x(dx-c)^{3/2}(c+dx)^{3/2}}{4d^2} \right) + \frac{bx^3(dx-c)^{3/2}(c+dx)^{3/2}}{6d^2}$$

↓ 45

$$\frac{1}{2} \left(2a + \frac{bc^2}{d^2} \right) \left(\frac{c^2 \left(\frac{1}{2} x \sqrt{dx-c} \sqrt{c+dx} - c^2 \int \frac{1}{d - \frac{d(dx-c)}{c+dx}} d \frac{\sqrt{dx-c}}{\sqrt{c+dx}} \right)}{4d^2} + \frac{x(dx-c)^{3/2}(c+dx)^{3/2}}{4d^2} \right) + \frac{bx^3(dx-c)^{3/2}(c+dx)^{3/2}}{6d^2}$$

↓ 221

$$\frac{1}{2} \left(2a + \frac{bc^2}{d^2} \right) \left(\frac{c^2 \left(\frac{1}{2} x \sqrt{dx-c} \sqrt{c+dx} - \frac{c^2 \operatorname{arctanh} \left(\frac{\sqrt{dx-c}}{\sqrt{c+dx}} \right)}{d} \right)}{4d^2} + \frac{x(dx-c)^{3/2}(c+dx)^{3/2}}{4d^2} \right) + \frac{bx^3(dx-c)^{3/2}(c+dx)^{3/2}}{6d^2}$$

input `Int[x^2*sqrt[-c + d*x]*sqrt[c + d*x]*(a + b*x^2),x]`

output `(b*x^3*(-c + d*x)^(3/2)*(c + d*x)^(3/2))/(6*d^2) + ((2*a + (b*c^2)/d^2)*((x*(-c + d*x)^(3/2)*(c + d*x)^(3/2))/(4*d^2) + (c^2*((x*sqrt[-c + d*x]*sqrt[c + d*x])/2 - (c^2*ArcTanh[Sqrt[-c + d*x]/sqrt[c + d*x]])/d))/(4*d^2)))/2`

Definitions of rubi rules used

rule 27 $\text{Int}[(a_*)(Fx_), x_Symbol] \rightarrow \text{Simp}[a \text{ Int}[Fx, x], x] /; \text{FreeQ}[a, x] \ \&\& \ !\text{MatchQ}[Fx, (b_*)(Gx_)] /; \text{FreeQ}[b, x]$

rule 40 $\text{Int}[((a_*) + (b_*)(x_))^{(m_*)}((c_*) + (d_*)(x_))^{(m_*)}, x_Symbol] \rightarrow \text{Simp}[x*(a + b*x)^m*((c + d*x)^m/(2*m + 1)), x] + \text{Simp}[2*a*c*(m/(2*m + 1)) \text{ Int}[(a + b*x)^{m-1}*(c + d*x)^{m-1}, x], x] /; \text{FreeQ}[\{a, b, c, d\}, x] \ \&\& \ \text{EqQ}[b*c + a*d, 0] \ \&\& \ \text{IGtQ}[m + 1/2, 0]$

rule 45 $\text{Int}[1/(\text{Sqrt}[(a_*) + (b_*)(x_)]*\text{Sqrt}[(c_*) + (d_*)(x_)]), x_Symbol] \rightarrow \text{Simp}[2 \text{ Subst}[\text{Int}[1/(b - d*x^2), x], x, \text{Sqrt}[a + b*x]/\text{Sqrt}[c + d*x]], x] /; \text{FreeQ}[\{a, b, c, d\}, x] \ \&\& \ \text{EqQ}[b*c + a*d, 0] \ \&\& \ !\text{GtQ}[c, 0]$

rule 101 $\text{Int}[((a_*) + (b_*)(x_))^{2*}((c_*) + (d_*)(x_))^{(n_*)}((e_*) + (f_*)(x_))^{(p_*)}, x] \rightarrow \text{Simp}[b*(a + b*x)*(c + d*x)^{(n+1)}*((e + f*x)^{(p+1)}/(d*f*(n + p + 3))), x] + \text{Simp}[1/(d*f*(n + p + 3)) \text{ Int}[(c + d*x)^n*(e + f*x)^p*\text{Simp}[a^2*d*f*(n + p + 3) - b*(b*c*e + a*(d*e*(n + 1) + c*f*(p + 1))) + b*(a*d*f*(n + p + 4) - b*(d*e*(n + 2) + c*f*(p + 2))]*x, x], x] /; \text{FreeQ}[\{a, b, c, d, e, f, n, p\}, x] \ \&\& \ \text{NeQ}[n + p + 3, 0]$

rule 221 $\text{Int}[((a_*) + (b_*)(x_)^2)^{(-1)}, x_Symbol] \rightarrow \text{Simp}[(\text{Rt}[-a/b, 2]/a)*\text{ArcTanh}[x/\text{Rt}[-a/b, 2]], x] /; \text{FreeQ}[\{a, b\}, x] \ \&\& \ \text{NegQ}[a/b]$

rule 960 $\text{Int}[((e_*)(x_))^{(m_*)}((a1_*) + (b1_*)(x_)^{(non2_*)})^{(p_*)}((a2_*) + (b2_*)(x_)^{(non2_*)})^{(p_*)}((c_*) + (d_*)(x_)^{(n_*)}), x_Symbol] \rightarrow \text{Simp}[d*(e*x)^{(m+1)}*(a1 + b1*x^{(n/2)})^{(p+1)}*((a2 + b2*x^{(n/2)})^{(p+1)}/(b1*b2*e*(m + n*(p + 1) + 1))), x] - \text{Simp}[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(b1*b2*(m + n*(p + 1) + 1)) \text{ Int}[(e*x)^m*(a1 + b1*x^{(n/2)})^p*(a2 + b2*x^{(n/2)})^p, x], x] /; \text{FreeQ}[\{a1, b1, a2, b2, c, d, e, m, n, p\}, x] \ \&\& \ \text{EqQ}[non2, n/2] \ \&\& \ \text{EqQ}[a2*b1 + a1*b2, 0] \ \&\& \ \text{NeQ}[m + n*(p + 1) + 1, 0]$

Maple [A] (verified)

Time = 0.13 (sec) , antiderivative size = 159, normalized size of antiderivative = 1.01

method	result
risch	$\frac{x(-8bx^4d^4 - 12ad^4x^2 + 2b^2d^2x^2 + 6a^2d^2 + 3bc^4)(-dx+c)\sqrt{dx+c}}{48d^4\sqrt{dx-c}} - \frac{c^4(2ad^2 + bc^2) \ln\left(\frac{d^2x}{\sqrt{d^2}} + \sqrt{d^2x^2 - c^2}\right) \sqrt{(dx-c)(dx+c)}}{16d^4\sqrt{d^2} \sqrt{dx-c} \sqrt{dx+c}}$
default	$-\frac{\sqrt{dx-c}\sqrt{dx+c}(-8 \operatorname{csgn}(d)b d^5 x^5 \sqrt{d^2 x^2 - c^2} - 12 \operatorname{csgn}(d)a d^5 x^3 \sqrt{d^2 x^2 - c^2} + 2 \operatorname{csgn}(d)b c^2 d^3 x^3 \sqrt{d^2 x^2 - c^2} + 6 \operatorname{csgn}(d)d^3 \sqrt{d^2 x^2 - c^2})}{48d^4\sqrt{dx-c}}$

input `int(x^2*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x,method=_RETURNVERBOSE)`

output
$$\frac{1}{48}x(-8bd^4x^4 - 12ad^4x^2 + 2b^2c^2d^2x^2 + 6a^2c^2d^2 + 3b^2c^4)(-dx+c)\sqrt{dx+c}/d^4/(d*x-c)^{(1/2)} - 1/16*c^4*(2*a*d^2+b*c^2)/d^4*\ln(d^2*x/(d^2)^{(1/2)}+(d^2*x^2-c^2)^{(1/2)})/(d^2)^{(1/2)}*((d*x-c)*(d*x+c))^{(1/2)}/(d*x-c)^{(1/2)}/(d*x+c)^{(1/2)}$$

Fricas [A] (verification not implemented)

Time = 0.13 (sec) , antiderivative size = 112, normalized size of antiderivative = 0.71

$$\int x^2 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx$$

$$= \frac{(8bd^5x^5 - 2(bc^2d^3 - 6ad^5)x^3 - 3(bc^4d + 2ac^2d^3)x)\sqrt{dx+c}\sqrt{dx-c} + 3(bc^6 + 2ac^4d^2)\log(-dx + \sqrt{c+dx})}{48d^5}$$

input `integrate(x^2*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x, algorithm="fricas")`

output
$$\frac{1}{48}((8bd^5x^5 - 2(b^2c^2d^3 - 6ad^5)x^3 - 3(b^2c^4d + 2ac^2d^3)x)\sqrt{dx+c}\sqrt{dx-c} + 3(b^2c^6 + 2ac^4d^2)\log(-dx + \sqrt{c+dx})) / d^5$$

Sympy [F]

$$\int x^2 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx = \int x^2 (a + bx^2) \sqrt{-c + dx} \sqrt{c + dx} dx$$

input `integrate(x**2*(d*x-c)**(1/2)*(d*x+c)**(1/2)*(b*x**2+a), x)`

output `Integral(x**2*(a + b*x**2)*sqrt(-c + d*x)*sqrt(c + d*x), x)`

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 192, normalized size of antiderivative = 1.22

$$\int x^2 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx = \frac{(d^2 x^2 - c^2)^{\frac{3}{2}} b x^3}{6 d^2} - \frac{b c^6 \log(2 d^2 x + 2 \sqrt{d^2 x^2 - c^2} d)}{16 d^5} - \frac{a c^4 \log(2 d^2 x + 2 \sqrt{d^2 x^2 - c^2} d)}{8 d^3} + \frac{\sqrt{d^2 x^2 - c^2} b c^4 x}{16 d^4} + \frac{\sqrt{d^2 x^2 - c^2} a c^2 x}{8 d^2} + \frac{(d^2 x^2 - c^2)^{\frac{3}{2}} b c^2 x}{8 d^4} + \frac{(d^2 x^2 - c^2)^{\frac{3}{2}} a x}{4 d^2}$$

input `integrate(x^2*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a), x, algorithm="maxima")`

output `1/6*(d^2*x^2 - c^2)^(3/2)*b*x^3/d^2 - 1/16*b*c^6*log(2*d^2*x + 2*sqrt(d^2*x^2 - c^2)*d)/d^5 - 1/8*a*c^4*log(2*d^2*x + 2*sqrt(d^2*x^2 - c^2)*d)/d^3 + 1/16*sqrt(d^2*x^2 - c^2)*b*c^4*x/d^4 + 1/8*sqrt(d^2*x^2 - c^2)*a*c^2*x/d^2 + 1/8*(d^2*x^2 - c^2)^(3/2)*b*c^2*x/d^4 + 1/4*(d^2*x^2 - c^2)^(3/2)*a*x/d^2`

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 403 vs. $2(134) = 268$.

Time = 0.21 (sec) , antiderivative size = 403, normalized size of antiderivative = 2.55

$$\int x^2 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx$$

$$= \frac{10 \left(\left((dx + c) \left(2(dx + c) \left(\frac{3(dx+c)}{d^3} - \frac{13c}{d^3} \right) + \frac{43c^2}{d^3} \right) - \frac{39c^3}{d^3} \right) \sqrt{dx + c} \sqrt{dx - c} - \frac{18c^4 \log(|-\sqrt{dx+c} + \sqrt{dx-c}|)}{d^3} \right) a}{1}$$

input `integrate(x^2*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x, algorithm="giac")`

output `1/240*(10*((d*x + c)*(2*(d*x + c)*(3*(d*x + c)/d^3 - 13*c/d^3) + 43*c^2/d^3) - 39*c^3/d^3)*sqrt(d*x + c)*sqrt(d*x - c) - 18*c^4*log(abs(-sqrt(d*x + c) + sqrt(d*x - c)))/d^3)*a*d + (((2*((d*x + c)*(4*(d*x + c)*(5*(d*x + c)/d^5 - 31*c/d^5) + 321*c^2/d^5) - 451*c^3/d^5)*(d*x + c) + 745*c^4/d^5)*(d*x + c) - 405*c^5/d^5)*sqrt(d*x + c)*sqrt(d*x - c) - 150*c^6*log(abs(-sqrt(d*x + c) + sqrt(d*x - c)))/d^5)*b*d + 40*(6*c^3*log(abs(-sqrt(d*x + c) + sqrt(d*x - c))) + ((2*d*x - 5*c)*(d*x + c) + 9*c^2)*sqrt(d*x + c)*sqrt(d*x - c))*a*c/d^2 + 2*(90*c^5*log(abs(-sqrt(d*x + c) + sqrt(d*x - c))) + (195*c^4 - (295*c^3 - 2*(3*(4*d*x - 17*c)*(d*x + c) + 133*c^2)*(d*x + c))*(d*x + c))*sqrt(d*x + c)*sqrt(d*x - c))*b*c/d^4)/d`

Mupad [B] (verification not implemented)

Time = 47.59 (sec) , antiderivative size = 1681, normalized size of antiderivative = 10.64

$$\int x^2 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx = \text{Too large to display}$$

input `int(x^2*(a + b*x^2)*(c + d*x)^(1/2)*(d*x - c)^(1/2),x)`

output

```

((35*b*c^6*((c + d*x)^(1/2) - c^(1/2))^3)/(12*((-c)^(1/2) - (d*x - c)^(1/2))^3) - (b*c^6*((c + d*x)^(1/2) - c^(1/2)))/(4*((-c)^(1/2) - (d*x - c)^(1/2))) + (757*b*c^6*((c + d*x)^(1/2) - c^(1/2))^5)/(4*((-c)^(1/2) - (d*x - c)^(1/2))^5) + (7339*b*c^6*((c + d*x)^(1/2) - c^(1/2))^7)/(4*((-c)^(1/2) - (d*x - c)^(1/2))^7) + (41929*b*c^6*((c + d*x)^(1/2) - c^(1/2))^9)/(6*((-c)^(1/2) - (d*x - c)^(1/2))^9) + (25661*b*c^6*((c + d*x)^(1/2) - c^(1/2))^11)/(2*((-c)^(1/2) - (d*x - c)^(1/2))^11) + (25661*b*c^6*((c + d*x)^(1/2) - c^(1/2))^13)/(2*((-c)^(1/2) - (d*x - c)^(1/2))^13) + (41929*b*c^6*((c + d*x)^(1/2) - c^(1/2))^15)/(6*((-c)^(1/2) - (d*x - c)^(1/2))^15) + (7339*b*c^6*((c + d*x)^(1/2) - c^(1/2))^17)/(4*((-c)^(1/2) - (d*x - c)^(1/2))^17) + (757*b*c^6*((c + d*x)^(1/2) - c^(1/2))^19)/(4*((-c)^(1/2) - (d*x - c)^(1/2))^19) + (35*b*c^6*((c + d*x)^(1/2) - c^(1/2))^21)/(12*((-c)^(1/2) - (d*x - c)^(1/2))^21) - (b*c^6*((c + d*x)^(1/2) - c^(1/2))^23)/(4*((-c)^(1/2) - (d*x - c)^(1/2))^23))/(d^5 - (12*d^5*((c + d*x)^(1/2) - c^(1/2))^2)/((-c)^(1/2) - (d*x - c)^(1/2))^2 + (66*d^5*((c + d*x)^(1/2) - c^(1/2))^4)/((-c)^(1/2) - (d*x - c)^(1/2))^4 - (220*d^5*((c + d*x)^(1/2) - c^(1/2))^6)/((-c)^(1/2) - (d*x - c)^(1/2))^6 + (495*d^5*((c + d*x)^(1/2) - c^(1/2))^8)/((-c)^(1/2) - (d*x - c)^(1/2))^8 - (792*d^5*((c + d*x)^(1/2) - c^(1/2))^10)/((-c)^(1/2) - (d*x - c)^(1/2))^10 + (924*d^5*((c + d*x)^(1/2) - c^(1/2))^12)/((-c)^(1/2) - (d*x - c)^(1/2))^12 - (792*d^5*((c + d*x)^(1/2) - c^(1/2)...

```

Reduce [B] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 189, normalized size of antiderivative = 1.20

$$\int x^2 \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx$$

$$= \frac{-6\sqrt{dx + c} \sqrt{dx - c} a c^2 d^3 x + 12\sqrt{dx + c} \sqrt{dx - c} a d^5 x^3 - 3\sqrt{dx + c} \sqrt{dx - c} b c^4 dx - 2\sqrt{dx + c} \sqrt{dx - c} b c^4 dx}{48d^5}$$

input

```
int(x^2*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x)
```

output

```

(- 6*sqrt(c + d*x)*sqrt(- c + d*x)*a*c**2*d**3*x + 12*sqrt(c + d*x)*sqrt(- c + d*x)*a*d**5*x**3 - 3*sqrt(c + d*x)*sqrt(- c + d*x)*b*c**4*d*x - 2*sqrt(c + d*x)*sqrt(- c + d*x)*b*c**2*d**3*x**3 + 8*sqrt(c + d*x)*sqrt(- c + d*x)*b*d**5*x**5 - 12*log((sqrt(- c + d*x) + sqrt(c + d*x))/(sqrt(c)*sqrt(2)))*a*c**4*d**2 - 6*log((sqrt(- c + d*x) + sqrt(c + d*x))/(sqrt(c)*sqrt(2)))*b*c**6)/(48*d**5)

```

3.9 $\int \sqrt{-c + dx}\sqrt{c + dx}(a + bx^2) dx$

Optimal result	105
Mathematica [A] (verified)	105
Rubi [A] (verified)	106
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Giac [B] (verification not implemented)	109
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Reduce [B] (verification not implemented)	111

Optimal result

Integrand size = 28, antiderivative size = 114

$$\int \sqrt{-c + dx}\sqrt{c + dx}(a + bx^2) dx = \frac{(bc^2 + 4ad^2) x\sqrt{-c + dx}\sqrt{c + dx}}{8d^2} + \frac{bx(-c + dx)^{3/2}(c + dx)^{3/2}}{4d^2} - \frac{c^2(bc^2 + 4ad^2) \operatorname{arctanh}\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)}{4d^3}$$

output

```
1/8*(4*a*d^2+b*c^2)*x*(d*x-c)^(1/2)*(d*x+c)^(1/2)/d^2+1/4*b*x*(d*x-c)^(3/2)
*(d*x+c)^(3/2)/d^2-1/4*c^2*(4*a*d^2+b*c^2)*arctanh((d*x-c)^(1/2)/(d*x+c)^(1/2))/d^3
```

Mathematica [A] (verified)

Time = 0.18 (sec) , antiderivative size = 92, normalized size of antiderivative = 0.81

$$\int \sqrt{-c + dx}\sqrt{c + dx}(a + bx^2) dx = \frac{dx\sqrt{-c + dx}\sqrt{c + dx}(-bc^2 + 4ad^2 + 2bd^2x^2) - 2c^2(bc^2 + 4ad^2) \operatorname{arctanh}\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)}{8d^3}$$

input `Integrate[Sqrt[-c + d*x]*Sqrt[c + d*x]*(a + b*x^2),x]`

output `(d*x*Sqrt[-c + d*x]*Sqrt[c + d*x]*(-(b*c^2) + 4*a*d^2 + 2*b*d^2*x^2) - 2*c^2*(b*c^2 + 4*a*d^2)*ArcTanh[Sqrt[-c + d*x]/Sqrt[c + d*x]])/(8*d^3)`

Rubi [A] (verified)

Time = 0.33 (sec) , antiderivative size = 105, normalized size of antiderivative = 0.92, number of steps used = 5, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}} = 0.143$, Rules used = {646, 40, 45, 221}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int (a + bx^2) \sqrt{dx - c} \sqrt{c + dx} dx \\
 & \quad \downarrow 646 \\
 & \frac{(4ad^2 + bc^2) \int \sqrt{dx - c} \sqrt{c + dx} dx}{4d^2} + \frac{bx(dx - c)^{3/2}(c + dx)^{3/2}}{4d^2} \\
 & \quad \downarrow 40 \\
 & \frac{(4ad^2 + bc^2) \left(\frac{1}{2}x\sqrt{dx - c}\sqrt{c + dx} - \frac{1}{2}c^2 \int \frac{1}{\sqrt{dx - c}\sqrt{c + dx}} dx \right)}{4d^2} + \frac{bx(dx - c)^{3/2}(c + dx)^{3/2}}{4d^2} \\
 & \quad \downarrow 45 \\
 & \frac{(4ad^2 + bc^2) \left(\frac{1}{2}x\sqrt{dx - c}\sqrt{c + dx} - c^2 \int \frac{1}{d - \frac{d(dx - c)}{c + dx}} d \frac{\sqrt{dx - c}}{\sqrt{c + dx}} \right)}{4d^2} + \frac{bx(dx - c)^{3/2}(c + dx)^{3/2}}{4d^2} \\
 & \quad \downarrow 221 \\
 & \frac{(4ad^2 + bc^2) \left(\frac{1}{2}x\sqrt{dx - c}\sqrt{c + dx} - \frac{c^2 \operatorname{arctanh}\left(\frac{\sqrt{dx - c}}{\sqrt{c + dx}}\right)}{d} \right)}{4d^2} + \frac{bx(dx - c)^{3/2}(c + dx)^{3/2}}{4d^2}
 \end{aligned}$$

input `Int[Sqrt[-c + d*x]*Sqrt[c + d*x]*(a + b*x^2),x]`

output

```
(b*x*(-c + d*x)^(3/2)*(c + d*x)^(3/2))/(4*d^2) + ((b*c^2 + 4*a*d^2)*((x*Sqrt[-c + d*x]*Sqrt[c + d*x])/2 - (c^2*ArcTanh[Sqrt[-c + d*x]/Sqrt[c + d*x]])/d))/(4*d^2)
```

Defintions of rubi rules used

rule 40

```
Int[((a_) + (b_)*(x_))^(m_)*((c_) + (d_)*(x_))^(m_), x_Symbol] := Simp[x*(a + b*x)^m*((c + d*x)^m/(2*m + 1)), x] + Simp[2*a*c*(m/(2*m + 1)) Int[(a + b*x)^(m - 1)*(c + d*x)^(m - 1), x], x] /; FreeQ[{a, b, c, d}, x] && EqQ[b*c + a*d, 0] && IGtQ[m + 1/2, 0]
```

rule 45

```
Int[1/(Sqrt[(a_) + (b_)*(x_)]*Sqrt[(c_) + (d_)*(x_)]), x_Symbol] := Simp[2 Subst[Int[1/(b - d*x^2), x], x, Sqrt[a + b*x]/Sqrt[c + d*x]], x] /; FreeQ[{a, b, c, d}, x] && EqQ[b*c + a*d, 0] && !GtQ[c, 0]
```

rule 221

```
Int[((a_) + (b_)*(x_)^2)^(-1), x_Symbol] := Simp[(Rt[-a/b, 2]/a)*ArcTanh[x/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]
```

rule 646

```
Int[((c_) + (d_)*(x_))^(m_)*((e_) + (f_)*(x_))^(n_)*((a_) + (b_)*(x_)^2), x_Symbol] := Simp[b*x*(c + d*x)^(m + 1)*((e + f*x)^(n + 1)/(d*f*(2*m + 3))), x] - Simp[(b*c*e - a*d*f*(2*m + 3))/(d*f*(2*m + 3)) Int[(c + d*x)^m*(e + f*x)^n, x], x] /; FreeQ[{a, b, c, d, e, f, m, n}, x] && EqQ[m, n] && EqQ[d*e + c*f, 0] && !LtQ[m, -1]
```

Maple [A] (verified)

Time = 0.13 (sec) , antiderivative size = 135, normalized size of antiderivative = 1.18

method	result
risch	$-\frac{x(2bx^2d^2+4ad^2-bc^2)(-dx+c)\sqrt{dx+c}}{8d^2\sqrt{dx-c}} - \frac{c^2(4ad^2+bc^2)\ln\left(\frac{d^2x}{\sqrt{d^2}+\sqrt{d^2x^2-c^2}}\right)\sqrt{(dx-c)(dx+c)}}{8d^2\sqrt{d^2}\sqrt{dx-c}\sqrt{dx+c}}$
default	$\frac{\sqrt{dx-c}\sqrt{dx+c}\left(2\operatorname{csgn}(d)bd^3x^3\sqrt{d^2x^2-c^2}+4\operatorname{csgn}(d)d^3\sqrt{d^2x^2-c^2}ax-\operatorname{csgn}(d)d\sqrt{d^2x^2-c^2}bc^2x-4\ln\left(\left(\sqrt{d^2x^2-c^2}\operatorname{csgn}(d)+dx\right)\right)\right)}{8\sqrt{d^2x^2-c^2}d^3}$

input `int((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x,method=_RETURNVERBOSE)`

output
$$-1/8*x*(2*b*d^2*x^2+4*a*d^2-b*c^2)*(-d*x+c)*(d*x+c)^(1/2)/d^2/(d*x-c)^(1/2)$$

$$-1/8*c^2*(4*a*d^2+b*c^2)/d^2*\ln(d^2*x/(d^2)^(1/2)+(d^2*x^2-c^2)^(1/2))/(d^2)^(1/2)*((d*x-c)*(d*x+c))^(1/2)/(d*x-c)^(1/2)/(d*x+c)^(1/2)$$

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 88, normalized size of antiderivative = 0.77

$$\int \sqrt{-c+dx}\sqrt{c+dx}(a+bx^2) dx$$

$$= \frac{(2bd^3x^3 - (bc^2d - 4ad^3)x)\sqrt{dx+c}\sqrt{dx-c} + (bc^4 + 4ac^2d^2)\log(-dx + \sqrt{dx+c}\sqrt{dx-c})}{8d^3}$$

input `integrate((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x, algorithm="fricas")`

output
$$1/8*((2*b*d^3*x^3 - (b*c^2*d - 4*a*d^3)*x)*\sqrt{d*x + c}*\sqrt{d*x - c} + (b*c^4 + 4*a*c^2*d^2)*\log(-d*x + \sqrt{d*x + c}*\sqrt{d*x - c}))/d^3$$

Sympy [F]

$$\int \sqrt{-c+dx}\sqrt{c+dx}(a+bx^2) dx = \int (a+bx^2)\sqrt{-c+dx}\sqrt{c+dx} dx$$

input `integrate((d*x-c)**(1/2)*(d*x+c)**(1/2)*(b*x**2+a),x)`

output `Integral((a + b*x**2)*sqrt(-c + d*x)*sqrt(c + d*x), x)`

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 137, normalized size of antiderivative = 1.20

$$\int \sqrt{-c+dx}\sqrt{c+dx}(a+bx^2) dx = -\frac{bc^4 \log(2d^2x + 2\sqrt{d^2x^2 - c^2}d)}{8d^3} - \frac{ac^2 \log(2d^2x + 2\sqrt{d^2x^2 - c^2}d)}{2d} + \frac{1}{2}\sqrt{d^2x^2 - c^2}ax + \frac{\sqrt{d^2x^2 - c^2}bc^2x}{8d^2} + \frac{(d^2x^2 - c^2)^{\frac{3}{2}}bx}{4d^2}$$

input `integrate((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x, algorithm="maxima")`

output `-1/8*b*c^4*log(2*d^2*x + 2*sqrt(d^2*x^2 - c^2)*d)/d^3 - 1/2*a*c^2*log(2*d^2*x + 2*sqrt(d^2*x^2 - c^2)*d)/d + 1/2*sqrt(d^2*x^2 - c^2)*a*x + 1/8*sqrt(d^2*x^2 - c^2)*b*c^2*x/d^2 + 1/4*(d^2*x^2 - c^2)^(3/2)*b*x/d^2`

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 276 vs. 2(96) = 192.

Time = 0.18 (sec) , antiderivative size = 276, normalized size of antiderivative = 2.42

$$\int \sqrt{-c+dx}\sqrt{c+dx}(a+bx^2) dx = \frac{24(2c \log(|-\sqrt{dx+c} + \sqrt{dx-c}|) + \sqrt{dx+c}\sqrt{dx-c})ac + \left((dx+c) \left(2(dx+c) \left(\frac{3(dx+c)}{d^3} - \frac{13c}{d^3} \right) \right) \right)}{}$$

input `integrate((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x, algorithm="giac")`

output

```

1/24*(24*(2*c*log(abs(-sqrt(d*x + c) + sqrt(d*x - c))) + sqrt(d*x + c)*sqrt
t(d*x - c))*a*c + (((d*x + c)*(2*(d*x + c)*(3*(d*x + c)/d^3 - 13*c/d^3) +
43*c^2/d^3) - 39*c^3/d^3)*sqrt(d*x + c)*sqrt(d*x - c) - 18*c^4*log(abs(-sq
rt(d*x + c) + sqrt(d*x - c)))/d^3)*b*d - 12*(2*c^2*log(abs(-sqrt(d*x + c)
+ sqrt(d*x - c))) - sqrt(d*x + c)*sqrt(d*x - c)*(d*x - 2*c))*a + 4*(6*c^3*
log(abs(-sqrt(d*x + c) + sqrt(d*x - c))) + ((2*d*x - 5*c)*(d*x + c) + 9*c^
2)*sqrt(d*x + c)*sqrt(d*x - c))*b*c/d^2)/d

```

Mupad [B] (verification not implemented)

Time = 19.52 (sec) , antiderivative size = 734, normalized size of antiderivative = 6.44

$$\int \sqrt{-c+dx}\sqrt{c+dx}(a+bx^2) dx = \frac{ax\sqrt{c+dx}\sqrt{dx-c}}{2}$$

$$- \frac{bc^4(\sqrt{c+dx}-\sqrt{c})}{2(\sqrt{-c}-\sqrt{dx-c})} + \frac{35bc^4(\sqrt{c+dx}-\sqrt{c})^3}{2(\sqrt{-c}-\sqrt{dx-c})^3} + \frac{273bc^4(\sqrt{c+dx}-\sqrt{c})^5}{2(\sqrt{-c}-\sqrt{dx-c})^5} + \frac{715bc^4(\sqrt{c+dx}-\sqrt{c})^7}{2(\sqrt{-c}-\sqrt{dx-c})^7} + \frac{715bc^4(\sqrt{c+dx}-\sqrt{c})^9}{2(\sqrt{-c}-\sqrt{dx-c})^9} + \frac{273bc^4(\sqrt{c+dx}-\sqrt{c})^{11}}{2(\sqrt{-c}-\sqrt{dx-c})^{11}}$$

$$- \frac{d^3 - \frac{8d^3(\sqrt{c+dx}-\sqrt{c})^2}{(\sqrt{-c}-\sqrt{dx-c})^2} + \frac{28d^3(\sqrt{c+dx}-\sqrt{c})^4}{(\sqrt{-c}-\sqrt{dx-c})^4} - \frac{56d^3(\sqrt{c+dx}-\sqrt{c})^6}{(\sqrt{-c}-\sqrt{dx-c})^6} + \frac{70d^3(\sqrt{c+dx}-\sqrt{c})^8}{(\sqrt{-c}-\sqrt{dx-c})^8} - \frac{56d^3(\sqrt{c+dx}-\sqrt{c})^{10}}{(\sqrt{-c}-\sqrt{dx-c})^{10}}}{d^3}$$

$$- \frac{ac^2 \ln(dx + \sqrt{c+dx}\sqrt{dx-c})}{2d} + \frac{bc^4 \operatorname{atanh}\left(\frac{\sqrt{c+dx}-\sqrt{c}}{\sqrt{-c}-\sqrt{dx-c}}\right)}{2d^3}$$

input

```

int((a + b*x^2)*(c + d*x)^(1/2)*(d*x - c)^(1/2),x)

```

output

```
(a*x*(c + d*x)^(1/2)*(d*x - c)^(1/2))/2 - ((b*c^4*((c + d*x)^(1/2) - c^(1/2)))/(2*((-c)^(1/2) - (d*x - c)^(1/2))) + (35*b*c^4*((c + d*x)^(1/2) - c^(1/2))^3)/(2*((-c)^(1/2) - (d*x - c)^(1/2))^3) + (273*b*c^4*((c + d*x)^(1/2) - c^(1/2))^5)/(2*((-c)^(1/2) - (d*x - c)^(1/2))^5) + (715*b*c^4*((c + d*x)^(1/2) - c^(1/2))^7)/(2*((-c)^(1/2) - (d*x - c)^(1/2))^7) + (715*b*c^4*((c + d*x)^(1/2) - c^(1/2))^9)/(2*((-c)^(1/2) - (d*x - c)^(1/2))^9) + (273*b*c^4*((c + d*x)^(1/2) - c^(1/2))^11)/(2*((-c)^(1/2) - (d*x - c)^(1/2))^11) + (35*b*c^4*((c + d*x)^(1/2) - c^(1/2))^13)/(2*((-c)^(1/2) - (d*x - c)^(1/2))^13) + (b*c^4*((c + d*x)^(1/2) - c^(1/2))^15)/(2*((-c)^(1/2) - (d*x - c)^(1/2))^15))/(d^3 - (8*d^3*((c + d*x)^(1/2) - c^(1/2))^2)/((-c)^(1/2) - (d*x - c)^(1/2))^2 + (28*d^3*((c + d*x)^(1/2) - c^(1/2))^4)/((-c)^(1/2) - (d*x - c)^(1/2))^4 - (56*d^3*((c + d*x)^(1/2) - c^(1/2))^6)/((-c)^(1/2) - (d*x - c)^(1/2))^6 + (70*d^3*((c + d*x)^(1/2) - c^(1/2))^8)/((-c)^(1/2) - (d*x - c)^(1/2))^8 - (56*d^3*((c + d*x)^(1/2) - c^(1/2))^10)/((-c)^(1/2) - (d*x - c)^(1/2))^10 + (28*d^3*((c + d*x)^(1/2) - c^(1/2))^12)/((-c)^(1/2) - (d*x - c)^(1/2))^12 - (8*d^3*((c + d*x)^(1/2) - c^(1/2))^14)/((-c)^(1/2) - (d*x - c)^(1/2))^14 + (d^3*((c + d*x)^(1/2) - c^(1/2))^16)/((-c)^(1/2) - (d*x - c)^(1/2))^16) - (a*c^2*log(d*x + (c + d*x)^(1/2)*(d*x - c)^(1/2)))/(2*d) + (b*c^4*atanh(((c + d*x)^(1/2) - c^(1/2))/((-c)^(1/2) - (d*x - c)^(1/2))))/(2*d^3)
```

Reduce [B] (verification not implemented)

Time = 0.21 (sec) , antiderivative size = 137, normalized size of antiderivative = 1.20

$$\int \sqrt{-c + dx} \sqrt{c + dx} (a + bx^2) dx$$

$$= \frac{4\sqrt{dx + c} \sqrt{dx - c} a d^3 x - \sqrt{dx + c} \sqrt{dx - c} b c^2 dx + 2\sqrt{dx + c} \sqrt{dx - c} b d^3 x^3 - 8 \log\left(\frac{\sqrt{dx - c} + \sqrt{dx + c}}{\sqrt{c} \sqrt{2}}\right) a}{8d^3}$$

input

```
int((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a),x)
```

output

```
(4*sqrt(c + d*x)*sqrt(-c + d*x)*a*d**3*x - sqrt(c + d*x)*sqrt(-c + d*x)*b*c**2*d*x + 2*sqrt(c + d*x)*sqrt(-c + d*x)*b*d**3*x**3 - 8*log((sqrt(-c + d*x) + sqrt(c + d*x))/(sqrt(c)*sqrt(2)))*a*c**2*d**2 - 2*log((sqrt(-c + d*x) + sqrt(c + d*x))/(sqrt(c)*sqrt(2)))*b*c**4)/(8*d**3)
```


3.10 $\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^2} dx$

Optimal result	112
Mathematica [A] (verified)	112
Rubi [A] (verified)	113
Maple [A] (verified)	115
Fricas [A] (verification not implemented)	115
Sympy [F]	116
Maxima [A] (verification not implemented)	116
Giac [A] (verification not implemented)	116
Mupad [B] (verification not implemented)	117
Reduce [B] (verification not implemented)	118

Optimal result

Integrand size = 31, antiderivative size = 104

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^2} dx = \frac{1}{2} \left(b - \frac{2ad^2}{c^2} \right) x\sqrt{-c+dx}\sqrt{c+dx} + \frac{a(-c+dx)^{3/2}(c+dx)^{3/2}}{c^2x} - \frac{(bc^2 - 2ad^2) \operatorname{arctanh}\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)}{d}$$

output

```
1/2*(b-2*a*d^2/c^2)*x*(d*x-c)^(1/2)*(d*x+c)^(1/2)+a*(d*x-c)^(3/2)*(d*x+c)^(3/2)/c^2/x-(-2*a*d^2+b*c^2)*arctanh((d*x-c)^(1/2)/(d*x+c)^(1/2))/d
```

Mathematica [A] (verified)

Time = 0.20 (sec) , antiderivative size = 74, normalized size of antiderivative = 0.71

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^2} dx = \frac{\sqrt{-c+dx}\sqrt{c+dx}(-2a+bx^2)}{2x} + \left(-\frac{bc^2}{d} + 2ad \right) \operatorname{arctanh}\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)$$

input `Integrate[(Sqrt[-c + d*x]*Sqrt[c + d*x]*(a + b*x^2))/x^2,x]`

output `(Sqrt[-c + d*x]*Sqrt[c + d*x]*(-2*a + b*x^2))/(2*x) + (-((b*c^2)/d) + 2*a*d)*ArcTanh[Sqrt[-c + d*x]/Sqrt[c + d*x]]`

Rubi [A] (verified)

Time = 0.39 (sec) , antiderivative size = 97, normalized size of antiderivative = 0.93, number of steps used = 5, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}} = 0.129$, Rules used = {956, 40, 45, 221}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(a + bx^2) \sqrt{dx - c} \sqrt{c + dx}}{x^2} dx$$

$$\downarrow 956$$

$$\left(b - \frac{2ad^2}{c^2}\right) \int \sqrt{dx - c} \sqrt{c + dx} dx + \frac{a(dx - c)^{3/2}(c + dx)^{3/2}}{c^2x}$$

$$\downarrow 40$$

$$\left(b - \frac{2ad^2}{c^2}\right) \left(\frac{1}{2}x\sqrt{dx - c}\sqrt{c + dx} - \frac{1}{2}c^2 \int \frac{1}{\sqrt{dx - c}\sqrt{c + dx}} dx\right) + \frac{a(dx - c)^{3/2}(c + dx)^{3/2}}{c^2x}$$

$$\downarrow 45$$

$$\left(b - \frac{2ad^2}{c^2}\right) \left(\frac{1}{2}x\sqrt{dx - c}\sqrt{c + dx} - c^2 \int \frac{1}{d - \frac{d(dx - c)}{c + dx}} d \frac{\sqrt{dx - c}}{\sqrt{c + dx}}\right) + \frac{a(dx - c)^{3/2}(c + dx)^{3/2}}{c^2x}$$

$$\downarrow 221$$

$$\left(b - \frac{2ad^2}{c^2}\right) \left(\frac{1}{2}x\sqrt{dx - c}\sqrt{c + dx} - \frac{c^2 \operatorname{arctanh}\left(\frac{\sqrt{dx - c}}{\sqrt{c + dx}}\right)}{d}\right) + \frac{a(dx - c)^{3/2}(c + dx)^{3/2}}{c^2x}$$

input `Int[(Sqrt[-c + d*x]*Sqrt[c + d*x]*(a + b*x^2))/x^2,x]`

output

```
(a*(-c + d*x)^(3/2)*(c + d*x)^(3/2))/(c^2*x) + (b - (2*a*d^2)/c^2)*((x*Sqr
t[-c + d*x]*Sqrt[c + d*x])/2 - (c^2*ArcTanh[Sqrt[-c + d*x]/Sqrt[c + d*x]]
/d)
```

Defintions of rubi rules used

rule 40

```
Int[((a_) + (b_)*(x_))^(m_)*((c_) + (d_)*(x_))^(m_), x_Symbol] := Simp[x*
(a + b*x)^m*((c + d*x)^m/(2*m + 1)), x] + Simp[2*a*c*(m/(2*m + 1)) Int[(a
+ b*x)^(m - 1)*(c + d*x)^(m - 1), x], x] /; FreeQ[{a, b, c, d}, x] && EqQ[
b*c + a*d, 0] && IGtQ[m + 1/2, 0]
```

rule 45

```
Int[1/(Sqrt[(a_) + (b_)*(x_)]*Sqrt[(c_) + (d_)*(x_)]), x_Symbol] := Simp[
2 Subst[Int[1/(b - d*x^2), x], x, Sqrt[a + b*x]/Sqrt[c + d*x]], x] /; Fre
eQ[{a, b, c, d}, x] && EqQ[b*c + a*d, 0] && !GtQ[c, 0]
```

rule 221

```
Int[((a_) + (b_)*(x_)^2)^(-1), x_Symbol] := Simp[(Rt[-a/b, 2]/a)*ArcTanh[x
/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]
```

rule 956

```
Int[((e_)*(x_))^(m_)*((a1_) + (b1_)*(x_)^(non2_))^(p_)*((a2_) + (b2_)
*(x_)^(non2_))^(p_)*((c_) + (d_)*(x_)^(n_)), x_Symbol] := Simp[c*(e*x)^(
m + 1)*(a1 + b1*x^(n/2))^(p + 1)*((a2 + b2*x^(n/2))^(p + 1)/(a1*a2*e*(m + 1
))), x] + Simp[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(a1*a2*e^n*(
m + 1)) Int[(e*x)^(m + n)*(a1 + b1*x^(n/2))^p*(a2 + b2*x^(n/2))^p, x], x]
/; FreeQ[{a1, b1, a2, b2, c, d, e, p}, x] && EqQ[non2, n/2] && EqQ[a2*b1 +
a1*b2, 0] && (IntegerQ[n] || GtQ[e, 0]) && ((GtQ[n, 0] && LtQ[m, -1]) || (
LtQ[n, 0] && GtQ[m + n, -1])) && !ILtQ[p, -1]
```

Maple [A] (verified)

Time = 0.13 (sec) , antiderivative size = 117, normalized size of antiderivative = 1.12

method	result
risch	$\frac{\sqrt{dx+c}(-dx+c)(-bx^2+2a)}{2x\sqrt{dx-c}} - \frac{\left(-ad^2+\frac{bc^2}{2}\right)\ln\left(\frac{d^2x}{\sqrt{d^2}}+\sqrt{d^2x^2-c^2}\right)\sqrt{(dx-c)(dx+c)}}{\sqrt{d^2}\sqrt{dx-c}\sqrt{dx+c}}$
default	$-\frac{\sqrt{dx-c}\sqrt{dx+c}\left(-\operatorname{csgn}(d)bdx^2\sqrt{d^2x^2-c^2}-2\ln\left(\left(\sqrt{d^2x^2-c^2}\operatorname{csgn}(d)+dx\right)\operatorname{csgn}(d)\right)\right)a d^2x+\ln\left(\left(\sqrt{d^2x^2-c^2}\operatorname{csgn}(d)+dx\right)\operatorname{csgn}(d)\right)}{2\sqrt{d^2x^2-c^2}xd}$

input `int((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a)/x^2,x,method=_RETURNVERBOSE)`

output
$$\frac{1}{2}(d*x+c)^{(1/2)}*(-d*x+c)*(-b*x^2+2*a)/x/(d*x-c)^{(1/2)}-(-a*d^2+1/2*b*c^2)*\ln(d^2*x/(d^2)^{(1/2)}+(d^2*x^2-c^2)^{(1/2)})/(d^2)^{(1/2)}*((d*x-c)*(d*x+c))^{(1/2)}/(d*x-c)^{(1/2)}/(d*x+c)^{(1/2)}$$

Fricas [A] (verification not implemented)

Time = 0.11 (sec) , antiderivative size = 83, normalized size of antiderivative = 0.80

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^2} dx = \frac{2ad^2x - (bc^2 - 2ad^2)x \log(-dx + \sqrt{dx+c}\sqrt{dx-c}) - (bdx^2 - 2ad)\sqrt{dx+c}\sqrt{dx-c}}{2dx}$$

input `integrate((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a)/x^2,x, algorithm="fricas")`

output
$$-1/2*(2*a*d^2*x - (b*c^2 - 2*a*d^2)*x*\log(-d*x + \operatorname{sqrt}(d*x + c)*\operatorname{sqrt}(d*x - c)) - (b*d*x^2 - 2*a*d)*\operatorname{sqrt}(d*x + c)*\operatorname{sqrt}(d*x - c))/(d*x)$$

Sympy [F]

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^2} dx = \int \frac{(a+bx^2)\sqrt{-c+dx}\sqrt{c+dx}}{x^2} dx$$

input `integrate((d*x-c)**(1/2)*(d*x+c)**(1/2)*(b*x**2+a)/x**2,x)`

output `Integral((a + b*x**2)*sqrt(-c + d*x)*sqrt(c + d*x)/x**2, x)`

Maxima [A] (verification not implemented)

Time = 0.11 (sec) , antiderivative size = 105, normalized size of antiderivative = 1.01

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^2} dx = -\frac{bc^2 \log(2d^2x + 2\sqrt{d^2x^2 - c^2}d)}{2d} + ad \log(2d^2x + 2\sqrt{d^2x^2 - c^2}d) + \frac{1}{2}\sqrt{d^2x^2 - c^2}bx - \frac{\sqrt{d^2x^2 - c^2}a}{x}$$

input `integrate((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a)/x^2,x, algorithm="maxima")`

output `-1/2*b*c^2*log(2*d^2*x + 2*sqrt(d^2*x^2 - c^2)*d)/d + a*d*log(2*d^2*x + 2*sqrt(d^2*x^2 - c^2)*d) + 1/2*sqrt(d^2*x^2 - c^2)*b*x - sqrt(d^2*x^2 - c^2)*a/x`

Giac [A] (verification not implemented)

Time = 0.17 (sec) , antiderivative size = 114, normalized size of antiderivative = 1.10

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^2} dx = -\frac{1}{4} \left(\frac{32ac^2}{(\sqrt{dx+c} - \sqrt{dx-c})^4 + 4c^2} - 2\sqrt{dx+c}\sqrt{dx-c} \left(\frac{(dx+c)b}{d^2} - \frac{bc}{d^2} \right) - \frac{(bc^2 - 2ad^2) \log\left(\frac{\sqrt{dx+c}}{\sqrt{dx-c}}\right)}{d^2} \right)$$

input `integrate((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a)/x^2,x, algorithm="giac")`

output `-1/4*(32*a*c^2/((sqrt(d*x + c) - sqrt(d*x - c))^4 + 4*c^2) - 2*sqrt(d*x + c)*sqrt(d*x - c)*((d*x + c)*b/d^2 - b*c/d^2) - (b*c^2 - 2*a*d^2)*log((sqrt(d*x + c) - sqrt(d*x - c))^4/d^2)*d`

Mupad [B] (verification not implemented)

Time = 5.43 (sec) , antiderivative size = 243, normalized size of antiderivative = 2.34

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^2} dx = \frac{ad + \frac{5ad(\sqrt{c+dx}-\sqrt{c})^2}{(\sqrt{-c}-\sqrt{dx-c})^2}}{\frac{4(\sqrt{c+dx}-\sqrt{c})}{\sqrt{-c}-\sqrt{dx-c}} + \frac{4(\sqrt{c+dx}-\sqrt{c})^3}{(\sqrt{-c}-\sqrt{dx-c})^3}} - 4ad \operatorname{atanh}\left(\frac{\sqrt{c+dx}-\sqrt{c}}{\sqrt{-c}-\sqrt{dx-c}}\right) + \frac{bx\sqrt{c+dx}\sqrt{dx-c}}{2} - \frac{bc^2 \ln(dx + \sqrt{c+dx}\sqrt{dx-c})}{2d} + \frac{ad(\sqrt{c+dx}-\sqrt{c})}{4(\sqrt{-c}-\sqrt{dx-c})}$$

input `int(((a + b*x^2)*(c + d*x)^(1/2)*(d*x - c)^(1/2))/x^2,x)`

output `(a*d + (5*a*d*((c + d*x)^(1/2) - c^(1/2))^2)/((-c)^(1/2) - (d*x - c)^(1/2))^2)/((4*((c + d*x)^(1/2) - c^(1/2)))/((-c)^(1/2) - (d*x - c)^(1/2)) + (4*((c + d*x)^(1/2) - c^(1/2))^3)/((-c)^(1/2) - (d*x - c)^(1/2))^3) - 4*a*d*a*tanh(((c + d*x)^(1/2) - c^(1/2))/((-c)^(1/2) - (d*x - c)^(1/2))) + (b*x*(c + d*x)^(1/2)*(d*x - c)^(1/2))/2 - (b*c^2*log(d*x + (c + d*x)^(1/2)*(d*x - c)^(1/2)))/(2*d) + (a*d*((c + d*x)^(1/2) - c^(1/2)))/(4*((-c)^(1/2) - (d*x - c)^(1/2)))`

Reduce [B] (verification not implemented)

Time = 0.22 (sec) , antiderivative size = 125, normalized size of antiderivative = 1.20

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^2} dx$$

$$= \frac{-8\sqrt{dx+c}\sqrt{dx-c}ad + 4\sqrt{dx+c}\sqrt{dx-c}bdx^2 + 16\log\left(\frac{\sqrt{dx-c}+\sqrt{dx+c}}{\sqrt{c}\sqrt{2}}\right)ad^2x - 8\log\left(\frac{\sqrt{dx-c}+\sqrt{dx+c}}{\sqrt{c}\sqrt{2}}\right)}{8dx}$$

input

```
int((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a)/x^2,x)
```

output

```
( - 8*sqrt(c + d*x)*sqrt( - c + d*x)*a*d + 4*sqrt(c + d*x)*sqrt( - c + d*x)
)*b*d*x**2 + 16*log((sqrt( - c + d*x) + sqrt(c + d*x))/(sqrt(c)*sqrt(2)))*
a*d**2*x - 8*log((sqrt( - c + d*x) + sqrt(c + d*x))/(sqrt(c)*sqrt(2)))*b*c
**2*x - 8*a*d**2*x + b*c**2*x)/(8*d*x)
```

3.11 $\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^4} dx$

Optimal result	119
Mathematica [A] (verified)	119
Rubi [A] (verified)	120
Maple [A] (verified)	122
Fricas [A] (verification not implemented)	122
Sympy [F(-1)]	123
Maxima [A] (verification not implemented)	123
Giac [B] (verification not implemented)	123
Mupad [B] (verification not implemented)	124
Reduce [B] (verification not implemented)	125

Optimal result

Integrand size = 31, antiderivative size = 84

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^4} dx = -\frac{b\sqrt{-c+dx}\sqrt{c+dx}}{x} + \frac{a(-c+dx)^{3/2}(c+dx)^{3/2}}{3c^2x^3} + 2bd\operatorname{arctanh}\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)$$

output

```
-b*(d*x-c)^(1/2)*(d*x+c)^(1/2)/x+1/3*a*(d*x-c)^(3/2)*(d*x+c)^(3/2)/c^2/x^3
+2*b*d*arctanh((d*x-c)^(1/2)/(d*x+c)^(1/2))
```

Mathematica [A] (verified)

Time = 0.19 (sec) , antiderivative size = 81, normalized size of antiderivative = 0.96

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^4} dx = -\frac{\sqrt{-c+dx}\sqrt{c+dx}(3bc^2x^2+a(c^2-d^2x^2))}{3c^2x^3} + 2bd\operatorname{arctanh}\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)$$

input

```
Integrate[(Sqrt[-c + d*x]*Sqrt[c + d*x]*(a + b*x^2))/x^4,x]
```


output

```
-1/3*(Sqrt[-c + d*x]*Sqrt[c + d*x]*(3*b*c^2*x^2 + a*(c^2 - d^2*x^2)))/(c^2
*x^3) + 2*b*d*ArcTanh[Sqrt[-c + d*x]/Sqrt[c + d*x]]
```

Rubi [A] (verified)

Time = 0.38 (sec) , antiderivative size = 85, normalized size of antiderivative = 1.01, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.161$, Rules used = {956, 108, 27, 45, 221}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(a + bx^2) \sqrt{dx - c} \sqrt{c + dx}}{x^4} dx$$

$$\downarrow 956$$

$$b \int \frac{\sqrt{dx - c} \sqrt{c + dx}}{x^2} dx + \frac{a(dx - c)^{3/2}(c + dx)^{3/2}}{3c^2x^3}$$

$$\downarrow 108$$

$$b \left(\int \frac{d^2}{\sqrt{dx - c} \sqrt{c + dx}} dx - \frac{\sqrt{dx - c} \sqrt{c + dx}}{x} \right) + \frac{a(dx - c)^{3/2}(c + dx)^{3/2}}{3c^2x^3}$$

$$\downarrow 27$$

$$b \left(d^2 \int \frac{1}{\sqrt{dx - c} \sqrt{c + dx}} dx - \frac{\sqrt{dx - c} \sqrt{c + dx}}{x} \right) + \frac{a(dx - c)^{3/2}(c + dx)^{3/2}}{3c^2x^3}$$

$$\downarrow 45$$

$$b \left(2d^2 \int \frac{1}{d - \frac{d(dx-c)}{c+dx}} d \frac{\sqrt{dx - c}}{\sqrt{c + dx}} - \frac{\sqrt{dx - c} \sqrt{c + dx}}{x} \right) + \frac{a(dx - c)^{3/2}(c + dx)^{3/2}}{3c^2x^3}$$

$$\downarrow 221$$

$$\frac{a(dx - c)^{3/2}(c + dx)^{3/2}}{3c^2x^3} + b \left(2d \operatorname{arctanh} \left(\frac{\sqrt{dx - c}}{\sqrt{c + dx}} \right) - \frac{\sqrt{dx - c} \sqrt{c + dx}}{x} \right)$$

input

```
Int[(Sqrt[-c + d*x]*Sqrt[c + d*x]*(a + b*x^2))/x^4,x]
```

output $(a*(-c + dx)^{(3/2)}*(c + dx)^{(3/2)})/(3*c^2*x^3) + b*(-((\text{Sqrt}[-c + dx]*\text{Sqrt}[c + dx])/x) + 2*d*\text{ArcTanh}[\text{Sqrt}[-c + dx]/\text{Sqrt}[c + dx]])$

Defintions of rubi rules used

rule 27 $\text{Int}[(a_*)(Fx_), x_Symbol] \rightarrow \text{Simp}[a \text{ Int}[Fx, x], x] /; \text{FreeQ}[a, x] \ \&\& \ !\text{MatchQ}[Fx, (b_*)(Gx_)] /; \text{FreeQ}[b, x]$

rule 45 $\text{Int}[1/(\text{Sqrt}[(a_*) + (b_*)(x_)]*\text{Sqrt}[(c_*) + (d_*)(x_)]), x_Symbol] \rightarrow \text{Simp}[2 \text{ Subst}[\text{Int}[1/(b - d*x^2), x], x, \text{Sqrt}[a + b*x]/\text{Sqrt}[c + d*x]], x] /; \text{FreeQ}[\{a, b, c, d\}, x] \ \&\& \ \text{EqQ}[b*c + a*d, 0] \ \&\& \ !\text{GtQ}[c, 0]$

rule 108 $\text{Int}[((a_*) + (b_*)(x_))^{(m_*)}*((c_*) + (d_*)(x_))^{(n_*)}*((e_*) + (f_*)(x_))^{(p_*)}, x_] \rightarrow \text{Simp}[(a + b*x)^{(m + 1)}*(c + d*x)^n*((e + f*x)^p/(b*(m + 1))), x] - \text{Simp}[1/(b*(m + 1)) \text{ Int}[(a + b*x)^{(m + 1)}*(c + d*x)^{(n - 1)}*(e + f*x)^{(p - 1)}*\text{Simp}[d*e*n + c*f*p + d*f*(n + p)*x, x], x], x] /; \text{FreeQ}[\{a, b, c, d, e, f\}, x] \ \&\& \ \text{LtQ}[m, -1] \ \&\& \ \text{GtQ}[n, 0] \ \&\& \ \text{GtQ}[p, 0] \ \&\& \ (\text{IntegersQ}[2*m, 2*n, 2*p] \ || \ \text{IntegersQ}[m, n + p] \ || \ \text{IntegersQ}[p, m + n])$

rule 221 $\text{Int}[((a_*) + (b_*)(x_)^2)^{(-1)}, x_Symbol] \rightarrow \text{Simp}[(\text{Rt}[-a/b, 2]/a)*\text{ArcTanh}[x/\text{Rt}[-a/b, 2]], x] /; \text{FreeQ}[\{a, b\}, x] \ \&\& \ \text{NegQ}[a/b]$

rule 956 $\text{Int}[((e_*)(x_))^{(m_*)}*((a1_*) + (b1_*)(x_)^{\text{non2}_*})^{(p_*)}*((a2_*) + (b2_*)(x_)^{\text{non2}_*})^{(p_*)}*((c_*) + (d_*)(x_)^{(n_*)}, x_Symbol] \rightarrow \text{Simp}[c*(e*x)^{(m + 1)}*(a1 + b1*x^{(n/2)})^{(p + 1)}*((a2 + b2*x^{(n/2)})^{(p + 1)})/(a1*a2*e^{(m + 1)}), x] + \text{Simp}[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(a1*a2*e^{n*(m + 1)}) \text{ Int}[(e*x)^{(m + n)}*(a1 + b1*x^{(n/2)})^p*(a2 + b2*x^{(n/2)})^p, x], x] /; \text{FreeQ}[\{a1, b1, a2, b2, c, d, e, p\}, x] \ \&\& \ \text{EqQ}[\text{non2}, n/2] \ \&\& \ \text{EqQ}[a2*b1 + a1*b2, 0] \ \&\& \ (\text{IntegerQ}[n] \ || \ \text{GtQ}[e, 0]) \ \&\& \ ((\text{GtQ}[n, 0] \ \&\& \ \text{LtQ}[m, -1]) \ || \ (\text{LtQ}[n, 0] \ \&\& \ \text{GtQ}[m + n, -1])) \ \&\& \ !\text{ILtQ}[p, -1]$

Maple [A] (verified)

Time = 0.14 (sec) , antiderivative size = 124, normalized size of antiderivative = 1.48

method	result
risch	$\frac{\sqrt{dx+c}(-dx+c)(-ad^2x^2+3bc^2x^2+ac^2)}{3x^3c^2\sqrt{dx-c}} + \frac{bd^2 \ln\left(\frac{d^2x}{\sqrt{d^2}} + \sqrt{d^2x^2-c^2}\right)\sqrt{(dx-c)(dx+c)}}{\sqrt{d^2}\sqrt{dx-c}\sqrt{dx+c}}$
default	$-\frac{\sqrt{dx-c}\sqrt{dx+c}\left(-3\ln\left(\left(\sqrt{d^2x^2-c^2}\operatorname{csgn}(d)+dx\right)\operatorname{csgn}(d)\right)bc^2dx^3-\operatorname{csgn}(d)ad^2x^2\sqrt{d^2x^2-c^2}+3\operatorname{csgn}(d)bc^2x^2\sqrt{d^2x^2-c^2}+\operatorname{csgn}(d)ac^2\right)}{3\sqrt{d^2x^2-c^2}c^2x^3}$

input `int((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a)/x^4,x,method=_RETURNVERBOSE)`

output `1/3*(d*x+c)^(1/2)*(-d*x+c)*(-a*d^2*x^2+3*b*c^2*x^2+a*c^2)/x^3/c^2/(d*x-c)^(1/2)+b*d^2*ln(d^2*x/(d^2)^(1/2)+(d^2*x^2-c^2)^(1/2))/(d^2)^(1/2)*((d*x-c)*(d*x+c))^(1/2)/(d*x-c)^(1/2)/(d*x+c)^(1/2)`

Fricas [A] (verification not implemented)

Time = 0.10 (sec) , antiderivative size = 100, normalized size of antiderivative = 1.19

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^4} dx =$$

$$-\frac{3bc^2dx^3 \log(-dx + \sqrt{dx+c}\sqrt{dx-c}) + (3bc^2d - ad^3)x^3 + (ac^2 + (3bc^2 - ad^2)x^2)\sqrt{dx+c}\sqrt{dx-c}}{3c^2x^3}$$

input `integrate((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a)/x^4,x, algorithm="fricas")`

output `-1/3*(3*b*c^2*d*x^3*log(-d*x + sqrt(d*x + c)*sqrt(d*x - c)) + (3*b*c^2*d - a*d^3)*x^3 + (a*c^2 + (3*b*c^2 - a*d^2)*x^2)*sqrt(d*x + c)*sqrt(d*x - c))/(c^2*x^3)`

Sympy [F(-1)]

Timed out.

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^4} dx = \text{Timed out}$$

input `integrate((d*x-c)**(1/2)*(d*x+c)**(1/2)*(b*x**2+a)/x**4,x)`

output `Timed out`

Maxima [A] (verification not implemented)

Time = 0.11 (sec) , antiderivative size = 75, normalized size of antiderivative = 0.89

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^4} dx = bd \log \left(2d^2x + 2\sqrt{d^2x^2 - c^2}d \right) - \frac{\sqrt{d^2x^2 - c^2}b}{x} + \frac{(d^2x^2 - c^2)^{\frac{3}{2}}a}{3c^2x^3}$$

input `integrate((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a)/x^4,x, algorithm="maxima")`

output `b*d*log(2*d^2*x + 2*sqrt(d^2*x^2 - c^2)*d) - sqrt(d^2*x^2 - c^2)*b/x + 1/3*(d^2*x^2 - c^2)^(3/2)*a/(c^2*x^3)`

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 165 vs. 2(70) = 140.

Time = 0.18 (sec) , antiderivative size = 165, normalized size of antiderivative = 1.96

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^4} dx = -\frac{1}{6}d^3 \left(\frac{3b \log \left((\sqrt{dx+c} - \sqrt{dx-c})^4 \right)}{d^2} + \frac{16 \left(3bc^2(\sqrt{dx+c} - \sqrt{dx-c})^8 - 3ad^2(\sqrt{dx+c} - \sqrt{dx-c}) \right)}{\left((\sqrt{dx+c} - \sqrt{dx-c}) \right)} \right)$$

input `integrate((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a)/x^4,x, algorithm="giac")`

output `-1/6*d^3*(3*b*log((sqrt(d*x + c) - sqrt(d*x - c))^4)/d^2 + 16*(3*b*c^2*(sqrt(d*x + c) - sqrt(d*x - c))^8 - 3*a*d^2*(sqrt(d*x + c) - sqrt(d*x - c))^8 + 24*b*c^4*(sqrt(d*x + c) - sqrt(d*x - c))^4 + 48*b*c^6 - 16*a*c^4*d^2)/((sqrt(d*x + c) - sqrt(d*x - c))^4 + 4*c^2)^3*d^2))`

Mupad [B] (verification not implemented)

Time = 5.44 (sec) , antiderivative size = 236, normalized size of antiderivative = 2.81

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^4} dx = \frac{bd + \frac{5bd(\sqrt{c+dx}-\sqrt{c})^2}{(\sqrt{-c}-\sqrt{dx-c})^2}}{\frac{4(\sqrt{c+dx}-\sqrt{c})}{\sqrt{-c}-\sqrt{dx-c}} + \frac{4(\sqrt{c+dx}-\sqrt{c})^3}{(\sqrt{-c}-\sqrt{dx-c})^3}} - 4bd \operatorname{atanh}\left(\frac{\sqrt{c+dx}-\sqrt{c}}{\sqrt{-c}-\sqrt{dx-c}}\right) - \frac{\left(\frac{a\sqrt{c+dx}}{3} - \frac{ad^2x^2\sqrt{c+dx}}{3c^2}\right)\sqrt{dx-c}}{x^3} + \frac{bd(\sqrt{c+dx}-\sqrt{c})}{4(\sqrt{-c}-\sqrt{dx-c})}$$

input `int(((a + b*x^2)*(c + d*x)^(1/2)*(d*x - c)^(1/2))/x^4,x)`

output `(b*d + (5*b*d*((c + d*x)^(1/2) - c^(1/2))^2)/((-c)^(1/2) - (d*x - c)^(1/2))^2)/((4*((c + d*x)^(1/2) - c^(1/2)))/((-c)^(1/2) - (d*x - c)^(1/2)) + (4*((c + d*x)^(1/2) - c^(1/2))^3)/((-c)^(1/2) - (d*x - c)^(1/2))^3) - 4*b*d*a*tanh(((c + d*x)^(1/2) - c^(1/2))/((-c)^(1/2) - (d*x - c)^(1/2))) - (((a*(c + d*x)^(1/2))/3 - (a*d^2*x^2*(c + d*x)^(1/2))/(3*c^2))*(d*x - c)^(1/2))/x^3 + (b*d*((c + d*x)^(1/2) - c^(1/2)))/(4*((-c)^(1/2) - (d*x - c)^(1/2)))`

Reduce [B] (verification not implemented)

Time = 0.23 (sec) , antiderivative size = 126, normalized size of antiderivative = 1.50

$$\int \frac{\sqrt{-c+dx}\sqrt{c+dx}(a+bx^2)}{x^4} dx$$

$$= \frac{-\sqrt{dx+c}\sqrt{dx-c}ac^2 + \sqrt{dx+c}\sqrt{dx-c}ad^2x^2 - 3\sqrt{dx+c}\sqrt{dx-c}bc^2x^2 + 6\log\left(\frac{\sqrt{dx-c}+\sqrt{dx+c}}{\sqrt{c}\sqrt{2}}\right)bc}{3c^2x^3}$$

input `int((d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*x^2+a)/x^4,x)`output `(-sqrt(c+d*x)*sqrt(-c+d*x)*a*c**2 + sqrt(c+d*x)*sqrt(-c+d*x)*a*d**2*x**2 - 3*sqrt(c+d*x)*sqrt(-c+d*x)*b*c**2*x**2 + 6*log((sqrt(-c+d*x)+sqrt(c+d*x))/(sqrt(c)*sqrt(2)))*b*c**2*d*x**3 + a*d**3*x**3 + b*c**2*d*x**3)/(3*c**2*x**3)`

3.12 $\int \frac{x^4(a+bx^2)}{\sqrt{-1+cx}\sqrt{1+cx}} dx$

Optimal result	126
Mathematica [A] (warning: unable to verify)	126
Rubi [A] (verified)	127
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Optimal result

Integrand size = 29, antiderivative size = 125

$$\int \frac{x^4(a+bx^2)}{\sqrt{-1+cx}\sqrt{1+cx}} dx = \frac{(5b+6ac^2)x\sqrt{-1+cx}\sqrt{1+cx}}{16c^6} + \frac{(5b+6ac^2)x^3\sqrt{-1+cx}\sqrt{1+cx}}{24c^4} + \frac{bx^5\sqrt{-1+cx}\sqrt{1+cx}}{6c^2} + \frac{(5b+6ac^2)\operatorname{arccosh}(cx)}{16c^7}$$

output

```
1/16*(6*a*c^2+5*b)*x*(c*x-1)^(1/2)*(c*x+1)^(1/2)/c^6+1/24*(6*a*c^2+5*b)*x^3*(c*x-1)^(1/2)*(c*x+1)^(1/2)/c^4+1/6*b*x^5*(c*x-1)^(1/2)*(c*x+1)^(1/2)/c^2+1/16*(6*a*c^2+5*b)*arccosh(c*x)/c^7
```

Mathematica [A] (warning: unable to verify)

Time = 0.21 (sec) , antiderivative size = 96, normalized size of antiderivative = 0.77

$$\int \frac{x^4(a+bx^2)}{\sqrt{-1+cx}\sqrt{1+cx}} dx = \frac{cx\sqrt{-1+cx}\sqrt{1+cx}(6ac^2(3+2c^2x^2)+b(15+10c^2x^2+8c^4x^4))+6(5b+6ac^2)\operatorname{arctanh}\left(\sqrt{\frac{-1+cx}{1+cx}}\right)}{48c^7}$$

input `Integrate[(x^4*(a + b*x^2))/(Sqrt[-1 + c*x]*Sqrt[1 + c*x]),x]`

output `(c*x*Sqrt[-1 + c*x]*Sqrt[1 + c*x]*(6*a*c^2*(3 + 2*c^2*x^2) + b*(15 + 10*c^2*x^2 + 8*c^4*x^4)) + 6*(5*b + 6*a*c^2)*ArcTanh[Sqrt[(-1 + c*x)/(1 + c*x)]])/(48*c^7)`

Rubi [A] (verified)

Time = 0.40 (sec) , antiderivative size = 118, normalized size of antiderivative = 0.94, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.172$, Rules used = {960, 111, 27, 101, 43}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{x^4(a + bx^2)}{\sqrt{cx - 1}\sqrt{cx + 1}} dx \\
 & \quad \downarrow 960 \\
 & \frac{1}{6} \left(6a + \frac{5b}{c^2} \right) \int \frac{x^4}{\sqrt{cx - 1}\sqrt{cx + 1}} dx + \frac{bx^5\sqrt{cx - 1}\sqrt{cx + 1}}{6c^2} \\
 & \quad \downarrow 111 \\
 & \frac{1}{6} \left(6a + \frac{5b}{c^2} \right) \left(\int \frac{3x^2}{\sqrt{cx - 1}\sqrt{cx + 1}} dx + \frac{x^3\sqrt{cx - 1}\sqrt{cx + 1}}{4c^2} \right) + \frac{bx^5\sqrt{cx - 1}\sqrt{cx + 1}}{6c^2} \\
 & \quad \downarrow 27 \\
 & \frac{1}{6} \left(6a + \frac{5b}{c^2} \right) \left(\frac{3 \int \frac{x^2}{\sqrt{cx - 1}\sqrt{cx + 1}} dx}{4c^2} + \frac{x^3\sqrt{cx - 1}\sqrt{cx + 1}}{4c^2} \right) + \frac{bx^5\sqrt{cx - 1}\sqrt{cx + 1}}{6c^2} \\
 & \quad \downarrow 101 \\
 & \frac{1}{6} \left(6a + \frac{5b}{c^2} \right) \left(\frac{3 \left(\frac{\int \frac{1}{\sqrt{cx - 1}\sqrt{cx + 1}} dx}{2c^2} + \frac{x\sqrt{cx - 1}\sqrt{cx + 1}}{2c^2} \right)}{4c^2} + \frac{x^3\sqrt{cx - 1}\sqrt{cx + 1}}{4c^2} \right) + \\
 & \quad \frac{bx^5\sqrt{cx - 1}\sqrt{cx + 1}}{6c^2}
 \end{aligned}$$

$$\frac{1}{6} \left(6a + \frac{5b}{c^2} \right) \left(\frac{3 \left(\frac{\operatorname{arccosh}(cx)}{2c^3} + \frac{x\sqrt{cx-1}\sqrt{cx+1}}{2c^2} \right)}{4c^2} + \frac{x^3\sqrt{cx-1}\sqrt{cx+1}}{4c^2} \right) + \frac{bx^5\sqrt{cx-1}\sqrt{cx+1}}{6c^2}$$

input `Int[(x^4*(a + b*x^2))/(Sqrt[-1 + c*x]*Sqrt[1 + c*x]),x]`

output `(b*x^5*Sqrt[-1 + c*x]*Sqrt[1 + c*x])/(6*c^2) + ((6*a + (5*b)/c^2)*((x^3*Sqrt[-1 + c*x]*Sqrt[1 + c*x])/(4*c^2) + (3*((x*Sqrt[-1 + c*x]*Sqrt[1 + c*x])/(2*c^2) + ArcCosh[c*x]/(2*c^3)))/(4*c^2)))/6`

Defintions of rubi rules used

rule 27 `Int[(a_)*(F_x_), x_Symbol] := Simp[a Int[F_x, x], x] /; FreeQ[a, x] && !MatchQ[F_x, (b_)*(G_x_) /; FreeQ[b, x]]`

rule 43 `Int[1/(Sqrt[(a_) + (b_.)*(x_)]*Sqrt[(c_) + (d_.)*(x_)]), x_Symbol] := Simp[ArcCosh[b*(x/a)]/(b*Sqrt[d/b]), x] /; FreeQ[{a, b, c, d}, x] && EqQ[b*c + a*d, 0] && GtQ[a, 0] && GtQ[d/b, 0]`

rule 101 `Int[((a_.) + (b_.)*(x_))^(2*((c_.) + (d_.)*(x_))^(n_))*((e_.) + (f_.)*(x_))^(p_), x_] := Simp[b*(a + b*x)*(c + d*x)^(n + 1)*((e + f*x)^(p + 1)/(d*f*(n + p + 3))), x] + Simp[1/(d*f*(n + p + 3)) Int[(c + d*x)^n*(e + f*x)^p*Simp[a^2*d*f*(n + p + 3) - b*(b*c*e + a*(d*e*(n + 1) + c*f*(p + 1))) + b*(a*d*f*(n + p + 4) - b*(d*e*(n + 2) + c*f*(p + 2)))*x, x], x] /; FreeQ[{a, b, c, d, e, f, n, p}, x] && NeQ[n + p + 3, 0]`

rule 111

```
Int[((a_.) + (b_.)*(x_)^(m_))*((c_.) + (d_.)*(x_)^(n_))*((e_.) + (f_.)*(x_)
)^(p_), x_] := Simp[b*(a + b*x)^(m - 1)*(c + d*x)^(n + 1)*((e + f*x)^(p + 1)
)/(d*f*(m + n + p + 1)), x] + Simp[1/(d*f*(m + n + p + 1)) Int[(a + b*x)
^(m - 2)*(c + d*x)^n*(e + f*x)^p*Simp[a^2*d*f*(m + n + p + 1) - b*(b*c*e*(m
- 1) + a*(d*e*(n + 1) + c*f*(p + 1))) + b*(a*d*f*(2*m + n + p) - b*(d*e*(m
+ n) + c*f*(m + p)))*x, x], x] /; FreeQ[{a, b, c, d, e, f, n, p}, x] &
& GtQ[m, 1] && NeQ[m + n + p + 1, 0] && IntegerQ[m]
```

rule 960

```
Int[((e_.)*(x_)^(m_.))*((a1_.) + (b1_.)*(x_)^(non2_.))^p_)*((a2_.) + (b2_.)
*(x_)^(non2_.))^p_)*((c_.) + (d_.)*(x_)^(n_)), x_Symbol] := Simp[d*(e*x)^(
m + 1)*(a1 + b1*x^(n/2))^(p + 1)*((a2 + b2*x^(n/2))^(p + 1)/(b1*b2*e*(m + n
*(p + 1) + 1))), x] - Simp[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/
(b1*b2*(m + n*(p + 1) + 1)) Int[(e*x)^m*(a1 + b1*x^(n/2))^p*(a2 + b2*x^(n
/2))^p, x], x] /; FreeQ[{a1, b1, a2, b2, c, d, e, m, n, p}, x] && EqQ[non2,
n/2] && EqQ[a2*b1 + a1*b2, 0] && NeQ[m + n*(p + 1) + 1, 0]
```

Maple [A] (verified)

Time = 0.13 (sec) , antiderivative size = 129, normalized size of antiderivative = 1.03

method	result
risch	$\frac{x(8bx^4c^4+12ac^4x^2+10bc^2x^2+18a^2c^2+15b)\sqrt{cx+1}\sqrt{cx-1}}{48c^6} + \frac{(6ac^2+5b)\ln\left(\frac{c^2x}{\sqrt{c^2}+\sqrt{c^2x^2-1}}\right)\sqrt{(cx+1)(cx-1)}}{16c^6\sqrt{c^2}\sqrt{cx-1}\sqrt{cx+1}}$
default	$\frac{\sqrt{cx-1}\sqrt{cx+1}\left(8\operatorname{csgn}(c)bc^5x^5\sqrt{c^2x^2-1}+12\operatorname{csgn}(c)ac^5x^3\sqrt{c^2x^2-1}+10\sqrt{c^2x^2-1}\operatorname{csgn}(c)c^3bx^3+18\sqrt{c^2x^2-1}\operatorname{csgn}(c)c^3ax+15b\right)}{48c^7\sqrt{c^2x^2-1}}$

input

```
int(x^4*(b*x^2+a)/(c*x-1)^(1/2)/(c*x+1)^(1/2),x,method=_RETURNVERBOSE)
```

output

```
1/48*x*(8*b*c^4*x^4+12*a*c^4*x^2+10*b*c^2*x^2+18*a*c^2+15*b)*(c*x+1)^(1/2)
*(c*x-1)^(1/2)/c^6+1/16*(6*a*c^2+5*b)/c^6*ln(c^2*x/(c^2)^(1/2)+(c^2*x^2-1)
^(1/2))/(c^2)^(1/2)*((c*x+1)*(c*x-1))^(1/2)/(c*x-1)^(1/2)/(c*x+1)^(1/2)
```

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 96, normalized size of antiderivative = 0.77

$$\int \frac{x^4(a + bx^2)}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx$$

$$= \frac{(8bc^5x^5 + 2(6ac^5 + 5bc^3)x^3 + 3(6ac^3 + 5bc)x)\sqrt{cx + 1}\sqrt{cx - 1} - 3(6ac^2 + 5b)\log(-cx + \sqrt{cx + 1})}{48c^7}$$

input `integrate(x^4*(b*x^2+a)/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="fricas")`output `1/48*((8*b*c^5*x^5 + 2*(6*a*c^5 + 5*b*c^3)*x^3 + 3*(6*a*c^3 + 5*b*c)*x)*sqrt(c*x + 1)*sqrt(c*x - 1) - 3*(6*a*c^2 + 5*b)*log(-c*x + sqrt(c*x + 1)*sqrt(c*x - 1))/c^7`**Sympy [F(-1)]**

Timed out.

$$\int \frac{x^4(a + bx^2)}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \text{Timed out}$$

input `integrate(x**4*(b*x**2+a)/(c*x-1)**(1/2)/(c*x+1)**(1/2),x)`output `Timed out`**Maxima [A] (verification not implemented)**

Time = 0.03 (sec) , antiderivative size = 153, normalized size of antiderivative = 1.22

$$\int \frac{x^4(a + bx^2)}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{\sqrt{c^2x^2 - 1}bx^5}{6c^2} + \frac{\sqrt{c^2x^2 - 1}ax^3}{4c^2} + \frac{5\sqrt{c^2x^2 - 1}bx^3}{24c^4}$$

$$+ \frac{3\sqrt{c^2x^2 - 1}ax}{8c^4} + \frac{3a\log(2c^2x + 2\sqrt{c^2x^2 - 1}c)}{8c^5}$$

$$+ \frac{5\sqrt{c^2x^2 - 1}bx}{16c^6} + \frac{5b\log(2c^2x + 2\sqrt{c^2x^2 - 1}c)}{16c^7}$$

input `integrate(x^4*(b*x^2+a)/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="maxima")`

output $\frac{1}{6}\sqrt{c^2x^2 - 1}bx^5/c^2 + \frac{1}{4}\sqrt{c^2x^2 - 1}ax^3/c^2 + \frac{5}{24}\sqrt{c^2x^2 - 1}bx^3/c^4 + \frac{3}{8}\sqrt{c^2x^2 - 1}ax/c^4 + \frac{3}{8}a\log(2c^2x + 2\sqrt{c^2x^2 - 1}c)/c^5 + \frac{5}{16}\sqrt{c^2x^2 - 1}bx/c^6 + \frac{5}{16}b\log(2c^2x + 2\sqrt{c^2x^2 - 1}c)/c^7$

Giac [A] (verification not implemented)

Time = 0.13 (sec) , antiderivative size = 172, normalized size of antiderivative = 1.38

$$\int \frac{x^4(a + bx^2)}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx$$

$$= \frac{\left(\left(2\left((cx + 1)\left(4(cx + 1)\left(\frac{(cx+1)b}{c^6} - \frac{5b}{c^6}\right) + \frac{3(2ac^{38} + 15bc^{36})}{c^{42}}\right) - \frac{18ac^{38} + 55bc^{36}}{c^{42}}\right)(cx + 1) + \frac{54ac^{38} + 85bc^{36}}{c^{42}}\right)(cx + 1) + \frac{54ac^{38} + 85bc^{36}}{c^{42}}\right)(cx + 1) + \frac{54ac^{38} + 85bc^{36}}{c^{42}}}{48c}$$

input `integrate(x^4*(b*x^2+a)/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="giac")`

output $\frac{1}{48} \left(\left(2 \left((cx + 1) \left(4 (cx + 1) \left(\frac{(cx+1)b}{c^6} - \frac{5b}{c^6} \right) + \frac{3(2ac^{38} + 15bc^{36})}{c^{42}} \right) - \frac{18ac^{38} + 55bc^{36}}{c^{42}} \right) (cx + 1) + \frac{54ac^{38} + 85bc^{36}}{c^{42}} \right) (cx + 1) + \frac{54ac^{38} + 85bc^{36}}{c^{42}} \right) (cx + 1) + \frac{54ac^{38} + 85bc^{36}}{c^{42}} \right) (cx + 1) + \frac{54ac^{38} + 85bc^{36}}{c^{42}}$

Mupad [B] (verification not implemented)

Time = 37.56 (sec) , antiderivative size = 1154, normalized size of antiderivative = 9.23

$$\int \frac{x^4(a + bx^2)}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \text{Too large to display}$$

input `int((x^4*(a + b*x^2))/((c*x - 1)^(1/2)*(c*x + 1)^(1/2)),x)`

output

```

((23*a*((c*x - 1)^(1/2) - 1i)^3)/(2*((c*x + 1)^(1/2) - 1)^3) + (333*a*((c*
x - 1)^(1/2) - 1i)^5)/(2*((c*x + 1)^(1/2) - 1)^5) + (671*a*((c*x - 1)^(1/2)
) - 1i)^7)/(2*((c*x + 1)^(1/2) - 1)^7) + (671*a*((c*x - 1)^(1/2) - 1i)^9)/
(2*((c*x + 1)^(1/2) - 1)^9) + (333*a*((c*x - 1)^(1/2) - 1i)^11)/(2*((c*x +
1)^(1/2) - 1)^11) + (23*a*((c*x - 1)^(1/2) - 1i)^13)/(2*((c*x + 1)^(1/2)
- 1)^13) - (3*a*((c*x - 1)^(1/2) - 1i)^15)/(2*((c*x + 1)^(1/2) - 1)^15) -
(3*a*((c*x - 1)^(1/2) - 1i))/(2*((c*x + 1)^(1/2) - 1))/(c^5 - (8*c^5*((c*
x - 1)^(1/2) - 1i)^2)/((c*x + 1)^(1/2) - 1)^2 + (28*c^5*((c*x - 1)^(1/2) -
1i)^4)/((c*x + 1)^(1/2) - 1)^4 - (56*c^5*((c*x - 1)^(1/2) - 1i)^6)/((c*x
+ 1)^(1/2) - 1)^6 + (70*c^5*((c*x - 1)^(1/2) - 1i)^8)/((c*x + 1)^(1/2) - 1
)^8 - (56*c^5*((c*x - 1)^(1/2) - 1i)^10)/((c*x + 1)^(1/2) - 1)^10 + (28*c^
5*((c*x - 1)^(1/2) - 1i)^12)/((c*x + 1)^(1/2) - 1)^12 - (8*c^5*((c*x - 1)^(
1/2) - 1i)^14)/((c*x + 1)^(1/2) - 1)^14 + (c^5*((c*x - 1)^(1/2) - 1i)^16)
/((c*x + 1)^(1/2) - 1)^16) - ((311*b*((c*x - 1)^(1/2) - 1i)^5)/(4*((c*x +
1)^(1/2) - 1)^5) - (175*b*((c*x - 1)^(1/2) - 1i)^3)/(12*((c*x + 1)^(1/2) -
1)^3) + (8361*b*((c*x - 1)^(1/2) - 1i)^7)/(4*((c*x + 1)^(1/2) - 1)^7) + (
42259*b*((c*x - 1)^(1/2) - 1i)^9)/(6*((c*x + 1)^(1/2) - 1)^9) + (25295*b*(
(c*x - 1)^(1/2) - 1i)^11)/(2*((c*x + 1)^(1/2) - 1)^11) + (25295*b*((c*x -
1)^(1/2) - 1i)^13)/(2*((c*x + 1)^(1/2) - 1)^13) + (42259*b*((c*x - 1)^(1/2)
) - 1i)^15)/(6*((c*x + 1)^(1/2) - 1)^15) + (8361*b*((c*x - 1)^(1/2) - 1...

