

Computer Algebra Independent Integration Tests

Summer 2024

4-Trig-functions/4.1-Sine/184-4.1.2.3

Nasser M. Abbasi

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CHAPTER 1

INTRODUCTION

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This report gives the result of running the computer algebra independent integration test. The download section in on the main webpage contains links to download the problems in plain text format used for all CAS systems. The number of integrals in this report is [51]. This is test number [184].

1.1 Listing of CAS systems tested

The following are the CAS systems tested:

1. Mathematica 14 (January 9, 2024) on windows 10 pro.
2. Rubi 4.17.3 (Sept 25, 2023) on Mathematica 14 on windows 10m pro.
3. Maple 2024 (March 1, 2024) on windows 10 pro.
4. Maxima 5.47 (June 1, 2023) using Lisp SBCL 2.4.0 on Linux Manjaro 23.1.2 KDE via sagemath 10.3.
5. FriCAS 1.3.10 built with sbcl 2.3.11 (January 10, 2024) on Linux Manjaro 23.1.2 KDE via sagemath 10.3.
6. Giac/Xcas 1.9.0-99 on Linux via sagemath 10.3.
7. Sympy 1.12 using Python 3.11.6 (Nov 14 2023, 09:36:21) [GCC 13.2.1 20230801] on Linux Manjaro 23.1.2 KDE.
8. Mupad using Matlab 2021a with Symbolic Math Toolbox Version 8.7 on windows 10.
9. Reduce CSL rev 6687 (January 9, 2024) on Linux Manjaro 23.1.2 KDE.

Maxima and Fricas and Giac are called using Sagemath. This was done using Sagemath `integrate` command by changing the name of the algorithm to use the different CAS systems.

Sympy was run directly in Python not via sagemath.

Reduce was called directly.

1.2 Results

Important note: A number of problems in this test suite have no antiderivative in closed form. This means the antiderivative of these integrals can not be expressed in terms of elementary, special functions or Hypergeometric2F1 functions. RootSum and RootOf are not allowed. If a CAS returns the above integral unevaluated within the time limit, then the result is counted as passed and assigned an A grade.

However, if CAS times out, then it is assigned an F grade even if the integral is not integrable, as this implies CAS could not determine that the integral is not integrable in the time limit.

If a CAS returns an antiderivative to such an integral, it is assigned an A grade automatically and this special result is listed in the introduction section of each individual test report to make it easy to identify as this can be important result to investigate.

The results given in in the table below reflects the above.

System	% solved	% Failed
Rubi	100.00 (51)	0.00 (0)
Maple	98.04 (50)	1.96 (1)
Mathematica	88.24 (45)	11.76 (6)
Fricas	60.78 (31)	39.22 (20)
Giac	39.22 (20)	60.78 (31)
Maxima	31.37 (16)	68.63 (35)
Mupad	25.49 (13)	74.51 (38)
Reduce	23.53 (12)	76.47 (39)
Sympy	7.84 (4)	92.16 (47)

Table 1.1: Percentage solved for each CAS

The table below gives additional break down of the grading of quality of the antiderivatives generated by each CAS. The grading is given using the letters A,B,C and F with A being the best quality. The grading is accomplished by comparing the antiderivative generated with the optimal antiderivatives included in the test suite. The following table describes the meaning of these grades.

grade	description
A	Integral was solved and antiderivative is optimal in quality and leaf size.
B	Integral was solved and antiderivative is optimal in quality but leaf size is larger than twice the optimal antiderivatives leaf size.
C	Integral was solved and antiderivative is non-optimal in quality. This can be due to one or more of the following reasons <ol style="list-style-type: none"> 1. antiderivative contains a hypergeometric function and the optimal antiderivative does not. 2. antiderivative contains a special function and the optimal antiderivative does not. 3. antiderivative contains the imaginary unit and the optimal antiderivative does not.
F	Integral was not solved. Either the integral was returned unevaluated within the time limit, or it timed out, or CAS hanged or crashed or an exception was raised.

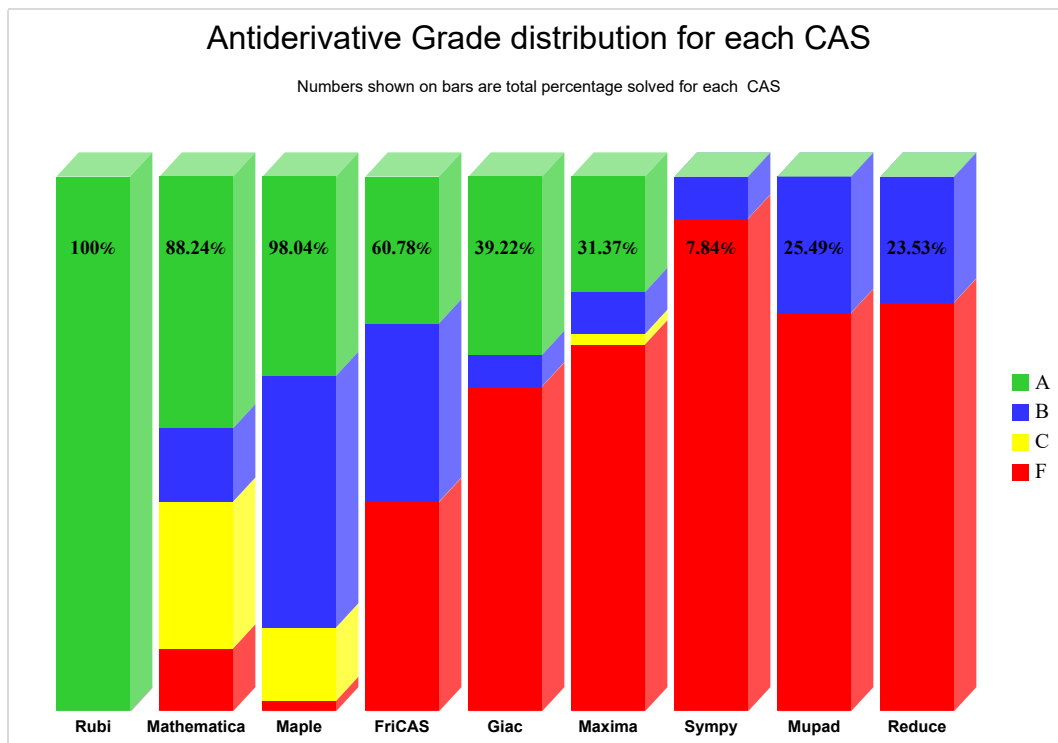
Table 1.2: Description of grading applied to integration result

Grading is implemented for all CAS systems. Based on the above, the following table summarizes the grading for this test suite.

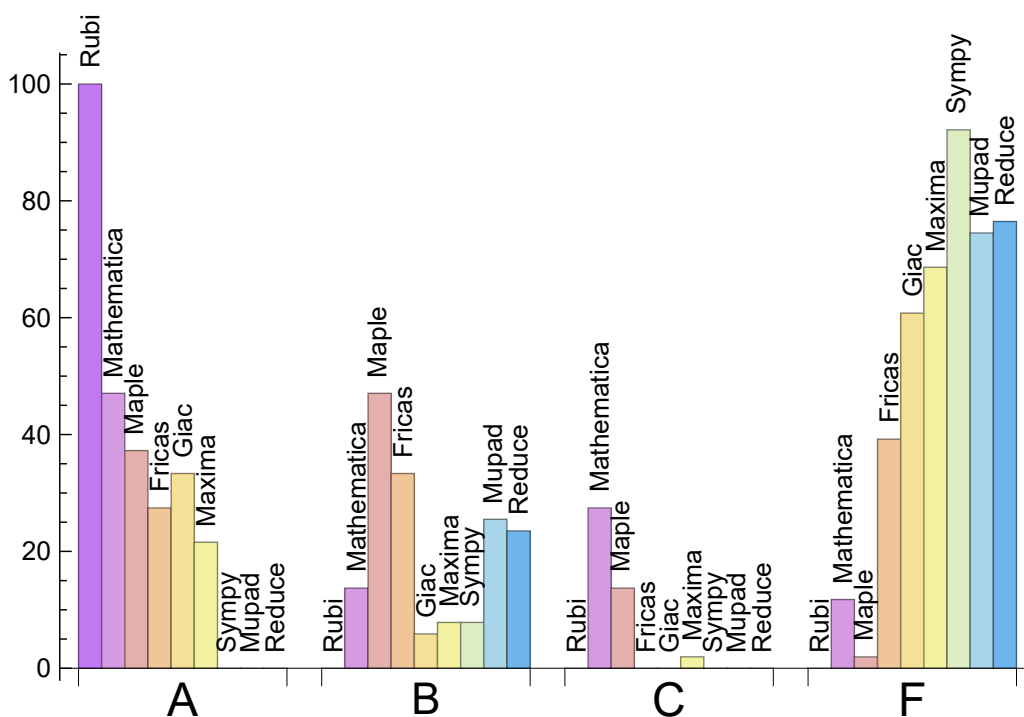
System	% A grade	% B grade	% C grade	% F grade
Rubi	100.000	0.000	0.000	0.000
Mathematica	47.059	13.725	27.451	11.765
Maple	37.255	47.059	13.725	1.961
Giac	33.333	5.882	0.000	60.784
Fricas	27.451	33.333	0.000	39.216
Maxima	21.569	7.843	1.961	68.627
Mupad	0.000	25.490	0.000	74.510
Reduce	0.000	23.529	0.000	76.471
Sympy	0.000	7.843	0.000	92.157

Table 1.3: Antiderivative Grade distribution of each CAS

The following is a Bar chart illustration of the data in the above table.



The figure below compares the grades of the CAS systems.



The following table shows the distribution of the different types of failures for each CAS. There are 3 types failures. The first is when CAS returns the input within the time limit, which means it could not solve it. This is the typical failure and given as **F**.

The second failure is due to time out. CAS could not solve the integral within the 3 minutes time limit which is assigned. This is assigned **F(-1)**.

The third is due to an exception generated, indicated as **F(-2)**. This most likely indicates an interface problem between sagemath and the CAS (applicable only to FriCAS, Maxima and Giac) or it could be an indication of an internal error in the CAS itself. This type of error requires more investigation to determine the cause.

System	Number failed	Percentage normal failure	Percentage time-out failure	Percentage exception failure
Rubi	0	0.00	0.00	0.00
Maple	1	100.00	0.00	0.00
Mathematica	6	100.00	0.00	0.00
Fricas	20	40.00	60.00	0.00
Giac	31	67.74	25.81	6.45
Maxima	35	97.14	0.00	2.86
Mupad	38	0.00	100.00	0.00
Reduce	39	100.00	0.00	0.00
Sympy	47	89.36	10.64	0.00

Table 1.4: Failure statistics for each CAS

1.3 Time and leaf size Performance

The table below summarizes the performance of each CAS system in terms of time used and leaf size of results.

Mean size is the average leaf size produced by the CAS (before any normalization). The Normalized mean is relative to the mean size of the optimal anti-derivative given in the input files.

For example, if CAS has **Normalized mean** of 3, then the mean size of its leaf size is 3 times as large as the mean size of the optimal leaf size.

Median size is value of leaf size where half the values are larger than this and half are smaller (before any normalization). i.e. The Middle value.

Similarly the **Normalized median** is relative to the median leaf size of the optimal.

For example, if a CAS has Normalized median of 1.2, then its median is 1.2 as large as the median leaf size of the optimal.

System	Mean time (sec)
Maxima	0.07
Reduce	0.19
Sympy	0.25
Giac	0.37
Rubi	0.74
Fricas	2.28
Mathematica	7.56
Maple	9.39
Mupad	37.51

Table 1.5: Time performance for each CAS

System	Mean size	Normalized mean	Median size	Normalized median
Giac	124.85	1.35	114.50	1.30
Maxima	142.81	1.98	121.50	1.56
Rubi	148.98	1.02	123.00	1.00
Reduce	198.42	1.61	80.50	1.02
Sympy	273.50	3.08	273.00	3.16
Mathematica	918.53	5.00	197.00	1.39
Fricas	952.03	7.60	240.00	3.31
Mupad	2002.92	12.16	163.00	2.21
Maple	7183.62	23.71	289.00	2.59

Table 1.6: Leaf size performance for each CAS

1.4 Performance based on number of rules Rubi used

This section shows how each CAS performed based on the number of rules Rubi needed to solve the same integral. One diagram is given for each CAS.

On the y axis is the percentage solved which Rubi itself needed the number of rules given the x axis. These plots show that as more rules are needed then most CAS system percentage of solving decreases which indicates the integral is becoming more complicated to solve.

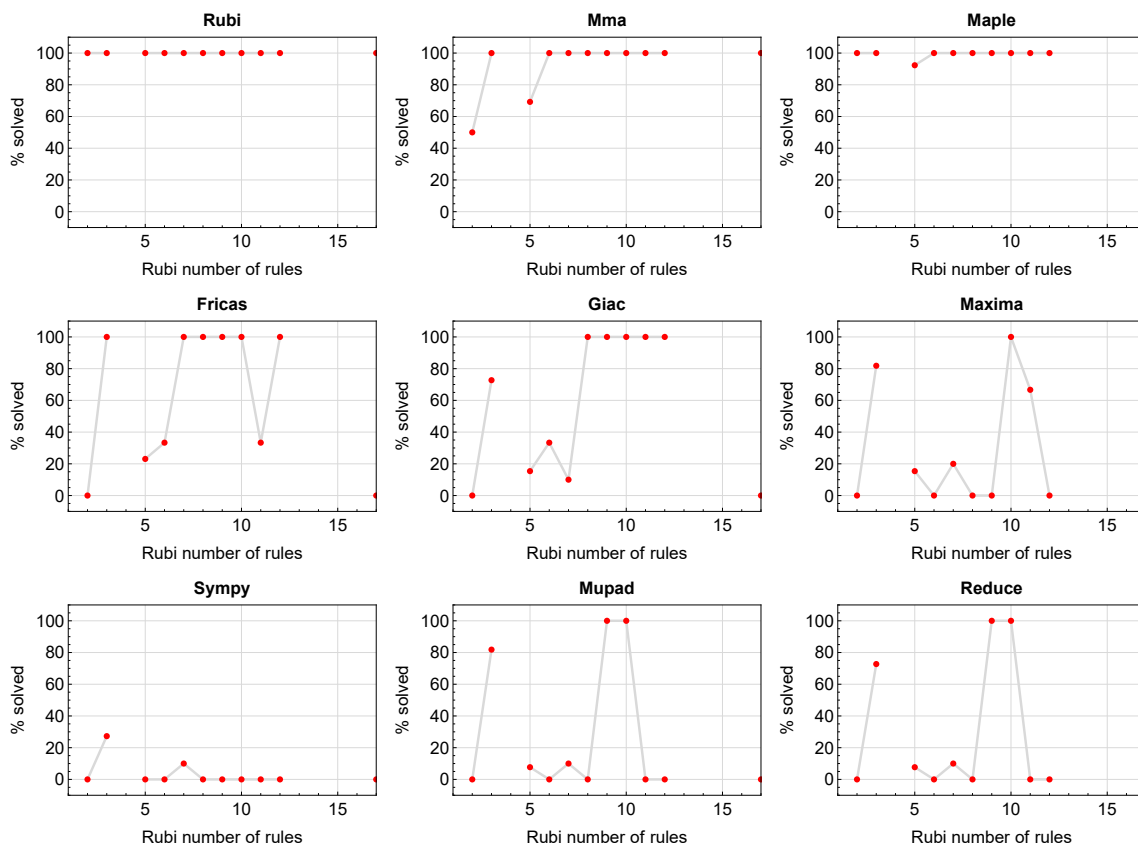


Figure 1.1: Solving statistics per number of Rubi rules used

1.5 Performance based on number of steps Rubi used

This section shows how each CAS performed based on the number of steps Rubi needed to solve the same integral. Note that the number of steps Rubi needed can be much higher than the number of rules, as the same rule could be used more than once.

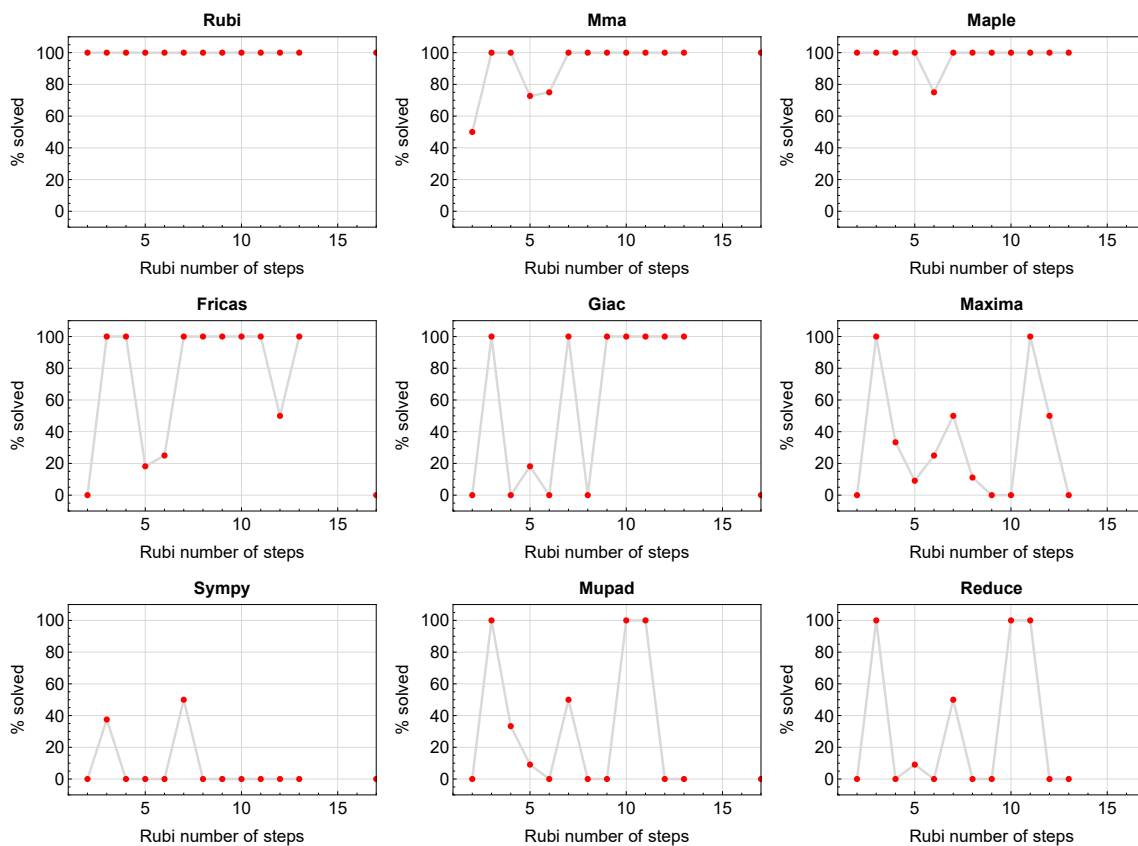


Figure 1.2: Solving statistics per number of Rubi steps used

The above diagram show that the percentage of solved intergals decreases for most CAS systems as the number of steps increases. As expected, for integrals that required less steps by Rubi, CAS systems had more success which indicates the integral was not as hard to solve. As Rubi needed more steps to solve the integral, the solved percentage decreased for most CAS systems which indicates the integral is becoming harder to solve.

1.6 Solved integrals histogram based on leaf size of result

The following shows the distribution of solved integrals for each CAS system based on leaf size of the antiderivatives produced by each CAS. It shows that most integrals solved produced leaf size less than about 100 to 150. The bin size used is 40.

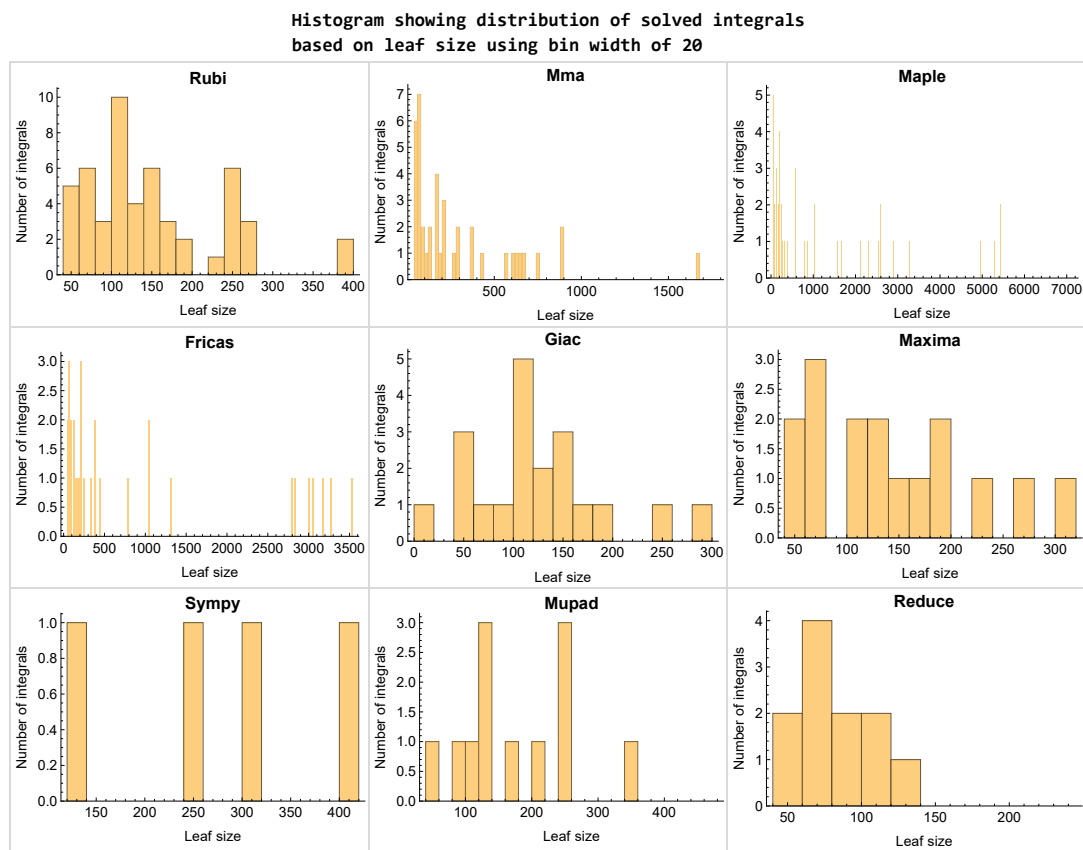


Figure 1.3: Solved integrals based on leaf size distribution

1.7 Solved integrals histogram based on CPU time used

The following shows the distribution of solved integrals for each CAS system based on CPU time used in seconds. The bin size used is 0.1 second.

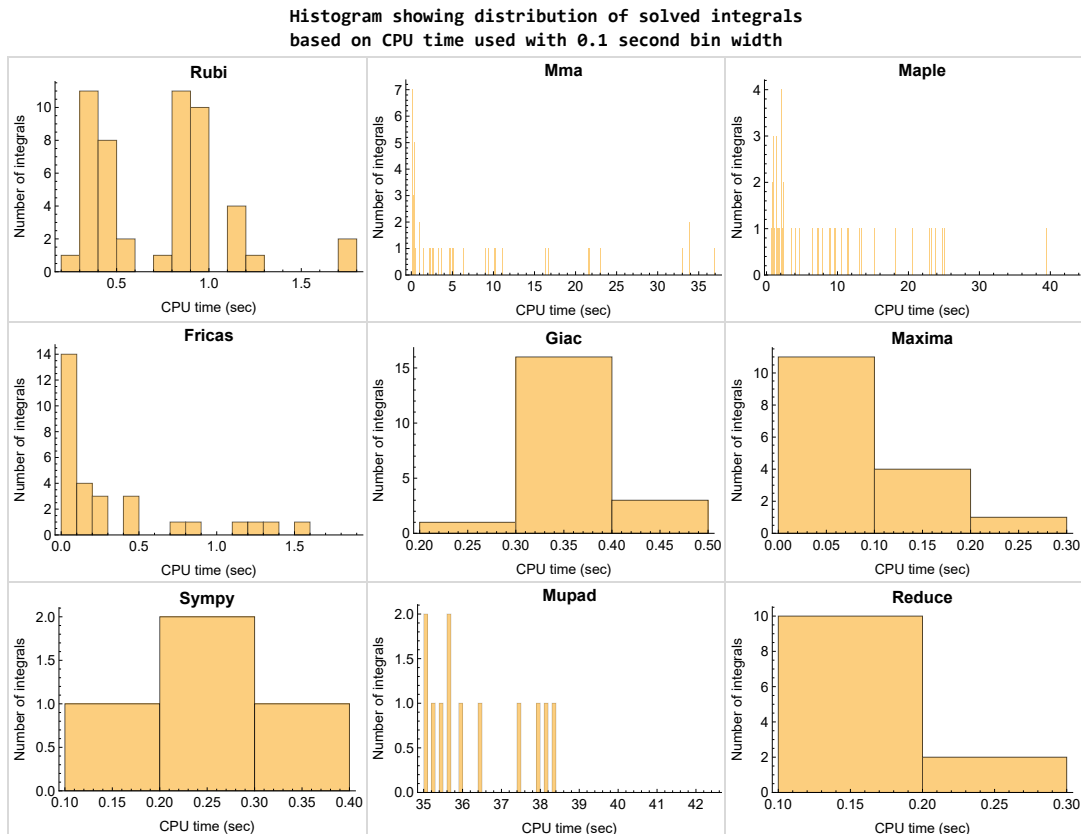


Figure 1.4: Solved integrals histogram based on CPU time used

1.8 Leaf size vs. CPU time used

The following shows the relation between the CPU time used to solve an integral and the leaf size of the antiderivative.

The result for Fricas, Maxima and Giac is shifted more to the right than the other CAS system due to the use of sagemath to call them, which causes an initial slight delay in the timing to start the integration due to overhead of starting a new process each time. This should also be taken into account when looking at the timing of these three CAS systems. Direct calls not using sagemath would result in faster timings, but current implementation uses sagemath as this makes testing much easier to do.

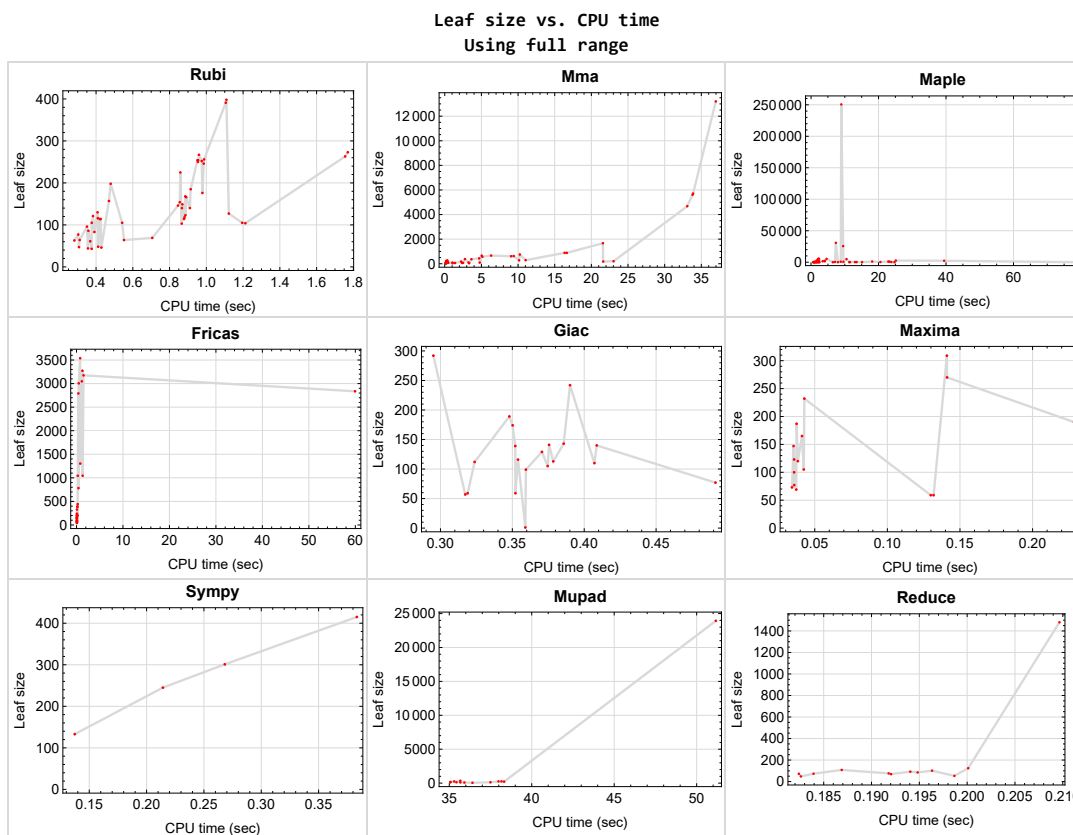


Figure 1.5: Leaf size vs. CPU time. Full range

1.9 list of integrals with no known antiderivative

{}

1.10 List of integrals solved by CAS but has no known antiderivative

Rubi {}

Mathematica {}

Maple {}

Maxima {}

Fricas {}

Sympy {}

Giac {}

Reduce {}

Mupad {}

1.11 list of integrals solved by CAS but failed verification

The following are integrals solved by CAS but the verification phase failed to verify the anti-derivative produced is correct. This does not necessarily mean that the anti-derivative is wrong as additional methods of verification might be needed, or more time is needed (3 minutes time limit was used). These integrals are listed here to make it possible to do further investigation to determine why the result could not be verified.

Rubi {}

Mathematica {24, 26, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 45}

Maple {15, 25, 27, 31, 32, 33, 34, 35, 36, 37, 38, 42, 43, 44, 45, 46, 47, 48, 49, 50}

Maxima Verification phase not currently implemented.

Fricas Verification phase not currently implemented.

Sympy Verification phase not currently implemented.

Giac Verification phase not currently implemented.

Reduce Verification phase not currently implemented.

Mupad Verification phase not currently implemented.

1.12 Timing

The command `AbsoluteTiming[]` was used in Mathematica to obtain the elapsed time for each integrate call. In Maple, the command `Usage` was used as in the following example

```
cpu_time := Usage(assign ('result_of_int',int(expr,x)),output='realtime')
```

For all other CAS systems, the elapsed time to complete each integral was found by taking the difference between the time after the call completed from the time before the call was made. This was done using Python's `time.time()` call.

All elapsed times shown are in seconds. A time limit of 3 CPU minutes was used for each integral. If the integrate command did not complete within this time limit, the integral was aborted and considered to have failed and assigned an F grade. The time used by failed integrals due to time out was not counted in the final statistics.

1.13 Verification

A verification phase was applied on the result of integration for **Rubi** and **Mathematica**.

Future version of this report will implement verification for the other CAS systems. For the integrals whose result was not run through a verification phase, it is assumed that the antiderivative was correct.

Verification phase also had 3 minutes time out. An integral whose result was not verified could still be correct, but further investigation is needed on those integrals. These integrals were marked in the summary table below and also in each integral separate section so they are easy to identify and locate.

1.14 Important notes about some of the results

Important note about Maxima results

Since tests were run in a batch mode, and using an automated script, then any integral where Maxima needed an interactive response from the user to answer a question during the evaluation of the integral will fail.

The exception raised is `ValueError`. Therefore Maxima results is lower than what would result if Maxima was run directly and each question was answered correctly.

The percentage of such failures were not counted for each test file, but for an example, for the `Timofeev` test file, there were about 14 such integrals out of total 705, or about 2 percent. This percentage can be higher or lower depending on the specific input test file.

Such integrals can be identified by looking at the output of the integration in each section for Maxima. The exception message will indicate the cause of error.

Maxima `integrate` was run using SageMath with the following settings set by default

```
'besselexpand : true'  
'display2d : false'  
'domain : complex'  
'keepfloat : true'  
'load(to_poly_solve)'  
'load(simplify_sum)'  
'load(abs_integrate)' 'load(diag)'
```

SageMath automatic loading of Maxima `abs_integrate` was found to cause some problems. So the following code was added to disable this effect.

```
from sage.interfaces.maxima_lib import maxima_lib  
maxima_lib.set('extra_definite_integration_methods', '[]')  
maxima_lib.set('extra_integration_methods', '[]')
```

See <https://ask.sagemath.org/question/43088/integrate-results-that-are-different-from-using-maxima/> for reference.

Important note about FriCAS result

There were few integrals which failed due to SageMath interface and not because FriCAS system could not do the integration.

These will fail With error `Exception raised: NotImplementedError`.

The number of such cases seems to be very small. About 1 or 2 percent of all integrals. These can be identified by looking at the exception message given in the result.

Important note about finding leaf size of antiderivative

For Mathematica, Rubi, and Maple, the builtin system function `LeafSize` was used to find the leaf size of each antiderivative.

The other CAS systems (SageMath and Sympy) do not have special builtin function for this purpose at this time. Therefore the leaf size for Fricas and Sympy antiderivative was determined using the following function, thanks to user `slelievre` at https://ask.sagemath.org/question/57123/could-we-have-a-leaf_count-function-in-base-sagemath/

```
def tree_size(expr):
    r"""
    Return the tree size of this expression.
    """
    if expr not in SR:
        # deal with lists, tuples, vectors
        return 1 + sum(tree_size(a) for a in expr)
    expr = SR(expr)
    x, aa = expr.operator(), expr.operands()
    if x is None:
        return 1
    else:
        return 1 + sum(tree_size(a) for a in aa)
```

For Sympy, which was called directly from Python, the following code was used to obtain the leafsize of its result

```
try:  
    # 1.7 is a fudge factor since it is low side from actual leaf count  
    leafCount = round(1.7*count_ops(anti))  
  
except Exception as ee:  
    leafCount = 1
```

Important note about Mupad results

Matlab's symbolic toolbox does not have a leaf count function to measure the size of the antiderivative. Maple was used to determine the leaf size of Mupad output by post processing Mupad result.

Currently no grading of the antiderivative for Mupad is implemented. If it can integrate the problem, it was assigned a B grade automatically as a placeholder. In the future, when grading function is implemented for Mupad, the tests will be rerun again.

The following is an example of using Matlab's symbolic toolbox (Mupad) to solve an integral

```
integrand = evalin(symengine, 'cos(x)*sin(x)')  
the_variable = evalin(symengine, 'x')  
anti = int(integrand,the_variable)
```

Which gives $\sin(x)^2/2$

1.15 Current tree layout of integration tests

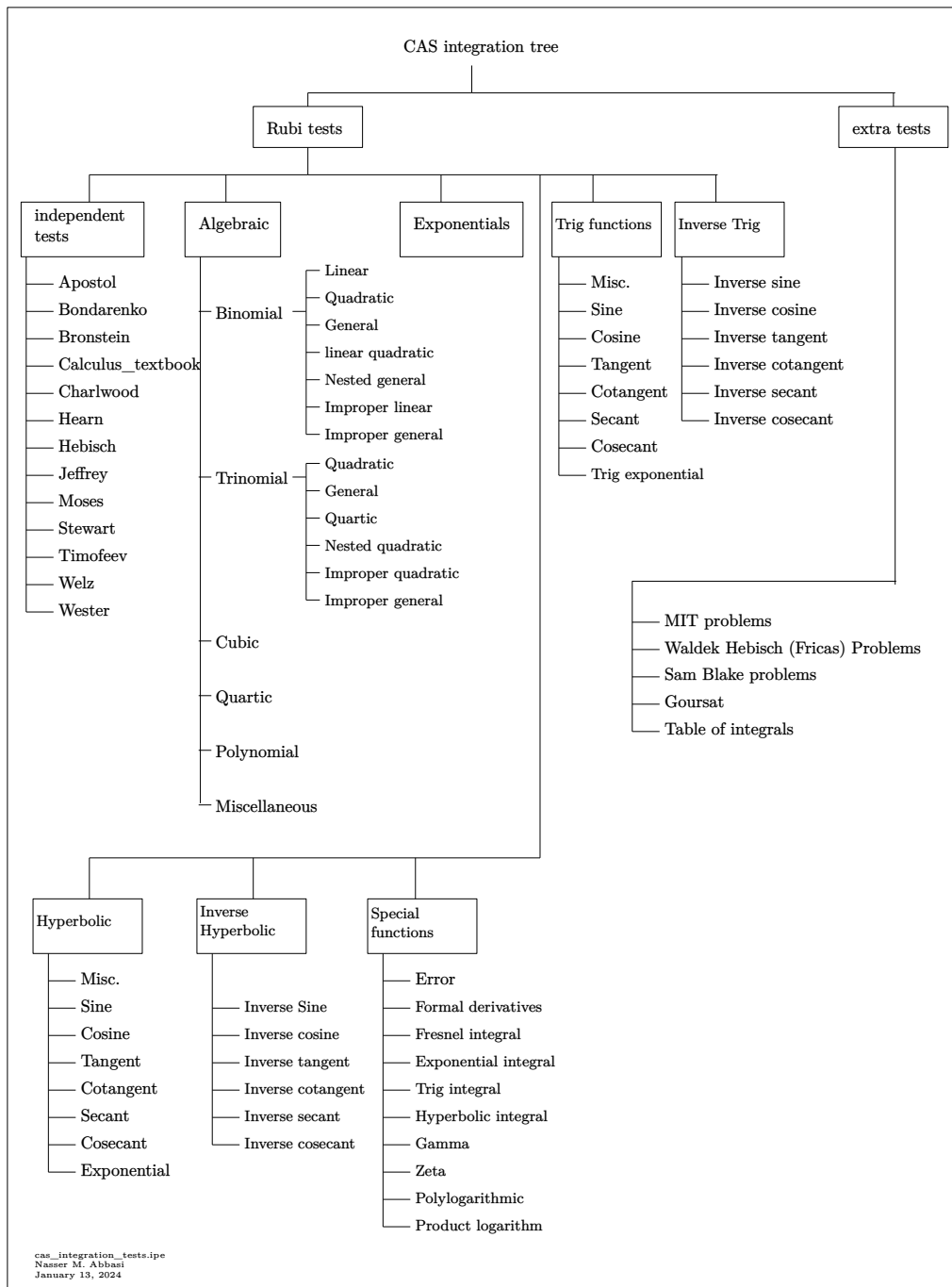
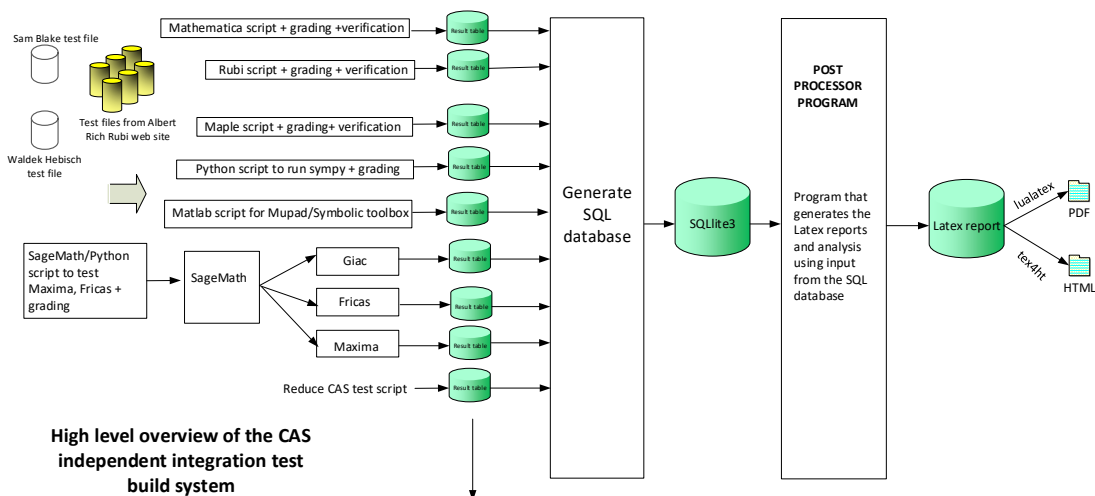


Figure 1.6: CAS integration tests tree

1.16 Design of the test system

The following diagram gives a high level view of the current test build system.



High level overview of the CAS independent integration test build system

One record (line) per one integral result. The line is CSV comma separated. This is description of each record

1. integer. the problem number.
2. integer. 0 for failed, 1 for passed, -1 for timeout, -2 for CAS specific exception. (this is not the grade field)
3. integer. Leaf size of result.
4. integer. Leaf size of the optimal antiderivative.
5. number. CPU time used to solve this integral. 0 if failed.
6. string. The integral in Latex format
7. string. The input used in CAS own syntax.
8. string. The result (antiderivative) produced by CAS in Latex format
9. string. The optimal antiderivative in Latex format.
10. integer. 0 or 1. Indicates if problem has known antiderivative or not
11. String. The result (antiderivative) in CAS own syntax.
12. String. The grade of the antiderivative. Can be "A", "B", "C", or "E"
13. String. Small string description of why the grade was given.
14. integer. 1 if result was verified or 0 if not verified. (For mma, rubi and maple only)

The following fields are present only in Rubi Table file

15. integer. Number of steps used.
16. integer. Number of rules used.
17. integer. Integrand leaf size.
18. real number. Ratio. Field 16 over field 17
19. String of form "{n,n,...}" which is list of the rules used by Rubi
20. String. The optimal antiderivative in Mathematica syntax

Nasser M. Abbasi
January 13, 2024
Design note

CHAPTER 2

DETAILED SUMMARY TABLES OF RESULTS

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2.1 List of integrals sorted by grade for each CAS

Rubi	25
Mma	25
Maple	26
Fricas	26
Maxima	26
Giac	27
Mupad	27
Sympy	27
Reduce	28

Rubi

A grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51 }

B grade { }

C grade { }

F normal fail { }

F(-1) timedout fail { }

F(-2) exception fail { }

Mma

A grade { 1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 15, 16, 17, 18, 19, 20, 21, 22, 27, 28, 39, 48, 49, 50 }

B grade { 8, 9, 32, 33, 37, 38, 45 }

C grade { 13, 14, 23, 24, 25, 26, 29, 30, 31, 34, 35, 36, 40, 41 }

F normal fail { 42, 43, 44, 46, 47, 51 }

F(-1) timedout fail { }

F(-2) exception fail { }

Maple

A grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 16, 23, 24, 39, 40, 41 }

B grade { 17, 18, 19, 20, 21, 22, 26, 27, 28, 29, 30, 32, 33, 34, 35, 36, 37, 38, 43, 44, 45, 47, 49, 50 }

C grade { 11, 15, 25, 31, 42, 46, 48 }

F normal fail { 51 }

F(-1) timedout fail { }

F(-2) exception fail { }

Fricas

A grade { 1, 2, 3, 4, 5, 6, 12, 15, 16, 17, 18, 19, 22, 27 }

B grade { 7, 8, 9, 10, 11, 13, 14, 23, 24, 25, 26, 28, 35, 36, 37, 38, 39 }

C grade { }

F normal fail { 20, 21, 31, 32, 33, 34, 45, 51 }

F(-1) timedout fail { 29, 30, 40, 41, 42, 43, 44, 46, 47, 48, 49, 50 }

F(-2) exception fail { }

Maxima

A grade { 1, 2, 3, 4, 5, 6, 7, 10, 11, 20, 21 }

B grade { 8, 9, 16, 17 }

C grade { 22 }

F normal fail { 12, 13, 14, 15, 18, 19, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51 }

F(-1) timedout fail { }

F(-2) exception fail { 39 }

Giac

A grade { 1, 2, 3, 4, 5, 7, 9, 10, 12, 13, 14, 19, 20, 21, 23, 24, 39 }

B grade { 6, 8, 11 }

C grade { }

F normal fail { 16, 26, 29, 30, 31, 32, 33, 34, 36, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51 }

F(-1) timedout fail { 15, 17, 18, 25, 27, 35, 37, 38 }

F(-2) exception fail { 22, 28 }

Mupad

A grade { }

B grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 16, 39 }

C grade { }

F normal fail { }

F(-1) timedout fail { 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51 }

F(-2) exception fail { }

Sympy

A grade { }

B grade { 1, 2, 3, 4 }

C grade { }

F normal fail { 5, 6, 7, 8, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50 }

F(-1) timedout fail { 9, 10, 11, 39, 51 }

F(-2) exception fail { }

Reduce

A grade { }

B grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 39 }

C grade { }

F normal fail { 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51 }

F(-1) timedout fail { }

F(-2) exception fail { }

2.2 Detailed conclusion table per each integral for all CAS systems

Detailed conclusion table per each integral is given by the table below. The elapsed time is in seconds. For failed result it is given as **F(-1)** if the failure was due to timeout. It is given as **F(-2)** if the failure was due to an exception being raised, which could indicate a bug in the system. If the failure was due to integral not being evaluated within the time limit, then it is given as **F**.

In this table, the column **N.S.** means **normalized size** and is defined as $\frac{\text{antiderivative leaf size}}{\text{optimal antiderivative leaf size}}$. To make the table fit the page, the name **Mathematica** was abbreviated to **MMA**.

Problem 1	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	B	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	121	121	77	79	147	91	415	113	101	256
N.S.	1	1.00	0.64	0.65	1.21	0.75	3.43	0.93	0.83	2.12
time (sec)	N/A	0.383	0.485	78.106	0.035	0.090	0.383	0.378	0.196	38.168

Problem 2	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	B	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	96	96	57	57	123	77	301	77	85	212
N.S.	1	1.00	0.59	0.59	1.28	0.80	3.14	0.80	0.89	2.21
time (sec)	N/A	0.351	0.204	24.715	0.036	0.087	0.268	0.491	0.195	38.325

Problem 3	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	B	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	77	77	47	46	100	63	245	59	69	250
N.S.	1	1.00	0.61	0.60	1.30	0.82	3.18	0.77	0.90	3.25
time (sec)	N/A	0.303	0.189	7.967	0.036	0.087	0.214	0.352	0.192	37.983

Problem 4	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	B	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	52	47	43	44	77	46	133	59	53	125
N.S.	1	0.90	0.83	0.85	1.48	0.88	2.56	1.13	1.02	2.40
time (sec)	N/A	0.307	0.181	11.343	0.036	0.091	0.137	0.319	0.199	37.497

Problem 5	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	63	63	44	45	73	75	0	105	48	88
N.S.	1	1.00	0.70	0.71	1.16	1.19	0.00	1.67	0.76	1.40
time (sec)	N/A	0.282	0.159	0.653	0.034	0.091	0.000	0.375	0.183	35.917

Problem 6	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	53	44	68	61	69	99	0	129	73	110
N.S.	1	0.83	1.28	1.15	1.30	1.87	0.00	2.43	1.38	2.08
time (sec)	N/A	0.356	0.152	1.089	0.037	0.091	0.000	0.371	0.184	35.671

Problem 7	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	B	F	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	64	64	95	73	105	138	0	116	72	163
N.S.	1	1.00	1.48	1.14	1.64	2.16	0.00	1.81	1.12	2.55
time (sec)	N/A	0.311	1.053	1.479	0.042	0.092	0.000	0.354	0.182	35.078

Problem 8	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	B	A	B	B	F	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	61	64	172	95	120	137	0	139	76	132
N.S.	1	1.05	2.82	1.56	1.97	2.25	0.00	2.28	1.25	2.16
time (sec)	N/A	0.552	0.351	1.868	0.038	0.090	0.000	0.352	0.192	35.437

Problem 9	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	B	A	B	B	F(-1)	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	86	86	179	95	165	166	0	140	92	133
N.S.	1	1.00	2.08	1.10	1.92	1.93	0.00	1.63	1.07	1.55
time (sec)	N/A	0.358	0.387	2.283	0.041	0.088	0.000	0.409	0.194	35.061

Problem 10	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	B	F(-1)	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	105	105	204	123	187	201	0	174	108	244
N.S.	1	1.00	1.94	1.17	1.78	1.91	0.00	1.66	1.03	2.32
time (sec)	N/A	0.376	0.178	2.327	0.038	0.087	0.000	0.350	0.187	35.295

Problem 11	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	A	B	F(-1)	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	130	130	204	171	232	240	0	242	124	340
N.S.	1	1.00	1.57	1.32	1.78	1.85	0.00	1.86	0.95	2.62
time (sec)	N/A	0.408	0.211	2.481	0.043	0.087	0.000	0.390	0.200	35.666

Problem 12	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	F	A	F	A	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	128	185	101	78	0	155	0	99	48	0
N.S.	1	1.45	0.79	0.61	0.00	1.21	0.00	0.77	0.38	0.00
time (sec)	N/A	0.915	3.239	1.559	0.000	0.074	0.000	0.359	0.186	0.000

Problem 13	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	C	A	F	B	F	A	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	69	69	46	79	0	202	0	110	35	0
N.S.	1	1.00	0.67	1.14	0.00	2.93	0.00	1.59	0.51	0.00
time (sec)	N/A	0.706	3.336	1.002	0.000	0.094	0.000	0.407	0.202	0.000

Problem 14	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	C	A	F	B	F	A	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	120	127	74	124	0	328	0	141	40	0
N.S.	1	1.06	0.62	1.03	0.00	2.73	0.00	1.18	0.33	0.00
time (sec)	N/A	1.122	2.501	0.840	0.000	0.103	0.000	0.376	0.188	0.000

Problem 15	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	F	A	F	F(-1)	F	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD	TBD
size	103	103	91	2555	0	442	0	0	38	0
N.S.	1	1.00	0.88	24.81	0.00	4.29	0.00	0.00	0.37	0.00
time (sec)	N/A	0.866	4.771	25.119	0.000	0.254	0.000	0.000	0.191	0.000

Problem 16	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	B	A	F	F	F	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	43	43	40	46	309	41	0	0	50	52
N.S.	1	1.00	0.93	1.07	7.19	0.95	0.00	0.00	1.16	1.21
time (sec)	N/A	0.376	1.067	6.470	0.141	0.083	0.000	0.000	0.201	36.406

Problem 17	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	B	B	A	F	F(-1)	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	114	114	133	309	270	385	0	0	43	0
N.S.	1	1.00	1.17	2.71	2.37	3.38	0.00	0.00	0.38	0.00
time (sec)	N/A	0.878	0.363	20.559	0.141	0.183	0.000	0.000	0.189	0.000

Problem 18	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	B	F	A	F	F(-1)	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	118	118	132	269	0	391	0	0	53	0
N.S.	1	1.00	1.12	2.28	0.00	3.31	0.00	0.00	0.45	0.00
time (sec)	N/A	0.881	0.305	23.799	0.000	0.193	0.000	0.000	0.194	0.000

Problem 19	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	B	F	A	F	A	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	46	48	62	162	0	202	0	57	34	0
N.S.	1	1.04	1.35	3.52	0.00	4.39	0.00	1.24	0.74	0.00
time (sec)	N/A	0.411	0.121	15.131	0.000	0.219	0.000	0.317	0.182	0.000

Problem 20	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	B	A	F	F	A	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	102	105	64	214	59	0	0	1	48	0
N.S.	1	1.03	0.63	2.10	0.58	0.00	0.00	0.01	0.47	0.00
time (sec)	N/A	1.195	2.355	13.020	0.132	0.000	0.000	0.359	0.182	0.000

Problem 21	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	B	A	F	F	A	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	100	104	61	207	59	0	0	112	47	0
N.S.	1	1.04	0.61	2.07	0.59	0.00	0.00	1.12	0.47	0.00
time (sec)	N/A	1.212	1.404	13.497	0.130	0.000	0.000	0.324	0.180	0.000

Problem 22	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	B	C	A	F	F(-2)	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	46	46	63	218	190	194	0	0	53	0
N.S.	1	1.00	1.37	4.74	4.13	4.22	0.00	0.00	1.15	0.00
time (sec)	N/A	0.429	0.186	11.551	0.229	0.160	0.000	0.000	0.182	0.000

Problem 23	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	C	A	F	B	F	A	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	105	105	746	120	0	781	0	143	33	0
N.S.	1	1.00	7.10	1.14	0.00	7.44	0.00	1.36	0.31	0.00
time (sec)	N/A	0.542	10.241	1.743	0.000	0.421	0.000	0.386	0.195	0.000

Problem 24	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	C	A	F	B	F	A	F	F(-1)
verified	N/A	Yes	No	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	165	176	654	208	0	1044	0	189	54	0
N.S.	1	1.07	3.96	1.26	0.00	6.33	0.00	1.15	0.33	0.00
time (sec)	N/A	0.979	5.042	1.414	0.000	1.324	0.000	0.348	0.180	0.000

Problem 25	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	C	C	F	B	F	F(-1)	F	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD	TBD
size	149	149	662	2303	0	3273	0	0	36	0
N.S.	1	1.00	4.44	15.46	0.00	21.97	0.00	0.00	0.24	0.00
time (sec)	N/A	0.870	6.328	39.453	0.000	1.278	0.000	0.000	0.188	0.000

Problem 26	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	C	B	F	B	F	F	F	F(-1)
verified	N/A	Yes	No	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	83	83	436	875	0	1303	0	0	48	0
N.S.	1	1.00	5.25	10.54	0.00	15.70	0.00	0.00	0.58	0.00
time (sec)	N/A	0.391	4.698	23.256	0.000	0.802	0.000	0.000	0.182	0.000

Problem 27	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	B	F	A	F	F(-1)	F	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD	TBD
size	166	166	280	1035	0	3048	0	0	57	0
N.S.	1	1.00	1.69	6.23	0.00	18.36	0.00	0.00	0.34	0.00
time (sec)	N/A	0.891	10.138	22.905	0.000	1.149	0.000	0.000	0.178	0.000

Problem 28	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	B	F	B	F	F(-2)	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	168	168	289	1031	0	3175	0	0	71	0
N.S.	1	1.00	1.72	6.14	0.00	18.90	0.00	0.00	0.42	0.00
time (sec)	N/A	0.885	11.034	18.129	0.000	1.505	0.000	0.000	0.185	0.000

Problem 29	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	C	B	F	F(-1)	F	F	F	F(-1)
verified	N/A	Yes	No	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	238	273	611	590	0	0	0	0	34	0
N.S.	1	1.15	2.57	2.48	0.00	0.00	0.00	0.00	0.14	0.00
time (sec)	N/A	1.770	9.056	1.412	0.000	0.000	0.000	0.000	0.189	0.000

Problem 30	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	C	B	F	F(-1)	F	F	F	F(-1)
verified	N/A	Yes	No	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	246	263	625	584	0	0	0	0	54	0
N.S.	1	1.07	2.54	2.37	0.00	0.00	0.00	0.00	0.22	0.00
time (sec)	N/A	1.755	9.410	1.149	0.000	0.000	0.000	0.000	0.191	0.000

Problem 31	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	C	C	F	F	F	F	F	F(-1)
verified	N/A	Yes	No	No	TBD	TBD	TBD	TBD	TBD	TBD
size	267	267	13199	5285	0	0	0	0	37	0
N.S.	1	1.00	49.43	19.79	0.00	0.00	0.00	0.00	0.14	0.00
time (sec)	N/A	0.960	36.978	4.608	0.000	0.000	0.000	0.000	0.187	0.000

Problem 32	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	B	B	F	F	F	F	F	F(-1)
verified	N/A	Yes	No	No	TBD	TBD	TBD	TBD	TBD	TBD
size	116	116	4679	1579	0	0	0	0	47	0
N.S.	1	1.00	40.34	13.61	0.00	0.00	0.00	0.00	0.41	0.00
time (sec)	N/A	0.409	33.082	3.480	0.000	0.000	0.000	0.000	0.180	0.000

Problem 33	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	B	B	F	F	F	F	F	F(-1)
verified	N/A	Yes	No	No	TBD	TBD	TBD	TBD	TBD	TBD
size	252	252	5708	1666	0	0	0	0	57	0
N.S.	1	1.00	22.65	6.61	0.00	0.00	0.00	0.00	0.23	0.00
time (sec)	N/A	0.975	33.875	4.101	0.000	0.000	0.000	0.000	0.192	0.000

Problem 34	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	C	B	F	F	F	F	F	F(-1)
verified	N/A	Yes	No	No	TBD	TBD	TBD	TBD	TBD	TBD
size	256	256	1667	2118	0	0	0	0	71	0
N.S.	1	1.00	6.51	8.27	0.00	0.00	0.00	0.00	0.28	0.00
time (sec)	N/A	0.988	21.574	4.044	0.000	0.000	0.000	0.000	0.182	0.000

Problem 35	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	C	B	F	B	F	F(-1)	F	F(-1)
verified	N/A	Yes	No	No	TBD	TBD	TBD	TBD	TBD	TBD
size	123	123	567	4967	0	3539	0	0	32	0
N.S.	1	1.00	4.61	40.38	0.00	28.77	0.00	0.00	0.26	0.00
time (sec)	N/A	0.886	5.130	10.510	0.000	0.780	0.000	0.000	0.202	0.000

Problem 36	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	C	B	F	B	F	F	F	F(-1)
verified	N/A	Yes	No	No	TBD	TBD	TBD	TBD	TBD	TBD
size	61	61	367	395	0	1044	0	0	44	0
N.S.	1	1.00	6.02	6.48	0.00	17.11	0.00	0.00	0.72	0.00
time (sec)	N/A	0.368	3.641	7.112	0.000	0.275	0.000	0.000	0.185	0.000

Problem 37	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	B	B	F	B	F	F(-1)	F	F(-1)
verified	N/A	Yes	No	No	TBD	TBD	TBD	TBD	TBD	TBD
size	140	140	885	791	0	2791	0	0	45	0
N.S.	1	1.00	6.32	5.65	0.00	19.94	0.00	0.00	0.32	0.00
time (sec)	N/A	0.911	16.349	8.815	0.000	0.407	0.000	0.000	0.189	0.000

Problem 38	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	B	B	F	B	F	F(-1)	F	F(-1)
verified	N/A	Yes	No	No	TBD	TBD	TBD	TBD	TBD	TBD
size	140	140	885	598	0	3005	0	0	65	0
N.S.	1	1.00	6.32	4.27	0.00	21.46	0.00	0.00	0.46	0.00
time (sec)	N/A	0.867	16.646	9.602	0.000	0.479	0.000	0.000	0.187	0.000

Problem 39	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	F(-2)	B	F(-1)	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	181	225	178	244	0	2837	0	292	1480	23933
N.S.	1	1.24	0.98	1.35	0.00	15.67	0.00	1.61	8.18	132.23
time (sec)	N/A	0.859	2.227	1.051	0.000	59.942	0.000	0.295	0.210	51.163

Problem 40	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	C	A	F	F(-1)	F	F	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	154	154	179	190	0	0	0	0	32	0
N.S.	1	1.00	1.16	1.23	0.00	0.00	0.00	0.00	0.21	0.00
time (sec)	N/A	0.856	21.612	0.881	0.000	0.000	0.000	0.000	0.191	0.000

Problem 41	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	C	A	F	F(-1)	F	F	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	146	146	203	254	0	0	0	0	55	0
N.S.	1	1.00	1.39	1.74	0.00	0.00	0.00	0.00	0.38	0.00
time (sec)	N/A	0.846	23.032	0.708	0.000	0.000	0.000	0.000	0.178	0.000

Problem 42	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	F	C	F	F(-1)	F	F	F	F(-1)
verified	N/A	Yes	N/A	No	TBD	TBD	TBD	TBD	TBD	TBD
size	254	254	0	5439	0	0	0	0	36	0
N.S.	1	1.00	0.00	21.41	0.00	0.00	0.00	0.00	0.14	0.00
time (sec)	N/A	0.952	0.000	2.372	0.000	0.000	0.000	0.000	0.195	0.000

Problem 43	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	F	B	F	F(-1)	F	F	F	F(-1)
verified	N/A	Yes	N/A	No	TBD	TBD	TBD	TBD	TBD	TBD
size	250	250	0	2893	0	0	0	0	48	0
N.S.	1	1.00	0.00	11.57	0.00	0.00	0.00	0.00	0.19	0.00
time (sec)	N/A	0.954	0.000	1.698	0.000	0.000	0.000	0.000	0.183	0.000

Problem 44	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	F	B	F	F(-1)	F	F	F	F(-1)
verified	N/A	Yes	N/A	No	TBD	TBD	TBD	TBD	TBD	TBD
size	114	114	0	2590	0	0	0	0	59	0
N.S.	1	1.00	0.00	22.72	0.00	0.00	0.00	0.00	0.52	0.00
time (sec)	N/A	0.420	0.000	2.158	0.000	0.000	0.000	0.000	0.187	0.000

Problem 45	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	B	B	F	F	F	F	F	F(-1)
verified	N/A	Yes	No	No	TBD	TBD	TBD	TBD	TBD	TBD
size	246	246	5612	3272	0	0	0	0	72	0
N.S.	1	1.00	22.81	13.30	0.00	0.00	0.00	0.00	0.29	0.00
time (sec)	N/A	0.986	33.802	2.187	0.000	0.000	0.000	0.000	0.182	0.000

Problem 46	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	F	C	F	F(-1)	F	F	F	F(-1)
verified	N/A	Yes	N/A	No	TBD	TBD	TBD	TBD	TBD	TBD
size	254	254	0	5439	0	0	0	0	36	0
N.S.	1	1.00	0.00	21.41	0.00	0.00	0.00	0.00	0.14	0.00
time (sec)	N/A	0.956	0.000	2.115	0.000	0.000	0.000	0.000	0.190	0.000

Problem 47	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	F	B	F	F(-1)	F	F	F	F(-1)
verified	N/A	Yes	N/A	No	TBD	TBD	TBD	TBD	TBD	TBD
size	114	114	0	2582	0	0	0	0	59	0
N.S.	1	1.00	0.00	22.65	0.00	0.00	0.00	0.00	0.52	0.00
time (sec)	N/A	0.425	0.000	2.129	0.000	0.000	0.000	0.000	0.195	0.000

Problem 48	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	F	F(-1)	F	F	F	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD	TBD
size	391	391	274	250622	0	0	0	0	31	0
N.S.	1	1.00	0.70	640.98	0.00	0.00	0.00	0.00	0.08	0.00
time (sec)	N/A	1.106	0.333	8.967	0.000	0.000	0.000	0.000	0.181	0.000

Problem 49	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	B	F	F(-1)	F	F	F	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD	TBD
size	198	198	197	30789	0	0	0	0	43	0
N.S.	1	1.00	0.99	155.50	0.00	0.00	0.00	0.00	0.22	0.00
time (sec)	N/A	0.480	0.200	7.352	0.000	0.000	0.000	0.000	0.185	0.000

Problem 50	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	B	F	F(-1)	F	F	F	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD	TBD
size	398	398	374	25572	0	0	0	0	66	0
N.S.	1	1.00	0.94	64.25	0.00	0.00	0.00	0.00	0.17	0.00
time (sec)	N/A	1.109	2.758	9.515	0.000	0.000	0.000	0.000	0.174	0.000

Problem 51	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	F	F	F	F	F(-1)	F	F	F(-1)
verified	N/A	Yes	N/A	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	157	157	0	0	0	0	0	0	40	0
N.S.	1	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.25	0.00
time (sec)	N/A	0.470	0.000	0.000	0.000	0.000	0.000	0.000	0.175	0.000

2.3 Detailed conclusion table specific for Rubi results

The following table is specific to Rubi only. It gives additional statistics for each integral. the column **steps** is the number of steps used by Rubi to obtain the antiderivative. The **rules** column is the number of unique rules used. The **integrand size** column is the leaf size of the integrand. Finally the ratio $\frac{\text{number of rules}}{\text{integrand size}}$ is also given. The larger this ratio is, the harder the integral is to solve. In this test file, problem number [29] had the largest ratio of [.515152000000000054]

Table 2.1: Rubi specific breakdown of results for each integral

#	grade	number of steps used	number of unique rules	normalized antiderivative leaf size	integrand leaf size	$\frac{\text{number of rules}}{\text{integrand leaf size}}$
1	A	3	3	1.00	32	0.094
2	A	3	3	1.00	32	0.094
3	A	3	3	1.00	30	0.100
4	A	7	7	0.90	24	0.292
5	A	3	3	1.00	30	0.100
6	A	5	5	0.83	32	0.156
7	A	3	3	1.00	32	0.094
8	A	11	10	1.05	32	0.312
9	A	3	3	1.00	32	0.094
10	A	3	3	1.00	32	0.094
11	A	3	3	1.00	32	0.094
12	A	12	12	1.45	34	0.353
13	A	9	8	1.00	34	0.235
14	A	13	12	1.06	34	0.353
15	A	8	7	1.00	40	0.175
16	A	4	3	1.00	40	0.075
17	A	8	7	1.00	40	0.175
18	A	8	7	1.00	40	0.175
19	A	5	5	1.04	36	0.139
20	A	12	11	1.03	36	0.306
21	A	12	11	1.04	36	0.306

Continued on next page

Table 2.1 – continued from previous page

#	grade	number of steps used	number of unique rules	normalized antiderivative leaf size	integrand leaf size	$\frac{\text{number of rules}}{\text{integrand leaf size}}$
22	A	6	5	1.00	36	0.139
23	A	7	6	1.00	33	0.182
24	A	12	11	1.07	33	0.333
25	A	8	7	1.00	39	0.179
26	A	4	3	1.00	39	0.077
27	A	8	7	1.00	39	0.179
28	A	8	7	1.00	39	0.179
29	A	17	17	1.15	33	0.515
30	A	17	17	1.07	33	0.515
31	A	5	5	1.00	39	0.128
32	A	2	2	1.00	39	0.051
33	A	5	5	1.00	39	0.128
34	A	5	5	1.00	39	0.128
35	A	8	7	1.00	35	0.200
36	A	4	3	1.00	35	0.086
37	A	8	7	1.00	35	0.200
38	A	8	7	1.00	35	0.200
39	A	10	9	1.24	33	0.273
40	A	6	6	1.00	33	0.182
41	A	6	6	1.00	33	0.182
42	A	5	5	1.00	39	0.128
43	A	5	5	1.00	39	0.128
44	A	2	2	1.00	39	0.051
45	A	5	5	1.00	39	0.128
46	A	5	5	1.00	39	0.128
47	A	2	2	1.00	39	0.051
48	A	5	5	1.00	35	0.143
49	A	2	2	1.00	35	0.057
50	A	5	5	1.00	35	0.143
51	A	6	5	1.00	38	0.132

CHAPTER 3

LISTING OF INTEGRALS

3.1	$\int \sin^3(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$	46
3.2	$\int \sin^2(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$	53
3.3	$\int \sin(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$	60
3.4	$\int (a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$	67
3.5	$\int \csc(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$	74
3.6	$\int \csc^2(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$	80
3.7	$\int \csc^3(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$	87
3.8	$\int \csc^4(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$	93
3.9	$\int \csc^5(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$	101
3.10	$\int \csc^6(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$	108
3.11	$\int \csc^7(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$	115
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3.15	$\int \frac{\sqrt{g \sin(e+fx)}\sqrt{a+a \sin(e+fx)}}{c-c \sin(e+fx)} dx$	148
3.16	$\int \frac{\sqrt{a+a \sin(e+fx)}}{\sqrt{g \sin(e+fx)}(c-c \sin(e+fx))} dx$	156
3.17	$\int \frac{\sqrt{g \sin(e+fx)}}{\sqrt{a+a \sin(e+fx)}(c-c \sin(e+fx))} dx$	162
3.18	$\int \frac{1}{\sqrt{g \sin(e+fx)}\sqrt{a+a \sin(e+fx)}(c-c \sin(e+fx))} dx$	171
3.19	$\int \csc(e + fx)\sqrt{a + a \sin(e + fx)}\sqrt{c - c \sin(e + fx)} dx$	179
3.20	$\int \frac{\csc(e+fx)\sqrt{a+a \sin(e+fx)}}{\sqrt{c-c \sin(e+fx)}} dx$	185
3.21	$\int \frac{\csc(e+fx)\sqrt{c-c \sin(e+fx)}}{\sqrt{a+a \sin(e+fx)}} dx$	193
3.22	$\int \frac{\csc(e+fx)}{\sqrt{a+a \sin(e+fx)}\sqrt{c-c \sin(e+fx)}} dx$	201
3.23	$\int \frac{\csc(e+fx)\sqrt{a+a \sin(e+fx)}}{c+d \sin(e+fx)} dx$	207