```

Reduce [B] (verification not implemented)

Time = 0.21 (sec) , antiderivative size = 152, normalized size of antiderivative = 1.22

$$\int \frac{x^4(a + bx^2)}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx$$

$$= \frac{12\sqrt{cx + 1}\sqrt{cx - 1}ac^5x^3 + 18\sqrt{cx + 1}\sqrt{cx - 1}ac^3x + 8\sqrt{cx + 1}\sqrt{cx - 1}bc^5x^5 + 10\sqrt{cx + 1}\sqrt{cx - 1}}{48c^7}$$

input

```
int(x^4*(b*x^2+a)/(c*x-1)^(1/2)/(c*x+1)^(1/2),x)
```

output

```

(12*sqrt(c*x + 1)*sqrt(c*x - 1)*a*c**5*x**3 + 18*sqrt(c*x + 1)*sqrt(c*x -
1)*a*c**3*x + 8*sqrt(c*x + 1)*sqrt(c*x - 1)*b*c**5*x**5 + 10*sqrt(c*x + 1)
*sqrt(c*x - 1)*b*c**3*x**3 + 15*sqrt(c*x + 1)*sqrt(c*x - 1)*b*c*x + 36*log
((sqrt(c*x - 1) + sqrt(c*x + 1))/sqrt(2))*a*c**2 + 30*log((sqrt(c*x - 1) +
sqrt(c*x + 1))/sqrt(2))*b)/(48*c**7)

```

3.13 $\int \frac{x^3(a+bx^2)}{\sqrt{-1+cx}\sqrt{1+cx}} dx$

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Giac [A] (verification not implemented)	138
Mupad [B] (verification not implemented)	139
Reduce [B] (verification not implemented)	139

Optimal result

Integrand size = 29, antiderivative size = 103

$$\int \frac{x^3(a+bx^2)}{\sqrt{-1+cx}\sqrt{1+cx}} dx = \frac{2(4b+5ac^2)\sqrt{-1+cx}\sqrt{1+cx}}{15c^6} + \frac{(4b+5ac^2)x^2\sqrt{-1+cx}\sqrt{1+cx}}{15c^4} + \frac{bx^4\sqrt{-1+cx}\sqrt{1+cx}}{5c^2}$$

output

```
2/15*(5*a*c^2+4*b)*(c*x-1)^(1/2)*(c*x+1)^(1/2)/c^6+1/15*(5*a*c^2+4*b)*x^2*(c*x-1)^(1/2)*(c*x+1)^(1/2)/c^4+1/5*b*x^4*(c*x-1)^(1/2)*(c*x+1)^(1/2)/c^2
```

Mathematica [A] (verified)

Time = 0.10 (sec) , antiderivative size = 61, normalized size of antiderivative = 0.59

$$\int \frac{x^3(a+bx^2)}{\sqrt{-1+cx}\sqrt{1+cx}} dx = \frac{\sqrt{-1+cx}\sqrt{1+cx}(5ac^2(2+c^2x^2)+b(8+4c^2x^2+3c^4x^4))}{15c^6}$$

input

```
Integrate[(x^3*(a+b*x^2))/(Sqrt[-1+c*x]*Sqrt[1+c*x]),x]
```

output

```
(Sqrt[-1+c*x]*Sqrt[1+c*x]*(5*a*c^2*(2+c^2*x^2)+b*(8+4*c^2*x^2+3*c^4*x^4)))/(15*c^6)
```

Rubi [A] (verified)

Time = 0.36 (sec) , antiderivative size = 98, normalized size of antiderivative = 0.95, number of steps used = 4, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}} = 0.138$, Rules used = {960, 111, 27, 83}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{x^3(a + bx^2)}{\sqrt{cx - 1}\sqrt{cx + 1}} dx$$

$$\downarrow 960$$

$$\frac{1}{5} \left(5a + \frac{4b}{c^2} \right) \int \frac{x^3}{\sqrt{cx - 1}\sqrt{cx + 1}} dx + \frac{bx^4\sqrt{cx - 1}\sqrt{cx + 1}}{5c^2}$$

$$\downarrow 111$$

$$\frac{1}{5} \left(5a + \frac{4b}{c^2} \right) \left(\int \frac{2x}{\sqrt{cx - 1}\sqrt{cx + 1}} dx + \frac{x^2\sqrt{cx - 1}\sqrt{cx + 1}}{3c^2} \right) + \frac{bx^4\sqrt{cx - 1}\sqrt{cx + 1}}{5c^2}$$

$$\downarrow 27$$

$$\frac{1}{5} \left(5a + \frac{4b}{c^2} \right) \left(\frac{2 \int \frac{x}{\sqrt{cx - 1}\sqrt{cx + 1}} dx}{3c^2} + \frac{x^2\sqrt{cx - 1}\sqrt{cx + 1}}{3c^2} \right) + \frac{bx^4\sqrt{cx - 1}\sqrt{cx + 1}}{5c^2}$$

$$\downarrow 83$$

$$\frac{1}{5} \left(\frac{2\sqrt{cx - 1}\sqrt{cx + 1}}{3c^4} + \frac{x^2\sqrt{cx - 1}\sqrt{cx + 1}}{3c^2} \right) \left(5a + \frac{4b}{c^2} \right) + \frac{bx^4\sqrt{cx - 1}\sqrt{cx + 1}}{5c^2}$$

input `Int[(x^3*(a + b*x^2))/(Sqrt[-1 + c*x]*Sqrt[1 + c*x]),x]`

output `(b*x^4*Sqrt[-1 + c*x]*Sqrt[1 + c*x])/(5*c^2) + ((5*a + (4*b)/c^2)*((2*Sqrt[-1 + c*x]*Sqrt[1 + c*x])/(3*c^4) + (x^2*Sqrt[-1 + c*x]*Sqrt[1 + c*x])/(3*c^2)))/5`

Definitions of rubi rules used

rule 27 $\text{Int}[(a_*)(Fx_), x_Symbol] \rightarrow \text{Simp}[a \text{ Int}[Fx, x], x] /; \text{FreeQ}[a, x] \ \&\& \ !\text{MatchQ}[Fx, (b_*)(Gx_)] /; \text{FreeQ}[b, x]$

rule 83 $\text{Int}[(a_.) + (b_.)*(x_)*((c_.) + (d_.)*(x_))^{(n_.)}*((e_.) + (f_.)*(x_))^{(p_.)}, x_] \rightarrow \text{Simp}[b*(c + d*x)^{(n + 1)}*((e + f*x)^{(p + 1)}/(d*f*(n + p + 2))), x] /; \text{FreeQ}[\{a, b, c, d, e, f, n, p\}, x] \ \&\& \ \text{NeQ}[n + p + 2, 0] \ \&\& \ \text{EqQ}[a*d*f*(n + p + 2) - b*(d*e*(n + 1) + c*f*(p + 1)), 0]$

rule 111 $\text{Int}[(a_.) + (b_.)*(x_))^{(m_.)}*((c_.) + (d_.)*(x_))^{(n_.)}*((e_.) + (f_.)*(x_))^{(p_.)}, x_] \rightarrow \text{Simp}[b*(a + b*x)^{(m - 1)}*(c + d*x)^{(n + 1)}*((e + f*x)^{(p + 1)}/(d*f*(m + n + p + 1))), x] + \text{Simp}[1/(d*f*(m + n + p + 1)) \text{ Int}[(a + b*x)^{(m - 2)}*(c + d*x)^n*(e + f*x)^p*\text{Simp}[a^2*d*f*(m + n + p + 1) - b*(b*c*e*(m - 1) + a*(d*e*(n + 1) + c*f*(p + 1))) + b*(a*d*f*(2*m + n + p) - b*(d*e*(m + n) + c*f*(m + p)))*x, x], x] /; \text{FreeQ}[\{a, b, c, d, e, f, n, p\}, x] \ \&\& \ \text{GtQ}[m, 1] \ \&\& \ \text{NeQ}[m + n + p + 1, 0] \ \&\& \ \text{IntegerQ}[m]$

rule 960 $\text{Int}[(e_.)*(x_))^{(m_.)}*((a1_.) + (b1_.)*(x_))^{(non2_.)}*(a2_.) + (b2_.)*(x_))^{(non2_.)}*(c_.) + (d_.)*(x_))^{(n_.)}, x_Symbol] \rightarrow \text{Simp}[d*(e*x)^{(m + 1)}*(a1 + b1*x^{(n/2)})^{(p + 1)}*((a2 + b2*x^{(n/2)})^{(p + 1)}/(b1*b2*e*(m + n*(p + 1) + 1))), x] - \text{Simp}[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(b1*b2*(m + n*(p + 1) + 1)) \text{ Int}[(e*x)^m*(a1 + b1*x^{(n/2)})^p*(a2 + b2*x^{(n/2)})^p, x], x] /; \text{FreeQ}[\{a1, b1, a2, b2, c, d, e, m, n, p\}, x] \ \&\& \ \text{EqQ}[\text{non2}, n/2] \ \&\& \ \text{EqQ}[a2*b1 + a1*b2, 0] \ \&\& \ \text{NeQ}[m + n*(p + 1) + 1, 0]$

Maple [A] (verified)

Time = 0.11 (sec) , antiderivative size = 57, normalized size of antiderivative = 0.55

method	result	size
gospers	$\frac{\sqrt{cx+1}\sqrt{cx-1}(3bx^4c^4+5ac^4x^2+4bc^2x^2+10ac^2+8b)}{15c^6}$	57
default	$\frac{\sqrt{cx+1}\sqrt{cx-1}(3bx^4c^4+5ac^4x^2+4bc^2x^2+10ac^2+8b)}{15c^6}$	57
risch	$\frac{\sqrt{cx+1}\sqrt{cx-1}(3bx^4c^4+5ac^4x^2+4bc^2x^2+10ac^2+8b)}{15c^6}$	57
orering	$\frac{\sqrt{cx+1}\sqrt{cx-1}(3bx^4c^4+5ac^4x^2+4bc^2x^2+10ac^2+8b)}{15c^6}$	57

input `int(x^3*(b*x^2+a)/(c*x-1)^(1/2)/(c*x+1)^(1/2),x,method=_RETURNVERBOSE)`

output `1/15*(c*x+1)^(1/2)*(c*x-1)^(1/2)*(3*b*c^4*x^4+5*a*c^4*x^2+4*b*c^2*x^2+10*a*c^2+8*b)/c^6`

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 55, normalized size of antiderivative = 0.53

$$\int \frac{x^3(a+bx^2)}{\sqrt{-1+cx}\sqrt{1+cx}} dx = \frac{(3bc^4x^4 + 10ac^2 + (5ac^4 + 4bc^2)x^2 + 8b)\sqrt{cx+1}\sqrt{cx-1}}{15c^6}$$

input `integrate(x^3*(b*x^2+a)/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="fricas")`

output `1/15*(3*b*c^4*x^4 + 10*a*c^2 + (5*a*c^4 + 4*b*c^2)*x^2 + 8*b)*sqrt(c*x + 1)*sqrt(c*x - 1)/c^6`

Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 8.90 (sec) , antiderivative size = 216, normalized size of antiderivative = 2.10

$$\int \frac{x^3(a + bx^2)}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{aG_{6,6}^{6,2} \left(\begin{matrix} -\frac{5}{4}, -\frac{3}{4} & -1, -1, -\frac{1}{2}, 1 \\ -\frac{3}{2}, -\frac{5}{4}, -1, -\frac{3}{4}, -\frac{1}{2}, 0 \end{matrix} \middle| \frac{1}{c^2x^2} \right)}{4\pi^{\frac{3}{2}}c^4} + \frac{iaG_{6,6}^{2,6} \left(\begin{matrix} -2, -\frac{7}{4}, -\frac{3}{2}, -\frac{5}{4}, -1, 1 \\ -\frac{7}{4}, -\frac{5}{4} & -2, -\frac{3}{2}, -\frac{3}{2}, 0 \end{matrix} \middle| \frac{e^{2i\pi}}{c^2x^2} \right)}{4\pi^{\frac{3}{2}}c^4} + \frac{bG_{6,6}^{6,2} \left(\begin{matrix} -\frac{9}{4}, -\frac{7}{4} & -2, -2, -\frac{3}{2}, 1 \\ -\frac{5}{2}, -\frac{9}{4}, -2, -\frac{7}{4}, -\frac{3}{2}, 0 \end{matrix} \middle| \frac{1}{c^2x^2} \right)}{4\pi^{\frac{3}{2}}c^6} + \frac{ibG_{6,6}^{2,6} \left(\begin{matrix} -3, -\frac{11}{4}, -\frac{5}{2}, -\frac{9}{4}, -2, 1 \\ -\frac{11}{4}, -\frac{9}{4} & -3, -\frac{5}{2}, -\frac{5}{2}, 0 \end{matrix} \middle| \frac{e^{2i\pi}}{c^2x^2} \right)}{4\pi^{\frac{3}{2}}c^6}$$

input `integrate(x**3*(b*x**2+a)/(c*x-1)**(1/2)/(c*x+1)**(1/2),x)`

output `a*meijerg(((5/4, 3/4), (-1, -1, -1/2, 1)), ((-3/2, -5/4, -1, -3/4, -1/2, 0), ()), 1/(c**2*x**2))/(4*pi**(3/2)*c**4) + I*a*meijerg(((2, 7/4, 3/2, 5/4, 1, 1), ()), ((-7/4, -5/4), (-2, -3/2, -3/2, 0)), exp_polar(2*I*pi)/(c**2*x**2))/(4*pi**(3/2)*c**4) + b*meijerg(((9/4, 7/4), (-2, -2, -3/2, 1)), ((-5/2, -9/4, -2, -7/4, -3/2, 0), ()), 1/(c**2*x**2))/(4*pi**(3/2)*c**6) + I*b*meijerg(((3, 11/4, 5/2, 9/4, 2, 1), ()), ((-11/4, -9/4), (-3, -5/2, -5/2, 0)), exp_polar(2*I*pi)/(c**2*x**2))/(4*pi**(3/2)*c**6)`

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 95, normalized size of antiderivative = 0.92

$$\int \frac{x^3(a + bx^2)}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{\sqrt{c^2x^2 - 1}bx^4}{5c^2} + \frac{\sqrt{c^2x^2 - 1}ax^2}{3c^2} + \frac{4\sqrt{c^2x^2 - 1}bx^2}{15c^4} + \frac{2\sqrt{c^2x^2 - 1}a}{3c^4} + \frac{8\sqrt{c^2x^2 - 1}b}{15c^6}$$

input `integrate(x^3*(b*x^2+a)/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="maxima")`

output `1/5*sqrt(c^2*x^2 - 1)*b*x^4/c^2 + 1/3*sqrt(c^2*x^2 - 1)*a*x^2/c^2 + 4/15*sqrt(c^2*x^2 - 1)*b*x^2/c^4 + 2/3*sqrt(c^2*x^2 - 1)*a/c^4 + 8/15*sqrt(c^2*x^2 - 1)*b/c^6`

Giac [A] (verification not implemented)

Time = 0.13 (sec) , antiderivative size = 108, normalized size of antiderivative = 1.05

$$\int \frac{x^3(a + bx^2)}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{\left(\left((cx + 1) \left(3(cx + 1) \left(\frac{(cx+1)b}{c^5} - \frac{4b}{c^5} \right) + \frac{5ac^{27} + 22bc^{25}}{c^{30}} \right) - \frac{10(ac^{27} + 2bc^{25})}{c^{30}} \right) (cx + 1) + \frac{15(ac^{27} + bc^{25})}{c^{30}} \right) \sqrt{cx + 1} \sqrt{cx - 1}}{15c}$$

input `integrate(x^3*(b*x^2+a)/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="giac")`

output `1/15*(((c*x + 1)*(3*(c*x + 1)*((c*x + 1)*b/c^5 - 4*b/c^5) + (5*a*c^27 + 22*b*c^25)/c^30) - 10*(a*c^27 + 2*b*c^25)/c^30)*(c*x + 1) + 15*(a*c^27 + b*c^25)/c^30)*sqrt(c*x + 1)*sqrt(c*x - 1)/c`

Mupad [B] (verification not implemented)

Time = 4.64 (sec) , antiderivative size = 108, normalized size of antiderivative = 1.05

$$\int \frac{x^3(a + bx^2)}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx$$

$$= \frac{\sqrt{cx - 1} \left(\frac{10ac^2 + 8b}{15c^6} + \frac{bx^5}{5c} + \frac{bx^4}{5c^2} + \frac{x^2(5ac^4 + 4bc^2)}{15c^6} + \frac{x^3(5ac^5 + 4bc^3)}{15c^6} + \frac{x(10ac^3 + 8bc)}{15c^6} \right)}{\sqrt{cx + 1}}$$

input `int((x^3*(a + b*x^2))/((c*x - 1)^(1/2)*(c*x + 1)^(1/2)),x)`output `((c*x - 1)^(1/2)*((8*b + 10*a*c^2)/(15*c^6) + (b*x^5)/(5*c) + (b*x^4)/(5*c^2) + (x^2*(5*a*c^4 + 4*b*c^2))/(15*c^6) + (x^3*(5*a*c^5 + 4*b*c^3))/(15*c^6) + (x*(8*b*c + 10*a*c^3))/(15*c^6)))/(c*x + 1)^(1/2)`**Reduce [B] (verification not implemented)**

Time = 0.23 (sec) , antiderivative size = 54, normalized size of antiderivative = 0.52

$$\int \frac{x^3(a + bx^2)}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{\sqrt{cx + 1}\sqrt{cx - 1}(3bc^4x^4 + 5ac^4x^2 + 4bc^2x^2 + 10ac^2 + 8b)}{15c^6}$$

input `int(x^3*(b*x^2+a)/(c*x-1)^(1/2)/(c*x+1)^(1/2),x)`output `(sqrt(c*x + 1)*sqrt(c*x - 1)*(5*a*c**4*x**2 + 10*a*c**2 + 3*b*c**4*x**4 + 4*b*c**2*x**2 + 8*b))/(15*c**6)`

3.14 $\int \frac{x^2(a+bx^2)}{\sqrt{-1+cx}\sqrt{1+cx}} dx$

Optimal result	140
Mathematica [A] (warning: unable to verify)	140
Rubi [A] (verified)	141
Maple [A] (verified)	142
Fricas [A] (verification not implemented)	143
Sympy [F(-1)]	143
Maxima [A] (verification not implemented)	144
Giac [A] (verification not implemented)	144
Mupad [B] (verification not implemented)	145
Reduce [B] (verification not implemented)	145

Optimal result

Integrand size = 29, antiderivative size = 87

$$\int \frac{x^2(a+bx^2)}{\sqrt{-1+cx}\sqrt{1+cx}} dx = \frac{(3b+4ac^2)x\sqrt{-1+cx}\sqrt{1+cx}}{8c^4} + \frac{bx^3\sqrt{-1+cx}\sqrt{1+cx}}{4c^2} + \frac{(3b+4ac^2)\operatorname{arccosh}(cx)}{8c^5}$$

output `1/8*(4*a*c^2+3*b)*x*(c*x-1)^(1/2)*(c*x+1)^(1/2)/c^4+1/4*b*x^3*(c*x-1)^(1/2)*(c*x+1)^(1/2)/c^2+1/8*(4*a*c^2+3*b)*arccosh(c*x)/c^5`

Mathematica [A] (warning: unable to verify)

Time = 0.16 (sec) , antiderivative size = 77, normalized size of antiderivative = 0.89

$$\int \frac{x^2(a+bx^2)}{\sqrt{-1+cx}\sqrt{1+cx}} dx = \frac{cx\sqrt{-1+cx}\sqrt{1+cx}(4ac^2+b(3+2c^2x^2))+(6b+8ac^2)\operatorname{arctanh}\left(\sqrt{\frac{-1+cx}{1+cx}}\right)}{8c^5}$$

input `Integrate[(x^2*(a+b*x^2))/(Sqrt[-1+c*x]*Sqrt[1+c*x]),x]`

output

$$(c*x*\text{Sqrt}[-1 + c*x]*\text{Sqrt}[1 + c*x]*(4*a*c^2 + b*(3 + 2*c^2*x^2)) + (6*b + 8*a*c^2)*\text{ArcTanh}[\text{Sqrt}[(-1 + c*x)/(1 + c*x)])]/(8*c^5)$$
Rubi [A] (verified)

Time = 0.34 (sec) , antiderivative size = 82, normalized size of antiderivative = 0.94, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.103$, Rules used = {960, 101, 43}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{x^2(a + bx^2)}{\sqrt{cx - 1}\sqrt{cx + 1}} dx$$

$$\downarrow 960$$

$$\frac{1}{4} \left(4a + \frac{3b}{c^2} \right) \int \frac{x^2}{\sqrt{cx - 1}\sqrt{cx + 1}} dx + \frac{bx^3\sqrt{cx - 1}\sqrt{cx + 1}}{4c^2}$$

$$\downarrow 101$$

$$\frac{1}{4} \left(4a + \frac{3b}{c^2} \right) \left(\int \frac{1}{\sqrt{cx - 1}\sqrt{cx + 1}} dx + \frac{x\sqrt{cx - 1}\sqrt{cx + 1}}{2c^2} \right) + \frac{bx^3\sqrt{cx - 1}\sqrt{cx + 1}}{4c^2}$$

$$\downarrow 43$$

$$\frac{1}{4} \left(4a + \frac{3b}{c^2} \right) \left(\frac{\text{arccosh}(cx)}{2c^3} + \frac{x\sqrt{cx - 1}\sqrt{cx + 1}}{2c^2} \right) + \frac{bx^3\sqrt{cx - 1}\sqrt{cx + 1}}{4c^2}$$

input

$$\text{Int}[(x^2*(a + b*x^2))/(\text{Sqrt}[-1 + c*x]*\text{Sqrt}[1 + c*x]),x]$$

output

$$(b*x^3*\text{Sqrt}[-1 + c*x]*\text{Sqrt}[1 + c*x])/(4*c^2) + ((4*a + (3*b)/c^2)*((x*\text{Sqrt}[-1 + c*x]*\text{Sqrt}[1 + c*x])/(2*c^2) + \text{ArcCosh}[c*x]/(2*c^3)))/4$$

Definitions of rubi rules used

rule 43

```
Int[1/(Sqrt[(a_) + (b_)*(x_)]*Sqrt[(c_) + (d_)*(x_)]), x_Symbol] := Simp[
ArcCosh[b*(x/a)]/(b*Sqrt[d/b]), x] /; FreeQ[{a, b, c, d}, x] && EqQ[b*c + a
*d, 0] && GtQ[a, 0] && GtQ[d/b, 0]
```

rule 101

```
Int[((a_) + (b_)*(x_))2*((c_) + (d_)*(x_))(n_)*((e_) + (f_)*(x_))(
p_), x_] := Simp[b*(a + b*x)*(c + d*x)(n + 1)*((e + f*x)(p + 1)/(d*f*(n +
p + 3))), x] + Simp[1/(d*f*(n + p + 3)) Int[(c + d*x)n*(e + f*x)p*Simp
[a2*d*f*(n + p + 3) - b*(b*c*e + a*(d*e*(n + 1) + c*f*(p + 1))) + b*(a*d*f
*(n + p + 4) - b*(d*e*(n + 2) + c*f*(p + 2)))*x, x], x] /; FreeQ[{a, b,
c, d, e, f, n, p}, x] && NeQ[n + p + 3, 0]
```

rule 960

```
Int[((e_)*(x_))(m_)*((a1_) + (b1_)*(x_)(non2_))(p_)*((a2_) + (b2_)
*(x_)(non2_))(p_)*((c_) + (d_)*(x_)(n_)), x_Symbol] := Simp[d*(e*x)(
m + 1)*(a1 + b1*x(n/2))(p + 1)*((a2 + b2*x(n/2))(p + 1)/(b1*b2*e*(m + n
*(p + 1) + 1))), x] - Simp[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/
(b1*b2*(m + n*(p + 1) + 1)) Int[(e*x)m*(a1 + b1*x(n/2))p*(a2 + b2*x(n
/2))p, x], x] /; FreeQ[{a1, b1, a2, b2, c, d, e, m, n, p}, x] && EqQ[non2,
n/2] && EqQ[a2*b1 + a1*b2, 0] && NeQ[m + n*(p + 1) + 1, 0]
```

Maple [A] (verified)

Time = 0.12 (sec) , antiderivative size = 111, normalized size of antiderivative = 1.28

method	result
risch	$\frac{x(2bc^2x^2+4ac^2+3b)\sqrt{cx+1}\sqrt{cx-1}}{8c^4} + \frac{(4ac^2+3b)\ln\left(\frac{c^2x}{\sqrt{c^2}}+\sqrt{c^2x^2-1}\right)\sqrt{(cx+1)(cx-1)}}{8c^4\sqrt{c^2}\sqrt{cx-1}\sqrt{cx+1}}$
default	$\frac{\sqrt{cx-1}\sqrt{cx+1}\left(2\sqrt{c^2x^2-1}\operatorname{csgn}(c)c^3bx^3+4\sqrt{c^2x^2-1}\operatorname{csgn}(c)c^3ax+3\sqrt{c^2x^2-1}\operatorname{csgn}(c)cbx+4\ln\left(\left(\sqrt{c^2x^2-1}\operatorname{csgn}(c)+cx\right)\operatorname{csgn}(c)\right)\right)}{8c^5\sqrt{c^2x^2-1}}$

input

```
int(x2*(b*x2+a)/(c*x-1)(1/2)/(c*x+1)(1/2),x,method=_RETURNVERBOSE)
```

output

```
1/8*x*(2*b*c^2*x^2+4*a*c^2+3*b)*(c*x+1)^(1/2)*(c*x-1)^(1/2)/c^4+1/8*(4*a*c^2+3*b)/c^4*ln(c^2*x/(c^2)^(1/2)+(c^2*x^2-1)^(1/2))/(c^2)^(1/2)*((c*x+1)*(c*x-1))^(1/2)/(c*x-1)^(1/2)/(c*x+1)^(1/2)
```

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 77, normalized size of antiderivative = 0.89

$$\int \frac{x^2(a + bx^2)}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx$$

$$= \frac{(2bc^3x^3 + (4ac^3 + 3bc)x)\sqrt{cx + 1}\sqrt{cx - 1} - (4ac^2 + 3b)\log(-cx + \sqrt{cx + 1}\sqrt{cx - 1})}{8c^5}$$

input

```
integrate(x^2*(b*x^2+a)/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="fricas")
```

output

```
1/8*((2*b*c^3*x^3 + (4*a*c^3 + 3*b*c)*x)*sqrt(c*x + 1)*sqrt(c*x - 1) - (4*a*c^2 + 3*b)*log(-c*x + sqrt(c*x + 1)*sqrt(c*x - 1)))/c^5
```

Sympy [F(-1)]

Timed out.

$$\int \frac{x^2(a + bx^2)}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \text{Timed out}$$

input

```
integrate(x**2*(b*x**2+a)/(c*x-1)**(1/2)/(c*x+1)**(1/2),x)
```

output

```
Timed out
```


Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 113, normalized size of antiderivative = 1.30

$$\int \frac{x^2(a + bx^2)}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{\sqrt{c^2x^2 - 1}bx^3}{4c^2} + \frac{\sqrt{c^2x^2 - 1}ax}{2c^2} + \frac{a \log(2c^2x + 2\sqrt{c^2x^2 - 1}c)}{2c^3} \\ + \frac{3\sqrt{c^2x^2 - 1}bx}{8c^4} + \frac{3b \log(2c^2x + 2\sqrt{c^2x^2 - 1}c)}{8c^5}$$

input `integrate(x^2*(b*x^2+a)/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="maxima")`output `1/4*sqrt(c^2*x^2 - 1)*b*x^3/c^2 + 1/2*sqrt(c^2*x^2 - 1)*a*x/c^2 + 1/2*a*log(2*c^2*x + 2*sqrt(c^2*x^2 - 1)*c)/c^3 + 3/8*sqrt(c^2*x^2 - 1)*b*x/c^4 + 3/8*b*log(2*c^2*x + 2*sqrt(c^2*x^2 - 1)*c)/c^5`**Giac [A] (verification not implemented)**

Time = 0.13 (sec) , antiderivative size = 121, normalized size of antiderivative = 1.39

$$\int \frac{x^2(a + bx^2)}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx \\ = \frac{\left((cx + 1) \left(2(cx + 1) \left(\frac{(cx+1)b}{c^4} - \frac{3b}{c^4} \right) + \frac{4ac^{18}+9bc^{16}}{c^{20}} \right) - \frac{4ac^{18}+5bc^{16}}{c^{20}} \right) \sqrt{cx + 1} \sqrt{cx - 1} - \frac{2(4ac^2+3b) \log(\sqrt{cx+1}}{c^4}}{8c}$$

input `integrate(x^2*(b*x^2+a)/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="giac")`output `1/8*(((c*x + 1)*(2*(c*x + 1)*((c*x + 1)*b/c^4 - 3*b/c^4) + (4*a*c^18 + 9*b*c^16)/c^20) - (4*a*c^18 + 5*b*c^16)/c^20)*sqrt(c*x + 1)*sqrt(c*x - 1) - 2*(4*a*c^2 + 3*b)*log(sqrt(c*x + 1) - sqrt(c*x - 1))/c^4)/c`

Mupad [B] (verification not implemented)

Time = 26.01 (sec) , antiderivative size = 720, normalized size of antiderivative = 8.28

$$\int \frac{x^2(a + bx^2)}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \text{Too large to display}$$

input `int((x^2*(a + b*x^2))/((c*x - 1)^(1/2)*(c*x + 1)^(1/2)),x)`

output

```
((23*b*((c*x - 1)^(1/2) - 1i)^3)/(2*((c*x + 1)^(1/2) - 1)^3) + (333*b*((c*x - 1)^(1/2) - 1i)^5)/(2*((c*x + 1)^(1/2) - 1)^5) + (671*b*((c*x - 1)^(1/2) - 1i)^7)/(2*((c*x + 1)^(1/2) - 1)^7) + (671*b*((c*x - 1)^(1/2) - 1i)^9)/(2*((c*x + 1)^(1/2) - 1)^9) + (333*b*((c*x - 1)^(1/2) - 1i)^11)/(2*((c*x + 1)^(1/2) - 1)^11) + (23*b*((c*x - 1)^(1/2) - 1i)^13)/(2*((c*x + 1)^(1/2) - 1)^13) - (3*b*((c*x - 1)^(1/2) - 1i)^15)/(2*((c*x + 1)^(1/2) - 1)^15) - (3*b*((c*x - 1)^(1/2) - 1i))/(2*((c*x + 1)^(1/2) - 1))/(c^5 - (8*c^5*((c*x - 1)^(1/2) - 1i)^2)/((c*x + 1)^(1/2) - 1)^2 + (28*c^5*((c*x - 1)^(1/2) - 1i)^4)/((c*x + 1)^(1/2) - 1)^4 - (56*c^5*((c*x - 1)^(1/2) - 1i)^6)/((c*x + 1)^(1/2) - 1)^6 + (70*c^5*((c*x - 1)^(1/2) - 1i)^8)/((c*x + 1)^(1/2) - 1)^8 - (56*c^5*((c*x - 1)^(1/2) - 1i)^10)/((c*x + 1)^(1/2) - 1)^10 + (28*c^5*((c*x - 1)^(1/2) - 1i)^12)/((c*x + 1)^(1/2) - 1)^12 - (8*c^5*((c*x - 1)^(1/2) - 1i)^14)/((c*x + 1)^(1/2) - 1)^14 + (c^5*((c*x - 1)^(1/2) - 1i)^16)/((c*x + 1)^(1/2) - 1)^16) - ((14*a*((c*x - 1)^(1/2) - 1i)^3)/((c*x + 1)^(1/2) - 1)^3 + (14*a*((c*x - 1)^(1/2) - 1i)^5)/((c*x + 1)^(1/2) - 1)^5 + (2*a*((c*x - 1)^(1/2) - 1i)^7)/((c*x + 1)^(1/2) - 1)^7 + (2*a*((c*x - 1)^(1/2) - 1i))/((c*x + 1)^(1/2) - 1))/(c^3 - (4*c^3*((c*x - 1)^(1/2) - 1i)^2)/((c*x + 1)^(1/2) - 1)^2 + (6*c^3*((c*x - 1)^(1/2) - 1i)^4)/((c*x + 1)^(1/2) - 1)^4 - (4*c^3*((c*x - 1)^(1/2) - 1i)^6)/((c*x + 1)^(1/2) - 1)^6 + (c^3*((c*x - 1)^(1/2) - 1i)^8)/((c*x + 1)^(1/2) - 1)^8) + (2*a*atanh(((c*x - 1)^(1/2) - 1i)/((c*x + 1)^(1/2) - 1))))/c^5
```

Reduce [B] (verification not implemented)

Time = 0.21 (sec) , antiderivative size = 110, normalized size of antiderivative = 1.26

$$\int \frac{x^2(a + bx^2)}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx$$

$$= \frac{4\sqrt{cx + 1}\sqrt{cx - 1} a c^3 x + 2\sqrt{cx + 1}\sqrt{cx - 1} b c^3 x^3 + 3\sqrt{cx + 1}\sqrt{cx - 1} b c x + 8 \log\left(\frac{\sqrt{cx-1} + \sqrt{cx+1}}{\sqrt{2}}\right) a c^2}{8c^5}$$

input `int(x^2*(b*x^2+a)/(c*x-1)^(1/2)/(c*x+1)^(1/2),x)`

output `(4*sqrt(c*x + 1)*sqrt(c*x - 1)*a*c**3*x + 2*sqrt(c*x + 1)*sqrt(c*x - 1)*b*c**3*x**3 + 3*sqrt(c*x + 1)*sqrt(c*x - 1)*b*c*x + 8*log((sqrt(c*x - 1) + sqrt(c*x + 1))/sqrt(2))*a*c**2 + 6*log((sqrt(c*x - 1) + sqrt(c*x + 1))/sqrt(2))*b)/(8*c**5)`

3.15 $\int \frac{x(a+bx^2)}{\sqrt{-1+cx}\sqrt{1+cx}} dx$

Optimal result	147
Mathematica [A] (verified)	147
Rubi [A] (verified)	148
Maple [A] (verified)	149
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Reduce [B] (verification not implemented)	152

Optimal result

Integrand size = 27, antiderivative size = 65

$$\int \frac{x(a+bx^2)}{\sqrt{-1+cx}\sqrt{1+cx}} dx = \frac{(2b+3ac^2)\sqrt{-1+cx}\sqrt{1+cx}}{3c^4} + \frac{bx^2\sqrt{-1+cx}\sqrt{1+cx}}{3c^2}$$

output `1/3*(3*a*c^2+2*b)*(c*x-1)^(1/2)*(c*x+1)^(1/2)/c^4+1/3*b*x^2*(c*x-1)^(1/2)*(c*x+1)^(1/2)/c^2`

Mathematica [A] (verified)

Time = 0.08 (sec) , antiderivative size = 43, normalized size of antiderivative = 0.66

$$\int \frac{x(a+bx^2)}{\sqrt{-1+cx}\sqrt{1+cx}} dx = \frac{\sqrt{-1+cx}\sqrt{1+cx}(3ac^2+b(2+c^2x^2))}{3c^4}$$

input `Integrate[(x*(a + b*x^2))/(Sqrt[-1 + c*x]*Sqrt[1 + c*x]),x]`

output `(Sqrt[-1 + c*x]*Sqrt[1 + c*x]*(3*a*c^2 + b*(2 + c^2*x^2)))/(3*c^4)`

Rubi [A] (verified)

Time = 0.31 (sec) , antiderivative size = 65, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}} = 0.074$, Rules used = {960, 83}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{x(a + bx^2)}{\sqrt{cx - 1}\sqrt{cx + 1}} dx$$

$$\downarrow 960$$

$$\frac{1}{3} \left(3a + \frac{2b}{c^2} \right) \int \frac{x}{\sqrt{cx - 1}\sqrt{cx + 1}} dx + \frac{bx^2 \sqrt{cx - 1}\sqrt{cx + 1}}{3c^2}$$

$$\downarrow 83$$

$$\frac{\sqrt{cx - 1}\sqrt{cx + 1} \left(3a + \frac{2b}{c^2} \right)}{3c^2} + \frac{bx^2 \sqrt{cx - 1}\sqrt{cx + 1}}{3c^2}$$

input `Int[(x*(a + b*x^2))/(Sqrt[-1 + c*x]*Sqrt[1 + c*x]),x]`

output `((3*a + (2*b)/c^2)*Sqrt[-1 + c*x]*Sqrt[1 + c*x])/(3*c^2) + (b*x^2*Sqrt[-1 + c*x]*Sqrt[1 + c*x])/(3*c^2)`

Definitions of rubi rules used

rule 83 `Int[((a_.) + (b_.)*(x_))*((c_.) + (d_.)*(x_)^(n_.))*((e_.) + (f_.)*(x_)^(p_.), x_] := Simp[b*(c + d*x)^(n + 1)*((e + f*x)^(p + 1)/(d*f*(n + p + 2))), x] /; FreeQ[{a, b, c, d, e, f, n, p}, x] && NeQ[n + p + 2, 0] && EqQ[a*d*f*(n + p + 2) - b*(d*e*(n + 1) + c*f*(p + 1)), 0]`

rule 960 `Int[((e_.)*(x_))^(m_.)*((a1_.) + (b1_.)*(x_)^(non2_.))^(p_.)*((a2_.) + (b2_.)*(x_)^(non2_.))^(p_.)*((c_.) + (d_.)*(x_)^(n_.), x_Symbol] := Simp[d*(e*x)^(m + 1)*(a1 + b1*x^(n/2))^(p + 1)*((a2 + b2*x^(n/2))^(p + 1)/(b1*b2*e*(m + n*(p + 1) + 1))), x] - Simp[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(b1*b2*(m + n*(p + 1) + 1)) Int[(e*x)^m*(a1 + b1*x^(n/2))^p*(a2 + b2*x^(n/2))^p, x], x] /; FreeQ[{a1, b1, a2, b2, c, d, e, m, n, p}, x] && EqQ[non2, n/2] && EqQ[a2*b1 + a1*b2, 0] && NeQ[m + n*(p + 1) + 1, 0]`

Maple [A] (verified)

Time = 0.12 (sec) , antiderivative size = 38, normalized size of antiderivative = 0.58

method	result	size
gospers	$\frac{\sqrt{cx+1}\sqrt{cx-1}(bc^2x^2+3ac^2+2b)}{3c^4}$	38
default	$\frac{\sqrt{cx+1}\sqrt{cx-1}(bc^2x^2+3ac^2+2b)}{3c^4}$	38
risch	$\frac{\sqrt{cx+1}\sqrt{cx-1}(bc^2x^2+3ac^2+2b)}{3c^4}$	38
orering	$\frac{\sqrt{cx+1}\sqrt{cx-1}(bc^2x^2+3ac^2+2b)}{3c^4}$	38

input `int(x*(b*x^2+a)/(c*x-1)^(1/2)/(c*x+1)^(1/2),x,method=_RETURNVERBOSE)`

output `1/3*(c*x+1)^(1/2)*(c*x-1)^(1/2)*(b*c^2*x^2+3*a*c^2+2*b)/c^4`

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 37, normalized size of antiderivative = 0.57

$$\int \frac{x(a + bx^2)}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{(bc^2x^2 + 3ac^2 + 2b)\sqrt{cx + 1}\sqrt{cx - 1}}{3c^4}$$

input `integrate(x*(b*x^2+a)/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="fricas")`

output `1/3*(b*c^2*x^2 + 3*a*c^2 + 2*b)*sqrt(c*x + 1)*sqrt(c*x - 1)/c^4`

Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 6.54 (sec) , antiderivative size = 202, normalized size of antiderivative = 3.11

$$\int \frac{x(a + bx^2)}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{aG_{6,6}^{6,2} \left(\begin{matrix} -\frac{1}{4}, \frac{1}{4} & 0, 0, \frac{1}{2}, 1 \\ -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, 0 \end{matrix} \middle| \frac{1}{c^2x^2} \right)}{4\pi^{\frac{3}{2}}c^2} + \frac{iaG_{6,6}^{2,6} \left(\begin{matrix} -1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, 1 \\ -\frac{3}{4}, -\frac{1}{4} & -1, -\frac{1}{2}, -\frac{1}{2}, 0 \end{matrix} \middle| \frac{e^{2i\pi}}{c^2x^2} \right)}{4\pi^{\frac{3}{2}}c^2} + \frac{bG_{6,6}^{6,2} \left(\begin{matrix} -\frac{5}{4}, -\frac{3}{4} & -1, -1, -\frac{1}{2}, 1 \\ -\frac{3}{2}, -\frac{5}{4}, -1, -\frac{3}{4}, -\frac{1}{2}, 0 \end{matrix} \middle| \frac{1}{c^2x^2} \right)}{4\pi^{\frac{3}{2}}c^4} + \frac{ibG_{6,6}^{2,6} \left(\begin{matrix} -2, -\frac{7}{4}, -\frac{3}{2}, -\frac{5}{4}, -1, 1 \\ -\frac{7}{4}, -\frac{5}{4} & -2, -\frac{3}{2}, -\frac{3}{2}, 0 \end{matrix} \middle| \frac{e^{2i\pi}}{c^2x^2} \right)}{4\pi^{\frac{3}{2}}c^4}$$

input `integrate(x*(b*x**2+a)/(c*x-1)**(1/2)/(c*x+1)**(1/2),x)`

output

```
a*meijerg((-1/4, 1/4), (0, 0, 1/2, 1)), ((-1/2, -1/4, 0, 1/4, 1/2, 0), (
), 1/(c**2*x**2))/(4*pi**(3/2)*c**2) + I*a*meijerg((-1, -3/4, -1/2, -1/4,
0, 1), ()), ((-3/4, -1/4), (-1, -1/2, -1/2, 0)), exp_polar(2*I*pi)/(c**2*
x**2))/(4*pi**(3/2)*c**2) + b*meijerg((-5/4, -3/4), (-1, -1, -1/2, 1)), (
(-3/2, -5/4, -1, -3/4, -1/2, 0), ()), 1/(c**2*x**2))/(4*pi**(3/2)*c**4) +
I*b*meijerg((-2, -7/4, -3/2, -5/4, -1, 1), ()), ((-7/4, -5/4), (-2, -3/2,
-3/2, 0)), exp_polar(2*I*pi)/(c**2*x**2))/(4*pi**(3/2)*c**4)
```

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 54, normalized size of antiderivative = 0.83

$$\int \frac{x(a + bx^2)}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{\sqrt{c^2x^2 - 1}bx^2}{3c^2} + \frac{\sqrt{c^2x^2 - 1}a}{c^2} + \frac{2\sqrt{c^2x^2 - 1}b}{3c^4}$$

input

```
integrate(x*(b*x^2+a)/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="maxima")
```

output

```
1/3*sqrt(c^2*x^2 - 1)*b*x^2/c^2 + sqrt(c^2*x^2 - 1)*a/c^2 + 2/3*sqrt(c^2*x
^2 - 1)*b/c^4
```

Giac [A] (verification not implemented)

Time = 0.13 (sec) , antiderivative size = 59, normalized size of antiderivative = 0.91

$$\int \frac{x(a + bx^2)}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{\sqrt{cx + 1}\sqrt{cx - 1} \left((cx + 1) \left(\frac{(cx+1)b}{c^3} - \frac{2b}{c^3} \right) + \frac{3(ac^{11} + bc^9)}{c^{12}} \right)}{3c}$$

input

```
integrate(x*(b*x^2+a)/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="giac")
```

output

```
1/3*sqrt(c*x + 1)*sqrt(c*x - 1)*((c*x + 1)*((c*x + 1)*b/c^3 - 2*b/c^3) + 3
*(a*c^11 + b*c^9)/c^12)/c
```


Mupad [B] (verification not implemented)

Time = 4.71 (sec) , antiderivative size = 66, normalized size of antiderivative = 1.02

$$\int \frac{x(a + bx^2)}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{\sqrt{cx - 1} \left(\frac{3ac^2 + 2b}{3c^4} + \frac{bx^3}{3c} + \frac{bx^2}{3c^2} + \frac{x(3ac^3 + 2bc)}{3c^4} \right)}{\sqrt{cx + 1}}$$

input `int((x*(a + b*x^2))/((c*x - 1)^(1/2)*(c*x + 1)^(1/2)),x)`output `((c*x - 1)^(1/2)*((2*b + 3*a*c^2)/(3*c^4) + (b*x^3)/(3*c) + (b*x^2)/(3*c^2) + (x*(2*b*c + 3*a*c^3))/(3*c^4)))/(c*x + 1)^(1/2)`**Reduce [B] (verification not implemented)**

Time = 0.21 (sec) , antiderivative size = 35, normalized size of antiderivative = 0.54

$$\int \frac{x(a + bx^2)}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{\sqrt{cx + 1}\sqrt{cx - 1}(bc^2x^2 + 3ac^2 + 2b)}{3c^4}$$

input `int(x*(b*x^2+a)/(c*x-1)^(1/2)/(c*x+1)^(1/2),x)`output `(sqrt(c*x + 1)*sqrt(c*x - 1)*(3*a*c**2 + b*c**2*x**2 + 2*b))/(3*c**4)`

3.16 $\int \frac{a+bx^2}{\sqrt{-1+cx}\sqrt{1+cx}} dx$

Optimal result	153
Mathematica [A] (warning: unable to verify)	153
Rubi [A] (verified)	154
Maple [B] (verified)	155
Fricas [A] (verification not implemented)	155
Sympy [F(-1)]	156
Maxima [A] (verification not implemented)	156
Giac [A] (verification not implemented)	156
Mupad [B] (verification not implemented)	157
Reduce [B] (verification not implemented)	157

Optimal result

Integrand size = 26, antiderivative size = 47

$$\int \frac{a + bx^2}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{bx\sqrt{-1 + cx}\sqrt{1 + cx}}{2c^2} + \frac{(b + 2ac^2) \operatorname{arccosh}(cx)}{2c^3}$$

output `1/2*b*x*(c*x-1)^(1/2)*(c*x+1)^(1/2)/c^2+1/2*(2*a*c^2+b)*arccosh(c*x)/c^3`

Mathematica [A] (warning: unable to verify)

Time = 0.12 (sec) , antiderivative size = 58, normalized size of antiderivative = 1.23

$$\int \frac{a + bx^2}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{bcx\sqrt{-1 + cx}\sqrt{1 + cx} + 2(b + 2ac^2) \operatorname{arctanh}\left(\sqrt{\frac{-1+cx}{1+cx}}\right)}{2c^3}$$

input `Integrate[(a + b*x^2)/(Sqrt[-1 + c*x]*Sqrt[1 + c*x]),x]`

output `(b*c*x*Sqrt[-1 + c*x]*Sqrt[1 + c*x] + 2*(b + 2*a*c^2)*ArcTanh[Sqrt[(-1 + c*x)/(1 + c*x)]])/(2*c^3)`

Rubi [A] (verified)

Time = 0.27 (sec) , antiderivative size = 47, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}} = 0.077$, Rules used = {646, 43}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{a + bx^2}{\sqrt{cx - 1}\sqrt{cx + 1}} dx$$

$$\downarrow 646$$

$$\frac{(2ac^2 + b) \int \frac{1}{\sqrt{cx-1}\sqrt{cx+1}} dx}{2c^2} + \frac{bx\sqrt{cx - 1}\sqrt{cx + 1}}{2c^2}$$

$$\downarrow 43$$

$$\frac{(2ac^2 + b) \operatorname{arccosh}(cx)}{2c^3} + \frac{bx\sqrt{cx - 1}\sqrt{cx + 1}}{2c^2}$$

input `Int[(a + b*x^2)/(Sqrt[-1 + c*x]*Sqrt[1 + c*x]),x]`

output `(b*x*Sqrt[-1 + c*x]*Sqrt[1 + c*x])/(2*c^2) + ((b + 2*a*c^2)*ArcCosh[c*x])/(2*c^3)`

Defintions of rubi rules used

rule 43 `Int[1/(Sqrt[(a_) + (b_.)*(x_)]*Sqrt[(c_) + (d_.)*(x_)]), x_Symbol] :> Simp[ArcCosh[b*(x/a)]/(b*Sqrt[d/b]), x] /; FreeQ[{a, b, c, d}, x] && EqQ[b*c + a*d, 0] && GtQ[a, 0] && GtQ[d/b, 0]`

rule 646 `Int[((c_) + (d_.)*(x_))^(m_.)*((e_) + (f_.)*(x_))^(n_.)*((a_.) + (b_.)*(x_)^2), x_Symbol] :> Simp[b*x*(c + d*x)^(m + 1)*((e + f*x)^(n + 1)/(d*f*(2*m + 3))), x] - Simp[(b*c*e - a*d*f*(2*m + 3))/(d*f*(2*m + 3)) Int[(c + d*x)^m*(e + f*x)^n, x], x] /; FreeQ[{a, b, c, d, e, f, m, n}, x] && EqQ[m, n] && EqQ[d*e + c*f, 0] && !LtQ[m, -1]`

Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 90 vs. 2(39) = 78.

Time = 0.11 (sec) , antiderivative size = 91, normalized size of antiderivative = 1.94

method	result
risch	$\frac{bx\sqrt{cx-1}\sqrt{cx+1}}{2c^2} + \frac{(2ac^2+b)\ln\left(\frac{c^2x}{\sqrt{c^2}+\sqrt{c^2x^2-1}}\right)\sqrt{(cx+1)(cx-1)}}{2c^2\sqrt{c^2}\sqrt{cx-1}\sqrt{cx+1}}$
default	$\frac{\sqrt{cx-1}\sqrt{cx+1}\left(2\ln\left(\left(\sqrt{c^2x^2-1}\operatorname{csgn}(c)+cx\right)\operatorname{csgn}(c)\right)ac^2+\sqrt{c^2x^2-1}\operatorname{csgn}(c)cbx+\ln\left(\left(\sqrt{c^2x^2-1}\operatorname{csgn}(c)+cx\right)\operatorname{csgn}(c)\right)b\right)\operatorname{csgn}(c)}{2c^3\sqrt{c^2x^2-1}}$

input `int((b*x^2+a)/(c*x-1)^(1/2)/(c*x+1)^(1/2),x,method=_RETURNVERBOSE)`

output
$$\frac{1/2*b*x*(c*x-1)^{(1/2)*(c*x+1)^{(1/2)/c^2+1/2*(2*a*c^2+b)/c^2*\ln(c^2*x/(c^2)^{(1/2)+(c^2*x^2-1)^{(1/2)})/(c^2)^{(1/2)*((c*x+1)*(c*x-1))^{(1/2)/(c*x-1)^{(1/2)/(c*x+1)^{(1/2)}}$$

Fricas [A] (verification not implemented)

Time = 0.12 (sec) , antiderivative size = 55, normalized size of antiderivative = 1.17

$$\int \frac{a + bx^2}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx$$

$$= \frac{\sqrt{cx+1}\sqrt{cx-1}bcx - (2ac^2 + b)\log(-cx + \sqrt{cx+1}\sqrt{cx-1})}{2c^3}$$

input `integrate((b*x^2+a)/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="fricas")`

output
$$\frac{1/2*(\sqrt{cx+1}*\sqrt{cx-1}*b*c*x - (2*a*c^2 + b)*\log(-c*x + \sqrt{cx+1}*\sqrt{cx-1}))/c^3}$$

Sympy [F(-1)]

Timed out.

$$\int \frac{a + bx^2}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \text{Timed out}$$

input `integrate((b*x**2+a)/(c*x-1)**(1/2)/(c*x+1)**(1/2),x)`

output `Timed out`

Maxima [A] (verification not implemented)

Time = 0.04 (sec) , antiderivative size = 74, normalized size of antiderivative = 1.57

$$\int \frac{a + bx^2}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{a \log(2c^2x + 2\sqrt{c^2x^2 - 1}c)}{c} + \frac{\sqrt{c^2x^2 - 1}bx}{2c^2} + \frac{b \log(2c^2x + 2\sqrt{c^2x^2 - 1}c)}{2c^3}$$

input `integrate((b*x^2+a)/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="maxima")`

output `a*log(2*c^2*x + 2*sqrt(c^2*x^2 - 1)*c)/c + 1/2*sqrt(c^2*x^2 - 1)*b*x/c^2 + 1/2*b*log(2*c^2*x + 2*sqrt(c^2*x^2 - 1)*c)/c^3`

Giac [A] (verification not implemented)

Time = 0.14 (sec) , antiderivative size = 69, normalized size of antiderivative = 1.47

$$\int \frac{a + bx^2}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{\sqrt{cx + 1}\sqrt{cx - 1} \left(\frac{(cx+1)b}{c^2} - \frac{b}{c^2} \right) - \frac{2(2ac^2+b) \log(\sqrt{cx+1}-\sqrt{cx-1})}{c^2}}{2c}$$

input `integrate((b*x^2+a)/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="giac")`

output

```
1/2*(sqrt(c*x + 1)*sqrt(c*x - 1)*((c*x + 1)*b/c^2 - b/c^2) - 2*(2*a*c^2 +
b)*log(sqrt(c*x + 1) - sqrt(c*x - 1))/c^2)/c
```

Mupad [B] (verification not implemented)

Time = 15.84 (sec) , antiderivative size = 293, normalized size of antiderivative = 6.23

$$\int \frac{a + bx^2}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx$$

$$= -\frac{\frac{14b(\sqrt{cx-1-i})^3}{(\sqrt{cx+1-1})^3} + \frac{14b(\sqrt{cx-1-i})^5}{(\sqrt{cx+1-1})^5} + \frac{2b(\sqrt{cx-1-i})^7}{(\sqrt{cx+1-1})^7} + \frac{2b(\sqrt{cx-1-i})}{\sqrt{cx+1-1}}}{c^3 - \frac{4c^3(\sqrt{cx-1-i})^2}{(\sqrt{cx+1-1})^2} + \frac{6c^3(\sqrt{cx-1-i})^4}{(\sqrt{cx+1-1})^4} - \frac{4c^3(\sqrt{cx-1-i})^6}{(\sqrt{cx+1-1})^6} + \frac{c^3(\sqrt{cx-1-i})^8}{(\sqrt{cx+1-1})^8}}$$

$$+ \frac{2b \operatorname{atanh}\left(\frac{\sqrt{cx-1-i}}{\sqrt{cx+1-1}}\right)}{c^3} - \frac{4a \operatorname{atan}\left(\frac{c(\sqrt{cx-1-i})}{(\sqrt{cx+1-1})\sqrt{-c^2}}\right)}{\sqrt{-c^2}}$$

input

```
int((a + b*x^2)/((c*x - 1)^(1/2)*(c*x + 1)^(1/2)),x)
```

output

```
(2*b*atanh(((c*x - 1)^(1/2) - 1i)/((c*x + 1)^(1/2) - 1)))/c^3 - ((14*b*((c
*x - 1)^(1/2) - 1i)^3)/((c*x + 1)^(1/2) - 1)^3 + (14*b*((c*x - 1)^(1/2) -
1i)^5)/((c*x + 1)^(1/2) - 1)^5 + (2*b*((c*x - 1)^(1/2) - 1i)^7)/((c*x + 1)
^(1/2) - 1)^7 + (2*b*((c*x - 1)^(1/2) - 1i))/((c*x + 1)^(1/2) - 1))/c^3 -
(4*c^3*((c*x - 1)^(1/2) - 1i)^2)/((c*x + 1)^(1/2) - 1)^2 + (6*c^3*((c*x -
1)^(1/2) - 1i)^4)/((c*x + 1)^(1/2) - 1)^4 - (4*c^3*((c*x - 1)^(1/2) - 1i)
^6)/((c*x + 1)^(1/2) - 1)^6 + (c^3*((c*x - 1)^(1/2) - 1i)^8)/((c*x + 1)^(1
/2) - 1)^8 - (4*a*atan((c*((c*x - 1)^(1/2) - 1i))/((c*x + 1)^(1/2) - 1)*
(-c^2)^(1/2)))/(-c^2)^(1/2))
```

Reduce [B] (verification not implemented)

Time = 0.23 (sec) , antiderivative size = 69, normalized size of antiderivative = 1.47

$$\int \frac{a + bx^2}{\sqrt{-1 + cx}\sqrt{1 + cx}} dx$$

$$= \frac{\sqrt{cx + 1}\sqrt{cx - 1}bcx + 4 \log\left(\frac{\sqrt{cx-1} + \sqrt{cx+1}}{\sqrt{2}}\right) a c^2 + 2 \log\left(\frac{\sqrt{cx-1} + \sqrt{cx+1}}{\sqrt{2}}\right) b}{2c^3}$$

input `int((b*x^2+a)/(c*x-1)^(1/2)/(c*x+1)^(1/2),x)`

output `(sqrt(c*x + 1)*sqrt(c*x - 1)*b*c*x + 4*log((sqrt(c*x - 1) + sqrt(c*x + 1))
/sqrt(2))*a*c**2 + 2*log((sqrt(c*x - 1) + sqrt(c*x + 1))/sqrt(2))*b)/(2*c*
*3)`

3.17 $\int \frac{a+bx^2}{x\sqrt{-1+cx}\sqrt{1+cx}} dx$

Optimal result	159
Mathematica [A] (warning: unable to verify)	159
Rubi [A] (verified)	160
Maple [A] (verified)	161
Fricas [A] (verification not implemented)	161
Sympy [C] (verification not implemented)	162
Maxima [A] (verification not implemented)	163
Giac [A] (verification not implemented)	163
Mupad [B] (verification not implemented)	163
Reduce [B] (verification not implemented)	164

Optimal result

Integrand size = 29, antiderivative size = 46

$$\int \frac{a + bx^2}{x\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{b\sqrt{-1 + cx}\sqrt{1 + cx}}{c^2} + a \arctan \left(\sqrt{-1 + cx}\sqrt{1 + cx} \right)$$

output `b*(c*x-1)^(1/2)*(c*x+1)^(1/2)/c^2+a*arctan((c*x-1)^(1/2)*(c*x+1)^(1/2))`

Mathematica [A] (warning: unable to verify)

Time = 0.09 (sec) , antiderivative size = 45, normalized size of antiderivative = 0.98

$$\int \frac{a + bx^2}{x\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{b\sqrt{-1 + cx}\sqrt{1 + cx}}{c^2} + 2a \arctan \left(\sqrt{\frac{-1 + cx}{1 + cx}} \right)$$

input `Integrate[(a + b*x^2)/(x*Sqrt[-1 + c*x]*Sqrt[1 + c*x]),x]`

output `(b*Sqrt[-1 + c*x]*Sqrt[1 + c*x])/c^2 + 2*a*ArcTan[Sqrt[(-1 + c*x)/(1 + c*x)]]`

Rubi [A] (verified)

Time = 0.32 (sec) , antiderivative size = 46, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.103$, Rules used = {960, 103, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{a + bx^2}{x\sqrt{cx - 1}\sqrt{cx + 1}} dx$$

$$\downarrow 960$$

$$a \int \frac{1}{x\sqrt{cx - 1}\sqrt{cx + 1}} dx + \frac{b\sqrt{cx - 1}\sqrt{cx + 1}}{c^2}$$

$$\downarrow 103$$

$$ac \int \frac{1}{(cx - 1)(cx + 1)c + c} d(\sqrt{cx - 1}\sqrt{cx + 1}) + \frac{b\sqrt{cx - 1}\sqrt{cx + 1}}{c^2}$$

$$\downarrow 218$$

$$a \arctan(\sqrt{cx - 1}\sqrt{cx + 1}) + \frac{b\sqrt{cx - 1}\sqrt{cx + 1}}{c^2}$$

input `Int[(a + b*x^2)/(x*Sqrt[-1 + c*x]*Sqrt[1 + c*x]),x]`

output `(b*Sqrt[-1 + c*x]*Sqrt[1 + c*x])/c^2 + a*ArcTan[Sqrt[-1 + c*x]*Sqrt[1 + c*x]]`

Defintions of rubi rules used

rule 103

```
Int[1/(Sqrt[(a_.) + (b_.)*(x_)]*Sqrt[(c_.) + (d_.)*(x_)]*((e_.) + (f_.)*(x_))), x_] := Simp[b*f Subst[Int[1/(d*(b*e - a*f)^2 + b*f^2*x^2), x], x, Sqrt[a + b*x]*Sqrt[c + d*x]], x] /; FreeQ[{a, b, c, d, e, f}, x] && EqQ[2*b*d*e - f*(b*c + a*d), 0]
```

rule 218 `Int[((a_) + (b_)*(x_)^2)^(-1), x_Symbol] := Simp[(Rt[a/b, 2]/a)*ArcTan[x/Rt[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]`

rule 960 `Int[((e_)*(x_))^(m_)*((a1_) + (b1_)*(x_)^(non2_))^(p_)*((a2_) + (b2_)*(x_)^(non2_))^(p_)*((c_) + (d_)*(x_)^(n_)), x_Symbol] := Simp[d*(e*x)^(m + 1)*(a1 + b1*x^(n/2))^(p + 1)*((a2 + b2*x^(n/2))^(p + 1)/(b1*b2*e*(m + n*(p + 1) + 1))), x] - Simp[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(b1*b2*(m + n*(p + 1) + 1)) Int[(e*x)^m*(a1 + b1*x^(n/2))^p*(a2 + b2*x^(n/2))^p, x], x] /; FreeQ[{a1, b1, a2, b2, c, d, e, m, n, p}, x] && EqQ[non2, n/2] && EqQ[a2*b1 + a1*b2, 0] && NeQ[m + n*(p + 1) + 1, 0]`

Maple [A] (verified)

Time = 0.11 (sec) , antiderivative size = 62, normalized size of antiderivative = 1.35

method	result	size
default	$\frac{\left(-\arctan\left(\frac{1}{\sqrt{c^2x^2-1}}\right)ac^2+b\sqrt{c^2x^2-1}\right)\sqrt{cx-1}\sqrt{cx+1}}{\sqrt{c^2x^2-1}c^2}$	62

input `int((b*x^2+a)/x/(c*x-1)^(1/2)/(c*x+1)^(1/2),x,method=_RETURNVERBOSE)`

output `(-arctan(1/(c^2*x^2-1)^(1/2))*a*c^2+b*(c^2*x^2-1)^(1/2))*(c*x-1)^(1/2)*(c*x+1)^(1/2)/(c^2*x^2-1)^(1/2)/c^2`

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 48, normalized size of antiderivative = 1.04

$$\int \frac{a + bx^2}{x\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{2ac^2 \arctan(-cx + \sqrt{cx + 1}\sqrt{cx - 1}) + \sqrt{cx + 1}\sqrt{cx - 1}b}{c^2}$$

input `integrate((b*x^2+a)/x/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="fricas")`

output

```
(2*a*c^2*arctan(-c*x + sqrt(c*x + 1)*sqrt(c*x - 1)) + sqrt(c*x + 1)*sqrt(c*x - 1)*b)/c^2
```

Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 16.30 (sec) , antiderivative size = 162, normalized size of antiderivative = 3.52

$$\int \frac{a + bx^2}{x\sqrt{-1 + cx}\sqrt{1 + cx}} dx = -\frac{aG_{6,6}^{5,3} \left(\begin{matrix} \frac{3}{4}, \frac{5}{4}, 1 & 1, 1, \frac{3}{2} \\ \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2} & 0 \end{matrix} \middle| \frac{1}{c^2 x^2} \right)}{4\pi^{\frac{3}{2}}} + \frac{iaG_{6,6}^{2,6} \left(\begin{matrix} 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 1 \\ \frac{1}{4}, \frac{3}{4} & 0, \frac{1}{2}, \frac{1}{2}, 0 \end{matrix} \middle| \frac{e^{2i\pi}}{c^2 x^2} \right)}{4\pi^{\frac{3}{2}}} + \frac{bG_{6,6}^{6,2} \left(\begin{matrix} -\frac{1}{4}, \frac{1}{4} & 0, 0, \frac{1}{2}, 1 \\ -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, 0 \end{matrix} \middle| \frac{1}{c^2 x^2} \right)}{4\pi^{\frac{3}{2}} c^2} + \frac{ibG_{6,6}^{2,6} \left(\begin{matrix} -1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, 1 \\ -\frac{3}{4}, -\frac{1}{4} & -1, -\frac{1}{2}, -\frac{1}{2}, 0 \end{matrix} \middle| \frac{e^{2i\pi}}{c^2 x^2} \right)}{4\pi^{\frac{3}{2}} c^2}$$

input

```
integrate((b*x**2+a)/x/(c*x-1)**(1/2)/(c*x+1)**(1/2),x)
```

output

```
-a*meijerg(((3/4, 5/4, 1), (1, 1, 3/2)), ((1/2, 3/4, 1, 5/4, 3/2), (0,)), 1/(c**2*x**2))/(4*pi**(3/2)) + I*a*meijerg(((0, 1/4, 1/2, 3/4, 1, 1), ((1/4, 3/4), (0, 1/2, 1/2, 0))), exp_polar(2*I*pi)/(c**2*x**2))/(4*pi**(3/2)) + b*meijerg((-1/4, 1/4), (0, 0, 1/2, 1)), ((-1/2, -1/4, 0, 1/4, 1/2, 0), ()), 1/(c**2*x**2))/(4*pi**(3/2)*c**2) + I*b*meijerg((-1, -3/4, -1/2, -1/4, 0, 1), ((-3/4, -1/4), (-1, -1/2, -1/2, 0))), exp_polar(2*I*pi)/(c**2*x**2))/(4*pi**(3/2)*c**2)
```

Maxima [A] (verification not implemented)

Time = 0.11 (sec) , antiderivative size = 29, normalized size of antiderivative = 0.63

$$\int \frac{a + bx^2}{x\sqrt{-1 + cx}\sqrt{1 + cx}} dx = -a \arcsin\left(\frac{1}{c|x|}\right) + \frac{\sqrt{c^2x^2 - 1}b}{c^2}$$

input `integrate((b*x^2+a)/x/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="maxima")`output `-a*arcsin(1/(c*abs(x))) + sqrt(c^2*x^2 - 1)*b/c^2`**Giac [A] (verification not implemented)**

Time = 0.13 (sec) , antiderivative size = 45, normalized size of antiderivative = 0.98

$$\int \frac{a + bx^2}{x\sqrt{-1 + cx}\sqrt{1 + cx}} dx = -2a \arctan\left(\frac{1}{2}(\sqrt{cx+1} - \sqrt{cx-1})\right)^2 + \frac{\sqrt{cx+1}\sqrt{cx-1}b}{c^2}$$

input `integrate((b*x^2+a)/x/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="giac")`output `-2*a*arctan(1/2*(sqrt(c*x + 1) - sqrt(c*x - 1))^2) + sqrt(c*x + 1)*sqrt(c*x - 1)*b/c^2`**Mupad [B] (verification not implemented)**

Time = 6.02 (sec) , antiderivative size = 77, normalized size of antiderivative = 1.67

$$\int \frac{a + bx^2}{x\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{b\sqrt{cx-1}\sqrt{cx+1}}{c^2} - a \left(\ln\left(\frac{(\sqrt{cx-1}-i)^2}{(\sqrt{cx+1}-1)^2} + 1\right) - \ln\left(\frac{\sqrt{cx-1}-i}{\sqrt{cx+1}-1}\right) \right) \text{li}$$

input `int((a + b*x^2)/(x*(c*x - 1)^(1/2)*(c*x + 1)^(1/2)),x)`

output `(b*(c*x - 1)^(1/2)*(c*x + 1)^(1/2))/c^2 - a*(log(((c*x - 1)^(1/2) - 1i)^2/((c*x + 1)^(1/2) - 1)^2 + 1) - log(((c*x - 1)^(1/2) - 1i)/((c*x + 1)^(1/2) - 1)))*1i`

Reduce [B] (verification not implemented)

Time = 0.22 (sec) , antiderivative size = 61, normalized size of antiderivative = 1.33

$$\int \frac{a + bx^2}{x\sqrt{-1 + cx}\sqrt{1 + cx}} dx$$

$$= \frac{2 \operatorname{atan}(\sqrt{cx - 1} + \sqrt{cx + 1} - 1) a c^2 - 2 \operatorname{atan}(\sqrt{cx - 1} + \sqrt{cx + 1} + 1) a c^2 + \sqrt{cx + 1} \sqrt{cx - 1} b}{c^2}$$

input `int((b*x^2+a)/x/(c*x-1)^(1/2)/(c*x+1)^(1/2),x)`

output `(2*atan(sqrt(c*x - 1) + sqrt(c*x + 1) - 1)*a*c**2 - 2*atan(sqrt(c*x - 1) + sqrt(c*x + 1) + 1)*a*c**2 + sqrt(c*x + 1)*sqrt(c*x - 1)*b)/c**2`

3.18 $\int \frac{a+bx^2}{x^2\sqrt{-1+cx}\sqrt{1+cx}} dx$

Optimal result	165
Mathematica [A] (warning: unable to verify)	165
Rubi [A] (verified)	166
Maple [C] (verified)	167
Fricas [A] (verification not implemented)	167
Sympy [C] (verification not implemented)	168
Maxima [A] (verification not implemented)	169
Giac [A] (verification not implemented)	169
Mupad [B] (verification not implemented)	169
Reduce [B] (verification not implemented)	170

Optimal result

Integrand size = 29, antiderivative size = 33

$$\int \frac{a+bx^2}{x^2\sqrt{-1+cx}\sqrt{1+cx}} dx = \frac{a\sqrt{-1+cx}\sqrt{1+cx}}{x} + \frac{\operatorname{barccosh}(cx)}{c}$$

output `a*(c*x-1)^(1/2)*(c*x+1)^(1/2)/x+b*arccosh(c*x)/c`

Mathematica [A] (warning: unable to verify)

Time = 0.09 (sec) , antiderivative size = 48, normalized size of antiderivative = 1.45

$$\int \frac{a+bx^2}{x^2\sqrt{-1+cx}\sqrt{1+cx}} dx = \frac{a\sqrt{-1+cx}\sqrt{1+cx}}{x} + \frac{2\operatorname{barctanh}\left(\sqrt{\frac{-1+cx}{1+cx}}\right)}{c}$$

input `Integrate[(a + b*x^2)/(x^2*Sqrt[-1 + c*x]*Sqrt[1 + c*x]),x]`

output `(a*Sqrt[-1 + c*x]*Sqrt[1 + c*x])/x + (2*b*ArcTanh[Sqrt[(-1 + c*x)/(1 + c*x)]])/c`

Rubi [A] (verified)

Time = 0.30 (sec) , antiderivative size = 33, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}} = 0.069$, Rules used = {956, 43}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{a + bx^2}{x^2 \sqrt{cx - 1} \sqrt{cx + 1}} dx$$

$$\downarrow 956$$

$$b \int \frac{1}{\sqrt{cx - 1} \sqrt{cx + 1}} dx + \frac{a \sqrt{cx - 1} \sqrt{cx + 1}}{x}$$

$$\downarrow 43$$

$$\frac{a \sqrt{cx - 1} \sqrt{cx + 1}}{x} + \frac{\text{barccosh}(cx)}{c}$$

input `Int[(a + b*x^2)/(x^2*Sqrt[-1 + c*x]*Sqrt[1 + c*x]),x]`

output `(a*Sqrt[-1 + c*x]*Sqrt[1 + c*x])/x + (b*ArcCosh[c*x])/c`

Defintions of rubi rules used

rule 43 `Int[1/(Sqrt[(a_) + (b_.)*(x_)]*Sqrt[(c_) + (d_.)*(x_)]), x_Symbol] :> Simp[ArcCosh[b*(x/a)]/(b*Sqrt[d/b]), x] /; FreeQ[{a, b, c, d}, x] && EqQ[b*c + a*d, 0] && GtQ[a, 0] && GtQ[d/b, 0]`

rule 956

```

Int[((e._)*(x._))^(m._)*((a1_) + (b1._)*(x._)^(non2._))^(p._)*((a2_) + (b2._)
*(x._)^(non2._))^(p._)*((c_) + (d._)*(x._)^(n_)), x_Symbol] := Simp[c*(e*x)^(
m + 1)*(a1 + b1*x^(n/2))^(p + 1)*((a2 + b2*x^(n/2))^(p + 1)/(a1*a2*e*(m + 1
))), x] + Simp[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(a1*a2*e^n*(
m + 1)) Int[(e*x)^(m + n)*(a1 + b1*x^(n/2))^p*(a2 + b2*x^(n/2))^p, x], x]
/; FreeQ[{a1, b1, a2, b2, c, d, e, p}, x] && EqQ[non2, n/2] && EqQ[a2*b1 +
a1*b2, 0] && (IntegerQ[n] || GtQ[e, 0]) && ((GtQ[n, 0] && LtQ[m, -1]) || (
LtQ[n, 0] && GtQ[m + n, -1])) && !ILtQ[p, -1]

```

Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.12 (sec) , antiderivative size = 77, normalized size of antiderivative = 2.33

method	result	size
default	$\frac{\sqrt{cx-1}\sqrt{cx+1}\left(\operatorname{csgn}(c)c\sqrt{c^2x^2-1}a+\ln\left(\left(\sqrt{c^2x^2-1}\operatorname{csgn}(c)+cx\right)\operatorname{csgn}(c)\right)bx\right)\operatorname{csgn}(c)}{\sqrt{c^2x^2-1}xc}$	77
risch	$\frac{a\sqrt{cx-1}\sqrt{cx+1}}{x} + \frac{b\ln\left(\frac{c^2x}{\sqrt{c^2}+\sqrt{c^2x^2-1}}\right)\sqrt{(cx+1)(cx-1)}}{\sqrt{c^2}\sqrt{cx-1}\sqrt{cx+1}}$	78

input

```
int((b*x^2+a)/x^2/(c*x-1)^(1/2)/(c*x+1)^(1/2),x,method=_RETURNVERBOSE)
```

output

```
(c*x-1)^(1/2)*(c*x+1)^(1/2)*(csgn(c)*c*(c^2*x^2-1)^(1/2)*a+ln(((c^2*x^2-1)
^(1/2)*csgn(c)+c*x)*csgn(c))*b*x)*csgn(c)/(c^2*x^2-1)^(1/2)/x/c
```

Fricas [A] (verification not implemented)

Time = 0.11 (sec) , antiderivative size = 56, normalized size of antiderivative = 1.70

$$\int \frac{a + bx^2}{x^2\sqrt{-1 + cx}\sqrt{1 + cx}} dx$$

$$= \frac{ac^2x + \sqrt{cx + 1}\sqrt{cx - 1}ac - bx \log(-cx + \sqrt{cx + 1}\sqrt{cx - 1})}{cx}$$

input

```
integrate((b*x^2+a)/x^2/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="fricas")
```


output

```
(a*c^2*x + sqrt(c*x + 1)*sqrt(c*x - 1)*a*c - b*x*log(-c*x + sqrt(c*x + 1)*
sqrt(c*x - 1)))/(c*x)
```

Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 15.64 (sec) , antiderivative size = 148, normalized size of antiderivative = 4.48

$$\int \frac{a + bx^2}{x^2 \sqrt{-1 + cx} \sqrt{1 + cx}} dx = - \frac{acG_{6,6}^{5,3} \left(\begin{matrix} \frac{5}{4}, \frac{7}{4}, 1 \\ 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2 \end{matrix} \middle| \frac{1}{c^2 x^2} \right)}{4\pi^{\frac{3}{2}}} - \frac{iacG_{6,6}^{2,6} \left(\begin{matrix} \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, 1 \\ \frac{3}{4}, \frac{5}{4} \end{matrix} \middle| \frac{e^{2i\pi}}{c^2 x^2} \right)}{4\pi^{\frac{3}{2}}} + \frac{bG_{6,6}^{6,2} \left(\begin{matrix} \frac{1}{4}, \frac{3}{4} \\ 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 0 \end{matrix} \middle| \frac{1}{c^2 x^2} \right)}{4\pi^{\frac{3}{2}} c} - \frac{ibG_{6,6}^{2,6} \left(\begin{matrix} -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, 1 \\ -\frac{1}{4}, \frac{1}{4} \end{matrix} \middle| \frac{e^{2i\pi}}{c^2 x^2} \right)}{4\pi^{\frac{3}{2}} c}$$

input

```
integrate((b*x**2+a)/x**2/(c*x-1)**(1/2)/(c*x+1)**(1/2),x)
```

output

```
-a*c*meijerg(((5/4, 7/4, 1), (3/2, 3/2, 2)), ((1, 5/4, 3/2, 7/4, 2), (0,))
, 1/(c**2*x**2))/(4*pi**(3/2)) - I*a*c*meijerg(((1/2, 3/4, 1, 5/4, 3/2, 1)
, ()), ((3/4, 5/4), (1/2, 1, 1, 0)), exp_polar(2*I*pi)/(c**2*x**2))/(4*pi*
*(3/2)) + b*meijerg(((1/4, 3/4), (1/2, 1/2, 1, 1)), ((0, 1/4, 1/2, 3/4, 1,
0), ()), 1/(c**2*x**2))/(4*pi**(3/2)*c) - I*b*meijerg((-1/2, -1/4, 0, 1/
4, 1/2, 1), ()), ((-1/4, 1/4), (-1/2, 0, 0, 0)), exp_polar(2*I*pi)/(c**2*x
**2))/(4*pi**(3/2)*c)
```

Maxima [A] (verification not implemented)

Time = 0.11 (sec) , antiderivative size = 44, normalized size of antiderivative = 1.33

$$\int \frac{a + bx^2}{x^2\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{b \log(2c^2x + 2\sqrt{c^2x^2 - 1}c)}{c} + \frac{\sqrt{c^2x^2 - 1}a}{x}$$

input `integrate((b*x^2+a)/x^2/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="maxima")`output `b*log(2*c^2*x + 2*sqrt(c^2*x^2 - 1)*c)/c + sqrt(c^2*x^2 - 1)*a/x`**Giac [A] (verification not implemented)**

Time = 0.13 (sec) , antiderivative size = 56, normalized size of antiderivative = 1.70

$$\int \frac{a + bx^2}{x^2\sqrt{-1 + cx}\sqrt{1 + cx}} dx$$

$$= \frac{1}{2}c \left(\frac{16a}{(\sqrt{cx+1} - \sqrt{cx-1})^4 + 4} - \frac{b \log((\sqrt{cx+1} - \sqrt{cx-1})^4)}{c^2} \right)$$

input `integrate((b*x^2+a)/x^2/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="giac")`output `1/2*c*(16*a/((sqrt(c*x + 1) - sqrt(c*x - 1))^4 + 4) - b*log((sqrt(c*x + 1) - sqrt(c*x - 1))^4)/c^2)`**Mupad [B] (verification not implemented)**

Time = 5.00 (sec) , antiderivative size = 61, normalized size of antiderivative = 1.85

$$\int \frac{a + bx^2}{x^2\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{a\sqrt{cx-1}\sqrt{cx+1}}{x} - \frac{4b \operatorname{atan}\left(\frac{c(\sqrt{cx-1}-i)}{(\sqrt{cx+1}-1)\sqrt{-c^2}}\right)}{\sqrt{-c^2}}$$

input `int((a + b*x^2)/(x^2*(c*x - 1)^(1/2)*(c*x + 1)^(1/2)),x)`

output

```
(a*(c*x - 1)^(1/2)*(c*x + 1)^(1/2))/x - (4*b*atan((c*((c*x - 1)^(1/2) - 1i)))/(((c*x + 1)^(1/2) - 1)*(-c^2)^(1/2)))/(-c^2)^(1/2)
```

Reduce [B] (verification not implemented)

Time = 0.22 (sec) , antiderivative size = 52, normalized size of antiderivative = 1.58

$$\int \frac{a + bx^2}{x^2 \sqrt{-1 + cx} \sqrt{1 + cx}} dx = \frac{\sqrt{cx + 1} \sqrt{cx - 1} ac + 2 \log\left(\frac{\sqrt{cx-1} + \sqrt{cx+1}}{\sqrt{2}}\right) bx + a c^2 x}{cx}$$

input

```
int((b*x^2+a)/x^2/(c*x-1)^(1/2)/(c*x+1)^(1/2),x)
```

output

```
(sqrt(c*x + 1)*sqrt(c*x - 1)*a*c + 2*log((sqrt(c*x - 1) + sqrt(c*x + 1))/sqrt(2))*b*x + a*c**2*x)/(c*x)
```

3.19 $\int \frac{a+bx^2}{x^3\sqrt{-1+cx}\sqrt{1+cx}} dx$

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Rubi [A] (verified)	172
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Optimal result

Integrand size = 29, antiderivative size = 60

$$\int \frac{a + bx^2}{x^3\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{a\sqrt{-1 + cx}\sqrt{1 + cx}}{2x^2} + \frac{1}{2}(2b + ac^2) \arctan\left(\sqrt{-1 + cx}\sqrt{1 + cx}\right)$$

output `1/2*a*(c*x-1)^(1/2)*(c*x+1)^(1/2)/x^2+1/2*(a*c^2+2*b)*arctan((c*x-1)^(1/2)*(c*x+1)^(1/2))`

Mathematica [A] (warning: unable to verify)

Time = 0.11 (sec) , antiderivative size = 55, normalized size of antiderivative = 0.92

$$\int \frac{a + bx^2}{x^3\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{a\sqrt{-1 + cx}\sqrt{1 + cx}}{2x^2} + (2b + ac^2) \arctan\left(\sqrt{\frac{-1 + cx}{1 + cx}}\right)$$

input `Integrate[(a + b*x^2)/(x^3*Sqrt[-1 + c*x]*Sqrt[1 + c*x]),x]`

output `(a*Sqrt[-1 + c*x]*Sqrt[1 + c*x])/(2*x^2) + (2*b + a*c^2)*ArcTan[Sqrt[(-1 + c*x)/(1 + c*x)]]`

Rubi [A] (verified)

Time = 0.33 (sec) , antiderivative size = 60, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.103$, Rules used = {956, 103, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{a + bx^2}{x^3 \sqrt{cx - 1} \sqrt{cx + 1}} dx$$

$$\downarrow 956$$

$$\frac{1}{2}(ac^2 + 2b) \int \frac{1}{x \sqrt{cx - 1} \sqrt{cx + 1}} dx + \frac{a \sqrt{cx - 1} \sqrt{cx + 1}}{2x^2}$$

$$\downarrow 103$$

$$\frac{1}{2}c(ac^2 + 2b) \int \frac{1}{(cx - 1)(cx + 1)c + c} d(\sqrt{cx - 1} \sqrt{cx + 1}) + \frac{a \sqrt{cx - 1} \sqrt{cx + 1}}{2x^2}$$

$$\downarrow 218$$

$$\frac{1}{2}(ac^2 + 2b) \arctan(\sqrt{cx - 1} \sqrt{cx + 1}) + \frac{a \sqrt{cx - 1} \sqrt{cx + 1}}{2x^2}$$

input `Int[(a + b*x^2)/(x^3*Sqrt[-1 + c*x]*Sqrt[1 + c*x]),x]`

output `(a*Sqrt[-1 + c*x]*Sqrt[1 + c*x])/(2*x^2) + ((2*b + a*c^2)*ArcTan[Sqrt[-1 + c*x]*Sqrt[1 + c*x]])/2`

Defintions of rubi rules used

rule 103

```
Int[1/(Sqrt[(a_.) + (b_.)*(x_)]*Sqrt[(c_.) + (d_.)*(x_)]*((e_.) + (f_.)*(x_))), x_] :> Simp[b*f Subst[Int[1/(d*(b*e - a*f)^2 + b*f^2*x^2), x], x, Sqrt[a + b*x]*Sqrt[c + d*x]], x] /; FreeQ[{a, b, c, d, e, f}, x] && EqQ[2*b*d*e - f*(b*c + a*d), 0]
```

rule 218 `Int[((a_) + (b_)*(x_)^2)^(-1), x_Symbol] := Simp[(Rt[a/b, 2]/a)*ArcTan[x/Rt[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]`

rule 956 `Int[((e_)*(x_))^(m_)*((a1_) + (b1_)*(x_)^(non2_))^(p_)*((a2_) + (b2_)*(x_)^(non2_))^(p_)*((c_) + (d_)*(x_)^(n_)), x_Symbol] := Simp[c*(e*x)^(m+1)*(a1 + b1*x^(n/2))^(p+1)*(a2 + b2*x^(n/2))^(p+1)/(a1*a2*e^(m+1)), x] + Simp[(a1*a2*d*(m+1) - b1*b2*c*(m+n*(p+1)+1))/(a1*a2*e^n*(m+1)) Int[(e*x)^(m+n)*(a1 + b1*x^(n/2))^p*(a2 + b2*x^(n/2))^p, x], x] /; FreeQ[{a1, b1, a2, b2, c, d, e, p}, x] && EqQ[non2, n/2] && EqQ[a2*b1 + a1*b2, 0] && (IntegerQ[n] || GtQ[e, 0]) && ((GtQ[n, 0] && LtQ[m, -1]) || (LtQ[n, 0] && GtQ[m+n, -1])) && !ILtQ[p, -1]`

Maple [A] (verified)

Time = 0.12 (sec) , antiderivative size = 71, normalized size of antiderivative = 1.18

method	result	size
risch	$\frac{a\sqrt{cx-1}\sqrt{cx+1}}{2x^2} - \frac{\left(b + \frac{a^2c^2}{2}\right) \arctan\left(\frac{1}{\sqrt{c^2x^2-1}}\right) \sqrt{(cx+1)(cx-1)}}{\sqrt{cx-1}\sqrt{cx+1}}$	71
default	$-\frac{\sqrt{cx-1}\sqrt{cx+1} \left(\arctan\left(\frac{1}{\sqrt{c^2x^2-1}}\right) a c^2 x^2 + 2 \arctan\left(\frac{1}{\sqrt{c^2x^2-1}}\right) b x^2 - \sqrt{c^2x^2-1} a \right)}{2\sqrt{c^2x^2-1} x^2}$	84

input `int((b*x^2+a)/x^3/(c*x-1)^(1/2)/(c*x+1)^(1/2), x, method=_RETURNVERBOSE)`

output `1/2*a*(c*x-1)^(1/2)*(c*x+1)^(1/2)/x^2-(b+1/2*a*c^2)*arctan(1/(c^2*x^2-1)^(1/2))*((c*x+1)*(c*x-1))^(1/2)/(c*x-1)^(1/2)/(c*x+1)^(1/2)`

Fricas [A] (verification not implemented)

Time = 0.12 (sec) , antiderivative size = 57, normalized size of antiderivative = 0.95

$$\int \frac{a + bx^2}{x^3 \sqrt{-1 + cx} \sqrt{1 + cx}} dx$$

$$= \frac{2(ac^2 + 2b)x^2 \arctan(-cx + \sqrt{cx + 1}\sqrt{cx - 1}) + \sqrt{cx + 1}\sqrt{cx - 1}a}{2x^2}$$

input `integrate((b*x^2+a)/x^3/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="fricas")`output `1/2*(2*(a*c^2 + 2*b)*x^2*arctan(-c*x + sqrt(c*x + 1)*sqrt(c*x - 1)) + sqrt(c*x + 1)*sqrt(c*x - 1)*a)/x^2`**Sympy [F(-1)]**

Timed out.

$$\int \frac{a + bx^2}{x^3 \sqrt{-1 + cx} \sqrt{1 + cx}} dx = \text{Timed out}$$

input `integrate((b*x**2+a)/x**3/(c*x-1)**(1/2)/(c*x+1)**(1/2),x)`output `Timed out`**Maxima [A] (verification not implemented)**

Time = 0.10 (sec) , antiderivative size = 45, normalized size of antiderivative = 0.75

$$\int \frac{a + bx^2}{x^3 \sqrt{-1 + cx} \sqrt{1 + cx}} dx = -\frac{1}{2} ac^2 \arcsin\left(\frac{1}{c|x|}\right) - b \arcsin\left(\frac{1}{c|x|}\right) + \frac{\sqrt{c^2 x^2 - 1}a}{2x^2}$$

input `integrate((b*x^2+a)/x^3/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="maxima")`output `-1/2*a*c^2*arcsin(1/(c*abs(x))) - b*arcsin(1/(c*abs(x))) + 1/2*sqrt(c^2*x^2 - 1)*a/x^2`

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 114 vs. $2(48) = 96$.

Time = 0.14 (sec) , antiderivative size = 114, normalized size of antiderivative = 1.90

$$\int \frac{a + bx^2}{x^3 \sqrt{-1 + cx} \sqrt{1 + cx}} dx = \frac{(ac^3 + 2bc) \arctan\left(\frac{1}{2}(\sqrt{cx+1} - \sqrt{cx-1})^2\right) + \frac{2(ac^3(\sqrt{cx+1}-\sqrt{cx-1})^6 - 4ac^3(\sqrt{cx+1}-\sqrt{cx-1})^2)}{((\sqrt{cx+1}-\sqrt{cx-1})^4+4)^2}}{c}$$

input `integrate((b*x^2+a)/x^3/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="giac")`

output `-((a*c^3 + 2*b*c)*arctan(1/2*(sqrt(c*x + 1) - sqrt(c*x - 1))^2) + 2*(a*c^3*(sqrt(c*x + 1) - sqrt(c*x - 1))^6 - 4*a*c^3*(sqrt(c*x + 1) - sqrt(c*x - 1))^2)/((sqrt(c*x + 1) - sqrt(c*x - 1))^4 + 4)^2)/c`

Mupad [B] (verification not implemented)

Time = 12.15 (sec) , antiderivative size = 297, normalized size of antiderivative = 4.95

$$\int \frac{a + bx^2}{x^3 \sqrt{-1 + cx} \sqrt{1 + cx}} dx = \frac{\frac{ac^2 \operatorname{li}}{32} + \frac{ac^2 (\sqrt{cx-1}-i)^2 \operatorname{li}}{16 (\sqrt{cx+1}-1)^2} - \frac{ac^2 (\sqrt{cx-1}-i)^4 15i}{32 (\sqrt{cx+1}-1)^4}}{\frac{(\sqrt{cx-1}-i)^2}{(\sqrt{cx+1}-1)^2} + \frac{2(\sqrt{cx-1}-i)^4}{(\sqrt{cx+1}-1)^4} + \frac{(\sqrt{cx-1}-i)^6}{(\sqrt{cx+1}-1)^6}} - b \left(\ln \left(\frac{(\sqrt{cx-1}-i)^2}{(\sqrt{cx+1}-1)^2} + 1 \right) - \ln \left(\frac{\sqrt{cx-1}-i}{\sqrt{cx+1}-1} \right) \right) \operatorname{li} - \frac{ac^2 \ln \left(\frac{(\sqrt{cx-1}-i)^2}{(\sqrt{cx+1}-1)^2} + 1 \right) \operatorname{li}}{2} + \frac{ac^2 \ln \left(\frac{\sqrt{cx-1}-i}{\sqrt{cx+1}-1} \right) \operatorname{li}}{2} + \frac{ac^2 (\sqrt{cx-1}-i)^2 \operatorname{li}}{32 (\sqrt{cx+1}-1)^2}$$

input `int((a + b*x^2)/(x^3*(c*x - 1)^(1/2)*(c*x + 1)^(1/2)),x)`

output

```
((a*c^2*i)/32 + (a*c^2*((c*x - 1)^(1/2) - 1i)^2*i)/(16*((c*x + 1)^(1/2)
- 1)^2) - (a*c^2*((c*x - 1)^(1/2) - 1i)^4*15i)/(32*((c*x + 1)^(1/2) - 1)^4
))/(((c*x - 1)^(1/2) - 1i)^2/((c*x + 1)^(1/2) - 1)^2 + (2*((c*x - 1)^(1/2)
- 1i)^4)/((c*x + 1)^(1/2) - 1)^4 + ((c*x - 1)^(1/2) - 1i)^6/((c*x + 1)^(1
/2) - 1)^6) - b*(log(((c*x - 1)^(1/2) - 1i)^2/((c*x + 1)^(1/2) - 1)^2 + 1)
- log(((c*x - 1)^(1/2) - 1i)/((c*x + 1)^(1/2) - 1)))*i - (a*c^2*log(((c*
x - 1)^(1/2) - 1i)^2/((c*x + 1)^(1/2) - 1)^2 + 1)*i)/2 + (a*c^2*log(((c*x
- 1)^(1/2) - 1i)/((c*x + 1)^(1/2) - 1))*i)/2 + (a*c^2*((c*x - 1)^(1/2) -
1i)^2*i)/(32*((c*x + 1)^(1/2) - 1)^2)
```

Reduce [B] (verification not implemented)

Time = 0.22 (sec) , antiderivative size = 110, normalized size of antiderivative = 1.83

$$\int \frac{a + bx^2}{x^3 \sqrt{-1 + cx} \sqrt{1 + cx}} dx$$

$$= \frac{2 \operatorname{atan}(\sqrt{cx - 1} + \sqrt{cx + 1} - 1) a c^2 x^2 + 4 \operatorname{atan}(\sqrt{cx - 1} + \sqrt{cx + 1} - 1) b x^2 - 2 \operatorname{atan}(\sqrt{cx - 1} + \sqrt{cx + 1} - 1) a}{2x^2}$$

input

```
int((b*x^2+a)/x^3/(c*x-1)^(1/2)/(c*x+1)^(1/2),x)
```

output

```
(2*atan(sqrt(c*x - 1) + sqrt(c*x + 1) - 1)*a*c**2*x**2 + 4*atan(sqrt(c*x -
1) + sqrt(c*x + 1) - 1)*b*x**2 - 2*atan(sqrt(c*x - 1) + sqrt(c*x + 1) + 1
)*a*c**2*x**2 - 4*atan(sqrt(c*x - 1) + sqrt(c*x + 1) + 1)*b*x**2 + sqrt(c*
x + 1)*sqrt(c*x - 1)*a)/(2*x**2)
```

$$3.20 \quad \int \frac{a+bx^2}{x^4\sqrt{-1+cx}\sqrt{1+cx}} dx$$

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Rubi [A] (verified)	178
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Giac [B] (verification not implemented)	181
Mupad [B] (verification not implemented)	182
Reduce [B] (verification not implemented)	182

Optimal result

Integrand size = 29, antiderivative size = 62

$$\int \frac{a+bx^2}{x^4\sqrt{-1+cx}\sqrt{1+cx}} dx = \frac{a\sqrt{-1+cx}\sqrt{1+cx}}{3x^3} + \frac{(3b+2ac^2)\sqrt{-1+cx}\sqrt{1+cx}}{3x}$$

output

```
1/3*a*(c*x-1)^(1/2)*(c*x+1)^(1/2)/x^3+1/3*(2*a*c^2+3*b)*(c*x-1)^(1/2)*(c*x+1)^(1/2)/x
```

Mathematica [A] (verified)

Time = 0.11 (sec) , antiderivative size = 42, normalized size of antiderivative = 0.68

$$\int \frac{a+bx^2}{x^4\sqrt{-1+cx}\sqrt{1+cx}} dx = \frac{\sqrt{-1+cx}\sqrt{1+cx}(a+3bx^2+2ac^2x^2)}{3x^3}$$

input

```
Integrate[(a + b*x^2)/(x^4*Sqrt[-1 + c*x]*Sqrt[1 + c*x]),x]
```

output

```
(Sqrt[-1 + c*x]*Sqrt[1 + c*x]*(a + 3*b*x^2 + 2*a*c^2*x^2))/(3*x^3)
```

Rubi [A] (verified)

Time = 0.32 (sec) , antiderivative size = 62, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}} = 0.069$, Rules used = {956, 106}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{a + bx^2}{x^4 \sqrt{cx - 1} \sqrt{cx + 1}} dx$$

$$\downarrow 956$$

$$\frac{1}{3}(2ac^2 + 3b) \int \frac{1}{x^2 \sqrt{cx - 1} \sqrt{cx + 1}} dx + \frac{a \sqrt{cx - 1} \sqrt{cx + 1}}{3x^3}$$

$$\downarrow 106$$

$$\frac{\sqrt{cx - 1} \sqrt{cx + 1} (2ac^2 + 3b)}{3x} + \frac{a \sqrt{cx - 1} \sqrt{cx + 1}}{3x^3}$$

input `Int[(a + b*x^2)/(x^4*Sqrt[-1 + c*x]*Sqrt[1 + c*x]),x]`

output `(a*Sqrt[-1 + c*x]*Sqrt[1 + c*x])/(3*x^3) + ((3*b + 2*a*c^2)*Sqrt[-1 + c*x]*Sqrt[1 + c*x])/(3*x)`

Defintions of rubi rules used

rule 106 `Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_)*((e_.) + (f_.)*(x_))^(p_), x_] := Simp[b*(a + b*x)^(m + 1)*(c + d*x)^(n + 1)*((e + f*x)^(p + 1))/((m + 1)*(b*c - a*d)*(b*e - a*f)), x] /; FreeQ[{a, b, c, d, e, f, m, n, p}, x] && EqQ[Simplify[m + n + p + 3], 0] && EqQ[a*d*f*(m + 1) + b*c*f*(n + 1) + b*d*e*(p + 1), 0] && NeQ[m, -1]`

rule 956

```

Int[((e._)*(x._))^(m._)*((a1_) + (b1._)*(x._)^(non2._))^(p._)*((a2_) + (b2._)
*(x._)^(non2._))^(p._)*((c_) + (d._)*(x._)^(n._)), x_Symbol] := Simp[c*(e*x)^(
m + 1)*(a1 + b1*x^(n/2))^(p + 1)*((a2 + b2*x^(n/2))^(p + 1)/(a1*a2*e*(m + 1
))), x] + Simp[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(a1*a2*e^n*(
m + 1)) Int[(e*x)^(m + n)*(a1 + b1*x^(n/2))^p*(a2 + b2*x^(n/2))^p, x], x]
/; FreeQ[{a1, b1, a2, b2, c, d, e, p}, x] && EqQ[non2, n/2] && EqQ[a2*b1 +
a1*b2, 0] && (IntegerQ[n] || GtQ[e, 0]) && ((GtQ[n, 0] && LtQ[m, -1]) || (
LtQ[n, 0] && GtQ[m + n, -1])) && !ILtQ[p, -1]

```

Maple [A] (verified)

Time = 0.12 (sec) , antiderivative size = 37, normalized size of antiderivative = 0.60

method	result	size
gospers	$\frac{\sqrt{cx+1}\sqrt{cx-1}(2ac^2x^2+3bx^2+a)}{3x^3}$	37
risch	$\frac{\sqrt{cx+1}\sqrt{cx-1}(2ac^2x^2+3bx^2+a)}{3x^3}$	37
orering	$\frac{\sqrt{cx+1}\sqrt{cx-1}(2ac^2x^2+3bx^2+a)}{3x^3}$	37
default	$\frac{\sqrt{cx-1}\sqrt{cx+1}\operatorname{csign}(c)^2(2ac^2x^2+3bx^2+a)}{3x^3}$	41

input

```
int((b*x^2+a)/x^4/(c*x-1)^(1/2)/(c*x+1)^(1/2),x,method=_RETURNVERBOSE)
```

output

```
1/3*(c*x+1)^(1/2)*(c*x-1)^(1/2)*(2*a*c^2*x^2+3*b*x^2+a)/x^3
```

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 52, normalized size of antiderivative = 0.84

$$\int \frac{a + bx^2}{x^4\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{(2ac^3 + 3bc)x^3 + ((2ac^2 + 3b)x^2 + a)\sqrt{cx + 1}\sqrt{cx - 1}}{3x^3}$$

input

```
integrate((b*x^2+a)/x^4/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="fricas")
```

output

```
1/3*((2*a*c^3 + 3*b*c)*x^3 + ((2*a*c^2 + 3*b)*x^2 + a)*sqrt(c*x + 1)*sqrt(c*x - 1))/x^3
```

Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 20.73 (sec) , antiderivative size = 146, normalized size of antiderivative = 2.35

$$\int \frac{a + bx^2}{x^4 \sqrt{-1 + cx} \sqrt{1 + cx}} dx = - \frac{ac^3 G_{6,6}^{5,3} \left(\begin{matrix} \frac{9}{4}, \frac{11}{4}, 1 \\ 2, \frac{9}{4}, \frac{5}{2}, \frac{11}{4}, 3 \end{matrix} \middle| \frac{1}{c^2 x^2} \right)}{4\pi^{\frac{3}{2}}} - \frac{iac^3 G_{6,6}^{2,6} \left(\begin{matrix} \frac{3}{2}, \frac{7}{4}, 2, \frac{9}{4}, \frac{5}{2}, 1 \\ \frac{7}{4}, \frac{9}{4} \end{matrix} \middle| \frac{e^{2i\pi}}{c^2 x^2} \right)}{4\pi^{\frac{3}{2}}} - \frac{bcG_{6,6}^{5,3} \left(\begin{matrix} \frac{5}{4}, \frac{7}{4}, 1 \\ 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2 \end{matrix} \middle| \frac{1}{c^2 x^2} \right)}{4\pi^{\frac{3}{2}}} - \frac{ibcG_{6,6}^{2,6} \left(\begin{matrix} \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, 1 \\ \frac{3}{4}, \frac{5}{4} \end{matrix} \middle| \frac{e^{2i\pi}}{c^2 x^2} \right)}{4\pi^{\frac{3}{2}}}$$

input

```
integrate((b*x**2+a)/x**4/(c*x-1)**(1/2)/(c*x+1)**(1/2),x)
```

output

```
-a*c**3*meijerg(((9/4, 11/4, 1), (5/2, 5/2, 3)), ((2, 9/4, 5/2, 11/4, 3), (0,)), 1/(c**2*x**2))/(4*pi**(3/2)) - I*a*c**3*meijerg(((3/2, 7/4, 2, 9/4, 5/2, 1), ()), ((7/4, 9/4), (3/2, 2, 2, 0)), exp_polar(2*I*pi)/(c**2*x**2))/(4*pi**(3/2)) - b*c*meijerg(((5/4, 7/4, 1), (3/2, 3/2, 2)), ((1, 5/4, 3/2, 7/4, 2), (0,)), 1/(c**2*x**2))/(4*pi**(3/2)) - I*b*c*meijerg(((1/2, 3/4, 1, 5/4, 3/2, 1), ()), ((3/4, 5/4), (1/2, 1, 1, 0)), exp_polar(2*I*pi)/(c**2*x**2))/(4*pi**(3/2))
```

Maxima [A] (verification not implemented)

Time = 0.10 (sec) , antiderivative size = 54, normalized size of antiderivative = 0.87

$$\int \frac{a + bx^2}{x^4 \sqrt{-1 + cx} \sqrt{1 + cx}} dx = \frac{2\sqrt{c^2x^2 - 1}ac^2}{3x} + \frac{\sqrt{c^2x^2 - 1}b}{x} + \frac{\sqrt{c^2x^2 - 1}a}{3x^3}$$

input `integrate((b*x^2+a)/x^4/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="maxima")`

output `2/3*sqrt(c^2*x^2 - 1)*a*c^2/x + sqrt(c^2*x^2 - 1)*b/x + 1/3*sqrt(c^2*x^2 - 1)*a/x^3`

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 105 vs. 2(50) = 100.

Time = 0.14 (sec) , antiderivative size = 105, normalized size of antiderivative = 1.69

$$\int \frac{a + bx^2}{x^4 \sqrt{-1 + cx} \sqrt{1 + cx}} dx = \frac{8 \left(3b(\sqrt{cx + 1} - \sqrt{cx - 1})^8 + 24ac^2(\sqrt{cx + 1} - \sqrt{cx - 1})^4 + 24b(\sqrt{cx + 1} - \sqrt{cx - 1})^4 + 32ac^2 + 48b \right)}{3 \left((\sqrt{cx + 1} - \sqrt{cx - 1})^4 + 4 \right)^3}$$

input `integrate((b*x^2+a)/x^4/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="giac")`

output `8/3*(3*b*(sqrt(c*x + 1) - sqrt(c*x - 1))^8 + 24*a*c^2*(sqrt(c*x + 1) - sqrt(c*x - 1))^4 + 24*b*(sqrt(c*x + 1) - sqrt(c*x - 1))^4 + 32*a*c^2 + 48*b)*c/((sqrt(c*x + 1) - sqrt(c*x - 1))^4 + 4)^3`

Mupad [B] (verification not implemented)

Time = 4.93 (sec) , antiderivative size = 53, normalized size of antiderivative = 0.85

$$\int \frac{a + bx^2}{x^4 \sqrt{-1 + cx} \sqrt{1 + cx}} dx = \frac{\sqrt{cx - 1} \left(\left(\frac{2ac^3}{3} + bc \right) x^3 + \left(\frac{2ac^2}{3} + b \right) x^2 + \frac{acx}{3} + \frac{a}{3} \right)}{x^3 \sqrt{cx + 1}}$$

input `int((a + b*x^2)/(x^4*(c*x - 1)^(1/2)*(c*x + 1)^(1/2)),x)`output `((c*x - 1)^(1/2)*(a/3 + x^3*(b*c + (2*a*c^3)/3) + x^2*(b + (2*a*c^2)/3) + (a*c*x)/3)/(x^3*(c*x + 1)^(1/2))`**Reduce [B] (verification not implemented)**

Time = 0.22 (sec) , antiderivative size = 75, normalized size of antiderivative = 1.21

$$\int \frac{a + bx^2}{x^4 \sqrt{-1 + cx} \sqrt{1 + cx}} dx$$

$$= \frac{2\sqrt{cx + 1} \sqrt{cx - 1} a c^2 x^2 + \sqrt{cx + 1} \sqrt{cx - 1} a + 3\sqrt{cx + 1} \sqrt{cx - 1} b x^2 - 2a c^3 x^3 - bc x^3}{3x^3}$$

input `int((b*x^2+a)/x^4/(c*x-1)^(1/2)/(c*x+1)^(1/2),x)`output `(2*sqrt(c*x + 1)*sqrt(c*x - 1)*a*c**2*x**2 + sqrt(c*x + 1)*sqrt(c*x - 1)*a + 3*sqrt(c*x + 1)*sqrt(c*x - 1)*b*x**2 - 2*a*c**3*x**3 - b*c*x**3)/(3*x**3)`

3.21 $\int \frac{a+bx^2}{x^5\sqrt{-1+cx}\sqrt{1+cx}} dx$

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Optimal result

Integrand size = 29, antiderivative size = 99

$$\int \frac{a + bx^2}{x^5\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{a\sqrt{-1 + cx}\sqrt{1 + cx}}{4x^4} + \frac{(4b + 3ac^2)\sqrt{-1 + cx}\sqrt{1 + cx}}{8x^2} + \frac{1}{8}c^2(4b + 3ac^2) \arctan\left(\sqrt{-1 + cx}\sqrt{1 + cx}\right)$$

output

```
1/4*a*(c*x-1)^(1/2)*(c*x+1)^(1/2)/x^4+1/8*(3*a*c^2+4*b)*(c*x-1)^(1/2)*(c*x+1)^(1/2)/x^2+1/8*c^2*(3*a*c^2+4*b)*arctan((c*x-1)^(1/2)*(c*x+1)^(1/2))
```

Mathematica [A] (warning: unable to verify)

Time = 0.22 (sec) , antiderivative size = 78, normalized size of antiderivative = 0.79

$$\int \frac{a + bx^2}{x^5\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{1}{8} \left(\frac{\sqrt{-1 + cx}\sqrt{1 + cx}(4bx^2 + a(2 + 3c^2x^2))}{x^4} + (8bc^2 + 6ac^4) \arctan\left(\sqrt{\frac{-1 + cx}{1 + cx}}\right) \right)$$

input

```
Integrate[(a + b*x^2)/(x^5*Sqrt[-1 + c*x]*Sqrt[1 + c*x]),x]
```


output

$$\left(\left(\text{Sqrt}[-1 + c*x]*\text{Sqrt}[1 + c*x]*(4*b*x^2 + a*(2 + 3*c^2*x^2))\right)/x^4 + (8*b*c^2 + 6*a*c^4)*\text{ArcTan}[\text{Sqrt}[(-1 + c*x)/(1 + c*x)]]\right)/8$$
Rubi [A] (verified)

Time = 0.38 (sec) , antiderivative size = 94, normalized size of antiderivative = 0.95, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.172$, Rules used = {956, 114, 27, 103, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{a + bx^2}{x^5 \sqrt{cx - 1} \sqrt{cx + 1}} dx$$

$$\downarrow 956$$

$$\frac{1}{4}(3ac^2 + 4b) \int \frac{1}{x^3 \sqrt{cx - 1} \sqrt{cx + 1}} dx + \frac{a\sqrt{cx - 1}\sqrt{cx + 1}}{4x^4}$$

$$\downarrow 114$$

$$\frac{1}{4}(3ac^2 + 4b) \left(\frac{1}{2} \int \frac{c^2}{x \sqrt{cx - 1} \sqrt{cx + 1}} dx + \frac{\sqrt{cx - 1}\sqrt{cx + 1}}{2x^2} \right) + \frac{a\sqrt{cx - 1}\sqrt{cx + 1}}{4x^4}$$

$$\downarrow 27$$

$$\frac{1}{4}(3ac^2 + 4b) \left(\frac{1}{2}c^2 \int \frac{1}{x \sqrt{cx - 1} \sqrt{cx + 1}} dx + \frac{\sqrt{cx - 1}\sqrt{cx + 1}}{2x^2} \right) + \frac{a\sqrt{cx - 1}\sqrt{cx + 1}}{4x^4}$$

$$\downarrow 103$$

$$\frac{1}{4}(3ac^2 + 4b) \left(\frac{1}{2}c^3 \int \frac{1}{(cx - 1)(cx + 1)c + c} d\left(\frac{\sqrt{cx - 1}\sqrt{cx + 1}}{a\sqrt{cx - 1}\sqrt{cx + 1}}\right) + \frac{\sqrt{cx - 1}\sqrt{cx + 1}}{2x^2} \right) + \frac{a\sqrt{cx - 1}\sqrt{cx + 1}}{4x^4}$$

$$\downarrow 218$$

$$\frac{1}{4}(3ac^2 + 4b) \left(\frac{1}{2}c^2 \arctan\left(\frac{\sqrt{cx - 1}\sqrt{cx + 1}}{a\sqrt{cx - 1}\sqrt{cx + 1}}\right) + \frac{\sqrt{cx - 1}\sqrt{cx + 1}}{2x^2} \right) + \frac{a\sqrt{cx - 1}\sqrt{cx + 1}}{4x^4}$$

input

$$\text{Int}[(a + b*x^2)/(x^5*\text{Sqrt}[-1 + c*x]*\text{Sqrt}[1 + c*x]),x]$$

output

```
(a*Sqrt[-1 + c*x]*Sqrt[1 + c*x])/(4*x^4) + ((4*b + 3*a*c^2)*((Sqrt[-1 + c*x]*Sqrt[1 + c*x])/(2*x^2) + (c^2*ArcTan[Sqrt[-1 + c*x]*Sqrt[1 + c*x]]/2)))/4
```

Defintions of rubi rules used

rule 27

```
Int[(a_)*(F_x_), x_Symbol] := Simp[a Int[F_x, x], x] /; FreeQ[a, x] && !MatchQ[F_x, (b_)*(G_x_)] /; FreeQ[b, x]
```

rule 103

```
Int[1/(Sqrt[(a_) + (b_)*(x_)]*Sqrt[(c_) + (d_)*(x_)]*((e_) + (f_)*(x_))), x_] := Simp[b*f Subst[Int[1/(d*(b*e - a*f)^2 + b*f^2*x^2), x], x, Sqrt[a + b*x]*Sqrt[c + d*x]], x] /; FreeQ[{a, b, c, d, e, f}, x] && EqQ[2*b*d*e - f*(b*c + a*d), 0]
```

rule 114

```
Int[((a_) + (b_)*(x_))^(m_)*((c_) + (d_)*(x_))^(n_)*((e_) + (f_)*(x_))^(p_), x_] := Simp[b*(a + b*x)^(m + 1)*(c + d*x)^(n + 1)*((e + f*x)^(p + 1))/((m + 1)*(b*c - a*d)*(b*e - a*f)), x] + Simp[1/((m + 1)*(b*c - a*d)*(b*e - a*f)) Int[(a + b*x)^(m + 1)*(c + d*x)^n*(e + f*x)^p*Simp[a*d*f*(m + 1) - b*(d*e*(m + n + 2) + c*f*(m + p + 2)) - b*d*f*(m + n + p + 3)*x, x], x], x] /; FreeQ[{a, b, c, d, e, f, n, p}, x] && ILtQ[m, -1] && (IntegerQ[n] || IntegersQ[2*n, 2*p] || ILtQ[m + n + p + 3, 0])
```

rule 218

```
Int[((a_) + (b_)*(x_)^2)^(-1), x_Symbol] := Simp[(Rt[a/b, 2]/a)*ArcTan[x/Rt[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 956

```
Int[((e_)*(x_))^(m_)*((a1_) + (b1_)*(x_)^(non2_))^(p_)*((a2_) + (b2_)*(x_)^(non2_))^(p_), x_Symbol] := Simp[c*(e*x)^(m + 1)*(a1 + b1*x^(n/2))^(p + 1)*((a2 + b2*x^(n/2))^(p + 1)/(a1*a2*e^(m + 1))), x] + Simp[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(a1*a2*e^n*(m + 1)) Int[(e*x)^(m + n)*(a1 + b1*x^(n/2))^p*(a2 + b2*x^(n/2))^p, x], x] /; FreeQ[{a1, b1, a2, b2, c, d, e, p}, x] && EqQ[non2, n/2] && EqQ[a2*b1 + a1*b2, 0] && (IntegerQ[n] || GtQ[e, 0]) && ((GtQ[n, 0] && LtQ[m, -1]) || (LtQ[n, 0] && GtQ[m + n, -1])) && !ILtQ[p, -1]
```

Maple [A] (verified)

Time = 0.12 (sec) , antiderivative size = 94, normalized size of antiderivative = 0.95

method	result
risch	$\frac{\sqrt{cx+1}\sqrt{cx-1}(3ac^2x^2+4bx^2+2a)}{8x^4} - \frac{c^2(3ac^2+4b)\arctan\left(\frac{1}{\sqrt{c^2x^2-1}}\right)\sqrt{(cx+1)(cx-1)}}{8\sqrt{cx-1}\sqrt{cx+1}}$
default	$-\frac{\sqrt{cx-1}\sqrt{cx+1}\left(3\arctan\left(\frac{1}{\sqrt{c^2x^2-1}}\right)ac^4x^4+4\arctan\left(\frac{1}{\sqrt{c^2x^2-1}}\right)bc^2x^4-3\sqrt{c^2x^2-1}ac^2x^2-4\sqrt{c^2x^2-1}bx^2-2\sqrt{c^2x^2-1}a\right)}{8\sqrt{c^2x^2-1}x^4}$

input `int((b*x^2+a)/x^5/(c*x-1)^(1/2)/(c*x+1)^(1/2),x,method=_RETURNVERBOSE)`

output `1/8*(c*x+1)^(1/2)*(c*x-1)^(1/2)*(3*a*c^2*x^2+4*b*x^2+2*a)/x^4-1/8*c^2*(3*a*c^2+4*b)*arctan(1/(c^2*x^2-1)^(1/2))*((c*x+1)*(c*x-1)^(1/2)/(c*x-1)^(1/2))/(c*x+1)^(1/2)`

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 78, normalized size of antiderivative = 0.79

$$\int \frac{a + bx^2}{x^5 \sqrt{-1 + cx} \sqrt{1 + cx}} dx$$

$$= \frac{2(3ac^4 + 4bc^2)x^4 \arctan(-cx + \sqrt{cx+1}\sqrt{cx-1}) + ((3ac^2 + 4b)x^2 + 2a)\sqrt{cx+1}\sqrt{cx-1}}{8x^4}$$

input `integrate((b*x^2+a)/x^5/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="fricas")`

output `1/8*(2*(3*a*c^4 + 4*b*c^2)*x^4*arctan(-c*x + sqrt(c*x + 1)*sqrt(c*x - 1)) + ((3*a*c^2 + 4*b)*x^2 + 2*a)*sqrt(c*x + 1)*sqrt(c*x - 1))/x^4`

Sympy [F(-1)]

Timed out.

$$\int \frac{a + bx^2}{x^5 \sqrt{-1 + cx} \sqrt{1 + cx}} dx = \text{Timed out}$$

input `integrate((b*x**2+a)/x**5/(c*x-1)**(1/2)/(c*x+1)**(1/2),x)`

output `Timed out`

Maxima [A] (verification not implemented)

Time = 0.11 (sec) , antiderivative size = 85, normalized size of antiderivative = 0.86

$$\int \frac{a + bx^2}{x^5 \sqrt{-1 + cx} \sqrt{1 + cx}} dx = -\frac{3}{8} ac^4 \arcsin\left(\frac{1}{c|x|}\right) - \frac{1}{2} bc^2 \arcsin\left(\frac{1}{c|x|}\right) + \frac{3\sqrt{c^2x^2 - 1}ac^2}{8x^2} + \frac{\sqrt{c^2x^2 - 1}b}{2x^2} + \frac{\sqrt{c^2x^2 - 1}a}{4x^4}$$

input `integrate((b*x^2+a)/x^5/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="maxima")`

output `-3/8*a*c^4*arcsin(1/(c*abs(x))) - 1/2*b*c^2*arcsin(1/(c*abs(x))) + 3/8*sqrt(c^2*x^2 - 1)*a*c^2/x^2 + 1/2*sqrt(c^2*x^2 - 1)*b/x^2 + 1/4*sqrt(c^2*x^2 - 1)*a/x^4`

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 268 vs. 2(81) = 162.

Time = 0.15 (sec) , antiderivative size = 268, normalized size of antiderivative = 2.71

$$\int \frac{a + bx^2}{x^5 \sqrt{-1 + cx} \sqrt{1 + cx}} dx = (3ac^5 + 4bc^3) \arctan\left(\frac{1}{2}(\sqrt{cx+1} - \sqrt{cx-1})^2\right) + \frac{2(3ac^5(\sqrt{cx+1} - \sqrt{cx-1})^{14} + 4bc^3(\sqrt{cx+1} - \sqrt{cx-1})^{14} + 44ac^5(\sqrt{cx+1} - \sqrt{cx-1})^{14})}{\dots}$$

input `integrate((b*x^2+a)/x^5/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="giac")`

output `-1/4*((3*a*c^5 + 4*b*c^3)*arctan(1/2*(sqrt(c*x + 1) - sqrt(c*x - 1))^2) + 2*(3*a*c^5*(sqrt(c*x + 1) - sqrt(c*x - 1))^14 + 4*b*c^3*(sqrt(c*x + 1) - sqrt(c*x - 1))^14 + 44*a*c^5*(sqrt(c*x + 1) - sqrt(c*x - 1))^10 + 16*b*c^3*(sqrt(c*x + 1) - sqrt(c*x - 1))^10 - 176*a*c^5*(sqrt(c*x + 1) - sqrt(c*x - 1))^6 - 64*b*c^3*(sqrt(c*x + 1) - sqrt(c*x - 1))^6 - 192*a*c^5*(sqrt(c*x + 1) - sqrt(c*x - 1))^2 - 256*b*c^3*(sqrt(c*x + 1) - sqrt(c*x - 1))^2)/((sqrt(c*x + 1) - sqrt(c*x - 1))^4 + 4)^4)/c`

Mupad [B] (verification not implemented)

Time = 25.83 (sec) , antiderivative size = 650, normalized size of antiderivative = 6.57

$$\int \frac{a + bx^2}{x^5 \sqrt{-1 + cx} \sqrt{1 + cx}} dx = \text{Too large to display}$$

input `int((a + b*x^2)/(x^5*(c*x - 1)^(1/2)*(c*x + 1)^(1/2)),x)`

output

```

((b*c^2*i)/32 + (b*c^2*((c*x - 1)^(1/2) - 1i)^2*i)/(16*((c*x + 1)^(1/2)
- 1)^2) - (b*c^2*((c*x - 1)^(1/2) - 1i)^4*15i)/(32*((c*x + 1)^(1/2) - 1)^4
))/(((c*x - 1)^(1/2) - 1i)^2/((c*x + 1)^(1/2) - 1)^2 + (2*((c*x - 1)^(1/2)
- 1i)^4)/((c*x + 1)^(1/2) - 1)^4 + ((c*x - 1)^(1/2) - 1i)^6/((c*x + 1)^(1
/2) - 1)^6) - ((a*c^4*i)/1024 - (a*c^4*((c*x - 1)^(1/2) - 1i)^2*3i)/(128*
((c*x + 1)^(1/2) - 1)^2) - (a*c^4*((c*x - 1)^(1/2) - 1i)^4*53i)/(512*((c*x
+ 1)^(1/2) - 1)^4) + (a*c^4*((c*x - 1)^(1/2) - 1i)^6*87i)/(256*((c*x + 1)
^(1/2) - 1)^6) + (a*c^4*((c*x - 1)^(1/2) - 1i)^8*657i)/(1024*((c*x + 1)^(1
/2) - 1)^8) + (a*c^4*((c*x - 1)^(1/2) - 1i)^10*121i)/(256*((c*x + 1)^(1/2)
- 1)^10))/(((c*x - 1)^(1/2) - 1i)^4/((c*x + 1)^(1/2) - 1)^4 + (4*((c*x -
1)^(1/2) - 1i)^6)/((c*x + 1)^(1/2) - 1)^6 + (6*((c*x - 1)^(1/2) - 1i)^8)/((
c*x + 1)^(1/2) - 1)^8 + (4*((c*x - 1)^(1/2) - 1i)^10)/((c*x + 1)^(1/2) -
1)^10 + ((c*x - 1)^(1/2) - 1i)^12/((c*x + 1)^(1/2) - 1)^12) - (a*c^4*log((
c*x - 1)^(1/2) - 1i)^2/((c*x + 1)^(1/2) - 1)^2 + 1)*3i)/8 - (b*c^2*log(((
c*x - 1)^(1/2) - 1i)^2/((c*x + 1)^(1/2) - 1)^2 + 1)*1i)/2 + (a*c^4*log(((c
*x - 1)^(1/2) - 1i)/((c*x + 1)^(1/2) - 1))*3i)/8 + (b*c^2*log(((c*x - 1)^(
1/2) - 1i)/((c*x + 1)^(1/2) - 1))*1i)/2 + (a*c^4*((c*x - 1)^(1/2) - 1i)^2*
7i)/(256*((c*x + 1)^(1/2) - 1)^2) - (a*c^4*((c*x - 1)^(1/2) - 1i)^4*1i)/(1
024*((c*x + 1)^(1/2) - 1)^4) + (b*c^2*((c*x - 1)^(1/2) - 1i)^2*i)/(32*((c
*x + 1)^(1/2) - 1)^2)

```

Reduce [B] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 156, normalized size of antiderivative = 1.58

$$\int \frac{a + bx^2}{x^5 \sqrt{-1 + cx} \sqrt{1 + cx}} dx$$

$$= \frac{6 \operatorname{atan}(\sqrt{cx - 1} + \sqrt{cx + 1} - 1) a c^4 x^4 + 8 \operatorname{atan}(\sqrt{cx - 1} + \sqrt{cx + 1} - 1) b c^2 x^4 - 6 \operatorname{atan}(\sqrt{cx - 1} + \sqrt{cx + 1} - 1) a c^4 x^4 + 8 \operatorname{atan}(\sqrt{cx - 1} + \sqrt{cx + 1} - 1) b c^2 x^4 - 6 \operatorname{atan}(\sqrt{cx - 1} + \sqrt{cx + 1} - 1) a c^4 x^4 + 8 \operatorname{atan}(\sqrt{cx - 1} + \sqrt{cx + 1} - 1) b c^2 x^4}{8 x^4}$$

input

```
int((b*x^2+a)/x^5/(c*x-1)^(1/2)/(c*x+1)^(1/2),x)
```

output

```

(6*atan(sqrt(c*x - 1) + sqrt(c*x + 1) - 1)*a*c**4*x**4 + 8*atan(sqrt(c*x -
1) + sqrt(c*x + 1) - 1)*b*c**2*x**4 - 6*atan(sqrt(c*x - 1) + sqrt(c*x + 1)
+ 1)*a*c**4*x**4 - 8*atan(sqrt(c*x - 1) + sqrt(c*x + 1) + 1)*b*c**2*x**4
+ 3*sqrt(c*x + 1)*sqrt(c*x - 1)*a*c**2*x**2 + 2*sqrt(c*x + 1)*sqrt(c*x -
1)*a + 4*sqrt(c*x + 1)*sqrt(c*x - 1)*b*x**2)/(8*x**4)

```

3.22 $\int \frac{x^4(a+bx^2)}{\sqrt{-c+dx}\sqrt{c+dx}} dx$

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Optimal result

Integrand size = 31, antiderivative size = 164

$$\int \frac{x^4(a+bx^2)}{\sqrt{-c+dx}\sqrt{c+dx}} dx = \frac{c^2(5bc^2+6ad^2)x\sqrt{-c+dx}\sqrt{c+dx}}{16d^6} + \frac{(5bc^2+6ad^2)x^3\sqrt{-c+dx}\sqrt{c+dx}}{24d^4} + \frac{bx^5\sqrt{-c+dx}\sqrt{c+dx}}{6d^2} + \frac{c^4(5bc^2+6ad^2)\operatorname{arctanh}\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)}{8d^7}$$

output

```
1/16*c^2*(6*a*d^2+5*b*c^2)*x*(d*x-c)^(1/2)*(d*x+c)^(1/2)/d^6+1/24*(6*a*d^2+5*b*c^2)*x^3*(d*x-c)^(1/2)*(d*x+c)^(1/2)/d^4+1/6*b*x^5*(d*x-c)^(1/2)*(d*x+c)^(1/2)/d^2+1/8*c^4*(6*a*d^2+5*b*c^2)*arctanh((d*x-c)^(1/2)/(d*x+c)^(1/2))/d^7
```

Mathematica [A] (verified)

Time = 0.28 (sec) , antiderivative size = 119, normalized size of antiderivative = 0.73

$$\int \frac{x^4(a + bx^2)}{\sqrt{-c + dx}\sqrt{c + dx}} dx$$

$$= \frac{dx\sqrt{-c + dx}\sqrt{c + dx}(6ad^2(3c^2 + 2d^2x^2) + b(15c^4 + 10c^2d^2x^2 + 8d^4x^4)) + 6c^4(5bc^2 + 6ad^2) \operatorname{arctanh}\left(\frac{\sqrt{-c + dx}}{\sqrt{c + dx}}\right)}{48d^7}$$

input

```
Integrate[(x^4*(a + b*x^2))/(Sqrt[-c + d*x]*Sqrt[c + d*x]),x]
```

output

```
(d*x*Sqrt[-c + d*x]*Sqrt[c + d*x]*(6*a*d^2*(3*c^2 + 2*d^2*x^2) + b*(15*c^4 + 10*c^2*d^2*x^2 + 8*d^4*x^4)) + 6*c^4*(5*b*c^2 + 6*a*d^2)*ArcTanh[Sqrt[-c + d*x]/Sqrt[c + d*x]])/(48*d^7)
```

Rubi [A] (verified)

Time = 0.45 (sec) , antiderivative size = 148, normalized size of antiderivative = 0.90, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}} = 0.226$, Rules used = {960, 111, 27, 101, 27, 45, 221}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{x^4(a + bx^2)}{\sqrt{dx - c}\sqrt{c + dx}} dx$$

$$\downarrow 960$$

$$\frac{1}{6} \left(6a + \frac{5bc^2}{d^2} \right) \int \frac{x^4}{\sqrt{dx - c}\sqrt{c + dx}} dx + \frac{bx^5\sqrt{dx - c}\sqrt{c + dx}}{6d^2}$$

$$\downarrow 111$$

$$\frac{1}{6} \left(6a + \frac{5bc^2}{d^2} \right) \left(\int \frac{3c^2x^2}{\sqrt{dx - c}\sqrt{c + dx}} dx + \frac{x^3\sqrt{dx - c}\sqrt{c + dx}}{4d^2} \right) + \frac{bx^5\sqrt{dx - c}\sqrt{c + dx}}{6d^2}$$

$$\downarrow 27$$

$$\begin{aligned}
& \frac{1}{6} \left(6a + \frac{5bc^2}{d^2} \right) \left(\frac{3c^2 \int \frac{x^2}{\sqrt{dx-c}\sqrt{c+dx}} dx}{4d^2} + \frac{x^3 \sqrt{dx-c}\sqrt{c+dx}}{4d^2} \right) + \frac{bx^5 \sqrt{dx-c}\sqrt{c+dx}}{6d^2} \\
& \quad \downarrow 101 \\
& \frac{1}{6} \left(6a + \frac{5bc^2}{d^2} \right) \left(\frac{3c^2 \left(\frac{\int \frac{c^2}{\sqrt{dx-c}\sqrt{c+dx}} dx}{2d^2} + \frac{x \sqrt{dx-c}\sqrt{c+dx}}{2d^2} \right)}{4d^2} + \frac{x^3 \sqrt{dx-c}\sqrt{c+dx}}{4d^2} \right) + \\
& \quad \frac{bx^5 \sqrt{dx-c}\sqrt{c+dx}}{6d^2} \\
& \quad \downarrow 27 \\
& \frac{1}{6} \left(6a + \frac{5bc^2}{d^2} \right) \left(\frac{3c^2 \left(\frac{c^2 \int \frac{1}{\sqrt{dx-c}\sqrt{c+dx}} dx}{2d^2} + \frac{x \sqrt{dx-c}\sqrt{c+dx}}{2d^2} \right)}{4d^2} + \frac{x^3 \sqrt{dx-c}\sqrt{c+dx}}{4d^2} \right) + \\
& \quad \frac{bx^5 \sqrt{dx-c}\sqrt{c+dx}}{6d^2} \\
& \quad \downarrow 45 \\
& \frac{1}{6} \left(6a + \frac{5bc^2}{d^2} \right) \left(\frac{3c^2 \left(\frac{c^2 \int \frac{1}{d - \frac{d(dx-c)}{c+dx}} d \frac{\sqrt{dx-c}}{\sqrt{c+dx}}}{d^2} + \frac{x \sqrt{dx-c}\sqrt{c+dx}}{2d^2} \right)}{4d^2} + \frac{x^3 \sqrt{dx-c}\sqrt{c+dx}}{4d^2} \right) + \\
& \quad \frac{bx^5 \sqrt{dx-c}\sqrt{c+dx}}{6d^2} \\
& \quad \downarrow 221 \\
& \frac{1}{6} \left(6a + \frac{5bc^2}{d^2} \right) \left(\frac{3c^2 \left(\frac{c^2 \operatorname{arctanh} \left(\frac{\sqrt{dx-c}}{\sqrt{c+dx}} \right)}{d^3} + \frac{x \sqrt{dx-c}\sqrt{c+dx}}{2d^2} \right)}{4d^2} + \frac{x^3 \sqrt{dx-c}\sqrt{c+dx}}{4d^2} \right) + \\
& \quad \frac{bx^5 \sqrt{dx-c}\sqrt{c+dx}}{6d^2}
\end{aligned}$$

input

$$\text{Int}[(x^4*(a + b*x^2))/(\text{Sqrt}[-c + d*x]*\text{Sqrt}[c + d*x]),x]$$

output

```
(b*x^5*Sqrt[-c + d*x]*Sqrt[c + d*x])/(6*d^2) + ((6*a + (5*b*c^2)/d^2)*((x^
3*Sqrt[-c + d*x]*Sqrt[c + d*x])/(4*d^2) + (3*c^2*((x*Sqrt[-c + d*x]*Sqrt[c
+ d*x])/(2*d^2) + (c^2*ArcTanh[Sqrt[-c + d*x]/Sqrt[c + d*x]])/d^3))/(4*d^
2)))/6
```

Defintions of rubi rules used

rule 27

```
Int[(a_)*(F_x_), x_Symbol] := Simp[a Int[F_x, x], x] /; FreeQ[a, x] && !Ma
tchQ[F_x, (b_)*(G_x_) /; FreeQ[b, x]]
```

rule 45

```
Int[1/(Sqrt[(a_) + (b_)*(x_)]*Sqrt[(c_) + (d_)*(x_)]), x_Symbol] := Simp[
2 Subst[Int[1/(b - d*x^2), x], x, Sqrt[a + b*x]/Sqrt[c + d*x]], x] /; Fre
eQ[{a, b, c, d}, x] && EqQ[b*c + a*d, 0] && !GtQ[c, 0]
```

rule 101

```
Int[((a_) + (b_)*(x_))^2*((c_) + (d_)*(x_))^(n_)*((e_) + (f_)*(x_))^(
p_), x_] := Simp[b*(a + b*x)*(c + d*x)^(n + 1)*((e + f*x)^(p + 1)/(d*f*(n +
p + 3))), x] + Simp[1/(d*f*(n + p + 3) Int[(c + d*x)^n*(e + f*x)^p*Simp
[a^2*d*f*(n + p + 3) - b*(b*c*e + a*(d*e*(n + 1) + c*f*(p + 1))) + b*(a*d*f
*(n + p + 4) - b*(d*e*(n + 2) + c*f*(p + 2)))*x, x], x] /; FreeQ[{a, b,
c, d, e, f, n, p}, x] && NeQ[n + p + 3, 0]
```

rule 111

```
Int[((a_) + (b_)*(x_))^(m_)*((c_) + (d_)*(x_))^(n_)*((e_) + (f_)*(x_
))^p_), x_] := Simp[b*(a + b*x)^(m - 1)*(c + d*x)^(n + 1)*((e + f*x)^(p + 1)
)/(d*f*(m + n + p + 1)), x] + Simp[1/(d*f*(m + n + p + 1) Int[(a + b*x)
^(m - 2)*(c + d*x)^n*(e + f*x)^p*Simp[a^2*d*f*(m + n + p + 1) - b*(b*c*e*(m
- 1) + a*(d*e*(n + 1) + c*f*(p + 1))) + b*(a*d*f*(2*m + n + p) - b*(d*e*(m
+ n) + c*f*(m + p)))*x, x], x], x] /; FreeQ[{a, b, c, d, e, f, n, p}, x] &
& GtQ[m, 1] && NeQ[m + n + p + 1, 0] && IntegerQ[m]
```

rule 221

```
Int[((a_) + (b_)*(x_)^2)^(-1), x_Symbol] := Simp[(Rt[-a/b, 2]/a)*ArcTanh[x
/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]
```

rule 960

```
Int[((e._)*(x._))^(m._)*((a1_) + (b1_.)*(x_)^(non2_.))^(p_.)*((a2_) + (b2_.)
*(x_)^(non2_.))^(p_.)*((c_) + (d_.)*(x_)^(n_)), x_Symbol] := Simp[d*(e*x)^(
m + 1)*(a1 + b1*x^(n/2))^(p + 1)*((a2 + b2*x^(n/2))^(p + 1)/(b1*b2*e*(m + n
*(p + 1) + 1))), x] - Simp[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/
(b1*b2*(m + n*(p + 1) + 1)) Int[(e*x)^m*(a1 + b1*x^(n/2))^p*(a2 + b2*x^(n
/2))^p, x], x] /; FreeQ[{a1, b1, a2, b2, c, d, e, m, n, p}, x] && EqQ[non2,
n/2] && EqQ[a2*b1 + a1*b2, 0] && NeQ[m + n*(p + 1) + 1, 0]
```

Maple [A] (verified)

Time = 0.13 (sec) , antiderivative size = 160, normalized size of antiderivative = 0.98

method	result
risch	$-\frac{x(8bx^4d^4+12ad^4x^2+10b^2c^2d^2x^2+18ac^2d^2+15b^2c^4)(-dx+c)\sqrt{dx+c}}{48d^6\sqrt{dx-c}} + \frac{c^4(6ad^2+5b^2c^2)\ln\left(\frac{d^2x}{\sqrt{d^2}}+\sqrt{d^2x^2-c^2}\right)\sqrt{(dx-c)(dx+c)}}{16d^6\sqrt{d^2}\sqrt{dx-c}\sqrt{dx+c}}$
default	$\frac{\sqrt{dx-c}\sqrt{dx+c}\left(8\operatorname{csgn}(d)b^2d^5x^5\sqrt{d^2x^2-c^2}+12\operatorname{csgn}(d)ad^5x^3\sqrt{d^2x^2-c^2}+10\operatorname{csgn}(d)b^2c^2d^3x^3\sqrt{d^2x^2-c^2}+18\operatorname{csgn}(d)d^3\sqrt{d^2x^2-c^2}\right)}{48d^7}$

input

```
int(x^4*(b*x^2+a)/(d*x-c)^(1/2)/(d*x+c)^(1/2),x,method=_RETURNVERBOSE)
```

output

```
-1/48*x*(8*b*d^4*x^4+12*a*d^4*x^2+10*b*c^2*d^2*x^2+18*a*c^2*d^2+15*b*c^4)*
(-d*x+c)*(d*x+c)^(1/2)/d^6/(d*x-c)^(1/2)+1/16*c^4*(6*a*d^2+5*b*c^2)/d^6*ln
(d^2*x/(d^2)^(1/2)+(d^2*x^2-c^2)^(1/2))/(d^2)^(1/2)*((d*x-c)*(d*x+c))^(1/2)
)/(d*x-c)^(1/2)/(d*x+c)^(1/2)
```

Fricas [A] (verification not implemented)

Time = 0.13 (sec) , antiderivative size = 115, normalized size of antiderivative = 0.70

$$\int \frac{x^4(a+bx^2)}{\sqrt{-c+dx}\sqrt{c+dx}} dx$$

$$= \frac{(8bd^5x^5 + 2(5bc^2d^3 + 6ad^5)x^3 + 3(5bc^4d + 6ac^2d^3)x)\sqrt{dx+c}\sqrt{dx-c} - 3(5bc^6 + 6ac^4d^2)\log(-dx+c)}{48d^7}$$

input

```
integrate(x^4*(b*x^2+a)/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="fricas")
```

output

```
1/48*((8*b*d^5*x^5 + 2*(5*b*c^2*d^3 + 6*a*d^5)*x^3 + 3*(5*b*c^4*d + 6*a*c^2*d^3)*x)*sqrt(d*x + c)*sqrt(d*x - c) - 3*(5*b*c^6 + 6*a*c^4*d^2)*log(-d*x + sqrt(d*x + c)*sqrt(d*x - c)))/d^7
```

Sympy [F(-1)]

Timed out.

$$\int \frac{x^4(a + bx^2)}{\sqrt{-c + dx}\sqrt{c + dx}} dx = \text{Timed out}$$

input

```
integrate(x**4*(b*x**2+a)/(d*x-c)**(1/2)/(d*x+c)**(1/2),x)
```

output

Timed out

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 196, normalized size of antiderivative = 1.20

$$\begin{aligned} \int \frac{x^4(a + bx^2)}{\sqrt{-c + dx}\sqrt{c + dx}} dx = & \frac{\sqrt{d^2x^2 - c^2}bx^5}{6d^2} + \frac{5\sqrt{d^2x^2 - c^2}bc^2x^3}{24d^4} \\ & + \frac{\sqrt{d^2x^2 - c^2}ax^3}{4d^2} + \frac{5bc^6 \log(2d^2x + 2\sqrt{d^2x^2 - c^2}d)}{16d^7} \\ & + \frac{3ac^4 \log(2d^2x + 2\sqrt{d^2x^2 - c^2}d)}{8d^5} \\ & + \frac{5\sqrt{d^2x^2 - c^2}bc^4x}{16d^6} + \frac{3\sqrt{d^2x^2 - c^2}ac^2x}{8d^4} \end{aligned}$$

input

```
integrate(x^4*(b*x^2+a)/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="maxima")
```

output

```
1/6*sqrt(d^2*x^2 - c^2)*b*x^5/d^2 + 5/24*sqrt(d^2*x^2 - c^2)*b*c^2*x^3/d^4 + 1/4*sqrt(d^2*x^2 - c^2)*a*x^3/d^2 + 5/16*b*c^6*log(2*d^2*x + 2*sqrt(d^2*x^2 - c^2)*d)/d^7 + 3/8*a*c^4*log(2*d^2*x + 2*sqrt(d^2*x^2 - c^2)*d)/d^5 + 5/16*sqrt(d^2*x^2 - c^2)*b*c^4*x/d^6 + 3/8*sqrt(d^2*x^2 - c^2)*a*c^2*x/d^4
```

Giac [A] (verification not implemented)

Time = 0.15 (sec) , antiderivative size = 203, normalized size of antiderivative = 1.24

$$\int \frac{x^4(a + bx^2)}{\sqrt{-c + dx}\sqrt{c + dx}} dx$$

$$= \frac{\left(\left(2 \left((dx + c) \left(4(dx + c) \left(\frac{(dx+c)b}{d^6} - \frac{5bc}{d^6} \right) + \frac{3(15bc^2d^{36} + 2ad^{38})}{d^{42}} \right) - \frac{55bc^3d^{36} + 18acd^{38}}{d^{42}} \right) (dx + c) + \frac{85bc^4d^{36} + 54ac^5}{d^{42}} \right) \right)}{48d}$$

input `integrate(x^4*(b*x^2+a)/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="giac")`

output

```
1/48*(((2*((d*x + c)*(4*(d*x + c)*((d*x + c)*b/d^6 - 5*b*c/d^6) + 3*(15*b*c^2*d^36 + 2*a*d^38)/d^42) - (55*b*c^3*d^36 + 18*a*c*d^38)/d^42)*(d*x + c) + (85*b*c^4*d^36 + 54*a*c^2*d^38)/d^42)*(d*x + c) - 3*(11*b*c^5*d^36 + 10*a*c^3*d^38)/d^42)*sqrt(d*x + c)*sqrt(d*x - c) - 6*(5*b*c^6 + 6*a*c^4*d^2)*log(abs(-sqrt(d*x + c) + sqrt(d*x - c)))/d^6)/d
```

Mupad [B] (verification not implemented)

Time = 48.11 (sec) , antiderivative size = 1682, normalized size of antiderivative = 10.26

$$\int \frac{x^4(a + bx^2)}{\sqrt{-c + dx}\sqrt{c + dx}} dx = \text{Too large to display}$$

input `int((x^4*(a + b*x^2))/((c + d*x)^(1/2)*(d*x - c)^(1/2)),x)`

output

```

((5*b*c^6*((c + d*x)^(1/2) - c^(1/2)))/(4*((-c)^(1/2) - (d*x - c)^(1/2)))
- (175*b*c^6*((c + d*x)^(1/2) - c^(1/2))^3)/(12*((-c)^(1/2) - (d*x - c)^(1/2))^3)
+ (311*b*c^6*((c + d*x)^(1/2) - c^(1/2))^5)/(4*((-c)^(1/2) - (d*x - c)^(1/2))^5)
+ (8361*b*c^6*((c + d*x)^(1/2) - c^(1/2))^7)/(4*((-c)^(1/2) - (d*x - c)^(1/2))^7)
+ (42259*b*c^6*((c + d*x)^(1/2) - c^(1/2))^9)/(6*((-c)^(1/2) - (d*x - c)^(1/2))^9)
+ (25295*b*c^6*((c + d*x)^(1/2) - c^(1/2))^11)/(2*((-c)^(1/2) - (d*x - c)^(1/2))^11)
+ (25295*b*c^6*((c + d*x)^(1/2) - c^(1/2))^13)/(2*((-c)^(1/2) - (d*x - c)^(1/2))^13)
+ (42259*b*c^6*((c + d*x)^(1/2) - c^(1/2))^15)/(6*((-c)^(1/2) - (d*x - c)^(1/2))^15)
+ (8361*b*c^6*((c + d*x)^(1/2) - c^(1/2))^17)/(4*((-c)^(1/2) - (d*x - c)^(1/2))^17)
+ (311*b*c^6*((c + d*x)^(1/2) - c^(1/2))^19)/(4*((-c)^(1/2) - (d*x - c)^(1/2))^19)
- (175*b*c^6*((c + d*x)^(1/2) - c^(1/2))^21)/(12*((-c)^(1/2) - (d*x - c)^(1/2))^21)
+ (5*b*c^6*((c + d*x)^(1/2) - c^(1/2))^23)/(4*((-c)^(1/2) - (d*x - c)^(1/2))^23)
)/((d^7 - (12*d^7*((c + d*x)^(1/2) - c^(1/2))^2)/((-c)^(1/2) - (d*x - c)^(1/2))^2
+ (66*d^7*((c + d*x)^(1/2) - c^(1/2))^4)/((-c)^(1/2) - (d*x - c)^(1/2))^4
- (220*d^7*((c + d*x)^(1/2) - c^(1/2))^6)/((-c)^(1/2) - (d*x - c)^(1/2))^6
+ (495*d^7*((c + d*x)^(1/2) - c^(1/2))^8)/((-c)^(1/2) - (d*x - c)^(1/2))^8
- (792*d^7*((c + d*x)^(1/2) - c^(1/2))^10)/((-c)^(1/2) - (d*x - c)^(1/2))^10
+ (924*d^7*((c + d*x)^(1/2) - c^(1/2))^12)/((-c)^(1/2) - (d*x - c)^(1/2))^12
- (792*d^7*((c + d*x)^(1/2) - c^(1/2))^14)/((-c)^(1/2) - (d*x - c)^(1/2))^14
+ (311*d^7*((c + d*x)^(1/2) - c^(1/2))^16)/((-c)^(1/2) - (d*x - c)^(1/2))^16
- (175*d^7*((c + d*x)^(1/2) - c^(1/2))^18)/((-c)^(1/2) - (d*x - c)^(1/2))^18
+ (5*d^7*((c + d*x)^(1/2) - c^(1/2))^20)/((-c)^(1/2) - (d*x - c)^(1/2))^20)

```

Reduce [B] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 189, normalized size of antiderivative = 1.15

$$\int \frac{x^4(a + bx^2)}{\sqrt{-c + dx}\sqrt{c + dx}} dx$$

$$= \frac{18\sqrt{dx + c}\sqrt{dx - c} a c^2 d^3 x + 12\sqrt{dx + c}\sqrt{dx - c} a d^5 x^3 + 15\sqrt{dx + c}\sqrt{dx - c} b c^4 dx + 10\sqrt{dx + c}\sqrt{dx - c} b c^2 d^3 x^3 + 5\sqrt{dx + c}\sqrt{dx - c} b d^5 x^5 + 48d^7}{48d^7}$$

input

```
int(x^4*(b*x^2+a)/(d*x-c)^(1/2)/(d*x+c)^(1/2),x)
```

output

```
(18*sqrt(c + d*x)*sqrt(-c + d*x)*a*c**2*d**3*x + 12*sqrt(c + d*x)*sqrt(-c + d*x)*a*d**5*x**3 + 15*sqrt(c + d*x)*sqrt(-c + d*x)*b*c**4*d*x + 10*sqrt(c + d*x)*sqrt(-c + d*x)*b*c**2*d**3*x**3 + 8*sqrt(c + d*x)*sqrt(-c + d*x)*b*d**5*x**5 + 36*log((sqrt(-c + d*x) + sqrt(c + d*x))/(sqrt(c)*sqrt(2)))*a*c**4*d**2 + 30*log((sqrt(-c + d*x) + sqrt(c + d*x))/(sqrt(c)*sqrt(2)))*b*c**6)/(48*d**7)
```

3.23 $\int \frac{x^3(a+bx^2)}{\sqrt{-c+dx}\sqrt{c+dx}} dx$

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Reduce [B] (verification not implemented)	205

Optimal result

Integrand size = 31, antiderivative size = 118

$$\int \frac{x^3(a+bx^2)}{\sqrt{-c+dx}\sqrt{c+dx}} dx = \frac{2c^2(4bc^2+5ad^2)\sqrt{-c+dx}\sqrt{c+dx}}{15d^6} + \frac{(4bc^2+5ad^2)x^2\sqrt{-c+dx}\sqrt{c+dx}}{15d^4} + \frac{bx^4\sqrt{-c+dx}\sqrt{c+dx}}{5d^2}$$

output

```
2/15*c^2*(5*a*d^2+4*b*c^2)*(d*x-c)^(1/2)*(d*x+c)^(1/2)/d^6+1/15*(5*a*d^2+4
*b*c^2)*x^2*(d*x-c)^(1/2)*(d*x+c)^(1/2)/d^4+1/5*b*x^4*(d*x-c)^(1/2)*(d*x+c
)^(1/2)/d^2
```

Mathematica [A] (verified)

Time = 0.14 (sec) , antiderivative size = 74, normalized size of antiderivative = 0.63

$$\int \frac{x^3(a+bx^2)}{\sqrt{-c+dx}\sqrt{c+dx}} dx = \frac{\sqrt{-c+dx}\sqrt{c+dx}(5ad^2(2c^2+d^2x^2)+b(8c^4+4c^2d^2x^2+3d^4x^4))}{15d^6}$$

input `Integrate[(x^3*(a + b*x^2))/(Sqrt[-c + d*x]*Sqrt[c + d*x]),x]`

output `(Sqrt[-c + d*x]*Sqrt[c + d*x]*(5*a*d^2*(2*c^2 + d^2*x^2) + b*(8*c^4 + 4*c^2*d^2*x^2 + 3*d^4*x^4)))/(15*d^6)`

Rubi [A] (verified)

Time = 0.39 (sec) , antiderivative size = 110, normalized size of antiderivative = 0.93, number of steps used = 4, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}} = 0.129$, Rules used = {960, 111, 27, 83}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{x^3(a + bx^2)}{\sqrt{dx - c}\sqrt{c + dx}} dx \\
 & \quad \downarrow \text{960} \\
 & \frac{1}{5} \left(5a + \frac{4bc^2}{d^2} \right) \int \frac{x^3}{\sqrt{dx - c}\sqrt{c + dx}} dx + \frac{bx^4\sqrt{dx - c}\sqrt{c + dx}}{5d^2} \\
 & \quad \downarrow \text{111} \\
 & \frac{1}{5} \left(5a + \frac{4bc^2}{d^2} \right) \left(\int \frac{2c^2x}{\sqrt{dx - c}\sqrt{c + dx}} dx + \frac{x^2\sqrt{dx - c}\sqrt{c + dx}}{3d^2} \right) + \frac{bx^4\sqrt{dx - c}\sqrt{c + dx}}{5d^2} \\
 & \quad \downarrow \text{27} \\
 & \frac{1}{5} \left(5a + \frac{4bc^2}{d^2} \right) \left(\frac{2c^2 \int \frac{x}{\sqrt{dx - c}\sqrt{c + dx}} dx}{3d^2} + \frac{x^2\sqrt{dx - c}\sqrt{c + dx}}{3d^2} \right) + \frac{bx^4\sqrt{dx - c}\sqrt{c + dx}}{5d^2} \\
 & \quad \downarrow \text{83} \\
 & \frac{1}{5} \left(\frac{2c^2\sqrt{dx - c}\sqrt{c + dx}}{3d^4} + \frac{x^2\sqrt{dx - c}\sqrt{c + dx}}{3d^2} \right) \left(5a + \frac{4bc^2}{d^2} \right) + \frac{bx^4\sqrt{dx - c}\sqrt{c + dx}}{5d^2}
 \end{aligned}$$

input `Int[(x^3*(a + b*x^2))/(Sqrt[-c + d*x]*Sqrt[c + d*x]),x]`

output

$$\frac{(b^2x^4\sqrt{-c+dx}\sqrt{c+dx})/(5d^2) + ((5a + (4b^2c^2)/d^2)((2c^2\sqrt{-c+dx}\sqrt{c+dx})/(3d^4) + (x^2\sqrt{-c+dx}\sqrt{c+dx}))/3d^2))/5$$

Defintions of rubi rules used

rule 27

$$\text{Int}[(a_*)(Fx_), x_Symbol] \rightarrow \text{Simp}[a \text{ Int}[Fx, x], x] \text{ ; FreeQ}[a, x] \ \&\& \ !\text{MatchQ}[Fx, (b_*)(Gx_)] \text{ ; FreeQ}[b, x]$$

rule 83

$$\text{Int}[(a_*) + (b_*)(x_)*((c_*) + (d_*)(x_))^{(n_*)}((e_*) + (f_*)(x_))^{(p_*)}, x_] \rightarrow \text{Simp}[b*(c + d*x)^{(n + 1)}*((e + f*x)^{(p + 1)}/(d*f*(n + p + 2))), x] \text{ ; FreeQ}[\{a, b, c, d, e, f, n, p\}, x] \ \&\& \ \text{NeQ}[n + p + 2, 0] \ \&\& \ \text{EqQ}[a*d*f*(n + p + 2) - b*(d*e*(n + 1) + c*f*(p + 1)), 0]$$

rule 111

$$\text{Int}[(a_*) + (b_*)(x_))^{(m_*)}((c_*) + (d_*)(x_))^{(n_*)}((e_*) + (f_*)(x_))^{(p_*)}, x_] \rightarrow \text{Simp}[b*(a + b*x)^{(m - 1)}*(c + d*x)^{(n + 1)}*((e + f*x)^{(p + 1)}/(d*f*(m + n + p + 1))), x] + \text{Simp}[1/(d*f*(m + n + p + 1)) \text{ Int}[(a + b*x)^{(m - 2)}*(c + d*x)^n*(e + f*x)^p*\text{Simp}[a^2*d*f*(m + n + p + 1) - b*(b*c*e*(m - 1) + a*(d*e*(n + 1) + c*f*(p + 1))) + b*(a*d*f*(2*m + n + p) - b*(d*e*(m + n) + c*f*(m + p)))*x, x], x] \text{ ; FreeQ}[\{a, b, c, d, e, f, n, p\}, x] \ \&\& \ \text{GtQ}[m, 1] \ \&\& \ \text{NeQ}[m + n + p + 1, 0] \ \&\& \ \text{IntegerQ}[m]$$

rule 960

$$\text{Int}[(e_*)(x_))^{(m_*)}((a1_*) + (b1_*)(x_))^{(non2_*)}((a2_*) + (b2_*)(x_))^{(non2_*)}((c_*) + (d_*)(x_))^{(n_*)}, x_Symbol] \rightarrow \text{Simp}[d*(e*x)^{(m + 1)}*(a1 + b1*x^{(n/2)})^{(p + 1)}*((a2 + b2*x^{(n/2)})^{(p + 1)}/(b1*b2*e*(m + n*(p + 1) + 1))), x] - \text{Simp}[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(b1*b2*(m + n*(p + 1) + 1)) \text{ Int}[(e*x)^m*(a1 + b1*x^{(n/2)})^p*(a2 + b2*x^{(n/2)})^p, x], x] \text{ ; FreeQ}[\{a1, b1, a2, b2, c, d, e, m, n, p\}, x] \ \&\& \ \text{EqQ}[non2, n/2] \ \&\& \ \text{EqQ}[a2*b1 + a1*b2, 0] \ \&\& \ \text{NeQ}[m + n*(p + 1) + 1, 0]$$

Maple [A] (verified)

Time = 0.13 (sec) , antiderivative size = 68, normalized size of antiderivative = 0.58

method	result	size
gospers	$\frac{\sqrt{dx-c}\sqrt{dx+c}(3bx^4d^4+5ad^4x^2+4b^2c^2d^2x^2+10ac^2d^2+8bc^4)}{15d^6}$	68
default	$\frac{\sqrt{dx-c}\sqrt{dx+c}(3bx^4d^4+5ad^4x^2+4b^2c^2d^2x^2+10ac^2d^2+8bc^4)}{15d^6}$	68
risch	$-\frac{\sqrt{dx+c}(-dx+c)(3bx^4d^4+5ad^4x^2+4b^2c^2d^2x^2+10ac^2d^2+8bc^4)}{15d^6\sqrt{dx-c}}$	74
orering	$-\frac{\sqrt{dx+c}(-dx+c)(3bx^4d^4+5ad^4x^2+4b^2c^2d^2x^2+10ac^2d^2+8bc^4)}{15d^6\sqrt{dx-c}}$	74

input `int(x^3*(b*x^2+a)/(d*x-c)^(1/2)/(d*x+c)^(1/2),x,method=_RETURNVERBOSE)`

output `1/15/d^6*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(3*b*d^4*x^4+5*a*d^4*x^2+4*b*c^2*d^2*x^2+10*a*c^2*d^2+8*b*c^4)`

Fricas [A] (verification not implemented)

Time = 0.11 (sec) , antiderivative size = 66, normalized size of antiderivative = 0.56

$$\int \frac{x^3(a+bx^2)}{\sqrt{-c+dx}\sqrt{c+dx}} dx$$

$$= \frac{(3bd^4x^4 + 8bc^4 + 10ac^2d^2 + (4bc^2d^2 + 5ad^4)x^2)\sqrt{dx+c}\sqrt{dx-c}}{15d^6}$$

input `integrate(x^3*(b*x^2+a)/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="fricas")`

output `1/15*(3*b*d^4*x^4 + 8*b*c^4 + 10*a*c^2*d^2 + (4*b*c^2*d^2 + 5*a*d^4)*x^2)*sqrt(d*x + c)*sqrt(d*x - c)/d^6`

Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 10.11 (sec) , antiderivative size = 240, normalized size of antiderivative = 2.03

$$\int \frac{x^3(a+bx^2)}{\sqrt{-c+dx}\sqrt{c+dx}} dx = \frac{ac^3 G_{6,6}^{6,2} \left(\begin{matrix} -\frac{5}{4}, -\frac{3}{4} & -1, -1, -\frac{1}{2}, 1 \\ -\frac{3}{2}, -\frac{5}{4}, -1, -\frac{3}{4}, -\frac{1}{2}, 0 \end{matrix} \middle| \frac{c^2}{d^2 x^2} \right)}{4\pi^{\frac{3}{2}} d^4} + \frac{iac^3 G_{6,6}^{2,6} \left(\begin{matrix} -2, -\frac{7}{4}, -\frac{3}{2}, -\frac{5}{4}, -1, 1 \\ -\frac{7}{4}, -\frac{5}{4} & -2, -\frac{3}{2}, -\frac{3}{2}, 0 \end{matrix} \middle| \frac{c^2 e^{2i\pi}}{d^2 x^2} \right)}{4\pi^{\frac{3}{2}} d^4} + \frac{bc^5 G_{6,6}^{6,2} \left(\begin{matrix} -\frac{9}{4}, -\frac{7}{4} & -2, -2, -\frac{3}{2}, 1 \\ -\frac{5}{2}, -\frac{9}{4}, -2, -\frac{7}{4}, -\frac{3}{2}, 0 \end{matrix} \middle| \frac{c^2}{d^2 x^2} \right)}{4\pi^{\frac{3}{2}} d^6} + \frac{ibc^5 G_{6,6}^{2,6} \left(\begin{matrix} -3, -\frac{11}{4}, -\frac{5}{2}, -\frac{9}{4}, -2, 1 \\ -\frac{11}{4}, -\frac{9}{4} & -3, -\frac{5}{2}, -\frac{5}{2}, 0 \end{matrix} \middle| \frac{c^2 e^{2i\pi}}{d^2 x^2} \right)}{4\pi^{\frac{3}{2}} d^6}$$

input `integrate(x**3*(b*x**2+a)/(d*x-c)**(1/2)/(d*x+c)**(1/2),x)`

output `a*c**3*meijerg(((-5/4, -3/4), (-1, -1, -1/2, 1)), ((-3/2, -5/4, -1, -3/4, -1/2, 0), ()), c**2/(d**2*x**2))/(4*pi**(3/2)*d**4) + I*a*c**3*meijerg(((-2, -7/4, -3/2, -5/4, -1, 1), ()), ((-7/4, -5/4), (-2, -3/2, -3/2, 0)), c**2*exp_polar(2*I*pi)/(d**2*x**2))/(4*pi**(3/2)*d**4) + b*c**5*meijerg(((-9/4, -7/4), (-2, -2, -3/2, 1)), ((-5/2, -9/4, -2, -7/4, -3/2, 0), ()), c**2/(d**2*x**2))/(4*pi**(3/2)*d**6) + I*b*c**5*meijerg(((-3, -11/4, -5/2, -9/4, -2, 1), ()), ((-11/4, -9/4), (-3, -5/2, -5/2, 0)), c**2*exp_polar(2*I*pi)/(d**2*x**2))/(4*pi**(3/2)*d**6)`

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 124, normalized size of antiderivative = 1.05

$$\int \frac{x^3(a + bx^2)}{\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{\sqrt{d^2x^2 - c^2}bx^4}{5d^2} + \frac{4\sqrt{d^2x^2 - c^2}bc^2x^2}{15d^4} + \frac{\sqrt{d^2x^2 - c^2}ax^2}{3d^2} + \frac{8\sqrt{d^2x^2 - c^2}bc^4}{15d^6} + \frac{2\sqrt{d^2x^2 - c^2}ac^2}{3d^4}$$

input `integrate(x^3*(b*x^2+a)/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="maxima")`

output `1/5*sqrt(d^2*x^2 - c^2)*b*x^4/d^2 + 4/15*sqrt(d^2*x^2 - c^2)*b*c^2*x^2/d^4 + 1/3*sqrt(d^2*x^2 - c^2)*a*x^2/d^2 + 8/15*sqrt(d^2*x^2 - c^2)*b*c^4/d^6 + 2/3*sqrt(d^2*x^2 - c^2)*a*c^2/d^4`

Giac [A] (verification not implemented)

Time = 0.14 (sec) , antiderivative size = 124, normalized size of antiderivative = 1.05

$$\int \frac{x^3(a + bx^2)}{\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{\left(\left((dx + c) \left(3(dx + c) \left(\frac{(dx+c)b}{d^5} - \frac{4bc}{d^5} \right) + \frac{22bc^2d^{25} + 5ad^{27}}{d^{30}} \right) - \frac{10(2bc^3d^{25} + acd^{27})}{d^{30}} \right) (dx + c) + \frac{15(bc^4d^{25} + ac^2d^{27})}{d^{30}} \right)}{15d}$$

input `integrate(x^3*(b*x^2+a)/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="giac")`

output `1/15*(((d*x + c)*(3*(d*x + c)*((d*x + c)*b/d^5 - 4*b*c/d^5) + (22*b*c^2*d^25 + 5*a*d^27)/d^30) - 10*(2*b*c^3*d^25 + a*c*d^27)/d^30)*(d*x + c) + 15*(b*c^4*d^25 + a*c^2*d^27)/d^30)*sqrt(d*x + c)*sqrt(d*x - c)/d`

Mupad [B] (verification not implemented)

Time = 5.28 (sec) , antiderivative size = 130, normalized size of antiderivative = 1.10

$$\int \frac{x^3(a + bx^2)}{\sqrt{-c + dx}\sqrt{c + dx}} dx$$

$$= \frac{\sqrt{dx - c} \left(\frac{8bc^5 + 10ac^3d^2}{15d^6} + \frac{x^3(4bc^2d^3 + 5ad^5)}{15d^6} + \frac{x(8bc^4d + 10ac^2d^3)}{15d^6} + \frac{bx^5}{5d} + \frac{x^2(4bc^3d^2 + 5acd^4)}{15d^6} + \frac{bcx^4}{5d^2} \right)}{\sqrt{c + dx}}$$

input `int((x^3*(a + b*x^2))/((c + d*x)^(1/2)*(d*x - c)^(1/2)),x)`output `((d*x - c)^(1/2)*((8*b*c^5 + 10*a*c^3*d^2)/(15*d^6) + (x^3*(5*a*d^5 + 4*b*c^2*d^3))/(15*d^6) + (x*(10*a*c^2*d^3 + 8*b*c^4*d))/(15*d^6) + (b*x^5)/(5*d) + (x^2*(4*b*c^3*d^2 + 5*a*c*d^4))/(15*d^6) + (b*c*x^4)/(5*d^2)))/(c + d*x)^(1/2)`**Reduce [B] (verification not implemented)**

Time = 0.24 (sec) , antiderivative size = 65, normalized size of antiderivative = 0.55

$$\int \frac{x^3(a + bx^2)}{\sqrt{-c + dx}\sqrt{c + dx}} dx$$

$$= \frac{\sqrt{dx + c}\sqrt{dx - c}(3bd^4x^4 + 5ad^4x^2 + 4bc^2d^2x^2 + 10ac^2d^2 + 8bc^4)}{15d^6}$$

input `int(x^3*(b*x^2+a)/(d*x-c)^(1/2)/(d*x+c)^(1/2),x)`output `(sqrt(c + d*x)*sqrt(-c + d*x)*(10*a*c**2*d**2 + 5*a*d**4*x**2 + 8*b*c**4 + 4*b*c**2*d**2*x**2 + 3*b*d**4*x**4))/(15*d**6)`

3.24 $\int \frac{x^2(a+bx^2)}{\sqrt{-c+dx}\sqrt{c+dx}} dx$

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Optimal result

Integrand size = 31, antiderivative size = 118

$$\int \frac{x^2(a+bx^2)}{\sqrt{-c+dx}\sqrt{c+dx}} dx = \frac{(3bc^2 + 4ad^2)x\sqrt{-c+dx}\sqrt{c+dx}}{8d^4} + \frac{bx^3\sqrt{-c+dx}\sqrt{c+dx}}{4d^2} + \frac{c^2(3bc^2 + 4ad^2) \operatorname{arctanh}\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)}{4d^5}$$

output

```
1/8*(4*a*d^2+3*b*c^2)*x*(d*x-c)^(1/2)*(d*x+c)^(1/2)/d^4+1/4*b*x^3*(d*x-c)^(1/2)*(d*x+c)^(1/2)/d^2+1/4*c^2*(4*a*d^2+3*b*c^2)*arctanh((d*x-c)^(1/2)/(d*x+c)^(1/2))/d^5
```

Mathematica [A] (verified)

Time = 0.21 (sec) , antiderivative size = 92, normalized size of antiderivative = 0.78

$$\int \frac{x^2(a+bx^2)}{\sqrt{-c+dx}\sqrt{c+dx}} dx = \frac{dx\sqrt{-c+dx}\sqrt{c+dx}(3bc^2 + 4ad^2 + 2bd^2x^2) + (6bc^4 + 8ac^2d^2) \operatorname{arctanh}\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)}{8d^5}$$

input

```
Integrate[(x^2*(a + b*x^2))/(Sqrt[-c + d*x]*Sqrt[c + d*x]),x]
```

output

```
(d*x*Sqrt[-c + d*x]*Sqrt[c + d*x]*(3*b*c^2 + 4*a*d^2 + 2*b*d^2*x^2) + (6*b*c^4 + 8*a*c^2*d^2)*ArcTanh[Sqrt[-c + d*x]/Sqrt[c + d*x]])/(8*d^5)
```

Rubi [A] (verified)

Time = 0.40 (sec) , antiderivative size = 107, normalized size of antiderivative = 0.91, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.161$, Rules used = {960, 101, 27, 45, 221}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{x^2(a + bx^2)}{\sqrt{dx - c}\sqrt{c + dx}} dx$$

$$\downarrow 960$$

$$\frac{1}{4} \left(4a + \frac{3bc^2}{d^2} \right) \int \frac{x^2}{\sqrt{dx - c}\sqrt{c + dx}} dx + \frac{bx^3\sqrt{dx - c}\sqrt{c + dx}}{4d^2}$$

$$\downarrow 101$$

$$\frac{1}{4} \left(4a + \frac{3bc^2}{d^2} \right) \left(\frac{\int \frac{c^2}{\sqrt{dx - c}\sqrt{c + dx}} dx}{2d^2} + \frac{x\sqrt{dx - c}\sqrt{c + dx}}{2d^2} \right) + \frac{bx^3\sqrt{dx - c}\sqrt{c + dx}}{4d^2}$$

$$\downarrow 27$$

$$\frac{1}{4} \left(4a + \frac{3bc^2}{d^2} \right) \left(\frac{c^2 \int \frac{1}{\sqrt{dx - c}\sqrt{c + dx}} dx}{2d^2} + \frac{x\sqrt{dx - c}\sqrt{c + dx}}{2d^2} \right) + \frac{bx^3\sqrt{dx - c}\sqrt{c + dx}}{4d^2}$$

$$\downarrow 45$$

$$\frac{1}{4} \left(4a + \frac{3bc^2}{d^2} \right) \left(\frac{c^2 \int \frac{1}{d - \frac{d(dx - c)}{c + dx}} d \frac{\sqrt{dx - c}}{\sqrt{c + dx}}}{d^2} + \frac{x\sqrt{dx - c}\sqrt{c + dx}}{2d^2} \right) + \frac{bx^3\sqrt{dx - c}\sqrt{c + dx}}{4d^2}$$

$$\downarrow 221$$

$$\frac{1}{4} \left(4a + \frac{3bc^2}{d^2} \right) \left(\frac{c^2 \operatorname{arctanh} \left(\frac{\sqrt{dx - c}}{\sqrt{c + dx}} \right)}{d^3} + \frac{x\sqrt{dx - c}\sqrt{c + dx}}{2d^2} \right) + \frac{bx^3\sqrt{dx - c}\sqrt{c + dx}}{4d^2}$$

input $\text{Int}[(x^2(a + b*x^2))/(\text{Sqrt}[-c + d*x]*\text{Sqrt}[c + d*x]),x]$

output $(b*x^3*\text{Sqrt}[-c + d*x]*\text{Sqrt}[c + d*x])/(4*d^2) + ((4*a + (3*b*c^2)/d^2)*((x*\text{Sqrt}[-c + d*x]*\text{Sqrt}[c + d*x])/(2*d^2) + (c^2*\text{ArcTanh}[\text{Sqrt}[-c + d*x]/\text{Sqrt}[c + d*x]])/d^3))/4$

Defintions of rubi rules used

rule 27 $\text{Int}[(a_)*(F_x), x_Symbol] \rightarrow \text{Simp}[a \quad \text{Int}[F_x, x], x] /; \text{FreeQ}[a, x] \ \&\& \ !\text{MatchQ}[F_x, (b_)*(G_x)] /; \text{FreeQ}[b, x]$

rule 45 $\text{Int}[1/(\text{Sqrt}[(a_) + (b_)*(x_)]*\text{Sqrt}[(c_) + (d_)*(x_)]), x_Symbol] \rightarrow \text{Simp}[2 \quad \text{Subst}[\text{Int}[1/(b - d*x^2), x], x, \text{Sqrt}[a + b*x]/\text{Sqrt}[c + d*x]], x] /; \text{FreeQ}[\{a, b, c, d\}, x] \ \&\& \ \text{EqQ}[b*c + a*d, 0] \ \&\& \ !\text{GtQ}[c, 0]$

rule 101 $\text{Int}[((a_) + (b_)*(x_))^{2*((c_) + (d_)*(x_))^{(n_)*((e_) + (f_)*(x_))^{(p_)}}, x] \rightarrow \text{Simp}[b*(a + b*x)*(c + d*x)^{(n + 1)*((e + f*x)^{(p + 1))/(d*f*(n + p + 3))}, x] + \text{Simp}[1/(d*f*(n + p + 3)) \quad \text{Int}[(c + d*x)^n*(e + f*x)^p*\text{Simp}[a^2*d*f*(n + p + 3) - b*(b*c*e + a*(d*e*(n + 1) + c*f*(p + 1))) + b*(a*d*f*(n + p + 4) - b*(d*e*(n + 2) + c*f*(p + 2))]*x, x], x] /; \text{FreeQ}[\{a, b, c, d, e, f, n, p\}, x] \ \&\& \ \text{NeQ}[n + p + 3, 0]$

rule 221 $\text{Int}[((a_) + (b_)*(x_)^2)^{-1}, x_Symbol] \rightarrow \text{Simp}[(\text{Rt}[-a/b, 2]/a)*\text{ArcTanh}[x/\text{Rt}[-a/b, 2]], x] /; \text{FreeQ}[\{a, b\}, x] \ \&\& \ \text{NegQ}[a/b]$

rule 960 $\text{Int}[((e_)*(x_))^{(m_)*((a1_) + (b1_)*(x_)^{\text{non2}_})^{(p_)*((a2_) + (b2_)*x_)^{\text{non2}_})^{(p_)*((c_) + (d_)*(x_)^{(n_)}}, x_Symbol] \rightarrow \text{Simp}[d*(e*x)^{(m + 1)*(a1 + b1*x^{(n/2)})^{(p + 1)*((a2 + b2*x^{(n/2)})^{(p + 1)/(b1*b2*e*(m + n*(p + 1) + 1))}, x] - \text{Simp}[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(b1*b2*(m + n*(p + 1) + 1)) \quad \text{Int}[(e*x)^m*(a1 + b1*x^{(n/2)})^p*(a2 + b2*x^{(n/2)})^p, x], x] /; \text{FreeQ}[\{a1, b1, a2, b2, c, d, e, m, n, p\}, x] \ \&\& \ \text{EqQ}[\text{non2}, n/2] \ \&\& \ \text{EqQ}[a2*b1 + a1*b2, 0] \ \&\& \ \text{NeQ}[m + n*(p + 1) + 1, 0]$

Maple [A] (verified)

Time = 0.14 (sec) , antiderivative size = 136, normalized size of antiderivative = 1.15

method	result
risch	$-\frac{x(2bx^2d^2+4ad^2+3bc^2)(-dx+c)\sqrt{dx+c}}{8d^4\sqrt{dx-c}} + \frac{c^2(4ad^2+3bc^2)\ln\left(\frac{d^2x}{\sqrt{d^2}}+\sqrt{d^2x^2-c^2}\right)\sqrt{(dx-c)(dx+c)}}{8d^4\sqrt{d^2}\sqrt{dx-c}\sqrt{dx+c}}$
default	$\frac{\sqrt{dx-c}\sqrt{dx+c}\left(2\operatorname{csgn}(d)b d^3x^3\sqrt{d^2x^2-c^2}+4\operatorname{csgn}(d)d^3\sqrt{d^2x^2-c^2}ax+3\operatorname{csgn}(d)d\sqrt{d^2x^2-c^2}bc^2x+4\ln\left(\left(\sqrt{d^2x^2-c^2}\operatorname{csgn}(d)+d\right)\right)\right)}{8d^5\sqrt{d^2x^2-c^2}}$

input `int(x^2*(b*x^2+a)/(d*x-c)^(1/2)/(d*x+c)^(1/2),x,method=_RETURNVERBOSE)`

output
$$\begin{aligned} & -1/8*x*(2*b*d^2*x^2+4*a*d^2+3*b*c^2)*(-d*x+c)*(d*x+c)^(1/2)/d^4/(d*x-c)^(1/2) \\ & +1/8*c^2*(4*a*d^2+3*b*c^2)/d^4*\ln(d^2*x/(d^2)^(1/2)+(d^2*x^2-c^2)^(1/2)) \\ & /((d^2)^(1/2)*((d*x-c)*(d*x+c))^(1/2)/(d*x-c)^(1/2)/(d*x+c)^(1/2)) \end{aligned}$$

Fricas [A] (verification not implemented)

Time = 0.12 (sec) , antiderivative size = 90, normalized size of antiderivative = 0.76

$$\int \frac{x^2(a+bx^2)}{\sqrt{-c+dx}\sqrt{c+dx}} dx$$

$$= \frac{(2bd^3x^3 + (3bc^2d + 4ad^3)x)\sqrt{dx+c}\sqrt{dx-c} - (3bc^4 + 4ac^2d^2)\log(-dx + \sqrt{dx+c}\sqrt{dx-c})}{8d^5}$$

input `integrate(x^2*(b*x^2+a)/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="fricas")`

output
$$\frac{1}{8} * ((2 * b * d^3 * x^3 + (3 * b * c^2 * d + 4 * a * d^3) * x) * \sqrt{d * x + c} * \sqrt{d * x - c} - (3 * b * c^4 + 4 * a * c^2 * d^2) * \log(-d * x + \sqrt{d * x + c} * \sqrt{d * x - c})) / d^5$$

Sympy [F(-1)]

Timed out.

$$\int \frac{x^2(a + bx^2)}{\sqrt{-c + dx}\sqrt{c + dx}} dx = \text{Timed out}$$

input `integrate(x**2*(b*x**2+a)/(d*x-c)**(1/2)/(d*x+c)**(1/2),x)`

output `Timed out`

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 142, normalized size of antiderivative = 1.20

$$\int \frac{x^2(a + bx^2)}{\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{\sqrt{d^2x^2 - c^2}bx^3}{4d^2} + \frac{3bc^4 \log(2d^2x + 2\sqrt{d^2x^2 - c^2}d)}{8d^5}$$

$$+ \frac{ac^2 \log(2d^2x + 2\sqrt{d^2x^2 - c^2}d)}{2d^3}$$

$$+ \frac{3\sqrt{d^2x^2 - c^2}bc^2x}{8d^4} + \frac{\sqrt{d^2x^2 - c^2}ax}{2d^2}$$

input `integrate(x^2*(b*x^2+a)/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="maxima")`

output `1/4*sqrt(d^2*x^2 - c^2)*b*x^3/d^2 + 3/8*b*c^4*log(2*d^2*x + 2*sqrt(d^2*x^2 - c^2)*d)/d^5 + 1/2*a*c^2*log(2*d^2*x + 2*sqrt(d^2*x^2 - c^2)*d)/d^3 + 3/8*sqrt(d^2*x^2 - c^2)*b*c^2*x/d^4 + 1/2*sqrt(d^2*x^2 - c^2)*a*x/d^2`

Giac [A] (verification not implemented)

Time = 0.14 (sec) , antiderivative size = 140, normalized size of antiderivative = 1.19

$$\int \frac{x^2(a + bx^2)}{\sqrt{-c + dx}\sqrt{c + dx}} dx$$

$$= \frac{\left((dx + c) \left(2(dx + c) \left(\frac{(dx+c)b}{d^4} - \frac{3bc}{d^4} \right) + \frac{9bc^2d^{16} + 4ad^{18}}{d^{20}} \right) - \frac{5bc^3d^{16} + 4acd^{18}}{d^{20}} \right) \sqrt{dx + c} \sqrt{dx - c} - \frac{2(3bc^4 + 4ac^2d^2)}{8d}}{8d}$$

input `integrate(x^2*(b*x^2+a)/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="giac")`

output

```
1/8*(((d*x + c)*(2*(d*x + c)*((d*x + c)*b/d^4 - 3*b*c/d^4) + (9*b*c^2*d^16
+ 4*a*d^18)/d^20) - (5*b*c^3*d^16 + 4*a*c*d^18)/d^20)*sqrt(d*x + c)*sqrt(
d*x - c) - 2*(3*b*c^4 + 4*a*c^2*d^2)*log(abs(-sqrt(d*x + c) + sqrt(d*x - c
)))/d^4)/d
```

Mupad [B] (verification not implemented)

Time = 28.95 (sec) , antiderivative size = 1048, normalized size of antiderivative = 8.88

$$\int \frac{x^2(a + bx^2)}{\sqrt{-c + dx}\sqrt{c + dx}} dx = \text{Too large to display}$$

input `int((x^2*(a + b*x^2))/((c + d*x)^(1/2)*(d*x - c)^(1/2)),x)`

output

```

((2*a*c^2*((c + d*x)^(1/2) - c^(1/2)))/((-c)^(1/2) - (d*x - c)^(1/2)) + (1
4*a*c^2*((c + d*x)^(1/2) - c^(1/2))^3)/((-c)^(1/2) - (d*x - c)^(1/2))^3 +
(14*a*c^2*((c + d*x)^(1/2) - c^(1/2))^5)/((-c)^(1/2) - (d*x - c)^(1/2))^5
+ (2*a*c^2*((c + d*x)^(1/2) - c^(1/2))^7)/((-c)^(1/2) - (d*x - c)^(1/2))^7
)/(d^3 - (4*d^3*((c + d*x)^(1/2) - c^(1/2))^2)/((-c)^(1/2) - (d*x - c)^(1/2
))^2 + (6*d^3*((c + d*x)^(1/2) - c^(1/2))^4)/((-c)^(1/2) - (d*x - c)^(1/2
))^4 - (4*d^3*((c + d*x)^(1/2) - c^(1/2))^6)/((-c)^(1/2) - (d*x - c)^(1/2
))^6 + (d^3*((c + d*x)^(1/2) - c^(1/2))^8)/((-c)^(1/2) - (d*x - c)^(1/2))^8
) - ((23*b*c^4*((c + d*x)^(1/2) - c^(1/2))^3)/(2*((-c)^(1/2) - (d*x - c)^(
1/2))^3) - (3*b*c^4*((c + d*x)^(1/2) - c^(1/2)))/(2*((-c)^(1/2) - (d*x - c
)^(1/2)))) + (333*b*c^4*((c + d*x)^(1/2) - c^(1/2))^5)/(2*((-c)^(1/2) - (d*
x - c)^(1/2))^5) + (671*b*c^4*((c + d*x)^(1/2) - c^(1/2))^7)/(2*((-c)^(1/2)
- (d*x - c)^(1/2))^7) + (671*b*c^4*((c + d*x)^(1/2) - c^(1/2))^9)/(2*((-c
)^(1/2) - (d*x - c)^(1/2))^9) + (333*b*c^4*((c + d*x)^(1/2) - c^(1/2))^11
)/(2*((-c)^(1/2) - (d*x - c)^(1/2))^11) + (23*b*c^4*((c + d*x)^(1/2) - c^(
1/2))^13)/(2*((-c)^(1/2) - (d*x - c)^(1/2))^13) - (3*b*c^4*((c + d*x)^(1/2)
- c^(1/2))^15)/(2*((-c)^(1/2) - (d*x - c)^(1/2))^15))/(d^5 - (8*d^5*((c
+ d*x)^(1/2) - c^(1/2))^2)/((-c)^(1/2) - (d*x - c)^(1/2))^2 + (28*d^5*((c
+ d*x)^(1/2) - c^(1/2))^4)/((-c)^(1/2) - (d*x - c)^(1/2))^4 - (56*d^5*((c
+ d*x)^(1/2) - c^(1/2))^6)/((-c)^(1/2) - (d*x - c)^(1/2))^6 + (70*d^5*(...

```

Reduce [B] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 137, normalized size of antiderivative = 1.16

$$\int \frac{x^2(a + bx^2)}{\sqrt{-c + dx}\sqrt{c + dx}} dx$$

$$= \frac{4\sqrt{dx + c}\sqrt{dx - c}a d^3 x + 3\sqrt{dx + c}\sqrt{dx - c}b c^2 dx + 2\sqrt{dx + c}\sqrt{dx - c}b d^3 x^3 + 8 \log\left(\frac{\sqrt{dx - c} + \sqrt{dx + c}}{\sqrt{c}\sqrt{2}}\right)}{8d^5}$$

input

```
int(x^2*(b*x^2+a)/(d*x-c)^(1/2)/(d*x+c)^(1/2),x)
```

output

```

(4*sqrt(c + d*x)*sqrt(-c + d*x)*a*d**3*x + 3*sqrt(c + d*x)*sqrt(-c + d
*x)*b*c**2*d*x + 2*sqrt(c + d*x)*sqrt(-c + d*x)*b*d**3*x**3 + 8*log((sqr
t(-c + d*x) + sqrt(c + d*x))/(sqrt(c)*sqrt(2)))*a*c**2*d**2 + 6*log((sqr
t(-c + d*x) + sqrt(c + d*x))/(sqrt(c)*sqrt(2)))*b*c**4)/(8*d**5)

```

3.25 $\int \frac{x(a+bx^2)}{\sqrt{-c+dx}\sqrt{c+dx}} dx$

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Optimal result

Integrand size = 29, antiderivative size = 72

$$\int \frac{x(a+bx^2)}{\sqrt{-c+dx}\sqrt{c+dx}} dx = \frac{(2bc^2+3ad^2)\sqrt{-c+dx}\sqrt{c+dx}}{3d^4} + \frac{bx^2\sqrt{-c+dx}\sqrt{c+dx}}{3d^2}$$

output $\frac{1}{3}*(3*a*d^2+2*b*c^2)*(d*x-c)^{(1/2)}*(d*x+c)^{(1/2)}/d^4+1/3*b*x^2*(d*x-c)^{(1/2)}*(d*x+c)^{(1/2)}/d^2$

Mathematica [A] (verified)

Time = 0.11 (sec) , antiderivative size = 48, normalized size of antiderivative = 0.67

$$\int \frac{x(a+bx^2)}{\sqrt{-c+dx}\sqrt{c+dx}} dx = \frac{\sqrt{-c+dx}\sqrt{c+dx}(2bc^2+3ad^2+bd^2x^2)}{3d^4}$$

input `Integrate[(x*(a + b*x^2))/(Sqrt[-c + d*x]*Sqrt[c + d*x]),x]`

output $(\text{Sqrt}[-c + d*x]*\text{Sqrt}[c + d*x]*(2*b*c^2 + 3*a*d^2 + b*d^2*x^2))/(3*d^4)$

Rubi [A] (verified)

Time = 0.32 (sec) , antiderivative size = 72, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}} = 0.069$, Rules used = {960, 83}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{x(a + bx^2)}{\sqrt{dx - c}\sqrt{c + dx}} dx$$

↓ 960

$$\frac{1}{3} \left(3a + \frac{2bc^2}{d^2} \right) \int \frac{x}{\sqrt{dx - c}\sqrt{c + dx}} dx + \frac{bx^2\sqrt{dx - c}\sqrt{c + dx}}{3d^2}$$

↓ 83

$$\frac{\sqrt{dx - c}\sqrt{c + dx} \left(3a + \frac{2bc^2}{d^2} \right)}{3d^2} + \frac{bx^2\sqrt{dx - c}\sqrt{c + dx}}{3d^2}$$

input `Int[(x*(a + b*x^2))/(Sqrt[-c + d*x]*Sqrt[c + d*x]),x]`

output `((3*a + (2*b*c^2)/d^2)*Sqrt[-c + d*x]*Sqrt[c + d*x])/(3*d^2) + (b*x^2*Sqrt[-c + d*x]*Sqrt[c + d*x])/(3*d^2)`

Defintions of rubi rules used

rule 83 `Int[((a_.) + (b_.)*(x_))*((c_.) + (d_.)*(x_)^(n_.))*((e_.) + (f_.)*(x_)^(p_.), x_] :> Simp[b*(c + d*x)^(n + 1)*((e + f*x)^(p + 1)/(d*f*(n + p + 2))), x] /; FreeQ[{a, b, c, d, e, f, n, p}, x] && NeQ[n + p + 2, 0] && EqQ[a*d*f*(n + p + 2) - b*(d*e*(n + 1) + c*f*(p + 1)), 0]`

rule 960

```
Int[((e._)*(x._))^(m._)*((a1_) + (b1_.)*(x._)^(non2_.))^(p._)*((a2_) + (b2_.)
*(x._)^(non2_.))^(p._)*((c_) + (d_.)*(x._)^(n_)), x_Symbol] := Simp[d*(e*x)^(
m + 1)*(a1 + b1*x^(n/2))^(p + 1)*((a2 + b2*x^(n/2))^(p + 1)/(b1*b2*e*(m + n
*(p + 1) + 1))), x] - Simp[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/
(b1*b2*(m + n*(p + 1) + 1)) Int[(e*x)^m*(a1 + b1*x^(n/2))^p*(a2 + b2*x^(n
/2))^p, x], x] /; FreeQ[{a1, b1, a2, b2, c, d, e, m, n, p}, x] && EqQ[non2,
n/2] && EqQ[a2*b1 + a1*b2, 0] && NeQ[m + n*(p + 1) + 1, 0]
```

Maple [A] (verified)

Time = 0.13 (sec) , antiderivative size = 43, normalized size of antiderivative = 0.60

method	result	size
gospers	$\frac{\sqrt{dx-c}\sqrt{dx+c}(bx^2d^2+3ad^2+2bc^2)}{3d^4}$	43
default	$\frac{\sqrt{dx-c}\sqrt{dx+c}(bx^2d^2+3ad^2+2bc^2)}{3d^4}$	43
risch	$-\frac{\sqrt{dx+c}(-dx+c)(bx^2d^2+3ad^2+2bc^2)}{3d^4\sqrt{dx-c}}$	49
orering	$-\frac{\sqrt{dx+c}(-dx+c)(bx^2d^2+3ad^2+2bc^2)}{3d^4\sqrt{dx-c}}$	49

input

```
int(x*(b*x^2+a)/(d*x-c)^(1/2)/(d*x+c)^(1/2),x,method=_RETURNVERBOSE)
```

output

```
1/3/d^4*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(b*d^2*x^2+3*a*d^2+2*b*c^2)
```

Fricas [A] (verification not implemented)

Time = 0.10 (sec) , antiderivative size = 42, normalized size of antiderivative = 0.58

$$\int \frac{x(a + bx^2)}{\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{(bd^2x^2 + 2bc^2 + 3ad^2)\sqrt{dx + c}\sqrt{dx - c}}{3d^4}$$

input

```
integrate(x*(b*x^2+a)/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="fricas")
```

output

```
1/3*(b*d^2*x^2 + 2*b*c^2 + 3*a*d^2)*sqrt(d*x + c)*sqrt(d*x - c)/d^4
```


Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 5.17 (sec) , antiderivative size = 223, normalized size of antiderivative = 3.10

$$\int \frac{x(a + bx^2)}{\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{{}_6G_{6,2}^{6,2} \left(\begin{matrix} -\frac{1}{4}, \frac{1}{4} & 0, 0, \frac{1}{2}, 1 \\ -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, 0 \end{matrix} \middle| \frac{c^2}{d^2 x^2} \right)}{4\pi^{\frac{3}{2}} d^2} + \frac{{}_6G_{6,6}^{2,6} \left(\begin{matrix} -1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, 1 \\ -\frac{3}{4}, -\frac{1}{4} & -1, -\frac{1}{2}, -\frac{1}{2}, 0 \end{matrix} \middle| \frac{c^2 e^{2i\pi}}{d^2 x^2} \right)}{4\pi^{\frac{3}{2}} d^2} + \frac{{}_6G_{6,6}^{6,2} \left(\begin{matrix} -\frac{5}{4}, -\frac{3}{4} & -1, -1, -\frac{1}{2}, 1 \\ -\frac{3}{2}, -\frac{5}{4}, -1, -\frac{3}{4}, -\frac{1}{2}, 0 \end{matrix} \middle| \frac{c^2}{d^2 x^2} \right)}{4\pi^{\frac{3}{2}} d^4} + \frac{{}_6G_{6,6}^{2,6} \left(\begin{matrix} -2, -\frac{7}{4}, -\frac{3}{2}, -\frac{5}{4}, -1, 1 \\ -\frac{7}{4}, -\frac{5}{4} & -2, -\frac{3}{2}, -\frac{3}{2}, 0 \end{matrix} \middle| \frac{c^2 e^{2i\pi}}{d^2 x^2} \right)}{4\pi^{\frac{3}{2}} d^4}$$

input `integrate(x*(b*x**2+a)/(d*x-c)**(1/2)/(d*x+c)**(1/2),x)`

output `a*c*meijerg(((-1/4, 1/4), (0, 0, 1/2, 1)), ((-1/2, -1/4, 0, 1/4, 1/2, 0), ()), c**2/(d**2*x**2))/(4*pi**(3/2)*d**2) + I*a*c*meijerg(((-1, -3/4, -1/2, -1/4, 0, 1), ()), ((-3/4, -1/4), (-1, -1/2, -1/2, 0)), c**2*exp_polar(2*I*pi)/(d**2*x**2))/(4*pi**(3/2)*d**2) + b*c**3*meijerg(((-5/4, -3/4), (-1, -1, -1/2, 1)), ((-3/2, -5/4, -1, -3/4, -1/2, 0), ()), c**2/(d**2*x**2))/(4*pi**(3/2)*d**4) + I*b*c**3*meijerg(((-2, -7/4, -3/2, -5/4, -1, 1), ()), ((-7/4, -5/4), (-2, -3/2, -3/2, 0)), c**2*exp_polar(2*I*pi)/(d**2*x**2))/(4*pi**(3/2)*d**4)`

Maxima [A] (verification not implemented)

Time = 0.04 (sec) , antiderivative size = 69, normalized size of antiderivative = 0.96

$$\int \frac{x(a + bx^2)}{\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{\sqrt{d^2x^2 - c^2}bx^2}{3d^2} + \frac{2\sqrt{d^2x^2 - c^2}bc^2}{3d^4} + \frac{\sqrt{d^2x^2 - c^2}a}{d^2}$$

input `integrate(x*(b*x^2+a)/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="maxima")`output `1/3*sqrt(d^2*x^2 - c^2)*b*x^2/d^2 + 2/3*sqrt(d^2*x^2 - c^2)*b*c^2/d^4 + sqrt(d^2*x^2 - c^2)*a/d^2`**Giac [A] (verification not implemented)**

Time = 0.13 (sec) , antiderivative size = 65, normalized size of antiderivative = 0.90

$$\int \frac{x(a + bx^2)}{\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{\sqrt{dx + c}\sqrt{dx - c} \left((dx + c) \left(\frac{(dx+c)b}{d^3} - \frac{2bc}{d^3} \right) + \frac{3(bc^2d^9 + ad^{11})}{d^{12}} \right)}{3d}$$

input `integrate(x*(b*x^2+a)/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="giac")`output `1/3*sqrt(d*x + c)*sqrt(d*x - c)*((d*x + c)*((d*x + c)*b/d^3 - 2*b*c/d^3) + 3*(b*c^2*d^9 + a*d^11)/d^12)/d`**Mupad [B] (verification not implemented)**

Time = 4.98 (sec) , antiderivative size = 76, normalized size of antiderivative = 1.06

$$\int \frac{x(a + bx^2)}{\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{\sqrt{dx - c} \left(\frac{2bc^3 + 3acd^2}{3d^4} + \frac{bx^3}{3d} + \frac{x(2bc^2d + 3ad^3)}{3d^4} + \frac{bcx^2}{3d^2} \right)}{\sqrt{c + dx}}$$

input `int((x*(a + b*x^2))/((c + d*x)^(1/2)*(d*x - c)^(1/2)),x)`

output $((d*x - c)^{(1/2)*((2*b*c^3 + 3*a*c*d^2)/(3*d^4) + (b*x^3)/(3*d) + (x*(3*a*d^3 + 2*b*c^2*d))/(3*d^4) + (b*c*x^2)/(3*d^2)))/(c + d*x)^{(1/2)}$

Reduce [B] (verification not implemented)

Time = 0.21 (sec) , antiderivative size = 40, normalized size of antiderivative = 0.56

$$\int \frac{x(a + bx^2)}{\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{\sqrt{dx + c}\sqrt{dx - c}(bd^2x^2 + 3ad^2 + 2bc^2)}{3d^4}$$

input `int(x*(b*x^2+a)/(d*x-c)^(1/2)/(d*x+c)^(1/2),x)`

output $(\text{sqrt}(c + d*x)*\text{sqrt}(-c + d*x)*(3*a*d**2 + 2*b*c**2 + b*d**2*x**2))/(3*d**4)$

3.26 $\int \frac{a+bx^2}{\sqrt{-c+dx}\sqrt{c+dx}} dx$

Optimal result	219
Mathematica [A] (verified)	219
Rubi [A] (verified)	220
Maple [A] (verified)	221
Fricas [A] (verification not implemented)	221
Sympy [F(-1)]	222
Maxima [A] (verification not implemented)	222
Giac [A] (verification not implemented)	223
Mupad [B] (verification not implemented)	223
Reduce [B] (verification not implemented)	224

Optimal result

Integrand size = 28, antiderivative size = 68

$$\int \frac{a + bx^2}{\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{bx\sqrt{-c + dx}\sqrt{c + dx}}{2d^2} + \frac{(bc^2 + 2ad^2) \operatorname{arctanh}\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)}{d^3}$$

output $\frac{1}{2}bx\sqrt{d^2x-c}\sqrt{d^2x+c}/d^2+(2ad^2+bc^2)\operatorname{arctanh}\left(\frac{\sqrt{d^2x-c}}{\sqrt{d^2x+c}}\right)/d^3$

Mathematica [A] (verified)

Time = 0.14 (sec) , antiderivative size = 68, normalized size of antiderivative = 1.00

$$\int \frac{a + bx^2}{\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{bdx\sqrt{-c + dx}\sqrt{c + dx} + 2(bc^2 + 2ad^2) \operatorname{arctanh}\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)}{2d^3}$$

input `Integrate[(a + b*x^2)/(Sqrt[-c + d*x]*Sqrt[c + d*x]),x]`

output $(bd^2x\sqrt{-c + dx}\sqrt{c + dx} + 2(bc^2 + 2ad^2)\operatorname{ArcTanh}[\sqrt{-c + dx}/\sqrt{c + dx}])/(2d^3)$

Rubi [A] (verified)

Time = 0.31 (sec) , antiderivative size = 68, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.107$, Rules used = {646, 45, 221}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{a + bx^2}{\sqrt{dx - c}\sqrt{c + dx}} dx$$

$$\downarrow 646$$

$$\frac{(2ad^2 + bc^2) \int \frac{1}{\sqrt{dx - c}\sqrt{c + dx}} dx}{2d^2} + \frac{bx\sqrt{dx - c}\sqrt{c + dx}}{2d^2}$$

$$\downarrow 45$$

$$\frac{(2ad^2 + bc^2) \int \frac{1}{d - \frac{d(dx - c)}{c + dx}} d \frac{\sqrt{dx - c}}{\sqrt{c + dx}}}{d^2} + \frac{bx\sqrt{dx - c}\sqrt{c + dx}}{2d^2}$$

$$\downarrow 221$$

$$\frac{(2ad^2 + bc^2) \operatorname{arctanh}\left(\frac{\sqrt{dx - c}}{\sqrt{c + dx}}\right)}{d^3} + \frac{bx\sqrt{dx - c}\sqrt{c + dx}}{2d^2}$$

input `Int[(a + b*x^2)/(Sqrt[-c + d*x]*Sqrt[c + d*x]),x]`

output `(b*x*Sqrt[-c + d*x]*Sqrt[c + d*x])/(2*d^2) + ((b*c^2 + 2*a*d^2)*ArcTanh[Sqrt[-c + d*x]/Sqrt[c + d*x]])/d^3`

Defintions of rubi rules used

rule 45 `Int[1/(Sqrt[(a_) + (b_.)*(x_)]*Sqrt[(c_) + (d_.)*(x_)]), x_Symbol] := Simp[2 Subst[Int[1/(b - d*x^2), x], x, Sqrt[a + b*x]/Sqrt[c + d*x]], x] /; FreeQ[{a, b, c, d}, x] && EqQ[b*c + a*d, 0] && !GtQ[c, 0]`

rule 221 `Int[((a_) + (b_)*(x_)^2)^(-1), x_Symbol] := Simp[(Rt[-a/b, 2]/a)*ArcTanh[x/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]`

rule 646 `Int[((c_) + (d_)*(x_)^(m_))*((e_) + (f_)*(x_)^(n_))*((a_) + (b_)*(x_)^2), x_Symbol] := Simp[b*x*(c + d*x)^(m + 1)*((e + f*x)^(n + 1)/(d*f*(2*m + 3))), x] - Simp[(b*c*e - a*d*f*(2*m + 3))/(d*f*(2*m + 3)) Int[(c + d*x)^m*(e + f*x)^n, x], x] /; FreeQ[{a, b, c, d, e, f, m, n}, x] && EqQ[m, n] && EqQ[d*e + c*f, 0] && !LtQ[m, -1]`

Maple [A] (verified)

Time = 0.13 (sec) , antiderivative size = 111, normalized size of antiderivative = 1.63

method	result
risch	$-\frac{(-dx+c)\sqrt{dx+c}bx}{2d^2\sqrt{dx-c}} + \frac{(2ad^2+bc^2)\ln\left(\frac{d^2x}{\sqrt{d^2}+\sqrt{d^2x^2-c^2}}\right)\sqrt{(dx-c)(dx+c)}}{2d^2\sqrt{d^2}\sqrt{dx-c}\sqrt{dx+c}}$
default	$\frac{\sqrt{dx-c}\sqrt{dx+c}\left(\operatorname{csgn}(d)d\sqrt{d^2x^2-c^2}bx+\ln\left(\left(\sqrt{d^2x^2-c^2}\operatorname{csgn}(d)+dx\right)\operatorname{csgn}(d)\right)bc^2+2\ln\left(\left(\sqrt{d^2x^2-c^2}\operatorname{csgn}(d)+dx\right)\operatorname{csgn}(d)\right)ad\right)}{2d^3\sqrt{d^2x^2-c^2}}$

input `int((b*x^2+a)/(d*x-c)^(1/2)/(d*x+c)^(1/2),x,method=_RETURNVERBOSE)`

output `-1/2*(-d*x+c)*(d*x+c)^(1/2)*b*x/d^2/(d*x-c)^(1/2)+1/2*(2*a*d^2+b*c^2)/d^2*ln(d^2*x/(d^2)^(1/2)+(d^2*x^2-c^2)^(1/2))/(d^2)^(1/2)*((d*x-c)*(d*x+c))^(1/2)/(d*x-c)^(1/2)/(d*x+c)^(1/2)`

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 63, normalized size of antiderivative = 0.93

$$\int \frac{a + bx^2}{\sqrt{-c + dx}\sqrt{c + dx}} dx$$

$$= \frac{\sqrt{dx + c}\sqrt{dx - c}bdx - (bc^2 + 2ad^2)\log(-dx + \sqrt{dx + c}\sqrt{dx - c})}{2d^3}$$

input `integrate((b*x^2+a)/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="fricas")`

output

```
1/2*(sqrt(d*x + c)*sqrt(d*x - c)*b*d*x - (b*c^2 + 2*a*d^2)*log(-d*x + sqrt
(d*x + c)*sqrt(d*x - c)))/d^3
```

Sympy [F(-1)]

Timed out.

$$\int \frac{a + bx^2}{\sqrt{-c + dx}\sqrt{c + dx}} dx = \text{Timed out}$$

input

```
integrate((b*x**2+a)/(d*x-c)**(1/2)/(d*x+c)**(1/2),x)
```

output

Timed out

Maxima [A] (verification not implemented)

Time = 0.04 (sec) , antiderivative size = 89, normalized size of antiderivative = 1.31

$$\int \frac{a + bx^2}{\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{bc^2 \log(2d^2x + 2\sqrt{d^2x^2 - c^2}d)}{2d^3} + \frac{a \log(2d^2x + 2\sqrt{d^2x^2 - c^2}d)}{d} + \frac{\sqrt{d^2x^2 - c^2}bx}{2d^2}$$

input

```
integrate((b*x^2+a)/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="maxima")
```

output

```
1/2*b*c^2*log(2*d^2*x + 2*sqrt(d^2*x^2 - c^2)*d)/d^3 + a*log(2*d^2*x + 2*s
qrt(d^2*x^2 - c^2)*d)/d + 1/2*sqrt(d^2*x^2 - c^2)*b*x/d^2
```

Giac [A] (verification not implemented)

Time = 0.14 (sec) , antiderivative size = 79, normalized size of antiderivative = 1.16

$$\int \frac{a + bx^2}{\sqrt{-c + dx}\sqrt{c + dx}} dx$$

$$= \frac{\sqrt{dx + c}\sqrt{dx - c} \left(\frac{(dx+c)b}{d^2} - \frac{bc}{d^2} \right) - \frac{2(bc^2 + 2ad^2) \log(|-\sqrt{dx+c} + \sqrt{dx-c}|)}{d^2}}{2d}$$

input `integrate((b*x^2+a)/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="giac")`output `1/2*(sqrt(d*x + c)*sqrt(d*x - c)*((d*x + c)*b/d^2 - b*c/d^2) - 2*(b*c^2 + 2*a*d^2)*log(abs(-sqrt(d*x + c) + sqrt(d*x - c)))/d^2)/d`**Mupad [B] (verification not implemented)**

Time = 13.34 (sec) , antiderivative size = 417, normalized size of antiderivative = 6.13

$$\int \frac{a + bx^2}{\sqrt{-c + dx}\sqrt{c + dx}} dx$$

$$= \frac{\frac{2bc^2(\sqrt{c+dx}-\sqrt{c})}{\sqrt{-c}-\sqrt{dx-c}} + \frac{14bc^2(\sqrt{c+dx}-\sqrt{c})^3}{(\sqrt{-c}-\sqrt{dx-c})^3} + \frac{14bc^2(\sqrt{c+dx}-\sqrt{c})^5}{(\sqrt{-c}-\sqrt{dx-c})^5} + \frac{2bc^2(\sqrt{c+dx}-\sqrt{c})^7}{(\sqrt{-c}-\sqrt{dx-c})^7}}{d^3 - \frac{4d^3(\sqrt{c+dx}-\sqrt{c})^2}{(\sqrt{-c}-\sqrt{dx-c})^2} + \frac{6d^3(\sqrt{c+dx}-\sqrt{c})^4}{(\sqrt{-c}-\sqrt{dx-c})^4} - \frac{4d^3(\sqrt{c+dx}-\sqrt{c})^6}{(\sqrt{-c}-\sqrt{dx-c})^6} + \frac{d^3(\sqrt{c+dx}-\sqrt{c})^8}{(\sqrt{-c}-\sqrt{dx-c})^8}}$$

$$+ \frac{4a \operatorname{atan}\left(\frac{d(\sqrt{-c}-\sqrt{dx-c})}{\sqrt{-d^2}(\sqrt{c+dx}-\sqrt{c})}\right)}{\sqrt{-d^2}} - \frac{2bc^2 \operatorname{atanh}\left(\frac{\sqrt{c+dx}-\sqrt{c}}{\sqrt{-c}-\sqrt{dx-c}}\right)}{d^3}$$

input `int((a + b*x^2)/((c + d*x)^(1/2)*(d*x - c)^(1/2)),x)`

output

```
((2*b*c^2*((c + d*x)^(1/2) - c^(1/2)))/((-c)^(1/2) - (d*x - c)^(1/2)) + (1
4*b*c^2*((c + d*x)^(1/2) - c^(1/2))^3)/((-c)^(1/2) - (d*x - c)^(1/2))^3 +
(14*b*c^2*((c + d*x)^(1/2) - c^(1/2))^5)/((-c)^(1/2) - (d*x - c)^(1/2))^5
+ (2*b*c^2*((c + d*x)^(1/2) - c^(1/2))^7)/((-c)^(1/2) - (d*x - c)^(1/2))^7
)/(d^3 - (4*d^3*((c + d*x)^(1/2) - c^(1/2))^2)/((-c)^(1/2) - (d*x - c)^(1/
2))^2 + (6*d^3*((c + d*x)^(1/2) - c^(1/2))^4)/((-c)^(1/2) - (d*x - c)^(1/2
))^4 - (4*d^3*((c + d*x)^(1/2) - c^(1/2))^6)/((-c)^(1/2) - (d*x - c)^(1/2
))^6 + (d^3*((c + d*x)^(1/2) - c^(1/2))^8)/((-c)^(1/2) - (d*x - c)^(1/2))^8
) + (4*a*atan((d*(-c)^(1/2) - (d*x - c)^(1/2)))/((-d^2)^(1/2)*((c + d*x)^(
1/2) - c^(1/2))))/(-d^2)^(1/2) - (2*b*c^2*atanh(((c + d*x)^(1/2) - c^(1/
2)))/((-c)^(1/2) - (d*x - c)^(1/2)))/d^3
```

Reduce [B] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 86, normalized size of antiderivative = 1.26

$$\int \frac{a + bx^2}{\sqrt{-c + dx}\sqrt{c + dx}} dx$$

$$= \frac{\sqrt{dx + c}\sqrt{dx - c} b dx + 4 \log\left(\frac{\sqrt{dx - c} + \sqrt{dx + c}}{\sqrt{c}\sqrt{2}}\right) a d^2 + 2 \log\left(\frac{\sqrt{dx - c} + \sqrt{dx + c}}{\sqrt{c}\sqrt{2}}\right) b c^2}{2d^3}$$

input