3.24	$\int \frac{\csc(e+fx)}{\sqrt{a+a \sin(e+fx)(c+d \sin(e+fx))}} dx$	214
3.25	$\int \frac{\sqrt{g \sin(e+fx)} \sqrt{a+a \sin(e+fx)}}{c+d \sin(e+fx)} dx$	223
3.26	$\int \frac{\sqrt{a+a \sin(e+fx)}}{\sqrt{g \sin(e+fx)(c+d \sin(e+fx))}} dx$	231
3.27	$\int \frac{\sqrt{g \sin(e+fx)}}{\sqrt{a+a \sin(e+fx)(c+d \sin(e+fx))}} dx$	238
3.28	$\int \frac{1}{\sqrt{g \sin(e+fx)} \sqrt{a+a \sin(e+fx)(c+d \sin(e+fx))}} dx$	246
3.29	$\int \frac{\csc(e+fx) \sqrt{a+b \sin(e+fx)}}{c+c \sin(e+fx)} dx$	254
3.30	$\int \frac{\csc(e+fx)}{\sqrt{a+b \sin(e+fx)(c+c \sin(e+fx))}} dx$	265
3.31	$\int \frac{\sqrt{g \sin(e+fx)} \sqrt{a+b \sin(e+fx)}}{c+c \sin(e+fx)} dx$	276
3.32	$\int \frac{\sqrt{a+b \sin(e+fx)}}{\sqrt{g \sin(e+fx)(c+c \sin(e+fx))}} dx$	283
3.33	$\int \frac{\sqrt{g \sin(e+fx)}}{\sqrt{a+b \sin(e+fx)(c+c \sin(e+fx))}} dx$	290
3.34	$\int \frac{1}{\sqrt{g \sin(e+fx)} \sqrt{a+b \sin(e+fx)(c+c \sin(e+fx))}} dx$	298
3.35	$\int \csc(e+fx) \sqrt{a+a \sin(e+fx)} \sqrt{c+d \sin(e+fx)} dx$	307
3.36	$\int \frac{\csc(e+fx) \sqrt{a+a \sin(e+fx)}}{\sqrt{c+d \sin(e+fx)}} dx$	315
3.37	$\int \frac{\csc(e+fx) \sqrt{c+d \sin(e+fx)}}{\sqrt{a+a \sin(e+fx)}} dx$	322
3.38	$\int \frac{\csc(e+fx)}{\sqrt{a+a \sin(e+fx)} \sqrt{c+d \sin(e+fx)}} dx$	331
3.39	$\int \frac{\sin^2(e+fx)}{(a+b \sin(e+fx))^2 (c+d \sin(e+fx))} dx$	340
3.40	$\int \frac{\csc(e+fx) \sqrt{c+d \sin(e+fx)}}{a+b \sin(e+fx)} dx$	350
3.41	$\int \frac{\csc(e+fx)}{(a+b \sin(e+fx)) \sqrt{c+d \sin(e+fx)}} dx$	357
3.42	$\int \frac{\sqrt{g \sin(e+fx)} \sqrt{a+b \sin(e+fx)}}{c+d \sin(e+fx)} dx$	364
3.43	$\int \frac{\sqrt{a+b \sin(e+fx)}}{\sqrt{g \sin(e+fx)(c+d \sin(e+fx))}} dx$	371
3.44	$\int \frac{\sqrt{g \sin(e+fx)}}{\sqrt{a+b \sin(e+fx)(c+d \sin(e+fx))}} dx$	378
3.45	$\int \frac{1}{\sqrt{g \sin(e+fx)} \sqrt{a+b \sin(e+fx)(c+d \sin(e+fx))}} dx$	384
3.46	$\int \frac{\sqrt{g \sin(e+fx)} \sqrt{c+d \sin(e+fx)}}{a+b \sin(e+fx)} dx$	392
3.47	$\int \frac{\sqrt{g \sin(e+fx)}}{(a+b \sin(e+fx)) \sqrt{c+d \sin(e+fx)}} dx$	399
3.48	$\int \csc(e+fx) \sqrt{a+b \sin(e+fx)} \sqrt{c+d \sin(e+fx)} dx$	405
3.49	$\int \frac{\csc(e+fx) \sqrt{a+b \sin(e+fx)}}{\sqrt{c+d \sin(e+fx)}} dx$	412
3.50	$\int \frac{\csc(e+fx)}{\sqrt{a+b \sin(e+fx)} \sqrt{c+d \sin(e+fx)}} dx$	418
3.51	$\int (a+a \sin(e+fx))^m (A+B \sin(e+fx))^p (c-c \sin(e+fx))^n dx$	425

3.1 $\int \sin^3(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$

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Optimal result

Integrand size = 32, antiderivative size = 121

$$\int \sin^3(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{1}{16}a^2cx - \frac{a^2c \cos^3(e + fx)}{3f} + \frac{a^2c \cos^5(e + fx)}{5f} - \frac{a^2c \cos(e + fx) \sin(e + fx)}{16f}$$

$$- \frac{a^2c \cos(e + fx) \sin^3(e + fx)}{24f} + \frac{a^2c \cos(e + fx) \sin^5(e + fx)}{6f}$$

output

```
1/16*a^2*c*x-1/3*a^2*c*cos(f*x+e)^3/f+1/5*a^2*c*cos(f*x+e)^5/f-1/16*a^2*c*cos(f*x+e)*sin(f*x+e)/f-1/24*a^2*c*cos(f*x+e)*sin(f*x+e)^3/f+1/6*a^2*c*cos(f*x+e)*sin(f*x+e)^5/f
```

Mathematica [A] (verified)

Time = 0.48 (sec) , antiderivative size = 77, normalized size of antiderivative = 0.64

$$\int \sin^3(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{a^2c(60e + 60fx - 120 \cos(e + fx) - 20 \cos(3(e + fx)) + 12 \cos(5(e + fx)) - 15 \sin(2(e + fx)) - 15 \sin(4(e + fx)) + 5 \sin(6(e + fx)))}{960f}$$

input

```
Integrate[Sin[e + f*x]^3*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output

```
(a^2*c*(60*e + 60*f*x - 120*Cos[e + f*x] - 20*Cos[3*(e + f*x)] + 12*Cos[5*(e + f*x)] - 15*Sin[2*(e + f*x)] - 15*Sin[4*(e + f*x)] + 5*Sin[6*(e + f*x)])/(960*f)
```

Rubi [A] (verified)

Time = 0.38 (sec) , antiderivative size = 121, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.094$, Rules used = {3042, 3445, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \sin^3(e + fx)(a \sin(e + fx) + a)^2(c - c \sin(e + fx)) dx$$

$$\downarrow \text{3042}$$

$$\int \sin(e + fx)^3(a \sin(e + fx) + a)^2(c - c \sin(e + fx)) dx$$

$$\downarrow \text{3445}$$

$$\int (-a^2c \sin^6(e + fx) - a^2c \sin^5(e + fx) + a^2c \sin^4(e + fx) + a^2c \sin^3(e + fx)) dx$$

$$\downarrow \text{2009}$$

$$\frac{a^2 c \cos^5(e + fx)}{5f} - \frac{a^2 c \cos^3(e + fx)}{3f} + \frac{a^2 c \sin^5(e + fx) \cos(e + fx)}{6f} - \frac{a^2 c \sin^3(e + fx) \cos(e + fx)}{24f} - \frac{a^2 c \sin(e + fx) \cos(e + fx)}{16f} + \frac{1}{16} a^2 c x$$

input `Int[Sin[e + f*x]^3*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]`

output `(a^2*c*x)/16 - (a^2*c*Cos[e + f*x]^3)/(3*f) + (a^2*c*Cos[e + f*x]^5)/(5*f) - (a^2*c*Cos[e + f*x]*Sin[e + f*x])/(16*f) - (a^2*c*Cos[e + f*x]*Sin[e + f*x]^3)/(24*f) + (a^2*c*Cos[e + f*x]*Sin[e + f*x]^5)/(6*f)`

Defintions of rubi rules used

rule 2009 `Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3445 `Int[sin[(e_.) + (f_.)*(x_)]^(n_.)*((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_.)*((A_.) + (B_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] := Int[ExpandTrig[sin[e + f*x]^n*(a + b*sin[e + f*x])^m*(A + B*sin[e + f*x]), x], x] /; FreeQ[{a, b, e, f, A, B}, x] && EqQ[A*b + a*B, 0] && EqQ[a^2 - b^2, 0] && IntegerQ[m] && IntegerQ[n]`

Maple [A] (verified)

Time = 78.11 (sec) , antiderivative size = 79, normalized size of antiderivative = 0.65

method	result
parallelrisch	$-\frac{a^2c(-60fx+120\cos(fx+e)-5\sin(6fx+6e)-12\cos(5fx+5e)+15\sin(4fx+4e)+20\cos(3fx+3e)+15\sin(2fx+2e)+128)}{960f}$
risch	$\frac{a^2cx}{16} - \frac{a^2c\cos(fx+e)}{8f} + \frac{a^2c\sin(6fx+6e)}{192f} + \frac{a^2c\cos(5fx+5e)}{80f} - \frac{a^2c\sin(4fx+4e)}{64f} - \frac{a^2c\cos(3fx+3e)}{48f} - \frac{a^2c\sin(2fx+2e)}{32f}$
derivativedivides	$-a^2c\left(-\frac{\left(\sin(fx+e)^5 + \frac{5\sin(fx+e)^3}{4} + \frac{15\sin(fx+e)}{8}\right)\cos(fx+e)}{6} + \frac{\frac{5fx}{16} + \frac{5e}{16}}{f}\right) + \frac{a^2c\left(\frac{8}{3} + \sin(fx+e)^4 + \frac{4\sin(fx+e)^2}{3}\right)\cos(fx+e)}{5}$
default	$-a^2c\left(-\frac{\left(\sin(fx+e)^5 + \frac{5\sin(fx+e)^3}{4} + \frac{15\sin(fx+e)}{8}\right)\cos(fx+e)}{6} + \frac{\frac{5fx}{16} + \frac{5e}{16}}{f}\right) + \frac{a^2c\left(\frac{8}{3} + \sin(fx+e)^4 + \frac{4\sin(fx+e)^2}{3}\right)\cos(fx+e)}{5}$
parts	$-\frac{a^2c(2+\sin(fx+e)^2)\cos(fx+e)}{3f} + \frac{a^2c\left(-\frac{\left(\sin(fx+e)^3 + \frac{3\sin(fx+e)}{2}\right)\cos(fx+e)}{4} + \frac{\frac{3fx}{8} + \frac{3e}{8}}{f}\right)}{f} + \frac{a^2c\left(\frac{8}{3} + \sin(fx+e)^4 + \frac{4\sin(fx+e)^2}{3}\right)\cos(fx+e)}{5}$
norman	$-\frac{4a^2c}{15f} + \frac{a^2cx}{16} - \frac{8a^2c\tan\left(\frac{fx}{2} + \frac{e}{2}\right)^6}{3f} - \frac{8a^2c\tan\left(\frac{fx}{2} + \frac{e}{2}\right)^2}{5f} - \frac{4a^2c\tan\left(\frac{fx}{2} + \frac{e}{2}\right)^8}{f} - \frac{a^2c\tan\left(\frac{fx}{2} + \frac{e}{2}\right)}{8f} - \frac{17a^2c\tan\left(\frac{fx}{2} + \frac{e}{2}\right)^3}{24f} + \frac{19a^2c\tan\left(\frac{fx}{2} + \frac{e}{2}\right)^5}{40f}$
orering	Expression too large to display

input `int(sin(f*x+e)^3*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x,method=_RETURNVERBOSE)`

output
$$-1/960*a^2*c*(-60*f*x+120*\cos(f*x+e)-5*\sin(6*f*x+6*e)-12*\cos(5*f*x+5*e)+15*\sin(4*f*x+4*e)+20*\cos(3*f*x+3*e)+15*\sin(2*f*x+2*e)+128)/f$$

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 91, normalized size of antiderivative = 0.75

$$\int \sin^3(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{48 a^2 c \cos(fx + e)^5 - 80 a^2 c \cos(fx + e)^3 + 15 a^2 c f x + 5 (8 a^2 c \cos(fx + e)^5 - 14 a^2 c \cos(fx + e)^3 + 3 a^2 c \cos(fx + e)) \sin(fx + e)}{240 f}$$

input `integrate(sin(f*x+e)^3*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="f
ricas")`

output `1/240*(48*a^2*c*cos(f*x + e)^5 - 80*a^2*c*cos(f*x + e)^3 + 15*a^2*c*f*x +
5*(8*a^2*c*cos(f*x + e)^5 - 14*a^2*c*cos(f*x + e)^3 + 3*a^2*c*cos(f*x + e)
) * sin(f*x + e))/f`

Sympy [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 415 vs. $2(110) = 220$.

Time = 0.38 (sec) , antiderivative size = 415, normalized size of antiderivative = 3.43

$$\int \sin^3(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \left\{ \begin{array}{l} -\frac{5a^2cx \sin^6(e+fx)}{16} - \frac{15a^2cx \sin^4(e+fx) \cos^2(e+fx)}{16} + \frac{3a^2cx \sin^4(e+fx)}{8} - \frac{15a^2cx \sin^2(e+fx) \cos^4(e+fx)}{16} + \frac{3a^2cx \sin^2(e+fx)}{4} \\ x(a \sin(e) + a)^2(-c \sin(e) + c) \sin^3(e) \end{array} \right.$$

input `integrate(sin(f*x+e)**3*(a+a*sin(f*x+e))**2*(c-c*sin(f*x+e)),x)`

output `Piecewise((-5*a**2*c*x*sin(e + f*x)**6/16 - 15*a**2*c*x*sin(e + f*x)**4*co
s(e + f*x)**2/16 + 3*a**2*c*x*sin(e + f*x)**4/8 - 15*a**2*c*x*sin(e + f*x)
2*cos(e + f*x)4/16 + 3*a**2*c*x*sin(e + f*x)**2*cos(e + f*x)**2/4 - 5*
a**2*c*x*cos(e + f*x)**6/16 + 3*a**2*c*x*cos(e + f*x)**4/8 + 11*a**2*c*sin
(e + f*x)**5*cos(e + f*x)/(16*f) + a**2*c*sin(e + f*x)**4*cos(e + f*x)/f +
5*a**2*c*sin(e + f*x)**3*cos(e + f*x)**3/(6*f) - 5*a**2*c*sin(e + f*x)**3
*cos(e + f*x)/(8*f) + 4*a**2*c*sin(e + f*x)**2*cos(e + f*x)**3/(3*f) - a**
2*c*sin(e + f*x)**2*cos(e + f*x)/f + 5*a**2*c*sin(e + f*x)*cos(e + f*x)**5
/(16*f) - 3*a**2*c*sin(e + f*x)*cos(e + f*x)**3/(8*f) + 8*a**2*c*cos(e + f
*x)**5/(15*f) - 2*a**2*c*cos(e + f*x)**3/(3*f), Ne(f, 0)), (x*(a*sin(e) +
a)**2*(-c*sin(e) + c)*sin(e)**3, True))`

Maxima [A] (verification not implemented)

Time = 0.04 (sec) , antiderivative size = 147, normalized size of antiderivative = 1.21

$$\int \sin^3(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{64(3 \cos(fx + e)^5 - 10 \cos(fx + e)^3 + 15 \cos(fx + e))a^2c + 320(\cos(fx + e)^3 - 3 \cos(fx + e))a^2c}{f}$$

input `integrate(sin(f*x+e)^3*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="maxima")`

output `1/960*(64*(3*cos(f*x + e)^5 - 10*cos(f*x + e)^3 + 15*cos(f*x + e))*a^2*c + 320*(cos(f*x + e)^3 - 3*cos(f*x + e))*a^2*c - 5*(4*sin(2*f*x + 2*e)^3 + 60*f*x + 60*e + 9*sin(4*f*x + 4*e) - 48*sin(2*f*x + 2*e))*a^2*c + 30*(12*f*x + 12*e + sin(4*f*x + 4*e) - 8*sin(2*f*x + 2*e))*a^2*c)/f`

Giac [A] (verification not implemented)

Time = 0.38 (sec) , antiderivative size = 113, normalized size of antiderivative = 0.93

$$\int \sin^3(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{1}{16} a^2 c x + \frac{a^2 c \cos(5fx + 5e)}{80f} - \frac{a^2 c \cos(3fx + 3e)}{48f} - \frac{a^2 c \cos(fx + e)}{8f}$$

$$+ \frac{a^2 c \sin(6fx + 6e)}{192f} - \frac{a^2 c \sin(4fx + 4e)}{64f} - \frac{a^2 c \sin(2fx + 2e)}{64f}$$

input `integrate(sin(f*x+e)^3*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="giac")`

output `1/16*a^2*c*x + 1/80*a^2*c*cos(5*f*x + 5*e)/f - 1/48*a^2*c*cos(3*f*x + 3*e)/f - 1/8*a^2*c*cos(f*x + e)/f + 1/192*a^2*c*sin(6*f*x + 6*e)/f - 1/64*a^2*c*sin(4*f*x + 4*e)/f - 1/64*a^2*c*sin(2*f*x + 2*e)/f`

Mupad [B] (verification not implemented)

Time = 38.17 (sec) , antiderivative size = 256, normalized size of antiderivative = 2.12

$$\int \sin^3(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{a^2 c \left(15e - 30 \tan\left(\frac{e}{2} + \frac{fx}{2}\right) - 384 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^2 - 170 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^3 + 1140 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^5 - 640 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^7 - 960 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^8 + 170 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^9 + 30 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^{11} + 15fx + 90 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^2(e + fx) + 225 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^4(e + fx) + 300 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^6(e + fx) + 225 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^8(e + fx) + 90 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^{10}(e + fx) + 15 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^{12}(e + fx) - 64 \right)}{(240f(\tan\left(\frac{e}{2} + \frac{fx}{2}\right)^2 + 1)^6)}$$

input

```
int(sin(e + f*x)^3*(a + a*sin(e + f*x))^2*(c - c*sin(e + f*x)),x)
```

output

```
(a^2*c*(15*e - 30*tan(e/2 + (f*x)/2) - 384*tan(e/2 + (f*x)/2)^2 - 170*tan(e/2 + (f*x)/2)^3 + 1140*tan(e/2 + (f*x)/2)^5 - 640*tan(e/2 + (f*x)/2)^6 - 1140*tan(e/2 + (f*x)/2)^7 - 960*tan(e/2 + (f*x)/2)^8 + 170*tan(e/2 + (f*x)/2)^9 + 30*tan(e/2 + (f*x)/2)^11 + 15*f*x + 90*tan(e/2 + (f*x)/2)^2*(e + f*x) + 225*tan(e/2 + (f*x)/2)^4*(e + f*x) + 300*tan(e/2 + (f*x)/2)^6*(e + f*x) + 225*tan(e/2 + (f*x)/2)^8*(e + f*x) + 90*tan(e/2 + (f*x)/2)^10*(e + f*x) + 15*tan(e/2 + (f*x)/2)^12*(e + f*x) - 64)/(240*f*(tan(e/2 + (f*x)/2)^2 + 1)^6)
```

Reduce [B] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 101, normalized size of antiderivative = 0.83

$$\int \sin^3(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{a^2 c (40 \cos(fx + e) \sin(fx + e)^5 + 48 \cos(fx + e) \sin(fx + e)^4 - 10 \cos(fx + e) \sin(fx + e)^3 - 16 \cos(fx + e) \sin(fx + e)^2 - 15 \cos(fx + e) \sin(fx + e) - 32 \cos(fx + e) + 15fx + 32)}{240f}$$

input

```
int(sin(f*x+e)^3*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x)
```

output

```
(a**2*c*(40*cos(e + f*x)*sin(e + f*x)**5 + 48*cos(e + f*x)*sin(e + f*x)**4 - 10*cos(e + f*x)*sin(e + f*x)**3 - 16*cos(e + f*x)*sin(e + f*x)**2 - 15*cos(e + f*x)*sin(e + f*x) - 32*cos(e + f*x) + 15*f*x + 32))/(240*f)
```

3.2 $\int \sin^2(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$

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Giac [A] (verification not implemented)	57
Mupad [B] (verification not implemented)	58
Reduce [B] (verification not implemented)	58

Optimal result

Integrand size = 32, antiderivative size = 96

$$\int \sin^2(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{1}{8}a^2cx - \frac{a^2c \cos^3(e + fx)}{3f} + \frac{a^2c \cos^5(e + fx)}{5f}$$

$$- \frac{a^2c \cos(e + fx) \sin(e + fx)}{8f} + \frac{a^2c \cos(e + fx) \sin^3(e + fx)}{4f}$$

```
output 1/8*a^2*c*x-1/3*a^2*c*cos(f*x+e)^3/f+1/5*a^2*c*cos(f*x+e)^5/f-1/8*a^2*c*cos(f*x+e)*sin(f*x+e)/f+1/4*a^2*c*cos(f*x+e)*sin(f*x+e)^3/f
```

Mathematica [A] (verified)

Time = 0.20 (sec) , antiderivative size = 57, normalized size of antiderivative = 0.59

$$\int \sin^2(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{a^2c(60e + 60fx - 60 \cos(e + fx) - 10 \cos(3(e + fx)) + 6 \cos(5(e + fx)) - 15 \sin(4(e + fx)))}{480f}$$

input `Integrate[Sin[e + f*x]^2*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]`

output `(a^2*c*(60*e + 60*f*x - 60*Cos[e + f*x] - 10*Cos[3*(e + f*x)] + 6*Cos[5*(e + f*x)] - 15*Sin[4*(e + f*x)])/(480*f)`

Rubi [A] (verified)

Time = 0.35 (sec) , antiderivative size = 96, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.094$, Rules used = {3042, 3445, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \sin^2(e + fx)(a \sin(e + fx) + a)^2(c - c \sin(e + fx)) dx$$

↓ 3042

$$\int \sin(e + fx)^2(a \sin(e + fx) + a)^2(c - c \sin(e + fx)) dx$$

↓ 3445

$$\int (-a^2 c \sin^5(e + fx) - a^2 c \sin^4(e + fx) + a^2 c \sin^3(e + fx) + a^2 c \sin^2(e + fx)) dx$$

↓ 2009

$$\frac{a^2 c \cos^5(e + fx)}{5f} - \frac{a^2 c \cos^3(e + fx)}{3f} + \frac{a^2 c \sin^3(e + fx) \cos(e + fx)}{4f} - \frac{a^2 c \sin(e + fx) \cos(e + fx)}{8f} + \frac{1}{8} a^2 c x$$

input `Int[Sin[e + f*x]^2*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]`

output `(a^2*c*x)/8 - (a^2*c*Cos[e + f*x]^3)/(3*f) + (a^2*c*Cos[e + f*x]^5)/(5*f) - (a^2*c*Cos[e + f*x]*Sin[e + f*x])/(8*f) + (a^2*c*Cos[e + f*x]*Sin[e + f*x]^3)/(4*f)`

Defintions of rubi rules used

```
rule 2009 Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]
```

```
rule 3042 Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]
```

```
rule 3445 Int[sin[(e_.) + (f_.)*(x_)]^(n_.)*((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_.)*((A_.) + (B_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] := Int[ExpandTrig[sin[e + f*x]^n*(a + b*sin[e + f*x])^m*(A + B*sin[e + f*x]), x], x] /; FreeQ[{a, b, e, f, A, B}, x] && EqQ[A*b + a*B, 0] && EqQ[a^2 - b^2, 0] && IntegerQ[m] && IntegerQ[n]
```

Maple [A] (verified)

Time = 24.72 (sec) , antiderivative size = 57, normalized size of antiderivative = 0.59

method	result
parallelrisc	$-\frac{a^2c(-60fx+60\cos(fx+e)-6\cos(5fx+5e)+15\sin(4fx+4e)+10\cos(3fx+3e)+64)}{480f}$
risc	$\frac{a^2cx}{8} - \frac{a^2c\cos(fx+e)}{8f} + \frac{a^2c\cos(5fx+5e)}{80f} - \frac{a^2c\sin(4fx+4e)}{32f} - \frac{a^2c\cos(3fx+3e)}{48f}$
derivativedivides	$\frac{a^2c\left(\frac{8}{3}+\sin(fx+e)^4+\frac{4\sin(fx+e)^2}{3}\right)\cos(fx+e)}{5} - a^2c\left(-\frac{\left(\sin(fx+e)^3+\frac{3\sin(fx+e)}{2}\right)\cos(fx+e)}{4} + \frac{3fx}{8} + \frac{3e}{8}\right) - \frac{a^2c(2+\sin(fx+e)^2)\cos(fx+e)}{f}$
default	$\frac{a^2c\left(\frac{8}{3}+\sin(fx+e)^4+\frac{4\sin(fx+e)^2}{3}\right)\cos(fx+e)}{5} - a^2c\left(-\frac{\left(\sin(fx+e)^3+\frac{3\sin(fx+e)}{2}\right)\cos(fx+e)}{4} + \frac{3fx}{8} + \frac{3e}{8}\right) - \frac{a^2c(2+\sin(fx+e)^2)\cos(fx+e)}{f}$
parts	$\frac{a^2c\left(-\frac{\sin(fx+e)\cos(fx+e)}{2} + \frac{fx}{2} + \frac{e}{2}\right)}{f} - \frac{a^2c(2+\sin(fx+e)^2)\cos(fx+e)}{3f} - \frac{a^2c\left(-\frac{\left(\sin(fx+e)^3+\frac{3\sin(fx+e)}{2}\right)\cos(fx+e)}{4} + \frac{3fx}{8} + \frac{3e}{8}\right)}{f}$
norman	$-\frac{4a^2c}{15f} + \frac{a^2cx}{8} - \frac{4a^2c\tan\left(\frac{fx}{2} + \frac{e}{2}\right)^6}{f} + \frac{4a^2c\tan\left(\frac{fx}{2} + \frac{e}{2}\right)^4}{3f} - \frac{4a^2c\tan\left(\frac{fx}{2} + \frac{e}{2}\right)^2}{3f} - \frac{a^2c\tan\left(\frac{fx}{2} + \frac{e}{2}\right)}{4f} + \frac{3a^2c\tan\left(\frac{fx}{2} + \frac{e}{2}\right)^3}{2f} - \frac{3a^2c}{2f}$
oring	Expression too large to display

input `int(sin(f*x+e)^2*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x,method=_RETURNVERBOSE)`

output `-1/480*a^2*c*(-60*f*x+60*cos(f*x+e)-6*cos(5*f*x+5*e)+15*sin(4*f*x+4*e)+10*cos(3*f*x+3*e)+64)/f`

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 77, normalized size of antiderivative = 0.80

$$\int \sin^2(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{24 a^2 c \cos(fx + e)^5 - 40 a^2 c \cos(fx + e)^3 + 15 a^2 c f x - 15 (2 a^2 c \cos(fx + e)^3 - a^2 c \cos(fx + e)) \sin(fx + e)}{120 f}$$

input `integrate(sin(f*x+e)^2*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="fricas")`

output `1/120*(24*a^2*c*cos(f*x + e)^5 - 40*a^2*c*cos(f*x + e)^3 + 15*a^2*c*f*x - 15*(2*a^2*c*cos(f*x + e)^3 - a^2*c*cos(f*x + e))*sin(f*x + e))/f`

Sympy [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 301 vs. 2(87) = 174.

Time = 0.27 (sec) , antiderivative size = 301, normalized size of antiderivative = 3.14

$$\int \sin^2(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \begin{cases} -\frac{3a^2cx \sin^4(e+fx)}{8} - \frac{3a^2cx \sin^2(e+fx) \cos^2(e+fx)}{4} + \frac{a^2cx \sin^2(e+fx)}{2} - \frac{3a^2cx \cos^4(e+fx)}{8} + \frac{a^2cx \cos^2(e+fx)}{2} + \frac{a^2c \sin^4(e+fx)}{8} \\ x(a \sin(e) + a)^2 (-c \sin(e) + c) \sin^2(e) \end{cases}$$

input `integrate(sin(f*x+e)**2*(a+a*sin(f*x+e))**2*(c-c*sin(f*x+e)),x)`

output

```
Piecewise((-3*a**2*c*x*sin(e + f*x)**4/8 - 3*a**2*c*x*sin(e + f*x)**2*cos(
e + f*x)**2/4 + a**2*c*x*sin(e + f*x)**2/2 - 3*a**2*c*x*cos(e + f*x)**4/8
+ a**2*c*x*cos(e + f*x)**2/2 + a**2*c*sin(e + f*x)**4*cos(e + f*x)/f + 5*a
**2*c*sin(e + f*x)**3*cos(e + f*x)/(8*f) + 4*a**2*c*sin(e + f*x)**2*cos(e
+ f*x)**3/(3*f) - a**2*c*sin(e + f*x)**2*cos(e + f*x)/f + 3*a**2*c*sin(e +
f*x)*cos(e + f*x)**3/(8*f) - a**2*c*sin(e + f*x)*cos(e + f*x)/(2*f) + 8*a
**2*c*cos(e + f*x)**5/(15*f) - 2*a**2*c*cos(e + f*x)**3/(3*f), Ne(f, 0)),
(x*(a*sin(e) + a)**2*(-c*sin(e) + c)*sin(e)**2, True))
```

Maxima [A] (verification not implemented)

Time = 0.04 (sec) , antiderivative size = 123, normalized size of antiderivative = 1.28

$$\int \sin^2(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{32 (3 \cos(fx + e)^5 - 10 \cos(fx + e)^3 + 15 \cos(fx + e))a^2c + 160 (\cos(fx + e)^3 - 3 \cos(fx + e))a^2c}{480}$$

input

```
integrate(sin(f*x+e)^2*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="m
axima")
```

output

```
1/480*(32*(3*cos(f*x + e)^5 - 10*cos(f*x + e)^3 + 15*cos(f*x + e))*a^2*c +
160*(cos(f*x + e)^3 - 3*cos(f*x + e))*a^2*c - 15*(12*f*x + 12*e + sin(4*f
*x + 4*e) - 8*sin(2*f*x + 2*e))*a^2*c + 120*(2*f*x + 2*e - sin(2*f*x + 2*
e))*a^2*c)/f
```

Giac [A] (verification not implemented)

Time = 0.49 (sec) , antiderivative size = 77, normalized size of antiderivative = 0.80

$$\int \sin^2(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{1}{8}a^2cx + \frac{a^2c \cos(5fx + 5e)}{80f} - \frac{a^2c \cos(3fx + 3e)}{48f}$$

$$- \frac{a^2c \cos(fx + e)}{8f} - \frac{a^2c \sin(4fx + 4e)}{32f}$$

input `integrate(sin(f*x+e)^2*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="giac")`

output $\frac{1}{8}a^2cx + \frac{1}{80}a^2c\cos(5fx + 5e)/f - \frac{1}{48}a^2c\cos(3fx + 3e)/f - \frac{1}{8}a^2c\cos(fx + e)/f - \frac{1}{32}a^2c\sin(4fx + 4e)/f$

Mupad [B] (verification not implemented)

Time = 38.32 (sec) , antiderivative size = 212, normalized size of antiderivative = 2.21

$$\int \sin^2(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{a^2c \left(15e - 30 \tan\left(\frac{e}{2} + \frac{fx}{2}\right) - 160 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^2 + 180 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^3 + 160 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^4 - 480 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^5 + 150 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^6 - 80 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^7 + 30 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^8 + 15fx + 75 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^2(e + fx) + 150 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^4(e + fx) + 150 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^6(e + fx) + 75 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^8(e + fx) + 15 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^{10}(e + fx) - 32 \right)}{(120f \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^2 + 1)^5}$$

input `int(sin(e + f*x)^2*(a + a*sin(e + f*x))^2*(c - c*sin(e + f*x)),x)`

output $(a^2c(15e - 30 \tan(e/2 + (fx)/2) - 160 \tan(e/2 + (fx)/2)^2 + 180 \tan(e/2 + (fx)/2)^3 + 160 \tan(e/2 + (fx)/2)^4 - 480 \tan(e/2 + (fx)/2)^5 + 150 \tan(e/2 + (fx)/2)^6 - 80 \tan(e/2 + (fx)/2)^7 + 30 \tan(e/2 + (fx)/2)^8 + 15fx + 75 \tan(e/2 + (fx)/2)^2(e + fx) + 150 \tan(e/2 + (fx)/2)^4(e + fx) + 150 \tan(e/2 + (fx)/2)^6(e + fx) + 75 \tan(e/2 + (fx)/2)^8(e + fx) + 15 \tan(e/2 + (fx)/2)^{10}(e + fx) - 32)/(120f \tan(e/2 + (fx)/2)^2 + 1)^5$

Reduce [B] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 85, normalized size of antiderivative = 0.89

$$\int \sin^2(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{a^2c(24 \cos(fx + e) \sin(fx + e)^4 + 30 \cos(fx + e) \sin(fx + e)^3 - 8 \cos(fx + e) \sin(fx + e)^2 - 15 \cos(fx + e) \sin(fx + e) + 15fx + 75 \sin(fx + e)^2(e + fx) + 150 \sin(fx + e)^4(e + fx) + 150 \sin(fx + e)^6(e + fx) + 75 \sin(fx + e)^8(e + fx) + 15 \sin(fx + e)^{10}(e + fx) - 32)}{120f}$$

input `int(sin(f*x+e)^2*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x)`

output

```
(a**2*c*(24*cos(e + f*x)*sin(e + f*x)**4 + 30*cos(e + f*x)*sin(e + f*x)**3  
- 8*cos(e + f*x)*sin(e + f*x)**2 - 15*cos(e + f*x)*sin(e + f*x) - 16*cos(  
e + f*x) + 15*f*x + 16))/(120*f)
```

3.3 $\int \sin(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$

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Optimal result

Integrand size = 30, antiderivative size = 77

$$\int \sin(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{1}{8}a^2cx - \frac{a^2c \cos^3(e + fx)}{3f} - \frac{a^2c \cos(e + fx) \sin(e + fx)}{8f}$$

$$+ \frac{a^2c \cos(e + fx) \sin^3(e + fx)}{4f}$$

```
output 1/8*a^2*c*x-1/3*a^2*c*cos(f*x+e)^3/f-1/8*a^2*c*cos(f*x+e)*sin(f*x+e)/f+1/4
*a^2*c*cos(f*x+e)*sin(f*x+e)^3/f
```

Mathematica [A] (verified)

Time = 0.19 (sec) , antiderivative size = 47, normalized size of antiderivative = 0.61

$$\int \sin(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{a^2c(12e + 12fx - 24 \cos(e + fx) - 8 \cos(3(e + fx)) - 3 \sin(4(e + fx)))}{96f}$$

input `Integrate[Sin[e + f*x]*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]`

output `(a^2*c*(12*e + 12*f*x - 24*Cos[e + f*x] - 8*Cos[3*(e + f*x)] - 3*Sin[4*(e + f*x)]))/(96*f)`

Rubi [A] (verified)

Time = 0.30 (sec) , antiderivative size = 77, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.100$, Rules used = {3042, 3445, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \sin(e + fx)(a \sin(e + fx) + a)^2(c - c \sin(e + fx)) dx$$

$$\downarrow 3042$$

$$\int \sin(e + fx)(a \sin(e + fx) + a)^2(c - c \sin(e + fx)) dx$$

$$\downarrow 3445$$

$$\int (-a^2 c \sin^4(e + fx) - a^2 c \sin^3(e + fx) + a^2 c \sin^2(e + fx) + a^2 c \sin(e + fx)) dx$$

$$\downarrow 2009$$

$$-\frac{a^2 c \cos^3(e + fx)}{3f} + \frac{a^2 c \sin^3(e + fx) \cos(e + fx)}{4f} - \frac{a^2 c \sin(e + fx) \cos(e + fx)}{8f} + \frac{1}{8} a^2 c x$$

input `Int[Sin[e + f*x]*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]`

output `(a^2*c*x)/8 - (a^2*c*Cos[e + f*x]^3)/(3*f) - (a^2*c*Cos[e + f*x]*Sin[e + f*x])/(8*f) + (a^2*c*Cos[e + f*x]*Sin[e + f*x]^3)/(4*f)`

Defintions of rubi rules used

```
rule 2009 Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]
```

```
rule 3042 Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]
```

```
rule 3445 Int[sin[(e_.) + (f_.)*(x_)]^(n_.)*((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_.)*((A_.) + (B_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] := Int[ExpandTrig[sin[e + f*x]^n*(a + b*sin[e + f*x])^m*(A + B*sin[e + f*x]), x], x] /; FreeQ[{a, b, e, f, A, B}, x] && EqQ[A*b + a*B, 0] && EqQ[a^2 - b^2, 0] && IntegerQ[m] && IntegerQ[n]
```

Maple [A] (verified)

Time = 7.97 (sec) , antiderivative size = 46, normalized size of antiderivative = 0.60

method	result
parallelrisc	$-\frac{a^2c(-12fx+24\cos(fx+e)+3\sin(4fx+4e)+8\cos(3fx+3e)+32)}{96f}$
risc	$\frac{a^2cx}{8} - \frac{a^2c\cos(fx+e)}{4f} - \frac{a^2c\sin(4fx+4e)}{32f} - \frac{a^2c\cos(3fx+3e)}{12f}$
derivativedivides	$-a^2c\left(-\frac{(\sin(fx+e)^3 + \frac{3\sin(\frac{fx+e}{2})}{2})\cos(fx+e)}{4} + \frac{3fx + \frac{3e}{8}}{8}\right) + \frac{a^2c(2+\sin(fx+e)^2)\cos(fx+e)}{3} + a^2c\left(-\frac{\sin(fx+e)\cos(fx+e)}{2}\right) + \frac{f}{f}$
default	$-a^2c\left(-\frac{(\sin(fx+e)^3 + \frac{3\sin(\frac{fx+e}{2})}{2})\cos(fx+e)}{4} + \frac{3fx + \frac{3e}{8}}{8}\right) + \frac{a^2c(2+\sin(fx+e)^2)\cos(fx+e)}{3} + a^2c\left(-\frac{\sin(fx+e)\cos(fx+e)}{2}\right) + \frac{f}{f}$
parts	$-\frac{a^2c\cos(fx+e)}{f} + \frac{a^2c\left(-\frac{\sin(fx+e)\cos(fx+e)}{2} + \frac{fx + \frac{e}{2}}{2}\right)}{f} + \frac{a^2c(2+\sin(fx+e)^2)\cos(fx+e)}{3f} - \frac{a^2c\left(-\frac{\sin(fx+e)\cos(fx+e)}{2}\right)}{f}$
norman	$\frac{-2a^2c + a^2cx}{3f} - \frac{2a^2c\tan\left(\frac{fx}{2} + \frac{e}{2}\right)^2}{3f} - \frac{2a^2c\tan\left(\frac{fx}{2} + \frac{e}{2}\right)^4}{f} - \frac{2a^2c\tan\left(\frac{fx}{2} + \frac{e}{2}\right)^6}{f} - \frac{a^2c\tan\left(\frac{fx}{2} + \frac{e}{2}\right)}{4f} + \frac{7a^2c\tan\left(\frac{fx}{2} + \frac{e}{2}\right)^3}{4f} - \frac{7a^2c}{4f} \left(1 + \tan\left(\frac{fx}{2} + \frac{e}{2}\right)\right)$

```
input int(sin(f*x+e)*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x,method=_RETURNVERBOSE)
```

output $-1/96*a^2*c*(-12*f*x+24*\cos(f*x+e)+3*\sin(4*f*x+4*e)+8*\cos(3*f*x+3*e)+32)/f$

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 63, normalized size of antiderivative = 0.82

$$\int \sin(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx =$$

$$\frac{8 a^2 c \cos(fx + e)^3 - 3 a^2 c f x + 3 (2 a^2 c \cos(fx + e)^3 - a^2 c \cos(fx + e)) \sin(fx + e)}{24 f}$$

input `integrate(sin(f*x+e)*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="fricas")`

output $-1/24*(8*a^2*c*\cos(f*x + e)^3 - 3*a^2*c*f*x + 3*(2*a^2*c*\cos(f*x + e)^3 - a^2*c*\cos(f*x + e))*\sin(f*x + e))/f$

Sympy [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 245 vs. $2(70) = 140$.

Time = 0.21 (sec) , antiderivative size = 245, normalized size of antiderivative = 3.18

$$\int \sin(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \left\{ \begin{array}{l} -\frac{3a^2cx \sin^4(e+fx)}{8} - \frac{3a^2cx \sin^2(e+fx) \cos^2(e+fx)}{4} + \frac{a^2cx \sin^2(e+fx)}{2} - \frac{3a^2cx \cos^4(e+fx)}{8} + \frac{a^2cx \cos^2(e+fx)}{2} + \frac{5a^2c \sin^3(e+fx)}{8} \\ x(a \sin(e) + a)^2(-c \sin(e) + c) \sin(e) \end{array} \right.$$

input `integrate(sin(f*x+e)*(a+a*sin(f*x+e))**2*(c-c*sin(f*x+e)),x)`

output

```
Piecewise((-3*a**2*c*x*sin(e + f*x)**4/8 - 3*a**2*c*x*sin(e + f*x)**2*cos(
e + f*x)**2/4 + a**2*c*x*sin(e + f*x)**2/2 - 3*a**2*c*x*cos(e + f*x)**4/8
+ a**2*c*x*cos(e + f*x)**2/2 + 5*a**2*c*sin(e + f*x)**3*cos(e + f*x)/(8*f)
+ a**2*c*sin(e + f*x)**2*cos(e + f*x)/f + 3*a**2*c*sin(e + f*x)*cos(e + f
*x)**3/(8*f) - a**2*c*sin(e + f*x)*cos(e + f*x)/(2*f) + 2*a**2*c*cos(e + f
*x)**3/(3*f) - a**2*c*cos(e + f*x)/f, Ne(f, 0)), (x*(a*sin(e) + a)**2*(-c*
sin(e) + c)*sin(e), True))
```

Maxima [A] (verification not implemented)

Time = 0.04 (sec) , antiderivative size = 100, normalized size of antiderivative = 1.30

$$\int \sin(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx = \frac{32(\cos(fx + e)^3 - 3 \cos(fx + e))a^2c + 3(12fx + 12e + \sin(4fx + 4e) - 8 \sin(2fx + 2e))a^2c - 96f}{96f}$$

input

```
integrate(sin(f*x+e)*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="max
ima")
```

output

```
-1/96*(32*(cos(f*x + e)^3 - 3*cos(f*x + e))*a^2*c + 3*(12*f*x + 12*e + sin
(4*f*x + 4*e) - 8*sin(2*f*x + 2*e))*a^2*c - 24*(2*f*x + 2*e - sin(2*f*x +
2*e))*a^2*c + 96*a^2*c*cos(f*x + e))/f
```

Giac [A] (verification not implemented)

Time = 0.35 (sec) , antiderivative size = 59, normalized size of antiderivative = 0.77

$$\int \sin(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx = \frac{1}{8}a^2cx - \frac{a^2c \cos(3fx + 3e)}{12f} - \frac{a^2c \cos(fx + e)}{4f} - \frac{a^2c \sin(4fx + 4e)}{32f}$$

input

```
integrate(sin(f*x+e)*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="gia
c")
```

output

$$\frac{1}{8}a^2cx - \frac{1}{12}a^2c\cos(3fx + 3e)/f - \frac{1}{4}a^2c\cos(fx + e)/f - \frac{1}{32}a^2c\sin(4fx + 4e)/f$$

Mupad [B] (verification not implemented)

Time = 37.98 (sec) , antiderivative size = 250, normalized size of antiderivative = 3.25

$$\int \sin(e + fx)(a + a\sin(e + fx))^2(c - c\sin(e + fx)) dx = \frac{a^2 cx}{8} \\ - \frac{\tan\left(\frac{e}{2} + \frac{fx}{2}\right)^2 \left(\frac{a^2 c(3e+3fx)}{6} - \frac{a^2 c(12e+12fx-16)}{24}\right) + \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^6 \left(\frac{a^2 c(3e+3fx)}{6} - \frac{a^2 c(12e+12fx-48)}{24}\right)}{1}$$

input

```
int(sin(e + f*x)*(a + a*sin(e + f*x))^2*(c - c*sin(e + f*x)),x)
```

output

$$\frac{(a^2cx)/8 - (\tan(e/2 + (fx)/2))^2((a^2c(3e + 3fx))/6 - (a^2c(12e + 12fx - 16))/24) + \tan(e/2 + (fx)/2)^6((a^2c(3e + 3fx))/6 - (a^2c(12e + 12fx - 48))/24) + \tan(e/2 + (fx)/2)^4((a^2c(3e + 3fx))/4 - (a^2c(18e + 18fx - 48))/24) + (a^2c\tan(e/2 + (fx)/2))/4 - (7a^2c\tan(e/2 + (fx)/2)^3)/4 + (7a^2c\tan(e/2 + (fx)/2)^5)/4 - (a^2c\tan(e/2 + (fx)/2)^7)/4 + (a^2c(3e + 3fx))/24 - (a^2c(3e + 3fx - 16))/24}{f(\tan(e/2 + (fx)/2)^2 + 1)^4}$$

Reduce [B] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 69, normalized size of antiderivative = 0.90

$$\int \sin(e + fx)(a + a\sin(e + fx))^2(c - c\sin(e + fx)) dx \\ = \frac{a^2c(6\cos(fx + e)\sin(fx + e)^3 + 8\cos(fx + e)\sin(fx + e)^2 - 3\cos(fx + e)\sin(fx + e) - 8\cos(fx + e))}{24f}$$

input

```
int(sin(f*x+e)*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x)
```

output

```
(a**2*c*(6*cos(e + f*x)*sin(e + f*x)**3 + 8*cos(e + f*x)*sin(e + f*x)**2 -  
3*cos(e + f*x)*sin(e + f*x) - 8*cos(e + f*x) + 3*f*x + 8))/(24*f)
```

3.4 $\int (a + a \sin(e + fx))^2 (c - c \sin(e + fx)) dx$

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Optimal result

Integrand size = 24, antiderivative size = 52

$$\int (a + a \sin(e + fx))^2 (c - c \sin(e + fx)) dx$$

$$= \frac{1}{2} a^2 c x - \frac{a^2 c \cos^3(e + fx)}{3f} + \frac{a^2 c \cos(e + fx) \sin(e + fx)}{2f}$$

output `1/2*a^2*c*x-1/3*a^2*c*cos(f*x+e)^3/f+1/2*a^2*c*cos(f*x+e)*sin(f*x+e)/f`

Mathematica [A] (verified)

Time = 0.18 (sec) , antiderivative size = 43, normalized size of antiderivative = 0.83

$$\int (a + a \sin(e + fx))^2 (c - c \sin(e + fx)) dx$$

$$= -\frac{a^2 c (3 \cos(e + fx) + \cos(3(e + fx)) - 3(2fx + \sin(2(e + fx))))}{12f}$$

input `Integrate[(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]`

output `-1/12*(a^2*c*(3*Cos[e + f*x] + Cos[3*(e + f*x)] - 3*(2*f*x + Sin[2*(e + f*x)])))/f`

Rubi [A] (verified)

Time = 0.31 (sec) , antiderivative size = 47, normalized size of antiderivative = 0.90, number of steps used = 7, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}} = 0.292$, Rules used = {3042, 3215, 3042, 3148, 3042, 3115, 24}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int (a \sin(e + fx) + a)^2 (c - c \sin(e + fx)) dx \\
 & \quad \downarrow \text{3042} \\
 & \int (a \sin(e + fx) + a)^2 (c - c \sin(e + fx)) dx \\
 & \quad \downarrow \text{3215} \\
 & ac \int \cos^2(e + fx) (\sin(e + fx)a + a) dx \\
 & \quad \downarrow \text{3042} \\
 & ac \int \cos(e + fx)^2 (\sin(e + fx)a + a) dx \\
 & \quad \downarrow \text{3148} \\
 & ac \left(a \int \cos^2(e + fx) dx - \frac{a \cos^3(e + fx)}{3f} \right) \\
 & \quad \downarrow \text{3042} \\
 & ac \left(a \int \sin \left(e + fx + \frac{\pi}{2} \right)^2 dx - \frac{a \cos^3(e + fx)}{3f} \right) \\
 & \quad \downarrow \text{3115} \\
 & ac \left(a \left(\frac{\int 1 dx}{2} + \frac{\sin(e + fx) \cos(e + fx)}{2f} \right) - \frac{a \cos^3(e + fx)}{3f} \right) \\
 & \quad \downarrow \text{24} \\
 & ac \left(a \left(\frac{\sin(e + fx) \cos(e + fx)}{2f} + \frac{x}{2} \right) - \frac{a \cos^3(e + fx)}{3f} \right)
 \end{aligned}$$

input

```
Int[(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output $a*c*(-1/3*(a*\cos[e + f*x]^3)/f + a*(x/2 + (\cos[e + f*x]*\sin[e + f*x])/(2*f)))$

Defintions of rubi rules used

rule 24 $\text{Int}[a_, x_Symbol] \rightarrow \text{Simp}[a*x, x] /; \text{FreeQ}[a, x]$

rule 3042 $\text{Int}[u_, x_Symbol] \rightarrow \text{Int}[\text{DeactivateTrig}[u, x], x] /; \text{FunctionOfTrigOfLinearQ}[u, x]$

rule 3115 $\text{Int}[(b_*\sin[c_*] + (d_*)*(x_*))^n, x_Symbol] \rightarrow \text{Simp}[(-b)*\cos[c + d*x]*(b*\sin[c + d*x])^{n-1}/(d*n), x] + \text{Simp}[b^2*((n-1)/n) \text{Int}[(b*\sin[c + d*x])^{n-2}, x], x] /; \text{FreeQ}\{b, c, d\}, x \ \&\& \ \text{GtQ}[n, 1] \ \&\& \ \text{IntegerQ}[2*n]$

rule 3148 $\text{Int}[(\cos[e_*] + (f_*)*(x_*))*(g_*)^p*((a_*) + (b_*)*\sin[e_*] + (f_*)*(x_*))], x_Symbol] \rightarrow \text{Simp}[(-b)*((g*\cos[e + f*x])^{p+1}/(f*g*(p+1))), x] + \text{Simp}[a \text{Int}[(g*\cos[e + f*x])^p, x], x] /; \text{FreeQ}\{a, b, e, f, g, p\}, x \ \&\& \ (\text{IntegerQ}[2*p] \ || \ \text{NeQ}[a^2 - b^2, 0])$

rule 3215 $\text{Int}[(a_* + (b_*)*\sin[e_*] + (f_*)*(x_*))^m*((c_*) + (d_*)*\sin[e_*] + (f_*)*(x_*))^n], x_Symbol] \rightarrow \text{Simp}[a^m*c^m \text{Int}[\cos[e + f*x]^{2*m}*(c + d*\sin[e + f*x])^{n-m}, x], x] /; \text{FreeQ}\{a, b, c, d, e, f, n\}, x \ \&\& \ \text{EqQ}[b*c + a*d, 0] \ \&\& \ \text{EqQ}[a^2 - b^2, 0] \ \&\& \ \text{IntegerQ}[m] \ \&\& \ !(\text{IntegerQ}[n] \ \&\& \ ((\text{LtQ}[m, 0] \ \&\& \ \text{GtQ}[n, 0]) \ || \ \text{LtQ}[0, n, m] \ || \ \text{LtQ}[m, n, 0]))$

Maple [A] (verified)

Time = 11.34 (sec) , antiderivative size = 44, normalized size of antiderivative = 0.85

method	result
parallelrisch	$-\frac{a^2c(-6fx+3\cos(fx+e)+\cos(3fx+3e)-3\sin(2fx+2e)+4)}{12f}$
risch	$\frac{a^2cx}{2} - \frac{a^2c\cos(fx+e)}{4f} - \frac{a^2c\cos(3fx+3e)}{12f} + \frac{a^2c\sin(2fx+2e)}{4f}$
derivativedivides	$\frac{\frac{a^2c(2+\sin(fx+e)^2)\cos(fx+e)}{3} - a^2c\left(-\frac{\sin(fx+e)\cos(fx+e)}{2} + \frac{fx}{2} + \frac{e}{2}\right) - \cos(fx+e)a^2c + a^2c(fx+e)}{f}$
default	$\frac{\frac{a^2c(2+\sin(fx+e)^2)\cos(fx+e)}{3} - a^2c\left(-\frac{\sin(fx+e)\cos(fx+e)}{2} + \frac{fx}{2} + \frac{e}{2}\right) - \cos(fx+e)a^2c + a^2c(fx+e)}{f}$
parts	$a^2cx - \frac{a^2c\cos(fx+e)}{f} - \frac{a^2c\left(-\frac{\sin(fx+e)\cos(fx+e)}{2} + \frac{fx}{2} + \frac{e}{2}\right)}{f} + \frac{a^2c(2+\sin(fx+e)^2)\cos(fx+e)}{3f}$
norman	$\frac{\frac{a^2c\tan\left(\frac{fx}{2} + \frac{e}{2}\right)}{f} - \frac{2a^2c}{3f} + \frac{a^2cx}{2} - \frac{2a^2c\tan\left(\frac{fx}{2} + \frac{e}{2}\right)^4}{f} - \frac{a^2c\tan\left(\frac{fx}{2} + \frac{e}{2}\right)^5}{f} + \frac{3a^2cx\tan\left(\frac{fx}{2} + \frac{e}{2}\right)^2}{2} + \frac{3a^2cx\tan\left(\frac{fx}{2} + \frac{e}{2}\right)^4}{2} + \frac{a^2cx}{2}}{\left(1+\tan\left(\frac{fx}{2} + \frac{e}{2}\right)\right)^3}$

input `int((a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x,method=_RETURNVERBOSE)`output `-1/12*a^2*c*(-6*f*x+3*cos(f*x+e)+cos(3*f*x+3*e)-3*sin(2*f*x+2*e)+4)/f`**Fricas [A] (verification not implemented)**

Time = 0.09 (sec) , antiderivative size = 46, normalized size of antiderivative = 0.88

$$\int (a + a \sin(e + fx))^2 (c - c \sin(e + fx)) dx$$

$$= -\frac{2a^2c\cos(fx+e)^3 - 3a^2cfx - 3a^2c\cos(fx+e)\sin(fx+e)}{6f}$$

input `integrate((a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="fricas")`output `-1/6*(2*a^2*c*cos(f*x + e)^3 - 3*a^2*c*f*x - 3*a^2*c*cos(f*x + e)*sin(f*x + e))/f`

Sympy [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 133 vs. $2(46) = 92$.

Time = 0.14 (sec) , antiderivative size = 133, normalized size of antiderivative = 2.56

$$\int (a + a \sin(e + fx))^2 (c - c \sin(e + fx)) dx$$

$$= \begin{cases} -\frac{a^2 c x \sin^2(e+fx)}{2} - \frac{a^2 c x \cos^2(e+fx)}{2} + a^2 c x + \frac{a^2 c \sin^2(e+fx) \cos(e+fx)}{f} + \frac{a^2 c \sin(e+fx) \cos(e+fx)}{2f} + \frac{2a^2 c \cos^3(e+fx)}{3f} \\ x(a \sin(e) + a)^2 (-c \sin(e) + c) \end{cases}$$

input `integrate((a+a*sin(f*x+e))**2*(c-c*sin(f*x+e)),x)`

output `Piecewise((-a**2*c*x*sin(e + f*x)**2/2 - a**2*c*x*cos(e + f*x)**2/2 + a**2*c*x + a**2*c*sin(e + f*x)**2*cos(e + f*x)/f + a**2*c*sin(e + f*x)*cos(e + f*x)/(2*f) + 2*a**2*c*cos(e + f*x)**3/(3*f) - a**2*c*cos(e + f*x)/f, Ne(f, 0)), (x*(a*sin(e) + a)**2*(-c*sin(e) + c), True))`

Maxima [A] (verification not implemented)

Time = 0.04 (sec) , antiderivative size = 77, normalized size of antiderivative = 1.48

$$\int (a + a \sin(e + fx))^2 (c - c \sin(e + fx)) dx =$$

$$\frac{4(\cos(fx + e)^3 - 3\cos(fx + e))a^2c + 3(2fx + 2e - \sin(2fx + 2e))a^2c - 12(fx + e)a^2c + 12a^2c}{12f}$$

input `integrate((a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="maxima")`

output `-1/12*(4*(cos(f*x + e)^3 - 3*cos(f*x + e))*a^2*c + 3*(2*f*x + 2*e - sin(2*f*x + 2*e))*a^2*c - 12*(f*x + e)*a^2*c + 12*a^2*c*cos(f*x + e))/f`

Giac [A] (verification not implemented)

Time = 0.32 (sec) , antiderivative size = 59, normalized size of antiderivative = 1.13

$$\int (a + a \sin(e + fx))^2 (c - c \sin(e + fx)) dx$$

$$= \frac{1}{2} a^2 c x - \frac{a^2 c \cos(3fx + 3e)}{12f} - \frac{a^2 c \cos(fx + e)}{4f} + \frac{a^2 c \sin(2fx + 2e)}{4f}$$

input `integrate((a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="giac")`

output `1/2*a^2*c*x - 1/12*a^2*c*cos(3*f*x + 3*e)/f - 1/4*a^2*c*cos(f*x + e)/f + 1/4*a^2*c*sin(2*f*x + 2*e)/f`

Mupad [B] (verification not implemented)

Time = 37.50 (sec) , antiderivative size = 125, normalized size of antiderivative = 2.40

$$\int (a + a \sin(e + fx))^2 (c - c \sin(e + fx)) dx = \frac{a^2 c x}{2}$$

$$\frac{\tan\left(\frac{e}{2} + \frac{fx}{2}\right)^4 \left(\frac{3a^2 c(e+fx)}{2} - \frac{a^2 c(9e+9fx-12)}{6}\right) - a^2 c \tan\left(\frac{e}{2} + \frac{fx}{2}\right) + \frac{a^2 c(e+fx)}{2} + a^2 c \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^5 - a^2 c \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^3}{f \left(\tan\left(\frac{e}{2} + \frac{fx}{2}\right)^2 + 1\right)^3}$$

input `int((a + a*sin(e + f*x))^2*(c - c*sin(e + f*x)),x)`

output `(a^2*c*x)/2 - (tan(e/2 + (f*x)/2)^4*((3*a^2*c*(e + f*x))/2 - (a^2*c*(9*e + 9*f*x - 12))/6) - a^2*c*tan(e/2 + (f*x)/2) + (a^2*c*(e + f*x))/2 + a^2*c*tan(e/2 + (f*x)/2)^5 - (a^2*c*(3*e + 3*f*x - 4))/6)/(f*(tan(e/2 + (f*x)/2)^2 + 1)^3)`

Reduce [B] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 53, normalized size of antiderivative = 1.02

$$\int (a + a \sin(e + fx))^2 (c - c \sin(e + fx)) dx$$

$$= \frac{a^2 c (2 \cos(fx + e) \sin(fx + e)^2 + 3 \cos(fx + e) \sin(fx + e) - 2 \cos(fx + e) + 3fx + 2)}{6f}$$

input `int((a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x)`

output `(a**2*c*(2*cos(e + f*x)*sin(e + f*x)**2 + 3*cos(e + f*x)*sin(e + f*x) - 2*cos(e + f*x) + 3*f*x + 2))/(6*f)`

3.5 $\int \csc(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$

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Giac [A] (verification not implemented)	78
Mupad [B] (verification not implemented)	79
Reduce [B] (verification not implemented)	79

Optimal result

Integrand size = 30, antiderivative size = 63

$$\int \csc(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{1}{2}a^2cx - \frac{a^2c \operatorname{arctanh}(\cos(e + fx))}{f} + \frac{a^2c \cos(e + fx)}{f} + \frac{a^2c \cos(e + fx) \sin(e + fx)}{2f}$$

output 1/2*a^2*c*x-a^2*c*arctanh(cos(f*x+e))/f+a^2*c*cos(f*x+e)/f+1/2*a^2*c*cos(f*x+e)*sin(f*x+e)/f

Mathematica [A] (verified)

Time = 0.16 (sec) , antiderivative size = 44, normalized size of antiderivative = 0.70

$$\int \csc(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{a^2c(-2e + 2fx - 4\operatorname{arctanh}(\cos(e + fx)) + 4\cos(e + fx) + \sin(2(e + fx)))}{4f}$$

input Integrate[Csc[e + f*x]*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]

output

$$\frac{(a^2c(-2e + 2fx - 4\text{ArcTanh}[\text{Cos}[e + fx]] + 4\text{Cos}[e + fx] + \text{Sin}[2(e + fx)])))/(4f)}$$

Rubi [A] (verified)

Time = 0.28 (sec) , antiderivative size = 63, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.100$, Rules used = {3042, 3445, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \csc(e + fx)(a \sin(e + fx) + a)^2(c - c \sin(e + fx)) dx \\ & \quad \downarrow \text{3042} \\ & \int \frac{(a \sin(e + fx) + a)^2(c - c \sin(e + fx))}{\sin(e + fx)} dx \\ & \quad \downarrow \text{3445} \\ & \int (a^2(-c) \sin^2(e + fx) - a^2c \sin(e + fx) + a^2c \csc(e + fx) + a^2c) dx \\ & \quad \downarrow \text{2009} \\ & -\frac{a^2 \text{arctanh}(\cos(e + fx))}{f} + \frac{a^2c \cos(e + fx)}{f} + \frac{a^2c \sin(e + fx) \cos(e + fx)}{2f} + \frac{1}{2}a^2cx \end{aligned}$$

input

$$\text{Int}[\text{Csc}[e + fx]*(a + a*\text{Sin}[e + fx])^2*(c - c*\text{Sin}[e + fx]),x]$$

output

$$\frac{(a^2c*x)}{2} - \frac{(a^2c*\text{ArcTanh}[\text{Cos}[e + fx]])}{f} + \frac{(a^2c*\text{Cos}[e + fx])}{f} + \frac{(a^2c*\text{Cos}[e + fx]*\text{Sin}[e + fx])}{(2*f)}$$

Definitions of rubi rules used

rule 2009 `Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3445 `Int[sin[(e_.) + (f_.)*(x_)]^(n_.)*((a_.) + (b_.)*sin[(e_.) + (f_.)*(x_)]^(m_.)*((A_.) + (B_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] := Int[ExpandTrig[sin[e + f*x]^n*(a + b*sin[e + f*x])^m*(A + B*sin[e + f*x]), x], x] /; FreeQ[{a, b, e, f, A, B}, x] && EqQ[A*b + a*B, 0] && EqQ[A^2 - b^2, 0] && IntegerQ[m] && IntegerQ[n]`

Maple [A] (verified)

Time = 0.65 (sec) , antiderivative size = 45, normalized size of antiderivative = 0.71

method	result
parallelrisc	$\frac{a^2c(2fx+4+4\ln(\tan(\frac{fx}{2}+\frac{e}{2}))+\sin(2fx+2e)+4\cos(fx+e))}{4f}$
derivativedivides	$\frac{-a^2c(-\frac{\sin(fx+e)\cos(fx+e)}{2}+\frac{fx}{2}+\frac{e}{2})+\cos(fx+e)a^2c+a^2c(fx+e)+a^2c\ln(\csc(fx+e)-\cot(fx+e))}{f}$
default	$\frac{-a^2c(-\frac{\sin(fx+e)\cos(fx+e)}{2}+\frac{fx}{2}+\frac{e}{2})+\cos(fx+e)a^2c+a^2c(fx+e)+a^2c\ln(\csc(fx+e)-\cot(fx+e))}{f}$
risc	$\frac{a^2cx}{2} + \frac{a^2ce^{i(fx+e)}}{2f} + \frac{a^2ce^{-i(fx+e)}}{2f} + \frac{a^2c\ln(e^{i(fx+e)}-1)}{f} - \frac{a^2c\ln(e^{i(fx+e)}+1)}{f} + \frac{a^2c\sin(2fx+2e)}{4f}$
norman	$\frac{a^2c\tan(\frac{fx}{2}+\frac{e}{2})}{f} + \frac{a^2cx}{2} - \frac{2a^2c\tan(\frac{fx}{2}+\frac{e}{2})^2}{f} - \frac{2a^2c\tan(\frac{fx}{2}+\frac{e}{2})^6}{f} - \frac{4a^2c\tan(\frac{fx}{2}+\frac{e}{2})^4}{f} - \frac{a^2c\tan(\frac{fx}{2}+\frac{e}{2})^5}{f} + \frac{3a^2cx\tan(\frac{fx}{2}+\frac{e}{2})}{2} \cdot \frac{1}{\left(1+\tan(\frac{fx}{2}+\frac{e}{2})\right)^3}$

input `int(csc(f*x+e)*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x,method=_RETURNVERBOSE)`

output `1/4*a^2*c*(2*f*x+4+4*ln(tan(1/2*f*x+1/2*e))+sin(2*f*x+2*e)+4*cos(f*x+e))/f`

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 75, normalized size of antiderivative = 1.19

$$\int \csc(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{a^2 c f x + a^2 c \cos(fx + e) \sin(fx + e) + 2 a^2 c \cos(fx + e) - a^2 c \log\left(\frac{1}{2} \cos(fx + e) + \frac{1}{2}\right) + a^2 c \log\left(-\frac{1}{2} \cos(fx + e) + \frac{1}{2}\right)}{2 f}$$

input `integrate(csc(f*x+e)*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="fricas")`

output `1/2*(a^2*c*f*x + a^2*c*cos(f*x + e)*sin(f*x + e) + 2*a^2*c*cos(f*x + e) - a^2*c*log(1/2*cos(f*x + e) + 1/2) + a^2*c*log(-1/2*cos(f*x + e) + 1/2))/f`

Sympy [F]

$$\int \csc(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= -a^2 c \left(\int (-\sin(e + fx) \csc(e + fx)) dx + \int \sin^2(e + fx) \csc(e + fx) dx \right. \\ \left. + \int \sin^3(e + fx) \csc(e + fx) dx + \int (-\csc(e + fx)) dx \right)$$

input `integrate(csc(f*x+e)*(a+a*sin(f*x+e))**2*(c-c*sin(f*x+e)),x)`

output `-a**2*c*(Integral(-sin(e + f*x)*csc(e + f*x), x) + Integral(sin(e + f*x)**2*csc(e + f*x), x) + Integral(sin(e + f*x)**3*csc(e + f*x), x) + Integral(-csc(e + f*x), x))`

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 73, normalized size of antiderivative = 1.16

$$\int \csc(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx =$$

$$\frac{(2fx + 2e - \sin(2fx + 2e))a^2c - 4(fx + e)a^2c - 4a^2c \cos(fx + e) + 4a^2c \log(\cot(fx + e) + \csc(fx + e))}{4f}$$

input

```
integrate(csc(f*x+e)*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="maxima")
```

output

```
-1/4*((2*f*x + 2*e - sin(2*f*x + 2*e))*a^2*c - 4*(f*x + e)*a^2*c - 4*a^2*c*cos(f*x + e) + 4*a^2*c*log(cot(f*x + e) + csc(f*x + e)))/f
```

Giac [A] (verification not implemented)

Time = 0.37 (sec) , antiderivative size = 105, normalized size of antiderivative = 1.67

$$\int \csc(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{(fx + e)a^2c + 2a^2c \log\left(\left|\tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)\right|\right) - \frac{2\left(a^2c \tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)^3 - 2a^2c \tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)^2 - a^2c \tan\left(\frac{1}{2}fx + \frac{1}{2}e\right) - 2a^2c\right)}{\left(\tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)^2 + 1\right)^2}}{2f}$$

input

```
integrate(csc(f*x+e)*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="giac")
```

output

```
1/2*((f*x + e)*a^2*c + 2*a^2*c*log(abs(tan(1/2*f*x + 1/2*e)))) - 2*(a^2*c*tan(1/2*f*x + 1/2*e)^3 - 2*a^2*c*tan(1/2*f*x + 1/2*e)^2 - a^2*c*tan(1/2*f*x + 1/2*e) - 2*a^2*c)/(tan(1/2*f*x + 1/2*e)^2 + 1)/f
```

Mupad [B] (verification not implemented)

Time = 35.92 (sec) , antiderivative size = 88, normalized size of antiderivative = 1.40

$$\int \csc(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{a^2 c \left(\cos(e + fx) + \ln \left(\frac{\sin\left(\frac{e}{2} + \frac{fx}{2}\right)}{\cos\left(\frac{e}{2} + \frac{fx}{2}\right)} \right) + \frac{\sin(2e + 2fx)}{4} + \operatorname{atan} \left(\frac{\sqrt{5} \left(\cos\left(\frac{e}{2} + \frac{fx}{2}\right) + 2 \sin\left(\frac{e}{2} + \frac{fx}{2}\right) \right)}{5 \cos\left(\frac{e}{2} + \operatorname{atan}\left(\frac{1}{2}\right) + \frac{fx}{2}\right)} \right) \right)}{f}$$

input `int(((a + a*sin(e + f*x))^2*(c - c*sin(e + f*x)))/sin(e + f*x),x)`

output `(a^2*c*(cos(e + f*x) + log(sin(e/2 + (f*x)/2)/cos(e/2 + (f*x)/2)) + sin(2*e + 2*f*x)/4 + atan((5^(1/2)*(cos(e/2 + (f*x)/2) + 2*sin(e/2 + (f*x)/2)))/(5*cos(e/2 + atan(1/2) + (f*x)/2))))/f`

Reduce [B] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 48, normalized size of antiderivative = 0.76

$$\int \csc(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{a^2 c (\cos(fx + e) \sin(fx + e) + 2 \cos(fx + e) + 2 \log(\tan(\frac{fx}{2} + \frac{e}{2})) + e + fx - 2)}{2f}$$

input `int(csc(f*x+e)*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x)`

output `(a**2*c*(cos(e + f*x)*sin(e + f*x) + 2*cos(e + f*x) + 2*log(tan((e + f*x)/2)) + e + f*x - 2))/(2*f)`

3.6 $\int \csc^2(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$

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Optimal result

Integrand size = 32, antiderivative size = 53

$$\int \csc^2(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= -a^2cx - \frac{a^2c \operatorname{arctanh}(\cos(e + fx))}{f} + \frac{a^2c \cos(e + fx)}{f} - \frac{a^2c \cot(e + fx)}{f}$$

output `-a^2*c*x-a^2*c*arctanh(cos(f*x+e))/f+a^2*c*cos(f*x+e)/f-a^2*c*cot(f*x+e)/f`