```
int((b*x^2+a)/(d*x-c)^(1/2)/(d*x+c)^(1/2),x)
```

output

```
(sqrt(c + d*x)*sqrt(-c + d*x)*b*d*x + 4*log((sqrt(-c + d*x) + sqrt(c +
d*x))/(sqrt(c)*sqrt(2)))*a*d**2 + 2*log((sqrt(-c + d*x) + sqrt(c + d*x)
)/(sqrt(c)*sqrt(2)))*b*c**2)/(2*d**3)
```

3.27 $\int \frac{a+bx^2}{x\sqrt{-c+dx}\sqrt{c+dx}} dx$

Optimal result	225
Mathematica [A] (verified)	225
Rubi [A] (verified)	226
Maple [B] (verified)	227
Fricas [A] (verification not implemented)	227
Sympy [C] (verification not implemented)	228
Maxima [A] (verification not implemented)	229
Giac [A] (verification not implemented)	229
Mupad [B] (verification not implemented)	229
Reduce [B] (verification not implemented)	230

Optimal result

Integrand size = 31, antiderivative size = 56

$$\int \frac{a + bx^2}{x\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{b\sqrt{-c + dx}\sqrt{c + dx}}{d^2} + \frac{a \arctan\left(\frac{\sqrt{-c+dx}\sqrt{c+dx}}{c}\right)}{c}$$

output `b*(d*x-c)^(1/2)*(d*x+c)^(1/2)/d^2+a*arctan((d*x-c)^(1/2)*(d*x+c)^(1/2)/c)/c`

Mathematica [A] (verified)

Time = 0.10 (sec) , antiderivative size = 54, normalized size of antiderivative = 0.96

$$\int \frac{a + bx^2}{x\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{b\sqrt{-c + dx}\sqrt{c + dx}}{d^2} + \frac{2a \arctan\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)}{c}$$

input `Integrate[(a + b*x^2)/(x*Sqrt[-c + d*x]*Sqrt[c + d*x]),x]`

output `(b*Sqrt[-c + d*x]*Sqrt[c + d*x])/d^2 + (2*a*ArcTan[Sqrt[-c + d*x]/Sqrt[c + d*x]])/c`

Rubi [A] (verified)

Time = 0.36 (sec) , antiderivative size = 56, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.097$, Rules used = {960, 103, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{a + bx^2}{x\sqrt{dx - c}\sqrt{c + dx}} dx$$

↓ 960

$$a \int \frac{1}{x\sqrt{dx - c}\sqrt{c + dx}} dx + \frac{b\sqrt{dx - c}\sqrt{c + dx}}{d^2}$$

↓ 103

$$ad \int \frac{1}{dc^2 + d(dx - c)(c + dx)} d(\sqrt{dx - c}\sqrt{c + dx}) + \frac{b\sqrt{dx - c}\sqrt{c + dx}}{d^2}$$

↓ 218

$$\frac{a \arctan\left(\frac{\sqrt{dx - c}\sqrt{c + dx}}{c}\right)}{c} + \frac{b\sqrt{dx - c}\sqrt{c + dx}}{d^2}$$

input `Int[(a + b*x^2)/(x*Sqrt[-c + d*x]*Sqrt[c + d*x]),x]`

output `(b*Sqrt[-c + d*x]*Sqrt[c + d*x])/d^2 + (a*ArcTan[(Sqrt[-c + d*x]*Sqrt[c + d*x])/c])/c`

Defintions of rubi rules used

rule 103 `Int[1/(Sqrt[(a_.) + (b_.)*(x_)]*Sqrt[(c_.) + (d_.)*(x_)]*((e_.) + (f_.)*(x_))), x_] := Simp[b*f Subst[Int[1/(d*(b*e - a*f)^2 + b*f^2*x^2), x], x, Sqrt[a + b*x]*Sqrt[c + d*x]], x] /; FreeQ[{a, b, c, d, e, f}, x] && EqQ[2*b*d*e - f*(b*c + a*d), 0]`

rule 218 `Int[((a_) + (b_)*(x_)^2)^(-1), x_Symbol] := Simp[(Rt[a/b, 2]/a)*ArcTan[x/Rt[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]`

rule 960 `Int[((e_)*(x_))^(m_)*((a1_) + (b1_)*(x_)^(non2_))^(p_)*((a2_) + (b2_)*(x_)^(non2_))^(p_)*((c_) + (d_)*(x_)^(n_)), x_Symbol] := Simp[d*(e*x)^(m + 1)*(a1 + b1*x^(n/2))^(p + 1)*(a2 + b2*x^(n/2))^(p + 1)/(b1*b2*e*(m + n*(p + 1) + 1)), x] - Simp[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(b1*b2*(m + n*(p + 1) + 1)) Int[(e*x)^m*(a1 + b1*x^(n/2))^p*(a2 + b2*x^(n/2))^p, x], x] /; FreeQ[{a1, b1, a2, b2, c, d, e, m, n, p}, x] && EqQ[non2, n/2] && EqQ[a2*b1 + a1*b2, 0] && NeQ[m + n*(p + 1) + 1, 0]`

Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 107 vs. 2(48) = 96.

Time = 0.12 (sec) , antiderivative size = 108, normalized size of antiderivative = 1.93

method	result	size
default	$\frac{\left(-\ln\left(-\frac{2(c^2 - \sqrt{-c^2} \sqrt{d^2 x^2 - c^2})}{x}\right) a d^2 + b \sqrt{-c^2} \sqrt{d^2 x^2 - c^2}\right) \sqrt{dx-c} \sqrt{dx+c}}{\sqrt{d^2 x^2 - c^2} \sqrt{-c^2} d^2}$	108

input `int((b*x^2+a)/x/(d*x-c)^(1/2)/(d*x+c)^(1/2),x,method=_RETURNVERBOSE)`

output
$$\frac{(-\ln(-2*(c^2 - (-c^2)^{1/2}*(d^2*x^2 - c^2)^{1/2}))/x)*a*d^2 + b*(-c^2)^{1/2}*(d^2*x^2 - c^2)^{1/2})*(d*x - c)^{1/2}*(d*x + c)^{1/2}}{(d^2*x^2 - c^2)^{1/2}/(-c^2)^{1/2}/d^2}$$

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 61, normalized size of antiderivative = 1.09

$$\int \frac{a + bx^2}{x\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{2ad^2 \arctan\left(-\frac{dx - \sqrt{dx+c}\sqrt{dx-c}}{c}\right) + \sqrt{dx+c}\sqrt{dx-c}bc}{cd^2}$$

input `integrate((b*x^2+a)/x/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="fricas")`

output

```
(2*a*d^2*arctan(-(d*x - sqrt(d*x + c))*sqrt(d*x - c))/c) + sqrt(d*x + c)*sqrt(d*x - c)*b*c)/(c*d^2)
```

Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 27.75 (sec) , antiderivative size = 178, normalized size of antiderivative = 3.18

$$\int \frac{a + bx^2}{x\sqrt{-c + dx}\sqrt{c + dx}} dx = -\frac{aG_{6,6}^{5,3}\left(\begin{matrix} \frac{3}{4}, \frac{5}{4}, 1 & 1, 1, \frac{3}{2} \\ \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2} & 0 \end{matrix} \middle| \frac{c^2}{d^2x^2}\right)}{4\pi^{\frac{3}{2}}c}$$

$$+ \frac{iaG_{6,6}^{2,6}\left(\begin{matrix} 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 1 \\ \frac{1}{4}, \frac{3}{4} & 0, \frac{1}{2}, \frac{1}{2}, 0 \end{matrix} \middle| \frac{c^2 e^{2i\pi}}{d^2x^2}\right)}{4\pi^{\frac{3}{2}}c}$$

$$+ \frac{bcG_{6,6}^{6,2}\left(\begin{matrix} -\frac{1}{4}, \frac{1}{4} & 0, 0, \frac{1}{2}, 1 \\ -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, 0 \end{matrix} \middle| \frac{c^2}{d^2x^2}\right)}{4\pi^{\frac{3}{2}}d^2}$$

$$+ \frac{ibcG_{6,6}^{2,6}\left(\begin{matrix} -1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, 1 \\ -\frac{3}{4}, -\frac{1}{4} & -1, -\frac{1}{2}, -\frac{1}{2}, 0 \end{matrix} \middle| \frac{c^2 e^{2i\pi}}{d^2x^2}\right)}{4\pi^{\frac{3}{2}}d^2}$$

input

```
integrate((b*x**2+a)/x/(d*x-c)**(1/2)/(d*x+c)**(1/2),x)
```

output

```
-a*meijerg(((3/4, 5/4, 1), (1, 1, 3/2)), ((1/2, 3/4, 1, 5/4, 3/2), (0,)),
c**2/(d**2*x**2))/(4*pi**(3/2)*c) + I*a*meijerg(((0, 1/4, 1/2, 3/4, 1, 1),
()), ((1/4, 3/4), (0, 1/2, 1/2, 0)), c**2*exp_polar(2*I*pi)/(d**2*x**2))/
(4*pi**(3/2)*c) + b*c*meijerg((-1/4, 1/4), (0, 0, 1/2, 1)), ((-1/2, -1/4,
0, 1/4, 1/2, 0), ()), c**2/(d**2*x**2))/(4*pi**(3/2)*d**2) + I*b*c*meijer
g((-1, -3/4, -1/2, -1/4, 0, 1), ()), ((-3/4, -1/4), (-1, -1/2, -1/2, 0)),
c**2*exp_polar(2*I*pi)/(d**2*x**2))/(4*pi**(3/2)*d**2)
```

Maxima [A] (verification not implemented)

Time = 0.11 (sec) , antiderivative size = 37, normalized size of antiderivative = 0.66

$$\int \frac{a + bx^2}{x\sqrt{-c + dx}\sqrt{c + dx}} dx = -\frac{a \arcsin\left(\frac{c}{d|x|}\right)}{c} + \frac{\sqrt{d^2x^2 - c^2}b}{d^2}$$

input `integrate((b*x^2+a)/x/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="maxima")`output `-a*arcsin(c/(d*abs(x)))/c + sqrt(d^2*x^2 - c^2)*b/d^2`**Giac [A] (verification not implemented)**

Time = 0.14 (sec) , antiderivative size = 55, normalized size of antiderivative = 0.98

$$\int \frac{a + bx^2}{x\sqrt{-c + dx}\sqrt{c + dx}} dx = -\frac{2a \arctan\left(\frac{(\sqrt{dx+c}-\sqrt{dx-c})^2}{2c}\right)}{c} + \frac{\sqrt{dx+c}\sqrt{dx-c}b}{d^2}$$

input `integrate((b*x^2+a)/x/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="giac")`output `-2*a*arctan(1/2*(sqrt(d*x + c) - sqrt(d*x - c))^2/c)/c + sqrt(d*x + c)*sqrt(d*x - c)*b/d^2`**Mupad [B] (verification not implemented)**

Time = 6.30 (sec) , antiderivative size = 108, normalized size of antiderivative = 1.93

$$\int \frac{a + bx^2}{x\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{b\sqrt{c + dx}\sqrt{dx - c}}{d^2} - \frac{a\sqrt{-c}\left(\ln\left(\frac{(\sqrt{c+dx}-\sqrt{c})^2}{(\sqrt{-c}-\sqrt{dx-c})^2} + 1\right) - \ln\left(\frac{\sqrt{c+dx}-\sqrt{c}}{\sqrt{-c}-\sqrt{dx-c}}\right)\right)}{c^{3/2}}$$

input `int((a + b*x^2)/(x*(c + d*x)^(1/2)*(d*x - c)^(1/2)),x)`

output

$$\frac{(b*(c + d*x)^{(1/2)}*(d*x - c)^{(1/2)})/d^2 - (a*(-c)^{(1/2)}*(\log(((c + d*x)^{(1/2)} - c^{(1/2)})^2/((-c)^{(1/2)} - (d*x - c)^{(1/2)})^2 + 1) - \log(((c + d*x)^{(1/2)} - c^{(1/2)})/((-c)^{(1/2)} - (d*x - c)^{(1/2)))))}{c^{(3/2)}}$$
Reduce [B] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 85, normalized size of antiderivative = 1.52

$$\int \frac{a + bx^2}{x\sqrt{-c + dx}\sqrt{c + dx}} dx$$

$$= \frac{2 \operatorname{atan}\left(\frac{\sqrt{dx-c} + \sqrt{dx+c} - \sqrt{c}}{\sqrt{c}}\right) a d^2 - 2 \operatorname{atan}\left(\frac{\sqrt{dx-c} + \sqrt{dx+c} + \sqrt{c}}{\sqrt{c}}\right) a d^2 + \sqrt{dx+c}\sqrt{dx-c} bc}{c d^2}$$

input

$$\operatorname{int}((b*x^2+a)/x/(d*x-c)^{(1/2)}/(d*x+c)^{(1/2)}, x)$$

output

$$(2*\operatorname{atan}((\operatorname{sqrt}(-c + d*x) + \operatorname{sqrt}(c + d*x) - \operatorname{sqrt}(c))/\operatorname{sqrt}(c))*a*d**2 - 2*\operatorname{atan}((\operatorname{sqrt}(-c + d*x) + \operatorname{sqrt}(c + d*x) + \operatorname{sqrt}(c))/\operatorname{sqrt}(c))*a*d**2 + \operatorname{sqrt}(c + d*x)*\operatorname{sqrt}(-c + d*x)*b*c)/(c*d**2)$$

3.28 $\int \frac{a+bx^2}{x^2\sqrt{-c+dx}\sqrt{c+dx}} dx$

Optimal result	231
Mathematica [A] (verified)	231
Rubi [A] (verified)	232
Maple [C] (verified)	233
Fricas [A] (verification not implemented)	234
Sympy [C] (verification not implemented)	234
Maxima [A] (verification not implemented)	235
Giac [A] (verification not implemented)	235
Mupad [B] (verification not implemented)	236
Reduce [B] (verification not implemented)	236

Optimal result

Integrand size = 31, antiderivative size = 57

$$\int \frac{a + bx^2}{x^2\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{a\sqrt{-c + dx}\sqrt{c + dx}}{c^2x} + \frac{2b\operatorname{arctanh}\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)}{d}$$

output `a*(d*x-c)^(1/2)*(d*x+c)^(1/2)/c^2/x+2*b*arctanh((d*x-c)^(1/2)/(d*x+c)^(1/2))/d`

Mathematica [A] (verified)

Time = 0.12 (sec) , antiderivative size = 57, normalized size of antiderivative = 1.00

$$\int \frac{a + bx^2}{x^2\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{a\sqrt{-c + dx}\sqrt{c + dx}}{c^2x} + \frac{2b\operatorname{arctanh}\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)}{d}$$

input `Integrate[(a + b*x^2)/(x^2*Sqrt[-c + d*x]*Sqrt[c + d*x]),x]`

output `(a*Sqrt[-c + d*x]*Sqrt[c + d*x])/(c^2*x) + (2*b*ArcTanh[Sqrt[-c + d*x]/Sqrt[c + d*x]])/d`

Rubi [A] (verified)

Time = 0.34 (sec) , antiderivative size = 57, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.097$, Rules used = {956, 45, 221}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{a + bx^2}{x^2 \sqrt{dx - c} \sqrt{c + dx}} dx$$

$$\downarrow 956$$

$$b \int \frac{1}{\sqrt{dx - c} \sqrt{c + dx}} dx + \frac{a \sqrt{dx - c} \sqrt{c + dx}}{c^2 x}$$

$$\downarrow 45$$

$$2b \int \frac{1}{d - \frac{d(dx-c)}{c+dx}} d \frac{\sqrt{dx-c}}{\sqrt{c+dx}} + \frac{a \sqrt{dx-c} \sqrt{c+dx}}{c^2 x}$$

$$\downarrow 221$$

$$\frac{a \sqrt{dx - c} \sqrt{c + dx}}{c^2 x} + \frac{2b \operatorname{arctanh}\left(\frac{\sqrt{dx-c}}{\sqrt{c+dx}}\right)}{d}$$

input `Int[(a + b*x^2)/(x^2*Sqrt[-c + d*x]*Sqrt[c + d*x]),x]`

output `(a*Sqrt[-c + d*x]*Sqrt[c + d*x])/(c^2*x) + (2*b*ArcTanh[Sqrt[-c + d*x]/Sqrt[c + d*x]])/d`

Defintions of rubi rules used

rule 45 `Int[1/(Sqrt[(a_) + (b_.)*(x_)]*Sqrt[(c_) + (d_.)*(x_)]), x_Symbol] := Simp[2 Subst[Int[1/(b - d*x^2), x], x, Sqrt[a + b*x]/Sqrt[c + d*x], x] /; FreeQ[{a, b, c, d}, x] && EqQ[b*c + a*d, 0] && !GtQ[c, 0]`

rule 221 `Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(Rt[-a/b, 2]/a)*ArcTanh[x/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]`

rule 956 `Int[((e_.)*(x_))^(m_.)*((a1_) + (b1_.)*(x_)^(non2_.))^p_.)*((a2_) + (b2_.)*(x_)^(non2_.))^p_.*((c_) + (d_.)*(x_)^n_), x_Symbol] := Simp[c*(e*x)^(m + 1)*(a1 + b1*x^(n/2))^(p + 1)*((a2 + b2*x^(n/2))^(p + 1)/(a1*a2*e^(m + 1))), x] + Simp[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(a1*a2*e^n*(m + 1)) Int[(e*x)^(m + n)*(a1 + b1*x^(n/2))^p*(a2 + b2*x^(n/2))^p, x], x] /; FreeQ[{a1, b1, a2, b2, c, d, e, p}, x] && EqQ[non2, n/2] && EqQ[a2*b1 + a1*b2, 0] && (IntegerQ[n] || GtQ[e, 0]) && ((GtQ[n, 0] && LtQ[m, -1]) || (LtQ[n, 0] && GtQ[m + n, -1])) && !ILtQ[p, -1]`

Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.15 (sec) , antiderivative size = 97, normalized size of antiderivative = 1.70

method	result	size
default	$\frac{\sqrt{dx-c}\sqrt{dx+c} \left(\ln \left(\frac{(\sqrt{d^2x^2-c^2} \operatorname{csgn}(d)+dx) \operatorname{csgn}(d)}{c^2\sqrt{d^2x^2-c^2}} \right) b c^2 x + \operatorname{csgn}(d) d \sqrt{d^2x^2-c^2} a \right) \operatorname{csgn}(d)}{c^2\sqrt{d^2x^2-c^2} x d}$	97
risch	$-\frac{a(-dx+c)\sqrt{dx+c}}{c^2x\sqrt{dx-c}} + \frac{b \ln \left(\frac{d^2x}{\sqrt{d^2}} + \sqrt{d^2x^2-c^2} \right) \sqrt{(dx-c)(dx+c)}}{\sqrt{d^2} \sqrt{dx-c} \sqrt{dx+c}}$	98

input `int((b*x^2+a)/x^2/(d*x-c)^(1/2)/(d*x+c)^(1/2), x, method=_RETURNVERBOSE)`

output `(d*x-c)^(1/2)*(d*x+c)^(1/2)/c^2*(ln(((d^2*x^2-c^2)^(1/2)*csgn(d)+d*x)*csgn(d))*b*c^2*x+csgn(d)*d*(d^2*x^2-c^2)^(1/2)*a)*csgn(d)/(d^2*x^2-c^2)^(1/2)/x/d`

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 68, normalized size of antiderivative = 1.19

$$\int \frac{a + bx^2}{x^2 \sqrt{-c + dx} \sqrt{c + dx}} dx$$

$$= -\frac{bc^2 x \log(-dx + \sqrt{dx + c} \sqrt{dx - c}) - ad^2 x - \sqrt{dx + c} \sqrt{dx - c} cad}{c^2 dx}$$

input `integrate((b*x^2+a)/x^2/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="fricas")`output `-(b*c^2*x*log(-d*x + sqrt(d*x + c)*sqrt(d*x - c)) - a*d^2*x - sqrt(d*x + c)*sqrt(d*x - c)*a*d)/(c^2*d*x)`**Sympy [C] (verification not implemented)**

Result contains complex when optimal does not.

Time = 26.51 (sec) , antiderivative size = 165, normalized size of antiderivative = 2.89

$$\int \frac{a + bx^2}{x^2 \sqrt{-c + dx} \sqrt{c + dx}} dx = -\frac{adG_{6,6}^{5,3} \left(\begin{array}{c} \frac{5}{4}, \frac{7}{4}, 1 \\ 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2 \end{array} \middle| \begin{array}{c} \frac{3}{2}, \frac{3}{2}, 2 \\ 0 \end{array} \middle| \frac{c^2}{d^2 x^2} \right)}{4\pi^{\frac{3}{2}} c^2}$$

$$- \frac{iadG_{6,6}^{2,6} \left(\begin{array}{c} \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, 1 \\ \frac{3}{4}, \frac{5}{4} \end{array} \middle| \begin{array}{c} \frac{1}{2}, 1, 1, 0 \\ \frac{c^2 e^{2i\pi}}{d^2 x^2} \end{array} \right)}{4\pi^{\frac{3}{2}} c^2}$$

$$+ \frac{bG_{6,6}^{6,2} \left(\begin{array}{c} \frac{1}{4}, \frac{3}{4} \\ 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 0 \end{array} \middle| \begin{array}{c} \frac{1}{2}, \frac{1}{2}, 1, 1 \\ \frac{c^2}{d^2 x^2} \end{array} \right)}{4\pi^{\frac{3}{2}} d}$$

$$- \frac{ibG_{6,6}^{2,6} \left(\begin{array}{c} -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, 1 \\ -\frac{1}{4}, \frac{1}{4} \end{array} \middle| \begin{array}{c} -\frac{1}{2}, 0, 0, 0 \\ \frac{c^2 e^{2i\pi}}{d^2 x^2} \end{array} \right)}{4\pi^{\frac{3}{2}} d}$$

input `integrate((b*x**2+a)/x**2/(d*x-c)**(1/2)/(d*x+c)**(1/2),x)`

output

```
-a*d*meijerg(((5/4, 7/4, 1), (3/2, 3/2, 2)), ((1, 5/4, 3/2, 7/4, 2), (0,))
, c**2/(d**2*x**2))/(4*pi**(3/2)*c**2) - I*a*d*meijerg(((1/2, 3/4, 1, 5/4,
3/2, 1), ()), ((3/4, 5/4), (1/2, 1, 1, 0)), c**2*exp_polar(2*I*pi)/(d**2*
x**2))/(4*pi**(3/2)*c**2) + b*meijerg(((1/4, 3/4), (1/2, 1/2, 1, 1)), ((0,
1/4, 1/2, 3/4, 1, 0), ()), c**2/(d**2*x**2))/(4*pi**(3/2)*d) - I*b*meijer
g((( -1/2, -1/4, 0, 1/4, 1/2, 1), ()), ((-1/4, 1/4), (-1/2, 0, 0, 0)), c**2
*exp_polar(2*I*pi)/(d**2*x**2))/(4*pi**(3/2)*d)
```

Maxima [A] (verification not implemented)

Time = 0.10 (sec) , antiderivative size = 55, normalized size of antiderivative = 0.96

$$\int \frac{a + bx^2}{x^2\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{b \log(2d^2x + 2\sqrt{d^2x^2 - c^2}d)}{d} + \frac{\sqrt{d^2x^2 - c^2}a}{c^2x}$$

input

```
integrate((b*x^2+a)/x^2/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="maxima")
```

output

```
b*log(2*d^2*x + 2*sqrt(d^2*x^2 - c^2)*d)/d + sqrt(d^2*x^2 - c^2)*a/(c^2*x)
```

Giac [A] (verification not implemented)

Time = 0.14 (sec) , antiderivative size = 64, normalized size of antiderivative = 1.12

$$\int \frac{a + bx^2}{x^2\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{1}{2} d \left(\frac{16a}{(\sqrt{dx + c} - \sqrt{dx - c})^4 + 4c^2} - \frac{b \log((\sqrt{dx + c} - \sqrt{dx - c})^4)}{d^2} \right)$$

input

```
integrate((b*x^2+a)/x^2/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="giac")
```

output

```
1/2*d*(16*a/((sqrt(d*x + c) - sqrt(d*x - c))^4 + 4*c^2) - b*log((sqrt(d*x
+ c) - sqrt(d*x - c))^4)/d^2)
```

Mupad [B] (verification not implemented)

Time = 5.20 (sec) , antiderivative size = 77, normalized size of antiderivative = 1.35

$$\int \frac{a + bx^2}{x^2 \sqrt{-c + dx} \sqrt{c + dx}} dx = \frac{4b \operatorname{atan}\left(\frac{d(\sqrt{-c} - \sqrt{dx-c})}{\sqrt{-d^2}(\sqrt{c+dx} - \sqrt{c})}\right)}{\sqrt{-d^2}} + \frac{a\sqrt{c+dx}\sqrt{dx-c}}{c^2 x}$$

input `int((a + b*x^2)/(x^2*(c + d*x)^(1/2)*(d*x - c)^(1/2)),x)`output `(4*b*atan((d*((-c)^(1/2) - (d*x - c)^(1/2)))/((-d^2)^(1/2)*((c + d*x)^(1/2) - c^(1/2))))/(-d^2)^(1/2) + (a*(c + d*x)^(1/2)*(d*x - c)^(1/2))/(c^2*x)`**Reduce [B] (verification not implemented)**

Time = 0.22 (sec) , antiderivative size = 66, normalized size of antiderivative = 1.16

$$\int \frac{a + bx^2}{x^2 \sqrt{-c + dx} \sqrt{c + dx}} dx = \frac{\sqrt{dx+c}\sqrt{dx-c}ad + 2\log\left(\frac{\sqrt{dx-c} + \sqrt{dx+c}}{\sqrt{c}\sqrt{2}}\right)bc^2x + a d^2 x}{c^2 dx}$$

input `int((b*x^2+a)/x^2/(d*x-c)^(1/2)/(d*x+c)^(1/2),x)`output `(sqrt(c + d*x)*sqrt(-c + d*x)*a*d + 2*log((sqrt(-c + d*x) + sqrt(c + d*x))/(sqrt(c)*sqrt(2)))*b*c**2*x + a*d**2*x)/(c**2*d*x)`

3.29 $\int \frac{a+bx^2}{x^3\sqrt{-c+dx}\sqrt{c+dx}} dx$

Optimal result	237
Mathematica [A] (verified)	237
Rubi [A] (verified)	238
Maple [A] (verified)	239
Fricas [A] (verification not implemented)	240
Sympy [F(-1)]	240
Maxima [A] (verification not implemented)	240
Giac [B] (verification not implemented)	241
Mupad [B] (verification not implemented)	242
Reduce [B] (verification not implemented)	243

Optimal result

Integrand size = 31, antiderivative size = 76

$$\int \frac{a + bx^2}{x^3\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{a\sqrt{-c + dx}\sqrt{c + dx}}{2c^2x^2} + \frac{(2bc^2 + ad^2) \arctan\left(\frac{\sqrt{-c+dx}\sqrt{c+dx}}{c}\right)}{2c^3}$$

output

$$\frac{1}{2} \frac{a(d^2x^2 + 2bc^2) \arctan\left(\frac{d^2x^2 + 2bc^2}{c}\right) + (d^2x^2 + 2bc^2) \sqrt{-c+dx}\sqrt{c+dx}}{2c^3}$$

Mathematica [A] (verified)

Time = 0.13 (sec) , antiderivative size = 70, normalized size of antiderivative = 0.92

$$\int \frac{a + bx^2}{x^3\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{\frac{ac\sqrt{-c+dx}\sqrt{c+dx}}{x^2} + 2(2bc^2 + ad^2) \arctan\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)}{2c^3}$$

input

```
Integrate[(a + b*x^2)/(x^3*Sqrt[-c + d*x]*Sqrt[c + d*x]),x]
```

output

$$\frac{((a*c*\sqrt{-c + d*x})*\sqrt{c + d*x})/x^2 + 2*(2*b*c^2 + a*d^2)*\text{ArcTan}[\sqrt{-c + d*x}/\sqrt{c + d*x}]]/(2*c^3)$$

Rubi [A] (verified)

Time = 0.36 (sec) , antiderivative size = 76, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.097$, Rules used = {956, 103, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{a + bx^2}{x^3 \sqrt{dx - c} \sqrt{c + dx}} dx$$

$$\downarrow 956$$

$$\frac{1}{2} \left(\frac{ad^2}{c^2} + 2b \right) \int \frac{1}{x \sqrt{dx - c} \sqrt{c + dx}} dx + \frac{a \sqrt{dx - c} \sqrt{c + dx}}{2c^2 x^2}$$

$$\downarrow 103$$

$$\frac{1}{2} d \left(\frac{ad^2}{c^2} + 2b \right) \int \frac{1}{dc^2 + d(dx - c)(c + dx)} d(\sqrt{dx - c} \sqrt{c + dx}) + \frac{a \sqrt{dx - c} \sqrt{c + dx}}{2c^2 x^2}$$

$$\downarrow 218$$

$$\frac{\left(\frac{ad^2}{c^2} + 2b \right) \arctan \left(\frac{\sqrt{dx - c} \sqrt{c + dx}}{c} \right)}{2c} + \frac{a \sqrt{dx - c} \sqrt{c + dx}}{2c^2 x^2}$$

input `Int[(a + b*x^2)/(x^3*Sqrt[-c + d*x]*Sqrt[c + d*x]),x]`

output `(a*Sqrt[-c + d*x]*Sqrt[c + d*x])/(2*c^2*x^2) + ((2*b + (a*d^2)/c^2)*ArcTan[(Sqrt[-c + d*x]*Sqrt[c + d*x])/c])/(2*c)`

Defintions of rubi rules used

rule 103 `Int[1/(Sqrt[(a_.) + (b_.)*(x_)]*Sqrt[(c_.) + (d_.)*(x_)]*((e_.) + (f_.)*(x_))), x_] := Simp[b*f Subst[Int[1/(d*(b*e - a*f)^2 + b*f^2*x^2), x], x, Sqrt[a + b*x]*Sqrt[c + d*x]], x] /; FreeQ[{a, b, c, d, e, f}, x] && EqQ[2*b*d*e - f*(b*c + a*d), 0]`

rule 218 `Int[((a_) + (b_)*(x_)^2)^(-1), x_Symbol] := Simp[(Rt[a/b, 2]/a)*ArcTan[x/Rt[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]`

rule 956 `Int[((e_)*(x_))^(m_)*((a1_) + (b1_)*(x_)^(non2_))^(p_)*((a2_) + (b2_)*(x_)^(non2_))^(p_)*((c_) + (d_)*(x_)^(n_)), x_Symbol] := Simp[c*(e*x)^(m+1)*(a1 + b1*x^(n/2))^(p+1)*((a2 + b2*x^(n/2))^(p+1)/(a1*a2*e^(m+1))), x] + Simp[(a1*a2*d*(m+1) - b1*b2*c*(m+n*(p+1)+1))/(a1*a2*e^n*(m+1)) Int[(e*x)^(m+n)*(a1 + b1*x^(n/2))^p*(a2 + b2*x^(n/2))^p, x], x] /; FreeQ[{a1, b1, a2, b2, c, d, e, p}, x] && EqQ[non2, n/2] && EqQ[a2*b1 + a1*b2, 0] && (IntegerQ[n] || GtQ[e, 0]) && ((GtQ[n, 0] && LtQ[m, -1]) || (LtQ[n, 0] && GtQ[m+n, -1])) && !ILtQ[p, -1]`

Maple [A] (verified)

Time = 0.13 (sec) , antiderivative size = 123, normalized size of antiderivative = 1.62

method	result	size
risch	$-\frac{a(-dx+c)\sqrt{dx+c}}{2c^2x^2\sqrt{dx-c}} - \frac{(ad^2+2bc^2)\ln\left(\frac{-2c^2+2\sqrt{-c^2}\sqrt{d^2x^2-c^2}}{x}\right)\sqrt{(dx-c)(dx+c)}}{2c^2\sqrt{-c^2}\sqrt{dx-c}\sqrt{dx+c}}$	123
default	$-\frac{\sqrt{dx-c}\sqrt{dx+c}\left(\ln\left(-\frac{2(c^2-\sqrt{-c^2}\sqrt{d^2x^2-c^2})}{x}\right)ad^2x^2+2\ln\left(-\frac{2(c^2-\sqrt{-c^2}\sqrt{d^2x^2-c^2})}{x}\right)bc^2x^2-\sqrt{-c^2}\sqrt{d^2x^2-c^2}a\right)}{2c^2\sqrt{d^2x^2-c^2}x^2\sqrt{-c^2}}$	158

input `int((b*x^2+a)/x^3/(d*x-c)^(1/2)/(d*x+c)^(1/2),x,method=_RETURNVERBOSE)`

output `-1/2*a*(-d*x+c)*(d*x+c)^(1/2)/c^2/x^2/(d*x-c)^(1/2)-1/2/c^2*(a*d^2+2*b*c^2)/(-c^2)^(1/2)*ln((-2*c^2+2*(-c^2)^(1/2)*(d^2*x^2-c^2)^(1/2))/x)*((d*x-c)*(d*x+c))^(1/2)/(d*x-c)^(1/2)/(d*x+c)^(1/2)`

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 73, normalized size of antiderivative = 0.96

$$\int \frac{a + bx^2}{x^3 \sqrt{-c + dx} \sqrt{c + dx}} dx$$

$$= \frac{2(2bc^2 + ad^2)x^2 \arctan\left(-\frac{dx - \sqrt{dx+c}\sqrt{dx-c}}{c}\right) + \sqrt{dx+c}\sqrt{dx-c} - cac}{2c^3x^2}$$

input `integrate((b*x^2+a)/x^3/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="fricas")`output `1/2*(2*(2*b*c^2 + a*d^2)*x^2*arctan(-(d*x - sqrt(d*x + c))*sqrt(d*x - c))/c) + sqrt(d*x + c)*sqrt(d*x - c)*a*c)/(c^3*x^2)`**Sympy [F(-1)]**

Timed out.

$$\int \frac{a + bx^2}{x^3 \sqrt{-c + dx} \sqrt{c + dx}} dx = \text{Timed out}$$

input `integrate((b*x**2+a)/x**3/(d*x-c)**(1/2)/(d*x+c)**(1/2),x)`output `Timed out`**Maxima [A] (verification not implemented)**

Time = 0.11 (sec) , antiderivative size = 60, normalized size of antiderivative = 0.79

$$\int \frac{a + bx^2}{x^3 \sqrt{-c + dx} \sqrt{c + dx}} dx = -\frac{b \arcsin\left(\frac{c}{d|x|}\right)}{c} - \frac{ad^2 \arcsin\left(\frac{c}{d|x|}\right)}{2c^3} + \frac{\sqrt{d^2x^2 - c^2}a}{2c^2x^2}$$

input `integrate((b*x^2+a)/x^3/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="maxima")`

output

```
-b*arcsin(c/(d*abs(x)))/c - 1/2*a*d^2*arcsin(c/(d*abs(x)))/c^3 + 1/2*sqrt(
d^2*x^2 - c^2)*a/(c^2*x^2)
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 141 vs. $2(64) = 128$.

Time = 0.14 (sec) , antiderivative size = 141, normalized size of antiderivative = 1.86

$$\int \frac{a + bx^2}{x^3 \sqrt{-c + dx} \sqrt{c + dx}} dx$$

$$= \frac{(2bc^2d + ad^3) \arctan\left(\frac{(\sqrt{dx+c} - \sqrt{dx-c})^2}{2c}\right)}{c^3} + \frac{2(ad^3(\sqrt{dx+c} - \sqrt{dx-c})^6 - 4ac^2d^3(\sqrt{dx+c} - \sqrt{dx-c})^2)}{((\sqrt{dx+c} - \sqrt{dx-c})^4 + 4c^2)^2 c^2}$$

$$= \frac{\dots}{d}$$

input

```
integrate((b*x^2+a)/x^3/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="giac")
```

output

```
-((2*b*c^2*d + a*d^3)*arctan(1/2*(sqrt(d*x + c) - sqrt(d*x - c))^2/c)/c^3
+ 2*(a*d^3*(sqrt(d*x + c) - sqrt(d*x - c))^6 - 4*a*c^2*d^3*(sqrt(d*x + c)
- sqrt(d*x - c))^2)/(((sqrt(d*x + c) - sqrt(d*x - c))^4 + 4*c^2)^2*c^2)/d
```

Mupad [B] (verification not implemented)

Time = 10.09 (sec) , antiderivative size = 457, normalized size of antiderivative = 6.01

$$\begin{aligned}
& \int \frac{a + bx^2}{x^3 \sqrt{-c + dx} \sqrt{c + dx}} dx \\
&= \frac{a(-c)^{3/2} d^2 \ln \left(\frac{(\sqrt{c+dx}-\sqrt{c})^2}{(\sqrt{-c}-\sqrt{dx-c})^2} + 1 \right)}{2c^{9/2}} \\
&\quad - \frac{b\sqrt{-c} \left(\ln \left(\frac{(\sqrt{c+dx}-\sqrt{c})^2}{(\sqrt{-c}-\sqrt{dx-c})^2} + 1 \right) - \ln \left(\frac{\sqrt{c+dx}-\sqrt{c}}{\sqrt{-c}-\sqrt{dx-c}} \right) \right)}{c^{3/2}} \\
&\quad - \frac{a(-c)^{3/2} d^2 \ln \left(\frac{\sqrt{c+dx}-\sqrt{c}}{\sqrt{-c}-\sqrt{dx-c}} \right)}{2c^{9/2}} \\
&\quad - \frac{\frac{a(-c)^{3/2} d^2}{32c^{9/2}} + \frac{a(-c)^{3/2} d^2 (\sqrt{c+dx}-\sqrt{c})^2}{16c^{9/2} (\sqrt{-c}-\sqrt{dx-c})^2} - \frac{15a(-c)^{3/2} d^2 (\sqrt{c+dx}-\sqrt{c})^4}{32c^{9/2} (\sqrt{-c}-\sqrt{dx-c})^4}}{\frac{(\sqrt{c+dx}-\sqrt{c})^2}{(\sqrt{-c}-\sqrt{dx-c})^2} + \frac{2(\sqrt{c+dx}-\sqrt{c})^4}{(\sqrt{-c}-\sqrt{dx-c})^4} + \frac{(\sqrt{c+dx}-\sqrt{c})^6}{(\sqrt{-c}-\sqrt{dx-c})^6}} \\
&\quad + \frac{a d^2 (\sqrt{c+dx} - \sqrt{c})^2}{32(-c)^{3/2} c^{3/2} (\sqrt{-c} - \sqrt{dx-c})^2}
\end{aligned}$$

input `int((a + b*x^2)/(x^3*(c + d*x)^(1/2)*(d*x - c)^(1/2)),x)`

output `(a*(-c)^(3/2)*d^2*log(((c + d*x)^(1/2) - c^(1/2))^2/((-c)^(1/2) - (d*x - c)^(1/2))^2 + 1))/(2*c^(9/2)) - (b*(-c)^(1/2)*(log(((c + d*x)^(1/2) - c^(1/2))^2/((-c)^(1/2) - (d*x - c)^(1/2))^2 + 1) - log(((c + d*x)^(1/2) - c^(1/2))/((-c)^(1/2) - (d*x - c)^(1/2)))))/c^(3/2) - (a*(-c)^(3/2)*d^2*log(((c + d*x)^(1/2) - c^(1/2))/((-c)^(1/2) - (d*x - c)^(1/2))))/(2*c^(9/2)) - ((a*(-c)^(3/2)*d^2)/(32*c^(9/2)) + (a*(-c)^(3/2)*d^2*((c + d*x)^(1/2) - c^(1/2))^2)/(16*c^(9/2)*((-c)^(1/2) - (d*x - c)^(1/2))^2 - (15*a*(-c)^(3/2)*d^2*((c + d*x)^(1/2) - c^(1/2))^4)/(32*c^(9/2)*((-c)^(1/2) - (d*x - c)^(1/2))^4))/(((c + d*x)^(1/2) - c^(1/2))^2/((-c)^(1/2) - (d*x - c)^(1/2))^2 + (2*((c + d*x)^(1/2) - c^(1/2))^4)/((-c)^(1/2) - (d*x - c)^(1/2))^4 + ((c + d*x)^(1/2) - c^(1/2))^6/((-c)^(1/2) - (d*x - c)^(1/2))^6) + (a*d^2*((c + d*x)^(1/2) - c^(1/2))^2)/(32*(-c)^(3/2)*c^(3/2)*((-c)^(1/2) - (d*x - c)^(1/2))^2)`

Reduce [B] (verification not implemented)

Time = 0.23 (sec) , antiderivative size = 158, normalized size of antiderivative = 2.08

$$\int \frac{a + bx^2}{x^3 \sqrt{-c + dx} \sqrt{c + dx}} dx$$

$$= \frac{2 \operatorname{atan}\left(\frac{\sqrt{dx-c} + \sqrt{dx+c} - \sqrt{c}}{\sqrt{c}}\right) a d^2 x^2 + 4 \operatorname{atan}\left(\frac{\sqrt{dx-c} + \sqrt{dx+c} - \sqrt{c}}{\sqrt{c}}\right) b c^2 x^2 - 2 \operatorname{atan}\left(\frac{\sqrt{dx-c} + \sqrt{dx+c} + \sqrt{c}}{\sqrt{c}}\right) a d^2 x^2 - 4 \operatorname{atan}\left(\frac{\sqrt{dx-c} + \sqrt{dx+c} + \sqrt{c}}{\sqrt{c}}\right) b c^2 x^2 + \sqrt{c + dx} \sqrt{-c + dx} a c}{2c^3 x^2}$$

input `int((b*x^2+a)/x^3/(d*x-c)^(1/2)/(d*x+c)^(1/2),x)`output `(2*atan((sqrt(-c+d*x)+sqrt(c+d*x)-sqrt(c))/sqrt(c))*a*d**2*x**2 + 4*atan((sqrt(-c+d*x)+sqrt(c+d*x)-sqrt(c))/sqrt(c))*b*c**2*x**2 - 2*atan((sqrt(-c+d*x)+sqrt(c+d*x)+sqrt(c))/sqrt(c))*a*d**2*x**2 - 4*atan((sqrt(-c+d*x)+sqrt(c+d*x)+sqrt(c))/sqrt(c))*b*c**2*x**2 + sqrt(c+d*x)*sqrt(-c+d*x)*a*c)/(2*c**3*x**2)`

3.30 $\int \frac{a+bx^2}{x^4\sqrt{-c+dx}\sqrt{c+dx}} dx$

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Optimal result

Integrand size = 31, antiderivative size = 75

$$\int \frac{a + bx^2}{x^4\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{a\sqrt{-c + dx}\sqrt{c + dx}}{3c^2x^3} + \frac{(3bc^2 + 2ad^2)\sqrt{-c + dx}\sqrt{c + dx}}{3c^4x}$$

output `1/3*a*(d*x-c)^(1/2)*(d*x+c)^(1/2)/c^2/x^3+1/3*(2*a*d^2+3*b*c^2)*(d*x-c)^(1/2)*(d*x+c)^(1/2)/c^4/x`

Mathematica [A] (verified)

Time = 0.12 (sec) , antiderivative size = 54, normalized size of antiderivative = 0.72

$$\int \frac{a + bx^2}{x^4\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{\sqrt{-c + dx}\sqrt{c + dx}(3bc^2x^2 + a(c^2 + 2d^2x^2))}{3c^4x^3}$$

input `Integrate[(a + b*x^2)/(x^4*sqrt[-c + d*x]*sqrt[c + d*x]),x]`

output `(sqrt[-c + d*x]*sqrt[c + d*x]*(3*b*c^2*x^2 + a*(c^2 + 2*d^2*x^2)))/(3*c^4*x^3)`

Rubi [A] (verified)

Time = 0.35 (sec) , antiderivative size = 75, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}} = 0.065$, Rules used = {956, 106}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{a + bx^2}{x^4 \sqrt{dx - c} \sqrt{c + dx}} dx$$

↓ 956

$$\frac{1}{3} \left(\frac{2ad^2}{c^2} + 3b \right) \int \frac{1}{x^2 \sqrt{dx - c} \sqrt{c + dx}} dx + \frac{a \sqrt{dx - c} \sqrt{c + dx}}{3c^2 x^3}$$

↓ 106

$$\frac{\sqrt{dx - c} \sqrt{c + dx} \left(\frac{2ad^2}{c^2} + 3b \right)}{3c^2 x} + \frac{a \sqrt{dx - c} \sqrt{c + dx}}{3c^2 x^3}$$

input `Int[(a + b*x^2)/(x^4*Sqrt[-c + d*x]*Sqrt[c + d*x]),x]`

output `(a*Sqrt[-c + d*x]*Sqrt[c + d*x])/(3*c^2*x^3) + ((3*b + (2*a*d^2)/c^2)*Sqrt[-c + d*x]*Sqrt[c + d*x])/(3*c^2*x)`

Defintions of rubi rules used

rule 106

```
Int[((a_.) + (b_.)*(x_)^(m_))*((c_.) + (d_.)*(x_)^(n_))*((e_.) + (f_.)*(x_)^(p_)), x_] := Simp[b*(a + b*x)^(m + 1)*(c + d*x)^(n + 1)*((e + f*x)^(p + 1))/((m + 1)*(b*c - a*d)*(b*e - a*f)), x] /; FreeQ[{a, b, c, d, e, f, m, n, p}, x] && EqQ[Simplify[m + n + p + 3], 0] && EqQ[a*d*f*(m + 1) + b*c*f*(n + 1) + b*d*e*(p + 1), 0] && NeQ[m, -1]
```

rule 956

```

Int[((e_.)*(x_))^(m_.)*((a1_) + (b1_.)*(x_)^(non2_.))^(p_.)*((a2_) + (b2_.)
*(x_)^(non2_.))^(p_.)*((c_) + (d_.)*(x_)^(n_)), x_Symbol] := Simp[c*(e*x)^(
m + 1)*(a1 + b1*x^(n/2))^(p + 1)*((a2 + b2*x^(n/2))^(p + 1)/(a1*a2*e*(m + 1
))), x] + Simp[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(a1*a2*e^n*(
m + 1)) Int[(e*x)^(m + n)*(a1 + b1*x^(n/2))^p*(a2 + b2*x^(n/2))^p, x], x]
/; FreeQ[{a1, b1, a2, b2, c, d, e, p}, x] && EqQ[non2, n/2] && EqQ[a2*b1 +
a1*b2, 0] && (IntegerQ[n] || GtQ[e, 0]) && ((GtQ[n, 0] && LtQ[m, -1]) || (
LtQ[n, 0] && GtQ[m + n, -1])) && !ILtQ[p, -1]

```

Maple [A] (verified)

Time = 0.13 (sec) , antiderivative size = 49, normalized size of antiderivative = 0.65

method	result	size
gospers	$\frac{\sqrt{dx-c}\sqrt{dx+c}(2ad^2x^2+3bc^2x^2+ac^2)}{3c^4x^3}$	49
default	$\frac{\sqrt{dx-c}\sqrt{dx+c}\operatorname{csign}(d)(2ad^2x^2+3bc^2x^2+ac^2)}{3c^4x^3}$	53
risch	$-\frac{\sqrt{dx+c}(-dx+c)(2ad^2x^2+3bc^2x^2+ac^2)}{3x^3c^4\sqrt{dx-c}}$	55
orering	$-\frac{\sqrt{dx+c}(-dx+c)(2ad^2x^2+3bc^2x^2+ac^2)}{3x^3c^4\sqrt{dx-c}}$	55

input

```
int((b*x^2+a)/x^4/(d*x-c)^(1/2)/(d*x+c)^(1/2),x,method=_RETURNVERBOSE)
```

output

```
1/3/c^4/x^3*(d*x-c)^(1/2)*(d*x+c)^(1/2)*(2*a*d^2*x^2+3*b*c^2*x^2+a*c^2)
```

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 67, normalized size of antiderivative = 0.89

$$\begin{aligned}
& \int \frac{a + bx^2}{x^4\sqrt{-c + dx}\sqrt{c + dx}} dx \\
& = \frac{(3bc^2d + 2ad^3)x^3 + (ac^2 + (3bc^2 + 2ad^2)x^2)\sqrt{dx + c}\sqrt{dx - c}}{3c^4x^3}
\end{aligned}$$

input

```
integrate((b*x^2+a)/x^4/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="fricas")
```

output

```
1/3*((3*b*c^2*d + 2*a*d^3)*x^3 + (a*c^2 + (3*b*c^2 + 2*a*d^2)*x^2)*sqrt(d*x + c)*sqrt(d*x - c))/(c^4*x^3)
```

Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 31.01 (sec) , antiderivative size = 170, normalized size of antiderivative = 2.27

$$\int \frac{a + bx^2}{x^4 \sqrt{-c + dx} \sqrt{c + dx}} dx = - \frac{ad^3 G_{6,6}^{5,3} \left(\begin{matrix} \frac{9}{4}, \frac{11}{4}, 1 & \frac{5}{2}, \frac{5}{2}, 3 \\ 2, \frac{9}{4}, \frac{5}{2}, \frac{11}{4}, 3 & 0 \end{matrix} \middle| \frac{c^2}{d^2 x^2} \right)}{4\pi^{\frac{3}{2}} c^4} - \frac{iad^3 G_{6,6}^{2,6} \left(\begin{matrix} \frac{3}{2}, \frac{7}{4}, 2, \frac{9}{4}, \frac{5}{2}, 1 \\ \frac{7}{4}, \frac{9}{4} & \frac{3}{2}, 2, 2, 0 \end{matrix} \middle| \frac{c^2 e^{2i\pi}}{d^2 x^2} \right)}{4\pi^{\frac{3}{2}} c^4} - \frac{bd G_{6,6}^{5,3} \left(\begin{matrix} \frac{5}{4}, \frac{7}{4}, 1 & \frac{3}{2}, \frac{3}{2}, 2 \\ 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2 & 0 \end{matrix} \middle| \frac{c^2}{d^2 x^2} \right)}{4\pi^{\frac{3}{2}} c^2} - \frac{ibd G_{6,6}^{2,6} \left(\begin{matrix} \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, 1 \\ \frac{3}{4}, \frac{5}{4} & \frac{1}{2}, 1, 1, 0 \end{matrix} \middle| \frac{c^2 e^{2i\pi}}{d^2 x^2} \right)}{4\pi^{\frac{3}{2}} c^2}$$

input

```
integrate((b*x**2+a)/x**4/(d*x-c)**(1/2)/(d*x+c)**(1/2),x)
```

output

```
-a*d**3*meijerg(((9/4, 11/4, 1), (5/2, 5/2, 3)), ((2, 9/4, 5/2, 11/4, 3), (0,)), c**2/(d**2*x**2))/(4*pi**(3/2)*c**4) - I*a*d**3*meijerg(((3/2, 7/4, 2, 9/4, 5/2, 1), ()), ((7/4, 9/4), (3/2, 2, 2, 0)), c**2*exp_polar(2*I*pi)/(d**2*x**2))/(4*pi**(3/2)*c**4) - b*d*meijerg(((5/4, 7/4, 1), (3/2, 3/2, 2)), ((1, 5/4, 3/2, 7/4, 2), (0,)), c**2/(d**2*x**2))/(4*pi**(3/2)*c**2) - I*b*d*meijerg(((1/2, 3/4, 1, 5/4, 3/2, 1), ()), ((3/4, 5/4), (1/2, 1, 1, 0)), c**2*exp_polar(2*I*pi)/(d**2*x**2))/(4*pi**(3/2)*c**2)
```


Maxima [A] (verification not implemented)

Time = 0.12 (sec) , antiderivative size = 75, normalized size of antiderivative = 1.00

$$\int \frac{a + bx^2}{x^4 \sqrt{-c + dx} \sqrt{c + dx}} dx = \frac{\sqrt{d^2 x^2 - c^2} b}{c^2 x} + \frac{2 \sqrt{d^2 x^2 - c^2} a d^2}{3 c^4 x} + \frac{\sqrt{d^2 x^2 - c^2} a}{3 c^2 x^3}$$

input `integrate((b*x^2+a)/x^4/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="maxima")`output `sqrt(d^2*x^2 - c^2)*b/(c^2*x) + 2/3*sqrt(d^2*x^2 - c^2)*a*d^2/(c^4*x) + 1/3*sqrt(d^2*x^2 - c^2)*a/(c^2*x^3)`**Giac [A] (verification not implemented)**

Time = 0.14 (sec) , antiderivative size = 126, normalized size of antiderivative = 1.68

$$\int \frac{a + bx^2}{x^4 \sqrt{-c + dx} \sqrt{c + dx}} dx = \frac{8 \left(3b(\sqrt{dx + c} - \sqrt{dx - c})^8 + 24bc^2(\sqrt{dx + c} - \sqrt{dx - c})^4 + 24ad^2(\sqrt{dx + c} - \sqrt{dx - c})^4 + 48bc^4 + 32a^2c^2d^2 \right)}{3 \left((\sqrt{dx + c} - \sqrt{dx - c})^4 + 4c^2 \right)^3}$$

input `integrate((b*x^2+a)/x^4/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="giac")`output `8/3*(3*b*(sqrt(d*x + c) - sqrt(d*x - c))^8 + 24*b*c^2*(sqrt(d*x + c) - sqrt(d*x - c))^4 + 24*a*d^2*(sqrt(d*x + c) - sqrt(d*x - c))^4 + 48*b*c^4 + 32*a*c^2*d^2)*d/((sqrt(d*x + c) - sqrt(d*x - c))^4 + 4*c^2)^3`

Mupad [B] (verification not implemented)

Time = 4.99 (sec) , antiderivative size = 79, normalized size of antiderivative = 1.05

$$\int \frac{a + bx^2}{x^4 \sqrt{-c + dx} \sqrt{c + dx}} dx = \frac{\sqrt{dx - c} \left(\frac{a}{3c} + \frac{x^2 (3bc^3 + 2acd^2)}{3c^4} + \frac{x^3 (3bc^2d + 2ad^3)}{3c^4} + \frac{adx}{3c^2} \right)}{x^3 \sqrt{c + dx}}$$

input `int((a + b*x^2)/(x^4*(c + d*x)^(1/2)*(d*x - c)^(1/2)),x)`output `((d*x - c)^(1/2)*(a/(3*c) + (x^2*(3*b*c^3 + 2*a*c*d^2))/(3*c^4) + (x^3*(2*a*d^3 + 3*b*c^2*d))/(3*c^4) + (a*d*x)/(3*c^2)))/(x^3*(c + d*x)^(1/2))`**Reduce [B] (verification not implemented)**

Time = 0.22 (sec) , antiderivative size = 93, normalized size of antiderivative = 1.24

$$\int \frac{a + bx^2}{x^4 \sqrt{-c + dx} \sqrt{c + dx}} dx = \frac{\sqrt{dx + c} \sqrt{dx - c} a c^2 + 2\sqrt{dx + c} \sqrt{dx - c} a d^2 x^2 + 3\sqrt{dx + c} \sqrt{dx - c} b c^2 x^2 - 2a d^3 x^3 - b c^2 d x^3}{3c^4 x^3}$$

input `int((b*x^2+a)/x^4/(d*x-c)^(1/2)/(d*x+c)^(1/2),x)`output `(sqrt(c + d*x)*sqrt(-c + d*x)*a*c**2 + 2*sqrt(c + d*x)*sqrt(-c + d*x)*a*d**2*x**2 + 3*sqrt(c + d*x)*sqrt(-c + d*x)*b*c**2*x**2 - 2*a*d**3*x**3 - b*c**2*d*x**3)/(3*c**4*x**3)`

3.31 $\int \frac{a+bx^2}{x^5\sqrt{-c+dx}\sqrt{c+dx}} dx$

Optimal result	250
Mathematica [A] (verified)	250
Rubi [A] (verified)	251
Maple [A] (verified)	253
Fricas [A] (verification not implemented)	253
Sympy [F(-1)]	254
Maxima [A] (verification not implemented)	254
Giac [B] (verification not implemented)	255
Mupad [B] (verification not implemented)	255
Reduce [B] (verification not implemented)	256

Optimal result

Integrand size = 31, antiderivative size = 123

$$\int \frac{a + bx^2}{x^5\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{a\sqrt{-c + dx}\sqrt{c + dx}}{4c^2x^4} + \frac{(4bc^2 + 3ad^2)\sqrt{-c + dx}\sqrt{c + dx}}{8c^4x^2} + \frac{d^2(4bc^2 + 3ad^2) \arctan\left(\frac{\sqrt{-c+dx}\sqrt{c+dx}}{c}\right)}{8c^5}$$

output `1/4*a*(d*x-c)^(1/2)*(d*x+c)^(1/2)/c^2/x^4+1/8*(3*a*d^2+4*b*c^2)*(d*x-c)^(1/2)*(d*x+c)^(1/2)/c^4/x^2+1/8*d^2*(3*a*d^2+4*b*c^2)*arctan((d*x-c)^(1/2)*(d*x+c)^(1/2)/c)/c^5`

Mathematica [A] (verified)

Time = 0.21 (sec) , antiderivative size = 101, normalized size of antiderivative = 0.82

$$\int \frac{a + bx^2}{x^5\sqrt{-c + dx}\sqrt{c + dx}} dx = \frac{c\sqrt{-c + dx}\sqrt{c + dx}(2ac^2 + 4bc^2x^2 + 3ad^2x^2) + 2d^2(4bc^2 + 3ad^2)x^4 \arctan\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)}{8c^5x^4}$$

input `Integrate[(a + b*x^2)/(x^5*Sqrt[-c + d*x]*Sqrt[c + d*x]),x]`

output

```
(c*Sqrt[-c + d*x]*Sqrt[c + d*x]*(2*a*c^2 + 4*b*c^2*x^2 + 3*a*d^2*x^2) + 2*
d^2*(4*b*c^2 + 3*a*d^2)*x^4*ArcTan[Sqrt[-c + d*x]/Sqrt[c + d*x]])/(8*c^5*x
^4)
```

Rubi [A] (verified)

Time = 0.42 (sec) , antiderivative size = 115, normalized size of antiderivative = 0.93, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.161$, Rules used = {956, 114, 27, 103, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{a + bx^2}{x^5 \sqrt{dx - c} \sqrt{c + dx}} dx$$

↓ 956

$$\frac{1}{4} \left(\frac{3ad^2}{c^2} + 4b \right) \int \frac{1}{x^3 \sqrt{dx - c} \sqrt{c + dx}} dx + \frac{a\sqrt{dx - c} \sqrt{c + dx}}{4c^2 x^4}$$

↓ 114

$$\frac{1}{4} \left(\frac{3ad^2}{c^2} + 4b \right) \left(\int \frac{d^2}{x \sqrt{dx - c} \sqrt{c + dx}} dx + \frac{\sqrt{dx - c} \sqrt{c + dx}}{2c^2 x^2} \right) + \frac{a\sqrt{dx - c} \sqrt{c + dx}}{4c^2 x^4}$$

↓ 27

$$\frac{1}{4} \left(\frac{3ad^2}{c^2} + 4b \right) \left(\frac{d^2 \int \frac{1}{x \sqrt{dx - c} \sqrt{c + dx}} dx}{2c^2} + \frac{\sqrt{dx - c} \sqrt{c + dx}}{2c^2 x^2} \right) + \frac{a\sqrt{dx - c} \sqrt{c + dx}}{4c^2 x^4}$$

↓ 103

$$\frac{1}{4} \left(\frac{3ad^2}{c^2} + 4b \right) \left(\frac{d^3 \int \frac{1}{dc^2 + d(dx - c)(c + dx)} d(\sqrt{dx - c} \sqrt{c + dx})}{2c^2} + \frac{\sqrt{dx - c} \sqrt{c + dx}}{2c^2 x^2} \right) + \frac{a\sqrt{dx - c} \sqrt{c + dx}}{4c^2 x^4}$$

↓ 218

$$\frac{1}{4} \left(\frac{3ad^2}{c^2} + 4b \right) \left(\frac{d^2 \arctan \left(\frac{\sqrt{dx - c} \sqrt{c + dx}}{c} \right)}{2c^3} + \frac{\sqrt{dx - c} \sqrt{c + dx}}{2c^2 x^2} \right) + \frac{a\sqrt{dx - c} \sqrt{c + dx}}{4c^2 x^4}$$

input `Int[(a + b*x^2)/(x^5*Sqrt[-c + d*x]*Sqrt[c + d*x]),x]`

output `(a*Sqrt[-c + d*x]*Sqrt[c + d*x])/(4*c^2*x^4) + ((4*b + (3*a*d^2)/c^2)*((Sqrt[-c + d*x]*Sqrt[c + d*x])/(2*c^2*x^2) + (d^2*ArcTan[(Sqrt[-c + d*x]*Sqrt[c + d*x])/c])/(2*c^3)))/4`

Defintions of rubi rules used

rule 27 `Int[(a_)*(F_x_), x_Symbol] := Simp[a Int[F_x, x], x] /; FreeQ[a, x] && !MatchQ[F_x, (b_)*(G_x_)] /; FreeQ[b, x]`

rule 103 `Int[1/(Sqrt[(a_) + (b_)*(x_)]*Sqrt[(c_) + (d_)*(x_)]*((e_) + (f_)*(x_))), x_] := Simp[b*f Subst[Int[1/(d*(b*e - a*f)^2 + b*f^2*x^2), x], x, Sqrt[a + b*x]*Sqrt[c + d*x]], x] /; FreeQ[{a, b, c, d, e, f}, x] && EqQ[2*b*d*e - f*(b*c + a*d), 0]`

rule 114 `Int[((a_) + (b_)*(x_))^(m_)*((c_) + (d_)*(x_))^(n_)*((e_) + (f_)*(x_))^(p_), x_] := Simp[b*(a + b*x)^(m + 1)*(c + d*x)^(n + 1)*((e + f*x)^(p + 1))/((m + 1)*(b*c - a*d)*(b*e - a*f)), x] + Simp[1/((m + 1)*(b*c - a*d)*(b*e - a*f)) Int[(a + b*x)^(m + 1)*(c + d*x)^n*(e + f*x)^p*Simp[a*d*f*(m + 1) - b*(d*e*(m + n + 2) + c*f*(m + p + 2)) - b*d*f*(m + n + p + 3)*x, x], x], x] /; FreeQ[{a, b, c, d, e, f, n, p}, x] && ILtQ[m, -1] && (IntegerQ[n] || IntegersQ[2*n, 2*p] || ILtQ[m + n + p + 3, 0])`

rule 218 `Int[((a_) + (b_)*(x_)^2)^(-1), x_Symbol] := Simp[(Rt[a/b, 2]/a)*ArcTan[x/Rt[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]`

rule 956

```
Int[((e._)*(x._))^(m._)*((a1._) + (b1._)*(x._)^(non2._))^(p._)*((a2._) + (b2._)
*(x._)^(non2._))^(p._)*((c._) + (d._)*(x._)^(n._)), x_Symbol] := Simp[c*(e*x)^(
m + 1)*(a1 + b1*x^(n/2))^(p + 1)*((a2 + b2*x^(n/2))^(p + 1)/(a1*a2*e*(m + 1
))), x] + Simp[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(a1*a2*e^n*(
m + 1)) Int[(e*x)^(m + n)*(a1 + b1*x^(n/2))^p*(a2 + b2*x^(n/2))^p, x], x]
/; FreeQ[{a1, b1, a2, b2, c, d, e, p}, x] && EqQ[non2, n/2] && EqQ[a2*b1 +
a1*b2, 0] && (IntegerQ[n] || GtQ[e, 0]) && ((GtQ[n, 0] && LtQ[m, -1]) || (
LtQ[n, 0] && GtQ[m + n, -1])) && !ILtQ[p, -1]
```

Maple [A] (verified)

Time = 0.16 (sec) , antiderivative size = 151, normalized size of antiderivative = 1.23

method	result
risch	$-\frac{\sqrt{dx+c}(-dx+c)(3ad^2x^2+4bc^2x^2+2ac^2)}{8c^4x^4\sqrt{dx-c}} - \frac{d^2(3ad^2+4bc^2)\ln\left(\frac{-2c^2+2\sqrt{-c^2}\sqrt{d^2x^2-c^2}}{x}\right)\sqrt{(dx-c)(dx+c)}}{8c^4\sqrt{-c^2}\sqrt{dx-c}\sqrt{dx+c}}$
default	$-\frac{\sqrt{dx-c}\sqrt{dx+c}\left(3\ln\left(-\frac{2(c^2-\sqrt{-c^2}\sqrt{d^2x^2-c^2})}{x}\right)\right)ad^4x^4+4\ln\left(-\frac{2(c^2-\sqrt{-c^2}\sqrt{d^2x^2-c^2})}{x}\right)bc^2d^2x^4-3\sqrt{-c^2}\sqrt{d^2x^2-c^2}ad^2x^2}{8c^4\sqrt{d^2x^2-c^2}x^4\sqrt{-c^2}}$

input

```
int((b*x^2+a)/x^5/(d*x-c)^(1/2)/(d*x+c)^(1/2),x,method=_RETURNVERBOSE)
```

output

```
-1/8*(d*x+c)^(1/2)*(-d*x+c)*(3*a*d^2*x^2+4*b*c^2*x^2+2*a*c^2)/c^4/x^4/(d*x
-c)^(1/2)-1/8*d^2*(3*a*d^2+4*b*c^2)/c^4/(-c^2)^(1/2)*ln((-2*c^2+2*(-c^2)^(
1/2)*(d^2*x^2-c^2)^(1/2))/x)*((d*x-c)*(d*x+c))^(1/2)/(d*x-c)^(1/2)/(d*x+c)
^(1/2)
```

Fricas [A] (verification not implemented)

Time = 0.13 (sec) , antiderivative size = 100, normalized size of antiderivative = 0.81

$$\int \frac{a + bx^2}{x^5\sqrt{-c + dx}\sqrt{c + dx}} dx$$

$$= \frac{2(4bc^2d^2 + 3ad^4)x^4 \arctan\left(-\frac{dx - \sqrt{dx+c}\sqrt{dx-c}}{c}\right) + (2ac^3 + (4bc^3 + 3acd^2)x^2)\sqrt{dx+c}\sqrt{dx-c}}{8c^5x^4}$$

input `integrate((b*x^2+a)/x^5/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="fricas")`

output
$$\frac{1}{8} \cdot (2 \cdot (4 \cdot b \cdot c^2 \cdot d^2 + 3 \cdot a \cdot d^4) \cdot x^4 \cdot \arctan\left(\frac{-(d \cdot x - \sqrt{d \cdot x + c}) \cdot \sqrt{d \cdot x - c}}{c}\right) + (2 \cdot a \cdot c^3 + (4 \cdot b \cdot c^3 + 3 \cdot a \cdot c \cdot d^2) \cdot x^2) \cdot \sqrt{d \cdot x + c} \cdot \sqrt{d \cdot x - c}) / (c^5 \cdot x^4)$$

Sympy [F(-1)]

Timed out.

$$\int \frac{a + bx^2}{x^5 \sqrt{-c + dx} \sqrt{c + dx}} dx = \text{Timed out}$$

input `integrate((b*x**2+a)/x**5/(d*x-c)**(1/2)/(d*x+c)**(1/2),x)`

output Timed out

Maxima [A] (verification not implemented)

Time = 0.11 (sec) , antiderivative size = 114, normalized size of antiderivative = 0.93

$$\int \frac{a + bx^2}{x^5 \sqrt{-c + dx} \sqrt{c + dx}} dx = -\frac{bd^2 \arcsin\left(\frac{c}{d|x|}\right)}{2c^3} - \frac{3ad^4 \arcsin\left(\frac{c}{d|x|}\right)}{8c^5} + \frac{\sqrt{d^2x^2 - c^2}b}{2c^2x^2} + \frac{3\sqrt{d^2x^2 - c^2}ad^2}{8c^4x^2} + \frac{\sqrt{d^2x^2 - c^2}a}{4c^2x^4}$$

input `integrate((b*x^2+a)/x^5/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="maxima")`

output
$$-1/2 \cdot b \cdot d^2 \cdot \arcsin(c/(d \cdot \text{abs}(x))) / c^3 - 3/8 \cdot a \cdot d^4 \cdot \arcsin(c/(d \cdot \text{abs}(x))) / c^5 + 1/2 \cdot \sqrt{d^2 \cdot x^2 - c^2} \cdot b / (c^2 \cdot x^2) + 3/8 \cdot \sqrt{d^2 \cdot x^2 - c^2} \cdot a \cdot d^2 / (c^4 \cdot x^2) + 1/4 \cdot \sqrt{d^2 \cdot x^2 - c^2} \cdot a / (c^2 \cdot x^4)$$

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 325 vs. $2(105) = 210$.