Mathematica [A] (verified)

Time = 0.15 (sec) , antiderivative size = 68, normalized size of antiderivative = 1.28

$$\int \csc^2(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= -a^2cx - \frac{a^2c \operatorname{arctanh}(\cos(e + fx))}{f} + \frac{a^2c \cos(e) \cos(fx)}{f}$$

$$- \frac{a^2c \cot(e + fx)}{f} - \frac{a^2c \sin(e) \sin(fx)}{f}$$

input `Integrate[Csc[e + f*x]^2*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]`

output $-(a^2*c*x) - (a^2*c*ArcTanh[Cos[e + f*x]])/f + (a^2*c*Cos[e]*Cos[f*x])/f - (a^2*c*Cot[e + f*x])/f - (a^2*c*Sin[e]*Sin[f*x])/f$

Rubi [A] (verified)

Time = 0.36 (sec) , antiderivative size = 44, normalized size of antiderivative = 0.83, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.156$, Rules used = {3042, 3429, 3042, 3189, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \csc^2(e + fx)(a \sin(e + fx) + a)^2(c - c \sin(e + fx)) dx \\ & \quad \downarrow \text{3042} \\ & \int \frac{(a \sin(e + fx) + a)^2(c - c \sin(e + fx))}{\sin(e + fx)^2} dx \\ & \quad \downarrow \text{3429} \\ & ac \int \cot^2(e + fx)(\sin(e + fx)a + a) dx \\ & \quad \downarrow \text{3042} \\ & ac \int \frac{\sin(e + fx)a + a}{\tan(e + fx)^2} dx \\ & \quad \downarrow \text{3189} \\ & ac \int (a \cot^2(e + fx) + a \cos(e + fx) \cot(e + fx)) dx \\ & \quad \downarrow \text{2009} \\ & ac \left(-\frac{a \operatorname{arctanh}(\cos(e + fx))}{f} + \frac{a \cos(e + fx)}{f} - \frac{a \cot(e + fx)}{f} - ax \right) \end{aligned}$$

input `Int[Csc[e + f*x]^2*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]`

output

```
a*c*(-(a*x) - (a*ArcTanh[Cos[e + f*x]])/f + (a*Cos[e + f*x])/f - (a*Cot[e + f*x])/f)
```

Defintions of rubi rules used

rule 2009

```
Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 3042

```
Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]
```

rule 3189

```
Int[((a_) + (b_)*sin[(e_) + (f_)*(x_)])^(m_)*((g_)*tan[(e_) + (f_)*(x_)])^(p_), x_Symbol] := Int[ExpandIntegrand[(g*Tan[e + f*x])^p, (a + b*Sin[e + f*x])^m, x], x] /; FreeQ[{a, b, e, f, g, p}, x] && EqQ[a^2 - b^2, 0] && IGtQ[m, 0]
```

rule 3429

```
Int[sin[(e_) + (f_)*(x_)]^(p_)*((a_) + (b_)*sin[(e_) + (f_)*(x_)])^(m_)*((c_) + (d_)*sin[(e_) + (f_)*(x_)])^(n_), x_Symbol] := Simp[a^n*c^n Int[Tan[e + f*x]^p*(a + b*Sin[e + f*x])^(m - n), x], x] /; FreeQ[{a, b, c, d, e, f, m}, x] && EqQ[b*c + a*d, 0] && EqQ[a^2 - b^2, 0] && EqQ[p + 2*n, 0] && IntegerQ[n]
```

Maple [A] (verified)

Time = 1.09 (sec) , antiderivative size = 61, normalized size of antiderivative = 1.15

method	result
derivativedivides	$\frac{\cos(fx+e)a^2c-a^2c(fx+e)+a^2c\ln(\csc(fx+e)-\cot(fx+e))-a^2c\cot(fx+e)}{f}$
default	$\frac{\cos(fx+e)a^2c-a^2c(fx+e)+a^2c\ln(\csc(fx+e)-\cot(fx+e))-a^2c\cot(fx+e)}{f}$
parallelrisc	$\frac{a^2c(-2fx-2+2\ln(\tan(\frac{fx}{2}+\frac{e}{2}))+2\cos(fx+e)+2\tan(\frac{fx}{2}+\frac{e}{2}))-sec(\frac{fx}{2}+\frac{e}{2})\csc(\frac{fx}{2}+\frac{e}{2})}{2f}$
risc	$-a^2cx + \frac{a^2ce^{i(fx+e)}}{2f} + \frac{a^2ce^{-i(fx+e)}}{2f} - \frac{2ia^2c}{f(e^{2i(fx+e)}-1)} - \frac{a^2c\ln(e^{i(fx+e)}+1)}{f} + \frac{a^2c\ln(e^{i(fx+e)}-1)}{f}$
norman	$\frac{\frac{a^2c\tan(\frac{fx}{2}+\frac{e}{2})^6}{f} - \frac{a^2c}{2f} - \frac{2a^2c\tan(\frac{fx}{2}+\frac{e}{2})^3}{f} - \frac{2a^2c\tan(\frac{fx}{2}+\frac{e}{2})^7}{f} - \frac{4a^2c\tan(\frac{fx}{2}+\frac{e}{2})^5}{f} - \frac{a^2c\tan(\frac{fx}{2}+\frac{e}{2})^2}{f} + \frac{a^2c\tan(\frac{fx}{2}+\frac{e}{2})}{2f}}{\tan(\frac{fx}{2}+\frac{e}{2})\left(1+\tan(\frac{fx}{2}+\frac{e}{2})\right)^2}$

input `int(csc(f*x+e)^2*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x,method=_RETURNVERBOSE)`

output `1/f*(cos(f*x+e)*a^2*c-a^2*c*(f*x+e)+a^2*c*ln(csc(f*x+e)-cot(f*x+e))-a^2*c*cot(f*x+e))`

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 99, normalized size of antiderivative = 1.87

$$\int \csc^2(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))dx = \frac{a^2c\log\left(\frac{1}{2}\cos(fx+e)+\frac{1}{2}\right)\sin(fx+e) - a^2c\log\left(-\frac{1}{2}\cos(fx+e)+\frac{1}{2}\right)\sin(fx+e) + 2a^2c\cos(fx+e)}{2f\sin(fx+e)}$$

input `integrate(csc(f*x+e)^2*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x,algorithm="fricas")`

output `-1/2*(a^2*c*log(1/2*cos(f*x + e) + 1/2)*sin(f*x + e) - a^2*c*log(-1/2*cos(f*x + e) + 1/2)*sin(f*x + e) + 2*a^2*c*cos(f*x + e) + 2*(a^2*c*f*x - a^2*c*cos(f*x + e))*sin(f*x + e))/(f*sin(f*x + e))`

Sympy [F]

$$\begin{aligned} & \int \csc^2(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx \\ &= -a^2c \left(\int (-\sin(e + fx) \csc^2(e + fx)) dx + \int \sin^2(e + fx) \csc^2(e + fx) dx \right. \\ & \quad \left. + \int \sin^3(e + fx) \csc^2(e + fx) dx + \int (-\csc^2(e + fx)) dx \right) \end{aligned}$$

input `integrate(csc(f*x+e)**2*(a+a*sin(f*x+e))**2*(c-c*sin(f*x+e)),x)`

output `-a**2*c*(Integral(-sin(e + f*x)*csc(e + f*x)**2, x) + Integral(sin(e + f*x)
)**2*csc(e + f*x)**2, x) + Integral(sin(e + f*x)**3*csc(e + f*x)**2, x) +
Integral(-csc(e + f*x)**2, x))`

Maxima [A] (verification not implemented)

Time = 0.04 (sec) , antiderivative size = 69, normalized size of antiderivative = 1.30

$$\begin{aligned} & \int \csc^2(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx = \\ & \frac{2(fx + e)a^2c + a^2c(\log(\cos(fx + e) + 1) - \log(\cos(fx + e) - 1)) - 2a^2c \cos(fx + e) + \frac{2a^2c}{\tan(fx + e)}}{2f} \end{aligned}$$

input `integrate(csc(f*x+e)^2*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="m
axima")`

output `-1/2*(2*(f*x + e)*a^2*c + a^2*c*(log(cos(f*x + e) + 1) - log(cos(f*x + e)
- 1)) - 2*a^2*c*cos(f*x + e) + 2*a^2*c/tan(f*x + e))/f`

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 129 vs. $2(53) = 106$.

Time = 0.37 (sec) , antiderivative size = 129, normalized size of antiderivative = 2.43

$$\int \csc^2(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx =$$

$$\frac{6(fx + e)a^2c - 6a^2c \log\left(\left|\tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)\right|\right) - 3a^2c \tan\left(\frac{1}{2}fx + \frac{1}{2}e\right) + \frac{2a^2c \tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)^3 + 3a^2c \tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)}{\tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)^3 + \tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)} + \frac{3a^2c \tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)^2}{\tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)^3 + \tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)} + \frac{3a^2c \tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)}{\tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)^3 + \tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)} + \frac{3a^2c}{\tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)^3 + \tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)}}{6f}$$

input `integrate(csc(f*x+e)^2*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="giac")`

output `-1/6*(6*(f*x + e)*a^2*c - 6*a^2*c*log(abs(tan(1/2*f*x + 1/2*e))) - 3*a^2*c*tan(1/2*f*x + 1/2*e) + (2*a^2*c*tan(1/2*f*x + 1/2*e)^3 + 3*a^2*c*tan(1/2*f*x + 1/2*e)^2 - 10*a^2*c*tan(1/2*f*x + 1/2*e) + 3*a^2*c)/(tan(1/2*f*x + 1/2*e)^3 + tan(1/2*f*x + 1/2*e)))/f`

Mupad [B] (verification not implemented)

Time = 35.67 (sec) , antiderivative size = 110, normalized size of antiderivative = 2.08

$$\int \csc^2(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{a^2 c \left(2 \operatorname{atan} \left(\frac{\sqrt{2} \left(\cos \left(\frac{e}{2} + \frac{fx}{2} \right) - \sin \left(\frac{e}{2} + \frac{fx}{2} \right) \right)}{2 \cos \left(\frac{e}{2} - \frac{\pi}{4} + \frac{fx}{2} \right)} \right) + \ln \left(\frac{\sin \left(\frac{e}{2} + \frac{fx}{2} \right)}{\cos \left(\frac{e}{2} + \frac{fx}{2} \right)} \right) \right)}{f}$$

$$= \frac{a^2 c \left(\cos(e + fx) - \frac{\sin(2e + 2fx)}{2} \right)}{f \sin(e + fx)}$$

input `int(((a + a*sin(e + f*x))^2*(c - c*sin(e + f*x)))/sin(e + f*x)^2,x)`

output `(a^2*c*(2*atan((2^(1/2)*(cos(e/2 + (f*x)/2) - sin(e/2 + (f*x)/2)))/(2*cos(e/2 - pi/4 + (f*x)/2))) + log(sin(e/2 + (f*x)/2)/cos(e/2 + (f*x)/2)))/f - (a^2*c*(cos(e + f*x) - sin(2*e + 2*f*x)/2))/(f*sin(e + f*x))`

Reduce [B] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 73, normalized size of antiderivative = 1.38

$$\int \csc^2(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{a^2 c (\cos(fx + e) \sin(fx + e) - \cos(fx + e) + \log(\tan(\frac{fx}{2} + \frac{e}{2})) \sin(fx + e) - \sin(fx + e) fx - \sin(fx + e))}{\sin(fx + e) f}$$

input `int(csc(f*x+e)^2*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x)`output `(a**2*c*(cos(e + f*x)*sin(e + f*x) - cos(e + f*x) + log(tan((e + f*x)/2)))*sin(e + f*x) - sin(e + f*x)*f*x - sin(e + f*x))/(sin(e + f*x)*f)`

3.7 $\int \csc^3(e+fx)(a+a \sin(e+fx))^2(c-c \sin(e+fx)) dx$

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Optimal result

Integrand size = 32, antiderivative size = 64

$$\int \csc^3(e+fx)(a+a \sin(e+fx))^2(c-c \sin(e+fx)) dx$$

$$= -a^2cx + \frac{a^2c \operatorname{arctanh}(\cos(e+fx))}{2f} - \frac{a^2c \cot(e+fx)}{f} - \frac{a^2c \cot(e+fx) \csc(e+fx)}{2f}$$

output

```
-a^2*c*x+1/2*a^2*c*arctanh(cos(f*x+e))/f-a^2*c*cot(f*x+e)/f-1/2*a^2*c*cot(f*x+e)*csc(f*x+e)/f
```

Mathematica [A] (verified)

Time = 1.05 (sec) , antiderivative size = 95, normalized size of antiderivative = 1.48

$$\int \csc^3(e+fx)(a+a \sin(e+fx))^2(c-c \sin(e+fx)) dx =$$

$$\frac{a^2c(8e+8fx+4 \cot(\frac{1}{2}(e+fx))+\csc^2(\frac{1}{2}(e+fx))-4 \log(\cos(\frac{1}{2}(e+fx))))+4 \log(\sin(\frac{1}{2}(e+fx)))}{8f}$$

input

```
Integrate[Csc[e+f*x]^3*(a+a*Sin[e+f*x])^2*(c-c*Sin[e+f*x]),x]
```


output

```
-1/8*(a^2*c*(8*e + 8*f*x + 4*Cot[(e + f*x)/2] + Csc[(e + f*x)/2]^2 - 4*Log
[Cos[(e + f*x)/2]] + 4*Log[Sin[(e + f*x)/2]] - Sec[(e + f*x)/2]^2 - 4*Tan[
(e + f*x)/2]))/f
```

Rubi [A] (verified)

Time = 0.31 (sec) , antiderivative size = 64, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.094$, Rules used = {3042, 3445, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \csc^3(e + fx)(a \sin(e + fx) + a)^2(c - c \sin(e + fx)) dx$$

$$\downarrow 3042$$

$$\int \frac{(a \sin(e + fx) + a)^2(c - c \sin(e + fx))}{\sin(e + fx)^3} dx$$

$$\downarrow 3445$$

$$\int (a^2 c \csc^3(e + fx) + a^2 c \csc^2(e + fx) - a^2 c \csc(e + fx) - a^2 c) dx$$

$$\downarrow 2009$$

$$\frac{a^2 c \operatorname{arctanh}(\cos(e + fx))}{2f} - \frac{a^2 c \cot(e + fx)}{f} - \frac{a^2 c \cot(e + fx) \csc(e + fx)}{2f} + a^2(-c)x$$

input

```
Int[Csc[e + f*x]^3*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output

```
-(a^2*c*x) + (a^2*c*ArcTanh[Cos[e + f*x]])/(2*f) - (a^2*c*Cot[e + f*x])/f
- (a^2*c*Cot[e + f*x]*Csc[e + f*x])/(2*f)
```

Defintions of rubi rules used

rule 2009 `Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3445 `Int[sin[(e_.) + (f_.)*(x_)]^(n_.)*((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_.)*((A_.) + (B_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] := Int[ExpandTrig[sin[e + f*x]^n*(a + b*sin[e + f*x])^m*(A + B*sin[e + f*x]), x], x] /; FreeQ[{a, b, e, f, A, B}, x] && EqQ[A*b + a*B, 0] && EqQ[a^2 - b^2, 0] && IntegerQ[m] && IntegerQ[n]`

Maple [A] (verified)

Time = 1.48 (sec) , antiderivative size = 73, normalized size of antiderivative = 1.14

method	result
parallelrisc	$-\frac{a^2c\left(\cot\left(\frac{fx}{2}+\frac{e}{2}\right)^2-\tan\left(\frac{fx}{2}+\frac{e}{2}\right)^2+8fx+4\ln\left(\tan\left(\frac{fx}{2}+\frac{e}{2}\right)\right)+4\cot\left(\frac{fx}{2}+\frac{e}{2}\right)-4\tan\left(\frac{fx}{2}+\frac{e}{2}\right)\right)}{8f}$
derivativedivides	$\frac{-a^2c(fx+e)-a^2c\ln(\csc(fx+e)-\cot(fx+e))-a^2c\cot(fx+e)+a^2c\left(-\frac{\csc(fx+e)\cot(fx+e)}{2}+\frac{\ln(\csc(fx+e)-\cot(fx+e))}{2}\right)}{f}$
default	$\frac{-a^2c(fx+e)-a^2c\ln(\csc(fx+e)-\cot(fx+e))-a^2c\cot(fx+e)+a^2c\left(-\frac{\csc(fx+e)\cot(fx+e)}{2}+\frac{\ln(\csc(fx+e)-\cot(fx+e))}{2}\right)}{f}$
risc	$-a^2cx + \frac{a^2c(e^{3i(fx+e)}+e^{i(fx+e)}-2ie^{2i(fx+e)}+2i)}{f(e^{2i(fx+e)}-1)^2} + \frac{a^2c\ln(e^{i(fx+e)}+1)}{2f} - \frac{a^2c\ln(e^{i(fx+e)}-1)}{2f}$
norman	$\frac{a^2c\tan\left(\frac{fx}{2}+\frac{e}{2}\right)^7}{f}-\frac{a^2c}{8f}-\frac{3a^2c\tan\left(\frac{fx}{2}+\frac{e}{2}\right)^2}{4f}-\frac{7a^2c\tan\left(\frac{fx}{2}+\frac{e}{2}\right)^6}{8f}-\frac{11a^2c\tan\left(\frac{fx}{2}+\frac{e}{2}\right)^4}{8f}-\frac{a^2c\tan\left(\frac{fx}{2}+\frac{e}{2}\right)}{2f}-\frac{a^2c\tan\left(\frac{fx}{2}+\frac{e}{2}\right)}{f}-\tan\left(\frac{fx}{2}+\frac{e}{2}\right)$

input `int(csc(f*x+e)^3*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x,method=_RETURNVERBOSE)`

output `-1/8*a^2*c*(cot(1/2*f*x+1/2*e)^2-tan(1/2*f*x+1/2*e)^2+8*f*x+4*ln(tan(1/2*f*x+1/2*e))+4*cot(1/2*f*x+1/2*e)-4*tan(1/2*f*x+1/2*e))/f`

Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 138 vs. $2(60) = 120$.

Time = 0.09 (sec) , antiderivative size = 138, normalized size of antiderivative = 2.16

$$\int \csc^3(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx = \frac{4a^2cfx \cos(fx + e)^2 - 4a^2cfx - 4a^2c \cos(fx + e) \sin(fx + e) - 2a^2c \cos(fx + e) - (a^2c \cos(fx + e) - a^2c \cos(fx + e))}{4(f \cos(fx + e))}$$

input `integrate(csc(f*x+e)^3*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="fricas")`

output `-1/4*(4*a^2*c*f*x*cos(f*x + e)^2 - 4*a^2*c*f*x - 4*a^2*c*cos(f*x + e)*sin(f*x + e) - 2*a^2*c*cos(f*x + e) - (a^2*c*cos(f*x + e)^2 - a^2*c)*log(1/2*cos(f*x + e) + 1/2) + (a^2*c*cos(f*x + e)^2 - a^2*c)*log(-1/2*cos(f*x + e) + 1/2))/(f*cos(f*x + e)^2 - f)`

Sympy [F]

$$\begin{aligned} & \int \csc^3(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx \\ &= -a^2c \left(\int (-\sin(e + fx) \csc^3(e + fx)) dx + \int \sin^2(e + fx) \csc^3(e + fx) dx \right. \\ & \quad \left. + \int \sin^3(e + fx) \csc^3(e + fx) dx + \int (-\csc^3(e + fx)) dx \right) \end{aligned}$$

input `integrate(csc(f*x+e)**3*(a+a*sin(f*x+e))**2*(c-c*sin(f*x+e)),x)`

output `-a**2*c*(Integral(-sin(e + f*x)*csc(e + f*x)**3, x) + Integral(sin(e + f*x)**2*csc(e + f*x)**3, x) + Integral(sin(e + f*x)**3*csc(e + f*x)**3, x) + Integral(-csc(e + f*x)**3, x))`

Maxima [A] (verification not implemented)

Time = 0.04 (sec) , antiderivative size = 105, normalized size of antiderivative = 1.64

$$\int \csc^3(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx =$$

$$\frac{4(fx + e)a^2c - a^2c\left(\frac{2 \cos(fx+e)}{\cos(fx+e)^2-1} - \log(\cos(fx + e) + 1) + \log(\cos(fx + e) - 1)\right) - 2a^2c(\log(\cos(fx + e) + 1) - \log(\cos(fx + e) - 1)) + 4a^2c/\tan(fx + e)}{4f}$$

input `integrate(csc(f*x+e)^3*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="maxima")`

output `-1/4*(4*(f*x + e)*a^2*c - a^2*c*(2*cos(f*x + e)/(cos(f*x + e)^2 - 1) - log(cos(f*x + e) + 1) + log(cos(f*x + e) - 1)) - 2*a^2*c*(log(cos(f*x + e) + 1) - log(cos(f*x + e) - 1)) + 4*a^2*c/tan(f*x + e))/f`

Giac [A] (verification not implemented)

Time = 0.35 (sec) , antiderivative size = 116, normalized size of antiderivative = 1.81

$$\int \csc^3(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{a^2c \tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)^2 - 8(fx + e)a^2c - 4a^2c \log\left(\left|\tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)\right|\right) + 4a^2c \tan\left(\frac{1}{2}fx + \frac{1}{2}e\right) + \frac{6a^2c \tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)}{\tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)^2}}{8f}$$

input `integrate(csc(f*x+e)^3*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="giac")`

output `1/8*(a^2*c*tan(1/2*f*x + 1/2*e)^2 - 8*(f*x + e)*a^2*c - 4*a^2*c*log(abs(tan(1/2*f*x + 1/2*e))) + 4*a^2*c*tan(1/2*f*x + 1/2*e) + (6*a^2*c*tan(1/2*f*x + 1/2*e)^2 - 4*a^2*c*tan(1/2*f*x + 1/2*e) - a^2*c)/tan(1/2*f*x + 1/2*e)^2)/f`

Mupad [B] (verification not implemented)

Time = 35.08 (sec) , antiderivative size = 163, normalized size of antiderivative = 2.55

$$\int \csc^3(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{a^2 c \tan\left(\frac{e}{2} + \frac{fx}{2}\right)}{2f} - \frac{a^2 c \ln\left(\frac{\sin\left(\frac{e}{2} + \frac{fx}{2}\right)}{\cos\left(\frac{e}{2} + \frac{fx}{2}\right)}\right)}{2f} - \frac{2 a^2 c \operatorname{atan}\left(\frac{2 \cos\left(\frac{e}{2} + \frac{fx}{2}\right) + \sin\left(\frac{e}{2} + \frac{fx}{2}\right)}{\cos\left(\frac{e}{2} + \frac{fx}{2}\right) - 2 \sin\left(\frac{e}{2} + \frac{fx}{2}\right)}\right)}{f}$$

$$- \frac{a^2 c \cot\left(\frac{e}{2} + \frac{fx}{2}\right)}{2f} - \frac{a^2 c \cot\left(\frac{e}{2} + \frac{fx}{2}\right)^2}{8f} + \frac{a^2 c \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^2}{8f}$$

input `int(((a + a*sin(e + f*x))^2*(c - c*sin(e + f*x)))/sin(e + f*x)^3,x)`output `(a^2*c*tan(e/2 + (f*x)/2))/(2*f) - (a^2*c*log(sin(e/2 + (f*x)/2)/cos(e/2 + (f*x)/2)))/(2*f) - (2*a^2*c*atan((2*cos(e/2 + (f*x)/2) + sin(e/2 + (f*x)/2))/(cos(e/2 + (f*x)/2) - 2*sin(e/2 + (f*x)/2))))/f - (a^2*c*cot(e/2 + (f*x)/2))/(2*f) - (a^2*c*cot(e/2 + (f*x)/2)^2)/(8*f) + (a^2*c*tan(e/2 + (f*x)/2)^2)/(8*f)`**Reduce [B] (verification not implemented)**

Time = 0.18 (sec) , antiderivative size = 72, normalized size of antiderivative = 1.12

$$\int \csc^3(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{a^2 c (-2 \cos(fx + e) \sin(fx + e) - \cos(fx + e) - \log(\tan(\frac{fx}{2} + \frac{e}{2})) \sin(fx + e)^2 - 2 \sin(fx + e)^2 fx)}{2 \sin(fx + e)^2 f}$$

input `int(csc(f*x+e)^3*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x)`output `(a**2*c*(-2*cos(e + f*x)*sin(e + f*x) - cos(e + f*x) - log(tan((e + f*x)/2))*sin(e + f*x)**2 - 2*sin(e + f*x)**2*f*x))/(2*sin(e + f*x)**2*f)`

3.8 $\int \csc^4(e+fx)(a+a \sin(e+fx))^2(c-c \sin(e+fx)) dx$

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Optimal result

Integrand size = 32, antiderivative size = 61

$$\int \csc^4(e+fx)(a+a \sin(e+fx))^2(c-c \sin(e+fx)) dx$$

$$= \frac{a^2 c \operatorname{arctanh}(\cos(e+fx))}{2f} - \frac{a^2 c \cot^3(e+fx)}{3f} - \frac{a^2 c \cot(e+fx) \csc(e+fx)}{2f}$$

```
output 1/2*a^2*c*arctanh(cos(f*x+e))/f-1/3*a^2*c*cot(f*x+e)^3/f-1/2*a^2*c*cot(f*x+e)*csc(f*x+e)/f
```

Mathematica [B] (verified)

Leaf count is larger than twice the leaf count of optimal. 172 vs. $2(61) = 122$.

Time = 0.35 (sec) , antiderivative size = 172, normalized size of antiderivative = 2.82

$$\int \csc^4(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= a^2 c \left(\frac{\cot\left(\frac{1}{2}(e + fx)\right)}{6f} - \frac{\csc^2\left(\frac{1}{2}(e + fx)\right)}{8f} - \frac{\cot\left(\frac{1}{2}(e + fx)\right) \csc^2\left(\frac{1}{2}(e + fx)\right)}{24f} \right. \\ \left. + \frac{\log\left(\cos\left(\frac{1}{2}(e + fx)\right)\right)}{2f} - \frac{\log\left(\sin\left(\frac{1}{2}(e + fx)\right)\right)}{2f} + \frac{\sec^2\left(\frac{1}{2}(e + fx)\right)}{8f} \right. \\ \left. - \frac{\tan\left(\frac{1}{2}(e + fx)\right)}{6f} + \frac{\sec^2\left(\frac{1}{2}(e + fx)\right) \tan\left(\frac{1}{2}(e + fx)\right)}{24f} \right)$$

input

```
Integrate[Csc[e + f*x]^4*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output

```
a^2*c*(Cot[(e + f*x)/2]/(6*f) - Csc[(e + f*x)/2]^2/(8*f) - (Cot[(e + f*x)/2]*Csc[(e + f*x)/2]^2)/(24*f) + Log[Cos[(e + f*x)/2]]/(2*f) - Log[Sin[(e + f*x)/2]]/(2*f) + Sec[(e + f*x)/2]^2/(8*f) - Tan[(e + f*x)/2]/(6*f) + (Sec[(e + f*x)/2]^2*Tan[(e + f*x)/2])/(24*f))
```

Rubi [A] (verified)

Time = 0.55 (sec) , antiderivative size = 64, normalized size of antiderivative = 1.05, number of steps used = 11, number of rules used = 10, $\frac{\text{number of rules}}{\text{integrand size}} = 0.312$, Rules used = {3042, 3429, 3042, 3185, 3042, 3087, 15, 3091, 3042, 4257}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \csc^4(e + fx)(a \sin(e + fx) + a)^2(c - c \sin(e + fx)) dx$$

$$\downarrow \text{3042}$$

$$\int \frac{(a \sin(e + fx) + a)^2(c - c \sin(e + fx))}{\sin(e + fx)^4} dx$$

$$\begin{aligned}
& \downarrow 3429 \\
& a^2 c^2 \int \frac{\cot^4(e+fx)}{c - c \sin(e+fx)} dx \\
& \downarrow 3042 \\
& a^2 c^2 \int \frac{1}{(c - c \sin(e+fx)) \tan(e+fx)^4} dx \\
& \downarrow 3185 \\
& a^2 c^2 \left(\frac{\int \cot^2(e+fx) \csc^2(e+fx) dx}{c} + \frac{\int \cot^2(e+fx) \csc(e+fx) dx}{c} \right) \\
& \downarrow 3042 \\
& a^2 c^2 \left(\frac{\int \sec(e+fx - \frac{\pi}{2}) \tan(e+fx - \frac{\pi}{2})^2 dx}{c} + \frac{\int \sec(e+fx - \frac{\pi}{2})^2 \tan(e+fx - \frac{\pi}{2})^2 dx}{c} \right) \\
& \downarrow 3087 \\
& a^2 c^2 \left(\frac{\int \cot^2(e+fx) d(-\cot(e+fx))}{cf} + \frac{\int \sec(e+fx - \frac{\pi}{2}) \tan(e+fx - \frac{\pi}{2})^2 dx}{c} \right) \\
& \downarrow 15 \\
& a^2 c^2 \left(\frac{\int \sec(e+fx - \frac{\pi}{2}) \tan(e+fx - \frac{\pi}{2})^2 dx}{c} - \frac{\cot^3(e+fx)}{3cf} \right) \\
& \downarrow 3091 \\
& a^2 c^2 \left(\frac{-\frac{1}{2} \int \csc(e+fx) dx - \frac{\cot(e+fx) \csc(e+fx)}{2f}}{c} - \frac{\cot^3(e+fx)}{3cf} \right) \\
& \downarrow 3042 \\
& a^2 c^2 \left(\frac{-\frac{1}{2} \int \csc(e+fx) dx - \frac{\cot(e+fx) \csc(e+fx)}{2f}}{c} - \frac{\cot^3(e+fx)}{3cf} \right) \\
& \downarrow 4257 \\
& a^2 c^2 \left(\frac{\frac{\operatorname{arctanh}(\cos(e+fx))}{2f}}{c} - \frac{\cot(e+fx) \csc(e+fx)}{2f} - \frac{\cot^3(e+fx)}{3cf} \right)
\end{aligned}$$

input `Int[Csc[e + f*x]^4*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]`

output `a^2*c^2*(-1/3*Cot[e + f*x]^3/(c*f) + (ArcTanh[Cos[e + f*x]]/(2*f) - (Cot[e + f*x]*Csc[e + f*x])/(2*f))/c)`

Defintions of rubi rules used

rule 15 `Int[(a_.)*(x_)^(m_.), x_Symbol] := Simp[a*(x^(m + 1)/(m + 1)), x] /; FreeQ[{a, m}, x] && NeQ[m, -1]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3087 `Int[sec[(e_.) + (f_.)*(x_)]^(m_.)*((b_.)*tan[(e_.) + (f_.)*(x_)]^(n_.), x_Symbol] := Simp[1/f Subst[Int[(b*x)^n*(1 + x^2)^(m/2 - 1), x], x, Tan[e + f*x]], x] /; FreeQ[{b, e, f, n}, x] && IntegerQ[m/2] && !(IntegerQ[(n - 1)/2] && LtQ[0, n, m - 1])`

rule 3091 `Int[((a_.)*sec[(e_.) + (f_.)*(x_)]^(m_.)*((b_.)*tan[(e_.) + (f_.)*(x_)]^(n_.), x_Symbol] := Simp[b*(a*Sec[e + f*x])^m*((b*Tan[e + f*x])^(n - 1)/(f*(m + n - 1))), x] - Simp[b^2*((n - 1)/(m + n - 1)) Int[(a*Sec[e + f*x])^m*(b*Tan[e + f*x])^(n - 2), x], x] /; FreeQ[{a, b, e, f, m}, x] && GtQ[n, 1] && NeQ[m + n - 1, 0] && IntegersQ[2*m, 2*n]`

rule 3185 `Int[((g_.)*tan[(e_.) + (f_.)*(x_)]^(p_.)/((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] := Simp[1/a Int[Sec[e + f*x]^2*(g*Tan[e + f*x])^p, x], x] - Simp[1/(b*g) Int[Sec[e + f*x]*(g*Tan[e + f*x])^(p + 1), x], x] /; FreeQ[{a, b, e, f, g, p}, x] && EqQ[a^2 - b^2, 0] && NeQ[p, -1]`

rule 3429

```
Int[sin[(e_.) + (f_.)*(x_)]^(p_)*((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^(m_
.)*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])^(n_), x_Symbol] := Simp[a^n*c^n
Int[Tan[e + f*x]^p*(a + b*Sin[e + f*x])^(m - n), x], x] /; FreeQ[{a, b, c
, d, e, f, m}, x] && EqQ[b*c + a*d, 0] && EqQ[a^2 - b^2, 0] && EqQ[p + 2*n,
0] && IntegerQ[n]
```

rule 4257

```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] := Simp[-ArcTanh[Cos[c + d*x]]/d, x]
/; FreeQ[{c, d}, x]
```

Maple [A] (verified)

Time = 1.87 (sec) , antiderivative size = 95, normalized size of antiderivative = 1.56

method	result
parallelrisch	$-\frac{a^2c \left(\cot\left(\frac{fx}{2} + \frac{e}{2}\right)^3 - \tan\left(\frac{fx}{2} + \frac{e}{2}\right)^3 + 3\cot\left(\frac{fx}{2} + \frac{e}{2}\right)^2 - 3\tan\left(\frac{fx}{2} + \frac{e}{2}\right)^2 - 3\cot\left(\frac{fx}{2} + \frac{e}{2}\right) + 12\ln\left(\tan\left(\frac{fx}{2} + \frac{e}{2}\right)\right) + 3\tan\left(\frac{fx}{2} + \frac{e}{2}\right) \right)}{24f}$
derivativedivides	$-\frac{a^2c \ln(\csc(fx+e)) - \cot(fx+e) + a^2c \cot(fx+e) + a^2c \left(-\frac{\csc(fx+e)\cot(fx+e)}{2} + \frac{\ln(\csc(fx+e) - \cot(fx+e))}{2} \right) + a^2c \left(-\frac{2}{3} - \frac{3}{3} \right)}{f}$
default	$-\frac{a^2c \ln(\csc(fx+e)) - \cot(fx+e) + a^2c \cot(fx+e) + a^2c \left(-\frac{\csc(fx+e)\cot(fx+e)}{2} + \frac{\ln(\csc(fx+e) - \cot(fx+e))}{2} \right) + a^2c \left(-\frac{2}{3} - \frac{3}{3} \right)}{f}$
risch	$\frac{a^2c(6ie^{4i(fx+e)} + 3e^{5i(fx+e)} + 2i - 3e^{i(fx+e)})}{3f(e^{2i(fx+e)} - 1)^3} + \frac{a^2c \ln(e^{i(fx+e)} + 1)}{2f} - \frac{a^2c \ln(e^{i(fx+e)} - 1)}{2f}$
norman	$-\frac{\frac{a^2c}{24f} - \frac{3a^2c \tan\left(\frac{fx}{2} + \frac{e}{2}\right)^3}{4f} - \frac{7a^2c \tan\left(\frac{fx}{2} + \frac{e}{2}\right)^7}{8f} - \frac{11a^2c \tan\left(\frac{fx}{2} + \frac{e}{2}\right)^5}{8f} - \frac{a^2c \tan\left(\frac{fx}{2} + \frac{e}{2}\right)}{8f} + \frac{a^2c \tan\left(\frac{fx}{2} + \frac{e}{2}\right)^4}{8f} - \frac{a^2c \tan\left(\frac{fx}{2} + \frac{e}{2}\right)}{8f}}{\tan\left(\frac{fx}{2} + \frac{e}{2}\right)^3 \left(1 + \tan\left(\frac{fx}{2} + \frac{e}{2}\right)^2\right)^3}$

input

```
int(csc(f*x+e)^4*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x,method=_RETURNVERBOSE)
```

output

```
-1/24*a^2*c*(cot(1/2*f*x+1/2*e)^3-tan(1/2*f*x+1/2*e)^3+3*cot(1/2*f*x+1/2*
e)^2-3*tan(1/2*f*x+1/2*e)^2-3*cot(1/2*f*x+1/2*e)+12*ln(tan(1/2*f*x+1/2*
e))+3*tan(1/2*f*x+1/2*e))/f
```

Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 137 vs. $2(55) = 110$.

Time = 0.09 (sec) , antiderivative size = 137, normalized size of antiderivative = 2.25

$$\int \csc^4(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{4a^2c \cos(fx + e)^3 + 6a^2c \cos(fx + e) \sin(fx + e) + 3(a^2c \cos(fx + e)^2 - a^2c) \log\left(\frac{1}{2} \cos(fx + e) + \frac{1}{2}\right) - 3(a^2c \cos(fx + e)^2 - a^2c) \log\left(-\frac{1}{2} \cos(fx + e) + \frac{1}{2}\right) \sin(fx + e)}{12(f \cos(fx + e)^2 - f) \sin(fx + e)}$$

input `integrate(csc(f*x+e)^4*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="fricas")`

output `1/12*(4*a^2*c*cos(f*x + e)^3 + 6*a^2*c*cos(f*x + e)*sin(f*x + e) + 3*(a^2*c*cos(f*x + e)^2 - a^2*c)*log(1/2*cos(f*x + e) + 1/2)*sin(f*x + e) - 3*(a^2*c*cos(f*x + e)^2 - a^2*c)*log(-1/2*cos(f*x + e) + 1/2)*sin(f*x + e))/((f*cos(f*x + e)^2 - f)*sin(f*x + e))`

Sympy [F]

$$\int \csc^4(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= -a^2c \left(\int (-\sin(e + fx) \csc^4(e + fx)) dx + \int \sin^2(e + fx) \csc^4(e + fx) dx \right. \\ \left. + \int \sin^3(e + fx) \csc^4(e + fx) dx + \int (-\csc^4(e + fx)) dx \right)$$

input `integrate(csc(f*x+e)**4*(a+a*sin(f*x+e))**2*(c-c*sin(f*x+e)),x)`

output `-a**2*c*(Integral(-sin(e + f*x)*csc(e + f*x)**4, x) + Integral(sin(e + f*x)**2*csc(e + f*x)**4, x) + Integral(sin(e + f*x)**3*csc(e + f*x)**4, x) + Integral(-csc(e + f*x)**4, x))`

Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 120 vs. $2(55) = 110$.

Time = 0.04 (sec) , antiderivative size = 120, normalized size of antiderivative = 1.97

$$\int \csc^4(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{3a^2c \left(\frac{2 \cos(fx+e)}{\cos(fx+e)^2-1} - \log(\cos(fx+e)+1) + \log(\cos(fx+e)-1) \right) + 6a^2c(\log(\cos(fx+e)+1) - \log(\cos(fx+e)-1)) + 12a^2c/\tan(fx+e) - 4(3\tan(fx+e)^2+1)a^2c/\tan(fx+e)^3}{12f}$$

input `integrate(csc(f*x+e)^4*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="maxima")`

output `1/12*(3*a^2*c*(2*cos(f*x + e)/(cos(f*x + e)^2 - 1) - log(cos(f*x + e) + 1) + log(cos(f*x + e) - 1)) + 6*a^2*c*(log(cos(f*x + e) + 1) - log(cos(f*x + e) - 1)) + 12*a^2*c/tan(f*x + e) - 4*(3*tan(f*x + e)^2 + 1)*a^2*c/tan(f*x + e)^3)/f`

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 139 vs. $2(55) = 110$.

Time = 0.35 (sec) , antiderivative size = 139, normalized size of antiderivative = 2.28

$$\int \csc^4(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{a^2c \tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)^3 + 3a^2c \tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)^2 - 12a^2c \log\left(\left|\tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)\right|\right) - 3a^2c \tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)}{24f}$$

input `integrate(csc(f*x+e)^4*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="giac")`

output `1/24*(a^2*c*tan(1/2*f*x + 1/2*e)^3 + 3*a^2*c*tan(1/2*f*x + 1/2*e)^2 - 12*a^2*c*log(abs(tan(1/2*f*x + 1/2*e))) - 3*a^2*c*tan(1/2*f*x + 1/2*e) + (22*a^2*c*tan(1/2*f*x + 1/2*e)^3 + 3*a^2*c*tan(1/2*f*x + 1/2*e)^2 - 3*a^2*c*tan(1/2*f*x + 1/2*e) - a^2*c)/tan(1/2*f*x + 1/2*e)^3)/f`

Mupad [B] (verification not implemented)

Time = 35.44 (sec) , antiderivative size = 132, normalized size of antiderivative = 2.16

$$\int \csc^4(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{a^2 c \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^2}{8f} - \frac{a^2 c \tan\left(\frac{e}{2} + \frac{fx}{2}\right)}{8f}$$

$$- \frac{\cot\left(\frac{e}{2} + \frac{fx}{2}\right)^3 \left(-c a^2 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^2 + c a^2 \tan\left(\frac{e}{2} + \frac{fx}{2}\right) + \frac{c a^2}{3}\right)}{8f}$$

$$+ \frac{a^2 c \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^3}{24f} - \frac{a^2 c \ln\left(\tan\left(\frac{e}{2} + \frac{fx}{2}\right)\right)}{2f}$$

input

```
int(((a + a*sin(e + f*x))^2*(c - c*sin(e + f*x)))/sin(e + f*x)^4,x)
```

output

```
(a^2*c*tan(e/2 + (f*x)/2)^2)/(8*f) - (a^2*c*tan(e/2 + (f*x)/2))/(8*f) - (c
ot(e/2 + (f*x)/2)^3*((a^2*c)/3 + a^2*c*tan(e/2 + (f*x)/2) - a^2*c*tan(e/2
+ (f*x)/2)^2))/(8*f) + (a^2*c*tan(e/2 + (f*x)/2)^3)/(24*f) - (a^2*c*log(ta
n(e/2 + (f*x)/2)))/(2*f)
```

Reduce [B] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 76, normalized size of antiderivative = 1.25

$$\int \csc^4(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{a^2 c (2 \cos(fx + e) \sin(fx + e))^2 - 3 \cos(fx + e) \sin(fx + e) - 2 \cos(fx + e) - 3 \log\left(\tan\left(\frac{fx}{2} + \frac{e}{2}\right)\right) \sin(fx + e)}{6 \sin(fx + e)^3 f}$$

input

```
int(csc(f*x+e)^4*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x)
```

output

```
(a**2*c*(2*cos(e + f*x)*sin(e + f*x)**2 - 3*cos(e + f*x)*sin(e + f*x) - 2*
cos(e + f*x) - 3*log(tan((e + f*x)/2))*sin(e + f*x)**3))/(6*sin(e + f*x)**
3*f)
```

3.9 $\int \csc^5(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx)) dx$

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Optimal result

Integrand size = 32, antiderivative size = 86

$$\begin{aligned} & \int \csc^5(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx)) dx \\ &= \frac{a^2 c \operatorname{arctanh}(\cos(e+fx))}{8f} - \frac{a^2 c \cot^3(e+fx)}{3f} \\ & \quad + \frac{a^2 c \cot(e+fx) \csc(e+fx)}{8f} - \frac{a^2 c \cot(e+fx) \csc^3(e+fx)}{4f} \end{aligned}$$

output

```
1/8*a^2*c*arctanh(cos(f*x+e))/f-1/3*a^2*c*cot(f*x+e)^3/f+1/8*a^2*c*cot(f*x+e)*csc(f*x+e)/f-1/4*a^2*c*cot(f*x+e)*csc(f*x+e)^3/f
```

Mathematica [B] (verified)

Leaf count is larger than twice the leaf count of optimal. 179 vs. $2(86) = 172$.

Time = 0.39 (sec) , antiderivative size = 179, normalized size of antiderivative = 2.08

$$\begin{aligned} & \int \csc^5(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx \\ &= \frac{a^2 c \cot(e + fx)}{3f} + \frac{a^2 c \csc^2\left(\frac{1}{2}(e + fx)\right)}{32f} - \frac{a^2 c \csc^4\left(\frac{1}{2}(e + fx)\right)}{64f} \\ & \quad - \frac{a^2 c \cot(e + fx) \csc^2(e + fx)}{3f} + \frac{a^2 c \log\left(\cos\left(\frac{1}{2}(e + fx)\right)\right)}{8f} \\ & \quad - \frac{a^2 c \log\left(\sin\left(\frac{1}{2}(e + fx)\right)\right)}{8f} - \frac{a^2 c \sec^2\left(\frac{1}{2}(e + fx)\right)}{32f} + \frac{a^2 c \sec^4\left(\frac{1}{2}(e + fx)\right)}{64f} \end{aligned}$$

input

```
Integrate[Csc[e + f*x]^5*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]
```

output

```
(a^2*c*Cot[e + f*x])/(3*f) + (a^2*c*Csc[(e + f*x)/2]^2)/(32*f) - (a^2*c*Csc
c[(e + f*x)/2]^4)/(64*f) - (a^2*c*Cot[e + f*x]*Csc[e + f*x]^2)/(3*f) + (a^
2*c*Log[Cos[(e + f*x)/2]])/(8*f) - (a^2*c*Log[Sin[(e + f*x)/2]])/(8*f) - (
a^2*c*Sec[(e + f*x)/2]^2)/(32*f) + (a^2*c*Sec[(e + f*x)/2]^4)/(64*f)
```

Rubi [A] (verified)

Time = 0.36 (sec) , antiderivative size = 86, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.094$, Rules used = {3042, 3445, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \csc^5(e + fx)(a \sin(e + fx) + a)^2(c - c \sin(e + fx)) dx \\ & \quad \downarrow \text{3042} \\ & \int \frac{(a \sin(e + fx) + a)^2(c - c \sin(e + fx))}{\sin(e + fx)^5} dx \\ & \quad \downarrow \text{3445} \\ & \int (a^2 c \csc^5(e + fx) + a^2 c \csc^4(e + fx) - a^2 c \csc^3(e + fx) - a^2 c \csc^2(e + fx)) dx \end{aligned}$$

$$\frac{a^2 \operatorname{arctanh}(\cos(e+fx))}{8f} - \frac{a^2 c \cot^3(e+fx)}{3f} - \frac{a^2 c \cot(e+fx) \csc^3(e+fx)}{4f} + \frac{a^2 c \cot(e+fx) \csc(e+fx)}{8f}$$

input `Int[Csc[e + f*x]^5*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]`

output `(a^2*c*ArcTanh[Cos[e + f*x]])/(8*f) - (a^2*c*Cot[e + f*x]^3)/(3*f) + (a^2*c*Cot[e + f*x]*Csc[e + f*x])/(8*f) - (a^2*c*Cot[e + f*x]*Csc[e + f*x]^3)/(4*f)`

Defintions of rubi rules used

rule 2009 `Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3445 `Int[sin[(e_.) + (f_.)*(x_)]^(n_.)*((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]^(m_.))*((A_.) + (B_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] := Int[ExpandTrig[sin[e + f*x]^n*(a + b*sin[e + f*x])^m*(A + B*sin[e + f*x]), x], x] /; FreeQ[{a, b, e, f, A, B}, x] && EqQ[A*b + a*B, 0] && EqQ[a^2 - b^2, 0] && IntegerQ[m] && IntegerQ[n]`

Maple [A] (verified)

Time = 2.28 (sec) , antiderivative size = 95, normalized size of antiderivative = 1.10

output

```
-1/48*(16*a^2*c*cos(f*x + e)^3*sin(f*x + e) + 6*a^2*c*cos(f*x + e)^3 + 6*a^2*c*cos(f*x + e) - 3*(a^2*c*cos(f*x + e)^4 - 2*a^2*c*cos(f*x + e)^2 + a^2*c)*log(1/2*cos(f*x + e) + 1/2) + 3*(a^2*c*cos(f*x + e)^4 - 2*a^2*c*cos(f*x + e)^2 + a^2*c)*log(-1/2*cos(f*x + e) + 1/2))/(f*cos(f*x + e)^4 - 2*f*cos(f*x + e)^2 + f)
```

Sympy [F(-1)]

Timed out.

$$\int \csc^5(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx = \text{Timed out}$$

input

```
integrate(csc(f*x+e)**5*(a+a*sin(f*x+e))**2*(c-c*sin(f*x+e)),x)
```

output

Timed out

Maxima [B] (verification not implemented)Leaf count of result is larger than twice the leaf count of optimal. 165 vs. $2(78) = 156$.

Time = 0.04 (sec) , antiderivative size = 165, normalized size of antiderivative = 1.92

$$\int \csc^5(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{3 a^2 c \left(\frac{2 (3 \cos(fx+e)^3 - 5 \cos(fx+e))}{\cos(fx+e)^4 - 2 \cos(fx+e)^2 + 1} - 3 \log(\cos(fx+e) + 1) + 3 \log(\cos(fx+e) - 1) \right) - 12 a^2 c \left(\frac{2 \cos(fx+e)}{\cos(fx+e)} \right)}{48 f}$$

input

```
integrate(csc(f*x+e)^5*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="maxima")
```

output

```
1/48*(3*a^2*c*(2*(3*cos(f*x + e)^3 - 5*cos(f*x + e)))/(cos(f*x + e)^4 - 2*cos(f*x + e)^2 + 1) - 3*log(cos(f*x + e) + 1) + 3*log(cos(f*x + e) - 1)) - 12*a^2*c*(2*cos(f*x + e)/(cos(f*x + e)^2 - 1) - log(cos(f*x + e) + 1) + log(cos(f*x + e) - 1)) + 48*a^2*c/tan(f*x + e) - 16*(3*tan(f*x + e)^2 + 1)*a^2*c/tan(f*x + e)^3)/f
```

Giac [A] (verification not implemented)

Time = 0.41 (sec) , antiderivative size = 140, normalized size of antiderivative = 1.63

$$\int \csc^5(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{3a^2c \tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)^4 + 8a^2c \tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)^3 - 24a^2c \log\left(\left|\tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)\right|\right) - 24a^2c \tan\left(\frac{1}{2}fx + \frac{1}{2}e\right)}{192f}$$

input `integrate(csc(f*x+e)^5*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="giac")`

output `1/192*(3*a^2*c*tan(1/2*f*x + 1/2*e)^4 + 8*a^2*c*tan(1/2*f*x + 1/2*e)^3 - 24*a^2*c*log(abs(tan(1/2*f*x + 1/2*e))) - 24*a^2*c*tan(1/2*f*x + 1/2*e) + (50*a^2*c*tan(1/2*f*x + 1/2*e)^4 + 24*a^2*c*tan(1/2*f*x + 1/2*e)^3 - 8*a^2*c*tan(1/2*f*x + 1/2*e) - 3*a^2*c)/tan(1/2*f*x + 1/2*e)^4)/f`

Mupad [B] (verification not implemented)

Time = 35.06 (sec) , antiderivative size = 133, normalized size of antiderivative = 1.55

$$\int \csc^5(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{a^2c \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^3}{24f} - \frac{a^2c \tan\left(\frac{e}{2} + \frac{fx}{2}\right)}{8f}$$

$$- \frac{\cot\left(\frac{e}{2} + \frac{fx}{2}\right)^4 \left(-2ca^2 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^3 + \frac{2ca^2 \tan\left(\frac{e}{2} + \frac{fx}{2}\right)}{3} + \frac{ca^2}{4}\right)}{16f}$$

$$+ \frac{a^2c \tan\left(\frac{e}{2} + \frac{fx}{2}\right)^4}{64f} - \frac{a^2c \ln\left(\tan\left(\frac{e}{2} + \frac{fx}{2}\right)\right)}{8f}$$

input `int(((a + a*sin(e + f*x))^2*(c - c*sin(e + f*x)))/sin(e + f*x)^5,x)`

output

```
(a^2*c*tan(e/2 + (f*x)/2)^3)/(24*f) - (a^2*c*tan(e/2 + (f*x)/2))/(8*f) - (
cot(e/2 + (f*x)/2)^4*((a^2*c)/4 + (2*a^2*c*tan(e/2 + (f*x)/2))/3 - 2*a^2*c
*tan(e/2 + (f*x)/2)^3))/(16*f) + (a^2*c*tan(e/2 + (f*x)/2)^4)/(64*f) - (a^
2*c*log(tan(e/2 + (f*x)/2)))/(8*f)
```

Reduce [B] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 92, normalized size of antiderivative = 1.07

$$\int \csc^5(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{a^2 c (8 \cos(fx + e) \sin(fx + e)^3 + 3 \cos(fx + e) \sin(fx + e)^2 - 8 \cos(fx + e) \sin(fx + e) - 6 \cos(fx + e) - 3 \log(\tan((e + fx)/2))) \sin(e + fx)^4}{24 \sin(fx + e)^4 f}$$

input

```
int(csc(f*x+e)^5*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x)
```

output

```
(a**2*c*(8*cos(e + f*x)*sin(e + f*x)**3 + 3*cos(e + f*x)*sin(e + f*x)**2 -
8*cos(e + f*x)*sin(e + f*x) - 6*cos(e + f*x) - 3*log(tan((e + f*x)/2))*si
n(e + f*x)**4))/(24*sin(e + f*x)**4*f)
```

3.10 $\int \csc^6(e+fx)(a+a \sin(e+fx))^2(c-c \sin(e+fx)) dx$

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Maxima [A] (verification not implemented)	112
Giac [A] (verification not implemented)	113
Mupad [B] (verification not implemented)	113
Reduce [B] (verification not implemented)	114

Optimal result

Integrand size = 32, antiderivative size = 105

$$\int \csc^6(e+fx)(a+a \sin(e+fx))^2(c-c \sin(e+fx)) dx$$

$$= \frac{a^2 c \operatorname{arctanh}(\cos(e+fx))}{8f} - \frac{a^2 c \cot^3(e+fx)}{3f} - \frac{a^2 c \cot^5(e+fx)}{5f}$$

$$+ \frac{a^2 c \cot(e+fx) \csc(e+fx)}{8f} - \frac{a^2 c \cot(e+fx) \csc^3(e+fx)}{4f}$$

output

```
1/8*a^2*c*arctanh(cos(f*x+e))/f-1/3*a^2*c*cot(f*x+e)^3/f-1/5*a^2*c*cot(f*x
+e)^5/f+1/8*a^2*c*cot(f*x+e)*csc(f*x+e)/f-1/4*a^2*c*cot(f*x+e)*csc(f*x+e)^
3/f
```

Mathematica [A] (verified)

Time = 0.18 (sec) , antiderivative size = 204, normalized size of antiderivative = 1.94

$$\begin{aligned} & \int \csc^6(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx \\ &= \frac{2a^2c \cot(e + fx)}{15f} + \frac{a^2c \csc^2\left(\frac{1}{2}(e + fx)\right)}{32f} - \frac{a^2c \csc^4\left(\frac{1}{2}(e + fx)\right)}{64f} \\ &+ \frac{a^2c \cot(e + fx) \csc^2(e + fx)}{15f} - \frac{a^2c \cot(e + fx) \csc^4(e + fx)}{5f} \\ &+ \frac{a^2c \log\left(\cos\left(\frac{1}{2}(e + fx)\right)\right)}{8f} - \frac{a^2c \log\left(\sin\left(\frac{1}{2}(e + fx)\right)\right)}{8f} \\ &- \frac{a^2c \sec^2\left(\frac{1}{2}(e + fx)\right)}{32f} + \frac{a^2c \sec^4\left(\frac{1}{2}(e + fx)\right)}{64f} \end{aligned}$$

input `Integrate[Csc[e + f*x]^6*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]`

output `(2*a^2*c*Cot[e + f*x])/(15*f) + (a^2*c*Csc[(e + f*x)/2]^2)/(32*f) - (a^2*c*Csc[(e + f*x)/2]^4)/(64*f) + (a^2*c*Cot[e + f*x]*Csc[e + f*x]^2)/(15*f) - (a^2*c*Cot[e + f*x]*Csc[e + f*x]^4)/(5*f) + (a^2*c*Log[Cos[(e + f*x)/2]])/(8*f) - (a^2*c*Log[Sin[(e + f*x)/2]])/(8*f) - (a^2*c*Sec[(e + f*x)/2]^2)/(32*f) + (a^2*c*Sec[(e + f*x)/2]^4)/(64*f)`

Rubi [A] (verified)

Time = 0.38 (sec) , antiderivative size = 105, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.094$, Rules used = {3042, 3445, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \csc^6(e + fx)(a \sin(e + fx) + a)^2(c - c \sin(e + fx)) dx \\ & \quad \downarrow \text{3042} \\ & \int \frac{(a \sin(e + fx) + a)^2(c - c \sin(e + fx))}{\sin(e + fx)^6} dx \end{aligned}$$

$$\int (a^2 c \csc^6(e + fx) + a^2 c \csc^5(e + fx) - a^2 c \csc^4(e + fx) - a^2 c \csc^3(e + fx)) dx$$

↓ 3445

$$\frac{a^2 c \operatorname{arctanh}(\cos(e + fx))}{8f} - \frac{a^2 c \cot^5(e + fx)}{5f} - \frac{a^2 c \cot^3(e + fx)}{3f} - \frac{a^2 c \cot(e + fx) \csc^3(e + fx)}{4f} + \frac{a^2 c \cot(e + fx) \csc(e + fx)}{8f}$$

↓ 2009

input `Int[Csc[e + f*x]^6*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]`

output `(a^2*c*ArcTanh[Cos[e + f*x]])/(8*f) - (a^2*c*Cot[e + f*x]^3)/(3*f) - (a^2*c*Cot[e + f*x]^5)/(5*f) + (a^2*c*Cot[e + f*x]*Csc[e + f*x])/(8*f) - (a^2*c*Cot[e + f*x]*Csc[e + f*x]^3)/(4*f)`

Defintions of rubi rules used

rule 2009 `Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3445 `Int[sin[(e_.) + (f_.)*(x_)]^(n_.)*((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]^(m_.))*((A_.) + (B_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] := Int[ExpandTrig[sin[e + f*x]^n*(a + b*sin[e + f*x])^m*(A + B*sin[e + f*x]), x], x] /; FreeQ[{a, b, e, f, A, B}, x] && EqQ[A*b + a*B, 0] && EqQ[a^2 - b^2, 0] && IntegerQ[m] && IntegerQ[n]`

Maple [A] (verified)

Time = 2.33 (sec) , antiderivative size = 123, normalized size of antiderivative = 1.17

method	result
parallelrisch	$-\frac{a^2c\left(6\cot\left(\frac{fx}{2}+\frac{e}{2}\right)^5-6\tan\left(\frac{fx}{2}+\frac{e}{2}\right)^5+15\cot\left(\frac{fx}{2}+\frac{e}{2}\right)^4-15\tan\left(\frac{fx}{2}+\frac{e}{2}\right)^4+10\cot\left(\frac{fx}{2}+\frac{e}{2}\right)^3-10\tan\left(\frac{fx}{2}+\frac{e}{2}\right)^3+12\cot\left(\frac{fx}{2}+\frac{e}{2}\right)^2-12\tan\left(\frac{fx}{2}+\frac{e}{2}\right)^2+6\cot\left(\frac{fx}{2}+\frac{e}{2}\right)-6\tan\left(\frac{fx}{2}+\frac{e}{2}\right)+12\ln\left(\tan\left(\frac{fx}{2}+\frac{e}{2}\right)\right)+60\cot\left(\frac{fx}{2}+\frac{e}{2}\right)+60\tan\left(\frac{fx}{2}+\frac{e}{2}\right)\right)}{960f}$
risch	$-\frac{a^2c\left(15e^{9i(fx+e)}+240ie^{6i(fx+e)}+90e^{7i(fx+e)}+80ie^{4i(fx+e)}+80ie^{2i(fx+e)}-90e^{3i(fx+e)}-16i-15e^{i(fx+e)}\right)}{60f\left(e^{2i(fx+e)}-1\right)^5}$
derivativedivides	$-a^2c\left(-\frac{\csc(fx+e)\cot(fx+e)}{2}+\frac{\ln(\csc(fx+e)-\cot(fx+e))}{2}\right)-a^2c\left(-\frac{2}{3}-\frac{\csc(fx+e)^2}{3}\right)\cot(fx+e)+a^2c\left(\left(-\frac{\csc(fx+e)^3}{4}-\frac{3c}{4}\right)\frac{1}{f}\right)$
default	$-a^2c\left(-\frac{\csc(fx+e)\cot(fx+e)}{2}+\frac{\ln(\csc(fx+e)-\cot(fx+e))}{2}\right)-a^2c\left(-\frac{2}{3}-\frac{\csc(fx+e)^2}{3}\right)\cot(fx+e)+a^2c\left(\left(-\frac{\csc(fx+e)^3}{4}-\frac{3c}{4}\right)\frac{1}{f}\right)$
norman	$-\frac{a^2c}{160f}-\frac{a^2c\tan\left(\frac{fx}{2}+\frac{e}{2}\right)^9}{8f}-\frac{3a^2c\tan\left(\frac{fx}{2}+\frac{e}{2}\right)^5}{32f}-\frac{5a^2c\tan\left(\frac{fx}{2}+\frac{e}{2}\right)^7}{32f}-\frac{a^2c\tan\left(\frac{fx}{2}+\frac{e}{2}\right)}{64f}-\frac{7a^2c\tan\left(\frac{fx}{2}+\frac{e}{2}\right)^2}{240f}-\frac{3a^2c\tan\left(\frac{fx}{2}+\frac{e}{2}\right)}{64f}$

input `int(csc(f*x+e)^6*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x,method=_RETURNVERBOSE)`

output
$$-1/960*a^2*c*(6*\cot(1/2*f*x+1/2*e)^5-6*\tan(1/2*f*x+1/2*e)^5+15*\cot(1/2*f*x+1/2*e)^4-15*\tan(1/2*f*x+1/2*e)^4+10*\cot(1/2*f*x+1/2*e)^3-10*\tan(1/2*f*x+1/2*e)^3+120*\ln(\tan(1/2*f*x+1/2*e))-60*\cot(1/2*f*x+1/2*e)+60*\tan(1/2*f*x+1/2*e))/f$$

Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 201 vs. 2(95) = 190.

Time = 0.09 (sec) , antiderivative size = 201, normalized size of antiderivative = 1.91

$$\int \csc^6(e+fx)(a+a\sin(e+fx))^2(c-c\sin(e+fx))dx$$

$$= \frac{32a^2c\cos(fx+e)^5-80a^2c\cos(fx+e)^3+15(a^2c\cos(fx+e)^4-2a^2c\cos(fx+e)^2+a^2c)\log\left(\frac{1}{2}\cos\right)}{f}$$

input `integrate(csc(f*x+e)^6*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="fricas")`

output `1/240*(32*a^2*c*cos(f*x + e)^5 - 80*a^2*c*cos(f*x + e)^3 + 15*(a^2*c*cos(f*x + e)^4 - 2*a^2*c*cos(f*x + e)^2 + a^2*c)*log(1/2*cos(f*x + e) + 1/2)*sin(f*x + e) - 15*(a^2*c*cos(f*x + e)^4 - 2*a^2*c*cos(f*x + e)^2 + a^2*c)*log(-1/2*cos(f*x + e) + 1/2)*sin(f*x + e) - 30*(a^2*c*cos(f*x + e)^3 + a^2*c*cos(f*x + e))*sin(f*x + e)/((f*cos(f*x + e)^4 - 2*f*cos(f*x + e)^2 + f)*sin(f*x + e))`

Sympy [F(-1)]

Timed out.

$$\int \csc^6(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx = \text{Timed out}$$

input `integrate(csc(f*x+e)**6*(a+a*sin(f*x+e))**2*(c-c*sin(f*x+e)),x)`

output Timed out

Maxima [A] (verification not implemented)

Time = 0.04 (sec) , antiderivative size = 187, normalized size of antiderivative = 1.78

$$\int \csc^6(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{15 a^2 c \left(\frac{2 \left(3 \cos^3(fx+e) - 5 \cos(fx+e) \right)}{\cos^4(fx+e) - 2 \cos^2(fx+e) + 1} - 3 \log(\cos(fx+e) + 1) + 3 \log(\cos(fx+e) - 1) \right) - 60 a^2 c \left(\frac{2 \cos(fx+e)}{\cos(fx+e)} \right)}{1}$$

input `integrate(csc(f*x+e)^6*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="maxima")`

output

```
1/240*(15*a^2*c*(2*(3*cos(f*x + e)^3 - 5*cos(f*x + e))/(cos(f*x + e)^4 - 2
*cos(f*x + e)^2 + 1) - 3*log(cos(f*x + e) + 1) + 3*log(cos(f*x + e) - 1))
- 60*a^2*c*(2*cos(f*x + e)/(cos(f*x + e)^2 - 1) - log(cos(f*x + e) + 1) +
log(cos(f*x + e) - 1)) + 80*(3*tan(f*x + e)^2 + 1)*a^2*c/tan(f*x + e)^3 -
16*(15*tan(f*x + e)^4 + 10*tan(f*x + e)^2 + 3)*a^2*c/tan(f*x + e)^5)/f
```

Giac [A] (verification not implemented)

Time = 0.35 (sec) , antiderivative size = 174, normalized size of antiderivative = 1.66

$$\int \csc^6(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{6 a^2 c \tan\left(\frac{1}{2} f x + \frac{1}{2} e\right)^5 + 15 a^2 c \tan\left(\frac{1}{2} f x + \frac{1}{2} e\right)^4 + 10 a^2 c \tan\left(\frac{1}{2} f x + \frac{1}{2} e\right)^3 - 120 a^2 c \log\left(\left|\tan\left(\frac{1}{2} f x + \frac{1}{2} e\right)\right|\right)}{f}$$

input

```
integrate(csc(f*x+e)^6*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="giac")
```

output

```
1/960*(6*a^2*c*tan(1/2*f*x + 1/2*e)^5 + 15*a^2*c*tan(1/2*f*x + 1/2*e)^4 +
10*a^2*c*tan(1/2*f*x + 1/2*e)^3 - 120*a^2*c*log(abs(tan(1/2*f*x + 1/2*e)))
- 60*a^2*c*tan(1/2*f*x + 1/2*e) + (274*a^2*c*tan(1/2*f*x + 1/2*e)^5 + 60*
a^2*c*tan(1/2*f*x + 1/2*e)^4 - 10*a^2*c*tan(1/2*f*x + 1/2*e)^2 - 15*a^2*c*
tan(1/2*f*x + 1/2*e) - 6*a^2*c)/tan(1/2*f*x + 1/2*e)^5)/f
```

Mupad [B] (verification not implemented)

Time = 35.29 (sec) , antiderivative size = 244, normalized size of antiderivative = 2.32

$$\int \csc^6(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx =$$

$$= \frac{a^2 c \left(6 \cos\left(\frac{e}{2} + \frac{fx}{2}\right)^{10} - 6 \sin\left(\frac{e}{2} + \frac{fx}{2}\right)^{10} - 15 \cos\left(\frac{e}{2} + \frac{fx}{2}\right) \sin\left(\frac{e}{2} + \frac{fx}{2}\right)^9 + 15 \cos\left(\frac{e}{2} + \frac{fx}{2}\right)^9 \sin\left(\frac{e}{2} + \frac{fx}{2}\right) \right)}{f}$$

input

```
int(((a + a*sin(e + f*x))^2*(c - c*sin(e + f*x)))/sin(e + f*x)^6,x)
```

output

```

-(a^2*c*(6*cos(e/2 + (f*x)/2)^10 - 6*sin(e/2 + (f*x)/2)^10 - 15*cos(e/2 +
(f*x)/2)*sin(e/2 + (f*x)/2)^9 + 15*cos(e/2 + (f*x)/2)^9*sin(e/2 + (f*x)/2)
- 10*cos(e/2 + (f*x)/2)^2*sin(e/2 + (f*x)/2)^8 + 60*cos(e/2 + (f*x)/2)^4*
sin(e/2 + (f*x)/2)^6 - 60*cos(e/2 + (f*x)/2)^6*sin(e/2 + (f*x)/2)^4 + 10*c
os(e/2 + (f*x)/2)^8*sin(e/2 + (f*x)/2)^2 + 120*cos(e/2 + (f*x)/2)^5*log(si
n(e/2 + (f*x)/2)/cos(e/2 + (f*x)/2))*sin(e/2 + (f*x)/2)^5))/(960*f*cos(e/2
+ (f*x)/2)^5*sin(e/2 + (f*x)/2)^5)

```

Reduce [B] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 108, normalized size of antiderivative = 1.03

$$\int \csc^6(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{a^2 c (16 \cos(fx + e) \sin(fx + e)^4 + 15 \cos(fx + e) \sin(fx + e)^3 + 8 \cos(fx + e) \sin(fx + e)^2 - 30 \cos(fx + e) \sin(fx + e) - 15 \log(\tan((e + fx)/2)) \sin(e + fx)^5)}{120 \sin(fx + e)^5 f}$$

input

```
int(csc(f*x+e)^6*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x)
```

output

```

(a**2*c*(16*cos(e + f*x)*sin(e + f*x)**4 + 15*cos(e + f*x)*sin(e + f*x)**3
+ 8*cos(e + f*x)*sin(e + f*x)**2 - 30*cos(e + f*x)*sin(e + f*x) - 24*cos(
e + f*x) - 15*log(tan((e + f*x)/2))*sin(e + f*x)**5))/(120*sin(e + f*x)**5
*f)

```

3.11 $\int \csc^7(e+fx)(a+a \sin(e+fx))^2(c-c \sin(e+fx)) dx$

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Optimal result

Integrand size = 32, antiderivative size = 130

$$\int \csc^7(e+fx)(a+a \sin(e+fx))^2(c-c \sin(e+fx)) dx$$

$$= \frac{a^2 \operatorname{arctanh}(\cos(e+fx))}{16f} - \frac{a^2 c \cot^3(e+fx)}{3f}$$

$$- \frac{a^2 c \cot^5(e+fx)}{5f} + \frac{a^2 c \cot(e+fx) \csc(e+fx)}{16f}$$

$$+ \frac{a^2 c \cot(e+fx) \csc^3(e+fx)}{24f} - \frac{a^2 c \cot(e+fx) \csc^5(e+fx)}{6f}$$

output

```
1/16*a^2*c*arctanh(cos(f*x+e))/f-1/3*a^2*c*cot(f*x+e)^3/f-1/5*a^2*c*cot(f*x+e)^5/f+1/16*a^2*c*cot(f*x+e)*csc(f*x+e)/f+1/24*a^2*c*cot(f*x+e)*csc(f*x+e)^3/f-1/6*a^2*c*cot(f*x+e)*csc(f*x+e)^5/f
```

Mathematica [A] (verified)

Time = 0.21 (sec) , antiderivative size = 204, normalized size of antiderivative = 1.57

$$\begin{aligned} & \int \csc^7(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx \\ &= \frac{2a^2c \cot(e + fx)}{15f} + \frac{a^2c \csc^2\left(\frac{1}{2}(e + fx)\right)}{64f} - \frac{a^2c \csc^6\left(\frac{1}{2}(e + fx)\right)}{384f} \\ &+ \frac{a^2c \cot(e + fx) \csc^2(e + fx)}{15f} - \frac{a^2c \cot(e + fx) \csc^4(e + fx)}{5f} \\ &+ \frac{a^2c \log\left(\cos\left(\frac{1}{2}(e + fx)\right)\right)}{16f} - \frac{a^2c \log\left(\sin\left(\frac{1}{2}(e + fx)\right)\right)}{16f} \\ &- \frac{a^2c \sec^2\left(\frac{1}{2}(e + fx)\right)}{64f} + \frac{a^2c \sec^6\left(\frac{1}{2}(e + fx)\right)}{384f} \end{aligned}$$

input `Integrate[Csc[e + f*x]^7*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]`

output $(2a^2c \cot[e + fx]) / (15f) + (a^2c \csc[(e + fx)/2]^2) / (64f) - (a^2c \csc[(e + fx)/2]^6) / (384f) + (a^2c \cot[e + fx] \csc[e + fx]^2) / (15f) - (a^2c \cot[e + fx] \csc[e + fx]^4) / (5f) + (a^2c \log[\cos[(e + fx)/2]]) / (16f) - (a^2c \log[\sin[(e + fx)/2]]) / (16f) - (a^2c \sec[(e + fx)/2]^2) / (64f) + (a^2c \sec[(e + fx)/2]^6) / (384f)$

Rubi [A] (verified)

Time = 0.41 (sec) , antiderivative size = 130, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.094$, Rules used = {3042, 3445, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \csc^7(e + fx)(a \sin(e + fx) + a)^2(c - c \sin(e + fx)) dx \\ & \quad \downarrow \text{3042} \\ & \int \frac{(a \sin(e + fx) + a)^2(c - c \sin(e + fx))}{\sin(e + fx)^7} dx \end{aligned}$$

$$\int (a^2 c \csc^7(e + fx) + a^2 c \csc^6(e + fx) - a^2 c \csc^5(e + fx) - a^2 c \csc^4(e + fx)) dx$$

$$\frac{a^2 \operatorname{arctanh}(\cos(e + fx))}{16f} - \frac{a^2 c \cot^5(e + fx)}{5f} - \frac{a^2 c \cot^3(e + fx)}{3f} - \frac{a^2 c \cot(e + fx) \csc^5(e + fx)}{6f} + \frac{a^2 c \cot(e + fx) \csc^3(e + fx)}{24f} + \frac{a^2 c \cot(e + fx) \csc(e + fx)}{16f}$$

input `Int[Csc[e + f*x]^7*(a + a*Sin[e + f*x])^2*(c - c*Sin[e + f*x]),x]`

output `(a^2*c*ArcTanh[Cos[e + f*x]])/(16*f) - (a^2*c*Cot[e + f*x]^3)/(3*f) - (a^2*c*Cot[e + f*x]^5)/(5*f) + (a^2*c*Cot[e + f*x]*Csc[e + f*x])/(16*f) + (a^2*c*Cot[e + f*x]*Csc[e + f*x]^3)/(24*f) - (a^2*c*Cot[e + f*x]*Csc[e + f*x]^5)/(6*f)`

Defintions of rubi rules used

rule 2009 `Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3445 `Int[sin[(e_.) + (f_.)*(x_)]^(n_.)*((a_.) + (b_.)*sin[(e_.) + (f_.)*(x_)]^(m_.))*((A_.) + (B_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] := Int[ExpandTrig[sin[e + f*x]^n*(a + b*sin[e + f*x])^m*(A + B*sin[e + f*x]), x], x] /; FreeQ[{a, b, e, f, A, B}, x] && EqQ[A*b + a*B, 0] && EqQ[a^2 - b^2, 0] && IntegerQ[m] && IntegerQ[n]`

Maple [C] (verified)

Result contains complex when optimal does not.

Time = 2.48 (sec) , antiderivative size = 171, normalized size of antiderivative = 1.32

method	result
risch	$-\frac{a^2c(15e^{11i(fx+e)}-85e^{9i(fx+e)}-570e^{7i(fx+e)}+480ie^{8i(fx+e)}-570e^{5i(fx+e)}-320ie^{6i(fx+e)}-85e^{3i(fx+e)}+15e^{i(fx+e)})}{120f(e^{2i(fx+e)}-1)^6}$
paralelrisch	$-\frac{\left(\cot\left(\frac{fx}{2}+\frac{e}{2}\right)^6-\tan\left(\frac{fx}{2}+\frac{e}{2}\right)^6+\frac{12\cot\left(\frac{fx}{2}+\frac{e}{2}\right)^5}{5}-\frac{12\tan\left(\frac{fx}{2}+\frac{e}{2}\right)^5}{5}+3\cot\left(\frac{fx}{2}+\frac{e}{2}\right)^4-3\tan\left(\frac{fx}{2}+\frac{e}{2}\right)^4+4\cot\left(\frac{fx}{2}+\frac{e}{2}\right)^3-4\tan\left(\frac{fx}{2}+\frac{e}{2}\right)^3\right)}{3}$
derivativedivides	$-a^2c\left(-\frac{2}{3}-\frac{\csc(fx+e)^2}{3}\right)\cot(fx+e)-a^2c\left(\left(-\frac{\csc(fx+e)^3}{4}-\frac{3\csc(fx+e)}{8}\right)\cot(fx+e)+\frac{3\ln(\csc(fx+e)-\cot(fx+e))}{8}\right)+a^2c$
default	$-a^2c\left(-\frac{2}{3}-\frac{\csc(fx+e)^2}{3}\right)\cot(fx+e)-a^2c\left(\left(-\frac{\csc(fx+e)^3}{4}-\frac{3\csc(fx+e)}{8}\right)\cot(fx+e)+\frac{3\ln(\csc(fx+e)-\cot(fx+e))}{8}\right)+a^2c$

```
input int(csc(f*x+e)^7*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x,method=_RETURNVERBOSE)
```

```
output -1/120*a^2*c*(15*exp(11*I*(f*x+e))-85*exp(9*I*(f*x+e))-570*exp(7*I*(f*x+e))+480*I*exp(8*I*(f*x+e))-570*exp(5*I*(f*x+e))-320*I*exp(6*I*(f*x+e))-85*exp(3*I*(f*x+e))+15*exp(I*(f*x+e))-192*I*exp(2*I*(f*x+e))+32*I)/f/(exp(2*I*(f*x+e))-1)^6+1/16*a^2*c/f*ln(exp(I*(f*x+e))+1)-1/16*a^2*c/f*ln(exp(I*(f*x+e))-1)
```

Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 240 vs. 2(118) = 236.

Time = 0.09 (sec) , antiderivative size = 240, normalized size of antiderivative = 1.85

$$\int \csc^7(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx =$$

$$-\frac{30 a^2 c \cos (f x+e)^5-80 a^2 c \cos (f x+e)^3-30 a^2 c \cos (f x+e)-15\left(a^2 c \cos (f x+e)^6-3 a^2 c \cos (f x+e)^4+3 a^2 c \cos (f x+e)^2-a^2 c\right)}{120 f\left(e^{2 i(f x+e)}-1\right)^6}$$

input `integrate(csc(f*x+e)^7*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="fricas")`

output
$$\begin{aligned} & -1/480*(30*a^2*c*\cos(f*x + e)^5 - 80*a^2*c*\cos(f*x + e)^3 - 30*a^2*c*\cos(f \\ & *x + e) - 15*(a^2*c*\cos(f*x + e)^6 - 3*a^2*c*\cos(f*x + e)^4 + 3*a^2*c*\cos(\\ & f*x + e)^2 - a^2*c)*\log(1/2*\cos(f*x + e) + 1/2) + 15*(a^2*c*\cos(f*x + e)^6 \\ & - 3*a^2*c*\cos(f*x + e)^4 + 3*a^2*c*\cos(f*x + e)^2 - a^2*c)*\log(-1/2*\cos(f \\ & *x + e) + 1/2) + 32*(2*a^2*c*\cos(f*x + e)^5 - 5*a^2*c*\cos(f*x + e)^3)*\sin(\\ & f*x + e))/(f*\cos(f*x + e)^6 - 3*f*\cos(f*x + e)^4 + 3*f*\cos(f*x + e)^2 - f) \end{aligned}$$

Sympy [F(-1)]

Timed out.

$$\int \csc^7(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx = \text{Timed out}$$

input `integrate(csc(f*x+e)**7*(a+a*sin(f*x+e))**2*(c-c*sin(f*x+e)),x)`

output Timed out

Maxima [A] (verification not implemented)

Time = 0.04 (sec) , antiderivative size = 232, normalized size of antiderivative = 1.78

$$\begin{aligned} & \int \csc^7(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx \\ & = \frac{5 a^2 c \left(\frac{2 (15 \cos(fx+e)^5 - 40 \cos(fx+e)^3 + 33 \cos(fx+e))}{\cos(fx+e)^6 - 3 \cos(fx+e)^4 + 3 \cos(fx+e)^2 - 1} - 15 \log(\cos(fx+e) + 1) + 15 \log(\cos(fx+e) - 1) \right)}{\dots} \end{aligned}$$

input `integrate(csc(f*x+e)^7*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="maxima")`

output

```
1/480*(5*a^2*c*(2*(15*cos(f*x + e)^5 - 40*cos(f*x + e)^3 + 33*cos(f*x + e)
))/(cos(f*x + e)^6 - 3*cos(f*x + e)^4 + 3*cos(f*x + e)^2 - 1) - 15*log(cos(
f*x + e) + 1) + 15*log(cos(f*x + e) - 1)) - 30*a^2*c*(2*(3*cos(f*x + e)^3
- 5*cos(f*x + e))/(cos(f*x + e)^4 - 2*cos(f*x + e)^2 + 1) - 3*log(cos(f*x
+ e) + 1) + 3*log(cos(f*x + e) - 1)) + 160*(3*tan(f*x + e)^2 + 1)*a^2*c/ta
n(f*x + e)^3 - 32*(15*tan(f*x + e)^4 + 10*tan(f*x + e)^2 + 3)*a^2*c/tan(f*
x + e)^5)/f
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 242 vs. $2(118) = 236$.

Time = 0.39 (sec) , antiderivative size = 242, normalized size of antiderivative = 1.86

$$\int \csc^7(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{5 a^2 c \tan\left(\frac{1}{2} f x + \frac{1}{2} e\right)^6 + 12 a^2 c \tan\left(\frac{1}{2} f x + \frac{1}{2} e\right)^5 + 15 a^2 c \tan\left(\frac{1}{2} f x + \frac{1}{2} e\right)^4 + 20 a^2 c \tan\left(\frac{1}{2} f x + \frac{1}{2} e\right)^3 - \dots}{\dots}$$

input

```
integrate(csc(f*x+e)^7*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x, algorithm="g
iac")
```

output

```
1/1920*(5*a^2*c*tan(1/2*f*x + 1/2*e)^6 + 12*a^2*c*tan(1/2*f*x + 1/2*e)^5 +
15*a^2*c*tan(1/2*f*x + 1/2*e)^4 + 20*a^2*c*tan(1/2*f*x + 1/2*e)^3 - 15*a^
2*c*tan(1/2*f*x + 1/2*e)^2 - 120*a^2*c*log(abs(tan(1/2*f*x + 1/2*e))) - 12
0*a^2*c*tan(1/2*f*x + 1/2*e) + (294*a^2*c*tan(1/2*f*x + 1/2*e)^6 + 120*a^2
*c*tan(1/2*f*x + 1/2*e)^5 + 15*a^2*c*tan(1/2*f*x + 1/2*e)^4 - 20*a^2*c*tan
(1/2*f*x + 1/2*e)^3 - 15*a^2*c*tan(1/2*f*x + 1/2*e)^2 - 12*a^2*c*tan(1/2*f
*x + 1/2*e) - 5*a^2*c)/tan(1/2*f*x + 1/2*e)^6)/f
```

Mupad [B] (verification not implemented)

Time = 35.67 (sec) , antiderivative size = 340, normalized size of antiderivative = 2.62

$$\int \csc^7(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx =$$

$$\frac{a^2 c \left(5 \cos\left(\frac{e}{2} + \frac{fx}{2}\right)^{12} - 5 \sin\left(\frac{e}{2} + \frac{fx}{2}\right)^{12} - 12 \cos\left(\frac{e}{2} + \frac{fx}{2}\right) \sin\left(\frac{e}{2} + \frac{fx}{2}\right)^{11} + 12 \cos\left(\frac{e}{2} + \frac{fx}{2}\right)^{11} \sin\left(\frac{e}{2} + \frac{fx}{2}\right) \right)}{1920 f \cos\left(\frac{e}{2} + \frac{fx}{2}\right)^6 \sin\left(\frac{e}{2} + \frac{fx}{2}\right)^6}$$

input

```
int(((a + a*sin(e + f*x))^2*(c - c*sin(e + f*x)))/sin(e + f*x)^7,x)
```

output

```
-(a^2*c*(5*cos(e/2 + (f*x)/2)^12 - 5*sin(e/2 + (f*x)/2)^12 - 12*cos(e/2 + (f*x)/2)*sin(e/2 + (f*x)/2)^11 + 12*cos(e/2 + (f*x)/2)^11*sin(e/2 + (f*x)/2) - 15*cos(e/2 + (f*x)/2)^2*sin(e/2 + (f*x)/2)^10 - 20*cos(e/2 + (f*x)/2)^3*sin(e/2 + (f*x)/2)^9 + 15*cos(e/2 + (f*x)/2)^4*sin(e/2 + (f*x)/2)^8 + 120*cos(e/2 + (f*x)/2)^5*sin(e/2 + (f*x)/2)^7 - 120*cos(e/2 + (f*x)/2)^7*sin(e/2 + (f*x)/2)^5 - 15*cos(e/2 + (f*x)/2)^8*sin(e/2 + (f*x)/2)^4 + 20*cos(e/2 + (f*x)/2)^9*sin(e/2 + (f*x)/2)^3 + 15*cos(e/2 + (f*x)/2)^10*sin(e/2 + (f*x)/2)^2 + 120*cos(e/2 + (f*x)/2)^6*log(sin(e/2 + (f*x)/2)/cos(e/2 + (f*x)/2))*sin(e/2 + (f*x)/2)^6)/(1920*f*cos(e/2 + (f*x)/2)^6*sin(e/2 + (f*x)/2)^6)
```

Reduce [B] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 124, normalized size of antiderivative = 0.95

$$\int \csc^7(e + fx)(a + a \sin(e + fx))^2(c - c \sin(e + fx)) dx$$

$$= \frac{a^2 c (32 \cos(fx + e) \sin(fx + e)^5 + 15 \cos(fx + e) \sin(fx + e)^4 + 16 \cos(fx + e) \sin(fx + e)^3 + 10 \cos(fx + e) \sin(fx + e)^2 + 5 \cos(fx + e) \sin(fx + e) + 5 \cos(fx + e))}{240 \sin(fx + e)^6}$$

input

```
int(csc(f*x+e)^7*(a+a*sin(f*x+e))^2*(c-c*sin(f*x+e)),x)
```

output

```
(a**2*c*(32*cos(e + f*x)*sin(e + f*x)**5 + 15*cos(e + f*x)*sin(e + f*x)**4
+ 16*cos(e + f*x)*sin(e + f*x)**3 + 10*cos(e + f*x)*sin(e + f*x)**2 - 48*
cos(e + f*x)*sin(e + f*x) - 40*cos(e + f*x) - 15*log(tan((e + f*x)/2))*sin
(e + f*x)**6))/(240*sin(e + f*x)**6*f)
```

3.12 $\int \sin^2(c + dx)(a + a \sin(c + dx))^{3/2}(c - c \sin(c + dx)) dx$

Optimal result	123
Mathematica [A] (verified)	124
Rubi [A] (verified)	124
Maple [A] (verified)	128
Fricas [A] (verification not implemented)	128
Sympy [F]	129
Maxima [F]	129
Giac [A] (verification not implemented)	130
Mupad [F(-1)]	130
Reduce [F]	131

Optimal result

Integrand size = 34, antiderivative size = 128

$$\int \sin^2(c + dx)(a + a \sin(c + dx))^{3/2}(c - c \sin(c + dx)) dx =$$

$$\frac{8a^3c \cos^3(c + dx)}{63d(a + a \sin(c + dx))^{3/2}} - \frac{2a^2c \cos^3(c + dx)}{21d\sqrt{a + a \sin(c + dx)}}$$

$$+ \frac{4ac \cos^3(c + dx)\sqrt{a + a \sin(c + dx)}}{21d} - \frac{2c \cos^3(c + dx)(a + a \sin(c + dx))^{3/2}}{9d}$$

```
output -8/63*a^3*c*cos(d*x+c)^3/d/(a+a*sin(d*x+c))^(3/2)-2/21*a^2*c*cos(d*x+c)^3/
d/(a+a*sin(d*x+c))^(1/2)+4/21*a*c*cos(d*x+c)^3*(a+a*sin(d*x+c))^(1/2)/d-2/
9*c*cos(d*x+c)^3*(a+a*sin(d*x+c))^(3/2)/d
```

Mathematica [A] (verified)

Time = 3.24 (sec) , antiderivative size = 101, normalized size of antiderivative = 0.79

$$\int \sin^2(c+dx)(a+a\sin(c+dx))^{3/2}(c-c\sin(c+dx))dx = \frac{ac(\cos(\frac{1}{2}(c+dx))-\sin(\frac{1}{2}(c+dx)))^3\sqrt{a(1+\sin(c+dx))}(-62+30\cos(2(c+dx)))}{126d(\cos(\frac{1}{2}(c+dx))+\sin(\frac{1}{2}(c+dx)))}$$

input

```
Integrate[Sin[c + d*x]^2*(a + a*Sin[c + d*x])^(3/2)*(c - c*Sin[c + d*x]),x]
```

output

```
(a*c*(Cos[(c + d*x)/2] - Sin[(c + d*x)/2])^3*Sqrt[a*(1 + Sin[c + d*x])]*(-62 + 30*Cos[2*(c + d*x)] - 69*Sin[c + d*x] + 7*Sin[3*(c + d*x)])/(126*d*(Cos[(c + d*x)/2] + Sin[(c + d*x)/2]))
```

Rubi [A] (verified)

Time = 0.91 (sec) , antiderivative size = 185, normalized size of antiderivative = 1.45, number of steps used = 12, number of rules used = 12, $\frac{\text{number of rules}}{\text{integrand size}} = 0.353$, Rules used = {3042, 3455, 27, 3042, 3460, 3042, 3238, 27, 3042, 3230, 3042, 3125}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \sin^2(c+dx)(c-c\sin(c+dx))(a\sin(c+dx)+a)^{3/2}dx \\ & \quad \downarrow \text{3042} \\ & \int \sin(c+dx)^2(c-c\sin(c+dx))(a\sin(c+dx)+a)^{3/2}dx \\ & \quad \downarrow \text{3455} \\ & \frac{2}{9} \int \frac{1}{2} \sin^2(c+dx) \sqrt{\sin(c+dx)a+a} (3ac-ac\sin(c+dx))dx + \\ & \quad \frac{2ac\sin^3(c+dx)\cos(c+dx)\sqrt{a\sin(c+dx)+a}}{9d} \\ & \quad \downarrow \text{27} \end{aligned}$$

$$\frac{1}{9} \int \sin^2(c+dx) \sqrt{\sin(c+dx)a+a} (3ac-ac\sin(c+dx)) dx + \frac{2ac \sin^3(c+dx) \cos(c+dx) \sqrt{a \sin(c+dx)+a}}{9d}$$

↓ 3042

$$\frac{1}{9} \int \sin(c+dx)^2 \sqrt{\sin(c+dx)a+a} (3ac-ac\sin(c+dx)) dx + \frac{2ac \sin^3(c+dx) \cos(c+dx) \sqrt{a \sin(c+dx)+a}}{9d}$$

↓ 3460

$$\frac{1}{9} \left(\frac{15}{7} ac \int \sin^2(c+dx) \sqrt{\sin(c+dx)a+adx} + \frac{2a^2c \sin^3(c+dx) \cos(c+dx)}{7d \sqrt{a \sin(c+dx)+a}} \right) + \frac{2ac \sin^3(c+dx) \cos(c+dx) \sqrt{a \sin(c+dx)+a}}{9d}$$

↓ 3042

$$\frac{1}{9} \left(\frac{15}{7} ac \int \sin(c+dx)^2 \sqrt{\sin(c+dx)a+adx} + \frac{2a^2c \sin^3(c+dx) \cos(c+dx)}{7d \sqrt{a \sin(c+dx)+a}} \right) + \frac{2ac \sin^3(c+dx) \cos(c+dx) \sqrt{a \sin(c+dx)+a}}{9d}$$

↓ 3238

$$\frac{1}{9} \left(\frac{15}{7} ac \left(\frac{2 \int \frac{1}{2} (3a-2a \sin(c+dx)) \sqrt{\sin(c+dx)a+adx}}{5a} - \frac{2 \cos(c+dx) (a \sin(c+dx)+a)^{3/2}}{5ad} \right) + \frac{2a^2c \sin^3(c+dx) \cos(c+dx)}{7d \sqrt{a \sin(c+dx)+a}} \right) + \frac{2ac \sin^3(c+dx) \cos(c+dx) \sqrt{a \sin(c+dx)+a}}{9d}$$

↓ 27

$$\frac{1}{9} \left(\frac{15}{7} ac \left(\frac{\int (3a-2a \sin(c+dx)) \sqrt{\sin(c+dx)a+adx}}{5a} - \frac{2 \cos(c+dx) (a \sin(c+dx)+a)^{3/2}}{5ad} \right) + \frac{2a^2c \sin^3(c+dx) \cos(c+dx)}{7d \sqrt{a \sin(c+dx)+a}} \right) + \frac{2ac \sin^3(c+dx) \cos(c+dx) \sqrt{a \sin(c+dx)+a}}{9d}$$

↓ 3042

$$\frac{1}{9} \left(\frac{15}{7} ac \left(\frac{\int (3a - 2a \sin(c + dx)) \sqrt{\sin(c + dx)a + adx}}{5a} - \frac{2 \cos(c + dx)(a \sin(c + dx) + a)^{3/2}}{5ad} \right) + \frac{2a^2 c \sin^3(c + dx)}{7d \sqrt{a \sin(c + dx) + a}} \right)$$

$$\frac{2ac \sin^3(c + dx) \cos(c + dx) \sqrt{a \sin(c + dx) + a}}{9d}$$

↓ 3230

$$\frac{1}{9} \left(\frac{15}{7} ac \left(\frac{\frac{7}{3} a \int \sqrt{\sin(c + dx)a + adx} + \frac{4a \cos(c+dx) \sqrt{a \sin(c+dx)+a}}{3d}}{5a} - \frac{2 \cos(c + dx)(a \sin(c + dx) + a)^{3/2}}{5ad} \right) + \frac{2a^2 c \sin^3(c + dx)}{7d \sqrt{a \sin(c + dx) + a}} \right)$$

$$\frac{2ac \sin^3(c + dx) \cos(c + dx) \sqrt{a \sin(c + dx) + a}}{9d}$$

↓ 3042

$$\frac{1}{9} \left(\frac{15}{7} ac \left(\frac{\frac{7}{3} a \int \sqrt{\sin(c + dx)a + adx} + \frac{4a \cos(c+dx) \sqrt{a \sin(c+dx)+a}}{3d}}{5a} - \frac{2 \cos(c + dx)(a \sin(c + dx) + a)^{3/2}}{5ad} \right) + \frac{2a^2 c \sin^3(c + dx)}{7d \sqrt{a \sin(c + dx) + a}} \right)$$

$$\frac{2ac \sin^3(c + dx) \cos(c + dx) \sqrt{a \sin(c + dx) + a}}{9d}$$

↓ 3125

$$\frac{1}{9} \left(\frac{2a^2 c \sin^3(c + dx) \cos(c + dx)}{7d \sqrt{a \sin(c + dx) + a}} + \frac{15}{7} ac \left(\frac{\frac{4a \cos(c+dx) \sqrt{a \sin(c+dx)+a}}{3d} - \frac{14a^2 \cos(c+dx)}{3d \sqrt{a \sin(c+dx)+a}}}{5a} - \frac{2 \cos(c + dx)(a \sin(c + dx) + a)^{3/2}}{5ad} \right) \right)$$

$$\frac{2ac \sin^3(c + dx) \cos(c + dx) \sqrt{a \sin(c + dx) + a}}{9d}$$

input `Int[Sin[c + d*x]^2*(a + a*SIN[c + d*x])^(3/2)*(c - c*SIN[c + d*x]),x]`

output `(2*a*c*cos[c + d*x]*sin[c + d*x]^3*sqrt[a + a*sin[c + d*x]])/(9*d) + ((2*a^2*c*cos[c + d*x]*sin[c + d*x]^3)/(7*d*sqrt[a + a*sin[c + d*x]]) + (15*a*c*((-2*cos[c + d*x]*(a + a*sin[c + d*x])^(3/2))/(5*a*d) + ((-14*a^2*cos[c + d*x])/(3*d*sqrt[a + a*sin[c + d*x]]) + (4*a*cos[c + d*x]*sqrt[a + a*sin[c + d*x]])/(3*d))/(5*a)))/7)/9`

Defintions of rubi rules used

- rule 27 $\text{Int}[(a_*)(Fx_), x_Symbol] \rightarrow \text{Simp}[a \text{ Int}[Fx, x], x] /; \text{FreeQ}[a, x] \ \&\& \ !\text{MatchQ}[Fx, (b_*)(Gx_)] /; \text{FreeQ}[b, x]$
- rule 3042 $\text{Int}[u_, x_Symbol] \rightarrow \text{Int}[\text{DeactivateTrig}[u, x], x] /; \text{FunctionOfTrigOfLinearQ}[u, x]$
- rule 3125 $\text{Int}[\text{Sqrt}[(a_) + (b_*)\sin[(c_) + (d_*)(x_)]], x_Symbol] \rightarrow \text{Simp}[-2*b*(\text{Cos}[c + d*x]/(d*\text{Sqrt}[a + b*\text{Sin}[c + d*x]])), x] /; \text{FreeQ}[\{a, b, c, d\}, x] \ \&\& \ \text{EqQ}[a^2 - b^2, 0]$
- rule 3230 $\text{Int}[(a_) + (b_*)\sin[(e_) + (f_*)(x_)]^{(m_)*((c_) + (d_*)\sin[(e_) + (f_*)(x_)]), x_Symbol] \rightarrow \text{Simp}[(-d)*\text{Cos}[e + f*x]*((a + b*\text{Sin}[e + f*x])^m/(f*(m + 1))), x] + \text{Simp}[(a*d*m + b*c*(m + 1))/(b*(m + 1)) \text{ Int}[(a + b*\text{Sin}[e + f*x])^m, x], x] /; \text{FreeQ}[\{a, b, c, d, e, f, m\}, x] \ \&\& \ \text{NeQ}[b*c - a*d, 0] \ \&\& \ \text{EqQ}[a^2 - b^2, 0] \ \&\& \ !\text{LtQ}[m, -2^{(-1)}]$
- rule 3238 $\text{Int}[\sin[(e_) + (f_*)(x_)]^2*((a_) + (b_*)\sin[(e_) + (f_*)(x_)]^{(m_)}, x_Symbol] \rightarrow \text{Simp}[(-\text{Cos}[e + f*x])*((a + b*\text{Sin}[e + f*x])^{(m + 1)})/(b*f*(m + 2))], x] + \text{Simp}[1/(b*(m + 2)) \text{ Int}[(a + b*\text{Sin}[e + f*x])^m*(b*(m + 1) - a*\text{Sin}[e + f*x]), x], x] /; \text{FreeQ}[\{a, b, e, f, m\}, x] \ \&\& \ \text{EqQ}[a^2 - b^2, 0] \ \&\& \ !\text{LtQ}[m, -2^{(-1)}]$
- rule 3455 $\text{Int}[(a_) + (b_*)\sin[(e_) + (f_*)(x_)]^{(m_)*((A_) + (B_*)\sin[(e_) + (f_*)(x_)]^{(n_)}, x_Symbol] \rightarrow \text{Simp}[(-b)*B*\text{Cos}[e + f*x]*(a + b*\text{Sin}[e + f*x])^{(m - 1)}*((c + d*\text{Sin}[e + f*x])^{(n + 1)})/(d*f*(m + n + 1))], x] + \text{Simp}[1/(d*(m + n + 1)) \text{ Int}[(a + b*\text{Sin}[e + f*x])^{(m - 1)}*(c + d*\text{Sin}[e + f*x])^n*\text{Simp}[a*A*d*(m + n + 1) + B*(a*c*(m - 1) + b*d*(n + 1)) + (A*b*d*(m + n + 1) - B*(b*c*m - a*d*(2*m + n)))*\text{Sin}[e + f*x], x], x], x] /; \text{FreeQ}[\{a, b, c, d, e, f, A, B, n\}, x] \ \&\& \ \text{NeQ}[b*c - a*d, 0] \ \&\& \ \text{EqQ}[a^2 - b^2, 0] \ \&\& \ \text{NeQ}[c^2 - d^2, 0] \ \&\& \ \text{GtQ}[m, 1/2] \ \&\& \ !\text{LtQ}[n, -1] \ \&\& \ \text{IntegerQ}[2*m] \ \&\& \ (\text{IntegerQ}[2*n] \ || \ \text{EqQ}[c, 0])]$

rule 3460

```
Int[Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)]]*((A_) + (B_)*sin[(e_) + (
f_)*(x_)])*((c_) + (d_)*sin[(e_) + (f_)*(x_)])^(n_), x_Symbol] :> Simp
[-2*b*B*Cos[e + f*x]*((c + d*Sin[e + f*x])^(n + 1)/(d*f*(2*n + 3)*Sqrt[a +
b*Sin[e + f*x]])), x] + Simp[(A*b*d*(2*n + 3) - B*(b*c - 2*a*d*(n + 1)))/(b
*d*(2*n + 3)) Int[Sqrt[a + b*Sin[e + f*x]]*(c + d*Sin[e + f*x])^n, x], x]
/; FreeQ[{a, b, c, d, e, f, A, B, n}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 -
b^2, 0] && NeQ[c^2 - d^2, 0] && !LtQ[n, -1]
```

Maple [A] (verified)

Time = 1.56 (sec) , antiderivative size = 78, normalized size of antiderivative = 0.61

method	result
default	$\frac{2(1+\sin(dx+c))a^2(\sin(dx+c)-1)^2c(7\sin(dx+c)^3+15\sin(dx+c)^2+12\sin(dx+c)+8)}{63\cos(dx+c)\sqrt{a+a\sin(dx+c)}d}$
parts	$\frac{2c(1+\sin(dx+c))a^2(\sin(dx+c)-1)(15\sin(dx+c)^3+39\sin(dx+c)^2+52\sin(dx+c)+104)}{105\cos(dx+c)\sqrt{a+a\sin(dx+c)}d} - \frac{2c(1+\sin(dx+c))a^2(\sin(dx+c)-1)(35\sin(dx+c)^3+39\sin(dx+c)^2+52\sin(dx+c)+104)}{105\cos(dx+c)\sqrt{a+a\sin(dx+c)}d}$

input

```
int(sin(d*x+c)^2*(a+a*sin(d*x+c))^(3/2)*(c-c*sin(d*x+c)),x,method=_RETURNV
ERBOSE)
```

output

```
-2/63*(1+sin(d*x+c))*a^2*(sin(d*x+c)-1)^2*c*(7*sin(d*x+c)^3+15*sin(d*x+c)^
2+12*sin(d*x+c)+8)/cos(d*x+c)/(a+a*sin(d*x+c))^(1/2)/d
```

Fricas [A] (verification not implemented)

Time = 0.07 (sec) , antiderivative size = 155, normalized size of antiderivative = 1.21

$$\int \sin^2(c + dx)(a + a \sin(c + dx))^{3/2}(c - c \sin(c + dx)) dx = \frac{2(7ac \cos(dx + c)^5 - ac \cos(dx + c)^4 - 11ac \cos(dx + c)^3 + ac \cos(dx + c)^2 - 4ac \cos(dx + c))}{105 \cos(dx + c) \sqrt{a + a \sin(dx + c)}} + \frac{2c(1 + \sin(dx + c))a^2(\sin(dx + c) - 1)(15 \sin(dx + c)^3 + 39 \sin(dx + c)^2 + 52 \sin(dx + c) + 104)}{105 \cos(dx + c) \sqrt{a + a \sin(dx + c)}} - \frac{2c(1 + \sin(dx + c))a^2(\sin(dx + c) - 1)(35 \sin(dx + c)^3 + 39 \sin(dx + c)^2 + 52 \sin(dx + c) + 104)}{105 \cos(dx + c) \sqrt{a + a \sin(dx + c)}}$$

input

```
integrate(sin(d*x+c)^2*(a+a*sin(d*x+c))^(3/2)*(c-c*sin(d*x+c)),x, algorith
m="fricas")
```

output

```
2/63*(7*a*c*cos(d*x + c)^5 - a*c*cos(d*x + c)^4 - 11*a*c*cos(d*x + c)^3 +
a*c*cos(d*x + c)^2 - 4*a*c*cos(d*x + c) - 8*a*c - (7*a*c*cos(d*x + c)^4 +
8*a*c*cos(d*x + c)^3 - 3*a*c*cos(d*x + c)^2 - 4*a*c*cos(d*x + c) - 8*a*c)*
sin(d*x + c))*sqrt(a*sin(d*x + c) + a)/(d*cos(d*x + c) + d*sin(d*x + c) +
d)
```

Sympy [F]

$$\int \sin^2(c + dx)(a + a \sin(c + dx))^{3/2}(c - c \sin(c + dx)) dx =$$

$$-c \left(\int \left(-a \sqrt{a \sin(c + dx) + a} \sin^2(c + dx) \right) dx \right.$$

$$\left. + \int a \sqrt{a \sin(c + dx) + a} \sin^4(c + dx) dx \right)$$

input

```
integrate(sin(d*x+c)**2*(a+a*sin(d*x+c))**(3/2)*(c-c*sin(d*x+c)),x)
```

output

```
-c*(Integral(-a*sqrt(a*sin(c + d*x) + a)*sin(c + d*x)**2, x) + Integral(a*
sqrt(a*sin(c + d*x) + a)*sin(c + d*x)**4, x))
```

Maxima [F]

$$\int \sin^2(c + dx)(a + a \sin(c + dx))^{3/2}(c$$

$$- c \sin(c + dx)) dx = \int -(a \sin(dx + c) + a)^{3/2}(c \sin(dx + c) - c) \sin(dx + c)^2 dx$$

input

```
integrate(sin(d*x+c)^2*(a+a*sin(d*x+c))^(3/2)*(c-c*sin(d*x+c)),x, algorithm
m="maxima")
```

output

```
-integrate((a*sin(d*x + c) + a)^(3/2)*(c*sin(d*x + c) - c)*sin(d*x + c)^2,
x)
```

Giac [A] (verification not implemented)

Time = 0.36 (sec) , antiderivative size = 99, normalized size of antiderivative = 0.77

$$\int \sin^2(c + dx)(a + a \sin(c + dx))^{3/2}(c - c \sin(c + dx)) dx = \frac{\sqrt{2}(126 a c \operatorname{sgn}(\cos(-\frac{1}{4} \pi + \frac{1}{2} dx + \frac{1}{2} c)) \sin(-\frac{1}{4} \pi + \frac{1}{2} dx + \frac{1}{2} c) - 9 a c \operatorname{sgn}(\cos(-\frac{1}{4} \pi + \frac{1}{2} dx + \frac{1}{2} c)) \sin(-\frac{7}{4} \pi + \frac{7}{2} dx + \frac{7}{2} c) - 7 a c \operatorname{sgn}(\cos(-\frac{1}{4} \pi + \frac{1}{2} dx + \frac{1}{2} c)) \sin(-\frac{9}{4} \pi + \frac{9}{2} dx + \frac{9}{2} c)) \sqrt{a}}{d}$$

input `integrate(sin(d*x+c)^2*(a+a*sin(d*x+c))^(3/2)*(c-c*sin(d*x+c)),x, algorithm m="giac")`

output `1/504*sqrt(2)*(126*a*c*sgn(cos(-1/4*pi + 1/2*d*x + 1/2*c))*sin(-1/4*pi + 1/2*d*x + 1/2*c) - 9*a*c*sgn(cos(-1/4*pi + 1/2*d*x + 1/2*c))*sin(-7/4*pi + 7/2*d*x + 7/2*c) - 7*a*c*sgn(cos(-1/4*pi + 1/2*d*x + 1/2*c))*sin(-9/4*pi + 9/2*d*x + 9/2*c))*sqrt(a)/d`

Mupad [F(-1)]

Timed out.

$$\int \sin^2(c + dx)(a + a \sin(c + dx))^{3/2}(c - c \sin(c + dx)) dx = \int \sin(c + dx)^2 (a + a \sin(c + dx))^{3/2} (c - c \sin(c + dx)) dx$$

input `int(sin(c + d*x)^2*(a + a*sin(c + d*x))^(3/2)*(c - c*sin(c + d*x)),x)`

output `int(sin(c + d*x)^2*(a + a*sin(c + d*x))^(3/2)*(c - c*sin(c + d*x)), x)`

Reduce [F]

$$\int \sin^2(c + dx)(a + a \sin(c + dx))^{3/2}(c - c \sin(c + dx)) dx = \sqrt{a} ac \left(- \left(\int \sqrt{\sin(dx + c) + 1} \sin(dx + c)^4 dx \right) + \int \sqrt{\sin(dx + c) + 1} \sin(dx + c)^2 dx \right)$$

input

```
int(sin(d*x+c)^2*(a+a*sin(d*x+c))^(3/2)*(c-c*sin(d*x+c)),x)
```

output

```
sqrt(a)*a*c*( - int(sqrt(sin(c + d*x) + 1)*sin(c + d*x)**4,x) + int(sqrt(s
in(c + d*x) + 1)*sin(c + d*x)**2,x))
```

3.13 $\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{c-c\sin(e+fx)} dx$

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Optimal result

Integrand size = 34, antiderivative size = 69

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{c-c\sin(e+fx)} dx = -\frac{2\sqrt{a}\operatorname{arctanh}\left(\frac{\sqrt{a}\cos(e+fx)}{\sqrt{a+a\sin(e+fx)}}\right)}{cf} + \frac{2\sec(e+fx)\sqrt{a+a\sin(e+fx)}}{cf}$$

output

```
-2*a^(1/2)*arctanh(a^(1/2)*cos(f*x+e)/(a+a*sin(f*x+e))^(1/2))/c/f+2*sec(f*x+e)*(a+a*sin(f*x+e))^(1/2)/c/f
```

Mathematica [C] (verified)

Result contains higher order function than in optimal. Order 5 vs. order 3 in optimal.

Time = 3.34 (sec) , antiderivative size = 46, normalized size of antiderivative = 0.67

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{c-c\sin(e+fx)} dx = \frac{2 \operatorname{Hypergeometric2F1}\left(-\frac{1}{2}, 1, \frac{1}{2}, 1 - \sin(e+fx)\right) \sec(e+fx)\sqrt{a(1+\sin(e+fx))}}{cf}$$

input `Integrate[(Csc[e + f*x]*Sqrt[a + a*Sin[e + f*x]])/(c - c*Sin[e + f*x]),x]`

output `(2*Hypergeometric2F1[-1/2, 1, 1/2, 1 - Sin[e + f*x]]*Sec[e + f*x]*Sqrt[a*(1 + Sin[e + f*x])])/(c*f)`

Rubi [A] (verified)

Time = 0.71 (sec) , antiderivative size = 69, normalized size of antiderivative = 1.00, number of steps used = 9, number of rules used = 8, $\frac{\text{number of rules}}{\text{integrand size}} = 0.235$, Rules used = {3042, 3413, 3042, 3215, 3042, 3152, 3252, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{\csc(e + fx) \sqrt{a \sin(e + fx) + a}}{c - c \sin(e + fx)} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{\sqrt{a \sin(e + fx) + a}}{\sin(e + fx)(c - c \sin(e + fx))} dx \\
 & \quad \downarrow \text{3413} \\
 & \int \frac{\sqrt{\sin(e + fx)a + a}}{c - c \sin(e + fx)} dx + \int \frac{\csc(e + fx) \sqrt{\sin(e + fx)a + a}}{c} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{\sqrt{\sin(e + fx)a + a}}{\sin(e + fx)} dx + \int \frac{\sqrt{\sin(e + fx)a + a}}{c - c \sin(e + fx)} dx \\
 & \quad \downarrow \text{3215} \\
 & \int \frac{\sqrt{\sin(e + fx)a + a}}{\sin(e + fx)} dx + \int \frac{\sec^2(e + fx) (\sin(e + fx)a + a)^{3/2}}{ac} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{\sqrt{\sin(e + fx)a + a}}{\sin(e + fx)} dx + \int \frac{(\sin(e + fx)a + a)^{3/2}}{\cos(e + fx)^2} dx \\
 & \quad \downarrow \text{3152}
 \end{aligned}$$

$$\begin{aligned} & \frac{\int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)} dx}{c} + \frac{2 \sec(e+fx) \sqrt{a \sin(e+fx) + a}}{cf} \\ & \quad \downarrow \text{3252} \\ & \frac{2 \sec(e+fx) \sqrt{a \sin(e+fx) + a}}{cf} - \frac{2a \int \frac{1}{a - \frac{a^2 \cos^2(e+fx)}{\sin(e+fx)a+a}} d \frac{a \cos(e+fx)}{\sqrt{\sin(e+fx)a+a}}}{cf} \\ & \quad \downarrow \text{219} \\ & \frac{2 \sec(e+fx) \sqrt{a \sin(e+fx) + a}}{cf} - \frac{2\sqrt{a} \operatorname{arctanh}\left(\frac{\sqrt{a} \cos(e+fx)}{\sqrt{a \sin(e+fx) + a}}\right)}{cf} \end{aligned}$$

input `Int[(Csc[e + f*x]*Sqrt[a + a*Sin[e + f*x]])/(c - c*Sin[e + f*x]),x]`

output `(-2*Sqrt[a]*ArcTanh[(Sqrt[a]*Cos[e + f*x])/Sqrt[a + a*Sin[e + f*x]])/(c*f) + (2*Sec[e + f*x]*Sqrt[a + a*Sin[e + f*x]])/(c*f)`

Defintions of rubi rules used

rule 219 `Int[((a_) + (b_)*(x_)^2)^(-1), x_Symbol] := Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (GtQ[a, 0] || LtQ[b, 0])`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3152 `Int[(cos[(e_.) + (f_.)*(x_)]*(g_.))^p]*((a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])^m, x_Symbol] := Simp[b*(g*Cos[e + f*x])^(p + 1)*((a + b*Sin[e + f*x])^(m - 1)/(f*g*(m - 1))), x] /; FreeQ[{a, b, e, f, g, m, p}, x] && EqQ[a^2 - b^2, 0] && EqQ[2*m + p - 1, 0] && NeQ[m, 1]`

rule 3215 $\text{Int}[(a_.) + (b_.)\sin[(e_.) + (f_.)x]^{(m_.)}((c_.) + (d_.)\sin[(e_.) + (f_.)x])^{(n_.)}, x_Symbol] \rightarrow \text{Simp}[a^m c^m \text{Int}[\text{Cos}[e + fx]^{(2m)}(c + d\text{Sin}[e + fx])^{(n-m)}, x], x] /;$ FreeQ[{a, b, c, d, e, f, n}, x] && EqQ[b*c + a*d, 0] && EqQ[a^2 - b^2, 0] && IntegerQ[m] && !(IntegerQ[n] && ((LtQ[m, 0] && GtQ[n, 0]) || LtQ[0, n, m] || LtQ[m, n, 0]))

rule 3252 $\text{Int}[\text{Sqrt}[(a_.) + (b_.)\sin[(e_.) + (f_.)x]]/((c_.) + (d_.)\sin[(e_.) + (f_.)x]), x_Symbol] \rightarrow \text{Simp}[-2*(b/f) \text{Subst}[\text{Int}[1/(b*c + a*d - d*x^2), x], x, b*(\text{Cos}[e + fx]/\text{Sqrt}[a + b*\text{Sin}[e + fx]])], x] /;$ FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]

rule 3413 $\text{Int}[\text{Sqrt}[(a_.) + (b_.)\sin[(e_.) + (f_.)x]]/(\sin[(e_.) + (f_.)x]((c_.) + (d_.)\sin[(e_.) + (f_.)x])), x_Symbol] \rightarrow \text{Simp}[1/c \text{Int}[\text{Sqrt}[a + b*\text{Sin}[e + fx]]/\text{Sin}[e + fx], x], x] - \text{Simp}[d/c \text{Int}[\text{Sqrt}[a + b*\text{Sin}[e + fx]]/(c + d*\text{Sin}[e + fx]), x], x] /;$ FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]

Maple [A] (verified)

Time = 1.00 (sec) , antiderivative size = 79, normalized size of antiderivative = 1.14

method	result	size
default	$-\frac{2(1+\sin(fx+e))\left(\operatorname{arctanh}\left(\frac{\sqrt{a-a\sin(fx+e)}}{\sqrt{a}}\right)a\sqrt{a-a\sin(fx+e)}-a^{\frac{3}{2}}\right)}{\sqrt{a}c\cos(fx+e)\sqrt{a+a\sin(fx+e)}f}$	79

input `int(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x,method=_RETURNVERBOSE)`

output
$$-2/a^{(1/2)}/c*(1+\sin(f*x+e))*(\operatorname{arctanh}((a-a*\sin(f*x+e))^{(1/2)}/a^{(1/2)})*a*(a-a*\sin(f*x+e))^{(1/2)}-a^{(3/2)})/\cos(f*x+e)/(a+a*\sin(f*x+e))^{(1/2)}/f$$

Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 202 vs. $2(61) = 122$.

Time = 0.09 (sec) , antiderivative size = 202, normalized size of antiderivative = 2.93

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{c-c\sin(e+fx)} dx$$

$$= \frac{\sqrt{a}\cos(fx+e)\log\left(\frac{a\cos(fx+e)^3-7a\cos(fx+e)^2-4(\cos(fx+e)^2+(\cos(fx+e)+3)\sin(fx+e)-2\cos(fx+e)-3)\sqrt{a\sin(fx+e)+a}}{\cos(fx+e)^3+\cos(fx+e)^2+(\cos(fx+e)^2-1)\sin(fx+e)-\cos(fx+e)-1}\right)}{2cf\cos(fx+e)}$$

input `integrate(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x, algorithm="fricas")`

output `1/2*(sqrt(a)*cos(f*x + e)*log((a*cos(f*x + e)^3 - 7*a*cos(f*x + e)^2 - 4*(cos(f*x + e)^2 + (cos(f*x + e) + 3)*sin(f*x + e) - 2*cos(f*x + e) - 3)*sqrt(a*sin(f*x + e) + a)*sqrt(a) - 9*a*cos(f*x + e) + (a*cos(f*x + e)^2 + 8*a*cos(f*x + e) - a)*sin(f*x + e) - a)/(cos(f*x + e)^3 + cos(f*x + e)^2 + (cos(f*x + e)^2 - 1)*sin(f*x + e) - cos(f*x + e) - 1)) + 4*sqrt(a*sin(f*x + e) + a))/(c*f*cos(f*x + e))`

Sympy [F]

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{c-c\sin(e+fx)} dx = -\frac{\int \frac{\sqrt{a\sin(e+fx)+a}\csc(e+fx)}{\sin(e+fx)-1} dx}{c}$$

input `integrate(csc(f*x+e)*(a+a*sin(f*x+e))**(1/2)/(c-c*sin(f*x+e)),x)`

output `-Integral(sqrt(a*sin(e + f*x) + a)*csc(e + f*x)/(sin(e + f*x) - 1), x)/c`

Maxima [F]

$$\int \frac{\csc(e + fx) \sqrt{a + a \sin(e + fx)}}{c - c \sin(e + fx)} dx = \int -\frac{\sqrt{a \sin(fx + e) + a} \csc(fx + e)}{c \sin(fx + e) - c} dx$$

input `integrate(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x, algorithm="maxima")`

output `-integrate(sqrt(a*sin(f*x + e) + a)*csc(f*x + e)/(c*sin(f*x + e) - c), x)`

Giac [A] (verification not implemented)

Time = 0.41 (sec) , antiderivative size = 110, normalized size of antiderivative = 1.59

$$\int \frac{\csc(e + fx) \sqrt{a + a \sin(e + fx)}}{c - c \sin(e + fx)} dx = \frac{\sqrt{2} \left(\frac{\sqrt{2} \log \left(\frac{|-2\sqrt{2}+4 \sin(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e)|}{|2\sqrt{2}+4 \sin(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e)|} \right) \operatorname{sgn}(\cos(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e))}{c} + \frac{2 \operatorname{sgn}(\cos(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e))}{c \sin(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e)} \right) \sqrt{a}}{2f}$$

input `integrate(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x, algorithm="giac")`

output `-1/2*sqrt(2)*(sqrt(2)*log(abs(-2*sqrt(2) + 4*sin(-1/4*pi + 1/2*f*x + 1/2*e))/abs(2*sqrt(2) + 4*sin(-1/4*pi + 1/2*f*x + 1/2*e)))*sgn(cos(-1/4*pi + 1/2*f*x + 1/2*e))/c + 2*sgn(cos(-1/4*pi + 1/2*f*x + 1/2*e))/(c*sin(-1/4*pi + 1/2*f*x + 1/2*e)))*sqrt(a)/f`

Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e + fx) \sqrt{a + a \sin(e + fx)}}{c - c \sin(e + fx)} dx = \int \frac{\sqrt{a + a \sin(e + fx)}}{\sin(e + fx) (c - c \sin(e + fx))} dx$$

input `int((a + a*sin(e + f*x))^(1/2)/(sin(e + f*x)*(c - c*sin(e + f*x))),x)`

output `int((a + a*sin(e + f*x))^(1/2)/(sin(e + f*x)*(c - c*sin(e + f*x))), x)`

Reduce [F]

$$\int \frac{\csc(e + fx) \sqrt{a + a \sin(e + fx)}}{c - c \sin(e + fx)} dx = -\frac{\sqrt{a} \left(\int \frac{\sqrt{\sin(fx+e)+1} \csc(fx+e)}{\sin(fx+e)-1} dx \right)}{c}$$

input `int(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x)`

output `(- sqrt(a)*int((sqrt(sin(e + f*x) + 1)*csc(e + f*x))/(sin(e + f*x) - 1),x))/c`

3.14
$$\int \frac{\csc(e+fx)}{\sqrt{a+a \sin(e+fx)}(c-c \sin(e+fx))} dx$$

Optimal result	139
Mathematica [C] (verified)	140
Rubi [A] (verified)	140
Maple [A] (verified)	144
Fricas [B] (verification not implemented)	144
Sympy [F]	145
Maxima [F]	145
Giac [A] (verification not implemented)	146
Mupad [F(-1)]	146
Reduce [F]	147

Optimal result

Integrand size = 34, antiderivative size = 120

$$\int \frac{\csc(e+fx)}{\sqrt{a+a \sin(e+fx)}(c-c \sin(e+fx))} dx$$

$$= -\frac{2\operatorname{arctanh}\left(\frac{\sqrt{a} \cos(e+fx)}{\sqrt{a+a \sin(e+fx)}}\right)}{\sqrt{ac}f} + \frac{\operatorname{arctanh}\left(\frac{\sqrt{a} \cos(e+fx)}{\sqrt{2}\sqrt{a+a \sin(e+fx)}}\right)}{\sqrt{2}\sqrt{ac}f}$$

$$+ \frac{\sec(e+fx)\sqrt{a+a \sin(e+fx)}}{acf}$$

output

```
-2*arctanh(a^(1/2)*cos(f*x+e)/(a+a*sin(f*x+e))^(1/2))/a^(1/2)/c/f+1/2*arctanh(1/2*a^(1/2)*cos(f*x+e)*2^(1/2)/(a+a*sin(f*x+e))^(1/2))*2^(1/2)/a^(1/2)/c/f+sec(f*x+e)*(a+a*sin(f*x+e))^(1/2)/a/c/f
```

Mathematica [C] (verified)

Result contains higher order function than in optimal. Order 5 vs. order 3 in optimal.

Time = 2.50 (sec) , antiderivative size = 74, normalized size of antiderivative = 0.62

$$\int \frac{\csc(e + fx)}{\sqrt{a + a \sin(e + fx)}(c - c \sin(e + fx))} dx =$$

$$\frac{(\text{Hypergeometric2F1}(-\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}(1 - \sin(e + fx))) - 2 \text{Hypergeometric2F1}(-\frac{1}{2}, 1, \frac{1}{2}, 1 - \sin(e + fx)))}{acf}$$

input `Integrate[Csc[e + f*x]/(Sqrt[a + a*Sin[e + f*x]]*(c - c*Sin[e + f*x])),x]`

output `-(((Hypergeometric2F1[-1/2, 1, 1/2, (1 - Sin[e + f*x])/2] - 2*Hypergeometric2F1[-1/2, 1, 1/2, 1 - Sin[e + f*x]])*Sec[e + f*x]*Sqrt[a*(1 + Sin[e + f*x])]))/(a*c*f)`

Rubi [A] (verified)

Time = 1.12 (sec) , antiderivative size = 127, normalized size of antiderivative = 1.06, number of steps used = 13, number of rules used = 12, $\frac{\text{number of rules}}{\text{integrand size}} = 0.353$, Rules used = {3042, 3419, 3042, 3215, 3042, 3152, 3464, 3042, 3128, 219, 3252, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e + fx)}{\sqrt{a \sin(e + fx) + a}(c - c \sin(e + fx))} dx$$

$$\downarrow 3042$$

$$\int \frac{1}{\sin(e + fx)\sqrt{a \sin(e + fx) + a}(c - c \sin(e + fx))} dx$$

$$\downarrow 3419$$

$$\frac{\int \frac{\csc(e+fx)(2ac+a \sin(e+fx)c)}{\sqrt{\sin(e+fx)a+a}} dx}{2ac^2} + \frac{\int \frac{\sqrt{\sin(e+fx)a+a}}{c-c \sin(e+fx)} dx}{2a}$$

$$\downarrow 3042$$

$$\begin{aligned}
& \frac{\int \frac{2ac+a \sin(e+fx)c}{\sin(e+fx)\sqrt{\sin(e+fx)a+a}} dx}{2ac^2} + \frac{\int \frac{\sqrt{\sin(e+fx)a+a}}{c-c \sin(e+fx)} dx}{2a} \\
& \quad \downarrow \text{3215} \\
& \frac{\int \sec^2(e+fx)(\sin(e+fx)a+a)^{3/2} dx}{2a^2c} + \frac{\int \frac{2ac+a \sin(e+fx)c}{\sin(e+fx)\sqrt{\sin(e+fx)a+a}} dx}{2ac^2} \\
& \quad \downarrow \text{3042} \\
& \frac{\int \frac{(\sin(e+fx)a+a)^{3/2}}{\cos(e+fx)^2} dx}{2a^2c} + \frac{\int \frac{2ac+a \sin(e+fx)c}{\sin(e+fx)\sqrt{\sin(e+fx)a+a}} dx}{2ac^2} \\
& \quad \downarrow \text{3152} \\
& \frac{\int \frac{2ac+a \sin(e+fx)c}{\sin(e+fx)\sqrt{\sin(e+fx)a+a}} dx}{2ac^2} + \frac{\sec(e+fx)\sqrt{a \sin(e+fx)+a}}{acf} \\
& \quad \downarrow \text{3464} \\
& \frac{2c \int \csc(e+fx)\sqrt{\sin(e+fx)a+adx} - ac \int \frac{1}{\sqrt{\sin(e+fx)a+a}} dx}{2ac^2} + \frac{\sec(e+fx)\sqrt{a \sin(e+fx)+a}}{acf} \\
& \quad \downarrow \text{3042} \\
& \frac{2c \int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)} dx - ac \int \frac{1}{\sqrt{\sin(e+fx)a+a}} dx}{2ac^2} + \frac{\sec(e+fx)\sqrt{a \sin(e+fx)+a}}{acf} \\
& \quad \downarrow \text{3128} \\
& \frac{2ac \int \frac{1}{2a - \frac{a^2 \cos^2(e+fx)}{\sin(e+fx)a+a}} d \frac{a \cos(e+fx)}{\sqrt{\sin(e+fx)a+a}}}{2ac^2} + 2c \int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)} dx + \frac{\sec(e+fx)\sqrt{a \sin(e+fx)+a}}{acf} \\
& \quad \downarrow \text{219} \\
& \frac{2c \int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)} dx + \frac{\sqrt{2}\sqrt{a} \operatorname{arctanh}\left(\frac{\sqrt{a} \cos(e+fx)}{\sqrt{2}\sqrt{a \sin(e+fx)+a}}\right)}{f}}{2ac^2} + \frac{\sec(e+fx)\sqrt{a \sin(e+fx)+a}}{acf} \\
& \quad \downarrow \text{3252}
\end{aligned}$$

$$\frac{\frac{\sqrt{2}\sqrt{a}\operatorname{arctanh}\left(\frac{\sqrt{a}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}}\right)}{f} - \frac{4ac \int \frac{1}{a - \frac{a^2 \cos^2(e+fx)}{\sin(e+fx)a+a}} d \frac{a \cos(e+fx)}{\sqrt{\sin(e+fx)a+a}}}{f}}{acf} + \frac{2ac^2}{\sec(e+fx)\sqrt{a\sin(e+fx)+a}}$$

↓ 219

$$\frac{\frac{\sqrt{2}\sqrt{a}\operatorname{arctanh}\left(\frac{\sqrt{a}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}}\right)}{f} - \frac{4\sqrt{a}\operatorname{arctanh}\left(\frac{\sqrt{a}\cos(e+fx)}{\sqrt{a\sin(e+fx)+a}}\right)}{f}}{acf} + \frac{2ac^2}{\sec(e+fx)\sqrt{a\sin(e+fx)+a}}$$

input `Int[Csc[e + f*x]/(Sqrt[a + a*Sin[e + f*x]]*(c - c*Sin[e + f*x])),x]`

output `((-4*Sqrt[a]*c*ArcTanh[(Sqrt[a]*Cos[e + f*x])/Sqrt[a + a*Sin[e + f*x]])/f + (Sqrt[2]*Sqrt[a]*c*ArcTanh[(Sqrt[a]*Cos[e + f*x])/(Sqrt[2]*Sqrt[a + a*Sin[e + f*x]])]/f)/(2*a*c^2) + (Sec[e + f*x]*Sqrt[a + a*Sin[e + f*x]])/(a*c*f)`

Defintions of rubi rules used

rule 219 `Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (GtQ[a, 0] || LtQ[b, 0])`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3128 `Int[1/Sqrt[(a_) + (b_.)*sin[(c_.) + (d_.)*(x_)]], x_Symbol] := Simp[-2/d Subst[Int[1/(2*a - x^2), x], x, b*(Cos[c + d*x]/Sqrt[a + b*Sin[c + d*x])]], x] /; FreeQ[{a, b, c, d}, x] && EqQ[a^2 - b^2, 0]`

rule 3152 $\text{Int}[(\cos[(e_)] + (f_)*(x_)]*(g_))^{\wedge}(p_)*((a_)+(b_)*\sin[(e_)+(f_)*(x_)]^{\wedge}(m_), x_Symbol] \rightarrow \text{Simp}[b*(g*\cos[e + f*x])^{\wedge}(p + 1)*((a + b*\sin[e + f*x])^{\wedge}(m - 1)/(f*g*(m - 1))), x] /;$ FreeQ[{a, b, e, f, g, m, p}, x] && EqQ[a^2 - b^2, 0] && EqQ[2*m + p - 1, 0] && NeQ[m, 1]

rule 3215 $\text{Int}(((a_)+(b_)*\sin[(e_)+(f_)*(x_)]^{\wedge}(m_))*((c_)+(d_)*\sin[(e_)+(f_)*(x_)]^{\wedge}(n_), x_Symbol] \rightarrow \text{Simp}[a^m*c^m \text{ Int}[\cos[e + f*x]^{\wedge}(2*m)*(c + d*\sin[e + f*x])^{\wedge}(n - m), x], x] /;$ FreeQ[{a, b, c, d, e, f, n}, x] && EqQ[b*c + a*d, 0] && EqQ[a^2 - b^2, 0] && IntegerQ[m] && !(IntegerQ[n] && ((LtQ[m, 0] && GtQ[n, 0]) || LtQ[0, n, m] || LtQ[m, n, 0]))

rule 3252 $\text{Int}[\text{Sqrt}[(a_)+(b_)*\sin[(e_)+(f_)*(x_)]]/((c_)+(d_)*\sin[(e_)+(f_)*(x_)]), x_Symbol] \rightarrow \text{Simp}[-2*(b/f) \text{ Subst}[\text{Int}[1/(b*c + a*d - d*x^2), x], x, b*(\cos[e + f*x]/\text{Sqrt}[a + b*\sin[e + f*x]])], x] /;$ FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]

rule 3419 $\text{Int}[1/(\sin[(e_)+(f_)*(x_)]*\text{Sqrt}[(a_)+(b_)*\sin[(e_)+(f_)*(x_)]*((c_)+(d_)*\sin[(e_)+(f_)*(x_)])), x_Symbol] \rightarrow \text{Simp}[d^2/(c*(b*c - a*d)) \text{ Int}[\text{Sqrt}[a + b*\sin[e + f*x]]/(c + d*\sin[e + f*x]), x], x] + \text{Simp}[1/(c*(b*c - a*d)) \text{ Int}[(b*c - a*d - b*d*\sin[e + f*x])/(sin[e + f*x]*\text{Sqrt}[a + b*\sin[e + f*x])], x], x] /;$ FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]

rule 3464 $\text{Int}(((A_)+(B_)*\sin[(e_)+(f_)*(x_)])/(\text{Sqrt}[(a_)+(b_)*\sin[(e_)+(f_)*(x_)]*((c_)+(d_)*\sin[(e_)+(f_)*(x_)])), x_Symbol] \rightarrow \text{Simp}[(A*b - a*B)/(b*c - a*d) \text{ Int}[1/\text{Sqrt}[a + b*\sin[e + f*x]], x], x] + \text{Simp}[(B*c - A*d)/(b*c - a*d) \text{ Int}[\text{Sqrt}[a + b*\sin[e + f*x]]/(c + d*\sin[e + f*x]), x], x] /;$ FreeQ[{a, b, c, d, e, f, A, B}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]

Maple [A] (verified)

Time = 0.84 (sec) , antiderivative size = 124, normalized size of antiderivative = 1.03

method	result
default	$\frac{(1+\sin(fx+e))\left(\sqrt{2}\operatorname{arctanh}\left(\frac{\sqrt{a-a\sin(fx+e)}\sqrt{2}}{2\sqrt{a}}\right)a^2\sqrt{a-a\sin(fx+e)}+2a^{\frac{5}{2}}-4\operatorname{arctanh}\left(\frac{\sqrt{a-a\sin(fx+e)}}{\sqrt{a}}\right)a^2\sqrt{a-a\sin(fx+e)}\right)}{2ca^{\frac{5}{2}}\cos(fx+e)\sqrt{a+a\sin(fx+e)}f}$

input `int(csc(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x,method=_RETURNVERBOSE)`

output
$$\frac{1/2/c*(1+\sin(f*x+e))*(2^{(1/2)}*\operatorname{arctanh}(1/2*(a-a*\sin(f*x+e))^{(1/2)}*2^{(1/2)}/a^{(1/2)})*a^2*(a-a*\sin(f*x+e))^{(1/2)}+2*a^{(5/2)}-4*\operatorname{arctanh}((a-a*\sin(f*x+e))^{(1/2)}/a^{(1/2)})*a^2*(a-a*\sin(f*x+e))^{(1/2)})/a^{(5/2)}/\cos(f*x+e)/(a+a*\sin(f*x+e))^{(1/2)}/f}$$

Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 328 vs. 2(104) = 208.