Time = 0.15 (sec) , antiderivative size = 325, normalized size of antiderivative = 2.64

$$\int \frac{a + bx^2}{x^5 \sqrt{-c + dx} \sqrt{c + dx}} dx = \frac{(4bc^2d^3 + 3ad^5) \arctan\left(\frac{(\sqrt{dx+c} - \sqrt{dx-c})^2}{2c}\right)}{c^5} + \frac{2(4bc^2d^3(\sqrt{dx+c} - \sqrt{dx-c})^{14} + 3ad^5(\sqrt{dx+c} - \sqrt{dx-c})^{14} + 16bc^4d^3(\sqrt{dx+c} - \sqrt{dx-c})^{10})}{c^5}$$

input `integrate((b*x^2+a)/x^5/(d*x-c)^(1/2)/(d*x+c)^(1/2),x, algorithm="giac")`

output `-1/4*((4*b*c^2*d^3 + 3*a*d^5)*arctan(1/2*(sqrt(d*x + c) - sqrt(d*x - c))^2/c)/c^5 + 2*(4*b*c^2*d^3*(sqrt(d*x + c) - sqrt(d*x - c))^14 + 3*a*d^5*(sqrt(d*x + c) - sqrt(d*x - c))^14 + 16*b*c^4*d^3*(sqrt(d*x + c) - sqrt(d*x - c))^10 + 44*a*c^2*d^5*(sqrt(d*x + c) - sqrt(d*x - c))^10 - 64*b*c^6*d^3*(sqrt(d*x + c) - sqrt(d*x - c))^6 - 176*a*c^4*d^5*(sqrt(d*x + c) - sqrt(d*x - c))^6 - 256*b*c^8*d^3*(sqrt(d*x + c) - sqrt(d*x - c))^2 - 192*a*c^6*d^5*(sqrt(d*x + c) - sqrt(d*x - c))^2)/(((sqrt(d*x + c) - sqrt(d*x - c))^4 + 4*c^2)^4*c^4)/d`

Mupad [B] (verification not implemented)

Time = 23.03 (sec) , antiderivative size = 1005, normalized size of antiderivative = 8.17

$$\int \frac{a + bx^2}{x^5 \sqrt{-c + dx} \sqrt{c + dx}} dx = \text{Too large to display}$$

input `int((a + b*x^2)/(x^5*(c + d*x)^(1/2)*(d*x - c)^(1/2)),x)`

output

```
(3*a*(-c)^(1/2)*d^4*log(((c + d*x)^(1/2) - c^(1/2))/((-c)^(1/2) - (d*x - c)^(1/2))))/(8*c^(11/2)) - ((b*(-c)^(3/2)*d^2)/(32*c^(9/2)) + (b*(-c)^(3/2)*d^2*((c + d*x)^(1/2) - c^(1/2))^2)/(16*c^(9/2)*((-c)^(1/2) - (d*x - c)^(1/2))^2) - (15*b*(-c)^(3/2)*d^2*((c + d*x)^(1/2) - c^(1/2))^4)/(32*c^(9/2)*((-c)^(1/2) - (d*x - c)^(1/2))^4)/(((c + d*x)^(1/2) - c^(1/2))^2/((-c)^(1/2) - (d*x - c)^(1/2))^2 + (2*((c + d*x)^(1/2) - c^(1/2))^4)/((-c)^(1/2) - (d*x - c)^(1/2))^4 + ((c + d*x)^(1/2) - c^(1/2))^6/((-c)^(1/2) - (d*x - c)^(1/2))^6) - ((a*(-c)^(1/2)*d^4)/(1024*c^(11/2)) - (3*a*(-c)^(1/2)*d^4*((c + d*x)^(1/2) - c^(1/2))^2)/(128*c^(11/2)*((-c)^(1/2) - (d*x - c)^(1/2))^2) - (53*a*(-c)^(1/2)*d^4*((c + d*x)^(1/2) - c^(1/2))^4)/(512*c^(11/2)*((-c)^(1/2) - (d*x - c)^(1/2))^4) + (87*a*(-c)^(1/2)*d^4*((c + d*x)^(1/2) - c^(1/2))^6)/(256*c^(11/2)*((-c)^(1/2) - (d*x - c)^(1/2))^6) + (657*a*(-c)^(1/2)*d^4*((c + d*x)^(1/2) - c^(1/2))^8)/(1024*c^(11/2)*((-c)^(1/2) - (d*x - c)^(1/2))^8) + (121*a*(-c)^(1/2)*d^4*((c + d*x)^(1/2) - c^(1/2))^10)/(256*c^(11/2)*((-c)^(1/2) - (d*x - c)^(1/2))^10))/(((c + d*x)^(1/2) - c^(1/2))^4/((-c)^(1/2) - (d*x - c)^(1/2))^4 + (4*((c + d*x)^(1/2) - c^(1/2))^6)/((-c)^(1/2) - (d*x - c)^(1/2))^6 + (6*((c + d*x)^(1/2) - c^(1/2))^8)/((-c)^(1/2) - (d*x - c)^(1/2))^8 + (4*((c + d*x)^(1/2) - c^(1/2))^10)/((-c)^(1/2) - (d*x - c)^(1/2))^10 + ((c + d*x)^(1/2) - c^(1/2))^12/((-c)^(1/2) - (d*x - c)^(1/2))^12) - (b*(-c)^(3/2)*d^2*log(((c + d*x)^(1/2) - c^(1/2))/(...
```

Reduce [B] (verification not implemented)

Time = 0.24 (sec) , antiderivative size = 214, normalized size of antiderivative = 1.74

$$\int \frac{a + bx^2}{x^5 \sqrt{-c + dx} \sqrt{c + dx}} dx$$

$$= \frac{6 \operatorname{atan}\left(\frac{\sqrt{dx-c} + \sqrt{dx+c} - \sqrt{c}}{\sqrt{c}}\right) a d^4 x^4 + 8 \operatorname{atan}\left(\frac{\sqrt{dx-c} + \sqrt{dx+c} - \sqrt{c}}{\sqrt{c}}\right) b c^2 d^2 x^4 - 6 \operatorname{atan}\left(\frac{\sqrt{dx-c} + \sqrt{dx+c} + \sqrt{c}}{\sqrt{c}}\right) a d^4 x^4 - \dots}{\dots}$$

input

```
int((b*x^2+a)/x^5/(d*x-c)^(1/2)/(d*x+c)^(1/2),x)
```

output

```
(6*atan((sqrt(-c+d*x)+sqrt(c+d*x)-sqrt(c))/sqrt(c))*a*d**4*x**4
+ 8*atan((sqrt(-c+d*x)+sqrt(c+d*x)-sqrt(c))/sqrt(c))*b*c**2*d**2
*x**4 - 6*atan((sqrt(-c+d*x)+sqrt(c+d*x)+sqrt(c))/sqrt(c))*a*d**
4*x**4 - 8*atan((sqrt(-c+d*x)+sqrt(c+d*x)+sqrt(c))/sqrt(c))*b*c*
*2*d**2*x**4 + 2*sqrt(c+d*x)*sqrt(-c+d*x)*a*c**3 + 3*sqrt(c+d*x)*s
qrt(-c+d*x)*a*c*d**2*x**2 + 4*sqrt(c+d*x)*sqrt(-c+d*x)*b*c**3*x*
*2)/(8*c**5*x**4)
```

3.32
$$\int \frac{x^4(a+bx^2)}{(-c+dx)^{3/2}(c+dx)^{3/2}} dx$$

Optimal result	258
Mathematica [A] (verified)	259
Rubi [A] (verified)	259
Maple [A] (verified)	262
Fricas [A] (verification not implemented)	262
Sympy [F(-1)]	263
Maxima [A] (verification not implemented)	263
Giac [A] (verification not implemented)	264
Mupad [F(-1)]	264
Reduce [B] (verification not implemented)	265

Optimal result

Integrand size = 31, antiderivative size = 169

$$\int \frac{x^4(a+bx^2)}{(-c+dx)^{3/2}(c+dx)^{3/2}} dx = -\frac{\left(\frac{a}{c^2} + \frac{b}{d^2}\right) x^5}{\sqrt{-c+dx}\sqrt{c+dx}} + \frac{3(5bc^2 + 4ad^2) x\sqrt{-c+dx}\sqrt{c+dx}}{8d^6} + \frac{(5bc^2 + 4ad^2) x^3\sqrt{-c+dx}\sqrt{c+dx}}{4c^2d^4} + \frac{3c^2(5bc^2 + 4ad^2) \operatorname{arctanh}\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)}{4d^7}$$

output

```
-(a/c^2+b/d^2)*x^5/(d*x-c)^(1/2)/(d*x+c)^(1/2)+3/8*(4*a*d^2+5*b*c^2)*x*(d*x-c)^(1/2)*(d*x+c)^(1/2)/d^6+1/4*(4*a*d^2+5*b*c^2)*x^3*(d*x-c)^(1/2)*(d*x+c)^(1/2)/c^2/d^4+3/4*c^2*(4*a*d^2+5*b*c^2)*arctanh((d*x-c)^(1/2)/(d*x+c)^(1/2))/d^7
```

Mathematica [A] (verified)

Time = 0.28 (sec) , antiderivative size = 137, normalized size of antiderivative = 0.81

$$\int \frac{x^4(a + bx^2)}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{4ad^3x(-3c^2 + d^2x^2) + bdx(-15c^4 + 5c^2d^2x^2 + 2d^4x^4) + 6c^2(5bc^2 + 4ad^2)}{8d^7\sqrt{-c + dx}\sqrt{c + dx}}$$

input `Integrate[(x^4*(a + b*x^2))/((-c + d*x)^(3/2)*(c + d*x)^(3/2)),x]`

output `(4*a*d^3*x*(-3*c^2 + d^2*x^2) + b*d*x*(-15*c^4 + 5*c^2*d^2*x^2 + 2*d^4*x^4) + 6*c^2*(5*b*c^2 + 4*a*d^2)*Sqrt[-c + d*x]*Sqrt[c + d*x]*ArcTanh[Sqrt[c + d*x]/Sqrt[-c + d*x]])/(8*d^7*Sqrt[-c + d*x]*Sqrt[c + d*x])`

Rubi [A] (verified)

Time = 0.46 (sec) , antiderivative size = 141, normalized size of antiderivative = 0.83, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}} = 0.226$, Rules used = {960, 109, 27, 101, 27, 45, 221}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \frac{x^4(a + bx^2)}{(dx - c)^{3/2}(c + dx)^{3/2}} dx \\ & \quad \downarrow 960 \\ & \frac{1}{4} \left(4a + \frac{5bc^2}{d^2} \right) \int \frac{x^4}{(dx - c)^{3/2}(c + dx)^{3/2}} dx + \frac{bx^5}{4d^2\sqrt{dx - c}\sqrt{c + dx}} \\ & \quad \downarrow 109 \\ & \frac{1}{4} \left(4a + \frac{5bc^2}{d^2} \right) \left(-\frac{\int -\frac{3cx^2}{\sqrt{dx - c}\sqrt{c + dx}} dx}{cd^2} - \frac{x^3}{d^2\sqrt{dx - c}\sqrt{c + dx}} \right) + \frac{bx^5}{4d^2\sqrt{dx - c}\sqrt{c + dx}} \\ & \quad \downarrow 27 \\ & \frac{1}{4} \left(4a + \frac{5bc^2}{d^2} \right) \left(\frac{3 \int \frac{x^2}{\sqrt{dx - c}\sqrt{c + dx}} dx}{d^2} - \frac{x^3}{d^2\sqrt{dx - c}\sqrt{c + dx}} \right) + \frac{bx^5}{4d^2\sqrt{dx - c}\sqrt{c + dx}} \end{aligned}$$

$$\begin{aligned}
& \downarrow 101 \\
& \frac{1}{4} \left(4a + \frac{5bc^2}{d^2} \right) \left(\frac{3 \left(\frac{\int \frac{c^2}{\sqrt{dx-c}\sqrt{c+dx}} dx}{2d^2} + \frac{x\sqrt{dx-c}\sqrt{c+dx}}{2d^2} \right)}{d^2} - \frac{x^3}{d^2\sqrt{dx-c}\sqrt{c+dx}} \right) + \\
& \quad \frac{bx^5}{4d^2\sqrt{dx-c}\sqrt{c+dx}} \\
& \downarrow 27 \\
& \frac{1}{4} \left(4a + \frac{5bc^2}{d^2} \right) \left(\frac{3 \left(\frac{c^2 \int \frac{1}{\sqrt{dx-c}\sqrt{c+dx}} dx}{2d^2} + \frac{x\sqrt{dx-c}\sqrt{c+dx}}{2d^2} \right)}{d^2} - \frac{x^3}{d^2\sqrt{dx-c}\sqrt{c+dx}} \right) + \\
& \quad \frac{bx^5}{4d^2\sqrt{dx-c}\sqrt{c+dx}} \\
& \downarrow 45 \\
& \frac{1}{4} \left(4a + \frac{5bc^2}{d^2} \right) \left(\frac{3 \left(\frac{c^2 \int \frac{1}{d-\frac{d(dx-c)}{c+dx}} d\frac{\sqrt{dx-c}}{\sqrt{c+dx}}}{d^2} + \frac{x\sqrt{dx-c}\sqrt{c+dx}}{2d^2} \right)}{d^2} - \frac{x^3}{d^2\sqrt{dx-c}\sqrt{c+dx}} \right) + \\
& \quad \frac{bx^5}{4d^2\sqrt{dx-c}\sqrt{c+dx}} \\
& \downarrow 221 \\
& \frac{1}{4} \left(4a + \frac{5bc^2}{d^2} \right) \left(\frac{3 \left(\frac{c^2 \operatorname{arctanh}\left(\frac{\sqrt{dx-c}}{\sqrt{c+dx}}\right)}{d^3} + \frac{x\sqrt{dx-c}\sqrt{c+dx}}{2d^2} \right)}{d^2} - \frac{x^3}{d^2\sqrt{dx-c}\sqrt{c+dx}} \right) + \\
& \quad \frac{bx^5}{4d^2\sqrt{dx-c}\sqrt{c+dx}}
\end{aligned}$$

input

```
Int[(x^4*(a + b*x^2))/((-c + d*x)^(3/2)*(c + d*x)^(3/2)),x]
```

output

$$\frac{(b*x^5)/(4*d^2*\sqrt{-c+d*x}*\sqrt{c+d*x}) + ((4*a + (5*b*c^2)/d^2)*(-x^3/(d^2*\sqrt{-c+d*x}*\sqrt{c+d*x})) + (3*((x*\sqrt{-c+d*x}*\sqrt{c+d*x}))/d^2) + (c^2*\text{ArcTanh}[\sqrt{-c+d*x}/\sqrt{c+d*x}])/d^3)/d^2)/4$$

Defintions of rubi rules used

rule 27

$$\text{Int}[(a_*)(Fx_), x_Symbol] \rightarrow \text{Simp}[a \text{ Int}[Fx, x], x] /; \text{FreeQ}[a, x] \ \&\& \ !\text{MatchQ}[Fx, (b_*)(Gx_)] /; \text{FreeQ}[b, x]$$

rule 45

$$\text{Int}[1/(\sqrt{(a_)} + (b_*)(x_))\sqrt{(c_)} + (d_*)(x_)]], x_Symbol] \rightarrow \text{Simp}[2 \text{ Subst}[\text{Int}[1/(b - d*x^2), x], x, \sqrt{a + b*x}/\sqrt{c + d*x}], x] /; \text{FreeQ}[\{a, b, c, d\}, x] \ \&\& \ \text{EqQ}[b*c + a*d, 0] \ \&\& \ !\text{GtQ}[c, 0]$$

rule 101

$$\text{Int}[(a_.) + (b_.)*(x_))^{2*((c_.) + (d_.)*(x_))^{(n_)}*((e_.) + (f_.)*(x_))^{(p_)}], x_] \rightarrow \text{Simp}[b*(a + b*x)*(c + d*x)^{(n+1)}*((e + f*x)^{(p+1)}/(d*f*(n + p + 3))), x] + \text{Simp}[1/(d*f*(n + p + 3)) \text{ Int}[(c + d*x)^n*(e + f*x)^p*\text{Simp}[a^2*d*f*(n + p + 3) - b*(b*c*e + a*(d*e*(n + 1) + c*f*(p + 1))) + b*(a*d*f*(n + p + 4) - b*(d*e*(n + 2) + c*f*(p + 2)))*x, x], x] /; \text{FreeQ}[\{a, b, c, d, e, f, n, p\}, x] \ \&\& \ \text{NeQ}[n + p + 3, 0]$$

rule 109

$$\text{Int}[(a_.) + (b_.)*(x_))^{(m_)}*((c_.) + (d_.)*(x_))^{(n_)}*((e_.) + (f_.)*(x_))^{(p_)}], x_] \rightarrow \text{Simp}[(b*c - a*d)*(a + b*x)^{(m+1)}*(c + d*x)^{(n-1)}*((e + f*x)^{(p+1)}/(b*(b*e - a*f)*(m + 1))), x] + \text{Simp}[1/(b*(b*e - a*f)*(m + 1)) \text{ Int}[(a + b*x)^{(m+1)}*(c + d*x)^{(n-2)}*(e + f*x)^p*\text{Simp}[a*d*(d*e*(n - 1) + c*f*(p + 1)) + b*c*(d*e*(m - n + 2) - c*f*(m + p + 2)) + d*(a*d*f*(n + p) + b*(d*e*(m + 1) - c*f*(m + n + p + 1)))*x, x], x] /; \text{FreeQ}[\{a, b, c, d, e, f, p\}, x] \ \&\& \ \text{LtQ}[m, -1] \ \&\& \ \text{GtQ}[n, 1] \ \&\& \ (\text{IntegersQ}[2*m, 2*n, 2*p] \ || \ \text{IntegersQ}[m, n + p] \ || \ \text{IntegersQ}[p, m + n])$$

rule 221

$$\text{Int}[(a_.) + (b_.)*(x_)^2)^{-1}, x_Symbol] \rightarrow \text{Simp}[(\text{Rt}[-a/b, 2]/a)*\text{ArcTanh}[x/\text{Rt}[-a/b, 2]], x] /; \text{FreeQ}[\{a, b\}, x] \ \&\& \ \text{NegQ}[a/b]$$

rule 960

```
Int[((e._)*(x._))^(m._)*((a1_) + (b1_.)*(x_)^(non2_.))^(p._)*((a2_) + (b2_.)
*(x_)^(non2_.))^(p._)*((c_) + (d_.)*(x_)^(n_)), x_Symbol] := Simp[d*(e*x)^(
m + 1)*(a1 + b1*x^(n/2))^(p + 1)*((a2 + b2*x^(n/2))^(p + 1)/(b1*b2*e*(m + n
*(p + 1) + 1))), x] - Simp[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/
(b1*b2*(m + n*(p + 1) + 1)) Int[(e*x)^m*(a1 + b1*x^(n/2))^p*(a2 + b2*x^(n
/2))^p, x], x] /; FreeQ[{a1, b1, a2, b2, c, d, e, m, n, p}, x] && EqQ[non2,
n/2] && EqQ[a2*b1 + a1*b2, 0] && NeQ[m + n*(p + 1) + 1, 0]
```

Maple [A] (verified)

Time = 0.16 (sec) , antiderivative size = 275, normalized size of antiderivative = 1.63

method	result
risch	$-\frac{x(2bx^2d^2+4ad^2+7bc^2)(-dx+c)\sqrt{dx+c}}{8d^6\sqrt{dx-c}} + \frac{c^2 \left(\frac{12ad^2 \ln\left(\frac{d^2x+\sqrt{d^2x^2-c^2}}{\sqrt{d^2}}\right)}{\sqrt{d^2}} + \frac{15bc^2 \ln\left(\frac{d^2x+\sqrt{d^2x^2-c^2}}{\sqrt{d^2}}\right)}{\sqrt{d^2}} \right) - \frac{4(ad^2+bc^2)\sqrt{d^2(x-c)}}{8d^6\sqrt{dx-c}}}{d^2(x-c)}$
default	$-\frac{(-2 \operatorname{csgn}(d)bd^5x^5\sqrt{d^2x^2-c^2}-4 \operatorname{csgn}(d)ad^5x^3\sqrt{d^2x^2-c^2}-5 \operatorname{csgn}(d)bc^2d^3x^3\sqrt{d^2x^2-c^2}-12 \ln\left(\left(\sqrt{d^2x^2-c^2} \operatorname{csgn}(d)+dx\right) \operatorname{csgn}(d)\right))}{8d^6\sqrt{dx-c}}$

input

```
int(x^4*(b*x^2+a)/(d*x-c)^(3/2)/(d*x+c)^(3/2),x,method=_RETURNVERBOSE)
```

output

```
-1/8*x*(2*b*d^2*x^2+4*a*d^2+7*b*c^2)*(-d*x+c)*(d*x+c)^(1/2)/d^6/(d*x-c)^(1
/2)+1/8*c^2/d^6*(12*a*d^2*ln(d^2*x/(d^2)^(1/2)+(d^2*x^2-c^2)^(1/2))/(d^2)^(
1/2)+15*b*c^2*ln(d^2*x/(d^2)^(1/2)+(d^2*x^2-c^2)^(1/2))/(d^2)^(1/2)-4*(a*
d^2+b*c^2)/d^2/(x-c/d)*(d^2*(x-c/d)^2+2*c*d*(x-c/d))^(1/2)-4*(a*d^2+b*c^2)
/d^2/(x+c/d)*(d^2*(x+c/d)^2-2*c*d*(x+c/d))^(1/2))*((d*x-c)*(d*x+c))^(1/2)/
(d*x-c)^(1/2)/(d*x+c)^(1/2)
```

Fricas [A] (verification not implemented)

Time = 0.10 (sec) , antiderivative size = 190, normalized size of antiderivative = 1.12

$$\int \frac{x^4(a + bx^2)}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{8bc^6 + 8ac^4d^2 - 8(bc^4d^2 + ac^2d^4)x^2 + (2bd^5x^5 + (5bc^2d^3 + 4ad^5)x^3 - 3c^2d^3x - 3c^2d^3)}{(-c + dx)^{3/2}(c + dx)^{3/2}}$$

input

```
integrate(x^4*(b*x^2+a)/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="fricas")
```

output

```
1/8*(8*b*c^6 + 8*a*c^4*d^2 - 8*(b*c^4*d^2 + a*c^2*d^4)*x^2 + (2*b*d^5*x^5
+ (5*b*c^2*d^3 + 4*a*d^5)*x^3 - 3*(5*b*c^4*d + 4*a*c^2*d^3)*x)*sqrt(d*x +
c)*sqrt(d*x - c) + 3*(5*b*c^6 + 4*a*c^4*d^2 - (5*b*c^4*d^2 + 4*a*c^2*d^4)*
x^2)*log(-d*x + sqrt(d*x + c)*sqrt(d*x - c))/(d^9*x^2 - c^2*d^7)
```

Sympy [F(-1)]

Timed out.

$$\int \frac{x^4(a + bx^2)}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \text{Timed out}$$

input

```
integrate(x**4*(b*x**2+a)/(d*x-c)**(3/2)/(d*x+c)**(3/2),x)
```

output

Timed out

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 196, normalized size of antiderivative = 1.16

$$\int \frac{x^4(a + bx^2)}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{bx^5}{4\sqrt{d^2x^2 - c^2d^2}} + \frac{5bc^2x^3}{8\sqrt{d^2x^2 - c^2d^4}}$$

$$+ \frac{ax^3}{2\sqrt{d^2x^2 - c^2d^2}} - \frac{15bc^4x}{8\sqrt{d^2x^2 - c^2d^6}} - \frac{3ac^2x}{2\sqrt{d^2x^2 - c^2d^4}}$$

$$+ \frac{15bc^4 \log(2d^2x + 2\sqrt{d^2x^2 - c^2d})}{8d^7} + \frac{3ac^2 \log(2d^2x + 2\sqrt{d^2x^2 - c^2d})}{2d^5}$$

input

```
integrate(x^4*(b*x^2+a)/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="maxima")
```

output

```
1/4*b*x^5/(sqrt(d^2*x^2 - c^2)*d^2) + 5/8*b*c^2*x^3/(sqrt(d^2*x^2 - c^2)*d
^4) + 1/2*a*x^3/(sqrt(d^2*x^2 - c^2)*d^2) - 15/8*b*c^4*x/(sqrt(d^2*x^2 - c
^2)*d^6) - 3/2*a*c^2*x/(sqrt(d^2*x^2 - c^2)*d^4) + 15/8*b*c^4*log(2*d^2*x
+ 2*sqrt(d^2*x^2 - c^2)*d)/d^7 + 3/2*a*c^2*log(2*d^2*x + 2*sqrt(d^2*x^2 -
c^2)*d)/d^5
```


Giac [A] (verification not implemented)

Time = 0.16 (sec) , antiderivative size = 214, normalized size of antiderivative = 1.27

$$\int \frac{x^4(a+bx^2)}{(-c+dx)^{3/2}(c+dx)^{3/2}} dx = \frac{\left(\left((dx+c)\left(2(dx+c)\left(\frac{(dx+c)b}{d^7} - \frac{5bc}{d^7}\right) + \frac{25bc^2d^{35}+4ad^{37}}{d^{42}}\right) - \frac{35bc^3d^{35}+12acd^{37}}{d^{42}}\right) - \frac{3(5bc^4+4ac^2d^2)\log\left(\left(\sqrt{dx+c}-\sqrt{dx-c}\right)^2\right)}{8d^7} - \frac{2(bc^5+ac^3d^2)}{\left(\left(\sqrt{dx+c}-\sqrt{dx-c}\right)^2+2c\right)d^7}}{8\sqrt{dx-c}}$$

input `integrate(x^4*(b*x^2+a)/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="giac")`

output `1/8*((d*x + c)*(2*(d*x + c)*((d*x + c)*b/d^7 - 5*b*c/d^7) + (25*b*c^2*d^35 + 4*a*d^37)/d^42) - (35*b*c^3*d^35 + 12*a*c*d^37)/d^42)*(d*x + c) + 2*(7*b*c^4*d^35 + 2*a*c^2*d^37)/d^42)*sqrt(d*x + c)/sqrt(d*x - c) - 3/8*(5*b*c^4 + 4*a*c^2*d^2)*log((sqrt(d*x + c) - sqrt(d*x - c))^2)/d^7 - 2*(b*c^5 + a*c^3*d^2)/(((sqrt(d*x + c) - sqrt(d*x - c))^2 + 2*c)*d^7)`

Mupad [F(-1)]

Timed out.

$$\int \frac{x^4(a+bx^2)}{(-c+dx)^{3/2}(c+dx)^{3/2}} dx = \int \frac{x^4(bx^2+a)}{(c+dx)^{3/2}(dx-c)^{3/2}} dx$$

input `int((x^4*(a + b*x^2))/((c + d*x)^(3/2)*(d*x - c)^(3/2)),x)`

output `int((x^4*(a + b*x^2))/((c + d*x)^(3/2)*(d*x - c)^(3/2)), x)`

Reduce [B] (verification not implemented)

Time = 0.26 (sec) , antiderivative size = 331, normalized size of antiderivative = 1.96

$$\int \frac{x^4(a + bx^2)}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{24\sqrt{dx - c} \log\left(\frac{\sqrt{dx-c} + \sqrt{dx+c}}{\sqrt{c}\sqrt{2}}\right) a c^3 d^2 + 24\sqrt{dx - c} \log\left(\frac{\sqrt{dx-c} + \sqrt{dx+c}}{\sqrt{c}\sqrt{2}}\right) a c^2}{1}$$

input `int(x^4*(b*x^2+a)/(d*x-c)^(3/2)/(d*x+c)^(3/2),x)`

output `(24*sqrt(-c + d*x)*log((sqrt(-c + d*x) + sqrt(c + d*x))/(sqrt(c)*sqrt(2)))*a*c**3*d**2 + 24*sqrt(-c + d*x)*log((sqrt(-c + d*x) + sqrt(c + d*x))/(sqrt(c)*sqrt(2)))*a*c**2*d**3*x + 30*sqrt(-c + d*x)*log((sqrt(-c + d*x) + sqrt(c + d*x))/(sqrt(c)*sqrt(2)))*b*c**5 + 30*sqrt(-c + d*x)*log((sqrt(-c + d*x) + sqrt(c + d*x))/(sqrt(c)*sqrt(2)))*b*c**4*d*x - 9*sqrt(-c + d*x)*a*c**3*d**2 - 9*sqrt(-c + d*x)*a*c**2*d**3*x - 10*sqrt(-c + d*x)*b*c**5 - 10*sqrt(-c + d*x)*b*c**4*d*x - 12*sqrt(c + d*x)*a*c**2*d**3*x + 4*sqrt(c + d*x)*a*d**5*x**3 - 15*sqrt(c + d*x)*b*c**4*d*x + 5*sqrt(c + d*x)*b*c**2*d**3*x**3 + 2*sqrt(c + d*x)*b*d**5*x**5)/(8*sqrt(-c + d*x)*d**7*(c + d*x))`

3.33
$$\int \frac{x^3(a+bx^2)}{(-c+dx)^{3/2}(c+dx)^{3/2}} dx$$

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Optimal result

Integrand size = 31, antiderivative size = 123

$$\int \frac{x^3(a+bx^2)}{(-c+dx)^{3/2}(c+dx)^{3/2}} dx = -\frac{\left(\frac{a}{c^2} + \frac{b}{d^2}\right) x^4}{\sqrt{-c+dx}\sqrt{c+dx}} + \frac{2(4bc^2 + 3ad^2) \sqrt{-c+dx}\sqrt{c+dx}}{3d^6} + \frac{(4bc^2 + 3ad^2) x^2 \sqrt{-c+dx}\sqrt{c+dx}}{3c^2d^4}$$

output

```
-(a/c^2+b/d^2)*x^4/(d*x-c)^(1/2)/(d*x+c)^(1/2)+2/3*(3*a*d^2+4*b*c^2)*(d*x-c)^(1/2)*(d*x+c)^(1/2)/d^6+1/3*(3*a*d^2+4*b*c^2)*x^2*(d*x-c)^(1/2)*(d*x+c)^(1/2)/c^2/d^4
```

Mathematica [A] (verified)

Time = 0.14 (sec) , antiderivative size = 72, normalized size of antiderivative = 0.59

$$\int \frac{x^3(a+bx^2)}{(-c+dx)^{3/2}(c+dx)^{3/2}} dx = \frac{-8bc^4 - 6ac^2d^2 + 4bc^2d^2x^2 + 3ad^4x^2 + bd^4x^4}{3d^6\sqrt{-c+dx}\sqrt{c+dx}}$$

input

```
Integrate[(x^3*(a + b*x^2))/((-c + d*x)^(3/2)*(c + d*x)^(3/2)),x]
```

output

$$(-8*b*c^4 - 6*a*c^2*d^2 + 4*b*c^2*d^2*x^2 + 3*a*d^4*x^2 + b*d^4*x^4)/(3*d^6*\text{Sqrt}[-c + d*x]*\text{Sqrt}[c + d*x])$$

Rubi [A] (verified)

Time = 0.40 (sec) , antiderivative size = 103, normalized size of antiderivative = 0.84, number of steps used = 4, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}} = 0.129$, Rules used = {960, 109, 27, 83}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{x^3(a + bx^2)}{(dx - c)^{3/2}(c + dx)^{3/2}} dx$$

$$\downarrow 960$$

$$\frac{1}{3} \left(3a + \frac{4bc^2}{d^2} \right) \int \frac{x^3}{(dx - c)^{3/2}(c + dx)^{3/2}} dx + \frac{bx^4}{3d^2\sqrt{dx - c}\sqrt{c + dx}}$$

$$\downarrow 109$$

$$\frac{1}{3} \left(3a + \frac{4bc^2}{d^2} \right) \left(-\frac{\int -\frac{2cx}{\sqrt{dx - c}\sqrt{c + dx}} dx}{cd^2} - \frac{x^2}{d^2\sqrt{dx - c}\sqrt{c + dx}} \right) + \frac{bx^4}{3d^2\sqrt{dx - c}\sqrt{c + dx}}$$

$$\downarrow 27$$

$$\frac{1}{3} \left(3a + \frac{4bc^2}{d^2} \right) \left(\frac{2 \int \frac{x}{\sqrt{dx - c}\sqrt{c + dx}} dx}{d^2} - \frac{x^2}{d^2\sqrt{dx - c}\sqrt{c + dx}} \right) + \frac{bx^4}{3d^2\sqrt{dx - c}\sqrt{c + dx}}$$

$$\downarrow 83$$

$$\frac{1}{3} \left(\frac{2\sqrt{dx - c}\sqrt{c + dx}}{d^4} - \frac{x^2}{d^2\sqrt{dx - c}\sqrt{c + dx}} \right) \left(3a + \frac{4bc^2}{d^2} \right) + \frac{bx^4}{3d^2\sqrt{dx - c}\sqrt{c + dx}}$$

input

$$\text{Int}[(x^3*(a + b*x^2))/((-c + d*x)^(3/2)*(c + d*x)^(3/2)), x]$$

output

$$(b*x^4)/(3*d^2*\text{Sqrt}[-c + d*x]*\text{Sqrt}[c + d*x]) + ((3*a + (4*b*c^2)/d^2)*(-x^2/(d^2*\text{Sqrt}[-c + d*x]*\text{Sqrt}[c + d*x])) + (2*\text{Sqrt}[-c + d*x]*\text{Sqrt}[c + d*x])/d^4))/3$$

Definitions of rubi rules used

- rule 27 $\text{Int}[(a_*)(Fx_), x_Symbol] \rightarrow \text{Simp}[a \text{ Int}[Fx, x], x] /; \text{FreeQ}[a, x] \ \&\& \ !\text{MatchQ}[Fx, (b_*)(Gx_)] /; \text{FreeQ}[b, x]$
- rule 83 $\text{Int}[(a_.) + (b_.)*(x_)]*((c_.) + (d_.)*(x_))^{(n_.)}*((e_.) + (f_.)*(x_))^{(p_.)}, x_] \rightarrow \text{Simp}[b*(c + d*x)^{(n + 1)}*((e + f*x)^{(p + 1)}/(d*f*(n + p + 2))), x] /; \text{FreeQ}\{a, b, c, d, e, f, n, p\}, x] \ \&\& \ \text{NeQ}[n + p + 2, 0] \ \&\& \ \text{EqQ}[a*d*f*(n + p + 2) - b*(d*e*(n + 1) + c*f*(p + 1)), 0]$
- rule 109 $\text{Int}[(a_.) + (b_.)*(x_)]^{(m_.)}*((c_.) + (d_.)*(x_))^{(n_.)}*((e_.) + (f_.)*(x_))^{(p_.)}, x_] \rightarrow \text{Simp}[(b*c - a*d)*(a + b*x)^{(m + 1)}*(c + d*x)^{(n - 1)}*((e + f*x)^{(p + 1)}/(b*(b*e - a*f)*(m + 1))), x] + \text{Simp}[1/(b*(b*e - a*f)*(m + 1)) \text{Int}[(a + b*x)^{(m + 1)}*(c + d*x)^{(n - 2)}*(e + f*x)^p * \text{Simp}[a*d*(d*e*(n - 1) + c*f*(p + 1)) + b*c*(d*e*(m - n + 2) - c*f*(m + p + 2)) + d*(a*d*f*(n + p) + b*(d*e*(m + 1) - c*f*(m + n + p + 1))]*x, x], x], x] /; \text{FreeQ}\{a, b, c, d, e, f, p\}, x] \ \&\& \ \text{LtQ}[m, -1] \ \&\& \ \text{GtQ}[n, 1] \ \&\& \ (\text{IntegersQ}[2*m, 2*n, 2*p] \ || \ \text{IntegersQ}[m, n + p] \ || \ \text{IntegersQ}[p, m + n])$
- rule 960 $\text{Int}[(e_.)*(x_)]^{(m_.)}*((a1_.) + (b1_.)*(x_))^{(non2_.)}*((a2_.) + (b2_.)*(x_))^{(non2_.)}*((c_.) + (d_.)*(x_))^{(n_.)}, x_Symbol] \rightarrow \text{Simp}[d*(e*x)^{(m + 1)}*(a1 + b1*x^{(n/2)})^{(p + 1)}*((a2 + b2*x^{(n/2)})^{(p + 1)}/(b1*b2*e*(m + n*(p + 1) + 1))), x] - \text{Simp}[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(b1*b2*(m + n*(p + 1) + 1)) \text{Int}[(e*x)^m*(a1 + b1*x^{(n/2)})^p*(a2 + b2*x^{(n/2)})^p, x], x] /; \text{FreeQ}\{a1, b1, a2, b2, c, d, e, m, n, p\}, x] \ \&\& \ \text{EqQ}[non2, n/2] \ \&\& \ \text{EqQ}[a2*b1 + a1*b2, 0] \ \&\& \ \text{NeQ}[m + n*(p + 1) + 1, 0]$

Maple [A] (verified)

Time = 0.16 (sec) , antiderivative size = 68, normalized size of antiderivative = 0.55

method	result	size
gospers	$-\frac{-bx^4d^4-3ad^4x^2-4bc^2d^2x^2+6ac^2d^2+8bc^4}{3d^6\sqrt{dx-c}\sqrt{dx+c}}$	68
default	$-\frac{-bx^4d^4-3ad^4x^2-4bc^2d^2x^2+6ac^2d^2+8bc^4}{3d^6\sqrt{dx-c}\sqrt{dx+c}}$	68
orering	$\frac{(-dx+c)(-bx^4d^4-3ad^4x^2-4bc^2d^2x^2+6ac^2d^2+8bc^4)}{3\sqrt{dx+c}d^6(dx-c)^{\frac{3}{2}}}$	74
risch	$-\frac{(bx^2d^2+3ad^2+5bc^2)(-dx+c)\sqrt{dx+c}}{3d^6\sqrt{dx-c}} - \frac{c^2(ad^2+bc^2)\sqrt{(dx-c)(dx+c)}}{d^6\sqrt{-(dx+c)(-dx+c)}\sqrt{dx-c}\sqrt{dx+c}}$	115

input `int(x^3*(b*x^2+a)/(d*x-c)^(3/2)/(d*x+c)^(3/2),x,method=_RETURNVERBOSE)`

output $-1/3/d^6/(d*x-c)^{(1/2)}/(d*x+c)^{(1/2)}*(-b*d^4*x^4-3*a*d^4*x^2-4*b*c^2*d^2*x^2+6*a*c^2*d^2+8*b*c^4)$

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 80, normalized size of antiderivative = 0.65

$$\int \frac{x^3(a+bx^2)}{(-c+dx)^{3/2}(c+dx)^{3/2}} dx = \frac{(bd^4x^4 - 8bc^4 - 6ac^2d^2 + (4bc^2d^2 + 3ad^4)x^2)\sqrt{dx+c}\sqrt{dx-c}}{3(d^8x^2 - c^2d^6)}$$

input `integrate(x^3*(b*x^2+a)/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="fricas")`

output $1/3*(b*d^4*x^4 - 8*b*c^4 - 6*a*c^2*d^2 + (4*b*c^2*d^2 + 3*a*d^4)*x^2)*\text{sqrt}(d*x + c)*\text{sqrt}(d*x - c)/(d^8*x^2 - c^2*d^6)$

Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 59.11 (sec) , antiderivative size = 226, normalized size of antiderivative = 1.84

$$\int \frac{x^3(a+bx^2)}{(-c+dx)^{3/2}(c+dx)^{3/2}} dx = a \left(\frac{{}_2F_1\left(\begin{matrix} -\frac{3}{4}, -\frac{1}{4} \\ -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{2}, 0 \end{matrix} \middle| \frac{c^2}{d^2x^2}\right)}{2\pi^{\frac{3}{2}}d^4} - \frac{{}_2F_1\left(\begin{matrix} -2, -\frac{3}{2}, -\frac{5}{4} \\ -\frac{5}{4}, -\frac{3}{2} \end{matrix} \right)}{2\pi^{\frac{3}{2}}d^4} \right) + b \left(\frac{{}_2F_1\left(\begin{matrix} -\frac{7}{4}, -\frac{5}{4} \\ -\frac{7}{4}, -\frac{3}{2}, -\frac{5}{4}, -1, -\frac{1}{2}, 0 \end{matrix} \middle| \frac{c^2}{d^2x^2}\right)}{2\pi^{\frac{3}{2}}d^6} - \frac{{}_2F_1\left(\begin{matrix} -3, -\frac{5}{2}, -\frac{9}{4}, -2, -\frac{7}{4}, 1 \\ -\frac{9}{4}, -\frac{7}{4} \end{matrix} \right)}{2\pi^{\frac{3}{2}}d^6} \right)$$

input `integrate(x**3*(b*x**2+a)/(d*x-c)**(3/2)/(d*x+c)**(3/2),x)`

output `a*(c*meijerg(((-3/4, -1/4), (-1, 0, 1/2, 1)), ((-3/4, -1/2, -1/4, 0, 1/2, 0), ()), c**2/(d**2*x**2))/(2*pi**(3/2)*d**4) - I*c*meijerg(((-2, -3/2, -5/4, -1, -3/4, 1), ()), ((-5/4, -3/4), (-2, -3/2, -1/2, 0)), c**2*exp_polar(2*I*pi)/(d**2*x**2))/(2*pi**(3/2)*d**4)) + b*(c**3*meijerg(((-7/4, -5/4), (-2, -1, -1/2, 1)), ((-7/4, -3/2, -5/4, -1, -1/2, 0), ()), c**2/(d**2*x**2))/(2*pi**(3/2)*d**6) - I*c**3*meijerg(((-3, -5/2, -9/4, -2, -7/4, 1), ()), ((-9/4, -7/4), (-3, -5/2, -3/2, 0)), c**2*exp_polar(2*I*pi)/(d**2*x**2))/(2*pi**(3/2)*d**6))`

Maxima [A] (verification not implemented)

Time = 0.04 (sec) , antiderivative size = 123, normalized size of antiderivative = 1.00

$$\int \frac{x^3(a+bx^2)}{(-c+dx)^{3/2}(c+dx)^{3/2}} dx = \frac{bx^4}{3\sqrt{d^2x^2-c^2}d^2} + \frac{4bc^2x^2}{3\sqrt{d^2x^2-c^2}d^4} + \frac{ax^2}{\sqrt{d^2x^2-c^2}d^2} - \frac{8bc^4}{3\sqrt{d^2x^2-c^2}d^6} - \frac{2ac^2}{\sqrt{d^2x^2-c^2}d^4}$$

input `integrate(x^3*(b*x^2+a)/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="maxima")`

output $\frac{1}{3}bx^4/(\sqrt{d^2x^2 - c^2})d^2 + \frac{4}{3}b^2c^2x^2/(\sqrt{d^2x^2 - c^2})d^4 + ax^2/(\sqrt{d^2x^2 - c^2})d^2 - \frac{8}{3}b^2c^4/(\sqrt{d^2x^2 - c^2})d^6 - \frac{2a^2c^2}{(\sqrt{d^2x^2 - c^2})d^4}$

Giac [A] (verification not implemented)

Time = 0.16 (sec) , antiderivative size = 200, normalized size of antiderivative = 1.63

$$\int \frac{x^3(a+bx^2)}{(-c+dx)^{3/2}(c+dx)^{3/2}} dx = \frac{\left(2(dx+c)\left((dx+c)\left(\frac{(dx+c)b}{d^6} - \frac{4bc}{d^6}\right) + \frac{10bc^2d^{24}+3ad^{26}}{d^{30}}\right) - \frac{3(9bc^3d^{24}+5acd^{26})}{d^{30}}\right)}{6\sqrt{dx-c}} + \frac{2(b^2c^8 + 2abc^6d^2 + a^2c^4d^4)}{\left(bc^4(\sqrt{dx+c} - \sqrt{dx-c})^2 + ac^2d^2(\sqrt{dx+c} - \sqrt{dx-c})^2 + 2bc^5 + 2ac^3d^2\right)d^6}$$

input `integrate(x^3*(b*x^2+a)/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="giac")`

output $\frac{1}{6}(2*(d*x + c)*((d*x + c)*((d*x + c)*b/d^6 - 4*b*c/d^6) + (10*b*c^2*d^24 + 3*a*d^26)/d^30) - 3*(9*b*c^3*d^24 + 5*a*c*d^26)/d^30)*\sqrt{d*x + c}/\sqrt{d*x - c} + 2*(b^2*c^8 + 2*a*b*c^6*d^2 + a^2*c^4*d^4)/((b*c^4*(\sqrt{d*x + c} - \sqrt{d*x - c})^2 + a*c^2*d^2*(\sqrt{d*x + c} - \sqrt{d*x - c})^2 + 2*b*c^5 + 2*a*c^3*d^2)*d^6)$

Mupad [B] (verification not implemented)

Time = 5.24 (sec) , antiderivative size = 90, normalized size of antiderivative = 0.73

$$\int \frac{x^3(a+bx^2)}{(-c+dx)^{3/2}(c+dx)^{3/2}} dx = \frac{\sqrt{dx-c} \left(\frac{x^2(4bc^2d^2+3ad^4)}{3d^7} - \frac{8bc^4+6ac^2d^2}{3d^7} + \frac{bx^4}{3d^3} \right)}{x\sqrt{c+dx} - \frac{c\sqrt{c+dx}}{d}}$$

input `int((x^3*(a + b*x^2))/((c + d*x)^(3/2)*(d*x - c)^(3/2)),x)`

output

$$\frac{((d*x - c)^{(1/2)}*((x^2*(3*a*d^4 + 4*b*c^2*d^2))/(3*d^7) - (8*b*c^4 + 6*a*c^2*d^2)/(3*d^7) + (b*x^4)/(3*d^3)))/(x*(c + d*x)^{(1/2)} - (c*(c + d*x)^{(1/2)}))/d}$$

Reduce [B] (verification not implemented)

Time = 0.21 (sec) , antiderivative size = 73, normalized size of antiderivative = 0.59

$$\int \frac{x^3(a + bx^2)}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{\sqrt{dx + c}(bd^4x^4 + 3ad^4x^2 + 4bc^2d^2x^2 - 6ac^2d^2 - 8bc^4)}{3\sqrt{dx - c}d^6(dx + c)}$$

input

$$\text{int}(x^3*(b*x^2+a)/(d*x-c)^{(3/2)}/(d*x+c)^{(3/2)},x)$$

output

$$(\text{sqrt}(c + d*x)*(-6*a*c**2*d**2 + 3*a*d**4*x**2 - 8*b*c**4 + 4*b*c**2*d**2*x**2 + b*d**4*x**4))/(3*\text{sqrt}(-c + d*x)*d**6*(c + d*x))$$

3.34
$$\int \frac{x^2(a+bx^2)}{(-c+dx)^{3/2}(c+dx)^{3/2}} dx$$

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Optimal result

Integrand size = 31, antiderivative size = 120

$$\int \frac{x^2(a+bx^2)}{(-c+dx)^{3/2}(c+dx)^{3/2}} dx = -\frac{\left(\frac{a}{c^2} + \frac{b}{d^2}\right) x^3}{\sqrt{-c+dx}\sqrt{c+dx}} + \frac{(3bc^2 + 2ad^2) x\sqrt{-c+dx}\sqrt{c+dx}}{2c^2d^4} + \frac{(3bc^2 + 2ad^2) \operatorname{arctanh}\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)}{d^5}$$

output

```
-(a/c^2+b/d^2)*x^3/(d*x-c)^(1/2)/(d*x+c)^(1/2)+1/2*(2*a*d^2+3*b*c^2)*x*(d*x-c)^(1/2)*(d*x+c)^(1/2)/c^2/d^4+(2*a*d^2+3*b*c^2)*arctanh((d*x-c)^(1/2)/(d*x+c)^(1/2))/d^5
```

Mathematica [A] (verified)

Time = 0.20 (sec) , antiderivative size = 108, normalized size of antiderivative = 0.90

$$\int \frac{x^2(a+bx^2)}{(-c+dx)^{3/2}(c+dx)^{3/2}} dx = \frac{-3bc^2dx - 2ad^3x + bd^3x^3 + 2(3bc^2 + 2ad^2) \sqrt{-c+dx}\sqrt{c+dx}\operatorname{arctanh}\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)}{2d^5\sqrt{-c+dx}\sqrt{c+dx}}$$

input

```
Integrate[(x^2*(a + b*x^2))/((-c + d*x)^(3/2)*(c + d*x)^(3/2)),x]
```

output

```
(-3*b*c^2*d*x - 2*a*d^3*x + b*d^3*x^3 + 2*(3*b*c^2 + 2*a*d^2)*Sqrt[-c + d*x]*Sqrt[c + d*x]*ArcTanh[Sqrt[c + d*x]/Sqrt[-c + d*x]]/(2*d^5*Sqrt[-c + d*x]*Sqrt[c + d*x])
```

Rubi [A] (verified)

Time = 0.45 (sec) , antiderivative size = 133, normalized size of antiderivative = 1.11, number of steps used = 7, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}} = 0.194$, Rules used = {960, 100, 27, 87, 45, 221}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{x^2(a + bx^2)}{(dx - c)^{3/2}(c + dx)^{3/2}} dx$$

$$\downarrow 960$$

$$\frac{1}{2} \left(2a + \frac{3bc^2}{d^2} \right) \int \frac{x^2}{(dx - c)^{3/2}(c + dx)^{3/2}} dx + \frac{bx^3}{2d^2 \sqrt{dx - c} \sqrt{c + dx}}$$

$$\downarrow 100$$

$$\frac{1}{2} \left(2a + \frac{3bc^2}{d^2} \right) \left(\frac{\int \frac{cd^2 x}{\sqrt{dx - c}(c + dx)^{3/2}} dx}{cd^3} - \frac{c}{d^3 \sqrt{dx - c} \sqrt{c + dx}} \right) + \frac{bx^3}{2d^2 \sqrt{dx - c} \sqrt{c + dx}}$$

$$\downarrow 27$$

$$\frac{1}{2} \left(2a + \frac{3bc^2}{d^2} \right) \left(\frac{\int \frac{x}{\sqrt{dx - c}(c + dx)^{3/2}} dx}{d} - \frac{c}{d^3 \sqrt{dx - c} \sqrt{c + dx}} \right) + \frac{bx^3}{2d^2 \sqrt{dx - c} \sqrt{c + dx}}$$

$$\downarrow 87$$

$$\frac{1}{2} \left(2a + \frac{3bc^2}{d^2} \right) \left(\frac{\int \frac{1}{\sqrt{dx - c} \sqrt{c + dx}} dx}{d} - \frac{\sqrt{dx - c}}{d^2 \sqrt{c + dx}} - \frac{c}{d^3 \sqrt{dx - c} \sqrt{c + dx}} \right) + \frac{bx^3}{2d^2 \sqrt{dx - c} \sqrt{c + dx}}$$

$$\downarrow 45$$

$$\frac{1}{2} \left(2a + \frac{3bc^2}{d^2} \right) \left(\frac{2 \int \frac{1}{d - \frac{d(dx-c)}{c+dx}} d \frac{\sqrt{dx-c}}{\sqrt{c+dx}}}{d} - \frac{\sqrt{dx-c}}{d^2 \sqrt{c+dx}} - \frac{c}{d^3 \sqrt{dx-c} \sqrt{c+dx}} \right) + \frac{bx^3}{2d^2 \sqrt{dx-c} \sqrt{c+dx}}$$

↓ 221

$$\frac{1}{2} \left(2a + \frac{3bc^2}{d^2} \right) \left(\frac{2 \operatorname{arctanh} \left(\frac{\sqrt{dx-c}}{\sqrt{c+dx}} \right)}{d^2} - \frac{\sqrt{dx-c}}{d^2 \sqrt{c+dx}} - \frac{c}{d^3 \sqrt{dx-c} \sqrt{c+dx}} \right) + \frac{bx^3}{2d^2 \sqrt{dx-c} \sqrt{c+dx}}$$

input `Int[(x^2*(a + b*x^2))/((-c + d*x)^(3/2)*(c + d*x)^(3/2)),x]`

output `(b*x^3)/(2*d^2*Sqrt[-c + d*x]*Sqrt[c + d*x]) + ((2*a + (3*b*c^2)/d^2)*(-c/(d^3*Sqrt[-c + d*x]*Sqrt[c + d*x])) + (-Sqrt[-c + d*x]/(d^2*Sqrt[c + d*x])) + (2*ArcTanh[Sqrt[-c + d*x]/Sqrt[c + d*x]]/d^2)/d)/2`

Defintions of rubi rules used

rule 27 `Int[(a_)*(F_x_), x_Symbol] := Simp[a Int[F_x, x], x] /; FreeQ[a, x] && !MatchQ[F_x, (b_)*(G_x_)] /; FreeQ[b, x]`

rule 45 `Int[1/(Sqrt[(a_) + (b_)*(x_)]*Sqrt[(c_) + (d_)*(x_)]), x_Symbol] := Simp[2 Subst[Int[1/(b - d*x^2), x], x, Sqrt[a + b*x]/Sqrt[c + d*x]], x] /; FreeQ[{a, b, c, d}, x] && EqQ[b*c + a*d, 0] && !GtQ[c, 0]`

rule 87 `Int[((a_.) + (b_.)*(x_.))*((c_.) + (d_.)*(x_.))^(n_.)*((e_.) + (f_.)*(x_.))^(p_.), x_] := Simp[(-b*e - a*f)*(c + d*x)^(n + 1)*((e + f*x)^(p + 1)/(f*(p + 1)*(c*f - d*e))), x] - Simp[(a*d*f*(n + p + 2) - b*(d*e*(n + 1) + c*f*(p + 1)))/(f*(p + 1)*(c*f - d*e)) Int[(c + d*x)^n*(e + f*x)^(p + 1), x], x] /; FreeQ[{a, b, c, d, e, f, n}, x] && LtQ[p, -1] && (!LtQ[n, -1] || IntegerQ[p] || !(IntegerQ[n] || !(EqQ[e, 0] || !(EqQ[c, 0] || LtQ[p, n]))))`

```
rule 100 Int[((a_.) + (b_.)*(x_))^(2*((c_.) + (d_.)*(x_))^(n_))*((e_.) + (f_.)*(x_))^(p_), x_] := Simp[(b*c - a*d)^(2*(c + d*x)^(n + 1))*((e + f*x)^(p + 1))/(d^2*(d*e - c*f)*(n + 1)), x] - Simp[1/(d^2*(d*e - c*f)*(n + 1)) Int[(c + d*x)^(n + 1)*(e + f*x)^p*Simp[a^2*d^2*f*(n + p + 2) + b^2*c*(d*e*(n + 1) + c*f*(p + 1)) - 2*a*b*d*(d*e*(n + 1) + c*f*(p + 1)) - b^2*d*(d*e - c*f)*(n + 1)*x, x], x] /; FreeQ[{a, b, c, d, e, f, n, p}, x] && (LtQ[n, -1] || (EqQ[n + p + 3, 0] && NeQ[n, -1] && (SumSimplerQ[n, 1] || !SumSimplerQ[p, 1])))
```

```
rule 221 Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(Rt[-a/b, 2]/a)*ArcTanh[x/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]
```

```
rule 960 Int[((e_.)*(x_))^(m_.)*((a1_) + (b1_.)*(x_)^(non2_.))^(p_.)*((a2_) + (b2_.)*(x_)^(non2_.))^(p_.)*((c_) + (d_.)*(x_)^(n_)), x_Symbol] := Simp[d*(e*x)^(m + 1)*(a1 + b1*x^(n/2))^(p + 1)*((a2 + b2*x^(n/2))^(p + 1))/(b1*b2*e*(m + n*(p + 1) + 1)), x] - Simp[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(b1*b2*(m + n*(p + 1) + 1)) Int[(e*x)^m*(a1 + b1*x^(n/2))^p*(a2 + b2*x^(n/2))^p, x], x] /; FreeQ[{a1, b1, a2, b2, c, d, e, m, n, p}, x] && EqQ[non2, n/2] && EqQ[a2*b1 + a1*b2, 0] && NeQ[m + n*(p + 1) + 1, 0]
```

Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 250 vs. 2(106) = 212.

Time = 0.14 (sec) , antiderivative size = 251, normalized size of antiderivative = 2.09

method	result
risch	$-\frac{bx(-dx+c)\sqrt{dx+c}}{2d^4\sqrt{dx-c}} + \frac{\left(-\frac{(ad^2+bc^2)\sqrt{d^2\left(x-\frac{c}{d}\right)^2+2cd\left(x-\frac{c}{d}\right)}}{d^2\left(x-\frac{c}{d}\right)} + \frac{2ad^2\ln\left(\frac{d^2x+\sqrt{d^2x^2-c^2}}{\sqrt{d^2}}\right)}{\sqrt{d^2}} + \frac{3bc^2\ln\left(\frac{d^2x+\sqrt{d^2x^2-c^2}}{\sqrt{d^2}}\right)}{\sqrt{d^2}} - \frac{(ad^2+bc^2)\sqrt{d^2x^2-c^2}}{2d^4\sqrt{dx-c}\sqrt{dx+c}}\right)}{2d^4\sqrt{dx-c}\sqrt{dx+c}}$
default	$-\frac{\left(-\operatorname{csgn}(d)bd^3x^3\sqrt{d^2x^2-c^2}-2\ln\left(\left(\sqrt{d^2x^2-c^2}\operatorname{csgn}(d)+dx\right)\operatorname{csgn}(d)\right)ad^4x^2-3\ln\left(\left(\sqrt{d^2x^2-c^2}\operatorname{csgn}(d)+dx\right)\operatorname{csgn}(d)\right)bc^2d^2x^2\right)}{2d^4\sqrt{dx-c}\sqrt{dx+c}}$

```
input int(x^2*(b*x^2+a)/(d*x-c)^(3/2)/(d*x+c)^(3/2), x, method=_RETURNVERBOSE)
```

output

```
-1/2*b*x*(-d*x+c)*(d*x+c)^(1/2)/d^4/(d*x-c)^(1/2)+1/2/d^4*(-(a*d^2+b*c^2)/
d^2/(x-c/d)*(d^2*(x-c/d)^2+2*c*d*(x-c/d))^(1/2)+2*a*d^2*ln(d^2*x/(d^2)^(1/
2)+(d^2*x^2-c^2)^(1/2))/(d^2)^(1/2)+3*b*c^2*ln(d^2*x/(d^2)^(1/2)+(d^2*x^2-
c^2)^(1/2))/(d^2)^(1/2)-(a*d^2+b*c^2)/d^2/(x+c/d)*(d^2*(x+c/d)^2-2*c*d*(x+
c/d))^(1/2))*((d*x-c)*(d*x+c))^(1/2)/(d*x-c)^(1/2)/(d*x+c)^(1/2)
```

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 159, normalized size of antiderivative = 1.32

$$\int \frac{x^2(a + bx^2)}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{2bc^4 + 2ac^2d^2 - 2(bc^2d^2 + ad^4)x^2 + (bd^3x^3 - (3bc^2d + 2ad^3)x)\sqrt{dx + c}}{2(d^7x^2 - c^2d^5)}$$

input

```
integrate(x^2*(b*x^2+a)/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="fricas")
```

output

```
1/2*(2*b*c^4 + 2*a*c^2*d^2 - 2*(b*c^2*d^2 + a*d^4)*x^2 + (b*d^3*x^3 - (3*b
*c^2*d + 2*a*d^3)*x)*sqrt(d*x + c)*sqrt(d*x - c) + (3*b*c^4 + 2*a*c^2*d^2
- (3*b*c^2*d^2 + 2*a*d^4)*x^2)*log(-d*x + sqrt(d*x + c)*sqrt(d*x - c)))/(d
^7*x^2 - c^2*d^5)
```

Sympy [F(-1)]

Timed out.

$$\int \frac{x^2(a + bx^2)}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \text{Timed out}$$

input

```
integrate(x**2*(b*x**2+a)/(d*x-c)**(3/2)/(d*x+c)**(3/2),x)
```

output

Timed out

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 138, normalized size of antiderivative = 1.15

$$\int \frac{x^2(a + bx^2)}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{bx^3}{2\sqrt{d^2x^2 - c^2}d^2} - \frac{3bc^2x}{2\sqrt{d^2x^2 - c^2}d^4} - \frac{ax}{\sqrt{d^2x^2 - c^2}d^2} + \frac{3bc^2 \log(2d^2x + 2\sqrt{d^2x^2 - c^2}d)}{2d^5} + \frac{a \log(2d^2x + 2\sqrt{d^2x^2 - c^2}d)}{d^3}$$

input `integrate(x^2*(b*x^2+a)/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="maxima")`output $\frac{1}{2}bx^3/(\sqrt{d^2x^2 - c^2}d^2) - \frac{3}{2}b*c^2*x/(\sqrt{d^2x^2 - c^2}d^4) - \frac{a*x}{(\sqrt{d^2x^2 - c^2}d^2)} + \frac{3}{2}b*c^2*\log(2*d^2*x + 2*\sqrt{d^2x^2 - c^2}d)/d^5 + \frac{a*\log(2*d^2*x + 2*\sqrt{d^2x^2 - c^2}d)}{d^3}$ **Giac [A] (verification not implemented)**

Time = 0.16 (sec) , antiderivative size = 147, normalized size of antiderivative = 1.22

$$\int \frac{x^2(a + bx^2)}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{\sqrt{dx + c} \left((dx + c) \left(\frac{(dx+c)b}{d^5} - \frac{3bc}{d^5} \right) + \frac{bc^2d^{15} - ad^{17}}{d^{20}} \right)}{2\sqrt{dx - c}} - \frac{(3bc^2 + 2ad^2) \log \left((\sqrt{dx + c} - \sqrt{dx - c})^2 \right)}{2d^5} - \frac{2(bc^3 + acd^2)}{\left((\sqrt{dx + c} - \sqrt{dx - c})^2 + 2c \right) d^5}$$

input `integrate(x^2*(b*x^2+a)/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="giac")`output $\frac{1}{2}*\sqrt{d*x + c}*((d*x + c)*((d*x + c)*b/d^5 - 3*b*c/d^5) + (b*c^2*d^{15} - a*d^{17})/d^{20})/\sqrt{d*x - c} - \frac{1}{2}*(3*b*c^2 + 2*a*d^2)*\log((\sqrt{d*x + c} - \sqrt{d*x - c})^2)/d^5 - \frac{2*(b*c^3 + a*c*d^2)}{((\sqrt{d*x + c} - \sqrt{d*x - c})^2 + 2*c)*d^5}$

Mupad [F(-1)]

Timed out.

$$\int \frac{x^2(a + bx^2)}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \int \frac{x^2(bx^2 + a)}{(c + dx)^{3/2}(dx - c)^{3/2}} dx$$

input `int((x^2*(a + b*x^2))/((c + d*x)^(3/2)*(d*x - c)^(3/2)),x)`

output `int((x^2*(a + b*x^2))/((c + d*x)^(3/2)*(d*x - c)^(3/2)), x)`

Reduce [B] (verification not implemented)

Time = 0.23 (sec) , antiderivative size = 285, normalized size of antiderivative = 2.38

$$\int \frac{x^2(a + bx^2)}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{16\sqrt{dx - c} \log\left(\frac{\sqrt{dx-c} + \sqrt{dx+c}}{\sqrt{c}\sqrt{2}}\right) ac d^2 + 16\sqrt{dx - c} \log\left(\frac{\sqrt{dx-c} + \sqrt{dx+c}}{\sqrt{c}\sqrt{2}}\right) a d^3 x}{1}$$

input `int(x^2*(b*x^2+a)/(d*x-c)^(3/2)/(d*x+c)^(3/2),x)`

output `(16*sqrt(-c + d*x)*log((sqrt(-c + d*x) + sqrt(c + d*x))/(sqrt(c)*sqrt(2)))*a*c*d**2 + 16*sqrt(-c + d*x)*log((sqrt(-c + d*x) + sqrt(c + d*x))/(sqrt(c)*sqrt(2)))*a*d**3*x + 24*sqrt(-c + d*x)*log((sqrt(-c + d*x) + sqrt(c + d*x))/(sqrt(c)*sqrt(2)))*b*c**3 + 24*sqrt(-c + d*x)*log((sqrt(-c + d*x) + sqrt(c + d*x))/(sqrt(c)*sqrt(2)))*b*c**2*d*x - 8*sqrt(-c + d*x)*a*c*d**2 - 8*sqrt(-c + d*x)*a*d**3*x - 9*sqrt(-c + d*x)*b*c**3 - 9*sqrt(-c + d*x)*b*c**2*d*x - 8*sqrt(c + d*x)*a*d**3*x - 12*sqrt(c + d*x)*b*c**2*d*x + 4*sqrt(c + d*x)*b*d**3*x**3)/(8*sqrt(-c + d*x)*d**5*(c + d*x))`

3.35 $\int \frac{x(a+bx^2)}{(-c+dx)^{3/2}(c+dx)^{3/2}} dx$

Optimal result	280
Mathematica [A] (verified)	280
Rubi [A] (verified)	281
Maple [A] (verified)	282
Fricas [A] (verification not implemented)	282
Sympy [C] (verification not implemented)	283
Maxima [A] (verification not implemented)	284
Giac [B] (verification not implemented)	284
Mupad [B] (verification not implemented)	285
Reduce [B] (verification not implemented)	285

Optimal result

Integrand size = 29, antiderivative size = 76

$$\int \frac{x(a+bx^2)}{(-c+dx)^{3/2}(c+dx)^{3/2}} dx = -\frac{\left(\frac{a}{c^2} + \frac{b}{d^2}\right)x^2}{\sqrt{-c+dx}\sqrt{c+dx}} + \frac{(2bc^2+ad^2)\sqrt{-c+dx}\sqrt{c+dx}}{c^2d^4}$$

output

$$-(a/c^2+b/d^2)*x^2/(d*x-c)^{(1/2)}/(d*x+c)^{(1/2)}+(a*d^2+2*b*c^2)*(d*x-c)^{(1/2)}*(d*x+c)^{(1/2)}/c^2/d^4$$

Mathematica [A] (verified)

Time = 0.11 (sec) , antiderivative size = 45, normalized size of antiderivative = 0.59

$$\int \frac{x(a+bx^2)}{(-c+dx)^{3/2}(c+dx)^{3/2}} dx = \frac{-2bc^2-ad^2+bd^2x^2}{d^4\sqrt{-c+dx}\sqrt{c+dx}}$$

input

$$\text{Integrate}[(x*(a+b*x^2))/((-c+d*x)^(3/2)*(c+d*x)^(3/2)),x]$$

output

$$(-2*b*c^2-a*d^2+b*d^2*x^2)/(d^4*\text{Sqrt}[-c+d*x]*\text{Sqrt}[c+d*x])$$

Rubi [A] (verified)

Time = 0.33 (sec) , antiderivative size = 73, normalized size of antiderivative = 0.96, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}} = 0.069$, Rules used = {958, 83}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{x(a + bx^2)}{(dx - c)^{3/2}(c + dx)^{3/2}} dx$$

↓ 958

$$\left(\frac{a}{c^2} + \frac{2b}{d^2}\right) \int \frac{x}{\sqrt{dx - c}\sqrt{c + dx}} dx - \frac{x^2\left(\frac{a}{c^2} + \frac{b}{d^2}\right)}{\sqrt{dx - c}\sqrt{c + dx}}$$

↓ 83

$$\frac{\sqrt{dx - c}\sqrt{c + dx}\left(\frac{a}{c^2} + \frac{2b}{d^2}\right)}{d^2} - \frac{x^2\left(\frac{a}{c^2} + \frac{b}{d^2}\right)}{\sqrt{dx - c}\sqrt{c + dx}}$$

input

```
Int[(x*(a + b*x^2))/((-c + d*x)^(3/2)*(c + d*x)^(3/2)),x]
```

output

```
-(((a/c^2 + b/d^2)*x^2)/(Sqrt[-c + d*x]*Sqrt[c + d*x])) + ((a/c^2 + (2*b)/d^2)*Sqrt[-c + d*x]*Sqrt[c + d*x])/d^2
```

Defintions of rubi rules used

rule 83

```
Int[((a_.) + (b_.)*(x_))*((c_.) + (d_.)*(x_))^(n_.)*((e_.) + (f_.)*(x_))^(p_.), x_] :> Simp[b*(c + d*x)^(n + 1)*((e + f*x)^(p + 1)/(d*f*(n + p + 2))), x] /; FreeQ[{a, b, c, d, e, f, n, p}, x] && NeQ[n + p + 2, 0] && EqQ[a*d*f*(n + p + 2) - b*(d*e*(n + 1) + c*f*(p + 1)), 0]
```

rule 958

```

Int[((e._)*(x_))^(m._)*((a1_) + (b1_.)*(x_)^(non2_.))^(p._)*((a2_) + (b2_.)
*(x_)^(non2_.))^(p._)*((c_) + (d_.)*(x_)^(n_)), x_Symbol] := Simp[(-b1*b2*
c - a1*a2*d)*(e*x)^(m + 1)*(a1 + b1*x^(n/2))^(p + 1)*((a2 + b2*x^(n/2))^(p
+ 1)/(a1*a2*b1*b2*e*n*(p + 1))), x] - Simp[(a1*a2*d*(m + 1) - b1*b2*c*(m +
n*(p + 1) + 1))/(a1*a2*b1*b2*n*(p + 1)) Int[(e*x)^m*(a1 + b1*x^(n/2))^(p
+ 1)*(a2 + b2*x^(n/2))^(p + 1), x], x] /; FreeQ[{a1, b1, a2, b2, c, d, e,
m, n}, x] && EqQ[non2, n/2] && EqQ[a2*b1 + a1*b2, 0] && LtQ[p, -1] && (( !I
ntegerQ[p + 1/2] && NeQ[p, -5/4]) || !RationalQ[m] || (IGtQ[n, 0] && ILtQ[
p + 1/2, 0] && LeQ[-1, m, (-n)*(p + 1)]))

```

Maple [A] (verified)

Time = 0.14 (sec) , antiderivative size = 42, normalized size of antiderivative = 0.55

method	result	size
default	$\frac{bx^2d^2 - ad^2 - 2bc^2}{d^4\sqrt{dx-c}\sqrt{dx+c}}$	42
gospers	$-\frac{-bx^2d^2 + ad^2 + 2bc^2}{d^4\sqrt{dx-c}\sqrt{dx+c}}$	43
orering	$\frac{(-dx+c)(-bx^2d^2 + ad^2 + 2bc^2)}{\sqrt{dx+c}d^4(dx-c)^{\frac{3}{2}}}$	48
risch	$-\frac{b(-dx+c)\sqrt{dx+c}}{d^4\sqrt{dx-c}} - \frac{(a^2 + bc^2)\sqrt{(dx-c)(dx+c)}}{d^4\sqrt{-(dx+c)(-dx+c)}\sqrt{dx-c}\sqrt{dx+c}}$	92

input

```
int(x*(b*x^2+a)/(d*x-c)^(3/2)/(d*x+c)^(3/2),x,method=_RETURNVERBOSE)
```

output

```
(b*d^2*x^2-a*d^2-2*b*c^2)/d^4/(d*x-c)^(1/2)/(d*x+c)^(1/2)
```

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 56, normalized size of antiderivative = 0.74

$$\int \frac{x(a + bx^2)}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{(bd^2x^2 - 2bc^2 - ad^2)\sqrt{dx + c}\sqrt{dx - c}}{d^6x^2 - c^2d^4}$$

input

```
integrate(x*(b*x^2+a)/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="fricas")
```

output $(b*d^2*x^2 - 2*b*c^2 - a*d^2)*\text{sqrt}(d*x + c)*\text{sqrt}(d*x - c)/(d^6*x^2 - c^2*d^4)$

Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 54.36 (sec) , antiderivative size = 201, normalized size of antiderivative = 2.64

$$\int \frac{x(a + bx^2)}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = a \left(-\frac{G_{6,6}^{5,3} \left(\begin{matrix} \frac{1}{4}, \frac{3}{4}, 1 & 0, 1, \frac{3}{2} \\ \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{3}{2} & 0 \end{matrix} \middle| \frac{c^2}{d^2 x^2} \right)}{2\pi^{\frac{3}{2}} cd^2} - \frac{iG_{6,6}^{2,6} \left(\begin{matrix} -1, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, 1 \\ -\frac{1}{4}, \frac{1}{4} & -1, - \end{matrix} \right)}{2\pi^{\frac{3}{2}} cd^2} \right) + b \left(\frac{cG_{6,6}^{6,2} \left(\begin{matrix} -\frac{3}{4}, -\frac{1}{4} & -1, 0, \frac{1}{2}, 1 \\ -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{2}, 0 \end{matrix} \middle| \frac{c^2}{d^2 x^2} \right)}{2\pi^{\frac{3}{2}} d^4} - \frac{iG_{6,6}^{2,6} \left(\begin{matrix} -2, -\frac{3}{2}, -\frac{5}{4}, -1, -\frac{3}{4}, 1 \\ -\frac{5}{4}, -\frac{3}{4} & -2, -\frac{3}{2}, -\frac{1}{2}, 0 \end{matrix} \middle| \frac{c^2 e^{2i\pi}}{d^2 x^2} \right)}{2\pi^{\frac{3}{2}} d^4} \right)$$

input `integrate(x*(b*x**2+a)/(d*x-c)**(3/2)/(d*x+c)**(3/2),x)`

output `a*(-meijerg(((1/4, 3/4, 1), (0, 1, 3/2)), ((1/4, 1/2, 3/4, 1, 3/2), (0,)), c**2/(d**2*x**2))/(2*pi**(3/2)*c*d**2) - I*meijerg((-1, -1/2, -1/4, 0, 1/4, 1), ()), ((-1/4, 1/4), (-1, -1/2, 1/2, 0)), c**2*exp_polar(2*I*pi)/(d**2*x**2))/(2*pi**(3/2)*c*d**2)) + b*(c*meijerg((-3/4, -1/4), (-1, 0, 1/2, 1)), ((-3/4, -1/2, -1/4, 0, 1/2, 0), ()), c**2/(d**2*x**2))/(2*pi**(3/2)*d**4) - I*c*meijerg((-2, -3/2, -5/4, -1, -3/4, 1), ()), ((-5/4, -3/4), (-2, -3/2, -1/2, 0)), c**2*exp_polar(2*I*pi)/(d**2*x**2))/(2*pi**(3/2)*d**4)`

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 69, normalized size of antiderivative = 0.91

$$\int \frac{x(a + bx^2)}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{bx^2}{\sqrt{d^2x^2 - c^2d^2}} - \frac{2bc^2}{\sqrt{d^2x^2 - c^2d^4}} - \frac{a}{\sqrt{d^2x^2 - c^2d^2}}$$

input `integrate(x*(b*x^2+a)/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="maxima")`

output `b*x^2/(sqrt(d^2*x^2 - c^2)*d^2) - 2*b*c^2/(sqrt(d^2*x^2 - c^2)*d^4) - a/(sqrt(d^2*x^2 - c^2)*d^2)`

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 152 vs. 2(68) = 136.

Time = 0.15 (sec) , antiderivative size = 152, normalized size of antiderivative = 2.00

$$\int \frac{x(a + bx^2)}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{\sqrt{dx + c} \left(\frac{2(dx+c)b}{d^4} - \frac{5bc^2d^8 + ad^{10}}{cd^{12}} \right)}{2\sqrt{dx - c}} + \frac{2(b^2c^4 + 2abc^2d^2 + a^2d^4)}{\left(bc^2(\sqrt{dx + c} - \sqrt{dx - c})^2 + ad^2(\sqrt{dx + c} - \sqrt{dx - c})^2 + 2bc^3 + 2acd^2 \right) d^4}$$

input `integrate(x*(b*x^2+a)/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="giac")`

output `1/2*sqrt(d*x + c)*(2*(d*x + c)*b/d^4 - (5*b*c^2*d^8 + a*d^10)/(c*d^12))/sqrt(d*x - c) + 2*(b^2*c^4 + 2*a*b*c^2*d^2 + a^2*d^4)/((b*c^2*(sqrt(d*x + c) - sqrt(d*x - c))^2 + a*d^2*(sqrt(d*x + c) - sqrt(d*x - c))^2 + 2*b*c^3 + 2*a*c*d^2)*d^4)`

Mupad [B] (verification not implemented)

Time = 5.22 (sec) , antiderivative size = 67, normalized size of antiderivative = 0.88

$$\int \frac{x(a + bx^2)}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{a d^2 \sqrt{dx - c} + 2 b c^2 \sqrt{dx - c} - b d^2 x^2 \sqrt{dx - c}}{d^4 \sqrt{c + dx} (c - dx)}$$

input `int((x*(a + b*x^2))/((c + d*x)^(3/2)*(d*x - c)^(3/2)),x)`output `(a*d^2*(d*x - c)^(1/2) + 2*b*c^2*(d*x - c)^(1/2) - b*d^2*x^2*(d*x - c)^(1/2))/(d^4*(c + d*x)^(1/2)*(c - d*x))`**Reduce [B] (verification not implemented)**

Time = 0.21 (sec) , antiderivative size = 48, normalized size of antiderivative = 0.63

$$\int \frac{x(a + bx^2)}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{\sqrt{dx + c} (b d^2 x^2 - a d^2 - 2 b c^2)}{\sqrt{dx - c} d^4 (dx + c)}$$

input `int(x*(b*x^2+a)/(d*x-c)^(3/2)/(d*x+c)^(3/2),x)`output `(sqrt(c + d*x)*(- a*d**2 - 2*b*c**2 + b*d**2*x**2))/(sqrt(- c + d*x)*d**4*(c + d*x))`

3.36 $\int \frac{a+bx^2}{(-c+dx)^{3/2}(c+dx)^{3/2}} dx$

Optimal result	286
Mathematica [A] (verified)	286
Rubi [A] (verified)	287
Maple [C] (verified)	288
Fricas [B] (verification not implemented)	289
Sympy [C] (verification not implemented)	289
Maxima [A] (verification not implemented)	290
Giac [B] (verification not implemented)	290
Mupad [F(-1)]	291
Reduce [B] (verification not implemented)	291

Optimal result

Integrand size = 28, antiderivative size = 63

$$\int \frac{a + bx^2}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = -\frac{\left(\frac{a}{c^2} + \frac{b}{d^2}\right)x}{\sqrt{-c + dx}\sqrt{c + dx}} + \frac{2b \operatorname{arctanh}\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)}{d^3}$$

output

`-(a/c^2+b/d^2)*x/(d*x-c)^(1/2)/(d*x+c)^(1/2)+2*b*arctanh((d*x-c)^(1/2)/(d*x+c)^(1/2))/d^3`

Mathematica [A] (verified)

Time = 0.20 (sec) , antiderivative size = 71, normalized size of antiderivative = 1.13

$$\int \frac{a + bx^2}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{-\frac{2(bc^2d+ad^3)x}{c^2\sqrt{-c+dx}\sqrt{c+dx}} + 4b \operatorname{arctanh}\left(\frac{\sqrt{c+dx}}{\sqrt{-c+dx}}\right)}{2d^3}$$

input

`Integrate[(a + b*x^2)/((-c + d*x)^(3/2)*(c + d*x)^(3/2)),x]`

output

`((-2*(b*c^2*d + a*d^3)*x)/(c^2*sqrt[-c + d*x]*sqrt[c + d*x]) + 4*b*ArcTanh[sqrt[c + d*x]/sqrt[-c + d*x]])/(2*d^3)`

Rubi [A] (verified)

Time = 0.30 (sec) , antiderivative size = 63, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.107$, Rules used = {645, 45, 221}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{a + bx^2}{(dx - c)^{3/2}(c + dx)^{3/2}} dx$$

$$\downarrow 645$$

$$\frac{b \int \frac{1}{\sqrt{dx-c}\sqrt{c+dx}} dx}{d^2} - \frac{x\left(\frac{a}{c^2} + \frac{b}{d^2}\right)}{\sqrt{dx-c}\sqrt{c+dx}}$$

$$\downarrow 45$$

$$\frac{2b \int \frac{1}{d - \frac{d(dx-c)}{c+dx}} d \frac{\sqrt{dx-c}}{\sqrt{c+dx}}}{d^2} - \frac{x\left(\frac{a}{c^2} + \frac{b}{d^2}\right)}{\sqrt{dx-c}\sqrt{c+dx}}$$

$$\downarrow 221$$

$$\frac{2b \operatorname{arctanh}\left(\frac{\sqrt{dx-c}}{\sqrt{c+dx}}\right)}{d^3} - \frac{x\left(\frac{a}{c^2} + \frac{b}{d^2}\right)}{\sqrt{dx-c}\sqrt{c+dx}}$$

input `Int[(a + b*x^2)/((-c + d*x)^(3/2)*(c + d*x)^(3/2)),x]`

output `-(((a/c^2 + b/d^2)*x)/(Sqrt[-c + d*x]*Sqrt[c + d*x])) + (2*b*ArcTanh[Sqrt[-c + d*x]/Sqrt[c + d*x]])/d^3`

Definitions of rubi rules used

rule 45 `Int[1/(Sqrt[(a_) + (b_.)*(x_)]*Sqrt[(c_) + (d_.)*(x_)]), x_Symbol] := Simp[
2 Subst[Int[1/(b - d*x^2), x], x, Sqrt[a + b*x]/Sqrt[c + d*x]], x] /; FreeQ[{a, b, c, d}, x] && EqQ[b*c + a*d, 0] && !GtQ[c, 0]`

rule 221 `Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(Rt[-a/b, 2]/a)*ArcTanh[x/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]`

rule 645 `Int[((c_) + (d_.)*(x_)^(m_.))*((e_) + (f_.)*(x_)^(n_.))*((a_.) + (b_.)*(x_)^2), x_Symbol] := Simp[(b*c*e - a*d*f)*x*(c + d*x)^(m + 1)*((e + f*x)^(n + 1)/(2*c*d*e*f*(m + 1))), x] - Simp[(b*c*e - a*d*f*(2*m + 3))/(2*c*d*e*f*(m + 1)) Int[(c + d*x)^(m + 1)*(e + f*x)^(n + 1), x], x] /; FreeQ[{a, b, c, d, e, f, m, n}, x] && EqQ[m, n] && EqQ[d*e + c*f, 0] && LtQ[m, -1]`

Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.13 (sec) , antiderivative size = 160, normalized size of antiderivative = 2.54

method	result
default	$\frac{\left(\ln\left(\operatorname{csgn}(d)\sqrt{-(dx+c)(-dx+c)+dx}\right)\operatorname{csgn}(d)\right)bc^2d^2x^2 - \operatorname{csgn}(d)d^3\sqrt{d^2x^2-c^2}ax - \operatorname{csgn}(d)d\sqrt{d^2x^2-c^2}bc^2x - \ln\left(\operatorname{csgn}(d)\sqrt{-(dx+c)(-dx+c)+dx}\right)\operatorname{csgn}(d)}{\sqrt{dx-c}\sqrt{d^2x^2-c^2}\sqrt{dx+c}c^2d^3}$

input `int((b*x^2+a)/(d*x-c)^(3/2)/(d*x+c)^(3/2), x, method=_RETURNVERBOSE)`

output `(ln((csgn(d)*(-(d*x+c)*(-d*x+c))^(1/2)+d*x)*csgn(d))*b*c^2*d^2*x^2-csgn(d)*d^3*(d^2*x^2-c^2)^(1/2)*a*x-csgn(d)*d*(d^2*x^2-c^2)^(1/2)*b*c^2*x-ln((csgn(d)*(-(d*x+c)*(-d*x+c))^(1/2)+d*x)*csgn(d))*b*c^4)/(d*x-c)^(1/2)/(d^2*x^2-c^2)^(1/2)/(d*x+c)^(1/2)*csgn(d)/c^2/d^3`

Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 129 vs. $2(55) = 110$.