Time = 0.10 (sec) , antiderivative size = 328, normalized size of antiderivative = 2.73

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}(c-c\sin(e+fx))} dx$$

$$= \frac{\sqrt{2}\sqrt{a}\cos(fx+e)\log\left(-\frac{\cos(fx+e)^2-(\cos(fx+e)-2)\sin(fx+e)+\frac{2\sqrt{2}\sqrt{a\sin(fx+e)+a}(\cos(fx+e)-\sin(fx+e)+1)}{\sqrt{a}}+3\cos(fx+e)+2}{\cos(fx+e)^2-(\cos(fx+e)+2)\sin(fx+e)-\cos(fx+e)-2}}\right)}{\dots}$$

input `integrate(csc(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x, algorithm="fricas")`

output

```
1/4*(sqrt(2)*sqrt(a)*cos(f*x + e)*log(-(cos(f*x + e)^2 - (cos(f*x + e) - 2)
)*sin(f*x + e) + 2*sqrt(2)*sqrt(a*sin(f*x + e) + a)*(cos(f*x + e) - sin(f*
x + e) + 1)/sqrt(a) + 3*cos(f*x + e) + 2)/(cos(f*x + e)^2 - (cos(f*x + e)
+ 2)*sin(f*x + e) - cos(f*x + e) - 2)) + 2*sqrt(a)*cos(f*x + e)*log((a*cos
(f*x + e)^3 - 7*a*cos(f*x + e)^2 - 4*(cos(f*x + e)^2 + (cos(f*x + e) + 3)*
sin(f*x + e) - 2*cos(f*x + e) - 3)*sqrt(a*sin(f*x + e) + a)*sqrt(a) - 9*a*
cos(f*x + e) + (a*cos(f*x + e)^2 + 8*a*cos(f*x + e) - a)*sin(f*x + e) - a)
/(cos(f*x + e)^3 + cos(f*x + e)^2 + (cos(f*x + e)^2 - 1)*sin(f*x + e) - co
s(f*x + e) - 1)) + 4*sqrt(a*sin(f*x + e) + a))/(a*c*f*cos(f*x + e))
```

Sympy [F]

$$\int \frac{\csc(e + fx)}{\sqrt{a + a \sin(e + fx)}(c - c \sin(e + fx))} dx$$

$$= -\frac{\int \frac{\csc(e + fx)}{\sqrt{a \sin(e + fx) + a} \sin(e + fx) - \sqrt{a \sin(e + fx) + a}} dx}{c}$$

input

```
integrate(csc(f*x+e)/(a+a*sin(f*x+e))**(1/2)/(c-c*sin(f*x+e)),x)
```

output

```
-Integral(csc(e + f*x)/(sqrt(a*sin(e + f*x) + a)*sin(e + f*x) - sqrt(a*sin
(e + f*x) + a)), x)/c
```

Maxima [F]

$$\int \frac{\csc(e + fx)}{\sqrt{a + a \sin(e + fx)}(c - c \sin(e + fx))} dx$$

$$= \int -\frac{\csc(fx + e)}{\sqrt{a \sin(fx + e) + a}(c \sin(fx + e) - c)} dx$$

input

```
integrate(csc(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x, algorithm=
"maxima")
```

output `-integrate(csc(f*x + e)/(sqrt(a*sin(f*x + e) + a)*(c*sin(f*x + e) - c)), x)`

Giac [A] (verification not implemented)

Time = 0.38 (sec) , antiderivative size = 141, normalized size of antiderivative = 1.18

$$\int \frac{\csc(e + fx)}{\sqrt{a + a \sin(e + fx)}(c - c \sin(e + fx))} dx =$$

$$\frac{\sqrt{2} \left(\frac{2\sqrt{2} \log\left(\frac{|-2\sqrt{2}+4 \sin(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e)|}{|2\sqrt{2}+4 \sin(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e)|}\right)}{c} + \frac{\log(\sin(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e)+1)}{c} - \frac{\log(-\sin(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e)+1)}{c} + \frac{2}{c \sin(-\frac{1}{4}\pi+\frac{1}{2}fx+\frac{1}{2}e)} \right)}{4\sqrt{a}f \operatorname{sgn}\left(\cos\left(-\frac{1}{4}\pi + \frac{1}{2}fx + \frac{1}{2}e\right)\right)}$$

input `integrate(csc(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x, algorithm="giac")`

output `-1/4*sqrt(2)*(2*sqrt(2)*log(abs(-2*sqrt(2) + 4*sin(-1/4*pi + 1/2*f*x + 1/2*e))/abs(2*sqrt(2) + 4*sin(-1/4*pi + 1/2*f*x + 1/2*e)))/c + log(sin(-1/4*pi + 1/2*f*x + 1/2*e) + 1)/c - log(-sin(-1/4*pi + 1/2*f*x + 1/2*e) + 1)/c + 2/(c*sin(-1/4*pi + 1/2*f*x + 1/2*e)))/(sqrt(a)*f*sgn(cos(-1/4*pi + 1/2*f*x + 1/2*e)))`

Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e + fx)}{\sqrt{a + a \sin(e + fx)}(c - c \sin(e + fx))} dx$$

$$= \int \frac{1}{\sin(e + fx) \sqrt{a + a \sin(e + fx)} (c - c \sin(e + fx))} dx$$

input `int(1/(sin(e + f*x)*(a + a*sin(e + f*x))^(1/2)*(c - c*sin(e + f*x))),x)`

output `int(1/(sin(e + f*x)*(a + a*sin(e + f*x))^(1/2)*(c - c*sin(e + f*x))), x)`

Reduce [F]

$$\int \frac{\csc(e + fx)}{\sqrt{a + a \sin(e + fx)}(c - c \sin(e + fx))} dx = -\frac{\sqrt{a} \left(\int \frac{\sqrt{\sin(fx+e)+1} \csc(fx+e)}{\sin(fx+e)^2-1} dx \right)}{ac}$$

input `int(csc(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x)`

output `(- sqrt(a)*int((sqrt(sin(e + f*x) + 1)*csc(e + f*x))/(sin(e + f*x)**2 - 1),x))/(a*c)`

3.15
$$\int \frac{\sqrt{g \sin(e+fx)} \sqrt{a+a \sin(e+fx)}}{c-c \sin(e+fx)} dx$$

Optimal result	148
Mathematica [A] (verified)	149
Rubi [A] (verified)	149
Maple [C] (warning: unable to verify)	152
Fricas [A] (verification not implemented)	153
Sympy [F]	154
Maxima [F]	154
Giac [F(-1)]	154
Mupad [F(-1)]	155
Reduce [F]	155

Optimal result

Integrand size = 40, antiderivative size = 103

$$\int \frac{\sqrt{g \sin(e+fx)} \sqrt{a+a \sin(e+fx)}}{c-c \sin(e+fx)} dx$$

$$= \frac{2\sqrt{a}\sqrt{g} \arctan\left(\frac{\sqrt{a}\sqrt{g} \cos(e+fx)}{\sqrt{g \sin(e+fx)} \sqrt{a+a \sin(e+fx)}}\right)}{cf} + \frac{2 \sec(e+fx) \sqrt{g \sin(e+fx)} \sqrt{a+a \sin(e+fx)}}{cf}$$

output `2*a^(1/2)*g^(1/2)*arctan(a^(1/2)*g^(1/2)*cos(f*x+e)/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2))/c/f+2*sec(f*x+e)*(g*sin(f*x+e))^(1/2)*(a+a*sin(f*x+e))^(1/2)/c/f`

Mathematica [A] (verified)

Time = 4.77 (sec) , antiderivative size = 91, normalized size of antiderivative = 0.88

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)}}{c - c \sin(e + fx)} dx$$

$$= \frac{2 \sec(e + fx) \left(\arcsin \left(\sqrt{1 - \sin(e + fx)} \right) \sqrt{1 - \sin(e + fx)} + \sqrt{\sin(e + fx)} \right) \sqrt{g \sin(e + fx)} \sqrt{a(1 + \sin(e + fx))}}{cf \sqrt{\sin(e + fx)}}$$

input

```
Integrate[(Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]])/(c - c*Sin[e + f*x]),x]
```

output

```
(2*Sec[e + f*x]*(ArcSin[Sqrt[1 - Sin[e + f*x]]]*Sqrt[1 - Sin[e + f*x]] + Sqrt[Sin[e + f*x]])*Sqrt[g*Sin[e + f*x]]*Sqrt[a*(1 + Sin[e + f*x])]/(c*f*Sqrt[Sin[e + f*x]])
```

Rubi [A] (verified)

Time = 0.87 (sec) , antiderivative size = 103, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}} = 0.175$, Rules used = {3042, 3407, 3042, 3254, 218, 3409, 15}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sqrt{a \sin(e + fx) + a} \sqrt{g \sin(e + fx)}}{c - c \sin(e + fx)} dx$$

$$\downarrow 3042$$

$$\int \frac{\sqrt{a \sin(e + fx) + a} \sqrt{g \sin(e + fx)}}{c - c \sin(e + fx)} dx$$

$$\downarrow 3407$$

$$g \int \frac{\sqrt{\sin(e + fx) a + a}}{\sqrt{g \sin(e + fx)} (c - c \sin(e + fx))} dx - \frac{g \int \frac{\sqrt{\sin(e + fx) a + a}}{\sqrt{g \sin(e + fx)}} dx}{c}$$

$$\begin{aligned}
 & \downarrow 3042 \\
 & g \int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)}(c-c \sin(e+fx))} dx - \frac{g \int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)}} dx}{c} \\
 & \downarrow 3254 \\
 & \frac{2ag \int \frac{1}{\frac{\cos(e+fx) \cot(e+fx)a^2}{\sin(e+fx)a+a} + a} d \frac{a \cos(e+fx)}{\sqrt{g \sin(e+fx)} \sqrt{\sin(e+fx)a+a}}}{cf} + \\
 & g \int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)}(c-c \sin(e+fx))} dx \\
 & \downarrow 218 \\
 & g \int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)}(c-c \sin(e+fx))} dx + \frac{2\sqrt{a}\sqrt{g} \arctan\left(\frac{\sqrt{a}\sqrt{g} \cos(e+fx)}{\sqrt{a \sin(e+fx)+a} \sqrt{g \sin(e+fx)}}\right)}{cf} \\
 & \downarrow 3409 \\
 & \frac{2\sqrt{a}\sqrt{g} \arctan\left(\frac{\sqrt{a}\sqrt{g} \cos(e+fx)}{\sqrt{a \sin(e+fx)+a} \sqrt{g \sin(e+fx)}}\right)}{cf} - \\
 & \frac{2ag \int \frac{\sec(e+fx)(\sin(e+fx)a+a) \tan(e+fx)}{a^2c} d \frac{a \cos(e+fx)}{\sqrt{g \sin(e+fx)} \sqrt{\sin(e+fx)a+a}}}{f} \\
 & \downarrow 15 \\
 & \frac{2\sqrt{a}\sqrt{g} \arctan\left(\frac{\sqrt{a}\sqrt{g} \cos(e+fx)}{\sqrt{a \sin(e+fx)+a} \sqrt{g \sin(e+fx)}}\right)}{cf} + \frac{2 \sec(e+fx) \sqrt{a \sin(e+fx)+a} \sqrt{g \sin(e+fx)}}{cf}
 \end{aligned}$$

input

```
Int[(Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]])/(c - c*Sin[e + f*x]),x
]
```

output

```
(2*Sqrt[a]*Sqrt[g]*ArcTan[(Sqrt[a]*Sqrt[g]*Cos[e + f*x])/(Sqrt[g*Sin[e + f
*x]]*Sqrt[a + a*Sin[e + f*x]])]/(c*f) + (2*Sec[e + f*x]*Sqrt[g*Sin[e + f*
x]]*Sqrt[a + a*Sin[e + f*x]])/(c*f)
```

Definitions of rubi rules used

- rule 15 $\text{Int}[(a_.)(x_)^{(m_.)}, x_Symbol] \rightarrow \text{Simp}[a*(x^{(m+1)})/(m+1), x] /; \text{FreeQ}[\{a, m\}, x] \ \&\& \ \text{NeQ}[m, -1]$
- rule 218 $\text{Int}[((a_) + (b_.)(x_)^2)^{-1}, x_Symbol] \rightarrow \text{Simp}[(\text{Rt}[a/b, 2]/a)*\text{ArcTan}[x/\text{Rt}[a/b, 2]], x] /; \text{FreeQ}[\{a, b\}, x] \ \&\& \ \text{PosQ}[a/b]$
- rule 3042 $\text{Int}[u_, x_Symbol] \rightarrow \text{Int}[\text{DeactivateTrig}[u, x], x] /; \text{FunctionOfTrigOfLinearQ}[u, x]$
- rule 3254 $\text{Int}[\text{Sqrt}[(a_) + (b_.)\sin[(e_.) + (f_.)(x_)]]/\text{Sqrt}[(c_.) + (d_.)\sin[(e_.) + (f_.)(x_)]], x_Symbol] \rightarrow \text{Simp}[-2*(b/f) \ \text{Subst}[\text{Int}[1/(b + d*x^2), x], x, b*(\text{Cos}[e + f*x]/(\text{Sqrt}[a + b*\text{Sin}[e + f*x]]*\text{Sqrt}[c + d*\text{Sin}[e + f*x]])]], x] /; \text{FreeQ}[\{a, b, c, d, e, f\}, x] \ \&\& \ \text{NeQ}[b*c - a*d, 0] \ \&\& \ \text{EqQ}[a^2 - b^2, 0] \ \&\& \ \text{NeQ}[c^2 - d^2, 0]$
- rule 3407 $\text{Int}[(\text{Sqrt}[(g_.)\sin[(e_.) + (f_.)(x_)]]*\text{Sqrt}[(a_) + (b_.)\sin[(e_.) + (f_.)(x_)]])/((c_) + (d_.)\sin[(e_.) + (f_.)(x_)]), x_Symbol] \rightarrow \text{Simp}[g/d \ \text{Int}[\text{Sqrt}[a + b*\text{Sin}[e + f*x]]/\text{Sqrt}[g*\text{Sin}[e + f*x]], x], x] - \text{Simp}[c*(g/d) \ \text{Int}[\text{Sqrt}[a + b*\text{Sin}[e + f*x]]/(\text{Sqrt}[g*\text{Sin}[e + f*x]]*(c + d*\text{Sin}[e + f*x])), x], x] /; \text{FreeQ}[\{a, b, c, d, e, f, g\}, x] \ \&\& \ \text{NeQ}[b*c - a*d, 0] \ \&\& \ (\text{EqQ}[a^2 - b^2, 0] \ || \ \text{EqQ}[c^2 - d^2, 0])$
- rule 3409 $\text{Int}[\text{Sqrt}[(a_) + (b_.)\sin[(e_.) + (f_.)(x_)]]/(\text{Sqrt}[(g_.)\sin[(e_.) + (f_.)(x_)]]*((c_) + (d_.)\sin[(e_.) + (f_.)(x_)])), x_Symbol] \rightarrow \text{Simp}[-2*(b/f) \ \text{Subst}[\text{Int}[1/(b*c + a*d + c*g*x^2), x], x, b*(\text{Cos}[e + f*x]/(\text{Sqrt}[g*\text{Sin}[e + f*x]]*\text{Sqrt}[a + b*\text{Sin}[e + f*x]])]], x] /; \text{FreeQ}[\{a, b, c, d, e, f, g\}, x] \ \&\& \ \text{NeQ}[b*c - a*d, 0] \ \&\& \ \text{EqQ}[a^2 - b^2, 0]$

Maple [C] (warning: unable to verify)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 25.12 (sec) , antiderivative size = 2555, normalized size of antiderivative = 24.81

method	result	size
default	Expression too large to display	2555

input

```
int((g*sin(f*x+e))^(1/2)*(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x,method=
_RETURNVERBOSE)
```

output

```
1/c/f*(4*(g^2)^(1/4)*2^(1/2)*(cot(1/2*f*x+1/2*e)*csc(1/2*f*x+1/2*e)^2*(cos
(1/2*f*x+1/2*e)-1)^2*g)^(1/2)*cos(1/2*f*x+1/2*e)+2*cos(1/2*f*x+1/2*e)*arct
an((cot(1/2*f*x+1/2*e)*2^(1/2)*(cot(1/2*f*x+1/2*e)*csc(1/2*f*x+1/2*e)^2*(c
os(1/2*f*x+1/2*e)-1)^2*g)^(1/2)+(g^2)^(1/4)+csc(1/2*f*x+1/2*e)*2^(1/2)*(co
t(1/2*f*x+1/2*e)*csc(1/2*f*x+1/2*e)^2*(cos(1/2*f*x+1/2*e)-1)^2*g)^(1/2))/(
g^2)^(1/4))*csgn(g)*g-2*csgn(g)*g*arctan((cot(1/2*f*x+1/2*e)*2^(1/2)*(cot(
1/2*f*x+1/2*e)*csc(1/2*f*x+1/2*e)^2*(cos(1/2*f*x+1/2*e)-1)^2*g)^(1/2)+(g^2
)^(1/4)+csc(1/2*f*x+1/2*e)*2^(1/2)*(cot(1/2*f*x+1/2*e)*csc(1/2*f*x+1/2*e)^
2*(cos(1/2*f*x+1/2*e)-1)^2*g)^(1/2))/(g^2)^(1/4))*sin(1/2*f*x+1/2*e)+cos(1
/2*f*x+1/2*e)*ln((cot(1/2*f*x+1/2*e)^2*g-2*cot(1/2*f*x+1/2*e)*csc(1/2*f*x+
1/2*e)*g+csc(1/2*f*x+1/2*e)^2*g-2*csgn(g)*g*csc(1/2*f*x+1/2*e)+2*csgn(g)*g
*cot(1/2*f*x+1/2*e)-2*(g^2)^(1/4)*2^(1/2)*(cot(1/2*f*x+1/2*e)*csc(1/2*f*x+
1/2*e)^2*(cos(1/2*f*x+1/2*e)-1)^2*g)^(1/2)-g)/(cot(1/2*f*x+1/2*e)^2*g-2*co
t(1/2*f*x+1/2*e)*csc(1/2*f*x+1/2*e)*g+csc(1/2*f*x+1/2*e)^2*g-2*(g^2)^(1/2)
*csc(1/2*f*x+1/2*e)+2*(g^2)^(1/2)*cot(1/2*f*x+1/2*e)+2*(g^2)^(1/4)*(g*(cot
(1/2*f*x+1/2*e)^3-cot(1/2*f*x+1/2*e)-3*cot(1/2*f*x+1/2*e)^2*csc(1/2*f*x+1/
2*e)+csc(1/2*f*x+1/2*e)+3*cot(1/2*f*x+1/2*e)*csc(1/2*f*x+1/2*e)^2-csc(1/2*
f*x+1/2*e)^3))^(1/2)-g))*csgn(g)*g-csgn(g)*g*ln((cot(1/2*f*x+1/2*e)^2*g-2*
cot(1/2*f*x+1/2*e)*csc(1/2*f*x+1/2*e)*g+csc(1/2*f*x+1/2*e)^2*g-2*csgn(g)*g
*csc(1/2*f*x+1/2*e)+2*csgn(g)*g*cot(1/2*f*x+1/2*e)-2*(g^2)^(1/4)*2^(1/2...
```

Fricas [A] (verification not implemented)

Time = 0.25 (sec) , antiderivative size = 442, normalized size of antiderivative = 4.29

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)}}{c - c \sin(e + fx)} dx$$

$$= \frac{\sqrt{-ag} \cos(fx + e) \log\left(\frac{128 ag \cos(fx+e)^5 - 128 ag \cos(fx+e)^4 - 416 ag \cos(fx+e)^3 + 128 ag \cos(fx+e)^2 + 289 ag \cos(fx+e) + 8}{16}\right) - \sqrt{ag} \arctan\left(\frac{\sqrt{ag}(8 \cos(fx+e)^2 + 8 \sin(fx+e) - 9) \sqrt{a \sin(fx+e) + a} \sqrt{g \sin(fx+e)}}{4(2 ag \cos(fx+e)^3 + ag \cos(fx+e) \sin(fx+e) - 2 ag \cos(fx+e))}\right) \cos(fx + e) - 4 \sqrt{a \sin(fx + e)}}{2 cf \cos(fx + e)}$$

input

```
integrate((g*sin(f*x+e))^(1/2)*(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x,
algorithm="fricas")
```

output

```
[1/4*(sqrt(-a*g)*cos(f*x + e)*log((128*a*g*cos(f*x + e)^5 - 128*a*g*cos(f*
x + e)^4 - 416*a*g*cos(f*x + e)^3 + 128*a*g*cos(f*x + e)^2 + 289*a*g*cos(f
*x + e) + 8*(16*cos(f*x + e)^4 - 24*cos(f*x + e)^3 - 66*cos(f*x + e)^2 + (
16*cos(f*x + e)^3 + 40*cos(f*x + e)^2 - 26*cos(f*x + e) - 51)*sin(f*x + e)
+ 25*cos(f*x + e) + 51)*sqrt(-a*g)*sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*
x + e)) + a*g + (128*a*g*cos(f*x + e)^4 + 256*a*g*cos(f*x + e)^3 - 160*a*g
*cos(f*x + e)^2 - 288*a*g*cos(f*x + e) + a*g)*sin(f*x + e))/(cos(f*x + e)
+ sin(f*x + e) + 1)) + 8*sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e)))/(c
*f*cos(f*x + e)), -1/2*(sqrt(a*g)*arctan(1/4*sqrt(a*g)*(8*cos(f*x + e)^2 +
8*sin(f*x + e) - 9)*sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e)))/(2*a*g*
cos(f*x + e)^3 + a*g*cos(f*x + e)*sin(f*x + e) - 2*a*g*cos(f*x + e)))*cos(
f*x + e) - 4*sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e)))/(c*f*cos(f*x +
e))]
```

Sympy [F]

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)}}{c - c \sin(e + fx)} dx = - \int \frac{\sqrt{g \sin(e + fx)} \sqrt{a \sin(e + fx) + a}}{\sin(e + fx) - 1} dx$$

input `integrate((g*sin(f*x+e))**(1/2)*(a+a*sin(f*x+e))**(1/2)/(c-c*sin(f*x+e)),x)`

output `-Integral(sqrt(g*sin(e + f*x))*sqrt(a*sin(e + f*x) + a)/(sin(e + f*x) - 1), x)/c`

Maxima [F]

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)}}{c - c \sin(e + fx)} dx = \int - \frac{\sqrt{a \sin(fx + e) + a} \sqrt{g \sin(fx + e)}}{c \sin(fx + e) - c} dx$$

input `integrate((g*sin(f*x+e))^(1/2)*(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x, algorithm="maxima")`

output `-integrate(sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))/(c*sin(f*x + e) - c), x)`

Giac [F(-1)]

Timed out.

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)}}{c - c \sin(e + fx)} dx = \text{Timed out}$$

input `integrate((g*sin(f*x+e))^(1/2)*(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x, algorithm="giac")`

output `Timed out`

Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)}}{c - c \sin(e + fx)} dx = \int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)}}{c - c \sin(e + fx)} dx$$

input `int(((g*sin(e + f*x))^(1/2)*(a + a*sin(e + f*x))^(1/2))/(c - c*sin(e + f*x))),x)`

output `int(((g*sin(e + f*x))^(1/2)*(a + a*sin(e + f*x))^(1/2))/(c - c*sin(e + f*x))), x)`

Reduce [F]

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)}}{c - c \sin(e + fx)} dx = -\frac{\sqrt{g} \sqrt{a} \left(\int \frac{\sqrt{\sin(fx+e)} \sqrt{\sin(fx+e)+1}}{\sin(fx+e)-1} dx \right)}{c}$$

input `int((g*sin(f*x+e))^(1/2)*(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x)`

output `(- sqrt(g)*sqrt(a)*int((sqrt(sin(e + f*x))*sqrt(sin(e + f*x) + 1))/(sin(e + f*x) - 1),x))/c`

3.16
$$\int \frac{\sqrt{a+a \sin(e+fx)}}{\sqrt{g \sin(e+fx)}(c-c \sin(e+fx))} dx$$

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Optimal result

Integrand size = 40, antiderivative size = 43

$$\int \frac{\sqrt{a+a \sin(e+fx)}}{\sqrt{g \sin(e+fx)}(c-c \sin(e+fx))} dx = \frac{2 \sec(e+fx) \sqrt{g \sin(e+fx)} \sqrt{a+a \sin(e+fx)}}{c f g}$$

output `2*sec(f*x+e)*(g*sin(f*x+e))^(1/2)*(a+a*sin(f*x+e))^(1/2)/c/f/g`

Mathematica [A] (verified)

Time = 1.07 (sec) , antiderivative size = 40, normalized size of antiderivative = 0.93

$$\int \frac{\sqrt{a+a \sin(e+fx)}}{\sqrt{g \sin(e+fx)}(c-c \sin(e+fx))} dx = \frac{2 \sqrt{a(1+\sin(e+fx))} \tan(e+fx)}{c f \sqrt{g \sin(e+fx)}}$$

input `Integrate[Sqrt[a + a*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c - c*Sin[e + f*x])),x]`

output `(2*Sqrt[a*(1 + Sin[e + f*x]])*Tan[e + f*x])/(c*f*Sqrt[g*Sin[e + f*x]])`

Rubi [A] (verified)

Time = 0.38 (sec) , antiderivative size = 43, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.075$, Rules used = {3042, 3409, 15}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{array}{c}
 \int \frac{\sqrt{a \sin(e + fx) + a}}{(c - c \sin(e + fx)) \sqrt{g \sin(e + fx)}} dx \\
 \downarrow 3042 \\
 \int \frac{\sqrt{a \sin(e + fx) + a}}{(c - c \sin(e + fx)) \sqrt{g \sin(e + fx)}} dx \\
 \downarrow 3409 \\
 - \frac{2a \int \frac{\sec(e+fx)(\sin(e+fx)a+a) \tan(e+fx)}{a^2c} d \frac{a \cos(e+fx)}{\sqrt{g \sin(e+fx)} \sqrt{\sin(e+fx)a+a}}}{f} \\
 \downarrow 15 \\
 \frac{2 \sec(e + fx) \sqrt{a \sin(e + fx) + a} \sqrt{g \sin(e + fx)}}{c f g}
 \end{array}$$

input

```
Int[Sqrt[a + a*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c - c*Sin[e + f*x])),x]
```

output

```
(2*Sec[e + f*x]*Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]])/(c*f*g)
```

Defintions of rubi rules used

rule 15

```
Int[(a_.)*(x_)^(m_.), x_Symbol] := Simp[a*(x^(m + 1)/(m + 1)), x] /; FreeQ[
{a, m}, x] && NeQ[m, -1]
```

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3409 `Int[Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)]]/(Sqrt[(g_)*sin[(e_) + (f_)*(x_)])*(c_) + (d_)*sin[(e_) + (f_)*(x_)])], x_Symbol] := Simp[-2*(b/f) Subst[Int[1/(b*c + a*d + c*g*x^2), x], x, b*(Cos[e + f*x]/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]])], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]`

Maple [A] (verified)

Time = 6.47 (sec) , antiderivative size = 46, normalized size of antiderivative = 1.07

method	result	size
default	$\frac{2\sqrt{2} \tan(fx+e) \sqrt{a \sin\left(\frac{\pi}{4} + \frac{fx}{2} + \frac{e}{2}\right)^2}}{cf\sqrt{g \sin(fx+e)}}$	46

input `int((a+a*sin(f*x+e))^(1/2)/(g*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x,method=_RETURNVERBOSE)`

output `2*2^(1/2)/c/f*tan(f*x+e)*(a*sin(1/4*Pi+1/2*f*x+1/2*e))^2^(1/2)/(g*sin(f*x+e))^(1/2)`

Fricas [A] (verification not implemented)

Time = 0.08 (sec) , antiderivative size = 41, normalized size of antiderivative = 0.95

$$\int \frac{\sqrt{a + a \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c - c \sin(e + fx))} dx = \frac{2 \sqrt{a \sin(fx + e) + a} \sqrt{g \sin(fx + e)}}{c f g \cos(fx + e)}$$

input `integrate((a+a*sin(f*x+e))^(1/2)/(g*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x,algorithm="fricas")`

output `2*sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))/(c*f*g*cos(f*x + e))`

Sympy [F]

$$\int \frac{\sqrt{a + a \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c - c \sin(e + fx))} dx = - \frac{\int \frac{\sqrt{a \sin(e + fx) + a}}{\sqrt{g \sin(e + fx) \sin(e + fx) - \sqrt{g \sin(e + fx)}}} dx}{c}$$

input `integrate((a+a*sin(f*x+e))**(1/2)/(g*sin(f*x+e))**(1/2)/(c-c*sin(f*x+e)),x)`

output `-Integral(sqrt(a*sin(e + f*x) + a)/(sqrt(g*sin(e + f*x))*sin(e + f*x) - sqrt(g*sin(e + f*x))), x)/c`

Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 309 vs. 2(39) = 78.

Time = 0.14 (sec) , antiderivative size = 309, normalized size of antiderivative = 7.19

$$\int \frac{\sqrt{a + a \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c - c \sin(e + fx))} dx = \frac{4 \left(\left(\frac{3 \sqrt{2} \sqrt{a} \sqrt{g} \sin(fx+e)}{\cos(fx+e)+1} - \frac{\sqrt{2} \sqrt{a} \sqrt{g} \sin(fx+e)^2}{(\cos(fx+e)+1)^2} \right) \sqrt{\frac{\sin(fx+e)}{\cos(fx+e)+1}} - \frac{2 \left(\frac{3 \sqrt{2} \sqrt{a} \sqrt{g} \sin(fx+e)}{\cos(fx+e)+1} + \frac{\sqrt{2} \sqrt{a} \sqrt{g} \sin(fx+e)^2}{(\cos(fx+e)+1)^2} \right) \sqrt{\frac{\sin(fx+e)}{\cos(fx+e)+1}}}{cg - \frac{cg \sin(fx+e)}{\cos(fx+e)+1}} - \frac{2 \sqrt{2} \sqrt{a} \left(\frac{\sin(fx+e)}{\cos(fx+e)+1} \right)}{c \sqrt{g}}$$

12 f

input `integrate((a+a*sin(f*x+e))^(1/2)/(g*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x, algorithm="maxima")`

output

```
-1/12*(4*((3*sqrt(2)*sqrt(a)*sqrt(g)*sin(f*x + e)/(cos(f*x + e) + 1) - sqrt(2)*sqrt(a)*sqrt(g)*sin(f*x + e)^2/(cos(f*x + e) + 1)^2)*sqrt(sin(f*x + e)/(cos(f*x + e) + 1)) - 2*(3*sqrt(2)*sqrt(a)*sqrt(g)*sin(f*x + e)/(cos(f*x + e) + 1) + sqrt(2)*sqrt(a)*sqrt(g)*sin(f*x + e)^2/(cos(f*x + e) + 1)^2)/sqrt(sin(f*x + e)/(cos(f*x + e) + 1)))/(c*g - c*g*sin(f*x + e)/(cos(f*x + e) + 1)) - (2*sqrt(2)*sqrt(a)*(sin(f*x + e)/(cos(f*x + e) + 1))^(3/2) + 3*sqrt(2)*sqrt(a)*sin(f*x + e)/(cos(f*x + e) + 1))/(c*sqrt(g)) - (2*sqrt(2)*sqrt(a)*(sin(f*x + e)/(cos(f*x + e) + 1))^(3/2) - 3*sqrt(2)*sqrt(a)*sin(f*x + e)/(cos(f*x + e) + 1))/(c*sqrt(g)))/f
```

Giac [F]

$$\int \frac{\sqrt{a + a \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c - c \sin(e + fx))} dx = \int -\frac{\sqrt{a \sin(fx + e) + a}}{(c \sin(fx + e) - c)\sqrt{g \sin(fx + e)}} dx$$

input

```
integrate((a+a*sin(f*x+e))^(1/2)/(g*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x,algorithm="giac")
```

output

```
integrate(-sqrt(a*sin(f*x + e) + a)/((c*sin(f*x + e) - c)*sqrt(g*sin(f*x + e))), x)
```

Mupad [B] (verification not implemented)

Time = 36.41 (sec) , antiderivative size = 52, normalized size of antiderivative = 1.21

$$\int \frac{\sqrt{a + a \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c - c \sin(e + fx))} dx = \frac{2 \sin(2e + 2fx) \sqrt{a (\sin(e + fx) + 1)}}{c f (\cos(2e + 2fx) + 1) \sqrt{g \sin(e + fx)}}$$

input

```
int((a + a*sin(e + f*x))^(1/2)/((g*sin(e + f*x))^(1/2)*(c - c*sin(e + f*x))),x)
```

output

```
(2*sin(2*e + 2*f*x)*(a*(sin(e + f*x) + 1))^(1/2))/(c*f*(cos(2*e + 2*f*x) + 1)*(g*sin(e + f*x))^(1/2))
```

Reduce [F]

$$\int \frac{\sqrt{a + a \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c - c \sin(e + fx))} dx = -\frac{\sqrt{g} \sqrt{a} \left(\int \frac{\sqrt{\sin(fx+e)} \sqrt{\sin(fx+e)+1}}{\sin(fx+e)^2 - \sin(fx+e)} dx \right)}{cg}$$

input `int((a+a*sin(f*x+e))^(1/2)/(g*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x)`

output `(- sqrt(g)*sqrt(a)*int((sqrt(sin(e + f*x))*sqrt(sin(e + f*x) + 1))/(sin(e + f*x)**2 - sin(e + f*x)),x))/(c*g)`

3.17
$$\int \frac{\sqrt{g \sin(e+fx)}}{\sqrt{a+a \sin(e+fx)}(c-c \sin(e+fx))} dx$$

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Mathematica [A] (verified)	163
Rubi [A] (verified)	163
Maple [B] (verified)	166
Fricas [A] (verification not implemented)	167
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Maxima [B] (verification not implemented)	168
Giac [F(-1)]	169
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Reduce [F]	170

Optimal result

Integrand size = 40, antiderivative size = 114

$$\int \frac{\sqrt{g \sin(e+fx)}}{\sqrt{a+a \sin(e+fx)}(c-c \sin(e+fx))} dx$$

$$= \frac{\sqrt{g} \arctan\left(\frac{\sqrt{a}\sqrt{g} \cos(e+fx)}{\sqrt{2}\sqrt{g \sin(e+fx)}\sqrt{a+a \sin(e+fx)}}\right)}{\sqrt{2}\sqrt{ac}f} + \frac{\sec(e+fx)\sqrt{g \sin(e+fx)}\sqrt{a+a \sin(e+fx)}}{acf}$$

output

```
1/2*g^(1/2)*arctan(1/2*a^(1/2)*g^(1/2)*cos(f*x+e)*2^(1/2)/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2))*2^(1/2)/a^(1/2)/c/f+sec(f*x+e)*(g*sin(f*x+e))^(1/2)*(a+a*sin(f*x+e))^(1/2)/a/c/f
```

Mathematica [A] (verified)

Time = 0.36 (sec) , antiderivative size = 133, normalized size of antiderivative = 1.17

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + a \sin(e + fx)}(c - c \sin(e + fx))} dx$$

$$= \frac{\operatorname{csc}(2(e + fx))\sqrt{\sin(e + fx)}\sqrt{g \sin(e + fx)}\sqrt{a(1 + \sin(e + fx))}\left(2\sqrt{c}\sqrt{\sin(e + fx)} - \sqrt{2} \arctan\left(\frac{\sqrt{2}\sqrt{c}}{\sqrt{c}}\right)\right)}{ac^{3/2}f}$$

input

```
Integrate[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + a*Sin[e + f*x]]*(c - c*Sin[e + f*x])),x]
```

output

```
(Csc[2*(e + f*x)]*Sqrt[Sin[e + f*x]]*Sqrt[g*Sin[e + f*x]]*Sqrt[a*(1 + Sin[e + f*x])]*(2*Sqrt[c]*Sqrt[Sin[e + f*x]] - Sqrt[2]*ArcTan[(Sqrt[2]*Sqrt[c]*Sqrt[Sin[e + f*x]])/Sqrt[c - c*Sin[e + f*x]]]*Sqrt[c - c*Sin[e + f*x]]))/
(a*c^(3/2)*f)
```

Rubi [A] (verified)

Time = 0.88 (sec) , antiderivative size = 114, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}} = 0.175$, Rules used = {3042, 3415, 3042, 3261, 218, 3409, 15}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a \sin(e + fx) + a}(c - c \sin(e + fx))} dx$$

$$\downarrow \text{3042}$$

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a \sin(e + fx) + a}(c - c \sin(e + fx))} dx$$

$$\downarrow \text{3415}$$

$$\frac{g \int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)}(c-c \sin(e+fx))} dx}{2a} - \frac{g \int \frac{1}{\sqrt{g \sin(e+fx)}\sqrt{\sin(e+fx)a+a}} dx}{2c}$$

$$\begin{aligned}
& \downarrow 3042 \\
& \frac{g \int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)(c-c \sin(e+fx))}} dx}{2a} - \frac{g \int \frac{1}{\sqrt{g \sin(e+fx)}\sqrt{\sin(e+fx)a+a}} dx}{2c} \\
& \downarrow 3261 \\
& \frac{ag \int \frac{1}{\frac{\cos(e+fx) \cot(e+fx)a^3+2a^2}{\sin(e+fx)a+a}} d \frac{a \cos(e+fx)}{\sqrt{g \sin(e+fx)}\sqrt{\sin(e+fx)a+a}}}{cf} + \frac{g \int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)(c-c \sin(e+fx))}} dx}{2a} \\
& \downarrow 218 \\
& \frac{g \int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)(c-c \sin(e+fx))}} dx}{2a} + \frac{\sqrt{g} \arctan \left(\frac{\sqrt{a}\sqrt{g} \cos(e+fx)}{\sqrt{2}\sqrt{a \sin(e+fx)+a}\sqrt{g \sin(e+fx)}} \right)}{\sqrt{2}\sqrt{acf}} \\
& \downarrow 3409 \\
& \frac{\sqrt{g} \arctan \left(\frac{\sqrt{a}\sqrt{g} \cos(e+fx)}{\sqrt{2}\sqrt{a \sin(e+fx)+a}\sqrt{g \sin(e+fx)}} \right)}{\sqrt{2}\sqrt{acf}} - \frac{g \int \frac{\sec(e+fx)(\sin(e+fx)a+a) \tan(e+fx)}{a^2c} d \frac{a \cos(e+fx)}{\sqrt{g \sin(e+fx)}\sqrt{\sin(e+fx)a+a}}}{f} \\
& \downarrow 15 \\
& \frac{\sqrt{g} \arctan \left(\frac{\sqrt{a}\sqrt{g} \cos(e+fx)}{\sqrt{2}\sqrt{a \sin(e+fx)+a}\sqrt{g \sin(e+fx)}} \right)}{\sqrt{2}\sqrt{acf}} + \frac{\sec(e+fx)\sqrt{a \sin(e+fx)+a}\sqrt{g \sin(e+fx)}}{acf}
\end{aligned}$$

input `Int[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + a*Sin[e + f*x]]*(c - c*Sin[e + f*x])),x]`

output `(Sqrt[g]*ArcTan[(Sqrt[a]*Sqrt[g]*Cos[e + f*x])/(Sqrt[2]*Sqrt[g*Sin[e + f*x]])*Sqrt[a + a*Sin[e + f*x]])/(Sqrt[2]*Sqrt[a]*c*f) + (Sec[e + f*x]*Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]])/(a*c*f)`

Defintions of rubi rules used

- rule 15 `Int[(a_.)*(x_)^(m_.), x_Symbol] := Simp[a*(x^(m + 1)/(m + 1)), x] /; FreeQ[{a, m}, x] && NeQ[m, -1]`
- rule 218 `Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(Rt[a/b, 2]/a)*ArcTan[x/Rt[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]`
- rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`
- rule 3261 `Int[1/(Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)])*Sqrt[(c_.) + (d_.)*sin[(e_.) + (f_.)*(x_)]]), x_Symbol] := Simp[-2*(a/f) Subst[Int[1/(2*b^2 - (a*c - b*d)*x^2), x], x, b*(Cos[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]])], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]`
- rule 3409 `Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] := Simp[-2*(b/f) Subst[Int[1/(b*c + a*d + c*g*x^2), x], x, b*(Cos[e + f*x]/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]])], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]`
- rule 3415 `Int[Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] := Simp[(-a)*(g/(b*c - a*d)) Int[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]), x], x] + Simp[c*(g/(b*c - a*d)) Int[Sqrt[a + b*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c + d*Sin[e + f*x])), x], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && (EqQ[a^2 - b^2, 0] || EqQ[c^2 - d^2, 0])`

Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 308 vs. $2(96) = 192$.

Time = 20.56 (sec) , antiderivative size = 309, normalized size of antiderivative = 2.71

method	result
default	$\sqrt{2} \left(\left(1 - 2 \cos\left(\frac{fx}{2} + \frac{e}{2}\right) \right)^2 \sqrt{g} \arctan \left(\frac{\sqrt{\frac{\cos\left(\frac{fx}{2} + \frac{e}{2}\right) \sin\left(\frac{fx}{2} + \frac{e}{2}\right) g}{\left(\cos\left(\frac{fx}{2} + \frac{e}{2}\right) + 1\right)^2}} \sin\left(\frac{fx}{2} + \frac{e}{2}\right)}{\left(\cos\left(\frac{fx}{2} + \frac{e}{2}\right) - 1\right) \sqrt{g}} \right) + \left(\left(\cos\left(\frac{fx}{2} + \frac{e}{2}\right) + 1\right) \sin\left(\frac{fx}{2} + \frac{e}{2}\right) + \cos\left(\frac{fx}{2} + \frac{e}{2}\right) \right) \right)$ $cf \left(\cos\left(\frac{fx}{2} + \frac{e}{2}\right)^2 - \cos\left(\frac{fx}{2} + \frac{e}{2}\right) \sin\left(\frac{fx}{2} + \frac{e}{2}\right) + \cos\left(\frac{fx}{2} + \frac{e}{2}\right) - \sin\left(\frac{fx}{2} + \frac{e}{2}\right) \right) \sqrt{\frac{\cos\left(\frac{fx}{2} + \frac{e}{2}\right) \sin\left(\frac{fx}{2} + \frac{e}{2}\right)}{\left(\cos\left(\frac{fx}{2} + \frac{e}{2}\right) + 1\right)}}$

input `int((g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x,method=_RETURNVERBOSE)`

output `2^(1/2)/c/f*((1-2*cos(1/2*f*x+1/2*e))^2)*g^(1/2)*arctan((cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/2*e)+1)^2*g)^(1/2)*sin(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/2*e)-1)/g^(1/2))+((cos(1/2*f*x+1/2*e)+1)*sin(1/2*f*x+1/2*e)+cos(1/2*f*x+1/2*e)*(cos(1/2*f*x+1/2*e)+1))*(cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/2*e)+1)^2*g)^(1/2)*(g*sin(1/2*f*x+1/2*e)*cos(1/2*f*x+1/2*e))^(1/2)/(cos(1/2*f*x+1/2*e)^2-cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1/2*e)+cos(1/2*f*x+1/2*e)-sin(1/2*f*x+1/2*e))/(cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/2*e)+1)^2*g)^(1/2)/((2*cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1/2*e)+1)*a)^(1/2)`

Fricas [A] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 385, normalized size of antiderivative = 3.38

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + a \sin(e + fx)}(c - c \sin(e + fx))} dx$$

$$= \frac{\sqrt{2}a\sqrt{-\frac{g}{a}} \cos(fx + e) \log\left(\frac{17g \cos(fx+e)^3 + 4\sqrt{2}(3 \cos(fx+e)^2 + (3 \cos(fx+e) + 4) \sin(fx+e) - \cos(fx+e) - 4) \sqrt{a \sin(fx+e) + a}}{\cos(fx+e)^3 + 3 \cos(fx+e)^2 + (\cos(fx+e) - 4) \sin(fx+e) - \cos(fx+e) - 4}\right) + \sqrt{2}a\sqrt{\frac{g}{a}} \arctan\left(\frac{\sqrt{2}\sqrt{a \sin(fx+e) + a}\sqrt{g \sin(fx+e)}\sqrt{\frac{g}{a}}(3 \sin(fx+e) - 1)}{4g \cos(fx+e) \sin(fx+e)}\right) \cos(fx + e) - 4 \sqrt{a \sin(fx + e) + a}\sqrt{g}}{4acf \cos(fx + e)}$$

input

```
integrate((g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x,
algorithm="fricas")
```

output

```
[1/8*(sqrt(2)*a*sqrt(-g/a)*cos(f*x + e)*log((17*g*cos(f*x + e)^3 + 4*sqrt(
2)*(3*cos(f*x + e)^2 + (3*cos(f*x + e) + 4)*sin(f*x + e) - cos(f*x + e) -
4)*sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))*sqrt(-g/a) + 3*g*cos(f*x
+ e)^2 - 18*g*cos(f*x + e) + (17*g*cos(f*x + e)^2 + 14*g*cos(f*x + e) - 4*
g)*sin(f*x + e) - 4*g)/(cos(f*x + e)^3 + 3*cos(f*x + e)^2 + (cos(f*x + e)^
2 - 2*cos(f*x + e) - 4)*sin(f*x + e) - 2*cos(f*x + e) - 4)) + 8*sqrt(a*sin
(f*x + e) + a)*sqrt(g*sin(f*x + e)))/(a*c*f*cos(f*x + e)), -1/4*(sqrt(2)*a
*sqrt(g/a)*arctan(1/4*sqrt(2)*sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e)
)*sqrt(g/a)*(3*sin(f*x + e) - 1)/(g*cos(f*x + e)*sin(f*x + e)))*cos(f*x +
e) - 4*sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e)))/(a*c*f*cos(f*x + e))
]
```


SymPy [F]

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + a \sin(e + fx)}(c - c \sin(e + fx))} dx$$

$$= -\frac{\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a \sin(e + fx) + a} \sin(e + fx) - \sqrt{a \sin(e + fx) + a}} dx}{c}$$

input `integrate((g*sin(f*x+e))**(1/2)/(a+a*sin(f*x+e))**(1/2)/(c-c*sin(f*x+e)),x)`

output `-Integral(sqrt(g*sin(e + f*x))/(sqrt(a*sin(e + f*x) + a)*sin(e + f*x) - sqrt(a*sin(e + f*x) + a)), x)/c`

Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 270 vs. $2(96) = 192$.

Time = 0.14 (sec) , antiderivative size = 270, normalized size of antiderivative = 2.37

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + a \sin(e + fx)}(c - c \sin(e + fx))} dx$$

$$= \frac{4\sqrt{2}\sqrt{g}\left(\frac{\sin(fx+e)}{\cos(fx+e)+1}\right)^{\frac{3}{2}}}{\sqrt{ac} + \frac{\sqrt{ac}\sin(fx+e)}{\cos(fx+e)+1} - \frac{(\sqrt{ac} + \frac{\sqrt{ac}\sin(fx+e)}{\cos(fx+e)+1})\sin(fx+e)}{\cos(fx+e)+1}} - \frac{2\sqrt{2}\sqrt{g}\arctan\left(\sqrt{\frac{\sin(fx+e)}{\cos(fx+e)+1}}\right)}{\sqrt{ac}} + \frac{\sqrt{2}\sqrt{g}\sqrt{\frac{\sin(fx+e)}{\cos(fx+e)+1}} + \sqrt{2}\sqrt{g}}{\sqrt{ac} + \frac{\sqrt{ac}\sin(fx+e)}{\cos(fx+e)+1}} + \frac{\sqrt{2}\sqrt{g}\sqrt{\frac{\sin(fx+e)}{\cos(fx+e)+1}}}{\sqrt{ac} + \frac{\sqrt{ac}\sin(fx+e)}{\cos(fx+e)+1}}$$

input `integrate((g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x, algorithm="maxima")`

output

```
1/2*(4*sqrt(2)*sqrt(g)*(sin(f*x + e)/(cos(f*x + e) + 1))^(3/2)/(sqrt(a)*c
+ sqrt(a)*c*sin(f*x + e)/(cos(f*x + e) + 1) - (sqrt(a)*c + sqrt(a)*c*sin(f
*x + e)/(cos(f*x + e) + 1))*sin(f*x + e)/(cos(f*x + e) + 1)) - 2*sqrt(2)*s
qrt(g)*arctan(sqrt(sin(f*x + e)/(cos(f*x + e) + 1)))/(sqrt(a)*c) + (sqrt(2)
)*sqrt(g)*sqrt(sin(f*x + e)/(cos(f*x + e) + 1)) + sqrt(2)*sqrt(g)/(sqrt(a)
)*c + sqrt(a)*c*sin(f*x + e)/(cos(f*x + e) + 1)) + (sqrt(2)*sqrt(g)*sqrt(s
in(f*x + e)/(cos(f*x + e) + 1)) - sqrt(2)*sqrt(g))/(sqrt(a)*c + sqrt(a)*c*
sin(f*x + e)/(cos(f*x + e) + 1))/f
```

Giac [F(-1)]

Timed out.

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + a \sin(e + fx)}(c - c \sin(e + fx))} dx = \text{Timed out}$$

input

```
integrate((g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x,
algorithm="giac")
```

output

Timed out

Mupad [F(-1)]

Timed out.

$$\begin{aligned} & \int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + a \sin(e + fx)}(c - c \sin(e + fx))} dx \\ &= \int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + a \sin(e + fx)}(c - c \sin(e + fx))} dx \end{aligned}$$

input

```
int((g*sin(e + f*x))^(1/2)/((a + a*sin(e + f*x))^(1/2)*(c - c*sin(e + f*x)
)),x)
```

output

```
int((g*sin(e + f*x))^(1/2)/((a + a*sin(e + f*x))^(1/2)*(c - c*sin(e + f*x)
)), x)
```

Reduce [F]

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + a \sin(e + fx)}(c - c \sin(e + fx))} dx = -\frac{\sqrt{g} \sqrt{a} \left(\int \frac{\sqrt{\sin(fx+e)} \sqrt{\sin(fx+e)+1}}{\sin(fx+e)^2-1} dx \right)}{ac}$$

input `int((g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x)`

output `(- sqrt(g)*sqrt(a)*int((sqrt(sin(e + f*x))*sqrt(sin(e + f*x) + 1))/(sin(e + f*x)**2 - 1),x))/(a*c)`

3.18
$$\int \frac{1}{\sqrt{g \sin(e+fx)} \sqrt{a+a \sin(e+fx)}(c-c \sin(e+fx))} dx$$

Optimal result	171
Mathematica [A] (verified)	172
Rubi [A] (verified)	172
Maple [B] (verified)	175
Fricas [A] (verification not implemented)	175
Sympy [F]	176
Maxima [F]	177
Giac [F(-1)]	177
Mupad [F(-1)]	177
Reduce [F]	178

Optimal result

Integrand size = 40, antiderivative size = 118

$$\int \frac{1}{\sqrt{g \sin(e+fx)} \sqrt{a+a \sin(e+fx)}(c-c \sin(e+fx))} dx$$

$$= -\frac{\arctan\left(\frac{\sqrt{a}\sqrt{g} \cos(e+fx)}{\sqrt{2}\sqrt{g \sin(e+fx)} \sqrt{a+a \sin(e+fx)}}\right)}{\sqrt{2}\sqrt{ac}f\sqrt{g}} + \frac{\sec(e+fx)\sqrt{g \sin(e+fx)} \sqrt{a+a \sin(e+fx)}}{acfg}$$

output `-1/2*arctan(1/2*a^(1/2)*g^(1/2)*cos(f*x+e)*2^(1/2)/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2))*2^(1/2)/a^(1/2)/c/f/g^(1/2)+sec(f*x+e)*(g*sin(f*x+e))^(1/2)*(a+a*sin(f*x+e))^(1/2)/a/c/f/g`

Mathematica [A] (verified)

Time = 0.30 (sec) , antiderivative size = 132, normalized size of antiderivative = 1.12

$$\int \frac{1}{\sqrt{g \sin(e+fx)} \sqrt{a+a \sin(e+fx)} (c-c \sin(e+fx))} dx$$

$$= \frac{\csc(2(e+fx)) \sin^{\frac{3}{2}}(e+fx) \sqrt{a(1+\sin(e+fx))} \left(2\sqrt{c} \sqrt{\sin(e+fx)} + \sqrt{2} \arctan \left(\frac{\sqrt{2}\sqrt{c}\sqrt{\sin(e+fx)}}{\sqrt{c-c \sin(e+fx)}} \right) \right) \sqrt{c}}{ac^{3/2} f \sqrt{g \sin(e+fx)}}$$

input

```
Integrate[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]]*(c - c*Sin[e + f*x])),x]
```

output

```
(Csc[2*(e + f*x)]*Sin[e + f*x]^(3/2)*Sqrt[a*(1 + Sin[e + f*x])]*(2*Sqrt[c]*Sqrt[Sin[e + f*x]] + Sqrt[2]*ArcTan[(Sqrt[2]*Sqrt[c]*Sqrt[Sin[e + f*x]])/Sqrt[c - c*Sin[e + f*x]])*Sqrt[c - c*Sin[e + f*x]])/(a*c^(3/2)*f*Sqrt[g*Sin[e + f*x]])
```

Rubi [A] (verified)

Time = 0.88 (sec) , antiderivative size = 118, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}} = 0.175$, Rules used = {3042, 3417, 3042, 3261, 218, 3409, 15}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{1}{\sqrt{a \sin(e+fx)} + a(c-c \sin(e+fx)) \sqrt{g \sin(e+fx)}} dx$$

$$\downarrow \text{3042}$$

$$\int \frac{1}{\sqrt{a \sin(e+fx)} + a(c-c \sin(e+fx)) \sqrt{g \sin(e+fx)}} dx$$

$$\downarrow \text{3417}$$

$$\frac{\int \frac{1}{\sqrt{g \sin(e+fx)} \sqrt{\sin(e+fx)a+a}} dx}{2c} + \frac{\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)}(c-c \sin(e+fx))} dx}{2a}$$

$$\begin{aligned}
 & \downarrow 3042 \\
 & \frac{\int \frac{1}{\sqrt{g \sin(e+fx)} \sqrt{\sin(e+fx)a+a}} dx}{2c} + \frac{\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)}(c-c \sin(e+fx))} dx}{2a} \\
 & \downarrow 3261 \\
 & \frac{\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)}(c-c \sin(e+fx))} dx}{2a} - \frac{a \int \frac{1}{\frac{\cos(e+fx) \cot(e+fx)a^3}{\sin(e+fx)a+a} + 2a^2} d \frac{a \cos(e+fx)}{\sqrt{g \sin(e+fx)} \sqrt{\sin(e+fx)a+a}}}{cf} \\
 & \downarrow 218 \\
 & \frac{\int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)}(c-c \sin(e+fx))} dx}{2a} - \frac{\arctan\left(\frac{\sqrt{a}\sqrt{g} \cos(e+fx)}{\sqrt{2}\sqrt{a \sin(e+fx)+a}\sqrt{g \sin(e+fx)}}\right)}{\sqrt{2}\sqrt{acf}\sqrt{g}} \\
 & \downarrow 3409 \\
 & - \frac{\int \frac{\sec(e+fx)(\sin(e+fx)a+a) \tan(e+fx)}{a^2c} d \frac{a \cos(e+fx)}{\sqrt{g \sin(e+fx)} \sqrt{\sin(e+fx)a+a}}}{\frac{f}{\sqrt{2}\sqrt{acf}\sqrt{g}}} \\
 & \downarrow 15 \\
 & \frac{\sec(e+fx)\sqrt{a \sin(e+fx)+a}\sqrt{g \sin(e+fx)}}{acfg} - \frac{\arctan\left(\frac{\sqrt{a}\sqrt{g} \cos(e+fx)}{\sqrt{2}\sqrt{a \sin(e+fx)+a}\sqrt{g \sin(e+fx)}}\right)}{\sqrt{2}\sqrt{acf}\sqrt{g}}
 \end{aligned}$$

input `Int[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]]*(c - c*Sin[e + f*x])),x]`

output `-(ArcTan[(Sqrt[a]*Sqrt[g]*Cos[e + f*x])/(Sqrt[2]*Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]])]/(Sqrt[2]*Sqrt[a]*c*f*Sqrt[g])) + (Sec[e + f*x]*Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]])/(a*c*f*g)`

Defintions of rubi rules used

- rule 15 $\text{Int}[(a_.)(x_)^{(m_.)}, x_Symbol] \rightarrow \text{Simp}[a*(x^{(m+1)})/(m+1), x] /; \text{FreeQ}[\{a, m\}, x] \ \&\& \ \text{NeQ}[m, -1]$
- rule 218 $\text{Int}[((a_) + (b_.)(x_)^2)^{-1}, x_Symbol] \rightarrow \text{Simp}[(\text{Rt}[a/b, 2]/a)*\text{ArcTan}[x/\text{Rt}[a/b, 2]], x] /; \text{FreeQ}[\{a, b\}, x] \ \&\& \ \text{PosQ}[a/b]$
- rule 3042 $\text{Int}[u_, x_Symbol] \rightarrow \text{Int}[\text{DeactivateTrig}[u, x], x] /; \text{FunctionOfTrigOfLinearQ}[u, x]$
- rule 3261 $\text{Int}[1/(\text{Sqrt}[(a_) + (b_.)\sin[(e_.) + (f_.)(x_)])*\text{Sqrt}[(c_.) + (d_.)\sin[(e_.) + (f_.)(x_)])], x_Symbol] \rightarrow \text{Simp}[-2*(a/f) \ \text{Subst}[\text{Int}[1/(2*b^2 - (a*c - b*d)*x^2), x], x, b*(\text{Cos}[e + f*x]/(\text{Sqrt}[a + b*\text{Sin}[e + f*x]]*\text{Sqrt}[c + d*\text{Sin}[e + f*x]])], x] /; \text{FreeQ}[\{a, b, c, d, e, f\}, x] \ \&\& \ \text{NeQ}[b*c - a*d, 0] \ \&\& \ \text{EqQ}[a^2 - b^2, 0] \ \&\& \ \text{NeQ}[c^2 - d^2, 0]$
- rule 3409 $\text{Int}[\text{Sqrt}[(a_) + (b_.)\sin[(e_.) + (f_.)(x_)]]/(\text{Sqrt}[(g_.)\sin[(e_.) + (f_.)(x_)])*((c_) + (d_.)\sin[(e_.) + (f_.)(x_)])], x_Symbol] \rightarrow \text{Simp}[-2*(b/f) \ \text{Subst}[\text{Int}[1/(b*c + a*d + c*g*x^2), x], x, b*(\text{Cos}[e + f*x]/(\text{Sqrt}[g*\text{Sin}[e + f*x]]*\text{Sqrt}[a + b*\text{Sin}[e + f*x]])], x] /; \text{FreeQ}[\{a, b, c, d, e, f, g\}, x] \ \&\& \ \text{NeQ}[b*c - a*d, 0] \ \&\& \ \text{EqQ}[a^2 - b^2, 0]$
- rule 3417 $\text{Int}[1/(\text{Sqrt}[(g_.)\sin[(e_.) + (f_.)(x_)])*\text{Sqrt}[(a_) + (b_.)\sin[(e_.) + (f_.)(x_)]]*((c_) + (d_.)\sin[(e_.) + (f_.)(x_)])], x_Symbol] \rightarrow \text{Simp}[b/(b*c - a*d) \ \text{Int}[1/(\text{Sqrt}[g*\text{Sin}[e + f*x]]*\text{Sqrt}[a + b*\text{Sin}[e + f*x]]), x], x] - \text{Simp}[d/(b*c - a*d) \ \text{Int}[\text{Sqrt}[a + b*\text{Sin}[e + f*x]]/(\text{Sqrt}[g*\text{Sin}[e + f*x]]*(c + d*\text{Sin}[e + f*x])), x], x] /; \text{FreeQ}[\{a, b, c, d, e, f, g\}, x] \ \&\& \ \text{NeQ}[b*c - a*d, 0] \ \&\& \ (\text{EqQ}[a^2 - b^2, 0] \ || \ \text{EqQ}[c^2 - d^2, 0])$

Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 268 vs. 2(99) = 198.

Time = 23.80 (sec) , antiderivative size = 269, normalized size of antiderivative = 2.28

method	result
default	$\frac{\sqrt{2} \left(\cos\left(\frac{fx}{2} + \frac{e}{2}\right) \left(\cos\left(\frac{fx}{2} + \frac{e}{2}\right) \sin\left(\frac{fx}{2} + \frac{e}{2}\right) + \sin\left(\frac{fx}{2} + \frac{e}{2}\right)^2 \right) g^{\frac{3}{2}} + \sqrt{\frac{\cos\left(\frac{fx}{2} + \frac{e}{2}\right) \sin\left(\frac{fx}{2} + \frac{e}{2}\right) g}{\left(\cos\left(\frac{fx}{2} + \frac{e}{2}\right) + 1\right)^2}} g \arctan\left(\frac{\sqrt{\frac{\cos\left(\frac{fx}{2} + \frac{e}{2}\right) \sin\left(\frac{fx}{2} + \frac{e}{2}\right)}{\left(\cos\left(\frac{fx}{2} + \frac{e}{2}\right) + 1\right)^2}}}{\cos\left(\frac{fx}{2} + \frac{e}{2}\right)}\right)}{cf\left(-\cos\left(\frac{fx}{2} + \frac{e}{2}\right) + \sin\left(\frac{fx}{2} + \frac{e}{2}\right)\right) \sqrt{g \sin\left(\frac{fx}{2} + \frac{e}{2}\right) \cos\left(\frac{fx}{2} + \frac{e}{2}\right)} \sqrt{\left(2 \cos\left(\frac{fx}{2} + \frac{e}{2}\right) + 1\right)^2}}$

input

```
int(1/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x,method=_RETURNVERBOSE)
```

output

```
-2^(1/2)/c/f*(cos(1/2*f*x+1/2*e)*(cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1/2*e)+sin(1/2*f*x+1/2*e)^2)*g^(3/2)+(cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/2*e)+1)^2*g)^(1/2)*g*arctan((cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/2*e)+1)^2*g)^(1/2)*sin(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/2*e)-1)/g^(1/2))*(2*cos(1/2*f*x+1/2*e)^3+2*cos(1/2*f*x+1/2*e)^2*cos(1/2*f*x+1/2*e)-1)/(-cos(1/2*f*x+1/2*e)+sin(1/2*f*x+1/2*e))/(g*sin(1/2*f*x+1/2*e)*cos(1/2*f*x+1/2*e))^(1/2)/((2*cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1/2*e)+1)*a)^(1/2)/g^(3/2)
```

Fricas [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 391, normalized size of antiderivative = 3.31

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)} (c - c \sin(e + fx))} dx$$

$$= \left[\frac{\sqrt{2} a g \sqrt{-\frac{1}{a g}} \cos(fx + e) \log\left(-\frac{4\sqrt{2}\left(3 \cos(fx+e)^2 + (3 \cos(fx+e)+4) \sin(fx+e) - \cos(fx+e) - 4\right) \sqrt{a \sin(fx+e)+a} \sqrt{g \sin(fx+e)}}{\cos(fx+e)^3 + 3 \cos(fx+e)^2 + (\cos(fx+e) + 1)^2}\right)}{\dots} \right]$$

input `integrate(1/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x, algorithm="fricas")`

output `[1/8*(sqrt(2)*a*g*sqrt(-1/(a*g))*cos(f*x + e)*log(-4*sqrt(2)*(3*cos(f*x + e)^2 + (3*cos(f*x + e) + 4)*sin(f*x + e) - cos(f*x + e) - 4)*sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))*sqrt(-1/(a*g)) - 17*cos(f*x + e)^3 - 3*cos(f*x + e)^2 - (17*cos(f*x + e)^2 + 14*cos(f*x + e) - 4)*sin(f*x + e) + 18*cos(f*x + e) + 4)/(cos(f*x + e)^3 + 3*cos(f*x + e)^2 + (cos(f*x + e)^2 - 2*cos(f*x + e) - 4)*sin(f*x + e) - 2*cos(f*x + e) - 4) + 8*sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e)))/(a*c*f*g*cos(f*x + e)), 1/4*(sqrt(2)*a*g*sqrt(1/(a*g))*arctan(1/4*sqrt(2)*sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))*sqrt(1/(a*g))*(3*sin(f*x + e) - 1)/(cos(f*x + e)*sin(f*x + e)))*cos(f*x + e) + 4*sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e)))/(a*c*f*g*cos(f*x + e))]`

Sympy [F]

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)} (c - c \sin(e + fx))} dx$$

$$= - \frac{\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a \sin(e + fx) + a \sin(e + fx) - \sqrt{g \sin(e + fx)} \sqrt{a \sin(e + fx) + a}}} dx}{c}$$

input `integrate(1/(g*sin(f*x+e))**(1/2)/(a+a*sin(f*x+e))**(1/2)/(c-c*sin(f*x+e)),x)`

output `-Integral(1/(sqrt(g*sin(e + f*x))*sqrt(a*sin(e + f*x) + a)*sin(e + f*x) - sqrt(g*sin(e + f*x))*sqrt(a*sin(e + f*x) + a)), x)/c`

Maxima [F]

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)} (c - c \sin(e + fx))} dx$$

$$= \int -\frac{1}{\sqrt{a \sin(fx + e) + a} (c \sin(fx + e) - c) \sqrt{g \sin(fx + e)}} dx$$

input `integrate(1/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x
, algorithm="maxima")`

output `-integrate(1/(sqrt(a*sin(f*x + e) + a)*(c*sin(f*x + e) - c)*sqrt(g*sin(f*x
+ e))), x)`

Giac [F(-1)]

Timed out.

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)} (c - c \sin(e + fx))} dx = \text{Timed out}$$

input `integrate(1/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x
, algorithm="giac")`

output `Timed out`

Mupad [F(-1)]

Timed out.

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)} (c - c \sin(e + fx))} dx$$

$$= \int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)} (c - c \sin(e + fx))} dx$$

input `int(1/((g*sin(e + f*x))^(1/2)*(a + a*sin(e + f*x))^(1/2)*(c - c*sin(e + f*x))),x)`

output `int(1/((g*sin(e + f*x))^(1/2)*(a + a*sin(e + f*x))^(1/2)*(c - c*sin(e + f*x))), x)`

Reduce [F]

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)} (c - c \sin(e + fx))} dx$$

$$= - \frac{\sqrt{g} \sqrt{a} \left(\int \frac{\sqrt{\sin(fx+e)} \sqrt{\sin(fx+e)+1}}{\sin(fx+e)^3 - \sin(fx+e)} dx \right)}{acg}$$

input `int(1/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e)),x)`

output `(- sqrt(g)*sqrt(a)*int((sqrt(sin(e + f*x))*sqrt(sin(e + f*x) + 1))/(sin(e + f*x)**3 - sin(e + f*x)),x))/(a*c*g)`

3.19 $\int \csc(e+fx) \sqrt{a+a \sin(e+fx)} \sqrt{c-c \sin(e+fx)} dx$

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Optimal result

Integrand size = 36, antiderivative size = 46

$$\int \csc(e+fx) \sqrt{a+a \sin(e+fx)} \sqrt{c-c \sin(e+fx)} dx$$

$$= \frac{\log(\sin(e+fx)) \sec(e+fx) \sqrt{a+a \sin(e+fx)} \sqrt{c-c \sin(e+fx)}}{f}$$

output `ln(sin(f*x+e))*sec(f*x+e)*(a+a*sin(f*x+e))^(1/2)*(c-c*sin(f*x+e))^(1/2)/f`

Mathematica [A] (verified)

Time = 0.12 (sec) , antiderivative size = 62, normalized size of antiderivative = 1.35

$$\int \csc(e+fx) \sqrt{a+a \sin(e+fx)} \sqrt{c-c \sin(e+fx)} dx$$

$$= \frac{(\log(\cos(\frac{1}{2}(e+fx))) + \log(\sin(\frac{1}{2}(e+fx)))) \sec(e+fx) \sqrt{a(1+\sin(e+fx))} \sqrt{c-c \sin(e+fx)}}{f}$$

input `Integrate[Csc[e + f*x]*Sqrt[a + a*Sin[e + f*x]]*Sqrt[c - c*Sin[e + f*x]],x]`

output $((\text{Log}[\text{Cos}[(e + f*x)/2]] + \text{Log}[\text{Sin}[(e + f*x)/2]])*\text{Sec}[e + f*x]*\text{Sqrt}[a*(1 + \text{Sin}[e + f*x]])*\text{Sqrt}[c - c*\text{Sin}[e + f*x]])/f$

Rubi [A] (verified)

Time = 0.41 (sec) , antiderivative size = 48, normalized size of antiderivative = 1.04, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.139$, Rules used = {3042, 3427, 3042, 25, 3956}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \csc(e + fx) \sqrt{a \sin(e + fx) + a} \sqrt{c - c \sin(e + fx)} dx \\ & \quad \downarrow 3042 \\ & \int \frac{\sqrt{a \sin(e + fx) + a} \sqrt{c - c \sin(e + fx)}}{\sin(e + fx)} dx \\ & \quad \downarrow 3427 \\ & \sec(e + fx) \sqrt{a \sin(e + fx) + a} \sqrt{c - c \sin(e + fx)} \int \cot(e + fx) dx \\ & \quad \downarrow 3042 \\ & \sec(e + fx) \sqrt{a \sin(e + fx) + a} \sqrt{c - c \sin(e + fx)} \int -\tan\left(e + fx + \frac{\pi}{2}\right) dx \\ & \quad \downarrow 25 \\ & -\sec(e + fx) \sqrt{a \sin(e + fx) + a} \sqrt{c - c \sin(e + fx)} \int \tan\left(\frac{1}{2}(2e + \pi) + fx\right) dx \\ & \quad \downarrow 3956 \\ & \frac{\sec(e + fx) \sqrt{a \sin(e + fx) + a} \sqrt{c - c \sin(e + fx)} \log(-\sin(e + fx))}{f} \end{aligned}$$

input $\text{Int}[\text{Csc}[e + f*x]*\text{Sqrt}[a + a*\text{Sin}[e + f*x]]*\text{Sqrt}[c - c*\text{Sin}[e + f*x]],x]$

output $(\text{Log}[-\text{Sin}[e + f*x]]*\text{Sec}[e + f*x]*\text{Sqrt}[a + a*\text{Sin}[e + f*x]]*\text{Sqrt}[c - c*\text{Sin}[e + f*x]])/f$

Defintions of rubi rules used

rule 25 $\text{Int}[-(F x), x_Symbol] \rightarrow \text{Simp}[\text{Identity}[-1] \text{ Int}[F x, x], x]$

rule 3042 $\text{Int}[u, x_Symbol] \rightarrow \text{Int}[\text{DeactivateTrig}[u, x], x] /; \text{FunctionOfTrigOfLinearQ}[u, x]$

rule 3427 $\text{Int}[(\text{Sqrt}[(a) + (b) * \sin[(e) + (f) * (x)]] * \text{Sqrt}[(c) + (d) * \sin[(e) + (f) * (x)])] / \sin[(e) + (f) * (x)], x_Symbol] \rightarrow \text{Simp}[\text{Sqrt}[a + b * \text{Sin}[e + f * x]] * (\text{Sqrt}[c + d * \text{Sin}[e + f * x]] / \text{Cos}[e + f * x]) \text{ Int}[\text{Cot}[e + f * x], x], x] /; \text{FreeQ}\{a, b, c, d, e, f\}, x \ \&\& \ \text{NeQ}[b * c - a * d, 0] \ \&\& \ \text{EqQ}[a^2 - b^2, 0] \ \&\& \ \text{EqQ}[c^2 - d^2, 0]$

rule 3956 $\text{Int}[\tan[(c) + (d) * (x)], x_Symbol] \rightarrow \text{Simp}[-\text{Log}[\text{RemoveContent}[\text{Cos}[c + d * x], x]] / d, x] /; \text{FreeQ}\{c, d\}, x]$

Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 161 vs. $2(42) = 84$.

Time = 15.13 (sec) , antiderivative size = 162, normalized size of antiderivative = 3.52

method	result
default	$\frac{\sqrt{\left(2 \cos\left(\frac{fx}{2} + \frac{e}{2}\right) \sin\left(\frac{fx}{2} + \frac{e}{2}\right) + 1\right) a} \sqrt{-\left(2 \cos\left(\frac{fx}{2} + \frac{e}{2}\right) \sin\left(\frac{fx}{2} + \frac{e}{2}\right) - 1\right) c} \left(\ln\left(-\cot\left(\frac{fx}{2} + \frac{e}{2}\right) + \csc\left(\frac{fx}{2} + \frac{e}{2}\right) - 1\right) + \ln\left(\csc\left(\frac{fx}{2} + \frac{e}{2}\right)\right)\right)}{f \left(-1 + 2 \cos\left(\frac{fx}{2} + \frac{e}{2}\right)\right)^2}$

input $\text{int}(\text{csc}(f*x+e)*(a+a*\text{sin}(f*x+e))^{(1/2)}*(c-c*\text{sin}(f*x+e))^{(1/2)}, x, \text{method}=_RET \text{URNVERBOSE})$

output

```
1/f*((2*cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1/2*e)+1)*a)^(1/2)*(-(2*cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1/2*e)-1)*c)^(1/2)*(ln(-cot(1/2*f*x+1/2*e)+csc(1/2*f*x+1/2*e)-1)+ln(csc(1/2*f*x+1/2*e)-cot(1/2*f*x+1/2*e))-2*ln(2/(cos(1/2*f*x+1/2*e)+1))+ln(-cot(1/2*f*x+1/2*e)+csc(1/2*f*x+1/2*e)+1))/(-1+2*cos(1/2*f*x+1/2*e)^2)
```

Fricas [A] (verification not implemented)

Time = 0.22 (sec) , antiderivative size = 202, normalized size of antiderivative = 4.39

$$\int \csc(e + fx) \sqrt{a + a \sin(e + fx)} \sqrt{c - c \sin(e + fx)} dx$$

$$= \left[\frac{\sqrt{ac} \log \left(\frac{4 \left(256 ac \cos(fx+e)^5 - 512 ac \cos(fx+e)^3 + 337 ac \cos(fx+e) + (256 \cos(fx+e)^4 - 512 \cos(fx+e)^2 + 175) \sqrt{ac} \sqrt{a \sin(fx+e) + a} \right)}{\cos(fx+e)^3 - \cos(fx+e)} \right)}{2f} \right. \\ \left. - \frac{\sqrt{-ac} \arctan \left(\frac{\sqrt{-ac} (16 \cos(fx+e)^2 - 7) \sqrt{a \sin(fx+e) + a} \sqrt{-c \sin(fx+e) + c}}{16 ac \cos(fx+e)^3 - 25 ac \cos(fx+e)} \right)}{f} \right]$$

input

```
integrate(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)*(c-c*sin(f*x+e))^(1/2),x, algorithm="fricas")
```

output

```
[1/2*sqrt(a*c)*log(4*(256*a*c*cos(f*x + e)^5 - 512*a*c*cos(f*x + e)^3 + 337*a*c*cos(f*x + e) + (256*cos(f*x + e)^4 - 512*cos(f*x + e)^2 + 175)*sqrt(a*c)*sqrt(a*sin(f*x + e) + a)*sqrt(-c*sin(f*x + e) + c))/(cos(f*x + e)^3 - cos(f*x + e)))/f, -sqrt(-a*c)*arctan(sqrt(-a*c)*(16*cos(f*x + e)^2 - 7)*sqrt(a*sin(f*x + e) + a)*sqrt(-c*sin(f*x + e) + c)/(16*a*c*cos(f*x + e)^3 - 25*a*c*cos(f*x + e)))/f]
```

Sympy [F]

$$\int \csc(e + fx) \sqrt{a + a \sin(e + fx)} \sqrt{c - c \sin(e + fx)} dx$$

$$= \int \sqrt{a(\sin(e + fx) + 1)} \sqrt{-c(\sin(e + fx) - 1)} \csc(e + fx) dx$$

input `integrate(csc(f*x+e)*(a+a*sin(f*x+e))**(1/2)*(c-c*sin(f*x+e))**(1/2),x)`

output `Integral(sqrt(a*(sin(e + f*x) + 1))*sqrt(-c*(sin(e + f*x) - 1))*csc(e + f*x), x)`

Maxima [F]

$$\int \csc(e + fx) \sqrt{a + a \sin(e + fx)} \sqrt{c - c \sin(e + fx)} dx$$

$$= \int \sqrt{a \sin(fx + e) + a} \sqrt{-c \sin(fx + e) + c} \csc(fx + e) dx$$

input `integrate(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)*(c-c*sin(f*x+e))^(1/2),x, algorith="maxima")`

output `integrate(sqrt(a*sin(f*x + e) + a)*sqrt(-c*sin(f*x + e) + c)*csc(f*x + e), x)`

Giac [A] (verification not implemented)

Time = 0.32 (sec) , antiderivative size = 57, normalized size of antiderivative = 1.24

$$\int \csc(e + fx) \sqrt{a + a \sin(e + fx)} \sqrt{c - c \sin(e + fx)} dx =$$

$$\frac{\sqrt{a}\sqrt{c} \log \left(\left| 2 \sin \left(-\frac{1}{4} \pi + \frac{1}{2} fx + \frac{1}{2} e \right)^2 - 1 \right| \right) \operatorname{sgn} \left(\cos \left(-\frac{1}{4} \pi + \frac{1}{2} fx + \frac{1}{2} e \right) \right) \operatorname{sgn} \left(\sin \left(-\frac{1}{4} \pi + \frac{1}{2} fx + \frac{1}{2} e \right) \right)}{f}$$

input `integrate(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)*(c-c*sin(f*x+e))^(1/2),x, algo
rithm="giac")`

output `-sqrt(a)*sqrt(c)*log(abs(2*sin(-1/4*pi + 1/2*f*x + 1/2*e)^2 - 1))*sgn(cos(
-1/4*pi + 1/2*f*x + 1/2*e))*sgn(sin(-1/4*pi + 1/2*f*x + 1/2*e))/f`

Mupad [F(-1)]

Timed out.

$$\int \csc(e + fx) \sqrt{a + a \sin(e + fx)} \sqrt{c - c \sin(e + fx)} dx$$

$$= \int \frac{\sqrt{a + a \sin(e + fx)} \sqrt{c - c \sin(e + fx)}}{\sin(e + fx)} dx$$

input `int(((a + a*sin(e + f*x))^(1/2)*(c - c*sin(e + f*x))^(1/2))/sin(e + f*x),x
)`

output `int(((a + a*sin(e + f*x))^(1/2)*(c - c*sin(e + f*x))^(1/2))/sin(e + f*x),
x)`

Reduce [F]

$$\int \csc(e + fx) \sqrt{a + a \sin(e + fx)} \sqrt{c - c \sin(e + fx)} dx$$

$$= \sqrt{c} \sqrt{a} \left(\int \sqrt{\sin(fx + e) + 1} \sqrt{-\sin(fx + e) + 1} \csc(fx + e) dx \right)$$

input `int(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)*(c-c*sin(f*x+e))^(1/2),x)`

output `sqrt(c)*sqrt(a)*int(sqrt(sin(e + f*x) + 1)*sqrt(- sin(e + f*x) + 1)*csc(e
+ f*x),x)`

3.20
$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c-c\sin(e+fx)}} dx$$

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Mupad [F(-1)]	191
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Optimal result

Integrand size = 36, antiderivative size = 102

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c-c\sin(e+fx)}} dx$$

$$= -\frac{a\cos(e+fx)\log(1-\sin(e+fx))}{f\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}} + \frac{\log(\sin(e+fx))\sec(e+fx)\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}}{cf}$$

output

```
-a*cos(f*x+e)*ln(1-sin(f*x+e))/f/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))^(1/2)+ln(sin(f*x+e))*sec(f*x+e)*(a+a*sin(f*x+e))^(1/2)*(c-c*sin(f*x+e))^(1/2)/c/f
```

Mathematica [A] (verified)

Time = 2.36 (sec) , antiderivative size = 64, normalized size of antiderivative = 0.63

$$\int \frac{\csc(e + fx) \sqrt{a + a \sin(e + fx)}}{\sqrt{c - c \sin(e + fx)}} dx = \frac{(\log(1 - \sin(e + fx)) - \log(\sin(e + fx))) \sec(e + fx) \sqrt{a(1 + \sin(e + fx))} \sqrt{c - c \sin(e + fx)}}{cf}$$

input

```
Integrate[(Csc[e + f*x]*Sqrt[a + a*Sin[e + f*x]])/Sqrt[c - c*Sin[e + f*x]],x]
```

output

```
-(((Log[1 - Sin[e + f*x]] - Log[Sin[e + f*x]])*Sec[e + f*x]*Sqrt[a*(1 + Sin[e + f*x])]*Sqrt[c - c*Sin[e + f*x]])/(c*f))
```

Rubi [A] (verified)

Time = 1.20 (sec) , antiderivative size = 105, normalized size of antiderivative = 1.03, number of steps used = 12, number of rules used = 11, $\frac{\text{number of rules}}{\text{integrand size}} = 0.306$, Rules used = {3042, 3421, 3042, 3216, 3042, 3146, 16, 3427, 3042, 25, 3956}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \frac{\csc(e + fx) \sqrt{a \sin(e + fx) + a}}{\sqrt{c - c \sin(e + fx)}} dx \\ & \quad \downarrow \text{3042} \\ & \int \frac{\sqrt{a \sin(e + fx) + a}}{\sin(e + fx) \sqrt{c - c \sin(e + fx)}} dx \\ & \quad \downarrow \text{3421} \\ & \int \frac{\sqrt{\sin(e + fx)a + a}}{\sqrt{c - c \sin(e + fx)}} dx + \frac{\int \csc(e + fx) \sqrt{\sin(e + fx)a + a} \sqrt{c - c \sin(e + fx)} dx}{c} \\ & \quad \downarrow \text{3042} \end{aligned}$$

$$\begin{aligned}
& \int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{c-c\sin(e+fx)}} dx + \frac{\int \frac{\sqrt{\sin(e+fx)a+a}\sqrt{c-c\sin(e+fx)}}{\sin(e+fx)} dx}{c} \\
& \quad \downarrow \text{3216} \\
& \frac{\int \frac{\sqrt{\sin(e+fx)a+a}\sqrt{c-c\sin(e+fx)}}{\sin(e+fx)} dx}{c} + \frac{ac \cos(e+fx) \int \frac{\cos(e+fx)}{c-c\sin(e+fx)} dx}{\sqrt{a \sin(e+fx) + a\sqrt{c-c\sin(e+fx)}}} \\
& \quad \downarrow \text{3042} \\
& \frac{\int \frac{\sqrt{\sin(e+fx)a+a}\sqrt{c-c\sin(e+fx)}}{\sin(e+fx)} dx}{c} + \frac{ac \cos(e+fx) \int \frac{\cos(e+fx)}{c-c\sin(e+fx)} dx}{\sqrt{a \sin(e+fx) + a\sqrt{c-c\sin(e+fx)}}} \\
& \quad \downarrow \text{3146} \\
& \frac{\int \frac{\sqrt{\sin(e+fx)a+a}\sqrt{c-c\sin(e+fx)}}{\sin(e+fx)} dx}{c} - \frac{a \cos(e+fx) \int \frac{1}{c-c\sin(e+fx)} d(-c\sin(e+fx))}{f \sqrt{a \sin(e+fx) + a\sqrt{c-c\sin(e+fx)}}} \\
& \quad \downarrow \text{16} \\
& \frac{\int \frac{\sqrt{\sin(e+fx)a+a}\sqrt{c-c\sin(e+fx)}}{\sin(e+fx)} dx}{c} - \frac{a \cos(e+fx) \log(c-c\sin(e+fx))}{f \sqrt{a \sin(e+fx) + a\sqrt{c-c\sin(e+fx)}}} \\
& \quad \downarrow \text{3427} \\
& \frac{\sec(e+fx) \sqrt{a \sin(e+fx) + a\sqrt{c-c\sin(e+fx)}} \int \cot(e+fx) dx}{c} - \\
& \quad \frac{a \cos(e+fx) \log(c-c\sin(e+fx))}{f \sqrt{a \sin(e+fx) + a\sqrt{c-c\sin(e+fx)}}} \\
& \quad \downarrow \text{3042} \\
& \frac{\sec(e+fx) \sqrt{a \sin(e+fx) + a\sqrt{c-c\sin(e+fx)}} \int -\tan(e+fx + \frac{\pi}{2}) dx}{c} - \\
& \quad \frac{a \cos(e+fx) \log(c-c\sin(e+fx))}{f \sqrt{a \sin(e+fx) + a\sqrt{c-c\sin(e+fx)}}} \\
& \quad \downarrow \text{25} \\
& \frac{\sec(e+fx) \sqrt{a \sin(e+fx) + a\sqrt{c-c\sin(e+fx)}} \int \tan(\frac{1}{2}(2e+\pi) + fx) dx}{c} - \\
& \quad \frac{a \cos(e+fx) \log(c-c\sin(e+fx))}{f \sqrt{a \sin(e+fx) + a\sqrt{c-c\sin(e+fx)}}} \\
& \quad \downarrow \text{3956}
\end{aligned}$$

$$\frac{\sec(e + fx)\sqrt{a\sin(e + fx) + a}\sqrt{c - c\sin(e + fx)}\log(-\sin(e + fx))}{cf} - \frac{a\cos(e + fx)\log(c - c\sin(e + fx))}{f\sqrt{a\sin(e + fx) + a}\sqrt{c - c\sin(e + fx)}}$$

input `Int[(Csc[e + f*x]*Sqrt[a + a*Sin[e + f*x]])/Sqrt[c - c*Sin[e + f*x]],x]`

output `-((a*cos[e + f*x]*Log[c - c*Sin[e + f*x]])/(f*Sqrt[a + a*Sin[e + f*x]]*Sqrt[c - c*Sin[e + f*x])) + (Log[-Sin[e + f*x]]*Sec[e + f*x]*Sqrt[a + a*Sin[e + f*x]]*Sqrt[c - c*Sin[e + f*x]])/(c*f)`

Defintions of rubi rules used

rule 16 `Int[(c_)/((a_) + (b_)*(x_)), x_Symbol] := Simp[c*(Log[RemoveContent[a + b*x, x]/b), x] /; FreeQ[{a, b, c}, x]`

rule 25 `Int[-(Fx_), x_Symbol] := Simp[Identity[-1] Int[Fx, x], x]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3146 `Int[cos[(e_) + (f_)*(x_)]^(p_)*((a_) + (b_)*sin[(e_) + (f_)*(x_)]^(m_)), x_Symbol] := Simp[1/(b^p*f) Subst[Int[(a + x)^(m + (p - 1)/2)*(a - x)^(p - 1)/2], x], x, b*Sin[e + f*x], x] /; FreeQ[{a, b, e, f, m}, x] && IntegerQ[(p - 1)/2] && EqQ[a^2 - b^2, 0] && (GeQ[p, -1] || !IntegerQ[m + 1/2])`

rule 3216 `Int[Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)]]/Sqrt[(c_) + (d_)*sin[(e_) + (f_)*(x_)]], x_Symbol] := Simp[a*c*(Cos[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]])) Int[Cos[e + f*x]/(c + d*Sin[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && EqQ[b*c + a*d, 0] && EqQ[a^2 - b^2, 0]`

rule 3421

```
Int[Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)]]/(sin[(e_) + (f_)*(x_)]*Sqrt[(c_) + (d_)*sin[(e_) + (f_)*(x_)]]), x_Symbol] := Simp[-d/c Int[Sqrt[a + b*Sin[e + f*x]]/Sqrt[c + d*Sin[e + f*x]], x], x] + Simp[1/c Int[Sqrt[a + b*Sin[e + f*x]]*(Sqrt[c + d*Sin[e + f*x]]/Sin[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0] && EqQ[b*c + a*d, 0]
```

rule 3427

```
Int[(Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)]]*Sqrt[(c_) + (d_)*sin[(e_) + (f_)*(x_)]])/sin[(e_) + (f_)*(x_)], x_Symbol] := Simp[Sqrt[a + b*Sin[e + f*x]]*(Sqrt[c + d*Sin[e + f*x]]/Cos[e + f*x]) Int[Cot[e + f*x], x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0] && EqQ[c^2 - d^2, 0]
```

rule 3956

```
Int[tan[(c_) + (d_)*(x_)], x_Symbol] := Simp[-Log[RemoveContent[Cos[c + d*x], x]]/d, x] /; FreeQ[{c, d}, x]
```

Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 213 vs. 2(94) = 188.

Time = 13.02 (sec) , antiderivative size = 214, normalized size of antiderivative = 2.10

method	result
default	$\sqrt{\left(2 \cos\left(\frac{fx}{2} + \frac{e}{2}\right) \sin\left(\frac{fx}{2} + \frac{e}{2}\right) + 1\right)} a \left(2 \ln\left(\frac{-2 \cos\left(\frac{fx}{2} + \frac{e}{2}\right) + 2 \sin\left(\frac{fx}{2} + \frac{e}{2}\right)}{\cos\left(\frac{fx}{2} + \frac{e}{2}\right) + 1}\right) - \ln\left(-\cot\left(\frac{fx}{2} + \frac{e}{2}\right) + \csc\left(\frac{fx}{2} + \frac{e}{2}\right) - 1\right) - \ln\left(-\cot\left(\frac{fx}{2} + \frac{e}{2}\right) + \csc\left(\frac{fx}{2} + \frac{e}{2}\right) + 1\right) \right) \sqrt{-\left(2 \cos\left(\frac{fx}{2} + \frac{e}{2}\right) \sin\left(\frac{fx}{2} + \frac{e}{2}\right)\right)}$

input

```
int(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))^(1/2),x,method=_RETURNVERBOSE)
```

output

```
1/f*((2*cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1/2*e)+1)*a)^(1/2)*(2*ln(2*(-cos(1/2*f*x+1/2*e)+sin(1/2*f*x+1/2*e))/(cos(1/2*f*x+1/2*e)+1))-ln(-cot(1/2*f*x+1/2*e)+csc(1/2*f*x+1/2*e)-1)-ln(-cot(1/2*f*x+1/2*e)+csc(1/2*f*x+1/2*e)+1)-ln(csc(1/2*f*x+1/2*e)-cot(1/2*f*x+1/2*e)))*(-cos(1/2*f*x+1/2*e)+sin(1/2*f*x+1/2*e))/(cos(1/2*f*x+1/2*e)+sin(1/2*f*x+1/2*e))/((-2*cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1/2*e)-1)*c)^(1/2)
```

Fricas [F]

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c-c\sin(e+fx)}} dx = \int \frac{\sqrt{a\sin(fx+e)+a}\csc(fx+e)}{\sqrt{-c\sin(fx+e)+c}} dx$$

input

```
integrate(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))^(1/2),x, algorith="fricas")
```

output

```
integral(-sqrt(a*sin(f*x + e) + a)*sqrt(-c*sin(f*x + e) + c)*csc(f*x + e)/(c*sin(f*x + e) - c), x)
```

Sympy [F]

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c-c\sin(e+fx)}} dx = \int \frac{\sqrt{a(\sin(e+fx)+1)}\csc(e+fx)}{\sqrt{-c(\sin(e+fx)-1)}} dx$$

input

```
integrate(csc(f*x+e)*(a+a*sin(f*x+e))**(1/2)/(c-c*sin(f*x+e))**(1/2),x)
```

output

```
Integral(sqrt(a*(sin(e + f*x) + 1))*csc(e + f*x)/sqrt(-c*(sin(e + f*x) - 1)), x)
```

Maxima [A] (verification not implemented)

Time = 0.13 (sec) , antiderivative size = 59, normalized size of antiderivative = 0.58

$$\int \frac{\csc(e + fx) \sqrt{a + a \sin(e + fx)}}{\sqrt{c - c \sin(e + fx)}} dx = \frac{2\sqrt{a} \log\left(\frac{\sin(fx+e)}{\cos(fx+e)+1} - 1\right)}{\sqrt{c}} - \frac{\sqrt{a} \log\left(\frac{\sin(fx+e)}{\cos(fx+e)+1}\right)}{\sqrt{c}}$$

input `integrate(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))^(1/2),x, algorith="maxima")`

output `(2*sqrt(a)*log(sin(f*x + e)/(cos(f*x + e) + 1) - 1)/sqrt(c) - sqrt(a)*log(sin(f*x + e)/(cos(f*x + e) + 1))/sqrt(c))/f`

Giac [A] (verification not implemented)

Time = 0.36 (sec) , antiderivative size = 1, normalized size of antiderivative = 0.01

$$\int \frac{\csc(e + fx) \sqrt{a + a \sin(e + fx)}}{\sqrt{c - c \sin(e + fx)}} dx = 0$$

input `integrate(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))^(1/2),x, algorith="giac")`

output 0

Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e + fx) \sqrt{a + a \sin(e + fx)}}{\sqrt{c - c \sin(e + fx)}} dx = \int \frac{\sqrt{a + a \sin(e + fx)}}{\sin(e + fx) \sqrt{c - c \sin(e + fx)}} dx$$

input `int((a + a*sin(e + f*x))^(1/2)/(sin(e + f*x)*(c - c*sin(e + f*x))^(1/2)),x)`

output `int((a + a*sin(e + f*x))^(1/2)/(sin(e + f*x)*(c - c*sin(e + f*x))^(1/2)), x)`

Reduce [F]

$$\int \frac{\csc(e + fx) \sqrt{a + a \sin(e + fx)}}{\sqrt{c - c \sin(e + fx)}} dx$$

$$= - \frac{\sqrt{c} \sqrt{a} \left(\int \frac{\sqrt{\sin(fx+e)+1} \sqrt{-\sin(fx+e)+1} \csc(fx+e)}{\sin(fx+e)-1} dx \right)}{c}$$

input `int(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))^(1/2),x)`

output `(- sqrt(c)*sqrt(a)*int((sqrt(sin(e + f*x) + 1)*sqrt(- sin(e + f*x) + 1)* csc(e + f*x))/(sin(e + f*x) - 1),x))/c`

3.21
$$\int \frac{\csc(e+fx)\sqrt{c-c\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}} dx$$

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Rubi [A] (verified)	194
Maple [B] (verified)	197
Fricas [F]	198
Sympy [F]	198
Maxima [A] (verification not implemented)	199
Giac [A] (verification not implemented)	199
Mupad [F(-1)]	200
Reduce [F]	200

Optimal result

Integrand size = 36, antiderivative size = 100

$$\int \frac{\csc(e+fx)\sqrt{c-c\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}} dx$$

$$= -\frac{c\cos(e+fx)\log(1+\sin(e+fx))}{f\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}} + \frac{\log(\sin(e+fx))\sec(e+fx)\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}}{af}$$

output

```
-c*cos(f*x+e)*ln(1+sin(f*x+e))/f/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))^(1/2)+ln(sin(f*x+e))*sec(f*x+e)*(a+a*sin(f*x+e))^(1/2)*(c-c*sin(f*x+e))^(1/2)/a/f
```

Mathematica [A] (verified)

Time = 1.40 (sec) , antiderivative size = 61, normalized size of antiderivative = 0.61

$$\int \frac{\csc(e + fx)\sqrt{c - c\sin(e + fx)}}{\sqrt{a + a\sin(e + fx)}} dx$$

$$= \frac{(\log(\sin(e + fx)) - \log(1 + \sin(e + fx))) \sec(e + fx)\sqrt{a(1 + \sin(e + fx))}\sqrt{c - c\sin(e + fx)}}{af}$$

input

```
Integrate[(Csc[e + f*x]*Sqrt[c - c*Sin[e + f*x]])/Sqrt[a + a*Sin[e + f*x]]
,x]
```

output

```
((Log[Sin[e + f*x]] - Log[1 + Sin[e + f*x]])*Sec[e + f*x]*Sqrt[a*(1 + Sin[
e + f*x]])*Sqrt[c - c*Sin[e + f*x]])/(a*f)
```

Rubi [A] (verified)

Time = 1.21 (sec) , antiderivative size = 104, normalized size of antiderivative = 1.04, number of steps used = 12, number of rules used = 11, $\frac{\text{number of rules}}{\text{integrand size}} = 0.306$, Rules used = {3042, 3421, 3042, 3216, 3042, 3146, 16, 3427, 3042, 25, 3956}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e + fx)\sqrt{c - c\sin(e + fx)}}{\sqrt{a\sin(e + fx) + a}} dx$$

$$\downarrow 3042$$

$$\int \frac{\sqrt{c - c\sin(e + fx)}}{\sin(e + fx)\sqrt{a\sin(e + fx) + a}} dx$$

$$\downarrow 3421$$

$$\frac{\int \csc(e + fx)\sqrt{\sin(e + fx)a + a}\sqrt{c - c\sin(e + fx)} dx}{a} - \int \frac{\sqrt{c - c\sin(e + fx)}}{\sqrt{\sin(e + fx)a + a}} dx$$

$$\downarrow 3042$$

$$\begin{aligned}
& \frac{\int \frac{\sqrt{\sin(e+fx)a+a\sqrt{c-c\sin(e+fx)}}}{\sin(e+fx)} dx}{a} - \int \frac{\sqrt{c-c\sin(e+fx)}}{\sqrt{\sin(e+fx)a+a}} dx \\
& \quad \downarrow \text{3216} \\
& \frac{\int \frac{\sqrt{\sin(e+fx)a+a\sqrt{c-c\sin(e+fx)}}}{\sin(e+fx)} dx}{a} - \frac{ac \cos(e+fx) \int \frac{\cos(e+fx)}{\sin(e+fx)a+a} dx}{\sqrt{a \sin(e+fx) + a\sqrt{c-c\sin(e+fx)}}} \\
& \quad \downarrow \text{3042} \\
& \frac{\int \frac{\sqrt{\sin(e+fx)a+a\sqrt{c-c\sin(e+fx)}}}{\sin(e+fx)} dx}{a} - \frac{ac \cos(e+fx) \int \frac{\cos(e+fx)}{\sin(e+fx)a+a} dx}{\sqrt{a \sin(e+fx) + a\sqrt{c-c\sin(e+fx)}}} \\
& \quad \downarrow \text{3146} \\
& \frac{\int \frac{\sqrt{\sin(e+fx)a+a\sqrt{c-c\sin(e+fx)}}}{\sin(e+fx)} dx}{a} - \frac{c \cos(e+fx) \int \frac{1}{\sin(e+fx)a+a} d(a \sin(e+fx))}{f \sqrt{a \sin(e+fx) + a\sqrt{c-c\sin(e+fx)}}} \\
& \quad \downarrow \text{16} \\
& \frac{\int \frac{\sqrt{\sin(e+fx)a+a\sqrt{c-c\sin(e+fx)}}}{\sin(e+fx)} dx}{a} - \frac{c \cos(e+fx) \log(a \sin(e+fx) + a)}{f \sqrt{a \sin(e+fx) + a\sqrt{c-c\sin(e+fx)}}} \\
& \quad \downarrow \text{3427} \\
& \frac{\sec(e+fx) \sqrt{a \sin(e+fx) + a\sqrt{c-c\sin(e+fx)}} \int \cot(e+fx) dx}{a} - \\
& \quad \frac{c \cos(e+fx) \log(a \sin(e+fx) + a)}{f \sqrt{a \sin(e+fx) + a\sqrt{c-c\sin(e+fx)}}} \\
& \quad \downarrow \text{3042} \\
& \frac{\sec(e+fx) \sqrt{a \sin(e+fx) + a\sqrt{c-c\sin(e+fx)}} \int -\tan(e+fx + \frac{\pi}{2}) dx}{a} - \\
& \quad \frac{c \cos(e+fx) \log(a \sin(e+fx) + a)}{f \sqrt{a \sin(e+fx) + a\sqrt{c-c\sin(e+fx)}}} \\
& \quad \downarrow \text{25} \\
& \frac{\sec(e+fx) \sqrt{a \sin(e+fx) + a\sqrt{c-c\sin(e+fx)}} \int \tan(\frac{1}{2}(2e+\pi) + fx) dx}{a} - \\
& \quad \frac{c \cos(e+fx) \log(a \sin(e+fx) + a)}{f \sqrt{a \sin(e+fx) + a\sqrt{c-c\sin(e+fx)}}} \\
& \quad \downarrow \text{3956}
\end{aligned}$$

$$\frac{\sec(e + fx)\sqrt{a \sin(e + fx) + a}\sqrt{c - c \sin(e + fx)} \log(-\sin(e + fx))}{af} - \frac{c \cos(e + fx) \log(a \sin(e + fx) + a)}{f\sqrt{a \sin(e + fx) + a}\sqrt{c - c \sin(e + fx)}}$$

input `Int[(Csc[e + f*x]*Sqrt[c - c*Sin[e + f*x]])/Sqrt[a + a*Sin[e + f*x]],x]`

output `-((c*Cos[e + f*x]*Log[a + a*Sin[e + f*x]])/(f*Sqrt[a + a*Sin[e + f*x]]*Sqrt[c - c*Sin[e + f*x])) + (Log[-Sin[e + f*x]]*Sec[e + f*x]*Sqrt[a + a*Sin[e + f*x]]*Sqrt[c - c*Sin[e + f*x]])/(a*f)`

Defintions of rubi rules used

rule 16 `Int[(c_)/((a_) + (b_)*(x_)), x_Symbol] := Simp[c*(Log[RemoveContent[a + b*x, x]]/b), x] /; FreeQ[{a, b, c}, x]`

rule 25 `Int[-(Fx_), x_Symbol] := Simp[Identity[-1] Int[Fx, x], x]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3146 `Int[cos[(e_) + (f_)*(x_)]^(p_)*((a_) + (b_)*sin[(e_) + (f_)*(x_)]^(m_)), x_Symbol] := Simp[1/(b^p*f) Subst[Int[(a + x)^(m + (p - 1)/2)*(a - x)^(p - 1)/2], x], x, b*Sin[e + f*x], x] /; FreeQ[{a, b, e, f, m}, x] && IntegerQ[(p - 1)/2] && EqQ[a^2 - b^2, 0] && (GeQ[p, -1] || !IntegerQ[m + 1/2])`

rule 3216 `Int[Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)]]/Sqrt[(c_) + (d_)*sin[(e_) + (f_)*(x_)]], x_Symbol] := Simp[a*c*(Cos[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]])) Int[Cos[e + f*x]/(c + d*Sin[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && EqQ[b*c + a*d, 0] && EqQ[a^2 - b^2, 0]`

rule 3421

```
Int[Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)]]/(sin[(e_) + (f_)*(x_)]*Sqrt[(c_) + (d_)*sin[(e_) + (f_)*(x_)]]), x_Symbol] := Simp[-d/c Int[Sqrt[a + b*Sin[e + f*x]]/Sqrt[c + d*Sin[e + f*x]], x], x] + Simp[1/c Int[Sqrt[a + b*Sin[e + f*x]]*(Sqrt[c + d*Sin[e + f*x]]/Sin[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0] && EqQ[b*c + a*d, 0]
```

rule 3427

```
Int[(Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)]]*Sqrt[(c_) + (d_)*sin[(e_) + (f_)*(x_)]])/sin[(e_) + (f_)*(x_)], x_Symbol] := Simp[Sqrt[a + b*Sin[e + f*x]]*(Sqrt[c + d*Sin[e + f*x]]/Cos[e + f*x]) Int[Cot[e + f*x], x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0] && EqQ[c^2 - d^2, 0]
```

rule 3956

```
Int[tan[(c_) + (d_)*(x_)], x_Symbol] := Simp[-Log[RemoveContent[Cos[c + d*x], x]]/d, x] /; FreeQ[{c, d}, x]
```

Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 206 vs. $2(92) = 184$.

Time = 13.50 (sec) , antiderivative size = 207, normalized size of antiderivative = 2.07

method	result
default	$-\frac{\sqrt{-\left(2\cos\left(\frac{fx}{2}+\frac{e}{2}\right)\sin\left(\frac{fx}{2}+\frac{e}{2}\right)-1\right)}c\left(\ln\left(-\cot\left(\frac{fx}{2}+\frac{e}{2}\right)+\csc\left(\frac{fx}{2}+\frac{e}{2}\right)+1\right)-2\ln\left(-\frac{2\left(\cos\left(\frac{fx}{2}+\frac{e}{2}\right)+\sin\left(\frac{fx}{2}+\frac{e}{2}\right)\right)}{\cos\left(\frac{fx}{2}+\frac{e}{2}\right)+1}\right)+\ln\left(\csc\left(\frac{fx}{2}+\frac{e}{2}\right)\right)}{f\left(-\cos\left(\frac{fx}{2}+\frac{e}{2}\right)+\sin\left(\frac{fx}{2}+\frac{e}{2}\right)\right)\sqrt{\left(2\cos\left(\frac{fx}{2}+\frac{e}{2}\right)\sin\left(\frac{fx}{2}+\frac{e}{2}\right)-1\right)}}$

input

```
int(csc(f*x+e)*(c-c*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2),x,method=_RETURNVERBOSE)
```

output

```
-1/f*(-(2*cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1/2*e)-1)*c)^(1/2)*(ln(-cot(1/2*f*x+1/2*e)+csc(1/2*f*x+1/2*e)+1)-2*ln(-2*(cos(1/2*f*x+1/2*e)+sin(1/2*f*x+1/2*e))/(cos(1/2*f*x+1/2*e)+1))+ln(csc(1/2*f*x+1/2*e)-cot(1/2*f*x+1/2*e))+ln(-cot(1/2*f*x+1/2*e)+csc(1/2*f*x+1/2*e)-1))*(cos(1/2*f*x+1/2*e)+sin(1/2*f*x+1/2*e))/(-cos(1/2*f*x+1/2*e)+sin(1/2*f*x+1/2*e))/((2*cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1/2*e)+1)*a)^(1/2)
```

Fricas [F]

$$\int \frac{\csc(e+fx)\sqrt{c-c\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}} dx = \int \frac{\sqrt{-c\sin(fx+e)+c}\csc(fx+e)}{\sqrt{a\sin(fx+e)+a}} dx$$

input

```
integrate(csc(f*x+e)*(c-c*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2),x, algo
rithm="fricas")
```

output

```
integral(sqrt(-c*sin(f*x + e) + c)*csc(f*x + e)/sqrt(a*sin(f*x + e) + a),
x)
```

Sympy [F]

$$\int \frac{\csc(e+fx)\sqrt{c-c\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}} dx = \int \frac{\sqrt{-c(\sin(e+fx)-1)}\csc(e+fx)}{\sqrt{a(\sin(e+fx)+1)}} dx$$

input

```
integrate(csc(f*x+e)*(c-c*sin(f*x+e))**(1/2)/(a+a*sin(f*x+e))**(1/2),x)
```

output

```
Integral(sqrt(-c*(sin(e + f*x) - 1))*csc(e + f*x)/sqrt(a*(sin(e + f*x) + 1
)), x)
```

Maxima [A] (verification not implemented)

Time = 0.13 (sec) , antiderivative size = 59, normalized size of antiderivative = 0.59

$$\int \frac{\csc(e + fx) \sqrt{c - c \sin(e + fx)}}{\sqrt{a + a \sin(e + fx)}} dx = \frac{2\sqrt{c} \log\left(\frac{\sin(fx+e)}{\cos(fx+e)+1} + 1\right)}{\sqrt{a}} - \frac{\sqrt{c} \log\left(\frac{\sin(fx+e)}{\cos(fx+e)+1}\right)}{\sqrt{a}}$$

input `integrate(csc(f*x+e)*(c-c*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2),x, algorith="maxima")`

output `(2*sqrt(c)*log(sin(f*x + e)/(cos(f*x + e) + 1) + 1)/sqrt(a) - sqrt(c)*log(sin(f*x + e)/(cos(f*x + e) + 1))/sqrt(a))/f`

Giac [A] (verification not implemented)

Time = 0.32 (sec) , antiderivative size = 112, normalized size of antiderivative = 1.12

$$\int \frac{\csc(e + fx) \sqrt{c - c \sin(e + fx)}}{\sqrt{a + a \sin(e + fx)}} dx$$

$$= \frac{\sqrt{2}\sqrt{a}\sqrt{c} \left(\frac{\sqrt{2} \log\left(-\sin\left(-\frac{1}{4}\pi + \frac{1}{2}fx + \frac{1}{2}e\right)^2 + 1\right)}{\operatorname{asgn}\left(\cos\left(-\frac{1}{4}\pi + \frac{1}{2}fx + \frac{1}{2}e\right)\right)} - \frac{\sqrt{2} \log\left(\left|2 \sin\left(-\frac{1}{4}\pi + \frac{1}{2}fx + \frac{1}{2}e\right)^2 - 1\right|\right)}{\operatorname{asgn}\left(\cos\left(-\frac{1}{4}\pi + \frac{1}{2}fx + \frac{1}{2}e\right)\right)} \right) \operatorname{sgn}\left(\sin\left(-\frac{1}{4}\pi + \frac{1}{2}fx + \frac{1}{2}e\right)\right)}{2f}$$

input `integrate(csc(f*x+e)*(c-c*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2),x, algorith="giac")`

output `1/2*sqrt(2)*sqrt(a)*sqrt(c)*(sqrt(2)*log(-sin(-1/4*pi + 1/2*f*x + 1/2*e)^2 + 1)/(a*sgn(cos(-1/4*pi + 1/2*f*x + 1/2*e)))) - sqrt(2)*log(abs(2*sin(-1/4*pi + 1/2*f*x + 1/2*e)^2 - 1))/(a*sgn(cos(-1/4*pi + 1/2*f*x + 1/2*e))))*sgn(sin(-1/4*pi + 1/2*f*x + 1/2*e))/f`

Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e + fx) \sqrt{c - c \sin(e + fx)}}{\sqrt{a + a \sin(e + fx)}} dx = \int \frac{\sqrt{c - c \sin(e + fx)}}{\sin(e + fx) \sqrt{a + a \sin(e + fx)}} dx$$

input `int((c - c*sin(e + f*x))^(1/2)/(sin(e + f*x)*(a + a*sin(e + f*x))^(1/2)),x)`

output `int((c - c*sin(e + f*x))^(1/2)/(sin(e + f*x)*(a + a*sin(e + f*x))^(1/2)),x)`

Reduce [F]

$$\int \frac{\csc(e + fx) \sqrt{c - c \sin(e + fx)}}{\sqrt{a + a \sin(e + fx)}} dx = \frac{\sqrt{c} \sqrt{a} \left(\int \frac{\sqrt{\sin(fx+e)+1} \sqrt{-\sin(fx+e)+1} \csc(fx+e)}{\sin(fx+e)+1} dx \right)}{a}$$

input `int(csc(f*x+e)*(c-c*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2),x)`

output `(sqrt(c)*sqrt(a)*int((sqrt(sin(e + f*x) + 1)*sqrt(- sin(e + f*x) + 1)*csc(e + f*x))/(sin(e + f*x) + 1),x))/a`

3.22
$$\int \frac{\csc(e+fx)}{\sqrt{a+a \sin(e+fx)}\sqrt{c-c \sin(e+fx)}} dx$$

Optimal result	201
Mathematica [A] (verified)	201
Rubi [A] (verified)	202
Maple [B] (verified)	203
Fricas [A] (verification not implemented)	204
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Giac [F(-2)]	205
Mupad [F(-1)]	206
Reduce [F]	206

Optimal result

Integrand size = 36, antiderivative size = 46

$$\int \frac{\csc(e+fx)}{\sqrt{a+a \sin(e+fx)}\sqrt{c-c \sin(e+fx)}} dx = \frac{\cos(e+fx) \log(\tan(e+fx))}{f \sqrt{a+a \sin(e+fx)}\sqrt{c-c \sin(e+fx)}}$$

output `cos(f*x+e)*ln(tan(f*x+e))/f/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))^(1/2)`

Mathematica [A] (verified)

Time = 0.19 (sec) , antiderivative size = 63, normalized size of antiderivative = 1.37

$$\int \frac{\csc(e+fx)}{\sqrt{a+a \sin(e+fx)}\sqrt{c-c \sin(e+fx)}} dx = \frac{(\log(\cos(e+fx)) - \log(\sin(e+fx))) \sec(e+fx) \sqrt{a(1+\sin(e+fx))} \sqrt{c-c \sin(e+fx)}}{acf}$$

input `Integrate[Csc[e + f*x]/(Sqrt[a + a*Sin[e + f*x]]*Sqrt[c - c*Sin[e + f*x]]),x]`

output

$$-\left(\left(\text{Log}[\text{Cos}[e + f*x]] - \text{Log}[\text{Sin}[e + f*x]]\right)*\text{Sec}[e + f*x]*\text{Sqrt}[a*(1 + \text{Sin}[e + f*x])]*\text{Sqrt}[c - c*\text{Sin}[e + f*x]]\right)/(a*c*f)$$
Rubi [A] (verified)

Time = 0.43 (sec) , antiderivative size = 46, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.139$, Rules used = {3042, 3425, 3042, 3100, 14}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \frac{\csc(e + fx)}{\sqrt{a \sin(e + fx) + a} \sqrt{c - c \sin(e + fx)}} dx \\ & \quad \downarrow \text{3042} \\ & \int \frac{1}{\sin(e + fx) \sqrt{a \sin(e + fx) + a} \sqrt{c - c \sin(e + fx)}} dx \\ & \quad \downarrow \text{3425} \\ & \frac{\cos(e + fx) \int \csc(e + fx) \sec(e + fx) dx}{\sqrt{a \sin(e + fx) + a} \sqrt{c - c \sin(e + fx)}} \\ & \quad \downarrow \text{3042} \\ & \frac{\cos(e + fx) \int \csc(e + fx) \sec(e + fx) dx}{\sqrt{a \sin(e + fx) + a} \sqrt{c - c \sin(e + fx)}} \\ & \quad \downarrow \text{3100} \\ & \frac{\cos(e + fx) \int \cot(e + fx) d \tan(e + fx)}{f \sqrt{a \sin(e + fx) + a} \sqrt{c - c \sin(e + fx)}} \\ & \quad \downarrow \text{14} \\ & \frac{\cos(e + fx) \log(\tan(e + fx))}{f \sqrt{a \sin(e + fx) + a} \sqrt{c - c \sin(e + fx)}} \end{aligned}$$

input

$$\text{Int}[\text{Csc}[e + f*x]/(\text{Sqrt}[a + a*\text{Sin}[e + f*x]]*\text{Sqrt}[c - c*\text{Sin}[e + f*x]]),x]$$

output $(\text{Cos}[e + f*x]*\text{Log}[\text{Tan}[e + f*x]])/(f*\text{Sqrt}[a + a*\text{Sin}[e + f*x]]*\text{Sqrt}[c - c*\text{Sin}[e + f*x]])$

Defintions of rubi rules used

rule 14 $\text{Int}[(a_)/(x_), x_Symbol] \rightarrow \text{Simp}[a*\text{Log}[x], x] /; \text{FreeQ}[a, x]$

rule 3042 $\text{Int}[u_, x_Symbol] \rightarrow \text{Int}[\text{DeactivateTrig}[u, x], x] /; \text{FunctionOfTrigOfLinearQ}[u, x]$

rule 3100 $\text{Int}[\text{csc}[e_] + (f_)*(x_)]^{(m_)}*\text{sec}[e_] + (f_)*(x_)]^{(n_)}, x_Symbol] \rightarrow \text{Simp}[1/f \text{ Subst}[\text{Int}[(1 + x^2)^{(m+n)/2 - 1}/x^m, x], x, \text{Tan}[e + f*x]] , x] /; \text{FreeQ}\{e, f\}, x] \&\& \text{IntegersQ}[m, n, (m+n)/2]$

rule 3425 $\text{Int}[1/(\text{sin}[e_] + (f_)*(x_))*\text{Sqrt}[(a_) + (b_)*\text{sin}[e_] + (f_)*(x_)]*\text{Sqrt}[(c_) + (d_)*\text{sin}[e_] + (f_)*(x_)]], x_Symbol] \rightarrow \text{Simp}[\text{Cos}[e + f*x]/(\text{Sqrt}[a + b*\text{Sin}[e + f*x]]*\text{Sqrt}[c + d*\text{Sin}[e + f*x]]) \text{ Int}[1/(\text{Cos}[e + f*x]*\text{Sin}[e + f*x]), x], x] /; \text{FreeQ}\{a, b, c, d, e, f\}, x] \&\& \text{NeQ}[b*c - a*d, 0] \&\& \text{EqQ}[a^2 - b^2, 0] \&\& \text{EqQ}[c^2 - d^2, 0]$

Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 217 vs. $2(42) = 84$.

Time = 11.55 (sec) , antiderivative size = 218, normalized size of antiderivative = 4.74

method	result
default	$\frac{\left(\ln\left(-\cot\left(\frac{fx}{2} + \frac{e}{2}\right) + \csc\left(\frac{fx}{2} + \frac{e}{2}\right) + 1\right) - \ln\left(-\frac{2\left(\cos\left(\frac{fx}{2} + \frac{e}{2}\right) + \sin\left(\frac{fx}{2} + \frac{e}{2}\right)\right)}{\cos\left(\frac{fx}{2} + \frac{e}{2}\right) + 1}\right) + \ln\left(-\cot\left(\frac{fx}{2} + \frac{e}{2}\right) + \csc\left(\frac{fx}{2} + \frac{e}{2}\right) - 1\right) - \ln\left(\frac{-2\cos\left(\frac{fx}{2} + \frac{e}{2}\right)}{\cos\left(\frac{fx}{2} + \frac{e}{2}\right) + 1}\right)\right)}{f\sqrt{\left(2\cos\left(\frac{fx}{2} + \frac{e}{2}\right)\sin\left(\frac{fx}{2} + \frac{e}{2}\right) + 1\right)}a\sqrt{-\left(2\cos\left(\frac{fx}{2} + \frac{e}{2}\right)\sin\left(\frac{fx}{2} + \frac{e}{2}\right) + 1\right)}}$

input $\text{int}(\text{csc}(f*x+e)/(a+a*\text{sin}(f*x+e))^{(1/2)}/(c-c*\text{sin}(f*x+e))^{(1/2)}, x, \text{method}=_RET \text{URNVERBOSE})$

output

```
1/f*(ln(-cot(1/2*f*x+1/2*e)+csc(1/2*f*x+1/2*e)+1)-ln(-2*(cos(1/2*f*x+1/2*e)
)+sin(1/2*f*x+1/2*e))/(cos(1/2*f*x+1/2*e)+1)+ln(-cot(1/2*f*x+1/2*e)+csc(1
/2*f*x+1/2*e)-1)-ln(2*(-cos(1/2*f*x+1/2*e)+sin(1/2*f*x+1/2*e))/(cos(1/2*f*
x+1/2*e)+1))+ln(csc(1/2*f*x+1/2*e)-cot(1/2*f*x+1/2*e)))*(-1+2*cos(1/2*f*x+
1/2*e)^2)/((2*cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1/2*e)+1)*a)^(1/2)/(-(2*cos(1
/2*f*x+1/2*e)*sin(1/2*f*x+1/2*e)-1)*c)^(1/2)
```

Fricas [A] (verification not implemented)

Time = 0.16 (sec) , antiderivative size = 194, normalized size of antiderivative = 4.22

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}} dx$$

$$= \left[\frac{\sqrt{ac} \log \left(-\frac{4(2ac\cos(fx+e)^5 - 2ac\cos(fx+e)^3 + ac\cos(fx+e) - \sqrt{ac}(2\cos(fx+e)^2 - 1))\sqrt{a\sin(fx+e)+a}\sqrt{-c\sin(fx+e)+c}}{\cos(fx+e)^5 - \cos(fx+e)^3} \right)}{2acf} \right], \sqrt{-}$$

input

```
integrate(csc(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))^(1/2),x, algo
rithm="fricas")
```

output

```
[1/2*sqrt(a*c)*log(-4*(2*a*c*cos(f*x + e)^5 - 2*a*c*cos(f*x + e)^3 + a*c*c
os(f*x + e) - sqrt(a*c)*(2*cos(f*x + e)^2 - 1)*sqrt(a*sin(f*x + e) + a)*sq
rt(-c*sin(f*x + e) + c))/(cos(f*x + e)^5 - cos(f*x + e)^3)/(a*c*f), sqrt(
-a*c)*arctan(sqrt(-a*c)*(2*cos(f*x + e)^2 - 1)*sqrt(a*sin(f*x + e) + a)*sq
rt(-c*sin(f*x + e) + c)/(a*c*cos(f*x + e)))/(a*c*f)]
```

Sympy [F]

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}\sqrt{c-c\sin(e+fx)}} dx$$

$$= \int \frac{\csc(e+fx)}{\sqrt{a(\sin(e+fx)+1)}\sqrt{-c(\sin(e+fx)-1)}} dx$$

input

```
integrate(csc(f*x+e)/(a+a*sin(f*x+e))**(1/2)/(c-c*sin(f*x+e))**(1/2),x)
```

output

```
Integral(csc(e + f*x)/(sqrt(a*(sin(e + f*x) + 1))*sqrt(-c*(sin(e + f*x) - 1))), x)
```

Maxima [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 0.23 (sec) , antiderivative size = 190, normalized size of antiderivative = 4.13

$$\int \frac{\csc(e + fx)}{\sqrt{a + a \sin(e + fx)} \sqrt{c - c \sin(e + fx)}} dx = \frac{(-1)^{4 \cos(2fx+2e)} \cosh(4\pi \sin(2fx + 2e)) \log\left(\frac{16(\cos(2fx+2e)^2 + \sin(2fx+2e)^2 + 2\cos(2fx+2e)+1)}{ac|e^{2i fx+2ie}-1|^2}\right) - 2i(-1)^{4 \cos(2fx+2e)}}{2\sqrt{a}\sqrt{c}f}$$

input

```
integrate(csc(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))^(1/2),x, algo
rithm="maxima")
```

output

```
-1/2*((-1)^(4*cos(2*f*x + 2*e))*cosh(4*pi*sin(2*f*x + 2*e))*log(16*(cos(2*
f*x + 2*e)^2 + sin(2*f*x + 2*e)^2 + 2*cos(2*f*x + 2*e) + 1)/(a*c*abs(e^(2*
I*f*x + 2*I*e) - 1)^2)) - 2*I*(-1)^(4*cos(2*f*x + 2*e))*arctan2(4*sin(2*f*
x + 2*e)/(sqrt(a)*sqrt(c)*abs(e^(2*I*f*x + 2*I*e) - 1)), 4*(cos(2*f*x + 2*
e) + 1)/(sqrt(a)*sqrt(c)*abs(e^(2*I*f*x + 2*I*e) - 1)))*sinh(4*pi*sin(2*f*
x + 2*e)))/(sqrt(a)*sqrt(c)*f)
```

Giac [F(-2)]

Exception generated.

$$\int \frac{\csc(e + fx)}{\sqrt{a + a \sin(e + fx)} \sqrt{c - c \sin(e + fx)}} dx = \text{Exception raised: TypeError}$$

input

```
integrate(csc(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))^(1/2),x, algo
rithm="giac")
```

output

```
Exception raised: TypeError >> an error occurred running a Giac command:IN
PUT:sage2:=int(sage0,sageVARx);;OUTPUT:sym2poly/r2sym(const gen & e,const
index_m & i,const vecteur & l) Error: Bad Argument Value
```

Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e + fx)}{\sqrt{a + a \sin(e + fx)} \sqrt{c - c \sin(e + fx)}} dx$$

$$= \int \frac{1}{\sin(e + fx) \sqrt{a + a \sin(e + fx)} \sqrt{c - c \sin(e + fx)}} dx$$

input

```
int(1/(sin(e + f*x)*(a + a*sin(e + f*x))^(1/2)*(c - c*sin(e + f*x))^(1/2))
,x)
```

output

```
int(1/(sin(e + f*x)*(a + a*sin(e + f*x))^(1/2)*(c - c*sin(e + f*x))^(1/2))
, x)
```

Reduce [F]

$$\int \frac{\csc(e + fx)}{\sqrt{a + a \sin(e + fx)} \sqrt{c - c \sin(e + fx)}} dx$$

$$= - \frac{\sqrt{c} \sqrt{a} \left(\int \frac{\sqrt{\sin(fx+e)+1} \sqrt{-\sin(fx+e)+1} \csc(fx+e) dx}{\sin(fx+e)^2 - 1} \right)}{ac}$$

input

```
int(csc(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c-c*sin(f*x+e))^(1/2),x)
```

output

```
( - sqrt(c)*sqrt(a)*int((sqrt(sin(e + f*x) + 1)*sqrt(- sin(e + f*x) + 1)*
csc(e + f*x))/(sin(e + f*x)**2 - 1),x))/(a*c)
```

3.23 $\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{c+d\sin(e+fx)} dx$

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Optimal result

Integrand size = 33, antiderivative size = 105

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{c+d\sin(e+fx)} dx = -\frac{2\sqrt{a}\operatorname{arctanh}\left(\frac{\sqrt{a}\cos(e+fx)}{\sqrt{a+a\sin(e+fx)}}\right)}{cf} + \frac{2\sqrt{a}\sqrt{d}\operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{d}\cos(e+fx)}{\sqrt{c+d}\sqrt{a+a\sin(e+fx)}}\right)}{c\sqrt{c+df}}$$

output `-2*a^(1/2)*arctanh(a^(1/2)*cos(f*x+e)/(a+a*sin(f*x+e))^(1/2))/c/f+2*a^(1/2)*d^(1/2)*arctanh(a^(1/2)*d^(1/2)*cos(f*x+e)/(c+d)^(1/2)/(a+a*sin(f*x+e))^(1/2))/c/(c+d)^(1/2)/f`

Mathematica [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.

Time = 10.24 (sec) , antiderivative size = 746, normalized size of antiderivative = 7.10

$$\int \frac{\csc(e + fx) \sqrt{a + a \sin(e + fx)}}{c + d \sin(e + fx)} dx =$$

$$\left(\frac{1}{8} - \frac{i}{8}\right) \left((4 + 4i) \sqrt{c + d} \left(\log \left(1 + \cos \left(\frac{1}{2}(e + fx) \right) - \sin \left(\frac{1}{2}(e + fx) \right) \right) - \log \left(1 - \cos \left(\frac{1}{2}(e + fx) \right) \right) + \dots \right) \right)$$

input `Integrate[(Csc[e + f*x]*Sqrt[a + a*Sin[e + f*x]])/(c + d*Sin[e + f*x]),x]`

output

```
((-1/8 + I/8)*((4 + 4*I)*Sqrt[c + d]*(Log[1 + Cos[(e + f*x)/2] - Sin[(e + f*x)/2]] - Log[1 - Cos[(e + f*x)/2] + Sin[(e + f*x)/2]]) + Sqrt[d]*RootSum[-d + (2*I)*c*E^(I*e)**1^2 + d*E^((2*I)*e)**1^4 & , ((1 + I)*d*Sqrt[E^((-I)*e)]*f*x - (2 - 2*I)*d*Sqrt[E^((-I)*e)]*Log[E^((I/2)*f*x) - #1] - I*Sqrt[d]*Sqrt[c + d]*f*x**1 + 2*Sqrt[d]*Sqrt[c + d]*Log[E^((I/2)*f*x) - #1]**1 + ((1 - I)*c*f*x**1^2)/Sqrt[E^((-I)*e)] + ((2 + 2*I)*c*Log[E^((I/2)*f*x) - #1]**1^2)/Sqrt[E^((-I)*e)] - Sqrt[d]*Sqrt[c + d]*E^(I*e)*f*x**1^3 - (2*I)*Sqrt[d]*Sqrt[c + d]*E^(I*e)*Log[E^((I/2)*f*x) - #1]**1^3)/((-I)*d - c*E^(I*e)**1^2) & ]*(Cos[e/2] + I*Sin[e/2]) + Sqrt[d]*RootSum[-d + (2*I)*c*E^(I*e)**1^2 + d*E^((2*I)*e)**1^4 & , ((1 - I)*d*Sqrt[E^((-I)*e)]*f*x + (2 + 2*I)*d*Sqrt[E^((-I)*e)]*Log[E^((I/2)*f*x) - #1] + Sqrt[d]*Sqrt[c + d]*f*x**1 + (2*I)*Sqrt[d]*Sqrt[c + d]*Log[E^((I/2)*f*x) - #1]**1 - ((1 + I)*c*f*x**1^2)/Sqrt[E^((-I)*e)] + ((2 - 2*I)*c*Log[E^((I/2)*f*x) - #1]**1^2)/Sqrt[E^((-I)*e)] - I*Sqrt[d]*Sqrt[c + d]*E^(I*e)*f*x**1^3 + 2*Sqrt[d]*Sqrt[c + d]*E^(I*e)*Log[E^((I/2)*f*x) - #1]**1^3)/(d - I*c*E^(I*e)**1^2) & ]*(Cos[e/2] + I*Sin[e/2]))*Sqrt[a*(1 + Sin[e + f*x])]/(c*Sqrt[c + d]*f*(Cos[(e + f*x)/2] + Sin[(e + f*x)/2]))
```

Rubi [A] (verified)

Time = 0.54 (sec) , antiderivative size = 105, normalized size of antiderivative = 1.00, number of steps used = 7, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}} = 0.182$, Rules used = {3042, 3413, 3042, 3252, 219, 221}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
& \int \frac{\csc(e+fx)\sqrt{a\sin(e+fx)+a}}{c+d\sin(e+fx)} dx \\
& \quad \downarrow 3042 \\
& \int \frac{\sqrt{a\sin(e+fx)+a}}{\sin(e+fx)(c+d\sin(e+fx))} dx \\
& \quad \downarrow 3413 \\
& \frac{\int \csc(e+fx)\sqrt{\sin(e+fx)a+a} dx}{c} - \frac{d \int \frac{\sqrt{\sin(e+fx)a+a}}{c+d\sin(e+fx)} dx}{c} \\
& \quad \downarrow 3042 \\
& \frac{\int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)} dx}{c} - \frac{d \int \frac{\sqrt{\sin(e+fx)a+a}}{c+d\sin(e+fx)} dx}{c} \\
& \quad \downarrow 3252 \\
& \frac{2ad \int \frac{1}{a(c+d) - \frac{a^2 d \cos^2(e+fx)}{\sin(e+fx)a+a}} d \frac{a \cos(e+fx)}{\sqrt{\sin(e+fx)a+a}}}{cf} - \frac{2a \int \frac{1}{a - \frac{a^2 \cos^2(e+fx)}{\sin(e+fx)a+a}} d \frac{a \cos(e+fx)}{\sqrt{\sin(e+fx)a+a}}}{cf} \\
& \quad \downarrow 219 \\
& \frac{2ad \int \frac{1}{a(c+d) - \frac{a^2 d \cos^2(e+fx)}{\sin(e+fx)a+a}} d \frac{a \cos(e+fx)}{\sqrt{\sin(e+fx)a+a}}}{cf} - \frac{2\sqrt{a} \operatorname{arctanh}\left(\frac{\sqrt{a} \cos(e+fx)}{\sqrt{a \sin(e+fx)+a}}\right)}{cf} \\
& \quad \downarrow 221 \\
& \frac{2\sqrt{a}\sqrt{d} \operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{d} \cos(e+fx)}{\sqrt{c+d}\sqrt{a \sin(e+fx)+a}}\right)}{cf\sqrt{c+d}} - \frac{2\sqrt{a} \operatorname{arctanh}\left(\frac{\sqrt{a} \cos(e+fx)}{\sqrt{a \sin(e+fx)+a}}\right)}{cf}
\end{aligned}$$

input `Int[(Csc[e + f*x]*Sqrt[a + a*Sin[e + f*x]])/(c + d*Sin[e + f*x]),x]`

output `(-2*Sqrt[a]*ArcTanh[(Sqrt[a]*Cos[e + f*x])/Sqrt[a + a*Sin[e + f*x]])/(c*f) + (2*Sqrt[a]*Sqrt[d]*ArcTanh[(Sqrt[a]*Sqrt[d]*Cos[e + f*x])/(Sqrt[c + d]*Sqrt[a + a*Sin[e + f*x]])]/(c*Sqrt[c + d]*f)`

Definitions of rubi rules used

rule 219 $\text{Int}[(a_ + (b_ \cdot)(x_)^2)^{-1}, x_Symbol] \rightarrow \text{Simp}[(1/(\text{Rt}[a, 2] \cdot \text{Rt}[-b, 2])) \cdot \text{ArcTanh}[\text{Rt}[-b, 2] \cdot (x/\text{Rt}[a, 2])], x] /; \text{FreeQ}\{a, b\}, x \ \&\& \ \text{NegQ}[a/b] \ \&\& \ (\text{GtQ}[a, 0] \ || \ \text{LtQ}[b, 0])$

rule 221 $\text{Int}[(a_ + (b_ \cdot)(x_)^2)^{-1}, x_Symbol] \rightarrow \text{Simp}[(\text{Rt}[-a/b, 2]/a) \cdot \text{ArcTanh}[x/\text{Rt}[-a/b, 2]], x] /; \text{FreeQ}\{a, b\}, x \ \&\& \ \text{NegQ}[a/b]$

rule 3042 $\text{Int}[u_, x_Symbol] \rightarrow \text{Int}[\text{DeactivateTrig}[u, x], x] /; \text{FunctionOfTrigOfLinearQ}[u, x]$

rule 3252 $\text{Int}[\text{Sqrt}[(a_ + (b_ \cdot)\sin[(e_) + (f_ \cdot)(x_)])]/((c_) + (d_ \cdot)\sin[(e_) + (f_ \cdot)(x_)]), x_Symbol] \rightarrow \text{Simp}[-2 \cdot (b/f) \ \text{Subst}[\text{Int}[1/(b \cdot c + a \cdot d - d \cdot x^2), x], x, b \cdot (\text{Cos}[e + f \cdot x]/\text{Sqrt}[a + b \cdot \text{Sin}[e + f \cdot x]])], x] /; \text{FreeQ}\{a, b, c, d, e, f\}, x \ \&\& \ \text{NeQ}[b \cdot c - a \cdot d, 0] \ \&\& \ \text{EqQ}[a^2 - b^2, 0] \ \&\& \ \text{NeQ}[c^2 - d^2, 0]$

rule 3413 $\text{Int}[\text{Sqrt}[(a_ + (b_ \cdot)\sin[(e_) + (f_ \cdot)(x_)])]/(\sin[(e_) + (f_ \cdot)(x_)] \cdot ((c_) + (d_ \cdot)\sin[(e_) + (f_ \cdot)(x_)])), x_Symbol] \rightarrow \text{Simp}[1/c \ \text{Int}[\text{Sqrt}[a + b \cdot \text{Sin}[e + f \cdot x]]/\text{Sin}[e + f \cdot x], x], x] - \text{Simp}[d/c \ \text{Int}[\text{Sqrt}[a + b \cdot \text{Sin}[e + f \cdot x]]/(c + d \cdot \text{Sin}[e + f \cdot x]), x], x] /; \text{FreeQ}\{a, b, c, d, e, f\}, x \ \&\& \ \text{NeQ}[b \cdot c - a \cdot d, 0] \ \&\& \ \text{EqQ}[a^2 - b^2, 0]$

Maple [A] (verified)

Time = 1.74 (sec) , antiderivative size = 120, normalized size of antiderivative = 1.14

method	result	s
default	$-\frac{2(1+\sin(fx+e))\sqrt{-a(\sin(fx+e)-1)}\left(-d \operatorname{arctanh}\left(\frac{\sqrt{-a(\sin(fx+e)-1)}d}{\sqrt{a(c+d)d}}\right)a^{\frac{3}{2}}+\operatorname{arctanh}\left(\frac{\sqrt{-a(\sin(fx+e)-1)}}{\sqrt{a}}\right)a\sqrt{a(c+d)d}\right)}{\sqrt{a}c\sqrt{a(c+d)d}\cos(fx+e)\sqrt{a+a\sin(fx+e)}f}$	1

input $\text{int}(\csc(f \cdot x + e) \cdot (a + a \cdot \sin(f \cdot x + e))^{1/2} / (c + d \cdot \sin(f \cdot x + e)), x, \text{method} = _RETURNVER \text{BOSE})$

output

```
-2/a^(1/2)*(1+sin(f*x+e))*(-a*(sin(f*x+e)-1))^(1/2)*(-d*arctanh((-a*(sin(f
*x+e)-1))^(1/2)*d/(a*(c+d)*d)^(1/2))*a^(3/2)+arctanh((-a*(sin(f*x+e)-1))^(
1/2)/a^(1/2))*a*(a*(c+d)*d)^(1/2))/c/(a*(c+d)*d)^(1/2)/cos(f*x+e)/(a+a*sin
(f*x+e))^(1/2)/f
```

Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 242 vs. $2(85) = 170$.

Time = 0.42 (sec) , antiderivative size = 781, normalized size of antiderivative = 7.44

$$\int \frac{\csc(e + fx) \sqrt{a + a \sin(e + fx)}}{c + d \sin(e + fx)} dx = \text{Too large to display}$$

input

```
integrate(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x, algorithm=
"fricas")
```

output

```
[1/2*(sqrt(a*d/(c + d))*log((a*d^2*cos(f*x + e)^3 - a*c^2 - 2*a*c*d - a*d^
2 - (6*a*c*d + 7*a*d^2)*cos(f*x + e)^2 + 4*((c*d + d^2)*cos(f*x + e)^2 - c
^2 - 4*c*d - 3*d^2 - (c^2 + 3*c*d + 2*d^2)*cos(f*x + e) + (c^2 + 4*c*d + 3
*d^2 + (c*d + d^2)*cos(f*x + e))*sin(f*x + e))*sqrt(a*sin(f*x + e) + a)*sq
rt(a*d/(c + d)) - (a*c^2 + 8*a*c*d + 9*a*d^2)*cos(f*x + e) + (a*d^2*cos(f*
x + e)^2 - a*c^2 - 2*a*c*d - a*d^2 + 2*(3*a*c*d + 4*a*d^2)*cos(f*x + e))*s
in(f*x + e))/(d^2*cos(f*x + e)^3 + (2*c*d + d^2)*cos(f*x + e)^2 - c^2 - 2*
c*d - d^2 - (c^2 + d^2)*cos(f*x + e) + (d^2*cos(f*x + e)^2 - 2*c*d*cos(f*x
+ e) - c^2 - 2*c*d - d^2)*sin(f*x + e))) + sqrt(a)*log((a*cos(f*x + e)^3
- 7*a*cos(f*x + e)^2 - 4*(cos(f*x + e)^2 + (cos(f*x + e) + 3)*sin(f*x + e)
- 2*cos(f*x + e) - 3)*sqrt(a*sin(f*x + e) + a)*sqrt(a) - 9*a*cos(f*x + e)
+ (a*cos(f*x + e)^2 + 8*a*cos(f*x + e) - a)*sin(f*x + e) - a)/(cos(f*x +
e)^3 + cos(f*x + e)^2 + (cos(f*x + e)^2 - 1)*sin(f*x + e) - cos(f*x + e) -
1)))/(c*f), 1/2*(2*sqrt(-a*d/(c + d))*arctan(1/2*sqrt(a*sin(f*x + e) + a)
*(d*sin(f*x + e) - c - 2*d)*sqrt(-a*d/(c + d))/(a*d*cos(f*x + e))) + sqrt(
a)*log((a*cos(f*x + e)^3 - 7*a*cos(f*x + e)^2 - 4*(cos(f*x + e)^2 + (cos(f
*x + e) + 3)*sin(f*x + e) - 2*cos(f*x + e) - 3)*sqrt(a*sin(f*x + e) + a)*s
qrt(a) - 9*a*cos(f*x + e) + (a*cos(f*x + e)^2 + 8*a*cos(f*x + e) - a)*sin(
f*x + e) - a)/(cos(f*x + e)^3 + cos(f*x + e)^2 + (cos(f*x + e)^2 - 1)*sin(
f*x + e) - cos(f*x + e) - 1)))/(c*f)]
```

Sympy [F]

$$\int \frac{\csc(e + fx) \sqrt{a + a \sin(e + fx)}}{c + d \sin(e + fx)} dx = \int \frac{\sqrt{a (\sin(e + fx) + 1)} \csc(e + fx)}{c + d \sin(e + fx)} dx$$

input `integrate(csc(f*x+e)*(a+a*sin(f*x+e))**(1/2)/(c+d*sin(f*x+e)),x)`

output `Integral(sqrt(a*(sin(e + f*x) + 1))*csc(e + f*x)/(c + d*sin(e + f*x)), x)`

Maxima [F]

$$\int \frac{\csc(e + fx) \sqrt{a + a \sin(e + fx)}}{c + d \sin(e + fx)} dx = \int \frac{\sqrt{a \sin(fx + e) + a} \csc(fx + e)}{d \sin(fx + e) + c} dx$$

input `integrate(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x, algorithm="maxima")`

output `integrate(sqrt(a*sin(f*x + e) + a)*csc(f*x + e)/(d*sin(f*x + e) + c), x)`

Giac [A] (verification not implemented)

Time = 0.39 (sec) , antiderivative size = 143, normalized size of antiderivative = 1.36

$$\int \frac{\csc(e + fx) \sqrt{a + a \sin(e + fx)}}{c + d \sin(e + fx)} dx$$

$$= \frac{\sqrt{2} \left(\frac{2\sqrt{2}d \arctan\left(\frac{\sqrt{2}d \sin\left(-\frac{1}{4}\pi + \frac{1}{2}fx + \frac{1}{2}e\right)}{\sqrt{-cd-d^2}}\right) \operatorname{sgn}\left(\cos\left(-\frac{1}{4}\pi + \frac{1}{2}fx + \frac{1}{2}e\right)\right)}{\sqrt{-cd-d^2}c} - \frac{\sqrt{2} \log\left(\frac{|-2\sqrt{2}+4 \sin\left(-\frac{1}{4}\pi + \frac{1}{2}fx + \frac{1}{2}e\right)|}{|2\sqrt{2}+4 \sin\left(-\frac{1}{4}\pi + \frac{1}{2}fx + \frac{1}{2}e\right)|}\right) \operatorname{sgn}\left(\cos\left(-\frac{1}{4}\pi + \frac{1}{2}fx + \frac{1}{2}e\right)\right)}{c} \right)}{2f}$$

input `integrate(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x, algorithm="giac")`

output

```
1/2*sqrt(2)*(2*sqrt(2)*d*arctan(sqrt(2)*d*sin(-1/4*pi + 1/2*f*x + 1/2*e)/sqrt(-c*d - d^2))*sgn(cos(-1/4*pi + 1/2*f*x + 1/2*e))/(sqrt(-c*d - d^2)*c) - sqrt(2)*log(abs(-2*sqrt(2) + 4*sin(-1/4*pi + 1/2*f*x + 1/2*e))/abs(2*sqrt(2) + 4*sin(-1/4*pi + 1/2*f*x + 1/2*e)))*sgn(cos(-1/4*pi + 1/2*f*x + 1/2*e))/c)*sqrt(a)/f
```

Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e + fx) \sqrt{a + a \sin(e + fx)}}{c + d \sin(e + fx)} dx = \int \frac{\sqrt{a + a \sin(e + fx)}}{\sin(e + fx) (c + d \sin(e + fx))} dx$$

input

```
int((a + a*sin(e + f*x))^(1/2)/(sin(e + f*x)*(c + d*sin(e + f*x))),x)
```

output

```
int((a + a*sin(e + f*x))^(1/2)/(sin(e + f*x)*(c + d*sin(e + f*x))), x)
```

Reduce [F]

$$\int \frac{\csc(e + fx) \sqrt{a + a \sin(e + fx)}}{c + d \sin(e + fx)} dx = \sqrt{a} \left(\int \frac{\sqrt{\sin(fx + e) + 1} \csc(fx + e)}{\sin(fx + e) d + c} dx \right)$$

input

```
int(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x)
```

output

```
sqrt(a)*int((sqrt(sin(e + f*x) + 1)*csc(e + f*x))/(sin(e + f*x)*d + c),x)
```

3.24
$$\int \frac{\csc(e+fx)}{\sqrt{a+a \sin(e+fx)}(c+d \sin(e+fx))} dx$$

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Optimal result

Integrand size = 33, antiderivative size = 165

$$\int \frac{\csc(e+fx)}{\sqrt{a+a \sin(e+fx)}(c+d \sin(e+fx))} dx$$

$$= -\frac{2 \operatorname{arctanh}\left(\frac{\sqrt{a} \cos(e+fx)}{\sqrt{a+a \sin(e+fx)}}\right)}{\sqrt{ac}f} + \frac{\sqrt{2} \operatorname{arctanh}\left(\frac{\sqrt{a} \cos(e+fx)}{\sqrt{2}\sqrt{a+a \sin(e+fx)}}\right)}{\sqrt{a}(c-d)f}$$

$$- \frac{2d^{3/2} \operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{d} \cos(e+fx)}{\sqrt{c+d}\sqrt{a+a \sin(e+fx)}}\right)}{\sqrt{ac}(c-d)\sqrt{c+d}f}$$

output

```
-2*arctanh(a^(1/2)*cos(f*x+e)/(a+a*sin(f*x+e))^(1/2))/a^(1/2)/c/f+2^(1/2)*
arctanh(1/2*a^(1/2)*cos(f*x+e)*2^(1/2)/(a+a*sin(f*x+e))^(1/2))/a^(1/2)/(c-
d)/f-2*d^(3/2)*arctanh(a^(1/2)*d^(1/2)*cos(f*x+e)/(c+d)^(1/2)/(a+a*sin(f*x
+e))^(1/2))/a^(1/2)/c/(c-d)/(c+d)^(1/2)/f
```

Mathematica [C] (warning: unable to verify)

Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.