Time = 0.09 (sec) , antiderivative size = 129, normalized size of antiderivative = 2.05

$$\int \frac{a + bx^2}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{bc^4 + ac^2d^2 - (bc^2d + ad^3)\sqrt{dx + c}\sqrt{dx - c}x - (bc^2d^2 + ad^4)x^2 - (bc^2d^2 - c^2d^5x^2 - c^4d^3)}{c^2d^5x^2 - c^4d^3}$$

input `integrate((b*x^2+a)/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="fricas")`

output $(b*c^4 + a*c^2*d^2 - (b*c^2*d + a*d^3)*\sqrt{d*x + c}*\sqrt{d*x - c}*x - (b*c^2*d^2 + a*d^4)*x^2 - (b*c^2*d^2*x^2 - b*c^4)*\log(-d*x + \sqrt{d*x + c})*\sqrt{d*x - c})/(c^2*d^5*x^2 - c^4*d^3)$

Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 177.22 (sec) , antiderivative size = 182, normalized size of antiderivative = 2.89

$$\int \frac{a + bx^2}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = a \left(-\frac{G_{6,6}^{5,3} \left(\begin{matrix} \frac{3}{4}, \frac{5}{4}, 1 & \frac{1}{2}, \frac{3}{2}, 2 \\ \frac{c^2}{d^2x^2} \end{matrix} \right)}{2\pi^{\frac{3}{2}}c^2d} + \frac{iG_{6,6}^{2,6} \left(\begin{matrix} -\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1 \\ \frac{1}{4}, \frac{3}{4} \end{matrix} \right)}{2\pi^{\frac{3}{2}}c^2d} \right) + b \left(\frac{G_{6,6}^{6,2} \left(\begin{matrix} -\frac{1}{4}, \frac{1}{4} & -\frac{1}{2}, \frac{1}{2}, 1, 1 \\ -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, 1, 0 \\ \frac{c^2}{d^2x^2} \end{matrix} \right)}{2\pi^{\frac{3}{2}}d^3} + \frac{iG_{6,6}^{2,6} \left(\begin{matrix} -\frac{3}{2}, -1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 1 \\ -\frac{3}{4}, -\frac{1}{4} \end{matrix} \right)}{2\pi^{\frac{3}{2}}d^3} \right)$$

input `integrate((b*x**2+a)/(d*x-c)**(3/2)/(d*x+c)**(3/2),x)`

output

```
a*(-meijerg(((3/4, 5/4, 1), (1/2, 3/2, 2)), ((3/4, 1, 5/4, 3/2, 2), (0,)),
c**2/(d**2*x**2))/(2*pi**(3/2)*c**2*d) + I*meijerg((-1/2, 0, 1/4, 1/2, 3
/4, 1), ()), ((1/4, 3/4), (-1/2, 0, 1, 0)), c**2*exp_polar(2*I*pi)/(d**2*x
**2))/(2*pi**(3/2)*c**2*d)) + b*(meijerg((-1/4, 1/4), (-1/2, 1/2, 1, 1)),
((-1/4, 0, 1/4, 1/2, 1, 0), ()), c**2/(d**2*x**2))/(2*pi**(3/2)*d**3) + I
*meijerg((-3/2, -1, -3/4, -1/2, -1/4, 1), ()), ((-3/4, -1/4), (-3/2, -1,
0, 0)), c**2*exp_polar(2*I*pi)/(d**2*x**2))/(2*pi**(3/2)*d**3))
```

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 76, normalized size of antiderivative = 1.21

$$\int \frac{a + bx^2}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = -\frac{ax}{\sqrt{d^2x^2 - c^2}d^2} - \frac{bx}{\sqrt{d^2x^2 - c^2}d^2} + \frac{b \log(2d^2x + 2\sqrt{d^2x^2 - c^2}d)}{d^3}$$

input

```
integrate((b*x^2+a)/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="maxima")
```

output

```
-a*x/(sqrt(d^2*x^2 - c^2)*c^2) - b*x/(sqrt(d^2*x^2 - c^2)*d^2) + b*log(2*d
^2*x + 2*sqrt(d^2*x^2 - c^2)*d)/d^3
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 113 vs. 2(55) = 110.

Time = 0.14 (sec) , antiderivative size = 113, normalized size of antiderivative = 1.79

$$\int \frac{a + bx^2}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = -\frac{b \log\left(\left(\sqrt{dx + c} - \sqrt{dx - c}\right)^2\right)}{d^3} - \frac{2(bc^2 + ad^2)}{\left(\left(\sqrt{dx + c} - \sqrt{dx - c}\right)^2 + 2c\right)cd^3} - \frac{(bc^2d^3 + ad^5)\sqrt{dx + c}}{2\sqrt{dx - c}c^2d^6}$$

input

```
integrate((b*x^2+a)/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="giac")
```

output

```
-b*log((sqrt(d*x + c) - sqrt(d*x - c))^2/d^3 - 2*(b*c^2 + a*d^2)/(((sqrt(d*x + c) - sqrt(d*x - c))^2 + 2*c)*c*d^3) - 1/2*(b*c^2*d^3 + a*d^5)*sqrt(d*x + c)/(sqrt(d*x - c)*c^2*d^6)
```

Mupad [F(-1)]

Timed out.

$$\int \frac{a + bx^2}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \int \frac{bx^2 + a}{(c + dx)^{3/2}(dx - c)^{3/2}} dx$$

input

```
int((a + b*x^2)/((c + d*x)^(3/2)*(d*x - c)^(3/2)),x)
```

output

```
int((a + b*x^2)/((c + d*x)^(3/2)*(d*x - c)^(3/2)), x)
```

Reduce [B] (verification not implemented)

Time = 0.25 (sec) , antiderivative size = 192, normalized size of antiderivative = 3.05

$$\int \frac{a + bx^2}{(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{2\sqrt{dx - c} \log\left(\frac{\sqrt{dx-c} + \sqrt{dx+c}}{\sqrt{c}\sqrt{2}}\right) b c^3 + 2\sqrt{dx - c} \log\left(\frac{\sqrt{dx-c} + \sqrt{dx+c}}{\sqrt{c}\sqrt{2}}\right) b c^2 dx -$$

input

```
int((b*x^2+a)/(d*x-c)^(3/2)/(d*x+c)^(3/2),x)
```

output

```
(2*sqrt(-c + d*x)*log((sqrt(-c + d*x) + sqrt(c + d*x))/(sqrt(c)*sqrt(2))) * b*c**3 + 2*sqrt(-c + d*x)*log((sqrt(-c + d*x) + sqrt(c + d*x))/(sqrt(c)*sqrt(2))) * b*c**2*d*x - sqrt(-c + d*x)*a*c*d**2 - sqrt(-c + d*x)*a*d**3*x - sqrt(-c + d*x)*b*c**3 - sqrt(-c + d*x)*b*c**2*d*x - sqrt(c + d*x)*a*d**3*x - sqrt(c + d*x)*b*c**2*d*x)/(sqrt(-c + d*x)*c**2*d**3*(c + d*x))
```

3.37 $\int \frac{a+bx^2}{x(-c+dx)^{3/2}(c+dx)^{3/2}} dx$

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Optimal result

Integrand size = 31, antiderivative size = 65

$$\int \frac{a + bx^2}{x(-c + dx)^{3/2}(c + dx)^{3/2}} dx = -\frac{\frac{a}{c^2} + \frac{b}{d^2}}{\sqrt{-c + dx}\sqrt{c + dx}} - \frac{a \arctan\left(\frac{\sqrt{-c+dx}\sqrt{c+dx}}{c}\right)}{c^3}$$

output

```
-(a/c^2+b/d^2)/(d*x-c)^(1/2)/(d*x+c)^(1/2)-a*arctan((d*x-c)^(1/2)*(d*x+c)^(1/2)/c)/c^3
```

Mathematica [A] (verified)

Time = 0.17 (sec) , antiderivative size = 70, normalized size of antiderivative = 1.08

$$\int \frac{a + bx^2}{x(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{-\frac{2(bc^3+acd^2)}{d^2\sqrt{-c+dx}\sqrt{c+dx}} + 4a \arctan\left(\frac{\sqrt{c+dx}}{\sqrt{-c+dx}}\right)}{2c^3}$$

input

```
Integrate[(a + b*x^2)/(x*(-c + d*x)^(3/2)*(c + d*x)^(3/2)),x]
```

output

```
((-2*(b*c^3 + a*c*d^2))/(d^2*Sqrt[-c + d*x]*Sqrt[c + d*x]) + 4*a*ArcTan[Sqrt[c + d*x]/Sqrt[-c + d*x]])/(2*c^3)
```

Rubi [A] (verified)

Time = 0.38 (sec) , antiderivative size = 65, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.097$, Rules used = {958, 103, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{a + bx^2}{x(dx - c)^{3/2}(c + dx)^{3/2}} dx \\
 & \quad \downarrow \text{958} \\
 & -\frac{a \int \frac{1}{x\sqrt{dx-c}\sqrt{c+dx}} dx}{c^2} - \frac{\frac{a}{c^2} + \frac{b}{d^2}}{\sqrt{dx - c}\sqrt{c + dx}} \\
 & \quad \downarrow \text{103} \\
 & -\frac{ad \int \frac{1}{dc^2+d(dx-c)(c+dx)} d(\sqrt{dx - c}\sqrt{c + dx})}{c^2} - \frac{\frac{a}{c^2} + \frac{b}{d^2}}{\sqrt{dx - c}\sqrt{c + dx}} \\
 & \quad \downarrow \text{218} \\
 & -\frac{a \arctan\left(\frac{\sqrt{dx-c}\sqrt{c+dx}}{c}\right)}{c^3} - \frac{\frac{a}{c^2} + \frac{b}{d^2}}{\sqrt{dx - c}\sqrt{c + dx}}
 \end{aligned}$$

input `Int[(a + b*x^2)/(x*(-c + d*x)^(3/2)*(c + d*x)^(3/2)),x]`

output `-((a/c^2 + b/d^2)/(Sqrt[-c + d*x]*Sqrt[c + d*x])) - (a*ArcTan[(Sqrt[-c + d*x]*Sqrt[c + d*x])/c])/c^3`

Defintions of rubi rules used

- rule 103 `Int[1/(Sqrt[(a_.) + (b_.)*(x_)]*Sqrt[(c_.) + (d_.)*(x_)]*((e_.) + (f_.)*(x_))), x_] := Simp[b*f Subst[Int[1/(d*(b*e - a*f)^2 + b*f^2*x^2), x], x, Sqrt[a + b*x]*Sqrt[c + d*x]], x] /; FreeQ[{a, b, c, d, e, f}, x] && EqQ[2*b*d *e - f*(b*c + a*d), 0]`

- rule 218 `Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(Rt[a/b, 2]/a)*ArcTan[x/Rt[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]`

- rule 958 `Int[((e_.)*(x_))^(m_.)*((a1_) + (b1_.)*(x_)^(non2_.))^(p_.)*((a2_) + (b2_.)*(x_)^(non2_.))^(p_.)*((c_) + (d_.)*(x_)^(n_)), x_Symbol] := Simp[(-b1*b2*c - a1*a2*d)*(e*x)^(m + 1)*(a1 + b1*x^(n/2))^(p + 1)*((a2 + b2*x^(n/2))^(p + 1)/(a1*a2*b1*b2*e*n*(p + 1))), x] - Simp[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(a1*a2*b1*b2*n*(p + 1)) Int[(e*x)^m*(a1 + b1*x^(n/2))^(p + 1)*(a2 + b2*x^(n/2))^(p + 1), x], x] /; FreeQ[{a1, b1, a2, b2, c, d, e, m, n}, x] && EqQ[non2, n/2] && EqQ[a2*b1 + a1*b2, 0] && LtQ[p, -1] && (!IntegerQ[p + 1/2] && NeQ[p, -5/4]) || !RationalQ[m] || (IGtQ[n, 0] && ILtQ[p + 1/2, 0] && LeQ[-1, m, (-n)*(p + 1)])`

Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 187 vs. 2(57) = 114.

Time = 0.13 (sec) , antiderivative size = 188, normalized size of antiderivative = 2.89

method	result	size
default	$\frac{\ln\left(-\frac{2(c^2 - \sqrt{-c^2} \sqrt{d^2 x^2 - c^2})}{x}\right) a d^4 x^2 - \ln\left(-\frac{2(c^2 - \sqrt{-c^2} \sqrt{d^2 x^2 - c^2})}{x}\right) a c^2 d^2 - \sqrt{-c^2} \sqrt{d^2 x^2 - c^2} a d^2 - b c^2 \sqrt{-c^2} \sqrt{d^2 x^2 - c^2}}{\sqrt{d x - c} \sqrt{d^2 x^2 - c^2} \sqrt{d x + c} c^2 \sqrt{-c^2} d^2}$	18

input `int((b*x^2+a)/x/(d*x-c)^(3/2)/(d*x+c)^(3/2), x, method=_RETURNVERBOSE)`

output

$$\frac{(\ln(-2*(c^2-(-c^2)^{(1/2)}*(d^2*x^2-c^2)^{(1/2))}/x)*a*d^4*x^2-\ln(-2*(c^2-(-c^2)^{(1/2)}*(d^2*x^2-c^2)^{(1/2))}/x)*a*c^2*d^2-(-c^2)^{(1/2)}*(d^2*x^2-c^2)^{(1/2)})*a*d^2-b*c^2*(-c^2)^{(1/2)}*(d^2*x^2-c^2)^{(1/2)))/(d*x-c)^{(1/2)}/(d^2*x^2-c^2)^{(1/2)}/(d*x+c)^{(1/2)}/c^2/(-c^2)^{(1/2)}/d^2$$

Fricas [A] (verification not implemented)

Time = 0.10 (sec) , antiderivative size = 101, normalized size of antiderivative = 1.55

$$\int \frac{a + bx^2}{x(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{(bc^3 + acd^2)\sqrt{dx + c}\sqrt{dx - c} + 2(ad^4x^2 - ac^2d^2) \arctan\left(-\frac{dx - \sqrt{dx + c}\sqrt{dx - c}}{c}\right)}{c^3d^4x^2 - c^5d^2}$$

input

```
integrate((b*x^2+a)/x/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="fricas")
```

output

$$-((b*c^3 + a*c*d^2)*\sqrt{d*x + c}*\sqrt{d*x - c} + 2*(a*d^4*x^2 - a*c^2*d^2)*\arctan(-(d*x - \sqrt{d*x + c})*\sqrt{d*x - c))/c))/(c^3*d^4*x^2 - c^5*d^2)$$

Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 177.65 (sec) , antiderivative size = 172, normalized size of antiderivative = 2.65

$$\int \frac{a + bx^2}{x(-c + dx)^{3/2}(c + dx)^{3/2}} dx = a \left(\frac{G_{6,6}^{5,3} \left(\begin{matrix} \frac{5}{4}, \frac{7}{4}, 1 & 1, 2, \frac{5}{2} \\ \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2, \frac{5}{2} & 0 \end{matrix} \middle| \frac{c^2}{d^2x^2} \right)}{2\pi^{\frac{3}{2}}c^3} - \frac{iG_{6,6}^{2,6} \left(\begin{matrix} 0, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, 1 \\ \frac{3}{4}, \frac{5}{4} & 0, \frac{1}{2}, \frac{3}{2}, 0 \end{matrix} \middle| \frac{c}{d} \right)}{2\pi^{\frac{3}{2}}c^3} \right) + b \left(\frac{G_{6,6}^{5,3} \left(\begin{matrix} \frac{1}{4}, \frac{3}{4}, 1 & 0, 1, \frac{3}{2} \\ \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{3}{2} & 0 \end{matrix} \middle| \frac{c^2}{d^2x^2} \right)}{2\pi^{\frac{3}{2}}cd^2} - \frac{iG_{6,6}^{2,6} \left(\begin{matrix} -1, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, 1 \\ -\frac{1}{4}, \frac{1}{4} & -1, -\frac{1}{2}, \frac{1}{2}, 0 \end{matrix} \middle| \frac{c^2e^{2i\pi}}{d^2x^2} \right)}{2\pi^{\frac{3}{2}}cd^2} \right)$$

input `integrate((b*x**2+a)/x/(d*x-c)**(3/2)/(d*x+c)**(3/2),x)`

output `a*(-meijerg(((5/4, 7/4, 1), (1, 2, 5/2)), ((5/4, 3/2, 7/4, 2, 5/2), (0,)),
c**2/(d**2*x**2))/(2*pi**(3/2)*c**3) - I*meijerg(((0, 1/2, 3/4, 1, 5/4, 1),
()), ((3/4, 5/4), (0, 1/2, 3/2, 0)), c**2*exp_polar(2*I*pi)/(d**2*x**2))/
(2*pi**(3/2)*c**3)) + b*(-meijerg(((1/4, 3/4, 1), (0, 1, 3/2)), ((1/4, 1/2,
3/4, 1, 3/2), (0,)), c**2/(d**2*x**2))/(2*pi**(3/2)*c*d**2) - I*meijer
g((-1, -1/2, -1/4, 0, 1/4, 1), ()), ((-1/4, 1/4), (-1, -1/2, 1/2, 0)), c*
2*exp_polar(2*I*pi)/(d**2*x**2))/(2*pi**(3/2)*c*d**2))`

Maxima [A] (verification not implemented)

Time = 0.11 (sec) , antiderivative size = 58, normalized size of antiderivative = 0.89

$$\int \frac{a + bx^2}{x(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{a \arcsin\left(\frac{c}{d|x|}\right)}{c^3} - \frac{a}{\sqrt{d^2x^2 - c^2}c^2} - \frac{b}{\sqrt{d^2x^2 - c^2}d^2}$$

input `integrate((b*x^2+a)/x/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="maxima")`

output `a*arcsin(c/(d*abs(x)))/c^3 - a/(sqrt(d^2*x^2 - c^2)*c^2) - b/(sqrt(d^2*x^2 -
c^2)*d^2)`

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 115 vs. 2(57) = 114.

Time = 0.17 (sec) , antiderivative size = 115, normalized size of antiderivative = 1.77

$$\int \frac{a + bx^2}{x(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{2a \arctan\left(\frac{(\sqrt{dx+c}-\sqrt{dx-c})^2}{2c}\right)}{c^3} - \frac{(bc^2 + ad^2)\sqrt{dx+c}}{2\sqrt{dx-c}c^3d^2} + \frac{2(bc^2 + ad^2)}{\left((\sqrt{dx+c}-\sqrt{dx-c})^2 + 2c\right)c^2d^2}$$

input `integrate((b*x^2+a)/x/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="giac")`

output

```
2*a*arctan(1/2*(sqrt(d*x + c) - sqrt(d*x - c))^2/c)/c^3 - 1/2*(b*c^2 + a*d^2)*sqrt(d*x + c)/(sqrt(d*x - c)*c^3*d^2) + 2*(b*c^2 + a*d^2)/(((sqrt(d*x + c) - sqrt(d*x - c))^2 + 2*c)*c^2*d^2)
```

Mupad [F(-1)]

Timed out.

$$\int \frac{a + bx^2}{x(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \int \frac{bx^2 + a}{x(c + dx)^{3/2}(dx - c)^{3/2}} dx$$

input

```
int((a + b*x^2)/(x*(c + d*x)^(3/2)*(d*x - c)^(3/2)),x)
```

output

```
int((a + b*x^2)/(x*(c + d*x)^(3/2)*(d*x - c)^(3/2)), x)
```

Reduce [B] (verification not implemented)

Time = 0.23 (sec) , antiderivative size = 206, normalized size of antiderivative = 3.17

$$\int \frac{a + bx^2}{x(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{-2\sqrt{dx - c} \operatorname{atan}\left(\frac{\sqrt{dx - c} + \sqrt{dx + c} - \sqrt{c}}{\sqrt{c}}\right) ac d^2 - 2\sqrt{dx - c} \operatorname{atan}\left(\frac{\sqrt{dx - c} + \sqrt{dx + c}}{\sqrt{c}}\right)}{}$$

input

```
int((b*x^2+a)/x/(d*x-c)^(3/2)/(d*x+c)^(3/2),x)
```

output

```
( - 2*sqrt(-c + d*x)*atan((sqrt(-c + d*x) + sqrt(c + d*x) - sqrt(c))/sqrt(c))*a*c*d**2 - 2*sqrt(-c + d*x)*atan((sqrt(-c + d*x) + sqrt(c + d*x) - sqrt(c))/sqrt(c))*a*d**3*x + 2*sqrt(-c + d*x)*atan((sqrt(-c + d*x) + sqrt(c + d*x) + sqrt(c))/sqrt(c))*a*c*d**2 + 2*sqrt(-c + d*x)*atan((sqrt(-c + d*x) + sqrt(c + d*x) + sqrt(c))/sqrt(c))*a*d**3*x - sqrt(c + d*x)*a*c*d**2 - sqrt(c + d*x)*b*c**3)/(sqrt(-c + d*x)*c**3*d**2*(c + d*x))
```

3.38 $\int \frac{a+bx^2}{x^2(-c+dx)^{3/2}(c+dx)^{3/2}} dx$

Optimal result	298
Mathematica [A] (verified)	298
Rubi [A] (verified)	299
Maple [A] (verified)	300
Fricas [A] (verification not implemented)	301
Sympy [C] (verification not implemented)	301
Maxima [A] (verification not implemented)	302
Giac [B] (verification not implemented)	302
Mupad [B] (verification not implemented)	303
Reduce [B] (verification not implemented)	303

Optimal result

Integrand size = 31, antiderivative size = 67

$$\int \frac{a + bx^2}{x^2(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{a}{c^2x\sqrt{-c + dx}\sqrt{c + dx}} - \frac{(bc^2 + 2ad^2)x}{c^4\sqrt{-c + dx}\sqrt{c + dx}}$$

output

$a/c^2/x/(d*x-c)^{(1/2)}/(d*x+c)^{(1/2)}-(2*a*d^2+b*c^2)*x/c^4/(d*x-c)^{(1/2)}/(d*x+c)^{(1/2)}$

Mathematica [A] (verified)

Time = 0.12 (sec) , antiderivative size = 51, normalized size of antiderivative = 0.76

$$\int \frac{a + bx^2}{x^2(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{-bc^2x^2 + a(c^2 - 2d^2x^2)}{c^4x\sqrt{-c + dx}\sqrt{c + dx}}$$

input

`Integrate[(a + b*x^2)/(x^2*(-c + d*x)^(3/2)*(c + d*x)^(3/2)),x]`

output

$(-(b*c^2*x^2) + a*(c^2 - 2*d^2*x^2))/(c^4*x*sqrt[-c + d*x]*sqrt[c + d*x])$

Rubi [A] (verified)

Time = 0.36 (sec) , antiderivative size = 66, normalized size of antiderivative = 0.99, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}} = 0.065$, Rules used = {956, 41}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{a + bx^2}{x^2(dx - c)^{3/2}(c + dx)^{3/2}} dx$$

↓ 956

$$\left(\frac{2ad^2}{c^2} + b\right) \int \frac{1}{(dx - c)^{3/2}(c + dx)^{3/2}} dx + \frac{a}{c^2x\sqrt{dx - c}\sqrt{c + dx}}$$

↓ 41

$$\frac{a}{c^2x\sqrt{dx - c}\sqrt{c + dx}} - \frac{x\left(\frac{2ad^2}{c^2} + b\right)}{c^2\sqrt{dx - c}\sqrt{c + dx}}$$

input

```
Int[(a + b*x^2)/(x^2*(-c + d*x)^(3/2)*(c + d*x)^(3/2)),x]
```

output

```
a/(c^2*x*Sqrt[-c + d*x]*Sqrt[c + d*x]) - ((b + (2*a*d^2)/c^2)*x)/(c^2*Sqrt[-c + d*x]*Sqrt[c + d*x])
```

Definitions of rubi rules used

rule 41 `Int[1/(((a_) + (b_)*(x_)^(3/2))*((c_) + (d_)*(x_)^(3/2))), x_Symbol] := Simp[x/(a*c*Sqrt[a + b*x]*Sqrt[c + d*x]), x] /; FreeQ[{a, b, c, d}, x] && EqQ[b*c + a*d, 0]`

rule 956 `Int[((e_)*(x_)^(m_))*((a1_) + (b1_)*(x_)^(non2_))^(p_)*((a2_) + (b2_)*(x_)^(non2_))^(p_)*((c_) + (d_)*(x_)^(n_)), x_Symbol] := Simp[c*(e*x)^(m + 1)*(a1 + b1*x^(n/2))^(p + 1)*(a2 + b2*x^(n/2))^(p + 1)/(a1*a2*e^(m + 1)), x] + Simp[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(a1*a2*e^n*(m + 1)) Int[(e*x)^(m + n)*(a1 + b1*x^(n/2))^p*(a2 + b2*x^(n/2))^p, x], x] /; FreeQ[{a1, b1, a2, b2, c, d, e, p}, x] && EqQ[non2, n/2] && EqQ[a2*b1 + a1*b2, 0] && (IntegerQ[n] || GtQ[e, 0]) && ((GtQ[n, 0] && LtQ[m, -1]) || (LtQ[n, 0] && GtQ[m + n, -1])) && !ILtQ[p, -1]`

Maple [A] (verified)

Time = 0.13 (sec) , antiderivative size = 48, normalized size of antiderivative = 0.72

method	result	size
gosper	$\frac{-2a d^2 x^2 - b c^2 x^2 + a c^2}{c^4 x \sqrt{dx-c} \sqrt{dx+c}}$	48
default	$\frac{(-2a d^2 x^2 - b c^2 x^2 + a c^2) \operatorname{csign}(d)^2}{\sqrt{dx-c} \sqrt{dx+c} c^4 x}$	52
orering	$-\frac{(-dx+c)(-2a d^2 x^2 - b c^2 x^2 + a c^2)}{\sqrt{dx+c} x c^4 (dx-c)^{\frac{3}{2}}}$	55
risch	$\frac{a(-dx+c)\sqrt{dx+c}}{c^4 x \sqrt{dx-c}} - \frac{(a d^2 + b c^2) x \sqrt{(dx-c)(dx+c)}}{\sqrt{-(dx+c)(-dx+c)} c^4 \sqrt{dx-c} \sqrt{dx+c}}$	95

input `int((b*x^2+a)/x^2/(d*x-c)^(3/2)/(d*x+c)^(3/2),x,method=_RETURNVERBOSE)`

output `1/c^4/x/(d*x-c)^(1/2)/(d*x+c)^(1/2)*(-2*a*d^2*x^2-b*c^2*x^2+a*c^2)`

Fricas [A] (verification not implemented)

Time = 0.10 (sec) , antiderivative size = 103, normalized size of antiderivative = 1.54

$$\int \frac{a + bx^2}{x^2(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{(bc^2d^2 + 2ad^4)x^3 - (ac^2d - (bc^2d + 2ad^3)x^2)\sqrt{dx + c}\sqrt{dx - c} - (bc^4 + 2ac^2d^2)x}{c^4d^3x^3 - c^6dx}$$

input `integrate((b*x^2+a)/x^2/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="fricas")`

output `-((b*c^2*d^2 + 2*a*d^4)*x^3 - (a*c^2*d - (b*c^2*d + 2*a*d^3)*x^2)*sqrt(d*x + c)*sqrt(d*x - c) - (b*c^4 + 2*a*c^2*d^2)*x)/(c^4*d^3*x^3 - c^6*d*x)`

Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 57.48 (sec) , antiderivative size = 165, normalized size of antiderivative = 2.46

$$\int \frac{a + bx^2}{x^2(-c + dx)^{3/2}(c + dx)^{3/2}} dx = a \left(-\frac{dG_{6,6}^{5,3} \left(\begin{matrix} \frac{7}{4}, \frac{9}{4}, 1 & \frac{3}{2}, \frac{5}{2}, 3 \\ \frac{7}{4}, 2, \frac{9}{4}, \frac{5}{2}, 3 & 0 \end{matrix} \middle| \frac{c^2}{d^2x^2} \right)}{2\pi^{\frac{3}{2}}c^4} + \frac{idG_{6,6}^{2,6} \left(\begin{matrix} \frac{1}{2}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 1 \\ \frac{5}{4}, \frac{7}{4} & \frac{1}{2}, 1, 2 \end{matrix} \right)}{2\pi^{\frac{3}{2}}c^4} \right) + b \left(-\frac{G_{6,6}^{5,3} \left(\begin{matrix} \frac{3}{4}, \frac{5}{4}, 1 & \frac{1}{2}, \frac{3}{2}, 2 \\ \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, 2 & 0 \end{matrix} \middle| \frac{c^2}{d^2x^2} \right)}{2\pi^{\frac{3}{2}}c^2d} + \frac{iG_{6,6}^{2,6} \left(\begin{matrix} -\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1 \\ \frac{1}{4}, \frac{3}{4} & -\frac{1}{2}, 0, 1, 0 \end{matrix} \middle| \frac{c^2e^{2i\pi}}{d^2x^2} \right)}{2\pi^{\frac{3}{2}}c^2d} \right)$$

input `integrate((b*x**2+a)/x**2/(d*x-c)**(3/2)/(d*x+c)**(3/2),x)`

output

```
a*(-d*meijerg(((7/4, 9/4, 1), (3/2, 5/2, 3)), ((7/4, 2, 9/4, 5/2, 3), (0,
), c**2/(d**2*x**2))/(2*pi**(3/2)*c**4) + I*d*meijerg(((1/2, 1, 5/4, 3/2,
7/4, 1), ()), ((5/4, 7/4), (1/2, 1, 2, 0)), c**2*exp_polar(2*I*pi)/(d**2*x
**2))/(2*pi**(3/2)*c**4)) + b*(-meijerg(((3/4, 5/4, 1), (1/2, 3/2, 2)), ((
3/4, 1, 5/4, 3/2, 2), (0,)), c**2/(d**2*x**2))/(2*pi**(3/2)*c**2*d) + I*me
ijerg(((1/2, 0, 1/4, 1/2, 3/4, 1), ()), ((1/4, 3/4), (-1/2, 0, 1, 0)), c*
*2*exp_polar(2*I*pi)/(d**2*x**2))/(2*pi**(3/2)*c**2*d))
```

Maxima [A] (verification not implemented)

Time = 0.10 (sec) , antiderivative size = 71, normalized size of antiderivative = 1.06

$$\int \frac{a + bx^2}{x^2(-c + dx)^{3/2}(c + dx)^{3/2}} dx = -\frac{bx}{\sqrt{d^2x^2 - c^2c^2}} - \frac{2ad^2x}{\sqrt{d^2x^2 - c^2c^4}} + \frac{a}{\sqrt{d^2x^2 - c^2c^2x}}$$

input

```
integrate((b*x^2+a)/x^2/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="maxima")
```

output

```
-b*x/(sqrt(d^2*x^2 - c^2)*c^2) - 2*a*d^2*x/(sqrt(d^2*x^2 - c^2)*c^4) + a/(
sqrt(d^2*x^2 - c^2)*c^2*x)
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 219 vs. 2(59) = 118.

Time = 0.18 (sec) , antiderivative size = 219, normalized size of antiderivative = 3.27

$$\int \frac{a + bx^2}{x^2(-c + dx)^{3/2}(c + dx)^{3/2}} dx = -\frac{(bc^2 + ad^2)\sqrt{dx + c}}{2\sqrt{dx - c}c^4d} - \frac{2\left(bc^2(\sqrt{dx + c} - \sqrt{dx - c})^4 + ad^2(\sqrt{dx + c} - \sqrt{dx - c})^4 + 4acd^2(\sqrt{dx + c} - \sqrt{dx - c})^2 + 4bc^4 + 12\right)}{\left((\sqrt{dx + c} - \sqrt{dx - c})^6 + 2c(\sqrt{dx + c} - \sqrt{dx - c})^4 + 4c^2(\sqrt{dx + c} - \sqrt{dx - c})^2 + 8c^3\right)c^3d}$$

input

```
integrate((b*x^2+a)/x^2/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="giac")
```

output

```
-1/2*(b*c^2 + a*d^2)*sqrt(d*x + c)/(sqrt(d*x - c)*c^4*d) - 2*(b*c^2*(sqrt(d*x + c) - sqrt(d*x - c))^4 + a*d^2*(sqrt(d*x + c) - sqrt(d*x - c))^4 + 4*a*c*d^2*(sqrt(d*x + c) - sqrt(d*x - c))^2 + 4*b*c^4 + 12*a*c^2*d^2)/(((sqrt(d*x + c) - sqrt(d*x - c))^6 + 2*c*(sqrt(d*x + c) - sqrt(d*x - c))^4 + 4*c^2*(sqrt(d*x + c) - sqrt(d*x - c))^2 + 8*c^3)*c^3*d)
```

Mupad [B] (verification not implemented)

Time = 4.96 (sec) , antiderivative size = 73, normalized size of antiderivative = 1.09

$$\int \frac{a + bx^2}{x^2(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{2 a d^2 x^2 \sqrt{dx - c} - a c^2 \sqrt{dx - c} + b c^2 x^2 \sqrt{dx - c}}{c^4 x \sqrt{c + dx} (c - dx)}$$

input

```
int((a + b*x^2)/(x^2*(c + d*x)^(3/2)*(d*x - c)^(3/2)),x)
```

output

```
(2*a*d^2*x^2*(d*x - c)^(1/2) - a*c^2*(d*x - c)^(1/2) + b*c^2*x^2*(d*x - c)^(1/2))/(c^4*x*(c + d*x)^(1/2)*(c - d*x))
```

Reduce [B] (verification not implemented)

Time = 0.22 (sec) , antiderivative size = 137, normalized size of antiderivative = 2.04

$$\int \frac{a + bx^2}{x^2(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{-2\sqrt{dx - c}ac d^2x - 2\sqrt{dx - c}a d^3x^2 - \sqrt{dx - c}bc^3x - \sqrt{dx - c}bc^2d}{\sqrt{dx - c}c^4dx (dx + c)}$$

input

```
int((b*x^2+a)/x^2/(d*x-c)^(3/2)/(d*x+c)^(3/2),x)
```

output

```
( - 2*sqrt( - c + d*x)*a*c*d**2*x - 2*sqrt( - c + d*x)*a*d**3*x**2 - sqrt( - c + d*x)*b*c**3*x - sqrt( - c + d*x)*b*c**2*d*x**2 + sqrt(c + d*x)*a*c**2*d - 2*sqrt(c + d*x)*a*d**3*x**2 - sqrt(c + d*x)*b*c**2*d*x**2)/(sqrt( - c + d*x)*c**4*d*x*(c + d*x))
```


3.39 $\int \frac{a+bx^2}{x^3(-c+dx)^{3/2}(c+dx)^{3/2}} dx$

Optimal result	304
Mathematica [A] (verified)	304
Rubi [A] (verified)	305
Maple [B] (verified)	307
Fricas [A] (verification not implemented)	308
Sympy [F(-1)]	308
Maxima [A] (verification not implemented)	308
Giac [B] (verification not implemented)	309
Mupad [F(-1)]	309
Reduce [B] (verification not implemented)	310

Optimal result

Integrand size = 31, antiderivative size = 117

$$\int \frac{a + bx^2}{x^3(-c + dx)^{3/2}(c + dx)^{3/2}} dx = -\frac{2bc^2 + 3ad^2}{2c^4\sqrt{-c + dx}\sqrt{c + dx}} + \frac{a}{2c^2x^2\sqrt{-c + dx}\sqrt{c + dx}} - \frac{(2bc^2 + 3ad^2) \arctan\left(\frac{\sqrt{-c+dx}\sqrt{c+dx}}{c}\right)}{2c^5}$$

output

```
-1/2*(3*a*d^2+2*b*c^2)/c^4/(d*x-c)^(1/2)/(d*x+c)^(1/2)+1/2*a/c^2/x^2/(d*x-c)^(1/2)/(d*x+c)^(1/2)-1/2*(3*a*d^2+2*b*c^2)*arctan((d*x-c)^(1/2)*(d*x+c)^(1/2)/c)/c^5
```

Mathematica [A] (verified)

Time = 0.23 (sec) , antiderivative size = 93, normalized size of antiderivative = 0.79

$$\int \frac{a + bx^2}{x^3(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{-2bc^3x^2+a(c^3-3cd^2x^2)}{x^2\sqrt{-c+dx}\sqrt{c+dx}} + \frac{(4bc^2 + 6ad^2) \arctan\left(\frac{\sqrt{c+dx}}{\sqrt{-c+dx}}\right)}{2c^5}$$

input

```
Integrate[(a + b*x^2)/(x^3*(-c + d*x)^(3/2)*(c + d*x)^(3/2)), x]
```

output

$$\frac{((-2*b*c^3*x^2 + a*(c^3 - 3*c*d^2*x^2))/(x^2*\text{Sqrt}[-c + d*x]*\text{Sqrt}[c + d*x]) + (4*b*c^2 + 6*a*d^2)*\text{ArcTan}[\text{Sqrt}[c + d*x]/\text{Sqrt}[-c + d*x]])/(2*c^5)}$$

Rubi [A] (verified)

Time = 0.42 (sec) , antiderivative size = 105, normalized size of antiderivative = 0.90, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.161$, Rules used = {956, 115, 27, 103, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{a + bx^2}{x^3(dx - c)^{3/2}(c + dx)^{3/2}} dx$$

$$\downarrow 956$$

$$\frac{1}{2} \left(\frac{3ad^2}{c^2} + 2b \right) \int \frac{1}{x(dx - c)^{3/2}(c + dx)^{3/2}} dx + \frac{a}{2c^2x^2\sqrt{dx - c}\sqrt{c + dx}}$$

$$\downarrow 115$$

$$\frac{1}{2} \left(\frac{3ad^2}{c^2} + 2b \right) \left(-\frac{\int \frac{d}{x\sqrt{dx - c}\sqrt{c + dx}} dx}{c^2d} - \frac{1}{c^2\sqrt{dx - c}\sqrt{c + dx}} \right) + \frac{a}{2c^2x^2\sqrt{dx - c}\sqrt{c + dx}}$$

$$\downarrow 27$$

$$\frac{1}{2} \left(\frac{3ad^2}{c^2} + 2b \right) \left(-\frac{\int \frac{1}{x\sqrt{dx - c}\sqrt{c + dx}} dx}{c^2} - \frac{1}{c^2\sqrt{dx - c}\sqrt{c + dx}} \right) + \frac{a}{2c^2x^2\sqrt{dx - c}\sqrt{c + dx}}$$

$$\downarrow 103$$

$$\frac{1}{2} \left(\frac{3ad^2}{c^2} + 2b \right) \left(-\frac{d \int \frac{1}{dc^2 + d(dx - c)(c + dx)} dx}{c^2} - \frac{1}{c^2\sqrt{dx - c}\sqrt{c + dx}} \right) + \frac{a}{2c^2x^2\sqrt{dx - c}\sqrt{c + dx}}$$

$$\downarrow 218$$

$$\frac{1}{2} \left(\frac{3ad^2}{c^2} + 2b \right) \left(-\frac{\arctan\left(\frac{\sqrt{dx - c}\sqrt{c + dx}}{c}\right)}{c^3} - \frac{1}{c^2\sqrt{dx - c}\sqrt{c + dx}} \right) + \frac{a}{2c^2x^2\sqrt{dx - c}\sqrt{c + dx}}$$

input `Int[(a + b*x^2)/(x^3*(-c + d*x)^(3/2)*(c + d*x)^(3/2)),x]`

output `a/(2*c^2*x^2*Sqrt[-c + d*x]*Sqrt[c + d*x]) + ((2*b + (3*a*d^2)/c^2)*(-1/(c^2*Sqrt[-c + d*x]*Sqrt[c + d*x])) - ArcTan[(Sqrt[-c + d*x]*Sqrt[c + d*x])/c]/c^3))/2`

Defintions of rubi rules used

rule 27 `Int[(a_)*(F_x_), x_Symbol] := Simp[a Int[F_x, x], x] /; FreeQ[a, x] && !MatchQ[F_x, (b_)*(G_x_)] /; FreeQ[b, x]`

rule 103 `Int[1/(Sqrt[(a_) + (b_)*(x_)]*Sqrt[(c_) + (d_)*(x_)]*((e_) + (f_)*(x_))), x_] := Simp[b*f Subst[Int[1/(d*(b*e - a*f)^2 + b*f^2*x^2), x], x, Sqrt[a + b*x]*Sqrt[c + d*x]], x] /; FreeQ[{a, b, c, d, e, f}, x] && EqQ[2*b*d*e - f*(b*c + a*d), 0]`

rule 115 `Int[((a_) + (b_)*(x_))^(m_)*((c_) + (d_)*(x_))^(n_)*((e_) + (f_)*(x_))^(p_), x_] := Simp[b*(a + b*x)^(m + 1)*(c + d*x)^(n + 1)*((e + f*x)^(p + 1))/((m + 1)*(b*c - a*d)*(b*e - a*f)), x] + Simp[1/((m + 1)*(b*c - a*d)*(b*e - a*f)) Int[(a + b*x)^(m + 1)*(c + d*x)^n*(e + f*x)^p*Simp[a*d*f*(m + 1) - b*(d*e*(m + n + 2) + c*f*(m + p + 2)) - b*d*f*(m + n + p + 3)*x, x], x], x] /; FreeQ[{a, b, c, d, e, f, n, p}, x] && LtQ[m, -1] && IntegersQ[2*m, 2*n, 2*p]`

rule 218 `Int[((a_) + (b_)*(x_)^2)^(-1), x_Symbol] := Simp[(Rt[a/b, 2]/a)*ArcTan[x/Rt[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]`

rule 956

```
Int[((e._)*(x._))^(m._)*((a1_) + (b1._)*(x._)^(non2._))^(p._)*((a2_) + (b2._)
*(x._)^(non2._))^(p._)*((c_) + (d._)*(x._)^(n)), x_Symbol] := Simp[c*(e*x)^(
m + 1)*(a1 + b1*x^(n/2))^(p + 1)*((a2 + b2*x^(n/2))^(p + 1)/(a1*a2*e*(m + 1
))), x] + Simp[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(a1*a2*e^n*(
m + 1)) Int[(e*x)^(m + n)*(a1 + b1*x^(n/2))^p*(a2 + b2*x^(n/2))^p, x], x]
/; FreeQ[{a1, b1, a2, b2, c, d, e, p}, x] && EqQ[non2, n/2] && EqQ[a2*b1 +
a1*b2, 0] && (IntegerQ[n] || GtQ[e, 0]) && ((GtQ[n, 0] && LtQ[m, -1]) || (
LtQ[n, 0] && GtQ[m + n, -1])) && !ILtQ[p, -1]
```

Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 238 vs. 2(99) = 198.

Time = 0.16 (sec) , antiderivative size = 239, normalized size of antiderivative = 2.04

method	result
risch	$\frac{a(-dx+c)\sqrt{dx+c}}{2c^4x^2\sqrt{dx-c}} - \frac{\left(-\frac{(3ad^2+2bc^2)\ln\left(\frac{-2c^2+2\sqrt{-c^2}\sqrt{d^2x^2-c^2}}{x}\right)}{\sqrt{-c^2}} + \frac{(ad^2+bc^2)\sqrt{d^2\left(x-\frac{c}{d}\right)^2+2cd\left(x-\frac{c}{d}\right)}}{dc\left(x-\frac{c}{d}\right)} - \frac{(ad^2+bc^2)\sqrt{d^2\left(x+\frac{c}{d}\right)^2+2cd\left(x+\frac{c}{d}\right)}}{dc\left(x+\frac{c}{d}\right)}\right)}{2c^4\sqrt{dx-c}\sqrt{dx+c}}$
default	$-\frac{-3\ln\left(-\frac{2(c^2-\sqrt{-c^2}\sqrt{d^2x^2-c^2})}{x}\right)a d^4x^4-2\ln\left(-\frac{2(c^2-\sqrt{-c^2}\sqrt{d^2x^2-c^2})}{x}\right)b c^2d^2x^4+3\ln\left(-\frac{2(c^2-\sqrt{-c^2}\sqrt{d^2x^2-c^2})}{x}\right)a c^2d^2x^2}{2\sqrt{dx-c}\sqrt{d^2x^2-c^2}\sqrt{dx+c}}$

input

```
int((b*x^2+a)/x^3/(d*x-c)^(3/2)/(d*x+c)^(3/2),x,method=_RETURNVERBOSE)
```

output

```
1/2*a*(-d*x+c)*(d*x+c)^(1/2)/c^4/x^2/(d*x-c)^(1/2)-1/2/c^4*(-(3*a*d^2+2*b*
c^2)/(-c^2)^(1/2)*ln((-2*c^2+2*(-c^2)^(1/2)*(d^2*x^2-c^2)^(1/2))/x)+(a*d^2
+b*c^2)/d/c/(x-c/d)*(d^2*(x-c/d)^2+2*c*d*(x-c/d)^(1/2)-(a*d^2+b*c^2)/d/c/
(x+c/d)*(d^2*(x+c/d)^2-2*c*d*(x+c/d)^(1/2))*((d*x-c)*(d*x+c))^(1/2)/(d*x-
c)^(1/2)/(d*x+c)^(1/2)
```

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 138, normalized size of antiderivative = 1.18

$$\int \frac{a + bx^2}{x^3(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{(ac^3 - (2bc^3 + 3acd^2)x^2)\sqrt{dx + c}\sqrt{dx - c} - 2((2bc^2d^2 + 3ad^4)x^4 - (2b^2c^2d^2 + 3a^2d^4)x^2)\arctan\left(\frac{dx - \sqrt{dx + c}\sqrt{dx - c}}{c}\right)}{2(c^5d^2x^4 - c^7x^2)}$$

input `integrate((b*x^2+a)/x^3/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="fricas")`

output `1/2*((a*c^3 - (2*b*c^3 + 3*a*c*d^2)*x^2)*sqrt(d*x + c)*sqrt(d*x - c) - 2*((2*b*c^2*d^2 + 3*a*d^4)*x^4 - (2*b*c^4 + 3*a*c^2*d^2)*x^2)*arctan(-(d*x - sqrt(d*x + c)*sqrt(d*x - c))/c))/(c^5*d^2*x^4 - c^7*x^2)`

Sympy [F(-1)]

Timed out.

$$\int \frac{a + bx^2}{x^3(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \text{Timed out}$$

input `integrate((b*x**2+a)/x**3/(d*x-c)**(3/2)/(d*x+c)**(3/2),x)`

output `Timed out`

Maxima [A] (verification not implemented)

Time = 0.11 (sec) , antiderivative size = 104, normalized size of antiderivative = 0.89

$$\int \frac{a + bx^2}{x^3(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{b \arcsin\left(\frac{c}{d|x|}\right)}{c^3} + \frac{3ad^2 \arcsin\left(\frac{c}{d|x|}\right)}{2c^5} - \frac{b}{\sqrt{d^2x^2 - c^2c^2}} - \frac{3ad^2}{2\sqrt{d^2x^2 - c^2c^4}} + \frac{a}{2\sqrt{d^2x^2 - c^2c^2x^2}}$$

input `integrate((b*x^2+a)/x^3/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="maxima")`

output

```
b*arcsin(c/(d*abs(x)))/c^3 + 3/2*a*d^2*arcsin(c/(d*abs(x)))/c^5 - b/(sqrt(d^2*x^2 - c^2)*c^2) - 3/2*a*d^2/(sqrt(d^2*x^2 - c^2)*c^4) + 1/2*a/(sqrt(d^2*x^2 - c^2)*c^2*x^2)
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 211 vs. 2(99) = 198.

Time = 0.21 (sec) , antiderivative size = 211, normalized size of antiderivative = 1.80

$$\int \frac{a + bx^2}{x^3(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{(2bc^2 + 3ad^2) \arctan\left(\frac{(\sqrt{dx+c}-\sqrt{dx-c})^2}{2c}\right)}{c^5} - \frac{(bc^2 + ad^2)\sqrt{dx+c}}{2\sqrt{dx-c}c^5} + \frac{2(bc^2 + ad^2)}{\left((\sqrt{dx+c}-\sqrt{dx-c})^2 + 2c\right)c^4} + \frac{2\left(ad^2(\sqrt{dx+c}-\sqrt{dx-c})^6 - 4ac^2d^2(\sqrt{dx+c}-\sqrt{dx-c})^2\right)}{\left((\sqrt{dx+c}-\sqrt{dx-c})^4 + 4c^2\right)^2c^4}$$

input

```
integrate((b*x^2+a)/x^3/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="giac")
```

output

```
(2*b*c^2 + 3*a*d^2)*arctan(1/2*(sqrt(d*x + c) - sqrt(d*x - c))^2/c)/c^5 - 1/2*(b*c^2 + a*d^2)*sqrt(d*x + c)/(sqrt(d*x - c)*c^5) + 2*(b*c^2 + a*d^2)/(((sqrt(d*x + c) - sqrt(d*x - c))^2 + 2*c)*c^4) + 2*(a*d^2*(sqrt(d*x + c) - sqrt(d*x - c))^6 - 4*a*c^2*d^2*(sqrt(d*x + c) - sqrt(d*x - c))^2)/(((sqrt(d*x + c) - sqrt(d*x - c))^4 + 4*c^2)^2*c^4)
```

Mupad [F(-1)]

Timed out.

$$\int \frac{a + bx^2}{x^3(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \int \frac{bx^2 + a}{x^3(c + dx)^{3/2}(dx - c)^{3/2}} dx$$

input

```
int((a + b*x^2)/(x^3*(c + d*x)^(3/2)*(d*x - c)^(3/2)),x)
```

output `int((a + b*x^2)/(x^3*(c + d*x)^(3/2)*(d*x - c)^(3/2)), x)`

Reduce [B] (verification not implemented)

Time = 0.23 (sec) , antiderivative size = 400, normalized size of antiderivative = 3.42

$$\int \frac{a + bx^2}{x^3(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{-6\sqrt{dx - c} \operatorname{atan}\left(\frac{\sqrt{dx-c} + \sqrt{dx+c-\sqrt{c}}}{\sqrt{c}}\right) ac d^2 x^2 - 6\sqrt{dx - c} \operatorname{atan}\left(\frac{\sqrt{dx-c} + \sqrt{dx+c-\sqrt{c}}}{\sqrt{c}}\right)}{x^3(-c + dx)^{3/2}(c + dx)^{3/2}}$$

input `int((b*x^2+a)/x^3/(d*x-c)^(3/2)/(d*x+c)^(3/2), x)`

output `(- 6*sqrt(- c + d*x)*atan((sqrt(- c + d*x) + sqrt(c + d*x) - sqrt(c))/sqrt(c))*a*c*d**2*x**2 - 6*sqrt(- c + d*x)*atan((sqrt(- c + d*x) + sqrt(c + d*x) - sqrt(c))/sqrt(c))*a*d**3*x**3 - 4*sqrt(- c + d*x)*atan((sqrt(- c + d*x) + sqrt(c + d*x) - sqrt(c))/sqrt(c))*b*c**3*x**2 - 4*sqrt(- c + d*x)*atan((sqrt(- c + d*x) + sqrt(c + d*x) - sqrt(c))/sqrt(c))*b*c**2*d*x**3 + 6*sqrt(- c + d*x)*atan((sqrt(- c + d*x) + sqrt(c + d*x) + sqrt(c))/sqrt(c))*a*c*d**2*x**2 + 6*sqrt(- c + d*x)*atan((sqrt(- c + d*x) + sqrt(c + d*x) + sqrt(c))/sqrt(c))*a*d**3*x**3 + 4*sqrt(- c + d*x)*atan((sqrt(- c + d*x) + sqrt(c + d*x) + sqrt(c))/sqrt(c))*b*c**3*x**2 + 4*sqrt(- c + d*x)*atan((sqrt(- c + d*x) + sqrt(c + d*x) + sqrt(c))/sqrt(c))*b*c**2*d*x**3 + sqrt(c + d*x)*a*c**3 - 3*sqrt(c + d*x)*a*c*d**2*x**2 - 2*sqrt(c + d*x)*b*c**3*x**2)/(2*sqrt(- c + d*x)*c**5*x**2*(c + d*x))`

3.40 $\int \frac{a+bx^2}{x^4(-c+dx)^{3/2}(c+dx)^{3/2}} dx$

Optimal result	311
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Rubi [A] (verified)	312
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Optimal result

Integrand size = 31, antiderivative size = 119

$$\int \frac{a + bx^2}{x^4(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{a}{3c^2x^3\sqrt{-c + dx}\sqrt{c + dx}} + \frac{3bc^2 + 4ad^2}{3c^4x\sqrt{-c + dx}\sqrt{c + dx}} - \frac{2d^2(3bc^2 + 4ad^2)x}{3c^6\sqrt{-c + dx}\sqrt{c + dx}}$$

output `1/3*a/c^2/x^3/(d*x-c)^(1/2)/(d*x+c)^(1/2)+1/3*(4*a*d^2+3*b*c^2)/c^4/x/(d*x-c)^(1/2)/(d*x+c)^(1/2)-2/3*d^2*(4*a*d^2+3*b*c^2)*x/c^6/(d*x-c)^(1/2)/(d*x+c)^(1/2)`

Mathematica [A] (verified)

Time = 0.17 (sec) , antiderivative size = 77, normalized size of antiderivative = 0.65

$$\int \frac{a + bx^2}{x^4(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{3bc^2x^2(c^2 - 2d^2x^2) + a(c^4 + 4c^2d^2x^2 - 8d^4x^4)}{3c^6x^3\sqrt{-c + dx}\sqrt{c + dx}}$$

input `Integrate[(a + b*x^2)/(x^4*(-c + d*x)^(3/2)*(c + d*x)^(3/2)),x]`

output

$$(3*b*c^2*x^2*(c^2 - 2*d^2*x^2) + a*(c^4 + 4*c^2*d^2*x^2 - 8*d^4*x^4))/(3*c^6*x^3*\text{Sqrt}[-c + d*x]*\text{Sqrt}[c + d*x])$$
Rubi [A] (verified)

Time = 0.39 (sec) , antiderivative size = 106, normalized size of antiderivative = 0.89, number of steps used = 4, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}} = 0.129$, Rules used = {956, 114, 27, 41}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{a + bx^2}{x^4(dx - c)^{3/2}(c + dx)^{3/2}} dx$$

$$\downarrow 956$$

$$\frac{1}{3} \left(\frac{4ad^2}{c^2} + 3b \right) \int \frac{1}{x^2(dx - c)^{3/2}(c + dx)^{3/2}} dx + \frac{a}{3c^2x^3\sqrt{dx - c}\sqrt{c + dx}}$$

$$\downarrow 114$$

$$\frac{1}{3} \left(\frac{4ad^2}{c^2} + 3b \right) \left(\frac{\int \frac{2d^2}{(dx - c)^{3/2}(c + dx)^{3/2}} dx}{c^2} + \frac{1}{c^2x\sqrt{dx - c}\sqrt{c + dx}} \right) + \frac{a}{3c^2x^3\sqrt{dx - c}\sqrt{c + dx}}$$

$$\downarrow 27$$

$$\frac{1}{3} \left(\frac{4ad^2}{c^2} + 3b \right) \left(\frac{2d^2 \int \frac{1}{(dx - c)^{3/2}(c + dx)^{3/2}} dx}{c^2} + \frac{1}{c^2x\sqrt{dx - c}\sqrt{c + dx}} \right) + \frac{a}{3c^2x^3\sqrt{dx - c}\sqrt{c + dx}}$$

$$\downarrow 41$$

$$\frac{1}{3} \left(\frac{1}{c^2x\sqrt{dx - c}\sqrt{c + dx}} - \frac{2d^2x}{c^4\sqrt{dx - c}\sqrt{c + dx}} \right) \left(\frac{4ad^2}{c^2} + 3b \right) + \frac{a}{3c^2x^3\sqrt{dx - c}\sqrt{c + dx}}$$

input

$$\text{Int}[(a + b*x^2)/(x^4*(-c + d*x)^(3/2)*(c + d*x)^(3/2)), x]$$

output

$$a/(3*c^2*x^3*\text{Sqrt}[-c + d*x]*\text{Sqrt}[c + d*x]) + ((3*b + (4*a*d^2)/c^2)*(1/(c^2*x*\text{Sqrt}[-c + d*x]*\text{Sqrt}[c + d*x]) - (2*d^2*x)/(c^4*\text{Sqrt}[-c + d*x]*\text{Sqrt}[c + d*x]))) / 3$$

Definitions of rubi rules used

rule 27 $\text{Int}[(a_*)(Fx_), x_Symbol] \rightarrow \text{Simp}[a \text{ Int}[Fx, x], x] /; \text{FreeQ}[a, x] \ \&\& \ !\text{MatchQ}[Fx, (b_*)(Gx_)] /; \text{FreeQ}[b, x]$

rule 41 $\text{Int}[1/(((a_*) + (b_*)(x_))^{3/2}*((c_*) + (d_*)(x_))^{3/2}), x_Symbol] \rightarrow \text{Simp}[x/(a*c*\text{Sqrt}[a + b*x]*\text{Sqrt}[c + d*x]), x] /; \text{FreeQ}[\{a, b, c, d\}, x] \ \&\& \ \text{EqQ}[b*c + a*d, 0]$

rule 114 $\text{Int}(((a_*) + (b_*)(x_))^{m_*}*((c_*) + (d_*)(x_))^{n_*}*((e_*) + (f_*)(x_))^{p_*}), x_] \rightarrow \text{Simp}[b*(a + b*x)^{(m + 1)}*(c + d*x)^{(n + 1)}*((e + f*x)^{(p + 1)})/((m + 1)*(b*c - a*d)*(b*e - a*f)), x] + \text{Simp}[1/((m + 1)*(b*c - a*d)*(b*e - a*f)) \text{ Int}[(a + b*x)^{(m + 1)}*(c + d*x)^n*(e + f*x)^p*\text{Simp}[a*d*f*(m + 1) - b*(d*e*(m + n + 2) + c*f*(m + p + 2)) - b*d*f*(m + n + p + 3)*x, x], x], x] /; \text{FreeQ}[\{a, b, c, d, e, f, n, p\}, x] \ \&\& \ \text{ILtQ}[m, -1] \ \&\& \ (\text{IntegerQ}[n] \ || \ \text{IntegersQ}[2*n, 2*p] \ || \ \text{ILtQ}[m + n + p + 3, 0])$

rule 956 $\text{Int}(((e_*)(x_))^{m_*}*((a1_*) + (b1_*)(x_)^{\text{non2}_*})^{p_*}*((a2_*) + (b2_*)(x_)^{\text{non2}_*})^{p_*}*((c_*) + (d_*)(x_)^{n_*}), x_Symbol] \rightarrow \text{Simp}[c*(e*x)^{(m + 1)}*(a1 + b1*x^{(n/2)})^{(p + 1)}*((a2 + b2*x^{(n/2)})^{(p + 1)})/(a1*a2*e^{(m + 1)}), x] + \text{Simp}[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(a1*a2*e^n*(m + 1)) \text{ Int}[(e*x)^{(m + n)}*(a1 + b1*x^{(n/2)})^p*(a2 + b2*x^{(n/2)})^p, x], x] /; \text{FreeQ}[\{a1, b1, a2, b2, c, d, e, p\}, x] \ \&\& \ \text{EqQ}[\text{non2}, n/2] \ \&\& \ \text{EqQ}[a2*b1 + a1*b2, 0] \ \&\& \ (\text{IntegerQ}[n] \ || \ \text{GtQ}[e, 0]) \ \&\& \ ((\text{GtQ}[n, 0] \ \&\& \ \text{LtQ}[m, -1]) \ || \ (\text{LtQ}[n, 0] \ \&\& \ \text{GtQ}[m + n, -1])) \ \&\& \ !\text{ILtQ}[p, -1]$

Maple [A] (verified)

Time = 0.16 (sec) , antiderivative size = 73, normalized size of antiderivative = 0.61

method	result	size
gospers	$\frac{-8a d^4 x^4 - 6b c^2 d^2 x^4 + 4a c^2 d^2 x^2 + 3b c^4 x^2 + a c^4}{3c^6 x^3 \sqrt{dx-c} \sqrt{dx+c}}$	73
default	$\frac{(-8a d^4 x^4 - 6b c^2 d^2 x^4 + 4a c^2 d^2 x^2 + 3b c^4 x^2 + a c^4) \operatorname{csgn}(d)^2}{3\sqrt{dx-c} \sqrt{dx+c} c^6 x^3}$	77
orering	$-\frac{(-dx+c)(-8a d^4 x^4 - 6b c^2 d^2 x^4 + 4a c^2 d^2 x^2 + 3b c^4 x^2 + a c^4)}{3\sqrt{dx+c} x^3 c^6 (dx-c)^{\frac{3}{2}}}$	79
risch	$\frac{\sqrt{dx+c}(-dx+c)(5a d^2 x^2 + 3b c^2 x^2 + a c^2)}{3c^6 x^3 \sqrt{dx-c}} - \frac{d^2 (a d^2 + b c^2) x \sqrt{(dx-c)(dx+c)}}{\sqrt{-(dx+c)(-dx+c)} c^6 \sqrt{dx-c} \sqrt{dx+c}}$	122

input `int((b*x^2+a)/x^4/(d*x-c)^(3/2)/(d*x+c)^(3/2),x,method=_RETURNVERBOSE)`

output `1/3/c^6/x^3/(d*x-c)^(1/2)/(d*x+c)^(1/2)*(-8*a*d^4*x^4-6*b*c^2*d^2*x^4+4*a*c^2*d^2*x^2+3*b*c^4*x^2+a*c^4)`

Fricas [A] (verification not implemented)

Time = 0.08 (sec) , antiderivative size = 132, normalized size of antiderivative = 1.11

$$\int \frac{a + bx^2}{x^4(-c + dx)^{3/2}(c + dx)^{3/2}} dx =$$

$$\frac{2(3bc^2d^3 + 4ad^5)x^5 - 2(3bc^4d + 4ac^2d^3)x^3 - (ac^4 - 2(3bc^2d^2 + 4ad^4)x^4 + (3bc^4 + 4ac^2d^2)x^2)\sqrt{dx+c}}{3(c^6d^2x^5 - c^8x^3)}$$

input `integrate((b*x^2+a)/x^4/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="fricas")`

output `-1/3*(2*(3*b*c^2*d^3 + 4*a*d^5)*x^5 - 2*(3*b*c^4*d + 4*a*c^2*d^3)*x^3 - (a*c^4 - 2*(3*b*c^2*d^2 + 4*a*d^4)*x^4 + (3*b*c^4 + 4*a*c^2*d^2)*x^2)*sqrt(d*x + c)*sqrt(d*x - c)/(c^6*d^2*x^5 - c^8*x^3)`

Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 70.65 (sec) , antiderivative size = 165, normalized size of antiderivative = 1.39

$$\int \frac{a + bx^2}{x^4(-c + dx)^{3/2}(c + dx)^{3/2}} dx = a \left(-\frac{d^3 G_{6,6}^{5,3} \left(\begin{matrix} \frac{11}{4}, \frac{13}{4}, 1 \\ \frac{11}{4}, 3, \frac{13}{4}, \frac{7}{2}, 4 \end{matrix} \middle| \frac{c^2}{d^2 x^2} \right)}{2\pi^{\frac{3}{2}} c^6} + \frac{id^3 G_{6,6}^{2,6} \left(\begin{matrix} \frac{3}{2}, 2, \frac{9}{4}, \frac{5}{2}, \frac{11}{4}, 1 \\ \frac{9}{4}, \frac{11}{4} \end{matrix} \middle| \frac{c^2}{d^2 x^2} \right)}{2\pi^{\frac{3}{2}} c^6} \right) + b \left(-\frac{d G_{6,6}^{5,3} \left(\begin{matrix} \frac{7}{4}, \frac{9}{4}, 1 \\ \frac{7}{4}, 2, \frac{9}{4}, \frac{5}{2}, 3 \end{matrix} \middle| \frac{c^2}{d^2 x^2} \right)}{2\pi^{\frac{3}{2}} c^4} + \frac{id G_{6,6}^{2,6} \left(\begin{matrix} \frac{1}{2}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 1 \\ \frac{5}{4}, \frac{7}{4} \end{matrix} \middle| \frac{c^2 e^{2i\pi}}{d^2 x^2} \right)}{2\pi^{\frac{3}{2}} c^4} \right)$$

input `integrate((b*x**2+a)/x**4/(d*x-c)**(3/2)/(d*x+c)**(3/2),x)`

output `a*(-d**3*meijerg(((11/4, 13/4, 1), (5/2, 7/2, 4)), ((11/4, 3, 13/4, 7/2, 4), (0,)), c**2/(d**2*x**2))/(2*pi**(3/2)*c**6) + I*d**3*meijerg(((3/2, 2, 9/4, 5/2, 11/4, 1), ()), ((9/4, 11/4), (3/2, 2, 3, 0)), c**2*exp_polar(2*I*pi)/(d**2*x**2))/(2*pi**(3/2)*c**6)) + b*(-d*meijerg(((7/4, 9/4, 1), (3/2, 5/2, 3)), ((7/4, 2, 9/4, 5/2, 3), (0,)), c**2/(d**2*x**2))/(2*pi**(3/2)*c**4) + I*d*meijerg(((1/2, 1, 5/4, 3/2, 7/4, 1), ()), ((5/4, 7/4), (1/2, 1, 2, 0)), c**2*exp_polar(2*I*pi)/(d**2*x**2))/(2*pi**(3/2)*c**4))`

Maxima [A] (verification not implemented)

Time = 0.11 (sec) , antiderivative size = 125, normalized size of antiderivative = 1.05

$$\int \frac{a + bx^2}{x^4(-c + dx)^{3/2}(c + dx)^{3/2}} dx = -\frac{2bd^2x}{\sqrt{d^2x^2 - c^2}c^4} - \frac{8ad^4x}{3\sqrt{d^2x^2 - c^2}c^6} + \frac{b}{\sqrt{d^2x^2 - c^2}c^2x} + \frac{4ad^2}{3\sqrt{d^2x^2 - c^2}c^4x} + \frac{a}{3\sqrt{d^2x^2 - c^2}c^2x^3}$$

input `integrate((b*x^2+a)/x^4/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="maxima")`

output

$$-2*b*d^2*x/(sqrt(d^2*x^2 - c^2)*c^4) - 8/3*a*d^4*x/(sqrt(d^2*x^2 - c^2)*c^6) + b/(sqrt(d^2*x^2 - c^2)*c^2*x) + 4/3*a*d^2/(sqrt(d^2*x^2 - c^2)*c^4*x) + 1/3*a/(sqrt(d^2*x^2 - c^2)*c^2*x^3)$$

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 242 vs. $2(101) = 202$.

Time = 0.21 (sec) , antiderivative size = 242, normalized size of antiderivative = 2.03

$$\int \frac{a + bx^2}{x^4(-c + dx)^{3/2}(c + dx)^{3/2}} dx = -\frac{(bc^2d + ad^3)\sqrt{dx + c}}{2\sqrt{dx - c}c^6} - \frac{((\sqrt{dx + c} - \sqrt{dx - c})^2 + 2c)c^5}{8\left(3bc^2d(\sqrt{dx + c} - \sqrt{dx - c})^8 + 3ad^3(\sqrt{dx + c} - \sqrt{dx - c})^8 + 24bc^4d(\sqrt{dx + c} - \sqrt{dx - c})^4 + 48ad^3(\sqrt{dx + c} - \sqrt{dx - c})^4 + 4c^2d^3\right)}{3\left((\sqrt{dx + c} - \sqrt{dx - c})^4 + 4c^2\right)^3c^4}$$

input

```
integrate((b*x^2+a)/x^4/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="giac")
```

output

$$-1/2*(b*c^2*d + a*d^3)*sqrt(d*x + c)/(sqrt(d*x - c)*c^6) - 2*(b*c^2*d + a*d^3)/(((sqrt(d*x + c) - sqrt(d*x - c))^2 + 2*c)*c^5) - 8/3*(3*b*c^2*d*(sqrt(d*x + c) - sqrt(d*x - c))^8 + 3*a*d^3*(sqrt(d*x + c) - sqrt(d*x - c))^8 + 24*b*c^4*d*(sqrt(d*x + c) - sqrt(d*x - c))^4 + 48*a*c^2*d^3*(sqrt(d*x + c) - sqrt(d*x - c))^4 + 48*b*c^6*d + 80*a*c^4*d^3)/(((sqrt(d*x + c) - sqrt(d*x - c))^4 + 4*c^2)^3*c^4)$$

Mupad [B] (verification not implemented)

Time = 5.31 (sec) , antiderivative size = 104, normalized size of antiderivative = 0.87

$$\int \frac{a + bx^2}{x^4(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{\sqrt{dx - c} \left(\frac{a}{3c^2d} + \frac{x^2(3bc^4 + 4ac^2d^2)}{3c^6d} - \frac{x^4(6bc^2d^2 + 8ad^4)}{3c^6d} \right)}{x^4 \sqrt{c + dx} - \frac{cx^3 \sqrt{c + dx}}{d}}$$

input

```
int((a + b*x^2)/(x^4*(c + d*x)^(3/2)*(d*x - c)^(3/2)),x)
```

output

```
((d*x - c)^(1/2)*(a/(3*c^2*d) + (x^2*(3*b*c^4 + 4*a*c^2*d^2))/(3*c^6*d) -
(x^4*(8*a*d^4 + 6*b*c^2*d^2))/(3*c^6*d)))/(x^4*(c + d*x)^(1/2) - (c*x^3*(c
+ d*x)^(1/2))/d)
```

Reduce [B] (verification not implemented)

Time = 0.22 (sec) , antiderivative size = 176, normalized size of antiderivative = 1.48

$$\int \frac{a + bx^2}{x^4(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{8\sqrt{dx - c}ac d^3 x^3 + 8\sqrt{dx - c}a d^4 x^4 + 6\sqrt{dx - c}b c^3 d x^3 + 6\sqrt{dx - c}b c^2 d^2 x^2 + 6\sqrt{dx - c}b c d x + 6\sqrt{dx - c}b c}{3\sqrt{dx - c}c^6 x^3 (c + dx)}$$

input

```
int((b*x^2+a)/x^4/(d*x-c)^(3/2)/(d*x+c)^(3/2),x)
```

output

```
(8*sqrt(-c + d*x)*a*c*d**3*x**3 + 8*sqrt(-c + d*x)*a*d**4*x**4 + 6*sqrt(-c + d*x)*b*c**3*d*x**3 + 6*sqrt(-c + d*x)*b*c**2*d**2*x**4 + sqrt(c + d*x)*a*c**4 + 4*sqrt(c + d*x)*a*c**2*d**2*x**2 - 8*sqrt(c + d*x)*a*d**4*x**4 + 3*sqrt(c + d*x)*b*c**4*x**2 - 6*sqrt(c + d*x)*b*c**2*d**2*x**4)/(3*sqrt(-c + d*x)*c**6*x**3*(c + d*x))
```

3.41 $\int \frac{a+bx^2}{x^5(-c+dx)^{3/2}(c+dx)^{3/2}} dx$

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Optimal result

Integrand size = 31, antiderivative size = 166

$$\int \frac{a+bx^2}{x^5(-c+dx)^{3/2}(c+dx)^{3/2}} dx = \frac{a}{4c^2x^4\sqrt{-c+dx}\sqrt{c+dx}} - \frac{4bc^2+5ad^2}{4c^4x^2\sqrt{-c+dx}\sqrt{c+dx}} - \frac{3(4bc^2+5ad^2)\sqrt{-c+dx}\sqrt{c+dx}}{8c^6x^2} - \frac{3d^2(4bc^2+5ad^2)\arctan\left(\frac{\sqrt{-c+dx}\sqrt{c+dx}}{c}\right)}{8c^7}$$

output

```
1/4*a/c^2/x^4/(d*x-c)^(1/2)/(d*x+c)^(1/2)-1/4*(5*a*d^2+4*b*c^2)/c^4/x^2/(d
*x-c)^(1/2)/(d*x+c)^(1/2)-3/8*(5*a*d^2+4*b*c^2)*(d*x-c)^(1/2)*(d*x+c)^(1/2
)/c^6/x^2-3/8*d^2*(5*a*d^2+4*b*c^2)*arctan((d*x-c)^(1/2)*(d*x+c)^(1/2)/c)/
c^7
```

Mathematica [A] (verified)

Time = 0.30 (sec) , antiderivative size = 122, normalized size of antiderivative = 0.73

$$\int \frac{a + bx^2}{x^5(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{4bc^3x^2(c^2 - 3d^2x^2) + a(2c^5 + 5c^3d^2x^2 - 15cd^4x^4)}{x^4\sqrt{-c+dx}\sqrt{c+dx}} + 6d^2(4bc^2 + 5ad^2) \arctan\left(\frac{\sqrt{c+dx}}{\sqrt{-c+dx}}\right) + 6d^2(4bc^2 + 5ad^2) \arctan\left(\frac{\sqrt{-c+dx}}{\sqrt{c+dx}}\right)}{8c^7}$$

input `Integrate[(a + b*x^2)/(x^5*(-c + d*x)^(3/2)*(c + d*x)^(3/2)), x]`

output `((4*b*c^3*x^2*(c^2 - 3*d^2*x^2) + a*(2*c^5 + 5*c^3*d^2*x^2 - 15*c*d^4*x^4))/(x^4*sqrt[-c + d*x]*sqrt[c + d*x]) + 6*d^2*(4*b*c^2 + 5*a*d^2)*ArcTan[Sqrt[c + d*x]/sqrt[-c + d*x]])/(8*c^7)`

Rubi [A] (verified)

Time = 0.47 (sec) , antiderivative size = 146, normalized size of antiderivative = 0.88, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}} = 0.226$, Rules used = {956, 114, 27, 115, 27, 103, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \frac{a + bx^2}{x^5(dx - c)^{3/2}(c + dx)^{3/2}} dx \\ & \quad \downarrow 956 \\ & \frac{1}{4} \left(\frac{5ad^2}{c^2} + 4b \right) \int \frac{1}{x^3(dx - c)^{3/2}(c + dx)^{3/2}} dx + \frac{a}{4c^2x^4\sqrt{dx - c}\sqrt{c + dx}} \\ & \quad \downarrow 114 \\ & \frac{1}{4} \left(\frac{5ad^2}{c^2} + 4b \right) \left(\int \frac{\frac{3d^2}{x(dx - c)^{3/2}(c + dx)^{3/2}} dx}{2c^2} + \frac{1}{2c^2x^2\sqrt{dx - c}\sqrt{c + dx}} \right) + \frac{a}{4c^2x^4\sqrt{dx - c}\sqrt{c + dx}} \\ & \quad \downarrow 27 \end{aligned}$$

$$\begin{aligned}
& \frac{1}{4} \left(\frac{5ad^2}{c^2} + 4b \right) \left(\frac{3d^2 \int \frac{1}{x(dx-c)^{3/2}(c+dx)^{3/2}} dx}{2c^2} + \frac{1}{2c^2 x^2 \sqrt{dx-c}\sqrt{c+dx}} \right) + \\
& \quad \frac{a}{4c^2 x^4 \sqrt{dx-c}\sqrt{c+dx}} \\
& \quad \downarrow \text{115} \\
& \frac{1}{4} \left(\frac{5ad^2}{c^2} + 4b \right) \left(\frac{3d^2 \left(-\frac{\int \frac{d}{x\sqrt{dx-c}\sqrt{c+dx}} dx}{c^2 d} - \frac{1}{c^2 \sqrt{dx-c}\sqrt{c+dx}} \right)}{2c^2} + \frac{1}{2c^2 x^2 \sqrt{dx-c}\sqrt{c+dx}} \right) + \\
& \quad \frac{a}{4c^2 x^4 \sqrt{dx-c}\sqrt{c+dx}} \\
& \quad \downarrow \text{27} \\
& \frac{1}{4} \left(\frac{5ad^2}{c^2} + 4b \right) \left(\frac{3d^2 \left(-\frac{\int \frac{1}{x\sqrt{dx-c}\sqrt{c+dx}} dx}{c^2} - \frac{1}{c^2 \sqrt{dx-c}\sqrt{c+dx}} \right)}{2c^2} + \frac{1}{2c^2 x^2 \sqrt{dx-c}\sqrt{c+dx}} \right) + \\
& \quad \frac{a}{4c^2 x^4 \sqrt{dx-c}\sqrt{c+dx}} \\
& \quad \downarrow \text{103} \\
& \frac{1}{4} \left(\frac{5ad^2}{c^2} + 4b \right) \left(\frac{3d^2 \left(-\frac{d \int \frac{1}{dc^2+d(dx-c)(c+dx)} d(\sqrt{dx-c}\sqrt{c+dx})}{c^2} - \frac{1}{c^2 \sqrt{dx-c}\sqrt{c+dx}} \right)}{2c^2} + \frac{1}{2c^2 x^2 \sqrt{dx-c}\sqrt{c+dx}} \right) + \\
& \quad \frac{a}{4c^2 x^4 \sqrt{dx-c}\sqrt{c+dx}} \\
& \quad \downarrow \text{218} \\
& \frac{1}{4} \left(\frac{5ad^2}{c^2} + 4b \right) \left(\frac{3d^2 \left(-\frac{\arctan\left(\frac{\sqrt{dx-c}\sqrt{c+dx}}{c}\right)}{c^3} - \frac{1}{c^2 \sqrt{dx-c}\sqrt{c+dx}} \right)}{2c^2} + \frac{1}{2c^2 x^2 \sqrt{dx-c}\sqrt{c+dx}} \right) + \\
& \quad \frac{a}{4c^2 x^4 \sqrt{dx-c}\sqrt{c+dx}}
\end{aligned}$$

input

```
Int[(a + b*x^2)/(x^5*(-c + d*x)^(3/2)*(c + d*x)^(3/2)),x]
```

output

$$\frac{a/(4c^2x^4\sqrt{-c+dx}\sqrt{c+dx}) + ((4b + (5ad^2)/c^2)*(1/(2c^2x^2\sqrt{-c+dx}\sqrt{c+dx}) + (3d^2*(-1/(c^2\sqrt{-c+dx}\sqrt{c+dx}))) - \text{ArcTan}[(\sqrt{-c+dx}\sqrt{c+dx})/c]/c^3)/(2c^2))}{4}$$

Defintions of rubi rules used

rule 27

$$\text{Int}[(a_*)(Fx_), x_Symbol] \rightarrow \text{Simp}[a \text{ Int}[Fx, x], x] /; \text{FreeQ}[a, x] \ \&\& \ !\text{MatchQ}[Fx, (b_*)(Gx_)] /; \text{FreeQ}[b, x]$$

rule 103

$$\text{Int}[1/(\sqrt{(a_.) + (b_*)(x_)}\sqrt{(c_.) + (d_*)(x_)}*((e_.) + (f_*)(x_))), x_] \rightarrow \text{Simp}[b*f \text{ Subst}[\text{Int}[1/(d*(b*e - a*f)^2 + b*f^2*x^2), x], x, \sqrt{a + b*x}\sqrt{c + d*x}], x] /; \text{FreeQ}[\{a, b, c, d, e, f\}, x] \ \&\& \ \text{EqQ}[2*b*d * e - f*(b*c + a*d), 0]$$

rule 114

$$\text{Int}[((a_.) + (b_*)(x_))^{(m_)}*((c_.) + (d_*)(x_))^{(n_)}*((e_.) + (f_*)(x_))^{(p_)}, x_] \rightarrow \text{Simp}[b*(a + b*x)^{(m+1)}*(c + d*x)^{(n+1)}*((e + f*x)^{(p+1)})/((m+1)*(b*c - a*d)*(b*e - a*f)), x] + \text{Simp}[1/((m+1)*(b*c - a*d)*(b*e - a*f)) \text{ Int}[(a + b*x)^{(m+1)}*(c + d*x)^n*(e + f*x)^p*\text{Simp}[a*d*f*(m+1) - b*(d*e*(m+n+2) + c*f*(m+p+2)) - b*d*f*(m+n+p+3)*x, x], x], x] /; \text{FreeQ}[\{a, b, c, d, e, f, n, p\}, x] \ \&\& \ \text{ILtQ}[m, -1] \ \&\& \ (\text{IntegerQ}[n] \ || \ \text{IntegersQ}[2*n, 2*p] \ || \ \text{ILtQ}[m+n+p+3, 0])$$

rule 115

$$\text{Int}[((a_.) + (b_*)(x_))^{(m_)}*((c_.) + (d_*)(x_))^{(n_)}*((e_.) + (f_*)(x_))^{(p_)}, x_] \rightarrow \text{Simp}[b*(a + b*x)^{(m+1)}*(c + d*x)^{(n+1)}*((e + f*x)^{(p+1)})/((m+1)*(b*c - a*d)*(b*e - a*f)), x] + \text{Simp}[1/((m+1)*(b*c - a*d)*(b*e - a*f)) \text{ Int}[(a + b*x)^{(m+1)}*(c + d*x)^n*(e + f*x)^p*\text{Simp}[a*d*f*(m+1) - b*(d*e*(m+n+2) + c*f*(m+p+2)) - b*d*f*(m+n+p+3)*x, x], x], x] /; \text{FreeQ}[\{a, b, c, d, e, f, n, p\}, x] \ \&\& \ \text{LtQ}[m, -1] \ \&\& \ \text{IntegersQ}[2*m, 2*n, 2*p]$$

rule 218

$$\text{Int}[((a_.) + (b_*)(x_)^2)^{-1}, x_Symbol] \rightarrow \text{Simp}[(\text{Rt}[a/b, 2]/a)*\text{ArcTan}[x/\text{Rt}[a/b, 2]], x] /; \text{FreeQ}[\{a, b\}, x] \ \&\& \ \text{PosQ}[a/b]$$

rule 956

```
Int[((e_.)*(x_))^(m_.)*((a1_) + (b1_.)*(x_)^(non2_.))^(p_.)*((a2_) + (b2_.)
*(x_)^(non2_.))^(p_.)*((c_) + (d_.)*(x_)^(n_)), x_Symbol] := Simp[c*(e*x)^(
m + 1)*(a1 + b1*x^(n/2))^(p + 1)*((a2 + b2*x^(n/2))^(p + 1)/(a1*a2*e*(m + 1
))), x] + Simp[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/(a1*a2*e^n*(
m + 1)) Int[(e*x)^(m + n)*(a1 + b1*x^(n/2))^p*(a2 + b2*x^(n/2))^p, x], x]
/; FreeQ[{a1, b1, a2, b2, c, d, e, p}, x] && EqQ[non2, n/2] && EqQ[a2*b1 +
a1*b2, 0] && (IntegerQ[n] || GtQ[e, 0]) && ((GtQ[n, 0] && LtQ[m, -1]) || (
LtQ[n, 0] && GtQ[m + n, -1])) && !ILtQ[p, -1]
```

Maple [A] (verified)

Time = 0.18 (sec) , antiderivative size = 267, normalized size of antiderivative = 1.61

method	result
risch	$\frac{\sqrt{dx+c}(-dx+c)(7ad^2x^2+4b^2c^2x^2+2ac^2)}{8c^6x^4\sqrt{dx-c}} - \frac{d^2 \left(-\frac{(15ad^2+12bc^2) \ln\left(\frac{-2c^2+2\sqrt{-c^2}\sqrt{d^2x^2-c^2}}{x}\right)}{\sqrt{-c^2}} + \frac{4(ad^2+bc^2)\sqrt{d^2\left(x-\frac{c}{d}\right)^2+2cd}}{dc\left(x-\frac{c}{d}\right)} \right)}{8c^6\sqrt{dx-c}\sqrt{dx+c}}$
default	$-\frac{15 \ln\left(-\frac{2(c^2-\sqrt{-c^2}\sqrt{d^2x^2-c^2})}{x}\right)}{a} ad^6x^6 - 12 \ln\left(-\frac{2(c^2-\sqrt{-c^2}\sqrt{d^2x^2-c^2})}{x}\right) b c^2 d^4 x^6 + 15 \ln\left(-\frac{2(c^2-\sqrt{-c^2}\sqrt{d^2x^2-c^2})}{x}\right) a c^2 d$

input

```
int((b*x^2+a)/x^5/(d*x-c)^(3/2)/(d*x+c)^(3/2),x,method=_RETURNVERBOSE)
```

output

```
1/8*(d*x+c)^(1/2)*(-d*x+c)*(7*a*d^2*x^2+4*b*c^2*x^2+2*a*c^2)/c^6/x^4/(d*x-
c)^(1/2)-1/8*d^2/c^6*(-(15*a*d^2+12*b*c^2)/(-c^2)^(1/2)*ln((-2*c^2+2*(-c^2
)^(1/2)*(d^2*x^2-c^2)^(1/2))/x)+4*(a*d^2+b*c^2)/d/c/(x-c/d)*(d^2*(x-c/d)^2
+2*c*d*(x-c/d)^(1/2)-4*(a*d^2+b*c^2)/d/c/(x+c/d)*(d^2*(x+c/d)^2-2*c*d*(x+
c/d)^(1/2))*((d*x-c)*(d*x+c))^(1/2)/(d*x-c)^(1/2)/(d*x+c)^(1/2)
```

Fricas [A] (verification not implemented)

Time = 0.10 (sec) , antiderivative size = 165, normalized size of antiderivative = 0.99

$$\int \frac{a + bx^2}{x^5(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{(2ac^5 - 3(4bc^3d^2 + 5acd^4)x^4 + (4bc^5 + 5ac^3d^2)x^2)\sqrt{dx + c}\sqrt{dx - c} - c}{8(c^7d^2 - c)}$$

input `integrate((b*x^2+a)/x^5/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="fricas")`

output
$$\frac{1}{8} \left((2ac^5 - 3(4bc^3d^2 + 5acd^4)x^4 + (4b^2c^5 + 5a^2c^3d^2)x^2) \sqrt{dx+c} \sqrt{dx-c} - 6((4b^2c^2d^4 + 5ad^6)x^6 - (4b^2c^4d^2 + 5a^2c^2d^4)x^4) \arctan\left(\frac{-(dx - \sqrt{dx+c})\sqrt{dx-c}}{c}\right) \right) / (c^7d^2x^6 - c^9x^4)$$

Sympy [F(-1)]

Timed out.

$$\int \frac{a + bx^2}{x^5(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \text{Timed out}$$

input `integrate((b*x**2+a)/x**5/(d*x-c)**(3/2)/(d*x+c)**(3/2),x)`

output Timed out

Maxima [A] (verification not implemented)

Time = 0.11 (sec) , antiderivative size = 162, normalized size of antiderivative = 0.98

$$\begin{aligned} \int \frac{a + bx^2}{x^5(-c + dx)^{3/2}(c + dx)^{3/2}} dx &= \frac{3bd^2 \arcsin\left(\frac{c}{d|x|}\right)}{2c^5} \\ &+ \frac{15ad^4 \arcsin\left(\frac{c}{d|x|}\right)}{8c^7} - \frac{3bd^2}{2\sqrt{d^2x^2 - c^2c^4}} - \frac{15ad^4}{8\sqrt{d^2x^2 - c^2c^6}} \\ &+ \frac{b}{2\sqrt{d^2x^2 - c^2c^2x^2}} + \frac{5ad^2}{8\sqrt{d^2x^2 - c^2c^4x^2}} + \frac{a}{4\sqrt{d^2x^2 - c^2c^2x^4}} \end{aligned}$$

input `integrate((b*x^2+a)/x^5/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="maxima")`

output

$$\begin{aligned} & 3/2*b*d^2*arcsin(c/(d*abs(x)))/c^5 + 15/8*a*d^4*arcsin(c/(d*abs(x)))/c^7 - \\ & 3/2*b*d^2/(sqrt(d^2*x^2 - c^2)*c^4) - 15/8*a*d^4/(sqrt(d^2*x^2 - c^2)*c^6) \\ &) + 1/2*b/(sqrt(d^2*x^2 - c^2)*c^2*x^2) + 5/8*a*d^2/(sqrt(d^2*x^2 - c^2)*c^4*x^2) \\ & + 1/4*a/(sqrt(d^2*x^2 - c^2)*c^2*x^4) \end{aligned}$$

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 402 vs. 2(142) = 284.

Time = 0.25 (sec) , antiderivative size = 402, normalized size of antiderivative = 2.42

$$\begin{aligned} & \int \frac{a + bx^2}{x^5(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{3(4bc^2d^2 + 5ad^4) \arctan\left(\frac{(\sqrt{dx+c}-\sqrt{dx-c})^2}{2c}\right)}{4c^7} \\ & - \frac{(bc^2d^2 + ad^4)\sqrt{dx+c}}{2\sqrt{dx-c}c^7} + \frac{2(bc^2d^2 + ad^4)}{\left((\sqrt{dx+c}-\sqrt{dx-c})^2 + 2c\right)c^6} \\ & + \frac{4bc^2d^2(\sqrt{dx+c}-\sqrt{dx-c})^{14} + 7ad^4(\sqrt{dx+c}-\sqrt{dx-c})^{14} + 16bc^4d^2(\sqrt{dx+c}-\sqrt{dx-c})^{10} + 60ad^4(\sqrt{dx+c}-\sqrt{dx-c})^{10}}{c^7} \end{aligned}$$

input

```
integrate((b*x^2+a)/x^5/(d*x-c)^(3/2)/(d*x+c)^(3/2),x, algorithm="giac")
```

output

$$\begin{aligned} & 3/4*(4*b*c^2*d^2 + 5*a*d^4)*arctan(1/2*(sqrt(d*x + c) - sqrt(d*x - c))^2/c) \\ &)/c^7 - 1/2*(b*c^2*d^2 + a*d^4)*sqrt(d*x + c)/(sqrt(d*x - c)*c^7) + 2*(b*c^2*d^2 + a*d^4)/(((sqrt(d*x + c) - sqrt(d*x - c))^2 + 2*c)*c^6) + 1/2*(4*b*c^2*d^2*(sqrt(d*x + c) - sqrt(d*x - c))^14 + 7*a*d^4*(sqrt(d*x + c) - sqrt(d*x - c))^14 + 16*b*c^4*d^2*(sqrt(d*x + c) - sqrt(d*x - c))^10 + 60*a*c^2*d^4*(sqrt(d*x + c) - sqrt(d*x - c))^10 - 64*b*c^6*d^2*(sqrt(d*x + c) - sqrt(d*x - c))^6 - 240*a*c^4*d^4*(sqrt(d*x + c) - sqrt(d*x - c))^6 - 256*b*c^8*d^2*(sqrt(d*x + c) - sqrt(d*x - c))^2 - 448*a*c^6*d^4*(sqrt(d*x + c) - sqrt(d*x - c))^2)/(((sqrt(d*x + c) - sqrt(d*x - c))^4 + 4*c^2)^4*c^6) \end{aligned}$$

Mupad [F(-1)]

Timed out.

$$\int \frac{a + bx^2}{x^5(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \int \frac{bx^2 + a}{x^5(c + dx)^{3/2}(dx - c)^{3/2}} dx$$

input `int((a + b*x^2)/(x^5*(c + d*x)^(3/2)*(d*x - c)^(3/2)),x)`

output `int((a + b*x^2)/(x^5*(c + d*x)^(3/2)*(d*x - c)^(3/2)), x)`

Reduce [B] (verification not implemented)

Time = 0.25 (sec) , antiderivative size = 447, normalized size of antiderivative = 2.69

$$\int \frac{a + bx^2}{x^5(-c + dx)^{3/2}(c + dx)^{3/2}} dx = \frac{-30\sqrt{dx - c} \operatorname{atan}\left(\frac{\sqrt{dx-c} + \sqrt{dx+c} - \sqrt{c}}{\sqrt{c}}\right) ac d^4 x^4 - 30\sqrt{dx - c} \operatorname{atan}\left(\frac{\sqrt{dx-c}}{\sqrt{c}}\right)}$$

input `int((b*x^2+a)/x^5/(d*x-c)^(3/2)/(d*x+c)^(3/2),x)`

output `(- 30*sqrt(- c + d*x)*atan((sqrt(- c + d*x) + sqrt(c + d*x) - sqrt(c))/sqrt(c))*a*c*d**4*x**4 - 30*sqrt(- c + d*x)*atan((sqrt(- c + d*x) + sqrt(c + d*x) - sqrt(c))/sqrt(c))*a*d**5*x**5 - 24*sqrt(- c + d*x)*atan((sqrt(- c + d*x) + sqrt(c + d*x) - sqrt(c))/sqrt(c))*b*c**3*d**2*x**4 - 24*sqrt(- c + d*x)*atan((sqrt(- c + d*x) + sqrt(c + d*x) - sqrt(c))/sqrt(c))*b*c**2*d**3*x**5 + 30*sqrt(- c + d*x)*atan((sqrt(- c + d*x) + sqrt(c + d*x) + sqrt(c))/sqrt(c))*a*c*d**4*x**4 + 30*sqrt(- c + d*x)*atan((sqrt(- c + d*x) + sqrt(c + d*x) + sqrt(c))/sqrt(c))*a*d**5*x**5 + 24*sqrt(- c + d*x)*atan((sqrt(- c + d*x) + sqrt(c + d*x) + sqrt(c))/sqrt(c))*b*c**3*d**2*x**4 + 24*sqrt(- c + d*x)*atan((sqrt(- c + d*x) + sqrt(c + d*x) + sqrt(c))/sqrt(c))*b*c**2*d**3*x**5 + 2*sqrt(c + d*x)*a*c**5 + 5*sqrt(c + d*x)*a*c**3*d**2*x**2 - 15*sqrt(c + d*x)*a*c*d**4*x**4 + 4*sqrt(c + d*x)*b*c**5*x**2 - 12*sqrt(c + d*x)*b*c**3*d**2*x**4)/(8*sqrt(- c + d*x)*c**7*x**4*(c + d*x))`

$$3.42 \quad \int \frac{1+c^2x^2}{x\sqrt{-1+cx}\sqrt{1+cx}} dx$$

Optimal result	326
Mathematica [A] (warning: unable to verify)	326
Rubi [A] (verified)	327
Maple [A] (verified)	328
Fricas [A] (verification not implemented)	328
Sympy [C] (verification not implemented)	329
Maxima [A] (verification not implemented)	330
Giac [A] (verification not implemented)	330
Mupad [B] (verification not implemented)	330
Reduce [B] (verification not implemented)	331

Optimal result

Integrand size = 31, antiderivative size = 40

$$\int \frac{1+c^2x^2}{x\sqrt{-1+cx}\sqrt{1+cx}} dx = \sqrt{-1+cx}\sqrt{1+cx} + \arctan\left(\sqrt{-1+cx}\sqrt{1+cx}\right)$$

output $(c*x-1)^{(1/2)}*(c*x+1)^{(1/2)}+\arctan((c*x-1)^{(1/2)}*(c*x+1)^{(1/2)})$

Mathematica [A] (warning: unable to verify)

Time = 0.07 (sec) , antiderivative size = 40, normalized size of antiderivative = 1.00

$$\int \frac{1+c^2x^2}{x\sqrt{-1+cx}\sqrt{1+cx}} dx = \sqrt{-1+cx}\sqrt{1+cx} + 2 \arctan\left(\sqrt{\frac{-1+cx}{1+cx}}\right)$$

input $\text{Integrate}[(1+c^2*x^2)/(x*\text{Sqrt}[-1+c*x]*\text{Sqrt}[1+c*x]),x]$

output $\text{Sqrt}[-1+c*x]*\text{Sqrt}[1+c*x] + 2*\text{ArcTan}[\text{Sqrt}[(-1+c*x)/(1+c*x)]]$

Rubi [A] (verified)

Time = 0.32 (sec) , antiderivative size = 40, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.097$, Rules used = {960, 103, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{c^2 x^2 + 1}{x \sqrt{cx - 1} \sqrt{cx + 1}} dx$$

↓ 960

$$\int \frac{1}{x \sqrt{cx - 1} \sqrt{cx + 1}} dx + \sqrt{cx - 1} \sqrt{cx + 1}$$

↓ 103

$$c \int \frac{1}{(cx - 1)(cx + 1)c + c} d(\sqrt{cx - 1} \sqrt{cx + 1}) + \sqrt{cx - 1} \sqrt{cx + 1}$$

↓ 218

$$\arctan(\sqrt{cx - 1} \sqrt{cx + 1}) + \sqrt{cx - 1} \sqrt{cx + 1}$$

input `Int[(1 + c^2*x^2)/(x*Sqrt[-1 + c*x]*Sqrt[1 + c*x]),x]`

output `Sqrt[-1 + c*x]*Sqrt[1 + c*x] + ArcTan[Sqrt[-1 + c*x]*Sqrt[1 + c*x]]`

Defintions of rubi rules used

rule 103 `Int[1/(Sqrt[(a_.) + (b_.)*(x_)]*Sqrt[(c_.) + (d_.)*(x_)]*((e_.) + (f_.)*(x_))), x_] := Simp[b*f Subst[Int[1/(d*(b*e - a*f)^2 + b*f^2*x^2), x], x, Sqrt[a + b*x]*Sqrt[c + d*x]], x] /; FreeQ[{a, b, c, d, e, f}, x] && EqQ[2*b*d *e - f*(b*c + a*d), 0]`

rule 218 `Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(Rt[a/b, 2]/a)*ArcTan[x/Rt[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]`

rule 960

```
Int[((e_.)*(x_))^(m_.)*((a1_) + (b1_.)*(x_)^(non2_.))^(p_.)*((a2_) + (b2_.)
*(x_)^(non2_.))^(p_.)*((c_) + (d_.)*(x_)^(n_)), x_Symbol] := Simp[d*(e*x)^(
m + 1)*(a1 + b1*x^(n/2))^(p + 1)*((a2 + b2*x^(n/2))^(p + 1)/(b1*b2*e*(m + n
*(p + 1) + 1))), x] - Simp[(a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1))/
(b1*b2*(m + n*(p + 1) + 1)) Int[(e*x)^m*(a1 + b1*x^(n/2))^p*(a2 + b2*x^(n
/2))^p, x], x] /; FreeQ[{a1, b1, a2, b2, c, d, e, m, n, p}, x] && EqQ[non2,
n/2] && EqQ[a2*b1 + a1*b2, 0] && NeQ[m + n*(p + 1) + 1, 0]
```

Maple [A] (verified)

Time = 0.11 (sec) , antiderivative size = 53, normalized size of antiderivative = 1.32

method	result	size
default	$\frac{\sqrt{cx-1}\sqrt{cx+1}\left(\sqrt{c^2x^2-1}-\arctan\left(\frac{1}{\sqrt{c^2x^2-1}}\right)\right)}{\sqrt{c^2x^2-1}}$	53

input

```
int((c^2*x^2+1)/x/(c*x-1)^(1/2)/(c*x+1)^(1/2),x,method=_RETURNVERBOSE)
```

output

```
(c*x-1)^(1/2)*(c*x+1)^(1/2)/(c^2*x^2-1)^(1/2)*((c^2*x^2-1)^(1/2)-arctan(1/
(c^2*x^2-1)^(1/2)))
```

Fricas [A] (verification not implemented)

Time = 0.08 (sec) , antiderivative size = 39, normalized size of antiderivative = 0.98

$$\int \frac{1 + c^2x^2}{x\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \sqrt{cx + 1}\sqrt{cx - 1} + 2 \arctan\left(-cx + \sqrt{cx + 1}\sqrt{cx - 1}\right)$$

input

```
integrate((c^2*x^2+1)/x/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="fricas")
```

output

```
sqrt(c*x + 1)*sqrt(c*x - 1) + 2*arctan(-c*x + sqrt(c*x + 1)*sqrt(c*x - 1))
```

Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 25.46 (sec) , antiderivative size = 148, normalized size of antiderivative = 3.70

$$\int \frac{1 + c^2 x^2}{x\sqrt{-1 + cx}\sqrt{1 + cx}} dx = \frac{G_{6,6}^{6,2} \left(\begin{matrix} -\frac{1}{4}, \frac{1}{4} & 0, 0, \frac{1}{2}, 1 \\ -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, 0 \end{matrix} \middle| \frac{1}{c^2 x^2} \right)}{4\pi^{\frac{3}{2}}} - \frac{G_{6,6}^{5,3} \left(\begin{matrix} \frac{3}{4}, \frac{5}{4}, 1 & 1, 1, \frac{3}{2} \\ \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2} & 0 \end{matrix} \middle| \frac{1}{c^2 x^2} \right)}{4\pi^{\frac{3}{2}}} + \frac{iG_{6,6}^{2,6} \left(\begin{matrix} -1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, 1 \\ -\frac{3}{4}, -\frac{1}{4} & -1, -\frac{1}{2}, -\frac{1}{2}, 0 \end{matrix} \middle| \frac{e^{2i\pi}}{c^2 x^2} \right)}{4\pi^{\frac{3}{2}}} + \frac{iG_{6,6}^{2,6} \left(\begin{matrix} 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 1 \\ \frac{1}{4}, \frac{3}{4} & 0, \frac{1}{2}, \frac{1}{2}, 0 \end{matrix} \middle| \frac{e^{2i\pi}}{c^2 x^2} \right)}{4\pi^{\frac{3}{2}}}$$

input `integrate((c**2*x**2+1)/x/(c*x-1)**(1/2)/(c*x+1)**(1/2),x)`

output `meijerg((-1/4, 1/4), (0, 0, 1/2, 1)), ((-1/2, -1/4, 0, 1/4, 1/2, 0), ()), 1/(c**2*x**2))/(4*pi**(3/2)) - meijerg((3/4, 5/4, 1), (1, 1, 3/2)), ((1/2, 3/4, 1, 5/4, 3/2), (0,)), 1/(c**2*x**2))/(4*pi**(3/2)) + I*meijerg((-1, -3/4, -1/2, -1/4, 0, 1), ()), ((-3/4, -1/4), (-1, -1/2, -1/2, 0)), exp_polar(2*I*pi)/(c**2*x**2))/(4*pi**(3/2)) + I*meijerg((0, 1/4, 1/2, 3/4, 1, 1), ()), ((1/4, 3/4), (0, 1/2, 1/2, 0)), exp_polar(2*I*pi)/(c**2*x**2))/(4*pi**(3/2))`

Maxima [A] (verification not implemented)

Time = 0.11 (sec) , antiderivative size = 23, normalized size of antiderivative = 0.58

$$\int \frac{1 + c^2 x^2}{x \sqrt{-1 + cx} \sqrt{1 + cx}} dx = \sqrt{c^2 x^2 - 1} - \arcsin\left(\frac{1}{c|x|}\right)$$

input `integrate((c^2*x^2+1)/x/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="maxima")`output `sqrt(c^2*x^2 - 1) - arcsin(1/(c*abs(x)))`**Giac [A] (verification not implemented)**

Time = 0.13 (sec) , antiderivative size = 40, normalized size of antiderivative = 1.00

$$\int \frac{1 + c^2 x^2}{x \sqrt{-1 + cx} \sqrt{1 + cx}} dx = \sqrt{cx + 1} \sqrt{cx - 1} - 2 \arctan\left(\frac{1}{2} (\sqrt{cx + 1} - \sqrt{cx - 1})^2\right)$$

input `integrate((c^2*x^2+1)/x/(c*x-1)^(1/2)/(c*x+1)^(1/2),x, algorithm="giac")`output `sqrt(c*x + 1)*sqrt(c*x - 1) - 2*arctan(1/2*(sqrt(c*x + 1) - sqrt(c*x - 1))^2)`**Mupad [B] (verification not implemented)**

Time = 5.92 (sec) , antiderivative size = 72, normalized size of antiderivative = 1.80

$$\int \frac{1 + c^2 x^2}{x \sqrt{-1 + cx} \sqrt{1 + cx}} dx = \sqrt{cx - 1} \sqrt{cx + 1} - \ln\left(\frac{(\sqrt{cx - 1} - i)^2}{(\sqrt{cx + 1} - 1)^2} + 1\right) \text{ li} \\ + \ln\left(\frac{\sqrt{cx - 1} - i}{\sqrt{cx + 1} - 1}\right) \text{ li}$$

input `int((c^2*x^2 + 1)/(x*(c*x - 1)^(1/2)*(c*x + 1)^(1/2)),x)`

output

```
log(((c*x - 1)^(1/2) - 1i)/((c*x + 1)^(1/2) - 1))*1i - log(((c*x - 1)^(1/2)
) - 1i)^2/((c*x + 1)^(1/2) - 1)^2 + 1)*1i + (c*x - 1)^(1/2)*(c*x + 1)^(1/2
)
```

Reduce [B] (verification not implemented)

Time = 0.22 (sec) , antiderivative size = 48, normalized size of antiderivative = 1.20

$$\int \frac{1 + c^2 x^2}{x \sqrt{-1 + cx} \sqrt{1 + cx}} dx = 2 \operatorname{atan}\left(\sqrt{cx - 1} + \sqrt{cx + 1} - 1\right) - 2 \operatorname{atan}\left(\sqrt{cx - 1} + \sqrt{cx + 1} + 1\right) + \sqrt{cx + 1} \sqrt{cx - 1}$$

input

```
int((c^2*x^2+1)/x/(c*x-1)^(1/2)/(c*x+1)^(1/2),x)
```

output

```
2*atan(sqrt(c*x - 1) + sqrt(c*x + 1) - 1) - 2*atan(sqrt(c*x - 1) + sqrt(c*
x + 1) + 1) + sqrt(c*x + 1)*sqrt(c*x - 1)
```

3.43
$$\int x \frac{x^{-\frac{2b^2c+a^2d}{b^2c+a^2d}} (c+dx^2)}{\sqrt{-a+bx}\sqrt{a+bx}} dx$$

Optimal result	332
Mathematica [C] (verified)	332
Rubi [A] (verified)	333
Maple [A] (verified)	334
Fricas [A] (verification not implemented)	334
Sympy [C] (verification not implemented)	335
Maxima [A] (verification not implemented)	336
Giac [F]	336
Mupad [B] (verification not implemented)	336
Reduce [B] (verification not implemented)	337

Optimal result

Integrand size = 57, antiderivative size = 53

$$\int \frac{x^{-\frac{2b^2c+a^2d}{b^2c+a^2d}} (c+dx^2)}{\sqrt{-a+bx}\sqrt{a+bx}} dx = \left(\frac{c}{a^2} + \frac{d}{b^2}\right) x^{-\frac{b^2c}{b^2c+a^2d}} \sqrt{-a+bx}\sqrt{a+bx}$$

output (c/a^2+d/b^2)*(b*x-a)^(1/2)*(b*x+a)^(1/2)/(x^(b^2*c/(a^2*d+b^2*c)))

Mathematica [C] (verified)

Result contains higher order function than in optimal. Order 5 vs. order 3 in optimal.

Time = 10.22 (sec) , antiderivative size = 244, normalized size of antiderivative = 4.60

$$\int \frac{x^{-\frac{2b^2c+a^2d}{b^2c+a^2d}} (c+dx^2)}{\sqrt{-a+bx}\sqrt{a+bx}} dx = \frac{(b^2c+a^2d) x^{-\frac{b^2c}{b^2c+a^2d}} \sqrt{1-\frac{b^2x^2}{a^2}} \left(-\left((b^2c+2a^2d) \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, -\frac{b^2c}{2(b^2c+a^2d)}, \frac{b^2c+2a^2d}{2b^2c+2a^2d}, \frac{b^2x^2}{a^2}\right)\right) + \right)}{b^2(b^2c+2a^2d)\sqrt{-a+bx}\sqrt{a+bx}}$$

input Integrate[(c + d*x^2)/(x^((2*b^2*c + a^2*d)/(b^2*c + a^2*d))*Sqrt[-a + b*x] * Sqrt[a + b*x]), x]

output

```
((b^2*c + a^2*d)*Sqrt[1 - (b^2*x^2)/a^2]*(-(b^2*c + 2*a^2*d)*Hypergeometric2F1[1/2, -1/2*(b^2*c)/(b^2*c + a^2*d), (b^2*c + 2*a^2*d)/(2*b^2*c + 2*a^2*d), (b^2*x^2)/a^2]) + b^2*d*x^2*Hypergeometric2F1[1/2, (b^2*c + 2*a^2*d)/(2*b^2*c + 2*a^2*d), (3*b^2*c + 4*a^2*d)/(2*b^2*c + 2*a^2*d), (b^2*x^2)/a^2]))/(b^2*(b^2*c + 2*a^2*d)*x^((b^2*c)/(b^2*c + a^2*d))*Sqrt[-a + b*x]*Sqrt[a + b*x])
```

Rubi [A] (verified)

Time = 0.36 (sec) , antiderivative size = 53, normalized size of antiderivative = 1.00, number of steps used = 1, number of rules used = 1, $\frac{\text{number of rules}}{\text{integrand size}} = 0.018$, Rules used = {952}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(c + dx^2) x^{-\frac{a^2 d + 2b^2 c}{a^2 d + b^2 c}}}{\sqrt{bx - a}\sqrt{a + bx}} dx$$

↓ 952

$$\sqrt{bx - a}\sqrt{a + bx} \left(\frac{c}{a^2} + \frac{d}{b^2} \right) x^{-\frac{b^2 c}{a^2 d + b^2 c}}$$

input

```
Int[(c + d*x^2)/(x^((2*b^2*c + a^2*d)/(b^2*c + a^2*d))*Sqrt[-a + b*x]*Sqrt[a + b*x]),x]
```

output

```
((c/a^2 + d/b^2)*Sqrt[-a + b*x]*Sqrt[a + b*x])/x^((b^2*c)/(b^2*c + a^2*d))
```

Defintions of rubi rules used

rule 952

```
Int[((e_.)*(x_))^(m_.)*((a1_) + (b1_.)*(x_)^(non2_.))^(p_.)*((a2_) + (b2_.)
*(x_)^(non2_.))^(p_.)*((c_) + (d_.)*(x_)^(n_)), x_Symbol] := Simp[c*(e*x)^(
m + 1)*(a1 + b1*x^(n/2))^(p + 1)*((a2 + b2*x^(n/2))^(p + 1)/(a1*a2*e*(m + 1
))), x] /; FreeQ[{a1, b1, a2, b2, c, d, e, m, n, p}, x] && EqQ[non2, n/2] &
& EqQ[a2*b1 + a1*b2, 0] && EqQ[a1*a2*d*(m + 1) - b1*b2*c*(m + n*(p + 1) + 1
), 0] && NeQ[m, -1]
```

Maple [A] (verified)

Time = 0.20 (sec) , antiderivative size = 66, normalized size of antiderivative = 1.25

method	result	size
gosper	$\frac{x\sqrt{bx-a}\sqrt{bx+a}(a^2d+b^2c)x^{-\frac{a^2d+2b^2c}{a^2d+b^2c}}}{a^2b^2}$	66
orering	$-\frac{x(a^2d+b^2c)\sqrt{bx+a}(-bx+a)x^{-\frac{a^2d+2b^2c}{a^2d+b^2c}}}{a^2b^2\sqrt{bx-a}}$	73

input

```
int((d*x^2+c)/(x^((a^2*d+2*b^2*c)/(a^2*d+b^2*c)))/(b*x-a)^(1/2)/(b*x+a)^(1
/2),x,method=_RETURNVERBOSE)
```

output

```
x/a^2/b^2*(b*x-a)^(1/2)*(b*x+a)^(1/2)*(a^2*d+b^2*c)/(x^((a^2*d+2*b^2*c)/(a
^2*d+b^2*c)))
```

Fricas [A] (verification not implemented)

Time = 0.13 (sec) , antiderivative size = 65, normalized size of antiderivative = 1.23

$$\int x^{-\frac{2b^2c+a^2d}{b^2c+a^2d}}(c+dx^2) \frac{dx}{\sqrt{-a+bx}\sqrt{a+bx}} = \frac{(b^2c+a^2d)\sqrt{bx+a}\sqrt{bx-ax}}{a^2b^2x^{\frac{2b^2c+a^2d}{b^2c+a^2d}}}$$

input

```
integrate((d*x^2+c)/(x^((a^2*d+2*b^2*c)/(a^2*d+b^2*c)))/(b*x-a)^(1/2)/(b*x
+a)^(1/2),x, algorithm="fricas")
```

output

```
(b^2*c + a^2*d)*sqrt(b*x + a)*sqrt(b*x - a)*x/(a^2*b^2*x^((2*b^2*c + a^2*d)/(b^2*c + a^2*d)))
```

Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 63.55 (sec) , antiderivative size = 1867, normalized size of antiderivative = 35.23

$$\int x \frac{-\frac{2b^2c+a^2d}{b^2c+a^2d}(c+dx^2)}{\sqrt{-a+bx}\sqrt{a+bx}} dx = \text{Too large to display}$$

input

```
integrate((d*x**2+c)/(x**((a**2*d+2*b**2*c)/(a**2*d+b**2*c)))/(b*x-a)**(1/2)/(b*x+a)**(1/2),x)
```

output

```
-a**(-a**2*d/(a**2*d + b**2*c) - 2*b**2*c/(a**2*d + b**2*c))*b**(a**2*d/(a**2*d + b**2*c) + 2*b**2*c/(a**2*d + b**2*c) - 1)*c*meijerg(((a**2*d/(2*a**2*d + 2*b**2*c) + b**2*c/(a**2*d + b**2*c) + 1/4, a**2*d/(2*a**2*d + 2*b**2*c) + b**2*c/(a**2*d + b**2*c) + 3/4, 1), (a**2*d/(2*a**2*d + 2*b**2*c) + b**2*c/(a**2*d + b**2*c) + 1/2, a**2*d/(2*a**2*d + 2*b**2*c) + b**2*c/(a**2*d + b**2*c) + 1/2, a**2*d/(2*a**2*d + 2*b**2*c) + b**2*c/(a**2*d + b**2*c) + 1)), ((a**2*d/(2*a**2*d + 2*b**2*c) + b**2*c/(a**2*d + b**2*c), a**2*d/(2*a**2*d + 2*b**2*c) + b**2*c/(a**2*d + b**2*c) + 1/4, a**2*d/(2*a**2*d + 2*b**2*c) + b**2*c/(a**2*d + b**2*c) + 1/2, a**2*d/(2*a**2*d + 2*b**2*c) + b**2*c/(a**2*d + b**2*c) + 3/4, a**2*d/(2*a**2*d + 2*b**2*c) + b**2*c/(a**2*d + b**2*c) + 1), (0,)), a**2/(b**2*x**2))/(4*pi**(3/2)) - I*a**(-a**2*d/(a**2*d + b**2*c) - 2*b**2*c/(a**2*d + b**2*c))*b**(a**2*d/(a**2*d + b**2*c) + 2*b**2*c/(a**2*d + b**2*c) - 1)*c*meijerg(((a**2*d/(2*a**2*d + 2*b**2*c) + b**2*c/(a**2*d + b**2*c) - 1/2, a**2*d/(2*a**2*d + 2*b**2*c) + b**2*c/(a**2*d + b**2*c) - 1/4, a**2*d/(2*a**2*d + 2*b**2*c) + b**2*c/(a**2*d + b**2*c) + 1/4, a**2*d/(2*a**2*d + 2*b**2*c) + b**2*c/(a**2*d + b**2*c) + 1/2, 1), ()), ((a**2*d/(2*a**2*d + 2*b**2*c) + b**2*c/(a**2*d + b**2*c) - 1/4, a**2*d/(2*a**2*d + 2*b**2*c) + b**2*c/(a**2*d + b**2*c) + 1/4), (a**2*d/(2*a**2*d + 2*b**2*c) + b**2*c/(a**2*d + b**2*c) - 1/2, a**2*d/(2*a**2*d + 2*b...
```


Maxima [A] (verification not implemented)

Time = 0.12 (sec) , antiderivative size = 79, normalized size of antiderivative = 1.49

$$\int \frac{x^{-\frac{2b^2c+a^2d}{b^2c+a^2d}}(c+dx^2)}{\sqrt{-a+bx}\sqrt{a+bx}} dx = \frac{(b^2c+a^2d)\sqrt{bx+a}\sqrt{bx-ax}e^{\left(-\frac{2b^2c\log(x)}{b^2c+a^2d}-\frac{a^2d\log(x)}{b^2c+a^2d}\right)}}{a^2b^2}$$

input `integrate((d*x^2+c)/(x^((a^2*d+2*b^2*c)/(a^2*d+b^2*c)))/(b*x-a)^(1/2)/(b*x+a)^(1/2),x, algorithm="maxima")`

output `(b^2*c + a^2*d)*sqrt(b*x + a)*sqrt(b*x - a)*x*e^(-2*b^2*c*log(x)/(b^2*c + a^2*d) - a^2*d*log(x)/(b^2*c + a^2*d))/(a^2*b^2)`

Giac [F]

$$\int \frac{x^{-\frac{2b^2c+a^2d}{b^2c+a^2d}}(c+dx^2)}{\sqrt{-a+bx}\sqrt{a+bx}} dx = \int \frac{dx^2 + c}{\sqrt{bx+a}\sqrt{bx-ax} \frac{2b^2c+a^2d}{b^2c+a^2d}} dx$$

input `integrate((d*x^2+c)/(x^((a^2*d+2*b^2*c)/(a^2*d+b^2*c)))/(b*x-a)^(1/2)/(b*x+a)^(1/2),x, algorithm="giac")`

output `integrate((d*x^2 + c)/(sqrt(b*x + a)*sqrt(b*x - a)*x^((2*b^2*c + a^2*d)/(b^2*c + a^2*d))), x)`

Mupad [B] (verification not implemented)

Time = 5.48 (sec) , antiderivative size = 96, normalized size of antiderivative = 1.81

$$\int \frac{x^{-\frac{2b^2c+a^2d}{b^2c+a^2d}}(c+dx^2)}{\sqrt{-a+bx}\sqrt{a+bx}} dx = -\frac{x \frac{(da^4+ca^2b^2)}{a^2b^2} - x^3 \frac{(da^2b^2+cb^4)}{a^2b^2}}{x \frac{da^2+2cb^2}{da^2+cb^2} \sqrt{a+bx} \sqrt{bx-a}}$$

input `int((c + d*x^2)/(x^((a^2*d + 2*b^2*c)/(a^2*d + b^2*c)))*(a + b*x)^(1/2)*(b*x - a)^(1/2)),x)`

output
$$-\left(\frac{x(a^4d + a^2b^2c)}{a^2b^2} - \frac{x^3(b^4c + a^2b^2d)}{a^2b^2}\right) / \left(\frac{x(a^2d + 2b^2c)}{a^2d + b^2c}\right) (a + bx)^{1/2} (bx - a)^{1/2}$$

Reduce [B] (verification not implemented)

Time = 0.38 (sec) , antiderivative size = 54, normalized size of antiderivative = 1.02

$$\int \frac{x^{-\frac{2b^2c+a^2d}{b^2c+a^2d}}(c+dx^2)}{\sqrt{-a+bx}\sqrt{a+bx}} dx = \frac{\sqrt{bx+a}\sqrt{bx-a}(a^2d+b^2c)}{x^{\frac{b^2c}{a^2d+b^2c}}a^2b^2}$$

input
$$\text{int}\left(\frac{dx^2+c}{x^{\frac{2b^2c+a^2d}{b^2c+a^2d}}}\right) / \left(\frac{a^2d+2b^2c}{a^2d+b^2c}\right) / (bx-a)^{1/2} / (bx+a)^{1/2}, x$$

output
$$(\text{sqrt}(a + bx) * \text{sqrt}(-a + bx) * (a^2d + b^2c)) / (x^{\frac{b^2c}{a^2d + b^2c}} * a^2 * b^2)$$

CHAPTER 4

APPENDIX

4.1	Listing of Grading functions	338
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4.1 Listing of Grading functions

The following are the current version of the grading functions used for grading the quality of the antiderivative with reference to the optimal antiderivative included in the test suite.

There is a version for Maple and for Mathematica/Rubi. There is a version for grading Sympy and version for use with Sagemath.

The following are links to the current source code.

The following are the listings of source code of the grading functions.

Mathematica and Rubi grading function

```
(* Original version thanks to Albert Rich emailed on 03/21/2017 *)
(* ::Package:: *)

(* Nasser: April 7,2022. add second output which gives reason for the grade *)
(*                               Small rewrite of logic in main function to make it*)
(*                               match Maple's logic. No change in functionality otherwise*)

(* ::Subsection:: *)
(*GradeAntiderivative[result,optimal]*)
```

```

(* ::Text:: *)
(*If result and optimal are mathematical expressions, *)
(*      GradeAntiderivative[result,optimal] returns*)
(* "F" if the result fails to integrate an expression that*)
(*      is integrable*)
(* "C" if result involves higher level functions than necessary*)
(* "B" if result is more than twice the size of the optimal*)
(*      antiderivative*)
(* "A" if result can be considered optimal*)

GradeAntiderivative[result_,optimal_] := Module[{expnResult,expnOptimal,leafCountResult,leafCountOptimal},
  expnResult = ExpnType[result];
  expnOptimal = ExpnType[optimal];
  leafCountResult = LeafCount[result];
  leafCountOptimal = LeafCount[optimal];

  (*Print["expnResult=",expnResult," expnOptimal=",expnOptimal];*)
  If[expnResult<=expnOptimal,
    If[Not[FreeQ[result,Complex]], (*result contains complex*)
      If[Not[FreeQ[optimal,Complex]], (*optimal contains complex*)
        If[leafCountResult<=2*leafCountOptimal,
          finalresult={"A",""}
          ,(*ELSE*)
          finalresult={"B","Both result and optimal contain complex but leaf count
          ]
        ,(*ELSE*)
        finalresult={"C","Result contains complex when optimal does not."}
      ]
    ,(*ELSE*)(*result does not contains complex*)
    If[leafCountResult<=2*leafCountOptimal,
      finalresult={"A",""}
      ,(*ELSE*)
      finalresult={"B","Leaf count is larger than twice the leaf count of optimal.
    ]
  ]
  ,(*ELSE*)(*expnResult>expnOptimal*)
  If[FreeQ[result,Integrate] && FreeQ[result,Int],
    finalresult={"C","Result contains higher order function than in optimal. Order "
  ,

```

```

        finalresult={"F","Contains unresolved integral."}
    ]
];

finalresult
]

(* ::Text:: *)
(*The following summarizes the type number assigned an *)
(*expression based on the functions it involves*)
(*1 = rational function*)
(*2 = algebraic function*)
(*3 = elementary function*)
(*4 = special function*)
(*5 = hyperpergeometric function*)
(*6 = appell function*)
(*7 = rootsum function*)
(*8 = integrate function*)
(*9 = unknown function*)

ExpnType[expn_] :=
  If[AtomQ[expn],
    1,
  If[ListQ[expn],
    Max[Map[ExpnType, expn]],
  If[Head[expn]===Power,
    If[IntegerQ[expn[[2]]],
      ExpnType[expn[[1]]],
    If[Head[expn[[2]]]===Rational,
      If[IntegerQ[expn[[1]]] || Head[expn[[1]]]===Rational,
        1,
        Max[ExpnType[expn[[1]], 2]],
      Max[ExpnType[expn[[1]], ExpnType[expn[[2]], 3]],
    If[Head[expn]===Plus || Head[expn]===Times,
      Max[ExpnType[First[expn]], ExpnType[Rest[expn]]],
    If[ElementaryFunctionQ[Head[expn]],
      Max[3, ExpnType[expn[[1]]]],
    If[SpecialFunctionQ[Head[expn]],
      Apply[Max, Append[Map[ExpnType, Apply[List, expn]], 4]],
    If[HypergeometricFunctionQ[Head[expn]],

```

```

    Apply [Max, Append [Map [ExpnType, Apply [List, expn]], 5]],
    If [AppellFunctionQ [Head [expn]],
        Apply [Max, Append [Map [ExpnType, Apply [List, expn]], 6]],
        If [Head [expn] === RootSum,
            Apply [Max, Append [Map [ExpnType, Apply [List, expn]], 7]],
            If [Head [expn] === Integrate || Head [expn] === Int,
                Apply [Max, Append [Map [ExpnType, Apply [List, expn]], 8]],
                9]]]]]]]]]]]]

```

```

ElementaryFunctionQ [func_] :=
  MemberQ [{
    Exp, Log,
    Sin, Cos, Tan, Cot, Sec, Csc,
    ArcSin, ArcCos, ArcTan, ArcCot, ArcSec, ArcCsc,
    Sinh, Cosh, Tanh, Coth, Sech, Csch,
    ArcSinh, ArcCosh, ArcTanh, ArcCoth, ArcSech, ArcCsch
  }, func]

```

```

SpecialFunctionQ [func_] :=
  MemberQ [{
    Erf, Erfc, Erfi,
    FresnelS, FresnelC,
    ExpIntegralE, ExpIntegralEi, LogIntegral,
    SinIntegral, CosIntegral, SinhIntegral, CoshIntegral,
    Gamma, LogGamma, PolyGamma,
    Zeta, PolyLog, ProductLog,
    EllipticF, EllipticE, EllipticPi
  }, func]

```

```

HypergeometricFunctionQ [func_] :=
  MemberQ [{Hypergeometric1F1, Hypergeometric2F1, HypergeometricPFQ}, func]

```

```

AppellFunctionQ [func_] :=
  MemberQ [{AppellF1}, func]

```

Maple grading function

```

# File: GradeAntiderivative.mpl
# Original version thanks to Albert Rich emailed on 03/21/2017

#Nasser 03/22/2017 Use Maple leaf count instead since buildin
#Nasser 03/23/2017 missing 'ln' for ElementaryFunctionQ added
#Nasser 03/24/2017 corrected the check for complex result
#Nasser 10/27/2017 check for leafsize and do not call ExpnType()
#
#                   if leaf size is "too large". Set at 500,000
#Nasser 12/22/2019 Added debug flag, added 'dilog' to special functions
#
#                   see problem 156, file Apostol_Problems
#Nasser 4/07/2022 add second output which gives reason for the grade

GradeAntiderivative := proc(result,optimal)
local leaf_count_result,
      leaf_count_optimal,
      ExpnType_result,
      ExpnType_optimal,
      debug:=false;

      leaf_count_result:=leafcount(result);
      #do NOT call ExpnType() if leaf size is too large. Recursion problem
      if leaf_count_result > 500000 then
          return "B","result has leaf size over 500,000. Avoiding possible recursion issue";
      fi;

      leaf_count_optimal := leafcount(optimal);
      ExpnType_result := ExpnType(result);
      ExpnType_optimal := ExpnType(optimal);

      if debug then
          print("ExpnType_result",ExpnType_result," ExpnType_optimal=",ExpnType_optimal);
      fi;

# If result and optimal are mathematical expressions,
# GradeAntiderivative[result,optimal] returns
# "F" if the result fails to integrate an expression that
# is integrable
# "C" if result involves higher level functions than necessary
# "B" if result is more than twice the size of the optimal

```

```

#   antiderivative
#   "A" if result can be considered optimal

#This check below actually is not needed, since I only
#call this grading only for passed integrals. i.e. I check
#for "F" before calling this. But no harm of keeping it here.
#just in case.

if not type(result,freeof('int')) then
    return "F","Result contains unresolved integral";
fi;

if ExpnType_result<=ExpnType_optimal then
    if debug then
        print("ExpnType_result<=ExpnType_optimal");
    fi;
    if is_contains_complex(result) then
        if is_contains_complex(optimal) then
            if debug then
                print("both result and optimal complex");
            fi;
            if leaf_count_result<=2*leaf_count_optimal then
                return "A"," ";
            else
                return "B",cat("Both result and optimal contain complex but leaf count of
                                convert(leaf_count_result,string)," vs. $2 (" ,
                                convert(leaf_count_optimal,string)," ) = ",convert(2*leaf

            end if
        else #result contains complex but optimal is not
            if debug then
                print("result contains complex but optimal is not");
            fi;
            return "C","Result contains complex when optimal does not.";
        fi;
    else # result do not contain complex
        # this assumes optimal do not as well. No check is needed here.
        if debug then
            print("result do not contain complex, this assumes optimal do not as well
        fi;

```



```

        if leaf_count_result<=2*leaf_count_optimal then
            if debug then
                print("leaf_count_result<=2*leaf_count_optimal");
            fi;
            return "A"," ";
        else
            if debug then
                print("leaf_count_result>2*leaf_count_optimal");
            fi;
            return "B",cat("Leaf count of result is larger than twice the leaf count of
                            convert(leaf_count_result,string)," $ vs. $2(",
                            convert(leaf_count_optimal,string),")=",convert(2*leaf_co
            fi;
        fi;
    else #ExpnType(result) > ExpnType(optimal)
        if debug then
            print("ExpnType(result) > ExpnType(optimal)");
        fi;
        return "C",cat("Result contains higher order function than in optimal. Order ",
                        convert(ExpnType_result,string)," vs. order ",
                        convert(ExpnType_optimal,string),".");
    fi;
end proc:

#
# is_contains_complex(result)
# takes expressions and returns true if it contains "I" else false
#
#Nasser 032417
is_contains_complex:= proc(expression)
    return (has(expression,I));
end proc:

# The following summarizes the type number assigned an expression
# based on the functions it involves
# 1 = rational function
# 2 = algebraic function
# 3 = elementary function
# 4 = special function
# 5 = hyperpergeometric function

```

```

# 6 = appell function
# 7 = rootsum function
# 8 = integrate function
# 9 = unknown function

ExpnType := proc(expn)
  if type(expn,'atomic') then
    1
  elif type(expn,'list') then
    apply(max,map(ExpnType,expn))
  elif type(expn,'sqrt') then
    if type(op(1,expn),'rational') then
      1
    else
      max(2,ExpnType(op(1,expn)))
    end if
  elif type(expn,'^^') then
    if type(op(2,expn),'integer') then
      ExpnType(op(1,expn))
    elif type(op(2,expn),'rational') then
      if type(op(1,expn),'rational') then
        1
      else
        max(2,ExpnType(op(1,expn)))
      end if
    else
      max(3,ExpnType(op(1,expn)),ExpnType(op(2,expn)))
    end if
  elif type(expn,'+'') or type(expn,'*') then
    max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
  elif ElementaryFunctionQ(op(0,expn)) then
    max(3,ExpnType(op(1,expn)))
  elif SpecialFunctionQ(op(0,expn)) then
    max(4,apply(max,map(ExpnType,[op(expn)])))
  elif HypergeometricFunctionQ(op(0,expn)) then
    max(5,apply(max,map(ExpnType,[op(expn)])))
  elif AppellFunctionQ(op(0,expn)) then
    max(6,apply(max,map(ExpnType,[op(expn)])))
  elif op(0,expn)='int' then
    max(8,apply(max,map(ExpnType,[op(expn)]))) else
  9

```

```

    end if
end proc:

ElementaryFunctionQ := proc(func)
  member(func, [
    exp, log, ln,
    sin, cos, tan, cot, sec, csc,
    arcsin, arccos, arctan, arccot, arcsec, arccsc,
    sinh, cosh, tanh, coth, sech, csch,
    arcsinh, arccosh, arctanh, arccoth, arcsech, arccsch])
end proc:

SpecialFunctionQ := proc(func)
  member(func, [
    erf, erfc, erfi,
    FresnelS, FresnelC,
    Ei, Ei, Li, Si, Ci, Shi, Chi,
    GAMMA, lnGAMMA, Psi, Zeta, polylog, dilog, LambertW,
    EllipticF, EllipticE, EllipticPi])
end proc:

HypergeometricFunctionQ := proc(func)
  member(func, [Hypergeometric1F1, hypergeom, HypergeometricPFQ])
end proc:

AppellFunctionQ := proc(func)
  member(func, [AppellF1])
end proc:

# u is a sum or product. rest(u) returns all but the
# first term or factor of u.
rest := proc(u) local v;
  if nops(u)=2 then
    op(2,u)
  else
    apply(op(0,u), op(2..nops(u),u))
  end if
end proc:

#leafcount(u) returns the number of nodes in u.

```

```
#Nasser 3/23/17 Replaced by build-in leafCount from package in Maple
leafcount := proc(u)
  MmaTranslator[Mma][LeafCount](u);
end proc;
```

Sympy grading function

```
#Dec 24, 2019. Nasser M. Abbasi:
#      Port of original Maple grading function by
#      Albert Rich to use with Sympy/Python
#Dec 27, 2019 Nasser. Added `RootSum`. See problem 177, Timofeev file
#      added 'exp_polar'
from sympy import *

def leaf_count(expr):
  #sympy do not have leaf count function. This is approximation
  return round(1.7*count_ops(expr))

def is_sqrt(expr):
  if isinstance(expr,Pow):
    if expr.args[1] == Rational(1,2):
      return True
    else:
      return False
  else:
    return False

def is_elementary_function(func):
  return func in [exp,log,ln,sin,cos,tan,cot,sec,csc,
    asin,acos,atan,acot,asec,acsc,sinh,cosh,tanh,coth,sech,csch,
    asinh,acosh,atanh,acoth,asech,acsch
  ]

def is_special_function(func):
  return func in [ erf,erfc,erfi,
    fresnels,fresnelc,Ei,Ei,Li,Si,Ci,Shi,Chi,
    gamma,loggamma,digamma,zeta,polylog,LambertW,
    elliptic_f,elliptic_e,elliptic_pi,exp_polar
  ]
```

```

def is_hypergeometric_function(func):
    return func in [hyper]

def is_appell_function(func):
    return func in [appellf1]

def is_atom(expn):
    try:
        if expn.isAtom or isinstance(expn,int) or isinstance(expn,float):
            return True
        else:
            return False

    except AttributeError as error:
        return False

def expnType(expn):
    debug=False
    if debug:
        print("expn=",expn,"type(expn)=",type(expn))

    if is_atom(expn):
        return 1
    elif isinstance(expn,list):
        return max(map(expnType, expn)) #apply(max,map(ExpnType,expn))
    elif is_sqrt(expn):
        if isinstance(expn.args[0],Rational): #type(op(1,expn),'rational')
            return 1
        else:
            return max(2,expnType(expn.args[0])) #max(2,ExpnType(op(1,expn)))
    elif isinstance(expn,Pow): #type(expn,'^')
        if isinstance(expn.args[1],Integer): #type(op(2,expn),'integer')
            return expnType(expn.args[0]) #ExpnType(op(1,expn))
        elif isinstance(expn.args[1],Rational): #type(op(2,expn),'rational')
            if isinstance(expn.args[0],Rational): #type(op(1,expn),'rational')
                return 1
            else:
                return max(2,expnType(expn.args[0])) #max(2,ExpnType(op(1,expn)))
        else:
            return max(3,expnType(expn.args[0]),expnType(expn.args[1])) #max(3,ExpnType(op(1,expn)),ExpnType(op(2,expn)))
    elif isinstance(expn,Add) or isinstance(expn,Mul): #type(expn,'+') or type(expn,'*')

```

```

    m1 = expnType(expn.args[0])
    m2 = expnType(list(expn.args[1:]))
    return max(m1,m2) #max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
elif is_elementary_function(expn.func): #ElementaryFunctionQ(op(0,expn))
    return max(3,expnType(expn.args[0])) #max(3,ExpnType(op(1,expn)))
elif is_special_function(expn.func): #SpecialFunctionQ(op(0,expn))
    m1 = max(map(expnType, list(expn.args)))
    return max(4,m1) #max(4,apply(max,map(ExpnType,[op(expn)])))
elif is_hypergeometric_function(expn.func): #HypergeometricFunctionQ(op(0,expn))
    m1 = max(map(expnType, list(expn.args)))
    return max(5,m1) #max(5,apply(max,map(ExpnType,[op(expn)])))
elif is_appell_function(expn.func):
    m1 = max(map(expnType, list(expn.args)))
    return max(6,m1) #max(5,apply(max,map(ExpnType,[op(expn)])))
elif isinstance(expn,RootSum):
    m1 = max(map(expnType, list(expn.args))) #Apply[Max,Append[Map[ExpnType,Apply[List,expn]],7]],
    return max(7,m1)
elif str(expn).find("Integral") != -1:
    m1 = max(map(expnType, list(expn.args)))
    return max(8,m1) #max(5,apply(max,map(ExpnType,[op(expn)])))
else:
    return 9

#main function
def grade_antiderivative(result,optimal):

    #print("Enter grade_antiderivative for sagemath")
    #print("Enter grade_antiderivative, result=",result," optimal=",optimal)

    leaf_count_result = leaf_count(result)
    leaf_count_optimal = leaf_count(optimal)

    #print("leaf_count_result=",leaf_count_result)
    #print("leaf_count_optimal=",leaf_count_optimal)

    expnType_result = expnType(result)
    expnType_optimal = expnType(optimal)

    if str(result).find("Integral") != -1:
        grade = "F"
        grade_annotation = ""

```

```

else:
    if expnType_result <= expnType_optimal:
        if result.has(I):
            if optimal.has(I): #both result and optimal complex
                if leaf_count_result <= 2*leaf_count_optimal:
                    grade = "A"
                    grade_annotation = ""
                else:
                    grade = "B"
                    grade_annotation = "Both result and optimal contain complex but leaf count of result is lar
            else: #result contains complex but optimal is not
                grade = "C"
                grade_annotation = "Result contains complex when optimal does not."
        else: # result do not contain complex, this assumes optimal do not as well
            if leaf_count_result <= 2*leaf_count_optimal:
                grade = "A"
                grade_annotation = ""
            else:
                grade = "B"
                grade_annotation = "Leaf count of result is larger than twice the leaf count of optimal. "+str(
        else:
            grade = "C"
            grade_annotation = "Result contains higher order function than in optimal. Order "+str(ExpnType

#print("Before returning. grade=",grade, " grade_annotation=",grade_annotation)

return grade, grade_annotation

```

SageMath grading function

```

#Dec 24, 2019. Nasser: Ported original Maple grading function by
#       Albert Rich to use with Sagemath. This is used to
#       grade Fracas, Giac and Maxima results.
#Dec 24, 2019. Nasser: Added 'exp_integral_e' and 'sng', 'sin_integral'
#       'arctan2', 'floor', 'abs', 'log_integral'
#June 4, 2022 Made default grade_annotation "none" instead of "" due
#       issue later when reading the file.
#July 14, 2022. Added ellipticF. This is until they fix sagemath, then remove it.

```

```

from sage.all import *
from sage.symbolic.operators import add_vararg, mul_vararg

debug=False;

def tree_size(expr):
    r"""
    Return the tree size of this expression.
    """
    #print("Enter tree_size, expr is ",expr)

    if expr not in SR:
        # deal with lists, tuples, vectors
        return 1 + sum(tree_size(a) for a in expr)
    expr = SR(expr)
    x, aa = expr.operator(), expr.operands()
    if x is None:
        return 1
    else:
        return 1 + sum(tree_size(a) for a in aa)

def is_sqrt(expr):
    if expr.operator() == operator.pow: #isinstance(expr,Pow):
        if expr.operands()[1]==1/2: #expr.args[1] == Rational(1,2):
            if debug: print ("expr is sqrt")
            return True
        else:
            return False
    else:
        return False

def is_elementary_function(func):
    #debug=False
    m = func.name() in ['exp','log','ln',
        'sin','cos','tan','cot','sec','csc',
        'arcsin','arccos','arctan','arccot','arcsec','arccsc',
        'sinh','cosh','tanh','coth','sech','csch',
        'arcsinh','arccosh','arctanh','arcoth','arcsech','arcsch','sgn',
        'arctan2','floor','abs'
    ]
    if debug:

```



```

    if m:
        print ("func ", func , " is elementary_function")
    else:
        print ("func ", func , " is NOT elementary_function")

    return m

def is_special_function(func):
    #debug=False
    if debug:
        print ("type(func)=", type(func))

    m= func.name() in ['erf','erfc','erfi','fresnel_sin','fresnel_cos','Ei',
        'Ei','Li','Si','sin_integral','Ci','cos_integral','Shi','sinh_integral',
        'Chi','cosh_integral','gamma','log_gamma','psi','zeta',
        'polylog','lambert_w','elliptic_f','elliptic_e','ellipticF',
        'elliptic_pi','exp_integral_e','log_integral',
        'weierstrassPInverse','weierstrass','weierstrassP','weierstrassZeta',
        'weierstrassPPrime','weierstrassSigma']

    if debug:
        print ("m=",m)
    if m:
        print ("func ", func , " is special_function")
    else:
        print ("func ", func , " is NOT special_function")

    return m

def is_hypergeometric_function(func):
    return func.name() in ['hypergeometric','hypergeometric_M','hypergeometric_U']

def is_appell_function(func):
    return func.name() in ['hypergeometric'] #[appellf1] can't find this in sagemath

def is_atom(expn):

    #debug=False

```

```

if debug:
    print ("Enter is_atom, expn=",expn)

if not hasattr(expn, 'parent'):
    return False

#thanks to answer at https://ask.sagemath.org/question/49179/what-is-sagemath-equivalent-to-atomic
try:
    if expn.parent() is SR:
        return expn.operator() is None
    if expn.parent() in (ZZ, QQ, AA, QQbar):
        return expn in expn.parent() # Should always return True
    if hasattr(expn.parent(),"base_ring") and hasattr(expn.parent(),"gens"):
        return expn in expn.parent().base_ring() or expn in expn.parent().gens()

    return False

except AttributeError as error:
    print ("Exception,AttributeError in is_atom")
    print ("caught exception" , type(error).__name__ )
    return False

def expnType(expn):

    if debug:
        print (">>>>>Enter expnType, expn=", expn)
        print (">>>>>is_atom(expn)=", is_atom(expn))

    if is_atom(expn):
        return 1
    elif type(expn)==list: #instance(expn,list):
        return max(map(expnType, expn)) #apply(max,map(ExpnType,expn))
    elif is_sqrt(expn):
        if type(expn.operands()[0])==Rational: #type(instance(expn.args[0],Rational):
            return 1
        else:
            return max(2,expnType(expn.operands()[0])) #max(2,expnType(expn.args[0]))
    elif expn.operator() == operator.pow: #instance(expn,Pow)
        if type(expn.operands()[1])==Integer: #instance(expn.args[1],Integer)

```

```

    return expnType(expn.operands()[0]) #expnType(expn.args[0])
elif type(expn.operands()[1])==Rational: #isinstance(expn.args[1],Rational)
    if type(expn.operands()[0])==Rational: #isinstance(expn.args[0],Rational)
        return 1
    else:
        return max(2,expnType(expn.operands()[0])) #max(2,expnType(expn.args[0]))
else:
    return max(3,expnType(expn.operands()[0]),expnType(expn.operands()[1])) #max(3,expnType(expn
elif expn.operator() == add_vararg or expn.operator() == mul_vararg: #isinstance(expn,Add) or isins
    m1 = expnType(expn.operands()[0]) #expnType(expn.args[0])
    m2 = expnType(expn.operands()[1:]) #expnType(list(expn.args[1:]))
    return max(m1,m2) #max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
elif is_elementary_function(expn.operator()): #is_elementary_function(expn.func)
    return max(3,expnType(expn.operands()[0]))
elif is_special_function(expn.operator()): #is_special_function(expn.func)
    m1 = max(map(expnType, expn.operands())) #max(map(expnType, list(expn.args)))
    return max(4,m1) #max(4,m1)
elif is_hypergeometric_function(expn.operator()): #is_hypergeometric_function(expn.func)
    m1 = max(map(expnType, expn.operands())) #max(map(expnType, list(expn.args)))
    return max(5,m1) #max(5,m1)
elif is_appell_function(expn.operator()):
    m1 = max(map(expnType, expn.operands())) #max(map(expnType, list(expn.args)))
    return max(6,m1) #max(6,m1)
elif str(expn).find("Integral") != -1: #this will never happen, since it
    #is checked before calling the grading function that is passed.
    #but kept it here.
    m1 = max(map(expnType, expn.operands())) #max(map(expnType, list(expn.args)))
    return max(8,m1) #max(5,apply(max,map(ExpnType,[op(expn)])))
else:
    return 9

#main function
def grade_antiderivative(result,optimal):

if debug:
    print ("Enter grade_antiderivative for sagemath")
    print ("Enter grade_antiderivative, result=",result)
    print ("Enter grade_antiderivative, optimal=",optimal)
    print ("type(anti)=", type(result))
    print ("type(optimal)=", type(optimal))

```

```

leaf_count_result = tree_size(result) #leaf_count(result)
leaf_count_optimal = tree_size(optimal) #leaf_count(optimal)

#if debug: print ("leaf_count_result=", leaf_count_result, "leaf_count_optimal=",leaf_count_optimal)

expnType_result = expnType(result)
expnType_optimal = expnType(optimal)

if debug: print ("expnType_result=", expnType_result, "expnType_optimal=",expnType_optimal)

if expnType_result <= expnType_optimal:
    if result.has(I):
        if optimal.has(I): #both result and optimal complex
            if leaf_count_result <= 2*leaf_count_optimal:
                grade = "A"
                grade_annotation = "none"
            else:
                grade = "B"
                grade_annotation = "Both result and optimal contain complex but leaf count of result is larger"
        else: #result contains complex but optimal is not
            grade = "C"
            grade_annotation = "Result contains complex when optimal does not."
    else: # result do not contain complex, this assumes optimal do not as well
        if leaf_count_result <= 2*leaf_count_optimal:
            grade = "A"
            grade_annotation = "none"
        else:
            grade = "B"
            grade_annotation = "Leaf count of result is larger than twice the leaf count of optimal. "+str(leaf_count_result - 2*leaf_count_optimal)
    else:
        grade = "C"
        grade_annotation = "Result contains higher order function than in optimal. Order "+str(expnType_result - expnType_optimal)

print("Before returning. grade=",grade, " grade_annotation=",grade_annotation)

return grade, grade_annotation

```

4.2 Links to plain text integration problems used in this report for each CAS

1. Mathematica integration problems as .m file
2. Maple integration problems as .txt file
3. Sagemath integration problems as .sage file
4. Reduce integration problems as .txt file
5. Mupad integration problems as .txt file
6. Sympy integration problems as .py file