Time = 5.04 (sec) , antiderivative size = 654, normalized size of antiderivative = 3.96

$$\int \frac{\csc(e + fx)}{\sqrt{a + a \sin(e + fx)}(c + d \sin(e + fx))} dx = \text{Too large to display}$$

input `Integrate[Csc[e + f*x]/(Sqrt[a + a*Sin[e + f*x]]*(c + d*Sin[e + f*x])),x]`

output

```
((-2*Sqrt[c + d]*((2 + 2*I)*(-1)^(3/4)*c*ArcTanh[(1/2 + I/2)*(-1)^(3/4)*(-1 + Tan[(e + f*x)/4])] + (c - d)*(Log[1 + Cos[(e + f*x)/2] - Sin[(e + f*x)/2]] - Log[1 - Cos[(e + f*x)/2] + Sin[(e + f*x)/2]])) - d^(3/2)*RootSum[c + 4*d*#1 + 2*c*#1^2 - 4*d*#1^3 + c*#1^4 & , (-d*Log[-#1 + Tan[(e + f*x)/4]]) + Sqrt[d]*Sqrt[c + d]*Log[-#1 + Tan[(e + f*x)/4]] - c*Log[-#1 + Tan[(e + f*x)/4]]*#1 + 2*Sqrt[d]*Sqrt[c + d]*Log[-#1 + Tan[(e + f*x)/4]]*#1 + 3*d*Log[-#1 + Tan[(e + f*x)/4]]*#1^2 - Sqrt[d]*Sqrt[c + d]*Log[-#1 + Tan[(e + f*x)/4]]*#1^2 - c*Log[-#1 + Tan[(e + f*x)/4]]*#1^3)/(-d - c*#1 + 3*d*#1^2 - c*#1^3) & ] + d^(3/2)*RootSum[c + 4*d*#1 + 2*c*#1^2 - 4*d*#1^3 + c*#1^4 & , (-d*Log[-#1 + Tan[(e + f*x)/4]]) - Sqrt[d]*Sqrt[c + d]*Log[-#1 + Tan[(e + f*x)/4]] - c*Log[-#1 + Tan[(e + f*x)/4]]*#1 - 2*Sqrt[d]*Sqrt[c + d]*Log[-#1 + Tan[(e + f*x)/4]]*#1 + 3*d*Log[-#1 + Tan[(e + f*x)/4]]*#1^2 + Sqrt[d]*Sqrt[c + d]*Log[-#1 + Tan[(e + f*x)/4]]*#1^2 - c*Log[-#1 + Tan[(e + f*x)/4]]*#1^3)/(-d - c*#1 + 3*d*#1^2 - c*#1^3) & ]*(Cos[(e + f*x)/2] + Sin[(e + f*x)/2]))/(2*c*(c - d)*Sqrt[c + d]*f*Sqrt[a*(1 + Sin[e + f*x])])
```

Rubi [A] (verified)

Time = 0.98 (sec) , antiderivative size = 176, normalized size of antiderivative = 1.07, number of steps used = 12, number of rules used = 11, $\frac{\text{number of rules}}{\text{integrand size}} = 0.333$, Rules used = {3042, 3419, 3042, 3252, 221, 3464, 3042, 3128, 219, 3252, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e + fx)}{\sqrt{a \sin(e + fx) + a(c + d \sin(e + fx))}} dx$$

$$\begin{aligned}
& \int \frac{1}{\sin(e+fx)\sqrt{a\sin(e+fx)+a(c+d\sin(e+fx))}} dx && \downarrow \text{3042} \\
& \frac{d^2 \int \frac{\sqrt{\sin(e+fx)a+a}}{c+d\sin(e+fx)} dx}{ac(c-d)} + \frac{\int \frac{\csc(e+fx)(a(c-d)-ad\sin(e+fx))}{\sqrt{\sin(e+fx)a+a}} dx}{ac(c-d)} && \downarrow \text{3419} \\
& \frac{d^2 \int \frac{\sqrt{\sin(e+fx)a+a}}{c+d\sin(e+fx)} dx}{ac(c-d)} + \frac{\int \frac{a(c-d)-ad\sin(e+fx)}{\sin(e+fx)\sqrt{\sin(e+fx)a+a}} dx}{ac(c-d)} && \downarrow \text{3042} \\
& \frac{\int \frac{a(c-d)-ad\sin(e+fx)}{\sin(e+fx)\sqrt{\sin(e+fx)a+a}} dx}{ac(c-d)} - \frac{2d^2 \int \frac{1}{a(c+d)-\frac{a^2 d \cos^2(e+fx)}{\sin(e+fx)a+a}} d \frac{a \cos(e+fx)}{\sqrt{\sin(e+fx)a+a}}}{cf(c-d)} && \downarrow \text{3252} \\
& \frac{\int \frac{a(c-d)-ad\sin(e+fx)}{\sin(e+fx)\sqrt{\sin(e+fx)a+a}} dx}{ac(c-d)} - \frac{2d^{3/2} \operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{d} \cos(e+fx)}{\sqrt{c+d}\sqrt{a\sin(e+fx)+a}}\right)}{\sqrt{ac}f(c-d)\sqrt{c+d}} && \downarrow \text{221} \\
& \frac{(c-d) \int \csc(e+fx)\sqrt{\sin(e+fx)a+ad} dx - ac \int \frac{1}{\sqrt{\sin(e+fx)a+a}} dx}{ac(c-d)} - \frac{2d^{3/2} \operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{d} \cos(e+fx)}{\sqrt{c+d}\sqrt{a\sin(e+fx)+a}}\right)}{\sqrt{ac}f(c-d)\sqrt{c+d}} && \downarrow \text{3464} \\
& \frac{(c-d) \int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)} dx - ac \int \frac{1}{\sqrt{\sin(e+fx)a+a}} dx}{ac(c-d)} - \frac{2d^{3/2} \operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{d} \cos(e+fx)}{\sqrt{c+d}\sqrt{a\sin(e+fx)+a}}\right)}{\sqrt{ac}f(c-d)\sqrt{c+d}} && \downarrow \text{3042} \\
& \frac{2ac \int \frac{1}{2a-\frac{a^2 \cos^2(e+fx)}{\sin(e+fx)a+a}} d \frac{a \cos(e+fx)}{\sqrt{\sin(e+fx)a+a}}}{f} + \frac{(c-d) \int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)} dx}{ac(c-d)} - \frac{2d^{3/2} \operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{d} \cos(e+fx)}{\sqrt{c+d}\sqrt{a\sin(e+fx)+a}}\right)}{\sqrt{ac}f(c-d)\sqrt{c+d}} && \downarrow \text{3128} \\
& \frac{2d^{3/2} \operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{d} \cos(e+fx)}{\sqrt{c+d}\sqrt{a\sin(e+fx)+a}}\right)}{\sqrt{ac}f(c-d)\sqrt{c+d}} &&
\end{aligned}$$

$$\begin{aligned}
 & \downarrow 219 \\
 & \frac{(c-d) \int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)} dx + \frac{\sqrt{2}\sqrt{ac} \operatorname{arctanh}\left(\frac{\sqrt{a} \cos(e+fx)}{\sqrt{2}\sqrt{a} \sin(e+fx)+a}\right)}{f}}{ac(c-d)} \\
 & \frac{2d^{3/2} \operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{d} \cos(e+fx)}{\sqrt{c+d}\sqrt{a} \sin(e+fx)+a}\right)}{\sqrt{ac}f(c-d)\sqrt{c+d}} \\
 & \downarrow 3252 \\
 & \frac{\frac{\sqrt{2}\sqrt{ac} \operatorname{arctanh}\left(\frac{\sqrt{a} \cos(e+fx)}{\sqrt{2}\sqrt{a} \sin(e+fx)+a}\right)}{f} - \frac{2a(c-d) \int \frac{1}{a - \frac{a^2 \cos^2(e+fx)}{\sin(e+fx)a+a}} d \frac{a \cos(e+fx)}{\sqrt{\sin(e+fx)a+a}}}{f}}{ac(c-d)} \\
 & \frac{2d^{3/2} \operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{d} \cos(e+fx)}{\sqrt{c+d}\sqrt{a} \sin(e+fx)+a}\right)}{\sqrt{ac}f(c-d)\sqrt{c+d}} \\
 & \downarrow 219 \\
 & \frac{\frac{\sqrt{2}\sqrt{ac} \operatorname{arctanh}\left(\frac{\sqrt{a} \cos(e+fx)}{\sqrt{2}\sqrt{a} \sin(e+fx)+a}\right)}{f} - \frac{2\sqrt{a}(c-d) \operatorname{arctanh}\left(\frac{\sqrt{a} \cos(e+fx)}{\sqrt{a} \sin(e+fx)+a}\right)}{f}}{ac(c-d)} \\
 & \frac{2d^{3/2} \operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{d} \cos(e+fx)}{\sqrt{c+d}\sqrt{a} \sin(e+fx)+a}\right)}{\sqrt{ac}f(c-d)\sqrt{c+d}}
 \end{aligned}$$

input `Int[Csc[e + f*x]/(Sqrt[a + a*Sin[e + f*x]]*(c + d*Sin[e + f*x])),x]`

output `((-2*Sqrt[a]*(c - d)*ArcTanh[(Sqrt[a]*Cos[e + f*x])/Sqrt[a + a*Sin[e + f*x]])]/f + (Sqrt[2]*Sqrt[a]*c*ArcTanh[(Sqrt[a]*Cos[e + f*x])/Sqrt[2]*Sqrt[a + a*Sin[e + f*x]])]/f)/(a*c*(c - d) - (2*d^(3/2)*ArcTanh[(Sqrt[a]*Sqrt[d]*Cos[e + f*x])/Sqrt[c + d]*Sqrt[a + a*Sin[e + f*x]])]/(Sqrt[a]*c*(c - d)*Sqrt[c + d]*f)`

Definitions of rubi rules used

- rule 219 $\text{Int}[(a_ + (b_ \cdot x)^2)^{-1}, x_Symbol] \rightarrow \text{Simp}[(1/(\text{Rt}[a, 2] \cdot \text{Rt}[-b, 2])) \cdot \text{ArcTanh}[\text{Rt}[-b, 2] \cdot (x/\text{Rt}[a, 2])], x] /; \text{FreeQ}\{a, b\}, x \ \&\& \ \text{NegQ}[a/b] \ \&\& \ (\text{GtQ}[a, 0] \ || \ \text{LtQ}[b, 0])$
- rule 221 $\text{Int}[(a_ + (b_ \cdot x)^2)^{-1}, x_Symbol] \rightarrow \text{Simp}[(\text{Rt}[-a/b, 2]/a) \cdot \text{ArcTanh}[x/\text{Rt}[-a/b, 2]], x] /; \text{FreeQ}\{a, b\}, x \ \&\& \ \text{NegQ}[a/b]$
- rule 3042 $\text{Int}[u_, x_Symbol] \rightarrow \text{Int}[\text{DeactivateTrig}[u, x], x] /; \text{FunctionOfTrigOfLinearQ}[u, x]$
- rule 3128 $\text{Int}[1/\text{Sqrt}[(a_ + (b_ \cdot \sin[(c_ + (d_ \cdot x)])], x_Symbol] \rightarrow \text{Simp}[-2/d \ \text{Subst}[\text{Int}[1/(2 \cdot a - x^2), x], x, b \cdot (\text{Cos}[c + d \cdot x]/\text{Sqrt}[a + b \cdot \text{Sin}[c + d \cdot x]])], x] /; \text{FreeQ}\{a, b, c, d\}, x \ \&\& \ \text{EqQ}[a^2 - b^2, 0]$
- rule 3252 $\text{Int}[\text{Sqrt}[(a_ + (b_ \cdot \sin[(e_ + (f_ \cdot x)])]/((c_ + (d_ \cdot \sin[(e_ + (f_ \cdot x)])], x_Symbol] \rightarrow \text{Simp}[-2 \cdot (b/f) \ \text{Subst}[\text{Int}[1/(b \cdot c + a \cdot d - d \cdot x^2), x], x, b \cdot (\text{Cos}[e + f \cdot x]/\text{Sqrt}[a + b \cdot \text{Sin}[e + f \cdot x]])], x] /; \text{FreeQ}\{a, b, c, d, e, f\}, x \ \&\& \ \text{NeQ}[b \cdot c - a \cdot d, 0] \ \&\& \ \text{EqQ}[a^2 - b^2, 0] \ \&\& \ \text{NeQ}[c^2 - d^2, 0]$
- rule 3419 $\text{Int}[1/(\sin[(e_ + (f_ \cdot x)] \cdot \text{Sqrt}[(a_ + (b_ \cdot \sin[(e_ + (f_ \cdot x)])] \cdot ((c_ + (d_ \cdot \sin[(e_ + (f_ \cdot x)])], x_Symbol] \rightarrow \text{Simp}[d^2/(c \cdot (b \cdot c - a \cdot d)) \ \text{Int}[\text{Sqrt}[a + b \cdot \text{Sin}[e + f \cdot x]]/(c + d \cdot \text{Sin}[e + f \cdot x]), x], x] + \text{Simp}[1/(c \cdot (b \cdot c - a \cdot d)) \ \text{Int}[(b \cdot c - a \cdot d - b \cdot d \cdot \text{Sin}[e + f \cdot x])]/(\text{Sin}[e + f \cdot x] \cdot \text{Sqrt}[a + b \cdot \text{Sin}[e + f \cdot x]]), x], x] /; \text{FreeQ}\{a, b, c, d, e, f\}, x \ \&\& \ \text{NeQ}[b \cdot c - a \cdot d, 0] \ \&\& \ \text{EqQ}[a^2 - b^2, 0]$
- rule 3464 $\text{Int}[(A_ + (B_ \cdot \sin[(e_ + (f_ \cdot x)])]/(\text{Sqrt}[(a_ + (b_ \cdot \sin[(e_ + (f_ \cdot x)])] \cdot ((c_ + (d_ \cdot \sin[(e_ + (f_ \cdot x)])], x_Symbol] \rightarrow \text{Simp}[(A \cdot b - a \cdot B)/(b \cdot c - a \cdot d) \ \text{Int}[1/\text{Sqrt}[a + b \cdot \text{Sin}[e + f \cdot x]], x], x] + \text{Simp}[(B \cdot c - A \cdot d)/(b \cdot c - a \cdot d) \ \text{Int}[\text{Sqrt}[a + b \cdot \text{Sin}[e + f \cdot x]]/(c + d \cdot \text{Sin}[e + f \cdot x]), x], x] /; \text{FreeQ}\{a, b, c, d, e, f, A, B\}, x \ \&\& \ \text{NeQ}[b \cdot c - a \cdot d, 0] \ \&\& \ \text{EqQ}[a^2 - b^2, 0] \ \&\& \ \text{NeQ}[c^2 - d^2, 0]$

Maple [A] (verified)

Time = 1.41 (sec) , antiderivative size = 208, normalized size of antiderivative = 1.26

method	result
default	$-\frac{(1+\sin(fx+e))\sqrt{-a(\sin(fx+e)-1)}\left(2d^2 \operatorname{arctanh}\left(\frac{\sqrt{a-a\sin(fx+e)}d}{\sqrt{acd+a^2d^2}}\right)a^{\frac{5}{2}}-\sqrt{2} \operatorname{arctanh}\left(\frac{\sqrt{a-a\sin(fx+e)}\sqrt{2}}{2\sqrt{a}}\right)ca^2\sqrt{a(c+d)d+2}\right)}{a^{\frac{5}{2}}c(c-d)\sqrt{a(c+d)d}\cos(fx+e)\sqrt{a+a\sin(fx+e)}}$

input `int(csc(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,method=_RETURNVERBOSE)`

output
$$\frac{-(1+\sin(fx+e))*(-a*(\sin(fx+e)-1))^{(1/2)}*(2*d^2*\operatorname{arctanh}((a-a*\sin(fx+e))^{(1/2)}*d/(a*c*d+a*d^2)^{(1/2))}*a^{(5/2)}-2^{(1/2)}*\operatorname{arctanh}(1/2*(a-a*\sin(fx+e))^{(1/2)}*2^{(1/2)}/a^{(1/2)})*c*a^2*(a*(c+d)*d)^{(1/2)}+2*\operatorname{arctanh}((a-a*\sin(fx+e))^{(1/2)}/a^{(1/2)})*a^2*(a*(c+d)*d)^{(1/2)}*c-2*\operatorname{arctanh}((a-a*\sin(fx+e))^{(1/2)}/a^{(1/2)})*a^2*(a*(c+d)*d)^{(1/2)}*d)/a^{(5/2)}/c/(c-d)/(a*(c+d)*d)^{(1/2)}/\cos(fx+e)/(a+a*\sin(fx+e))^{(1/2)}/f}{1}$$

Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 380 vs. 2(136) = 272.

Time = 1.32 (sec) , antiderivative size = 1044, normalized size of antiderivative = 6.33

$$\int \frac{\csc(e+fx)}{\sqrt{a+a\sin(e+fx)}(c+d\sin(e+fx))} dx = \text{Too large to display}$$

input `integrate(csc(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x, algorithm="fricas")`

output

```

[-1/2*(a*d*sqrt(d/(a*c + a*d))*log((d^2*cos(f*x + e)^3 - (6*c*d + 7*d^2)*c
os(f*x + e)^2 - c^2 - 2*c*d - d^2 + 4*((c*d + d^2)*cos(f*x + e)^2 - c^2 -
4*c*d - 3*d^2 - (c^2 + 3*c*d + 2*d^2)*cos(f*x + e) + (c^2 + 4*c*d + 3*d^2
+ (c*d + d^2)*cos(f*x + e))*sin(f*x + e))*sqrt(a*sin(f*x + e) + a)*sqrt(d/
(a*c + a*d)) - (c^2 + 8*c*d + 9*d^2)*cos(f*x + e) + (d^2*cos(f*x + e)^2 -
c^2 - 2*c*d - d^2 + 2*(3*c*d + 4*d^2)*cos(f*x + e))*sin(f*x + e))/(d^2*cos
(f*x + e)^3 + (2*c*d + d^2)*cos(f*x + e)^2 - c^2 - 2*c*d - d^2 - (c^2 + d^
2)*cos(f*x + e) + (d^2*cos(f*x + e)^2 - 2*c*d*cos(f*x + e) - c^2 - 2*c*d -
d^2)*sin(f*x + e))) + sqrt(2)*sqrt(a)*c*log(-(cos(f*x + e)^2 - (cos(f*x +
e) - 2)*sin(f*x + e) - 2*sqrt(2)*sqrt(a*sin(f*x + e) + a)*(cos(f*x + e) -
sin(f*x + e) + 1)/sqrt(a) + 3*cos(f*x + e) + 2)/(cos(f*x + e)^2 - (cos(f*
x + e) + 2)*sin(f*x + e) - cos(f*x + e) - 2)) - sqrt(a)*(c - d)*log((a*cos
(f*x + e)^3 - 7*a*cos(f*x + e)^2 - 4*(cos(f*x + e)^2 + (cos(f*x + e) + 3)*
sin(f*x + e) - 2*cos(f*x + e) - 3)*sqrt(a*sin(f*x + e) + a)*sqrt(a) - 9*a*
cos(f*x + e) + (a*cos(f*x + e)^2 + 8*a*cos(f*x + e) - a)*sin(f*x + e) - a)
/(cos(f*x + e)^3 + cos(f*x + e)^2 + (cos(f*x + e)^2 - 1)*sin(f*x + e) - co
s(f*x + e) - 1)))/((a*c^2 - a*c*d)*f), -1/2*(2*a*d*sqrt(-d/(a*c + a*d))*ar
ctan(1/2*sqrt(a*sin(f*x + e) + a)*(d*sin(f*x + e) - c - 2*d)*sqrt(-d/(a*c
+ a*d))/(d*cos(f*x + e))) + sqrt(2)*sqrt(a)*c*log(-(cos(f*x + e)^2 - (cos(
f*x + e) - 2)*sin(f*x + e) - 2*sqrt(2)*sqrt(a*sin(f*x + e) + a)*(cos(f*...

```

Sympy [F]

$$\begin{aligned}
& \int \frac{\csc(e + fx)}{\sqrt{a + a \sin(e + fx)}(c + d \sin(e + fx))} dx \\
&= \int \frac{\csc(e + fx)}{\sqrt{a(\sin(e + fx) + 1)}(c + d \sin(e + fx))} dx
\end{aligned}$$

input

```
integrate(csc(f*x+e)/(a+a*sin(f*x+e))**(1/2)/(c+d*sin(f*x+e)),x)
```

output

```
Integral(csc(e + f*x)/(sqrt(a*(sin(e + f*x) + 1))*(c + d*sin(e + f*x))), x)
```

Maxima [F]

$$\int \frac{\csc(e + fx)}{\sqrt{a + a \sin(e + fx)}(c + d \sin(e + fx))} dx$$

$$= \int \frac{\csc(fx + e)}{\sqrt{a \sin(fx + e) + a}(d \sin(fx + e) + c)} dx$$

input `integrate(csc(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x, algorithm="maxima")`

output `integrate(csc(f*x + e)/(sqrt(a*sin(f*x + e) + a)*(d*sin(f*x + e) + c)), x)`

Giac [A] (verification not implemented)

Time = 0.35 (sec) , antiderivative size = 189, normalized size of antiderivative = 1.15

$$\int \frac{\csc(e + fx)}{\sqrt{a + a \sin(e + fx)}(c + d \sin(e + fx))} dx =$$

$$\frac{\sqrt{2} \left(\frac{2\sqrt{2}d^2 \arctan\left(\frac{\sqrt{2}d \sin\left(-\frac{1}{4}\pi + \frac{1}{2}fx + \frac{1}{2}e\right)}{\sqrt{-cd-d^2}}\right)}{(c^2-cd)\sqrt{-cd-d^2}} + \frac{\sqrt{2} \log\left(\frac{|-2\sqrt{2}+4 \sin\left(-\frac{1}{4}\pi + \frac{1}{2}fx + \frac{1}{2}e\right)|}{2\sqrt{2}+4 \sin\left(-\frac{1}{4}\pi + \frac{1}{2}fx + \frac{1}{2}e\right)}\right)}{c} + \frac{\log(\sin(-\frac{1}{4}\pi + \frac{1}{2}fx + \frac{1}{2}e)+1)}{c-d} - \frac{\log(-\sin(-\frac{1}{4}\pi + \frac{1}{2}fx + \frac{1}{2}e)+1)}{c-d} \right)}{2\sqrt{a}f \operatorname{sgn}\left(\cos\left(-\frac{1}{4}\pi + \frac{1}{2}fx + \frac{1}{2}e\right)\right)}$$

input `integrate(csc(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x, algorithm="giac")`

output `-1/2*sqrt(2)*(2*sqrt(2)*d^2*arctan(sqrt(2)*d*sin(-1/4*pi + 1/2*f*x + 1/2*e)/sqrt(-c*d - d^2))/((c^2 - c*d)*sqrt(-c*d - d^2)) + sqrt(2)*log(abs(-2*sqrt(2) + 4*sin(-1/4*pi + 1/2*f*x + 1/2*e))/abs(2*sqrt(2) + 4*sin(-1/4*pi + 1/2*f*x + 1/2*e)))/c + log(sin(-1/4*pi + 1/2*f*x + 1/2*e) + 1)/(c - d) - log(-sin(-1/4*pi + 1/2*f*x + 1/2*e) + 1)/(c - d)/(sqrt(a)*f*sgn(cos(-1/4*pi + 1/2*f*x + 1/2*e)))`

Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e + fx)}{\sqrt{a + a \sin(e + fx)}(c + d \sin(e + fx))} dx$$

$$= \int \frac{1}{\sin(e + fx) \sqrt{a + a \sin(e + fx)} (c + d \sin(e + fx))} dx$$

input `int(1/(sin(e + f*x)*(a + a*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))),x)`

output `int(1/(sin(e + f*x)*(a + a*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))), x)`

Reduce [F]

$$\int \frac{\csc(e + fx)}{\sqrt{a + a \sin(e + fx)}(c + d \sin(e + fx))} dx$$

$$= \frac{\sqrt{a} \left(\int \frac{\sqrt{\sin(fx+e)+1} \csc(fx+e)}{\sin(fx+e)^2 d + \sin(fx+e)c + \sin(fx+e)d+c} dx \right)}{a}$$

input `int(csc(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x)`

output `(sqrt(a)*int((sqrt(sin(e + f*x) + 1)*csc(e + f*x))/(sin(e + f*x)**2*d + sin(e + f*x)*c + sin(e + f*x)*d + c),x))/a`

3.25
$$\int \frac{\sqrt{g \sin(e+fx)} \sqrt{a+a \sin(e+fx)}}{c+d \sin(e+fx)} dx$$

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Optimal result

Integrand size = 39, antiderivative size = 149

$$\int \frac{\sqrt{g \sin(e+fx)} \sqrt{a+a \sin(e+fx)}}{c+d \sin(e+fx)} dx$$

$$= -\frac{2\sqrt{a}\sqrt{g} \arctan\left(\frac{\sqrt{a}\sqrt{g} \cos(e+fx)}{\sqrt{g \sin(e+fx)} \sqrt{a+a \sin(e+fx)}}\right)}{df}$$

$$+ \frac{2\sqrt{a}\sqrt{c}\sqrt{g} \arctan\left(\frac{\sqrt{a}\sqrt{c}\sqrt{g} \cos(e+fx)}{\sqrt{c+d}\sqrt{g \sin(e+fx)} \sqrt{a+a \sin(e+fx)}}\right)}{d\sqrt{c+d}f}$$

output

```
-2*a^(1/2)*g^(1/2)*arctan(a^(1/2)*g^(1/2)*cos(f*x+e)/(g*sin(f*x+e))^(1/2)/
(a+a*sin(f*x+e))^(1/2))/d/f+2*a^(1/2)*c^(1/2)*g^(1/2)*arctan(a^(1/2)*c^(1/2)*g^(1/2)*cos(f*x+e)/(c+d)^(1/2)/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2))/d/(c+d)^(1/2)/f
```


Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 6.33 (sec) , antiderivative size = 662, normalized size of antiderivative = 4.44

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)}}{c + d \sin(e + fx)} dx$$

$$\left(\frac{1}{2} + \frac{i}{2}\right) e^{-\frac{5}{2}i(e+fx)} (-1 + e^{2i(e+fx)})^{5/2} \left(-2i\sqrt{-1 + e^{2i(e+fx)}} + \left(i + \frac{c-d}{\sqrt{-c^2+d^2}}\right) \sqrt{-1 + e^{2i(e+fx)}} + \left(i + \frac{c+d}{\sqrt{-c^2+d^2}}\right) \sqrt{-1 + e^{2i(e+fx)}} \right)$$

input

```
Integrate[(Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]])/(c + d*Sin[e + f*x]),x]
```

output

```
((1/2 + I/2)*(-1 + E^((2*I)*(e + f*x)))^(5/2)*((-2*I)*Sqrt[-1 + E^((2*I)*(e + f*x))] + (I + (c - d)/Sqrt[-c^2 + d^2])*Sqrt[-1 + E^((2*I)*(e + f*x))] + (I + (-c + d)/Sqrt[-c^2 + d^2])*Sqrt[-1 + E^((2*I)*(e + f*x))] + (2*I)*ArcTan[Sqrt[-1 + E^((2*I)*(e + f*x))]]) + ((I + (-c + d)/Sqrt[-c^2 + d^2])*(Sqrt[2]*Sqrt[c]*Sqrt[c + I*Sqrt[-c^2 + d^2]]*ArcTan[(d - ((-I)*c + Sqrt[-c^2 + d^2])*E^(I*(e + f*x))]/(Sqrt[2]*Sqrt[c]*Sqrt[c + I*Sqrt[-c^2 + d^2]]*Sqrt[-1 + E^((2*I)*(e + f*x))]]) + ((-I)*c + Sqrt[-c^2 + d^2])*ArcTanh[E^(I*(e + f*x))/Sqrt[-1 + E^((2*I)*(e + f*x))]])/d + ((I + (c - d)/Sqrt[-c^2 + d^2])*(Sqrt[2]*Sqrt[c]*Sqrt[c - I*Sqrt[-c^2 + d^2]]*ArcTan[(d + (I*c + Sqrt[-c^2 + d^2])*E^(I*(e + f*x))]/(Sqrt[2]*Sqrt[c]*Sqrt[c - I*Sqrt[-c^2 + d^2]]*Sqrt[-1 + E^((2*I)*(e + f*x))]]) - (I*c + Sqrt[-c^2 + d^2])*ArcTanh[E^(I*(e + f*x))/Sqrt[-1 + E^((2*I)*(e + f*x))]])/d)*Sqrt[g*Sin[e + f*x]]*Sqrt[a*(1 + Sin[e + f*x])]/(Sqrt[2]*d*E^(((5*I)/2)*(e + f*x))*((-I)*(-1 + E^((2*I)*(e + f*x))))/E^(I*(e + f*x)))^(5/2)*f*(Cos[(e + f*x)/2] + Sin[(e + f*x)/2])*Sqrt[Sin[e + f*x]])
```

Rubi [A] (verified)

Time = 0.87 (sec) , antiderivative size = 149, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}} = 0.179$, Rules used = {3042, 3407, 3042, 3254, 218, 3409, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{\sqrt{a \sin(e+fx) + a} \sqrt{g \sin(e+fx)}}{c + d \sin(e+fx)} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{\sqrt{a \sin(e+fx) + a} \sqrt{g \sin(e+fx)}}{c + d \sin(e+fx)} dx \\
 & \quad \downarrow \text{3407} \\
 & \frac{g \int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)}} dx}{d} - \frac{cg \int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)(c+d \sin(e+fx))}} dx}{d} \\
 & \quad \downarrow \text{3042} \\
 & \frac{g \int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)}} dx}{d} - \frac{cg \int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)(c+d \sin(e+fx))}} dx}{d} \\
 & \quad \downarrow \text{3254} \\
 & \frac{2ag \int \frac{1}{\frac{\cos(e+fx) \cot(e+fx)a^2}{\sin(e+fx)a+a} + a} df}{df} - \frac{cg \int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)(c+d \sin(e+fx))}} dx}{d} \\
 & \quad \downarrow \text{218} \\
 & \frac{cg \int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)(c+d \sin(e+fx))}} dx}{d} - \frac{2\sqrt{a}\sqrt{g} \arctan\left(\frac{\sqrt{a}\sqrt{g} \cos(e+fx)}{\sqrt{a \sin(e+fx)+a} \sqrt{g \sin(e+fx)}}\right)}{df} \\
 & \quad \downarrow \text{3409} \\
 & \frac{2acg \int \frac{1}{\frac{c \cos(e+fx) \cot(e+fx)a^2}{\sin(e+fx)a+a} + (c+d)a} df}{df} - \frac{2\sqrt{a}\sqrt{g} \arctan\left(\frac{\sqrt{a}\sqrt{g} \cos(e+fx)}{\sqrt{a \sin(e+fx)+a} \sqrt{g \sin(e+fx)}}\right)}{df}
 \end{aligned}$$

$$\begin{array}{c} \downarrow 218 \\ \frac{2\sqrt{a}\sqrt{c}\sqrt{g} \arctan\left(\frac{\sqrt{a}\sqrt{c}\sqrt{g} \cos(e+fx)}{\sqrt{c+d}\sqrt{a \sin(e+fx)+a\sqrt{g} \sin(e+fx)}}\right)}{df\sqrt{c+d}} - \\ \frac{2\sqrt{a}\sqrt{g} \arctan\left(\frac{\sqrt{a}\sqrt{g} \cos(e+fx)}{\sqrt{a \sin(e+fx)+a\sqrt{g} \sin(e+fx)}}\right)}{df} \end{array}$$

input `Int[(Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]])/(c + d*Sin[e + f*x]),x]`

output `(-2*Sqrt[a]*Sqrt[g]*ArcTan[(Sqrt[a]*Sqrt[g]*Cos[e + f*x])/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]])]/(d*f) + (2*Sqrt[a]*Sqrt[c]*Sqrt[g]*ArcTan[(Sqrt[a]*Sqrt[c]*Sqrt[g]*Cos[e + f*x])/(Sqrt[c + d]*Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]])]/(d*Sqrt[c + d]*f)`

Defintions of rubi rules used

rule 218 `Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(Rt[a/b, 2]/a)*ArcTan[x/Rt[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3254 `Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/Sqrt[(c_.) + (d_.)*sin[(e_.) + (f_.)*(x_)]], x_Symbol] := Simp[-2*(b/f) Subst[Int[1/(b + d*x^2), x], x, b*(Cos[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]])]], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]`

rule 3407

```
Int[(Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.
)*(x_)])]/((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] :> Simp[g/d I
nt[Sqrt[a + b*Sin[e + f*x]]/Sqrt[g*Sin[e + f*x]], x], x] - Simp[c*(g/d) I
nt[Sqrt[a + b*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c + d*Sin[e + f*x])), x]
, x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && (EqQ[a^2 -
b^2, 0] || EqQ[c^2 - d^2, 0])
```

rule 3409

```
Int[Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(g_.)*sin[(e_.) + (f_.
)*(x_)])*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[-2*(b/f
) Subst[Int[1/(b*c + a*d + c*g*x^2), x], x, b*(Cos[e + f*x]/(Sqrt[g*Sin[e
+ f*x]]*Sqrt[a + b*Sin[e + f*x]])], x] /; FreeQ[{a, b, c, d, e, f, g}, x]
&& NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]
```

Maple [C] (warning: unable to verify)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 39.45 (sec) , antiderivative size = 2303, normalized size of antiderivative = 15.46

method	result	size
default	Expression too large to display	2303

input

```
int((g*sin(f*x+e))^(1/2)*(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,method=
_RETURNVERBOSE)
```

output

```

1/2/f*(g*sin(1/2*f*x+1/2*e)*cos(1/2*f*x+1/2*e))^(1/2)*((2*cos(1/2*f*x+1/2*
e)*sin(1/2*f*x+1/2*e)+1)*a)^(1/2)*g*(2*arctanh((cos(1/2*f*x+1/2*e)*sin(1/2
*f*x+1/2*e)/(cos(1/2*f*x+1/2*e)+1)^2*g)^(1/2)*sin(1/2*f*x+1/2*e)/(cos(1/2*
f*x+1/2*e)-1)*c/((-d*g+(-g^2*(c-d)*(c+d))^(1/2))*c)^(1/2))*2^(1/2)*((d*g+(
-g^2*(c-d)*(c+d))^(1/2))*c)^(1/2)*(g^2)^(1/4)*c^2*g-2*arctanh((cos(1/2*f*x
+1/2*e)*sin(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/2*e)+1)^2*g)^(1/2)*sin(1/2*f*x+1
/2*e)/(cos(1/2*f*x+1/2*e)-1)*c/((-d*g+(-g^2*(c-d)*(c+d))^(1/2))*c)^(1/2))*
2^(1/2)*((d*g+(-g^2*(c-d)*(c+d))^(1/2))*c)^(1/2)*(g^2)^(1/4)*c*d*g+2*arcta
n((cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/2*e)+1)^2*g)^(1/2)
*sin(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/2*e)-1)*c/((d*g+(-g^2*(c-d)*(c+d))^(1/2)
))*c)^(1/2))*2^(1/2)*((-d*g+(-g^2*(c-d)*(c+d))^(1/2))*c)^(1/2)*(g^2)^(1/4)
*c^2*g-2*arctan((cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/2*e)
+1)^2*g)^(1/2)*sin(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/2*e)-1)*c/((d*g+(-g^2*(c-
d)*(c+d))^(1/2))*c)^(1/2))*2^(1/2)*((-d*g+(-g^2*(c-d)*(c+d))^(1/2))*c)^(1/
2)*(g^2)^(1/4)*c*d*g+2*arctanh((cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1/2*e)/(cos
(1/2*f*x+1/2*e)+1)^2*g)^(1/2)*sin(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/2*e)-1)*c/
((-d*g+(-g^2*(c-d)*(c+d))^(1/2))*c)^(1/2))*(-g^2*(c-d)*(c+d))^(1/2)*2^(1/2)
)*((d*g+(-g^2*(c-d)*(c+d))^(1/2))*c)^(1/2)*(g^2)^(1/4)*c-2*arctan((cos(1/2
*f*x+1/2*e)*sin(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/2*e)+1)^2*g)^(1/2)*sin(1/2*f
*x+1/2*e)/(cos(1/2*f*x+1/2*e)-1)*c/((d*g+(-g^2*(c-d)*(c+d))^(1/2))*c)^(...

```

Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 271 vs. $2(117) = 234$.

Time = 1.28 (sec) , antiderivative size = 3273, normalized size of antiderivative = 21.97

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)}}{c + d \sin(e + fx)} dx = \text{Too large to display}$$

input

```

integrate((g*sin(f*x+e))^(1/2)*(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,
algorithm="fricas")

```

output

Too large to include

Sympy [F]

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)}}{c + d \sin(e + fx)} dx = \int \frac{\sqrt{a(\sin(e + fx) + 1)} \sqrt{g \sin(e + fx)}}{c + d \sin(e + fx)} dx$$

input `integrate((g*sin(f*x+e))**(1/2)*(a+a*sin(f*x+e))**(1/2)/(c+d*sin(f*x+e)),x)`

output `Integral(sqrt(a*(sin(e + f*x) + 1))*sqrt(g*sin(e + f*x))/(c + d*sin(e + f*x)), x)`

Maxima [F]

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)}}{c + d \sin(e + fx)} dx = \int \frac{\sqrt{a \sin(fx + e) + a} \sqrt{g \sin(fx + e)}}{d \sin(fx + e) + c} dx$$

input `integrate((g*sin(f*x+e))^(1/2)*(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x, algorithm="maxima")`

output `integrate(sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))/(d*sin(f*x + e) + c), x)`

Giac [F(-1)]

Timed out.

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)}}{c + d \sin(e + fx)} dx = \text{Timed out}$$

input `integrate((g*sin(f*x+e))^(1/2)*(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x, algorithm="giac")`

output `Timed out`

Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)}}{c + d \sin(e + fx)} dx = \int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)}}{c + d \sin(e + fx)} dx$$

input `int(((g*sin(e + f*x))^(1/2)*(a + a*sin(e + f*x))^(1/2))/(c + d*sin(e + f*x))),x)`

output `int(((g*sin(e + f*x))^(1/2)*(a + a*sin(e + f*x))^(1/2))/(c + d*sin(e + f*x))), x)`

Reduce [F]

$$\begin{aligned} & \int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)}}{c + d \sin(e + fx)} dx \\ &= \sqrt{g} \sqrt{a} \left(\int \frac{\sqrt{\sin(fx + e)} \sqrt{\sin(fx + e) + 1}}{\sin(fx + e) d + c} dx \right) \end{aligned}$$

input `int((g*sin(f*x+e))^(1/2)*(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x)`

output `sqrt(g)*sqrt(a)*int((sqrt(sin(e + f*x))*sqrt(sin(e + f*x) + 1))/(sin(e + f*x)*d + c),x)`

3.26
$$\int \frac{\sqrt{a+a \sin(e+fx)}}{\sqrt{g \sin(e+fx)}(c+d \sin(e+fx))} dx$$

Optimal result	231
Mathematica [C] (warning: unable to verify)	231
Rubi [A] (verified)	232
Maple [B] (verified)	233
Fricas [B] (verification not implemented)	234
Sympy [F]	235
Maxima [F]	236
Giac [F]	236
Mupad [F(-1)]	236
Reduce [F]	237

Optimal result

Integrand size = 39, antiderivative size = 83

$$\int \frac{\sqrt{a+a \sin(e+fx)}}{\sqrt{g \sin(e+fx)}(c+d \sin(e+fx))} dx = -\frac{2\sqrt{a} \arctan\left(\frac{\sqrt{a}\sqrt{c}\sqrt{g} \cos(e+fx)}{\sqrt{c+d}\sqrt{g \sin(e+fx)}\sqrt{a+a \sin(e+fx)}}\right)}{\sqrt{c}\sqrt{c+df}\sqrt{g}}$$

output -2*a^(1/2)*arctan(a^(1/2)*c^(1/2)*g^(1/2)*cos(f*x+e)/(c+d)^(1/2)/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2))/c^(1/2)/(c+d)^(1/2)/f/g^(1/2)

Mathematica [C] (warning: unable to verify)

Result contains complex when optimal does not.

Time = 4.70 (sec) , antiderivative size = 436, normalized size of antiderivative = 5.25

$$\int \frac{\sqrt{a+a \sin(e+fx)}}{\sqrt{g \sin(e+fx)}(c+d \sin(e+fx))} dx = \left(\frac{1}{4} + \frac{i}{4}\right) g \left(\sqrt{c+i\sqrt{-c^2+d^2}}(ic-id+\sqrt{-c^2+d^2}) \arctan\left(\frac{d-(-ic+\sqrt{-c^2+d^2})(\cos(e+fx)+i \sin(e+fx))}{\sqrt{2}\sqrt{c+i\sqrt{-c^2+d^2}}\sqrt{-1+\cos(2(e+fx))+i \sin(2(e+fx))}}\right)\right)$$

input

```
Integrate[Sqrt[a + a*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c + d*Sin[e + f*x])),x]
```

output

```
((1/4 + I/4)*g*(Sqrt[c + I*Sqrt[-c^2 + d^2]]*(I*c - I*d + Sqrt[-c^2 + d^2])
)*ArcTan[(d - ((-I)*c + Sqrt[-c^2 + d^2])*(Cos[e + f*x] + I*Sin[e + f*x]))
]/(Sqrt[2]*Sqrt[c]*Sqrt[c + I*Sqrt[-c^2 + d^2]]*Sqrt[-1 + Cos[2*(e + f*x)]
+ I*Sin[2*(e + f*x)]]) + Sqrt[c - I*Sqrt[-c^2 + d^2]]*((-I)*c + I*d + Sqr
t[-c^2 + d^2])*ArcTan[(d + (I*c + Sqrt[-c^2 + d^2])*(Cos[e + f*x] + I*Sin[
e + f*x]))/(Sqrt[2]*Sqrt[c]*Sqrt[c - I*Sqrt[-c^2 + d^2]]*Sqrt[-1 + Cos[2*(
e + f*x)] + I*Sin[2*(e + f*x)]])]*Sqrt[a*(1 + Sin[e + f*x])]*(Cos[(3*(e +
f*x))/2] - I*Sin[(3*(e + f*x))/2])*(-1 + Cos[2*(e + f*x)] + I*Sin[2*(e +
f*x)])^(3/2))/(Sqrt[2]*Sqrt[c]*d*Sqrt[-c^2 + d^2]*f*(Cos[(e + f*x)/2] + Si
n[(e + f*x)/2])*(g*Sin[e + f*x])^(3/2))
```

Rubi [A] (verified)

Time = 0.39 (sec) , antiderivative size = 83, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.077$, Rules used = {3042, 3409, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{\sqrt{a \sin(e + fx) + a}}{\sqrt{g \sin(e + fx)}(c + d \sin(e + fx))} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{\sqrt{a \sin(e + fx) + a}}{\sqrt{g \sin(e + fx)}(c + d \sin(e + fx))} dx \\
 & \quad \downarrow \text{3409} \\
 & \frac{2a \int \frac{1}{\frac{c \cos(e+fx) \cot(e+fx)a^2}{\sin(e+fx)a+a} + (c+d)a} d \frac{a \cos(e+fx)}{\sqrt{g \sin(e+fx)} \sqrt{\sin(e+fx)a+a}}}{f} \\
 & \quad \downarrow \text{218} \\
 & \frac{2\sqrt{a} \arctan\left(\frac{\sqrt{a}\sqrt{c}\sqrt{g} \cos(e+fx)}{\sqrt{c+d}\sqrt{a \sin(e+fx)+a}\sqrt{g \sin(e+fx)}}\right)}{\sqrt{c}f\sqrt{g}\sqrt{c+d}}
 \end{aligned}$$

input `Int[Sqrt[a + a*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c + d*Sin[e + f*x])),x]`

output `(-2*Sqrt[a]*ArcTan[(Sqrt[a]*Sqrt[c]*Sqrt[g]*Cos[e + f*x])/(Sqrt[c + d]*Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]])]/(Sqrt[c]*Sqrt[c + d]*f*Sqrt[g]))`

Defintions of rubi rules used

rule 218 `Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(Rt[a/b, 2]/a)*ArcTan[x/Rt[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3409 `Int[Sqrt[(a_) + (b_.)*sin[(e_) + (f_.)*(x_)]]/(Sqrt[(g_.)*sin[(e_) + (f_.)*(x_)]]*(c_) + (d_.)*sin[(e_) + (f_.)*(x_)])], x_Symbol] := Simp[-2*(b/f) Subst[Int[1/(b*c + a*d + c*g*x^2), x], x, b*(Cos[e + f*x]/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]])], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0]`

Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 874 vs. $2(63) = 126$.

Time = 23.26 (sec) , antiderivative size = 875, normalized size of antiderivative = 10.54

method	result	size
default	Expression too large to display	875

input `int((a+a*sin(f*x+e))^(1/2)/(g*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,method=_RETURNVERBOSE)`

output

```

2^(1/2)/f*(-((-d*g+(-g^2*(c-d)*(c+d))^(1/2))*c)^(1/2)*arctan((cos(1/2*f*x+
1/2*e)*sin(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/2*e)+1)^2*g)^(1/2)*sin(1/2*f*x+1/
2*e)/(cos(1/2*f*x+1/2*e)-1)*c/((d*g+(-g^2*(c-d)*(c+d))^(1/2))*c)^(1/2))*c*
g+((-d*g+(-g^2*(c-d)*(c+d))^(1/2))*c)^(1/2)*arctan((cos(1/2*f*x+1/2*e)*sin
(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/2*e)+1)^2*g)^(1/2)*sin(1/2*f*x+1/2*e)/(cos(
1/2*f*x+1/2*e)-1)*c/((d*g+(-g^2*(c-d)*(c+d))^(1/2))*c)^(1/2))*d*g-((d*g+(-
g^2*(c-d)*(c+d))^(1/2))*c)^(1/2)*arctanh((cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1
/2*e)/(cos(1/2*f*x+1/2*e)+1)^2*g)^(1/2)*sin(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/
2*e)-1)*c/((-d*g+(-g^2*(c-d)*(c+d))^(1/2))*c)^(1/2))*c*g+((d*g+(-g^2*(c-d)
*(c+d))^(1/2))*c)^(1/2)*arctanh((cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1/2*e)/(co
s(1/2*f*x+1/2*e)+1)^2*g)^(1/2)*sin(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/2*e)-1)*c
/((-d*g+(-g^2*(c-d)*(c+d))^(1/2))*c)^(1/2))*d*g+((-d*g+(-g^2*(c-d)*(c+d))
^(1/2))*c)^(1/2)*(-g^2*(c-d)*(c+d))^(1/2)*arctan((cos(1/2*f*x+1/2*e)*sin(1/
2*f*x+1/2*e)/(cos(1/2*f*x+1/2*e)+1)^2*g)^(1/2)*sin(1/2*f*x+1/2*e)/(cos(1/2
*f*x+1/2*e)-1)*c/((d*g+(-g^2*(c-d)*(c+d))^(1/2))*c)^(1/2))-((d*g+(-g^2*(c-
d)*(c+d))^(1/2))*c)^(1/2)*(-g^2*(c-d)*(c+d))^(1/2)*arctanh((cos(1/2*f*x+1/
2*e)*sin(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/2*e)+1)^2*g)^(1/2)*sin(1/2*f*x+1/2*
e)/(cos(1/2*f*x+1/2*e)-1)*c/((-d*g+(-g^2*(c-d)*(c+d))^(1/2))*c)^(1/2))*((
2*cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1/2*e)+1)*a)^(1/2)*(cos(1/2*f*x+1/2*e)*si
n(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/2*e)+1)^2*g)^(1/2)*(cos(1/2*f*x+1/2*e)+...

```

Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 168 vs. $2(63) = 126$.

Time = 0.80 (sec) , antiderivative size = 1303, normalized size of antiderivative = 15.70

$$\int \frac{\sqrt{a + a \sin(e + fx)}}{\sqrt{g \sin(e + fx)(c + d \sin(e + fx))}} dx = \text{Too large to display}$$

input

```

integrate((a+a*sin(f*x+e))^(1/2)/(g*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,
algorithm="fricas")

```

output

```
[1/4*sqrt(-a/((c^2 + c*d)*g))*log(((128*a*c^4 + 256*a*c^3*d + 160*a*c^2*d^2 + 32*a*c*d^3 + a*d^4)*cos(f*x + e)^5 + a*c^4 + 4*a*c^3*d + 6*a*c^2*d^2 + 4*a*c*d^3 + a*d^4 - (128*a*c^4 + 192*a*c^3*d + 64*a*c^2*d^2 - 4*a*c*d^3 - a*d^4)*cos(f*x + e)^4 - 2*(208*a*c^4 + 368*a*c^3*d + 195*a*c^2*d^2 + 32*a*c*d^3 + a*d^4)*cos(f*x + e)^3 + 2*(64*a*c^4 + 94*a*c^3*d + 29*a*c^2*d^2 - 4*a*c*d^3 - a*d^4)*cos(f*x + e)^2 - 8*(51*c^5 + 110*c^4*d + 76*c^3*d^2 + 18*c^2*d^3 + c*d^4 + (16*c^5 + 40*c^4*d + 34*c^3*d^2 + 11*c^2*d^3 + c*d^4)*cos(f*x + e)^4 - (24*c^5 + 52*c^4*d + 35*c^3*d^2 + 7*c^2*d^3)*cos(f*x + e)^3 - (66*c^5 + 149*c^4*d + 110*c^3*d^2 + 29*c^2*d^3 + 2*c*d^4)*cos(f*x + e)^2 + (25*c^5 + 53*c^4*d + 35*c^3*d^2 + 7*c^2*d^3)*cos(f*x + e) - (51*c^5 + 110*c^4*d + 76*c^3*d^2 + 18*c^2*d^3 + c*d^4 - (16*c^5 + 40*c^4*d + 34*c^3*d^2 + 11*c^2*d^3 + c*d^4)*cos(f*x + e)^3 - (40*c^5 + 92*c^4*d + 69*c^3*d^2 + 18*c^2*d^3 + c*d^4)*cos(f*x + e)^2 + (26*c^5 + 57*c^4*d + 41*c^3*d^2 + 11*c^2*d^3 + c*d^4)*cos(f*x + e))*sqrt(a*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))*sqrt(-a/((c^2 + c*d)*g)) + (289*a*c^4 + 480*a*c^3*d + 230*a*c^2*d^2 + 32*a*c*d^3 + a*d^4)*cos(f*x + e) + (a*c^4 + 4*a*c^3*d + 6*a*c^2*d^2 + 4*a*c*d^3 + a*d^4 + (128*a*c^4 + 256*a*c^3*d + 160*a*c^2*d^2 + 32*a*c*d^3 + a*d^4)*cos(f*x + e)^4 + 4*(64*a*c^4 + 112*a*c^3*d + 56*a*c^2*d^2 + 7*a*c*d^3)*cos(f*x + e)^3 - 2*(80*a*c^4 + 144*a*c^3*d + 83*a*c^2*d^2 + 18*a*c*d^3 + a*d^4)*cos(f*x + e)^2 - 4*(72*a*c^4 + 119*a*c^3*d + ...
```

Sympy [F]

$$\int \frac{\sqrt{a + a \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c + d \sin(e + fx))} dx = \int \frac{\sqrt{a (\sin(e + fx) + 1)}}{\sqrt{g \sin(e + fx)}(c + d \sin(e + fx))} dx$$

input

```
integrate((a+a*sin(f*x+e))**(1/2)/(g*sin(f*x+e))**(1/2)/(c+d*sin(f*x+e)),x)
```

output

```
Integral(sqrt(a*(sin(e + f*x) + 1))/(sqrt(g*sin(e + f*x))*(c + d*sin(e + f*x))), x)
```

Maxima [F]

$$\int \frac{\sqrt{a + a \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c + d \sin(e + fx))} dx = \int \frac{\sqrt{a \sin(fx + e) + a}}{(d \sin(fx + e) + c)\sqrt{g \sin(fx + e)}} dx$$

input `integrate((a+a*sin(f*x+e))^(1/2)/(g*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,
algorithm="maxima")`

output `integrate(sqrt(a*sin(f*x + e) + a)/((d*sin(f*x + e) + c)*sqrt(g*sin(f*x +
e))), x)`

Giac [F]

$$\int \frac{\sqrt{a + a \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c + d \sin(e + fx))} dx = \int \frac{\sqrt{a \sin(fx + e) + a}}{(d \sin(fx + e) + c)\sqrt{g \sin(fx + e)}} dx$$

input `integrate((a+a*sin(f*x+e))^(1/2)/(g*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,
algorithm="giac")`

output `integrate(sqrt(a*sin(f*x + e) + a)/((d*sin(f*x + e) + c)*sqrt(g*sin(f*x +
e))), x)`

Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{a + a \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c + d \sin(e + fx))} dx = \int \frac{\sqrt{a + a \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c + d \sin(e + fx))} dx$$

input `int((a + a*sin(e + f*x))^(1/2)/((g*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x)
)),x)`

output `int((a + a*sin(e + f*x))^(1/2)/((g*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))), x)`

Reduce [F]

$$\int \frac{\sqrt{a + a \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c + d \sin(e + fx))} dx = \frac{\sqrt{g} \sqrt{a} \left(\int \frac{\sqrt{\sin(fx+e)} \sqrt{\sin(fx+e)+1}}{\sin(fx+e)^2 d + \sin(fx+e) c} dx \right)}{g}$$

input `int((a+a*sin(f*x+e))^(1/2)/(g*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x)`

output `(sqrt(g)*sqrt(a)*int((sqrt(sin(e + f*x))*sqrt(sin(e + f*x) + 1))/(sin(e + f*x)**2*d + sin(e + f*x)*c),x))/g`

$$3.27 \quad \int \frac{\sqrt{g \sin(e+fx)}}{\sqrt{a+a \sin(e+fx)}(c+d \sin(e+fx))} dx$$

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Optimal result

Integrand size = 39, antiderivative size = 166

$$\begin{aligned} & \int \frac{\sqrt{g \sin(e+fx)}}{\sqrt{a+a \sin(e+fx)}(c+d \sin(e+fx))} dx \\ &= \frac{\sqrt{2}\sqrt{g} \arctan\left(\frac{\sqrt{a}\sqrt{g} \cos(e+fx)}{\sqrt{2}\sqrt{g \sin(e+fx)}\sqrt{a+a \sin(e+fx)}}\right)}{\sqrt{a}(c-d)f} \\ & \quad - \frac{2\sqrt{c}\sqrt{g} \arctan\left(\frac{\sqrt{a}\sqrt{c}\sqrt{g} \cos(e+fx)}{\sqrt{c+d}\sqrt{g \sin(e+fx)}\sqrt{a+a \sin(e+fx)}}\right)}{\sqrt{a}(c-d)\sqrt{c+d}f} \end{aligned}$$

output

```
2^(1/2)*g^(1/2)*arctan(1/2*a^(1/2)*g^(1/2)*cos(f*x+e)*2^(1/2)/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2))/a^(1/2)/(c-d)/f-2*c^(1/2)*g^(1/2)*arctan(a^(1/2)*c^(1/2)*g^(1/2)*cos(f*x+e)/(c+d)^(1/2)/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2))/a^(1/2)/(c-d)/(c+d)^(1/2)/f
```

Mathematica [A] (verified)

Time = 10.14 (sec) , antiderivative size = 280, normalized size of antiderivative = 1.69

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + a \sin(e + fx)(c + d \sin(e + fx))}} dx =$$

$$\frac{\left(2\sqrt{c}\sqrt{-c^2 + d^2} \arctan\left(\sqrt{\tan\left(\frac{1}{2}(e + fx)\right)}\right) - \sqrt{d - \sqrt{-c^2 + d^2}}(-c + d + \sqrt{-c^2 + d^2}) \arctan\left(\frac{\sqrt{c}}{\sqrt{d - \sqrt{-c^2 + d^2}}}\right) \right)}{\sqrt{c}(c - d)\sqrt{-c^2 + d^2}}$$

input

```
Integrate[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + a*Sin[e + f*x]]*(c + d*Sin[e + f*x])),x]
```

output

```
-(((2*Sqrt[c]*Sqrt[-c^2 + d^2]*ArcTan[Sqrt[Tan[(e + f*x)/2]]] - Sqrt[d - Sqrt[-c^2 + d^2]]*(-c + d + Sqrt[-c^2 + d^2])*ArcTan[(Sqrt[c]*Sqrt[Tan[(e + f*x)/2]])/Sqrt[d - Sqrt[-c^2 + d^2]]] - (c - d + Sqrt[-c^2 + d^2])*Sqrt[d + Sqrt[-c^2 + d^2]]*ArcTan[(Sqrt[c]*Sqrt[Tan[(e + f*x)/2]])/Sqrt[d + Sqrt[-c^2 + d^2]])]*Sqrt[g*Sin[e + f*x]]*(1 + Tan[(e + f*x)/2]))/(Sqrt[c]*(c - d)*Sqrt[-c^2 + d^2]*f*Sqrt[a*(1 + Sin[e + f*x])]*Sqrt[Tan[(e + f*x)/2]]))
```

Rubi [A] (verified)

Time = 0.89 (sec) , antiderivative size = 166, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}} = 0.179$, Rules used = {3042, 3415, 3042, 3261, 218, 3409, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a \sin(e + fx) + a(c + d \sin(e + fx))}} dx$$

$$\downarrow \text{3042}$$

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a \sin(e + fx) + a(c + d \sin(e + fx))}} dx$$

$$\downarrow \text{3415}$$

$$\begin{aligned}
& \frac{cg \int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)(c+d \sin(e+fx))}} dx}{a(c-d)} - \frac{g \int \frac{1}{\sqrt{g \sin(e+fx)} \sqrt{\sin(e+fx)a+a}} dx}{c-d} \\
& \quad \downarrow \text{3042} \\
& \frac{cg \int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)(c+d \sin(e+fx))}} dx}{a(c-d)} - \frac{g \int \frac{1}{\sqrt{g \sin(e+fx)} \sqrt{\sin(e+fx)a+a}} dx}{c-d} \\
& \quad \downarrow \text{3261} \\
& \frac{2ag \int \frac{1}{\frac{\cos(e+fx) \cot(e+fx)a^3}{\sin(e+fx)a+a} + 2a^2}}{f(c-d)} d \frac{a \cos(e+fx)}{\sqrt{g \sin(e+fx)} \sqrt{\sin(e+fx)a+a}} + \frac{cg \int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)(c+d \sin(e+fx))}} dx}{a(c-d)} \\
& \quad \downarrow \text{218} \\
& \frac{cg \int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)(c+d \sin(e+fx))}} dx}{a(c-d)} + \frac{\sqrt{2}\sqrt{g} \arctan\left(\frac{\sqrt{a}\sqrt{g} \cos(e+fx)}{\sqrt{2}\sqrt{a \sin(e+fx)+a}\sqrt{g \sin(e+fx)}}\right)}{\sqrt{a}f(c-d)} \\
& \quad \downarrow \text{3409} \\
& \frac{\sqrt{2}\sqrt{g} \arctan\left(\frac{\sqrt{a}\sqrt{g} \cos(e+fx)}{\sqrt{2}\sqrt{a \sin(e+fx)+a}\sqrt{g \sin(e+fx)}}\right)}{\sqrt{a}f(c-d)} - \\
& \frac{2cg \int \frac{1}{\frac{c \cos(e+fx) \cot(e+fx)a^2}{\sin(e+fx)a+a} + (c+d)a}}{f(c-d)} d \frac{a \cos(e+fx)}{\sqrt{g \sin(e+fx)} \sqrt{\sin(e+fx)a+a}} \\
& \quad \downarrow \text{218} \\
& \frac{\sqrt{2}\sqrt{g} \arctan\left(\frac{\sqrt{a}\sqrt{g} \cos(e+fx)}{\sqrt{2}\sqrt{a \sin(e+fx)+a}\sqrt{g \sin(e+fx)}}\right)}{\sqrt{a}f(c-d)} - \frac{2\sqrt{c}\sqrt{g} \arctan\left(\frac{\sqrt{a}\sqrt{c}\sqrt{g} \cos(e+fx)}{\sqrt{c+d}\sqrt{a \sin(e+fx)+a}\sqrt{g \sin(e+fx)}}\right)}{\sqrt{a}f(c-d)\sqrt{c+d}}
\end{aligned}$$

input `Int[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + a*Sin[e + f*x]]*(c + d*Sin[e + f*x])),x]`

output `(Sqrt[2]*Sqrt[g]*ArcTan[(Sqrt[a]*Sqrt[g]*Cos[e + f*x])/(Sqrt[2]*Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]])]/(Sqrt[a]*(c - d)*f) - (2*Sqrt[c]*Sqrt[g]*ArcTan[(Sqrt[a]*Sqrt[c]*Sqrt[g]*Cos[e + f*x])/(Sqrt[c + d]*Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]])]/(Sqrt[a]*(c - d)*Sqrt[c + d]*f))`

Definitions of rubi rules used

rule 218 $\text{Int}[(a_ + (b_ \cdot x)^2)^{-1}, x_Symbol] \rightarrow \text{Simp}[(\text{Rt}[a/b, 2]/a) \cdot \text{ArcTan}[x/\text{Rt}[a/b, 2]], x] \text{ ; FreeQ}\{a, b\}, x \ \&\& \ \text{PosQ}[a/b]$

rule 3042 $\text{Int}[u, x_Symbol] \rightarrow \text{Int}[\text{DeactivateTrig}[u, x], x] \text{ ; FunctionOfTrigOfLinearQ}[u, x]$

rule 3261 $\text{Int}[1/(\text{Sqrt}[a_ + (b_ \cdot \sin[e_ + (f_ \cdot x)])] \cdot \text{Sqrt}[c_ + (d_ \cdot \sin[e_ + (f_ \cdot x)])]), x_Symbol] \rightarrow \text{Simp}[-2 \cdot (a/f) \text{ Subst}[\text{Int}[1/(2 \cdot b^2 - (a \cdot c - b \cdot d) \cdot x^2), x], x, b \cdot (\text{Cos}[e + f \cdot x]/(\text{Sqrt}[a + b \cdot \text{Sin}[e + f \cdot x]] \cdot \text{Sqrt}[c + d \cdot \text{Sin}[e + f \cdot x]])]), x] \text{ ; FreeQ}\{a, b, c, d, e, f\}, x \ \&\& \ \text{NeQ}[b \cdot c - a \cdot d, 0] \ \&\& \ \text{EqQ}[a^2 - b^2, 0] \ \&\& \ \text{NeQ}[c^2 - d^2, 0]$

rule 3409 $\text{Int}[\text{Sqrt}[a_ + (b_ \cdot \sin[e_ + (f_ \cdot x)])]/(\text{Sqrt}[g_ \cdot \sin[e_ + (f_ \cdot x)] \cdot (x)] \cdot ((c_ + (d_ \cdot \sin[e_ + (f_ \cdot x)]))), x_Symbol] \rightarrow \text{Simp}[-2 \cdot (b/f) \text{ Subst}[\text{Int}[1/(b \cdot c + a \cdot d + c \cdot g \cdot x^2), x], x, b \cdot (\text{Cos}[e + f \cdot x]/(\text{Sqrt}[g \cdot \text{Sin}[e + f \cdot x]] \cdot \text{Sqrt}[a + b \cdot \text{Sin}[e + f \cdot x]])]), x] \text{ ; FreeQ}\{a, b, c, d, e, f, g\}, x \ \&\& \ \text{NeQ}[b \cdot c - a \cdot d, 0] \ \&\& \ \text{EqQ}[a^2 - b^2, 0]$

rule 3415 $\text{Int}[\text{Sqrt}[g_ \cdot \sin[e_ + (f_ \cdot x)]]/(\text{Sqrt}[a_ + (b_ \cdot \sin[e_ + (f_ \cdot x)] \cdot (x)] \cdot ((c_ + (d_ \cdot \sin[e_ + (f_ \cdot x)]))), x_Symbol] \rightarrow \text{Simp}[(-a) \cdot (g/(b \cdot c - a \cdot d)) \text{ Int}[1/(\text{Sqrt}[g \cdot \text{Sin}[e + f \cdot x]] \cdot \text{Sqrt}[a + b \cdot \text{Sin}[e + f \cdot x]]), x], x] + \text{Simp}[c \cdot (g/(b \cdot c - a \cdot d)) \text{ Int}[\text{Sqrt}[a + b \cdot \text{Sin}[e + f \cdot x]]/(\text{Sqrt}[g \cdot \text{Sin}[e + f \cdot x]] \cdot (c + d \cdot \text{Sin}[e + f \cdot x])), x], x] \text{ ; FreeQ}\{a, b, c, d, e, f, g\}, x \ \&\& \ \text{NeQ}[b \cdot c - a \cdot d, 0] \ \&\& \ (\text{EqQ}[a^2 - b^2, 0] \ || \ \text{EqQ}[c^2 - d^2, 0])$

Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 1034 vs. $2(131) = 262$.

Time = 22.90 (sec) , antiderivative size = 1035, normalized size of antiderivative = 6.23

method	result	size
default	Expression too large to display	1035

input

```
int((g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,method=
_RETURNVERBOSE)
```

output

```
-2^(1/2)/f*(g*sin(1/2*f*x+1/2*e)*cos(1/2*f*x+1/2*e))^(1/2)*(g^(5/2)*((-d*g
+(-g^2*(c-d)*(c+d))^(1/2))*c)^(1/2)*arctan((cos(1/2*f*x+1/2*e)*sin(1/2*f*x
+1/2*e)/(cos(1/2*f*x+1/2*e)+1)^2*g)^(1/2)*sin(1/2*f*x+1/2*e)/(cos(1/2*f*x+
1/2*e)-1)*c/((d*g+(-g^2*(c-d)*(c+d))^(1/2))*c)^(1/2))*c^2-g^(5/2)*((-d*g+(
-g^2*(c-d)*(c+d))^(1/2))*c)^(1/2)*arctan((cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1
/2*e)/(cos(1/2*f*x+1/2*e)+1)^2*g)^(1/2)*sin(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/2*
e)-1)*c/((d*g+(-g^2*(c-d)*(c+d))^(1/2))*c)^(1/2))*c*d+g^(5/2)*((d*g+(-g^
2*(c-d)*(c+d))^(1/2))*c)^(1/2)*arctanh((cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1/2*
e)/(cos(1/2*f*x+1/2*e)+1)^2*g)^(1/2)*sin(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/2*
e)-1)*c/((-d*g+(-g^2*(c-d)*(c+d))^(1/2))*c)^(1/2))*c^2-g^(5/2)*((d*g+(-g^2
*(c-d)*(c+d))^(1/2))*c)^(1/2)*arctanh((cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1/2*
e)/(cos(1/2*f*x+1/2*e)+1)^2*g)^(1/2)*sin(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/2*
e)-1)*c/((-d*g+(-g^2*(c-d)*(c+d))^(1/2))*c)^(1/2))*c*d-g^(3/2)*((-d*g+(-g^2
*(c-d)*(c+d))^(1/2))*c)^(1/2)*(-g^2*(c-d)*(c+d))^(1/2)*arctan((cos(1/2*f*x
+1/2*e)*sin(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/2*e)+1)^2*g)^(1/2)*sin(1/2*f*x+1
/2*e)/(cos(1/2*f*x+1/2*e)-1)*c/((d*g+(-g^2*(c-d)*(c+d))^(1/2))*c)^(1/2))*c
+g^(3/2)*((d*g+(-g^2*(c-d)*(c+d))^(1/2))*c)^(1/2)*(-g^2*(c-d)*(c+d))^(1/2)
*arctanh((cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/2*e)+1)^2*g)
^(1/2)*sin(1/2*f*x+1/2*e)/(cos(1/2*f*x+1/2*e)-1)*c/((-d*g+(-g^2*(c-d)*(c+
d))^(1/2))*c)^(1/2))*c+2*arctan((cos(1/2*f*x+1/2*e)*sin(1/2*f*x+1/2*e)/...
```

Fricas [A] (verification not implemented)

Time = 1.15 (sec) , antiderivative size = 3048, normalized size of antiderivative = 18.36

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + a \sin(e + fx)}(c + d \sin(e + fx))} dx = \text{Too large to display}$$

input

```
integrate((g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,
algorithm="fricas")
```

output

Too large to include

Sympy [F]

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + a \sin(e + fx)}(c + d \sin(e + fx))} dx$$

$$= \int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a(\sin(e + fx) + 1)}(c + d \sin(e + fx))} dx$$

input `integrate((g*sin(f*x+e))**(1/2)/(a+a*sin(f*x+e))**(1/2)/(c+d*sin(f*x+e)),x)`

output `Integral(sqrt(g*sin(e + f*x))/(sqrt(a*(sin(e + f*x) + 1))*(c + d*sin(e + f*x))), x)`

Maxima [F]

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + a \sin(e + fx)}(c + d \sin(e + fx))} dx$$

$$= \int \frac{\sqrt{g \sin(fx + e)}}{\sqrt{a \sin(fx + e) + a}(d \sin(fx + e) + c)} dx$$

input `integrate((g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x, algorithm="maxima")`

output `integrate(sqrt(g*sin(f*x + e))/(sqrt(a*sin(f*x + e) + a)*(d*sin(f*x + e) + c)), x)`

Giac [F(-1)]

Timed out.

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + a \sin(e + fx)}(c + d \sin(e + fx))} dx = \text{Timed out}$$

input `integrate((g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,
algorithm="giac")`

output Timed out

Mupad [F(-1)]

Timed out.

$$\begin{aligned} & \int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + a \sin(e + fx)}(c + d \sin(e + fx))} dx \\ &= \int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + a \sin(e + fx)}(c + d \sin(e + fx))} dx \end{aligned}$$

input `int((g*sin(e + f*x))^(1/2)/((a + a*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))`

output `int((g*sin(e + f*x))^(1/2)/((a + a*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))`

Reduce [F]

$$\begin{aligned} & \int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + a \sin(e + fx)}(c + d \sin(e + fx))} dx \\ &= \frac{\sqrt{g} \sqrt{a} \left(\int \frac{\sqrt{\sin(fx+e)} \sqrt{\sin(fx+e)+1}}{\sin(fx+e)^2 d + \sin(fx+e)c + \sin(fx+e)d+c} dx \right)}{a} \end{aligned}$$

input `int((g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x)`

output `(sqrt(g)*sqrt(a)*int((sqrt(sin(e + f*x))*sqrt(sin(e + f*x) + 1))/(sin(e + f*x)**2*d + sin(e + f*x)*c + sin(e + f*x)*d + c),x))/a`

3.28
$$\int \frac{1}{\sqrt{g \sin(e+fx)} \sqrt{a+a \sin(e+fx)}(c+d \sin(e+fx))} dx$$

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Optimal result

Integrand size = 39, antiderivative size = 168

$$\int \frac{1}{\sqrt{g \sin(e+fx)} \sqrt{a+a \sin(e+fx)}(c+d \sin(e+fx))} dx$$

$$= -\frac{\sqrt{2} \arctan\left(\frac{\sqrt{a}\sqrt{g} \cos(e+fx)}{\sqrt{2}\sqrt{g \sin(e+fx)} \sqrt{a+a \sin(e+fx)}}\right)}{\sqrt{a}(c-d)f\sqrt{g}} + \frac{2d \arctan\left(\frac{\sqrt{a}\sqrt{c}\sqrt{g} \cos(e+fx)}{\sqrt{c+d}\sqrt{g \sin(e+fx)} \sqrt{a+a \sin(e+fx)}}\right)}{\sqrt{a}\sqrt{c}(c-d)\sqrt{c+df}\sqrt{g}}$$

output

```
-2^(1/2)*arctan(1/2*a^(1/2)*g^(1/2)*cos(f*x+e)*2^(1/2)/(g*sin(f*x+e))^(1/2)
)/(a+a*sin(f*x+e))^(1/2)/a^(1/2)/(c-d)/f/g^(1/2)+2*d*arctan(a^(1/2)*c^(1/2)
)*g^(1/2)*cos(f*x+e)/(c+d)^(1/2)/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1
/2)/a^(1/2)/c^(1/2)/(c-d)/(c+d)^(1/2)/f/g^(1/2)
```

Mathematica [A] (verified)

Time = 11.03 (sec) , antiderivative size = 289, normalized size of antiderivative = 1.72

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)} (c + d \sin(e + fx))} dx =$$

$$\frac{2 \left(-2 \arctan \left(\sqrt{\tan \left(\frac{1}{2}(e + fx) \right)} \right) + \frac{d \left(1 + \frac{c-d}{\sqrt{-c^2+d^2}} \right) \arctan \left(\frac{\sqrt{c} \sqrt{\tan \left(\frac{1}{2}(e + fx) \right)}}{\sqrt{d - \sqrt{-c^2+d^2}}} \right)}{\sqrt{c} \sqrt{d - \sqrt{-c^2+d^2}}} + \frac{d(-c+d+\sqrt{-c^2+d^2}) \arctan \left(\frac{\sqrt{c}}{\sqrt{d + \sqrt{-c^2+d^2}}} \right)}{\sqrt{c} \sqrt{-c^2+d^2} \sqrt{d + \sqrt{-c^2+d^2}}} \right)}{(c-d)f \sqrt{g \sin(e + fx)} \sqrt{a(1 + \sin(e + fx))}}$$

input

```
Integrate[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]]*(c + d*Sin[e + f*x])),x]
```

output

```
(-2*(-2*ArcTan[Sqrt[Tan[(e + f*x)/2]])] + (d*(1 + (c - d)/Sqrt[-c^2 + d^2])
*ArcTan[(Sqrt[c]*Sqrt[Tan[(e + f*x)/2]])/Sqrt[d - Sqrt[-c^2 + d^2]])/(Sqr
t[c]*Sqrt[d - Sqrt[-c^2 + d^2]]) + (d*(-c + d + Sqrt[-c^2 + d^2])*ArcTan[(
Sqrt[c]*Sqrt[Tan[(e + f*x)/2]])/Sqrt[d + Sqrt[-c^2 + d^2]])/(Sqrt[c]*Sqrt
[-c^2 + d^2]*Sqrt[d + Sqrt[-c^2 + d^2]])*Cos[(e + f*x)/2]*(Cos[(e + f*x)/
2] + Sin[(e + f*x)/2])*Sqrt[Tan[(e + f*x)/2]])/((c - d)*f*Sqrt[g*Sin[e + f
*x]]*Sqrt[a*(1 + Sin[e + f*x])])
```

Rubi [A] (verified)Time = 0.89 (sec) , antiderivative size = 168, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}} = 0.179$, Rules used = {3042, 3417, 3042, 3261, 218, 3409, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{1}{\sqrt{a \sin(e + fx) + a} \sqrt{g \sin(e + fx)} (c + d \sin(e + fx))} dx$$

↓ 3042

$$\begin{aligned}
& \int \frac{1}{\sqrt{a \sin(e+fx) + a} \sqrt{g \sin(e+fx)(c+d \sin(e+fx))}} dx \\
& \quad \downarrow \text{3417} \\
& \frac{\int \frac{1}{\sqrt{g \sin(e+fx)} \sqrt{\sin(e+fx)a+a}} dx}{c-d} - \frac{d \int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)(c+d \sin(e+fx))}} dx}{a(c-d)} \\
& \quad \downarrow \text{3042} \\
& \frac{\int \frac{1}{\sqrt{g \sin(e+fx)} \sqrt{\sin(e+fx)a+a}} dx}{c-d} - \frac{d \int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)(c+d \sin(e+fx))}} dx}{a(c-d)} \\
& \quad \downarrow \text{3261} \\
& - \frac{2a \int \frac{1}{\frac{\cos(e+fx) \cot(e+fx)a^3}{\sin(e+fx)a+a} + 2a^2} dx}{f(c-d)} - \frac{d \int \frac{a \cos(e+fx)}{\sqrt{g \sin(e+fx)} \sqrt{\sin(e+fx)a+a}} dx}{a(c-d)} - \frac{d \int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)(c+d \sin(e+fx))}} dx}{a(c-d)} \\
& \quad \downarrow \text{218} \\
& - \frac{d \int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{g \sin(e+fx)(c+d \sin(e+fx))}} dx}{a(c-d)} - \frac{\sqrt{2} \arctan \left(\frac{\sqrt{a} \sqrt{g} \cos(e+fx)}{\sqrt{2} \sqrt{a \sin(e+fx) + a} \sqrt{g \sin(e+fx)}} \right)}{\sqrt{a} f \sqrt{g} (c-d)} \\
& \quad \downarrow \text{3409} \\
& \frac{2d \int \frac{1}{\frac{c \cos(e+fx) \cot(e+fx)a^2}{\sin(e+fx)a+a} + (c+d)a} dx}{f(c-d)} - \frac{d \int \frac{a \cos(e+fx)}{\sqrt{g \sin(e+fx)} \sqrt{\sin(e+fx)a+a}} dx}{a(c-d)} - \frac{\sqrt{2} \arctan \left(\frac{\sqrt{a} \sqrt{g} \cos(e+fx)}{\sqrt{2} \sqrt{a \sin(e+fx) + a} \sqrt{g \sin(e+fx)}} \right)}{\sqrt{a} f \sqrt{g} (c-d)} \\
& \quad \downarrow \text{218} \\
& \frac{2d \arctan \left(\frac{\sqrt{a} \sqrt{c} \sqrt{g} \cos(e+fx)}{\sqrt{c+d} \sqrt{a \sin(e+fx) + a} \sqrt{g \sin(e+fx)}} \right)}{\sqrt{a} \sqrt{c} f \sqrt{g} (c-d) \sqrt{c+d}} - \frac{\sqrt{2} \arctan \left(\frac{\sqrt{a} \sqrt{g} \cos(e+fx)}{\sqrt{2} \sqrt{a \sin(e+fx) + a} \sqrt{g \sin(e+fx)}} \right)}{\sqrt{a} f \sqrt{g} (c-d)}
\end{aligned}$$

input

```
Int[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + a*Sin[e + f*x]]*(c + d*Sin[e + f*x]))
,x]
```

output

$$-\left(\frac{\sqrt{2} \operatorname{ArcTan}\left[\frac{\sqrt{a} \sqrt{g} \cos[e + f x]}{\sqrt{2} \sqrt{g \sin[e + f x]}}\right] \sqrt{a + a \sin[e + f x]}}{\sqrt{a} (c - d) f \sqrt{g}}\right) + \left(2 d \operatorname{ArcTan}\left[\frac{\sqrt{a} \sqrt{c} \sqrt{g} \cos[e + f x]}{\sqrt{c + d} \sqrt{g \sin[e + f x]}}\right] \sqrt{a + a \sin[e + f x]}\right) / \left(\sqrt{a} \sqrt{c} (c - d) \sqrt{c + d} f \sqrt{g}\right)$$

Defintions of rubi rules used

rule 218

$$\operatorname{Int}\left[\left(\frac{a}{b} + (b x)^2\right)^{-1}, x_{\text{Symbol}}\right] \rightarrow \operatorname{Simp}\left[\frac{\operatorname{Rt}[a/b, 2]}{a} \operatorname{ArcTan}\left[\frac{x}{\operatorname{Rt}[a/b, 2]}\right], x\right] /; \operatorname{FreeQ}\{a, b\}, x \ \&\& \operatorname{PosQ}[a/b]$$

rule 3042

$$\operatorname{Int}[u, x_{\text{Symbol}}] \rightarrow \operatorname{Int}[\operatorname{DeactivateTrig}[u, x], x] /; \operatorname{FunctionOfTrigOfLinearQ}[u, x]$$

rule 3261

$$\operatorname{Int}\left[\frac{1}{\sqrt{(a) + (b) \sin[(e) + (f)(x)]} \sqrt{(c) + (d) \sin[(e) + (f)(x)]}}, x_{\text{Symbol}}\right] \rightarrow \operatorname{Simp}\left[-2 \frac{a}{f} \operatorname{Subst}\left[\operatorname{Int}\left[\frac{1}{2 b^2 - (a c - b d) x^2}\right], x\right], x, b \frac{\cos[e + f x]}{\sqrt{a + b \sin[e + f x]}} \sqrt{c + d \sin[e + f x]}\right], x] /; \operatorname{FreeQ}\{a, b, c, d, e, f\}, x \ \&\& \operatorname{NeQ}[b c - a d, 0] \ \&\& \operatorname{EqQ}[a^2 - b^2, 0] \ \&\& \operatorname{NeQ}[c^2 - d^2, 0]$$

rule 3409

$$\operatorname{Int}\left[\frac{\sqrt{(a) + (b) \sin[(e) + (f)(x)]}}{\sqrt{(g) \sin[(e) + (f)(x)]} (c) + (d) \sin[(e) + (f)(x)]}, x_{\text{Symbol}}\right] \rightarrow \operatorname{Simp}\left[-2 \frac{b}{f} \operatorname{Subst}\left[\operatorname{Int}\left[\frac{1}{b c + a d + c g x^2}\right], x\right], x, b \frac{\cos[e + f x]}{\sqrt{g \sin[e + f x]}} \sqrt{a + b \sin[e + f x]}\right], x] /; \operatorname{FreeQ}\{a, b, c, d, e, f, g\}, x \ \&\& \operatorname{NeQ}[b c - a d, 0] \ \&\& \operatorname{EqQ}[a^2 - b^2, 0]$$

rule 3417

$$\operatorname{Int}\left[\frac{1}{\sqrt{(g) \sin[(e) + (f)(x)]} \sqrt{(a) + (b) \sin[(e) + (f)(x)]} (c) + (d) \sin[(e) + (f)(x)]}, x_{\text{Symbol}}\right] \rightarrow \operatorname{Simp}\left[\frac{b}{b c - a d} \operatorname{Int}\left[\frac{1}{\sqrt{g \sin[e + f x]} \sqrt{a + b \sin[e + f x]}}\right], x\right] - \operatorname{Simp}\left[\frac{d}{b c - a d} \operatorname{Int}\left[\frac{\sqrt{a + b \sin[e + f x]}}{\sqrt{g \sin[e + f x]} (c + d \sin[e + f x])}\right], x\right] /; \operatorname{FreeQ}\{a, b, c, d, e, f, g\}, x \ \&\& \operatorname{NeQ}[b c - a d, 0] \ \&\& (\operatorname{EqQ}[a^2 - b^2, 0] \ || \ \operatorname{EqQ}[c^2 - d^2, 0])$$

Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 1030 vs. $2(133) = 266$.

Time = 18.13 (sec) , antiderivative size = 1031, normalized size of antiderivative = 6.14

method	result	size
default	Expression too large to display	1031

input `int(1/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,method=_RETURNVERBOSE)`

output
$$2^{(1/2)}/f*(\cos(1/2*f*x+1/2*e)*\sin(1/2*f*x+1/2*e)/(\cos(1/2*f*x+1/2*e)+1)^{2*g})^{(1/2)}*(g^{(3/2)}*((d*g+(-g^2*(c-d)*(c+d))^{(1/2)})^c)^{(1/2)}*\operatorname{arctanh}((\cos(1/2*f*x+1/2*e)*\sin(1/2*f*x+1/2*e)/(\cos(1/2*f*x+1/2*e)+1)^{2*g})^{(1/2)}*\sin(1/2*f*x+1/2*e)/(\cos(1/2*f*x+1/2*e)-1)*c/((-d*g+(-g^2*(c-d)*(c+d))^{(1/2)})^c)^{(1/2)}))^{(1/2)})*c*d-g^{(3/2)}*((d*g+(-g^2*(c-d)*(c+d))^{(1/2)})^c)^{(1/2)}*\operatorname{arctanh}((\cos(1/2*f*x+1/2*e)*\sin(1/2*f*x+1/2*e)/(\cos(1/2*f*x+1/2*e)+1)^{2*g})^{(1/2)}*\sin(1/2*f*x+1/2*e)/(\cos(1/2*f*x+1/2*e)-1)*c/((-d*g+(-g^2*(c-d)*(c+d))^{(1/2)})^c)^{(1/2)}))*d^2+g^{(3/2)}*((d*g+(-g^2*(c-d)*(c+d))^{(1/2)})^c)^{(1/2)}*\operatorname{arctan}((\cos(1/2*f*x+1/2*e)*\sin(1/2*f*x+1/2*e)/(\cos(1/2*f*x+1/2*e)+1)^{2*g})^{(1/2)}*\sin(1/2*f*x+1/2*e)/(\cos(1/2*f*x+1/2*e)-1)*c/((d*g+(-g^2*(c-d)*(c+d))^{(1/2)})^c)^{(1/2)}))*c*d-g^{(3/2)}*((d*g+(-g^2*(c-d)*(c+d))^{(1/2)})^c)^{(1/2)}*\operatorname{arctan}((\cos(1/2*f*x+1/2*e)*\sin(1/2*f*x+1/2*e)/(\cos(1/2*f*x+1/2*e)+1)^{2*g})^{(1/2)}*\sin(1/2*f*x+1/2*e)/(\cos(1/2*f*x+1/2*e)-1)*c/((d*g+(-g^2*(c-d)*(c+d))^{(1/2)})^c)^{(1/2)}))*d^2+g^{(1/2)}*((d*g+(-g^2*(c-d)*(c+d))^{(1/2)})^c)^{(1/2)}*(-g^2*(c-d)*(c+d))^{(1/2)}*\operatorname{arctanh}((\cos(1/2*f*x+1/2*e)*\sin(1/2*f*x+1/2*e)/(\cos(1/2*f*x+1/2*e)+1)^{2*g})^{(1/2)}*\sin(1/2*f*x+1/2*e)/(\cos(1/2*f*x+1/2*e)-1)*c/((-d*g+(-g^2*(c-d)*(c+d))^{(1/2)})^c)^{(1/2)}))*d-g^{(1/2)}*((d*g+(-g^2*(c-d)*(c+d))^{(1/2)})^c)^{(1/2)}*(-g^2*(c-d)*(c+d))^{(1/2)}*\operatorname{arctan}((\cos(1/2*f*x+1/2*e)*\sin(1/2*f*x+1/2*e)/(\cos(1/2*f*x+1/2*e)+1)^{2*g})^{(1/2)}*\sin(1/2*f*x+1/2*e)/(\cos(1/2*f*x+1/2*e)-1)*c/((d*g+(-g^2*(c-d)*(c+d))^{(1/2)})^c)^{(1/2)}))*d+2*\operatorname{arctan}((\cos(1/2*f*x+1/...$$

Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 310 vs. 2(133) = 266.

Time = 1.51 (sec) , antiderivative size = 3175, normalized size of antiderivative = 18.90

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)} (c + d \sin(e + fx))} dx = \text{Too large to display}$$

input `integrate(1/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x
, algorithm="fricas")`

output Too large to include

Sympy [F]

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)} (c + d \sin(e + fx))} dx$$

$$= \int \frac{1}{\sqrt{a (\sin(e + fx) + 1)} \sqrt{g \sin(e + fx)} (c + d \sin(e + fx))} dx$$

input `integrate(1/(g*sin(f*x+e))**(1/2)/(a+a*sin(f*x+e))**(1/2)/(c+d*sin(f*x+e))
,x)`

output `Integral(1/(sqrt(a*(sin(e + f*x) + 1))*sqrt(g*sin(e + f*x))*(c + d*sin(e +
f*x))), x)`

Maxima [F]

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)} (c + d \sin(e + fx))} dx$$

$$= \int \frac{1}{\sqrt{a \sin(fx + e) + a} (d \sin(fx + e) + c) \sqrt{g \sin(fx + e)}} dx$$

input `integrate(1/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x
, algorithm="maxima")`

output `integrate(1/(sqrt(a*sin(f*x + e) + a)*(d*sin(f*x + e) + c)*sqrt(g*sin(f*x
+ e))), x)`

Giac [F(-2)]

Exception generated.

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)} (c + d \sin(e + fx))} dx$$

= Exception raised: TypeError

input `integrate(1/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x
, algorithm="giac")`

output `Exception raised: TypeError >> an error occurred running a Giac command:IN
PUT:sage2:=int(sage0,sageVARx):;OUTPUT:index.cc index_m operator + Error:
Bad Argument Valueindex.cc index_m operator + Error: Bad Argument ValueDon
e`

Mupad [F(-1)]

Timed out.

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)} (c + d \sin(e + fx))} dx$$

$$= \int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)} (c + d \sin(e + fx))} dx$$

input

```
int(1/((g*sin(e + f*x))^(1/2)*(a + a*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))),x)
```

output

```
int(1/((g*sin(e + f*x))^(1/2)*(a + a*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))), x)
```

Reduce [F]

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + a \sin(e + fx)} (c + d \sin(e + fx))} dx$$

$$= \frac{\sqrt{g} \sqrt{a} \left(\int \frac{\sqrt{\sin(fx+e)} \sqrt{\sin(fx+e)+1}}{\sin(fx+e)^3 d + \sin(fx+e)^2 c + \sin(fx+e)^2 d + \sin(fx+e) c} dx \right)}{ag}$$

input

```
int(1/(g*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x)
```

output

```
(sqrt(g)*sqrt(a)*int((sqrt(sin(e + f*x))*sqrt(sin(e + f*x) + 1))/(sin(e + f*x)**3*d + sin(e + f*x)**2*c + sin(e + f*x)**2*d + sin(e + f*x)*c),x))/(a*g)
```

3.29
$$\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{c+c\sin(e+fx)} dx$$

Optimal result	254
Mathematica [C] (warning: unable to verify)	255
Rubi [A] (verified)	256
Maple [B] (verified)	261
Fricas [F(-1)]	262
Sympy [F]	263
Maxima [F]	263
Giac [F]	263
Mupad [F(-1)]	264
Reduce [F]	264

Optimal result

Integrand size = 33, antiderivative size = 238

$$\begin{aligned} & \int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{c+c\sin(e+fx)} dx \\ &= \frac{E\left(\frac{1}{2}\left(e-\frac{\pi}{2}+fx\right)\middle|\frac{2b}{a+b}\right)\sqrt{a+b\sin(e+fx)}}{cf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}} \\ & \quad - \frac{(a-b)\operatorname{EllipticF}\left(\frac{1}{2}\left(e-\frac{\pi}{2}+fx\right),\frac{2b}{a+b}\right)\sqrt{\frac{a+b\sin(e+fx)}{a+b}}}{cf\sqrt{a+b\sin(e+fx)}} \\ & \quad + \frac{2a\operatorname{EllipticPi}\left(2,\frac{1}{2}\left(e-\frac{\pi}{2}+fx\right),\frac{2b}{a+b}\right)\sqrt{\frac{a+b\sin(e+fx)}{a+b}}}{cf\sqrt{a+b\sin(e+fx)}} \\ & \quad + \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(c+c\sin(e+fx))} \end{aligned}$$

output

```
-EllipticE(cos(1/2*e+1/4*Pi+1/2*f*x),2^(1/2)*(b/(a+b))^(1/2))*(a+b*sin(f*x+e))^(1/2)/c/f/((a+b*sin(f*x+e))/(a+b))^(1/2)-(a-b)*InverseJacobiAM(1/2*e-1/4*Pi+1/2*f*x,2^(1/2)*(b/(a+b))^(1/2))*((a+b*sin(f*x+e))/(a+b))^(1/2)/c/f/(a+b*sin(f*x+e))^(1/2)-2*a*EllipticPi(cos(1/2*e+1/4*Pi+1/2*f*x),2,2^(1/2)*(b/(a+b))^(1/2))*((a+b*sin(f*x+e))/(a+b))^(1/2)/c/f/(a+b*sin(f*x+e))^(1/2)+cos(f*x+e)*(a+b*sin(f*x+e))^(1/2)/f/(c+c*sin(f*x+e))
```

Mathematica [C] (warning: unable to verify)

Result contains complex when optimal does not.

Time = 9.06 (sec) , antiderivative size = 611, normalized size of antiderivative = 2.57

$$\int \frac{\csc(e + fx) \sqrt{a + b \sin(e + fx)}}{c + c \sin(e + fx)} dx$$

$$= -\frac{2 \sin\left(\frac{1}{2}(e + fx)\right) \left(\cos\left(\frac{1}{2}(e + fx)\right) + \sin\left(\frac{1}{2}(e + fx)\right)\right) \sqrt{a + b \sin(e + fx)}}{f(c + c \sin(e + fx))}$$

$$+ \frac{\left(\cos\left(\frac{1}{2}(e + fx)\right) + \sin\left(\frac{1}{2}(e + fx)\right)\right)^2 \left(-\frac{4b \operatorname{EllipticF}\left(\frac{1}{2}(-e + \frac{\pi}{2} - fx), \frac{2b}{a+b}\right) \sqrt{\frac{a+b \sin(e+fx)}{a+b}}}{\sqrt{a+b \sin(e+fx)}} + \frac{2(-4a-b) \operatorname{EllipticPi}\left(2, \frac{1}{2}(-e + \frac{\pi}{2} - fx)\right)}{\sqrt{a+b \sin(e+fx)}}\right)}{f(c + c \sin(e + fx))}$$

input

```
Integrate[(Csc[e + f*x]*Sqrt[a + b*Sin[e + f*x]])/(c + c*Sin[e + f*x]),x]
```

output

```
(-2*Sin[(e + f*x)/2]*(Cos[(e + f*x)/2] + Sin[(e + f*x)/2])*Sqrt[a + b*Sin[e + f*x]])/(f*(c + c*Sin[e + f*x])) + ((Cos[(e + f*x)/2] + Sin[(e + f*x)/2])^2*((-4*b*EllipticF[(-e + Pi/2 - f*x)/2, (2*b)/(a + b)]*Sqrt[(a + b*Sin[e + f*x])/(a + b)]/Sqrt[a + b*Sin[e + f*x]] + (2*(-4*a - b)*EllipticPi[2, (-e + Pi/2 - f*x)/2, (2*b)/(a + b)]*Sqrt[(a + b*Sin[e + f*x])/(a + b)]/Sqrt[a + b*Sin[e + f*x]] + ((2*I)*b*Cos[e + f*x]*Cos[2*(e + f*x)]*(2*a*(a - b)*EllipticE[I*ArcSinh[Sqrt[-(a + b)^(-1)]*Sqrt[a + b*Sin[e + f*x]]], (a + b)/(a - b)] + b*(2*a*EllipticF[I*ArcSinh[Sqrt[-(a + b)^(-1)]*Sqrt[a + b*Sin[e + f*x]]], (a + b)/(a - b)] - b*EllipticPi[(a + b)/a, I*ArcSinh[Sqrt[-(a + b)^(-1)]*Sqrt[a + b*Sin[e + f*x]]], (a + b)/(a - b)])))*Sqrt[(b - b*Sin[e + f*x])/(a + b)]*Sqrt[-((b + b*Sin[e + f*x])/(a - b))])/(a*Sqrt[-(a + b)^(-1)]*Sqrt[1 - Sin[e + f*x]^2]*(-2*a^2 + b^2 + 4*a*(a + b*Sin[e + f*x]) - 2*(a + b*Sin[e + f*x])^2)*Sqrt[-((a^2 - b^2 - 2*a*(a + b*Sin[e + f*x]) + (a + b*Sin[e + f*x])^2)/b^2)]) + (2*Cot[e + f*x]*Sqrt[a + b*Sin[e + f*x]]*Sin[2*(e + f*x)]/(1 - Sin[e + f*x]^2)))/(4*f*(c + c*Sin[e + f*x]))
```


Rubi [A] (verified)

Time = 1.77 (sec) , antiderivative size = 273, normalized size of antiderivative = 1.15, number of steps used = 17, number of rules used = 17, $\frac{\text{number of rules}}{\text{integrand size}} = 0.515$, Rules used = {3042, 3414, 3042, 3247, 27, 3042, 3231, 3042, 3134, 3042, 3132, 3142, 3042, 3140, 3286, 3042, 3284}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{c\sin(e+fx)+c} dx$$

$$\downarrow \text{3042}$$

$$\int \frac{\sqrt{a+b\sin(e+fx)}}{\sin(e+fx)(c\sin(e+fx)+c)} dx$$

$$\downarrow \text{3414}$$

$$\frac{a \int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}} dx}{c} - (a-b) \int \frac{1}{\sqrt{a+b\sin(e+fx)}(\sin(e+fx)c+c)} dx$$

$$\downarrow \text{3042}$$

$$\frac{a \int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}} dx}{c} - (a-b) \int \frac{1}{\sqrt{a+b\sin(e+fx)}(\sin(e+fx)c+c)} dx$$

$$\downarrow \text{3247}$$

$$\frac{a \int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}} dx}{c} - (a -$$

$$b) \left(\frac{b \int -\frac{\sin(e+fx)c+c}{2\sqrt{a+b\sin(e+fx)}} dx}{c^2(a-b)} - \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)} \right)$$

$$\downarrow \text{27}$$

$$\frac{a \int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}} dx}{c} - (a -$$

$$b) \left(-\frac{b \int \frac{\sin(e+fx)c+c}{\sqrt{a+b\sin(e+fx)}} dx}{2c^2(a-b)} - \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)} \right)$$

$$\downarrow \text{3042}$$

$$b) \left(\frac{a \int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}} dx}{c} - (a - \frac{b \int \frac{\sin(e+fx)c+c}{\sqrt{a+b\sin(e+fx)}} dx}{2c^2(a-b)} - \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}) \right)$$

↓ 3231

$$b) \left(\frac{a \int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}} dx}{c} - (a - \frac{b \left(\frac{c \int \sqrt{a+b\sin(e+fx)} dx}{b} - \frac{c(a-b) \int \frac{1}{\sqrt{a+b\sin(e+fx)}} dx}{b} \right)}{2c^2(a-b)} - \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}) \right)$$

↓ 3042

$$b) \left(\frac{a \int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}} dx}{c} - (a - \frac{b \left(\frac{c \int \sqrt{a+b\sin(e+fx)} dx}{b} - \frac{c(a-b) \int \frac{1}{\sqrt{a+b\sin(e+fx)}} dx}{b} \right)}{2c^2(a-b)} - \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}) \right)$$

↓ 3134

$$b) \left(\frac{a \int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}} dx}{c} - (a - \frac{b \left(\frac{c\sqrt{a+b\sin(e+fx)} \int \sqrt{\frac{a}{a+b} + \frac{b\sin(e+fx)}{a+b}} dx}{b\sqrt{\frac{a+b\sin(e+fx)}{a+b}}} - \frac{c(a-b) \int \frac{1}{\sqrt{a+b\sin(e+fx)}} dx}{b} \right)}{2c^2(a-b)} - \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}) \right)$$

↓ 3042

$$b) \left(\frac{a \int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}} dx}{c} - (a - \frac{b \left(\frac{c\sqrt{a+b\sin(e+fx)} \int \sqrt{\frac{a}{a+b} + \frac{b\sin(e+fx)}{a+b}} dx}{b\sqrt{\frac{a+b\sin(e+fx)}{a+b}}} - \frac{c(a-b) \int \frac{1}{\sqrt{a+b\sin(e+fx)}} dx}{b} \right)}{2c^2(a-b)} - \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)}) \right)$$

↓ 3132

$$b) \left(\frac{a \int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}} dx - (a - \frac{c}{b} \left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}(e+fx-\frac{\pi}{2})\middle|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}} - \frac{c(a-b) \int \frac{1}{\sqrt{a+b\sin(e+fx)}} dx \right)}{2c^2(a-b)} - \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)} \right)$$

↓ 3142

$$b) \left(\frac{a \int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}} dx - (a - \frac{c}{b} \left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}(e+fx-\frac{\pi}{2})\middle|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}} - \frac{c(a-b)\sqrt{\frac{a+b\sin(e+fx)}{a+b}} \int \frac{1}{\sqrt{\frac{a}{a+b} + \frac{b\sin(e+fx)}{a+b}}} dx \right)}{2c^2(a-b)} - \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)} \right)$$

↓ 3042

$$b) \left(\frac{a \int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}} dx - (a - \frac{c}{b} \left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}(e+fx-\frac{\pi}{2})\middle|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}} - \frac{c(a-b)\sqrt{\frac{a+b\sin(e+fx)}{a+b}} \int \frac{1}{\sqrt{\frac{a}{a+b} + \frac{b\sin(e+fx)}{a+b}}} dx \right)}{2c^2(a-b)} - \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)} \right)$$

↓ 3140

$$b) \left(\frac{a \int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}} dx - (a - \frac{c}{b} \left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}(e+fx-\frac{\pi}{2})\middle|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}} - \frac{2c(a-b)\sqrt{\frac{a+b\sin(e+fx)}{a+b}} \text{EllipticF}\left(\frac{1}{2}(e+fx-\frac{\pi}{2}), \frac{2b}{a+b}\right)}{bf\sqrt{a+b\sin(e+fx)}} \right)}{2c^2(a-b)} - \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)} \right)$$

↓ 3286

$$b) \left(\frac{a\sqrt{\frac{a+b\sin(e+fx)}{a+b}} \int \frac{\csc(e+fx)}{\sqrt{\frac{a}{a+b} + \frac{b\sin(e+fx)}{a+b}}} dx}{c\sqrt{a+b\sin(e+fx)}} - (a - \frac{b \left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}(e+fx-\frac{\pi}{2})\middle|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}} - \frac{2c(a-b)\sqrt{\frac{a+b\sin(e+fx)}{a+b}} \operatorname{EllipticF}\left(\frac{1}{2}(e+fx-\frac{\pi}{2}), \frac{2b}{a+b}\right)}{bf\sqrt{a+b\sin(e+fx)}}} \right)}{2c^2(a-b)} - \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx))} \right)$$

3042

$$b) \left(\frac{a\sqrt{\frac{a+b\sin(e+fx)}{a+b}} \int \frac{1}{\sin(e+fx)\sqrt{\frac{a}{a+b} + \frac{b\sin(e+fx)}{a+b}}} dx}{c\sqrt{a+b\sin(e+fx)}} - (a - \frac{b \left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}(e+fx-\frac{\pi}{2})\middle|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}} - \frac{2c(a-b)\sqrt{\frac{a+b\sin(e+fx)}{a+b}} \operatorname{EllipticF}\left(\frac{1}{2}(e+fx-\frac{\pi}{2}), \frac{2b}{a+b}\right)}{bf\sqrt{a+b\sin(e+fx)}}} \right)}{2c^2(a-b)} - \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx))} \right)$$

3284

$$b) \left(\frac{2a\sqrt{\frac{a+b\sin(e+fx)}{a+b}} \operatorname{EllipticPi}\left(2, \frac{1}{2}(e+fx-\frac{\pi}{2}), \frac{2b}{a+b}\right)}{cf\sqrt{a+b\sin(e+fx)}} - (a - \frac{b \left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}(e+fx-\frac{\pi}{2})\middle|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}} - \frac{2c(a-b)\sqrt{\frac{a+b\sin(e+fx)}{a+b}} \operatorname{EllipticF}\left(\frac{1}{2}(e+fx-\frac{\pi}{2}), \frac{2b}{a+b}\right)}{bf\sqrt{a+b\sin(e+fx)}}} \right)}{2c^2(a-b)} - \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx))} \right)$$

input `Int[(Csc[e + f*x]*Sqrt[a + b*Sin[e + f*x]])/(c + c*Sin[e + f*x]),x]`

output `(2*a*EllipticPi[2, (e - Pi/2 + f*x)/2, (2*b)/(a + b)]*Sqrt[(a + b*Sin[e + f*x])/(a + b)]/(c*f*Sqrt[a + b*Sin[e + f*x]]) - (a - b)*(-(Cos[e + f*x]*Sqrt[a + b*Sin[e + f*x]])/((a - b)*f*(c + c*Sin[e + f*x]))) - (b*((2*c*EllipticE[(e - Pi/2 + f*x)/2, (2*b)/(a + b)]*Sqrt[a + b*Sin[e + f*x]])/(b*f*Sqrt[(a + b*Sin[e + f*x])/(a + b)]) - (2*(a - b)*c*EllipticF[(e - Pi/2 + f*x)/2, (2*b)/(a + b)]*Sqrt[(a + b*Sin[e + f*x])/(a + b)]/(b*f*Sqrt[a + b*Sin[e + f*x]])))/(2*(a - b)*c^2)`

Defintions of rubi rules used

- rule 27 `Int[(a_)*(Fx_), x_Symbol] := Simp[a Int[Fx, x], x] /; FreeQ[a, x] && !MatchQ[Fx, (b_)*(Gx_)] /; FreeQ[b, x]`
- rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`
- rule 3132 `Int[Sqrt[(a_) + (b_)*sin[(c_) + (d_)*(x_)]], x_Symbol] := Simp[2*(Sqrt[a + b]/d)*EllipticE[(1/2)*(c - Pi/2 + d*x), 2*(b/(a + b))], x] /; FreeQ[{a, b, c, d}, x] && NeQ[a^2 - b^2, 0] && GtQ[a + b, 0]`
- rule 3134 `Int[Sqrt[(a_) + (b_)*sin[(c_) + (d_)*(x_)]], x_Symbol] := Simp[Sqrt[a + b*Sin[c + d*x]]/Sqrt[(a + b*Sin[c + d*x])/(a + b)] Int[Sqrt[a/(a + b) + (b/(a + b))*Sin[c + d*x]], x], x] /; FreeQ[{a, b, c, d}, x] && NeQ[a^2 - b^2, 0] && !GtQ[a + b, 0]`
- rule 3140 `Int[1/Sqrt[(a_) + (b_)*sin[(c_) + (d_)*(x_)]], x_Symbol] := Simp[(2/(d*Sqrt[a + b]))*EllipticF[(1/2)*(c - Pi/2 + d*x), 2*(b/(a + b))], x] /; FreeQ[{a, b, c, d}, x] && NeQ[a^2 - b^2, 0] && GtQ[a + b, 0]`
- rule 3142 `Int[1/Sqrt[(a_) + (b_)*sin[(c_) + (d_)*(x_)]], x_Symbol] := Simp[Sqrt[(a + b*Sin[c + d*x])/(a + b)]/Sqrt[a + b*Sin[c + d*x]] Int[1/Sqrt[a/(a + b) + (b/(a + b))*Sin[c + d*x]], x], x] /; FreeQ[{a, b, c, d}, x] && NeQ[a^2 - b^2, 0] && !GtQ[a + b, 0]`
- rule 3231 `Int[((c_) + (d_)*sin[(e_) + (f_)*(x_)])/Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)]], x_Symbol] := Simp[(b*c - a*d)/b Int[1/Sqrt[a + b*Sin[e + f*x]], x], x] + Simp[d/b Int[Sqrt[a + b*Sin[e + f*x]], x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0]`

rule 3247

```
Int[((c_.) + (d_.)*sin[(e_.) + (f_.)*(x_)])^(n_)/((a_.) + (b_.)*sin[(e_.) +
(f_.)*(x_)]), x_Symbol] := Simp[(-b^2)*Cos[e + f*x]*((c + d*Sin[e + f*x])^(
n + 1)/(a*f*(b*c - a*d)*(a + b*Sin[e + f*x]))], x] + Simp[d/(a*(b*c - a*d))
  Int[(c + d*Sin[e + f*x])^n*(a*n - b*(n + 1)*Sin[e + f*x]), x], x] /; Fre
eQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0] && NeQ[
c^2 - d^2, 0] && LtQ[n, 0] && (IntegerQ[2*n] || EqQ[c, 0])
```

rule 3284

```
Int[1/(((a_.) + (b_.)*sin[(e_.) + (f_.)*(x_)])*Sqrt[(c_.) + (d_.)*sin[(e_.)
+ (f_.)*(x_)]]), x_Symbol] := Simp[(2/(f*(a + b)*Sqrt[c + d]))*EllipticPi[
2*(b/(a + b)), (1/2)*(e - Pi/2 + f*x), 2*(d/(c + d))], x] /; FreeQ[{a, b, c
, d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2,
0] && GtQ[c + d, 0]
```

rule 3286

```
Int[1/(((a_.) + (b_.)*sin[(e_.) + (f_.)*(x_)])*Sqrt[(c_.) + (d_.)*sin[(e_.)
+ (f_.)*(x_)]]), x_Symbol] := Simp[Sqrt[(c + d*Sin[e + f*x])/(c + d)]/Sqrt
[c + d*Sin[e + f*x]] Int[1/((a + b*Sin[e + f*x])*Sqrt[c/(c + d) + (d/(c +
d))*Sin[e + f*x]]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*
d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0] && !GtQ[c + d, 0]
```

rule 3414

```
Int[Sqrt[(a_.) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(sin[(e_.) + (f_.)*(x_)]*((c
_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] := Simp[a/c Int[1/(Sin[e
+ f*x]*Sqrt[a + b*Sin[e + f*x]]), x], x] + Simp[(b*c - a*d)/c Int[1/(Sqrt
[a + b*Sin[e + f*x]]*(c + d*Sin[e + f*x])), x], x] /; FreeQ[{a, b, c, d, e,
f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0]
```

Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 589 vs. $2(233) = 466$.

Time = 1.41 (sec) , antiderivative size = 590, normalized size of antiderivative = 2.48

method	result
default	$\frac{\sqrt{-(-a-b\sin(fx+e))\cos(fx+e)}^2 \left(-\frac{2\left(\frac{a}{b}-1\right)\sqrt{\frac{a+b\sin(fx+e)}{a-b}}\sqrt{\frac{(1-\sin(fx+e))b}{a+b}}\sqrt{-\frac{b(1+\sin(fx+e))}{a-b}} b \operatorname{EllipticPi}\left(\sqrt{\frac{a+b\sin(fx+e)}{a-b}}, -\left(\frac{a+b\sin(fx+e)}{a-b}\right)\right)}{\sqrt{-(-a-b\sin(fx+e))\cos(fx+e)}^2} \right)}{\sqrt{-(-a-b\sin(fx+e))\cos(fx+e)}^2}$

input `int(csc(f*x+e)*(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x,method=_RETURNVERBOSE)`

output
$$\begin{aligned} & \frac{(-(-a-b\sin(fx+e))\cos(fx+e)^2)^{1/2}/c(-2(1/ba-1)((a+b\sin(fx+e))/(a-b))^{1/2}(1/(a+b)(1-\sin(fx+e))b)^{1/2}(-b(1+\sin(fx+e))/(a-b))^{1/2})/(-(-a-b\sin(fx+e))\cos(fx+e)^2)^{1/2}b\text{EllipticPi}(((a+b\sin(fx+e))/(a-b))^{1/2},-(-1/ba+1)b/a,((a-b)/(a+b))^{1/2})+(-a+b)*(-(-\sin(fx+e)^2*b-a\sin(fx+e)+b\sin(fx+e)+a)/(a-b)/((1+\sin(fx+e))*(\sin(fx+e)-1)*(-a-b\sin(fx+e)))^{1/2}-2*b/(2*a-2*b)*(1/ba-1)((a+b\sin(fx+e))/(a-b))^{1/2})*(1/(a+b)(1-\sin(fx+e))b)^{1/2}(-b(1+\sin(fx+e))/(a-b))^{1/2}/(-(-a-b\sin(fx+e))\cos(fx+e)^2)^{1/2}\text{EllipticF}(((a+b\sin(fx+e))/(a-b))^{1/2},((a-b)/(a+b))^{1/2})-b/(a-b)*(1/ba-1)((a+b\sin(fx+e))/(a-b))^{1/2}(1/(a+b)(1-\sin(fx+e))b)^{1/2}(-b(1+\sin(fx+e))/(a-b))^{1/2}/(-(-a-b\sin(fx+e))\cos(fx+e)^2)^{1/2}((-1/ba-1)\text{EllipticE}(((a+b\sin(fx+e))/(a-b))^{1/2},((a-b)/(a+b))^{1/2})+\text{EllipticF}(((a+b\sin(fx+e))/(a-b))^{1/2},((a-b)/(a+b))^{1/2})))/\cos(fx+e)/(a+b\sin(fx+e))^{1/2}/f \end{aligned}$$

Fricas [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{c+c\sin(e+fx)} dx = \text{Timed out}$$

input `integrate(csc(f*x+e)*(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x, algorithm="fricas")`

output Timed out

Sympy [F]

$$\int \frac{\csc(e + fx) \sqrt{a + b \sin(e + fx)}}{c + c \sin(e + fx)} dx = \frac{\int \frac{\sqrt{a + b \sin(e + fx)} \csc(e + fx)}{\sin(e + fx) + 1} dx}{c}$$

input `integrate(csc(f*x+e)*(a+b*sin(f*x+e))**(1/2)/(c+c*sin(f*x+e)),x)`

output `Integral(sqrt(a + b*sin(e + f*x))*csc(e + f*x)/(sin(e + f*x) + 1), x)/c`

Maxima [F]

$$\int \frac{\csc(e + fx) \sqrt{a + b \sin(e + fx)}}{c + c \sin(e + fx)} dx = \int \frac{\sqrt{b \sin(fx + e) + a} \csc(fx + e)}{c \sin(fx + e) + c} dx$$

input `integrate(csc(f*x+e)*(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x, algorithm="maxima")`

output `integrate(sqrt(b*sin(f*x + e) + a)*csc(f*x + e)/(c*sin(f*x + e) + c), x)`

Giac [F]

$$\int \frac{\csc(e + fx) \sqrt{a + b \sin(e + fx)}}{c + c \sin(e + fx)} dx = \int \frac{\sqrt{b \sin(fx + e) + a} \csc(fx + e)}{c \sin(fx + e) + c} dx$$

input `integrate(csc(f*x+e)*(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x, algorithm="giac")`

output `integrate(sqrt(b*sin(f*x + e) + a)*csc(f*x + e)/(c*sin(f*x + e) + c), x)`

Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e + fx) \sqrt{a + b \sin(e + fx)}}{c + c \sin(e + fx)} dx = \int \frac{\sqrt{a + b \sin(e + fx)}}{\sin(e + fx) (c + c \sin(e + fx))} dx$$

input `int((a + b*sin(e + f*x))^(1/2)/(sin(e + f*x)*(c + c*sin(e + f*x))),x)`

output `int((a + b*sin(e + f*x))^(1/2)/(sin(e + f*x)*(c + c*sin(e + f*x))), x)`

Reduce [F]

$$\int \frac{\csc(e + fx) \sqrt{a + b \sin(e + fx)}}{c + c \sin(e + fx)} dx = \frac{\int \frac{\sqrt{\sin(fx+e)b+a} \csc(fx+e)}{\sin(fx+e)+1} dx}{c}$$

input `int(csc(f*x+e)*(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x)`

output `int((sqrt(sin(e + f*x)*b + a)*csc(e + f*x))/(sin(e + f*x) + 1),x)/c`

3.30
$$\int \frac{\csc(e+fx)}{\sqrt{a+b \sin(e+fx)}(c+c \sin(e+fx))} dx$$

Optimal result	265
Mathematica [C] (warning: unable to verify)	266
Rubi [A] (verified)	267
Maple [B] (verified)	272
Fricas [F(-1)]	273
Sympy [F]	273
Maxima [F]	274
Giac [F]	274
Mupad [F(-1)]	275
Reduce [F]	275

Optimal result

Integrand size = 33, antiderivative size = 246

$$\begin{aligned} & \int \frac{\csc(e+fx)}{\sqrt{a+b \sin(e+fx)}(c+c \sin(e+fx))} dx \\ &= \frac{E\left(\frac{1}{2}\left(e-\frac{\pi}{2}+fx\right)\middle|\frac{2b}{a+b}\right)\sqrt{a+b \sin(e+fx)}}{(a-b)cf\sqrt{\frac{a+b \sin(e+fx)}{a+b}}} \\ & \quad - \frac{\text{EllipticF}\left(\frac{1}{2}\left(e-\frac{\pi}{2}+fx\right),\frac{2b}{a+b}\right)\sqrt{\frac{a+b \sin(e+fx)}{a+b}}}{cf\sqrt{a+b \sin(e+fx)}} \\ & \quad + \frac{2\text{EllipticPi}\left(2,\frac{1}{2}\left(e-\frac{\pi}{2}+fx\right),\frac{2b}{a+b}\right)\sqrt{\frac{a+b \sin(e+fx)}{a+b}}}{cf\sqrt{a+b \sin(e+fx)}} \\ & \quad + \frac{\cos(e+fx)\sqrt{a+b \sin(e+fx)}}{(a-b)f(c+c \sin(e+fx))} \end{aligned}$$

output

```
-EllipticE(cos(1/2*e+1/4*Pi+1/2*f*x),2^(1/2)*(b/(a+b))^(1/2))*(a+b*sin(f*x+e))^(1/2)/(a-b)/c/f/((a+b*sin(f*x+e))/(a+b))^(1/2)-InverseJacobiAM(1/2*e-1/4*Pi+1/2*f*x,2^(1/2)*(b/(a+b))^(1/2))*((a+b*sin(f*x+e))/(a+b))^(1/2)/c/f/(a+b*sin(f*x+e))^(1/2)-2*EllipticPi(cos(1/2*e+1/4*Pi+1/2*f*x),2,2^(1/2)*(b/(a+b))^(1/2))*((a+b*sin(f*x+e))/(a+b))^(1/2)/c/f/(a+b*sin(f*x+e))^(1/2)+cos(f*x+e)*(a+b*sin(f*x+e))^(1/2)/(a-b)/f/(c+c*sin(f*x+e))
```

Mathematica [C] (warning: unable to verify)

Result contains complex when optimal does not.

Time = 9.41 (sec) , antiderivative size = 625, normalized size of antiderivative = 2.54

$$\int \frac{\csc(e + fx)}{\sqrt{a + b \sin(e + fx)}(c + c \sin(e + fx))} dx$$

$$= -\frac{2 \sin\left(\frac{1}{2}(e + fx)\right) \left(\cos\left(\frac{1}{2}(e + fx)\right) + \sin\left(\frac{1}{2}(e + fx)\right)\right) \sqrt{a + b \sin(e + fx)}}{(a - b)f(c + c \sin(e + fx))}$$

$$- \frac{\left(\cos\left(\frac{1}{2}(e + fx)\right) + \sin\left(\frac{1}{2}(e + fx)\right)\right)^2 \left(\frac{4b \operatorname{EllipticF}\left(\frac{1}{2}(-e + \frac{\pi}{2} - fx), \frac{2b}{a+b}\right) \sqrt{\frac{a+b \sin(e+fx)}{a+b}}}{\sqrt{a+b \sin(e+fx)}} - \frac{2(-4a+3b) \operatorname{EllipticPi}\left(2, \frac{1}{2}(-e + \frac{\pi}{2} - fx), \frac{2b}{a+b}\right)}{\sqrt{a+b \sin(e+fx)}}}\right)}{(a - b)f(c + c \sin(e + fx))}$$

input

```
Integrate[Csc[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*(c + c*Sin[e + f*x])),x]
```

output

```
(-2*Sin[(e + f*x)/2]*(Cos[(e + f*x)/2] + Sin[(e + f*x)/2])*Sqrt[a + b*Sin[e + f*x]]/((a - b)*f*(c + c*Sin[e + f*x])) - ((Cos[(e + f*x)/2] + Sin[(e + f*x)/2])^2*((4*b*EllipticF[(-e + Pi/2 - f*x)/2, (2*b)/(a + b)]*Sqrt[(a + b*Sin[e + f*x])/(a + b)])/Sqrt[a + b*Sin[e + f*x]] - (2*(-4*a + 3*b)*EllipticPi[2, (-e + Pi/2 - f*x)/2, (2*b)/(a + b)]*Sqrt[(a + b*Sin[e + f*x])/(a + b)])/Sqrt[a + b*Sin[e + f*x]] - ((2*I)*b*Cos[e + f*x]*Cos[2*(e + f*x)]*(2*a*(a - b)*EllipticE[I*ArcSinh[Sqrt[-(a + b)^(-1)]]*Sqrt[a + b*Sin[e + f*x]]], (a + b)/(a - b)] + b*(2*a*EllipticF[I*ArcSinh[Sqrt[-(a + b)^(-1)]]*Sqrt[a + b*Sin[e + f*x]]], (a + b)/(a - b)] - b*EllipticPi[(a + b)/a, I*ArcSinh[Sqrt[-(a + b)^(-1)]]*Sqrt[a + b*Sin[e + f*x]]], (a + b)/(a - b)))*Sqrt[(b - b*Sin[e + f*x])/(a + b)]*Sqrt[-((b + b*Sin[e + f*x])/(a - b))]/(a*Sqrt[-(a + b)^(-1)]]*Sqrt[1 - Sin[e + f*x]^2]*(-2*a^2 + b^2 + 4*a*(a + b*Sin[e + f*x]) - 2*(a + b*Sin[e + f*x])^2)*Sqrt[-((a^2 - b^2 - 2*a*(a + b*Sin[e + f*x]) + (a + b*Sin[e + f*x])^2)/b^2)]) - (2*Cot[e + f*x]*Sqrt[a + b*Sin[e + f*x]]*Sin[2*(e + f*x)]/(1 - Sin[e + f*x]^2)))/(4*(a - b)*f*(c + c*Sin[e + f*x]))
```

Rubi [A] (verified)

Time = 1.76 (sec) , antiderivative size = 263, normalized size of antiderivative = 1.07, number of steps used = 17, number of rules used = 17, $\frac{\text{number of rules}}{\text{integrand size}} = 0.515$, Rules used = {3042, 3420, 3042, 3247, 27, 3042, 3231, 3042, 3134, 3042, 3132, 3142, 3042, 3140, 3286, 3042, 3284}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{\csc(e+fx)}{(c\sin(e+fx)+c)\sqrt{a+b\sin(e+fx)}} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{1}{\sin(e+fx)(c\sin(e+fx)+c)\sqrt{a+b\sin(e+fx)}} dx \\
 & \quad \downarrow \text{3420} \\
 & \frac{\int \frac{\csc(e+fx)}{\sqrt{a+b\sin(e+fx)}} dx}{c} - \int \frac{1}{\sqrt{a+b\sin(e+fx)}(\sin(e+fx)c+c)} dx \\
 & \quad \downarrow \text{3042} \\
 & \frac{\int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}} dx}{c} - \int \frac{1}{\sqrt{a+b\sin(e+fx)}(\sin(e+fx)c+c)} dx \\
 & \quad \downarrow \text{3247} \\
 & -\frac{b \int -\frac{\sin(e+fx)c+c}{2\sqrt{a+b\sin(e+fx)}} dx}{c^2(a-b)} + \frac{\int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}} dx}{c} + \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)} \\
 & \quad \downarrow \text{27} \\
 & \frac{b \int \frac{\sin(e+fx)c+c}{\sqrt{a+b\sin(e+fx)}} dx}{2c^2(a-b)} + \frac{\int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}} dx}{c} + \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)} \\
 & \quad \downarrow \text{3042} \\
 & \frac{b \int \frac{\sin(e+fx)c+c}{\sqrt{a+b\sin(e+fx)}} dx}{2c^2(a-b)} + \frac{\int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}} dx}{c} + \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)} \\
 & \quad \downarrow \text{3231}
 \end{aligned}$$

$$\begin{aligned}
& \frac{b \left(\frac{c \int \sqrt{a+b \sin(e+fx)} dx}{b} - \frac{c(a-b) \int \frac{1}{\sqrt{a+b \sin(e+fx)}} dx}{b} \right)}{2c^2(a-b)} + \frac{\int \frac{1}{\sin(e+fx)\sqrt{a+b \sin(e+fx)}} dx}{c} + \\
& \quad \frac{\cos(e+fx)\sqrt{a+b \sin(e+fx)}}{f(a-b)(c \sin(e+fx) + c)} \\
& \quad \downarrow \text{3042} \\
& \frac{b \left(\frac{c \int \sqrt{a+b \sin(e+fx)} dx}{b} - \frac{c(a-b) \int \frac{1}{\sqrt{a+b \sin(e+fx)}} dx}{b} \right)}{2c^2(a-b)} + \frac{\int \frac{1}{\sin(e+fx)\sqrt{a+b \sin(e+fx)}} dx}{c} + \\
& \quad \frac{\cos(e+fx)\sqrt{a+b \sin(e+fx)}}{f(a-b)(c \sin(e+fx) + c)} \\
& \quad \downarrow \text{3134} \\
& \frac{b \left(\frac{c \sqrt{a+b \sin(e+fx)} \int \sqrt{\frac{a}{a+b} + \frac{b \sin(e+fx)}{a+b}} dx}{b \sqrt{\frac{a+b \sin(e+fx)}{a+b}}} - \frac{c(a-b) \int \frac{1}{\sqrt{a+b \sin(e+fx)}} dx}{b} \right)}{2c^2(a-b)} + \frac{\int \frac{1}{\sin(e+fx)\sqrt{a+b \sin(e+fx)}} dx}{c} + \\
& \quad \frac{\cos(e+fx)\sqrt{a+b \sin(e+fx)}}{f(a-b)(c \sin(e+fx) + c)} \\
& \quad \downarrow \text{3042} \\
& \frac{b \left(\frac{c \sqrt{a+b \sin(e+fx)} \int \sqrt{\frac{a}{a+b} + \frac{b \sin(e+fx)}{a+b}} dx}{b \sqrt{\frac{a+b \sin(e+fx)}{a+b}}} - \frac{c(a-b) \int \frac{1}{\sqrt{a+b \sin(e+fx)}} dx}{b} \right)}{2c^2(a-b)} + \frac{\int \frac{1}{\sin(e+fx)\sqrt{a+b \sin(e+fx)}} dx}{c} + \\
& \quad \frac{\cos(e+fx)\sqrt{a+b \sin(e+fx)}}{f(a-b)(c \sin(e+fx) + c)} \\
& \quad \downarrow \text{3132} \\
& \frac{b \left(\frac{2c \sqrt{a+b \sin(e+fx)} E\left(\frac{1}{2}(e+fx - \frac{\pi}{2}) \mid \frac{2b}{a+b}\right)}{b f \sqrt{\frac{a+b \sin(e+fx)}{a+b}}} - \frac{c(a-b) \int \frac{1}{\sqrt{a+b \sin(e+fx)}} dx}{b} \right)}{2c^2(a-b)} + \frac{\int \frac{1}{\sin(e+fx)\sqrt{a+b \sin(e+fx)}} dx}{c} + \\
& \quad \frac{\cos(e+fx)\sqrt{a+b \sin(e+fx)}}{f(a-b)(c \sin(e+fx) + c)} \\
& \quad \downarrow \text{3142}
\end{aligned}$$

$$\begin{aligned}
 & b \left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}(e+fx-\frac{\pi}{2})\middle|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}} - \frac{c(a-b)\sqrt{\frac{a+b\sin(e+fx)}{a+b}} \int \frac{1}{\sqrt{\frac{a}{a+b} + \frac{b\sin(e+fx)}{a+b}}} dx}{b\sqrt{a+b\sin(e+fx)}} \right) + \\
 & \frac{2c^2(a-b)}{\int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}} dx} + \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)} \\
 & \quad \downarrow \text{3042} \\
 & b \left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}(e+fx-\frac{\pi}{2})\middle|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}} - \frac{c(a-b)\sqrt{\frac{a+b\sin(e+fx)}{a+b}} \int \frac{1}{\sqrt{\frac{a}{a+b} + \frac{b\sin(e+fx)}{a+b}}} dx}{b\sqrt{a+b\sin(e+fx)}} \right) + \\
 & \frac{2c^2(a-b)}{\int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}} dx} + \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)} \\
 & \quad \downarrow \text{3140} \\
 & \frac{\int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}} dx}{c} + \\
 & b \left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}(e+fx-\frac{\pi}{2})\middle|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}} - \frac{2c(a-b)\sqrt{\frac{a+b\sin(e+fx)}{a+b}} \text{EllipticF}\left(\frac{1}{2}(e+fx-\frac{\pi}{2}), \frac{2b}{a+b}\right)}{bf\sqrt{a+b\sin(e+fx)}} \right) + \\
 & \frac{2c^2(a-b)}{\cos(e+fx)\sqrt{a+b\sin(e+fx)}} + \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)} \\
 & \quad \downarrow \text{3286} \\
 & \frac{\sqrt{\frac{a+b\sin(e+fx)}{a+b}} \int \frac{\csc(e+fx)}{\sqrt{\frac{a}{a+b} + \frac{b\sin(e+fx)}{a+b}}} dx}{c\sqrt{a+b\sin(e+fx)}} + \\
 & b \left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}(e+fx-\frac{\pi}{2})\middle|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}} - \frac{2c(a-b)\sqrt{\frac{a+b\sin(e+fx)}{a+b}} \text{EllipticF}\left(\frac{1}{2}(e+fx-\frac{\pi}{2}), \frac{2b}{a+b}\right)}{bf\sqrt{a+b\sin(e+fx)}} \right) + \\
 & \frac{2c^2(a-b)}{\cos(e+fx)\sqrt{a+b\sin(e+fx)}} + \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)} \\
 & \quad \downarrow \text{3042}
 \end{aligned}$$

$$\begin{aligned}
& \frac{\sqrt{\frac{a+b\sin(e+fx)}{a+b}} \int \frac{1}{\sin(e+fx)\sqrt{\frac{a}{a+b} + \frac{b\sin(e+fx)}{a+b}}} dx}{c\sqrt{a+b\sin(e+fx)}} + \\
& b \left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}(e+fx-\frac{\pi}{2})\middle|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}} - \frac{2c(a-b)\sqrt{\frac{a+b\sin(e+fx)}{a+b}} \operatorname{EllipticF}\left(\frac{1}{2}(e+fx-\frac{\pi}{2}), \frac{2b}{a+b}\right)}{bf\sqrt{a+b\sin(e+fx)}} \right) \\
& \frac{2c^2(a-b)}{\cos(e+fx)\sqrt{a+b\sin(e+fx)}} + \\
& \frac{f(a-b)(c\sin(e+fx)+c)}{f(a-b)(c\sin(e+fx)+c)} \\
& \quad \downarrow \text{3284} \\
& b \left(\frac{2c\sqrt{a+b\sin(e+fx)}E\left(\frac{1}{2}(e+fx-\frac{\pi}{2})\middle|\frac{2b}{a+b}\right)}{bf\sqrt{\frac{a+b\sin(e+fx)}{a+b}}} - \frac{2c(a-b)\sqrt{\frac{a+b\sin(e+fx)}{a+b}} \operatorname{EllipticF}\left(\frac{1}{2}(e+fx-\frac{\pi}{2}), \frac{2b}{a+b}\right)}{bf\sqrt{a+b\sin(e+fx)}} \right) \\
& \frac{2c^2(a-b)}{\cos(e+fx)\sqrt{a+b\sin(e+fx)}} + \\
& \frac{\cos(e+fx)\sqrt{a+b\sin(e+fx)}}{f(a-b)(c\sin(e+fx)+c)} + \frac{2\sqrt{\frac{a+b\sin(e+fx)}{a+b}} \operatorname{EllipticPi}\left(2, \frac{1}{2}(e+fx-\frac{\pi}{2}), \frac{2b}{a+b}\right)}{cf\sqrt{a+b\sin(e+fx)}}
\end{aligned}$$

input `Int[Csc[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*(c + c*Sin[e + f*x])),x]`

output `(2*EllipticPi[2, (e - Pi/2 + f*x)/2, (2*b)/(a + b)]*Sqrt[(a + b*Sin[e + f*x])/(a + b)]/(c*f*Sqrt[a + b*Sin[e + f*x]]) + (Cos[e + f*x]*Sqrt[a + b*Sin[e + f*x]])/((a - b)*f*(c + c*Sin[e + f*x])) + (b*((2*c*EllipticE[(e - Pi/2 + f*x)/2, (2*b)/(a + b)]*Sqrt[a + b*Sin[e + f*x]])/(b*f*Sqrt[(a + b*Sin[e + f*x])/(a + b)]) - (2*(a - b)*c*EllipticF[(e - Pi/2 + f*x)/2, (2*b)/(a + b)]*Sqrt[(a + b*Sin[e + f*x])/(a + b)])/(b*f*Sqrt[a + b*Sin[e + f*x]])))/(2*(a - b)*c^2)`

Defintions of rubi rules used

rule 27 `Int[(a_)*(F_x_), x_Symbol] := Simp[a Int[F_x, x], x] /; FreeQ[a, x] && !MatchQ[F_x, (b_)*(G_x_)] /; FreeQ[b, x]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3132 `Int[Sqrt[(a_) + (b_)*sin[(c_) + (d_)*(x_)]], x_Symbol] := Simp[2*(Sqrt[a + b]/d)*EllipticE[(1/2)*(c - Pi/2 + d*x), 2*(b/(a + b))], x] /; FreeQ[{a, b, c, d}, x] && NeQ[a^2 - b^2, 0] && GtQ[a + b, 0]`

rule 3134 `Int[Sqrt[(a_) + (b_)*sin[(c_) + (d_)*(x_)]], x_Symbol] := Simp[Sqrt[a + b*Sin[c + d*x]]/Sqrt[(a + b*Sin[c + d*x])/(a + b)] Int[Sqrt[a/(a + b) + (b/(a + b))*Sin[c + d*x]], x], x] /; FreeQ[{a, b, c, d}, x] && NeQ[a^2 - b^2, 0] && !GtQ[a + b, 0]`

rule 3140 `Int[1/Sqrt[(a_) + (b_)*sin[(c_) + (d_)*(x_)]], x_Symbol] := Simp[(2/(d*Sqrt[a + b]))*EllipticF[(1/2)*(c - Pi/2 + d*x), 2*(b/(a + b))], x] /; FreeQ[{a, b, c, d}, x] && NeQ[a^2 - b^2, 0] && GtQ[a + b, 0]`

rule 3142 `Int[1/Sqrt[(a_) + (b_)*sin[(c_) + (d_)*(x_)]], x_Symbol] := Simp[Sqrt[(a + b*Sin[c + d*x])/(a + b)]/Sqrt[a + b*Sin[c + d*x]] Int[1/Sqrt[a/(a + b) + (b/(a + b))*Sin[c + d*x]], x], x] /; FreeQ[{a, b, c, d}, x] && NeQ[a^2 - b^2, 0] && !GtQ[a + b, 0]`

rule 3231 `Int[((c_) + (d_)*sin[(e_) + (f_)*(x_)])/Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)]], x_Symbol] := Simp[(b*c - a*d)/b Int[1/Sqrt[a + b*Sin[e + f*x]], x], x] + Simp[d/b Int[Sqrt[a + b*Sin[e + f*x]], x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0]`

rule 3247 `Int[((c_) + (d_)*sin[(e_) + (f_)*(x_)])^n/((a_) + (b_)*sin[(e_) + (f_)*(x_)]), x_Symbol] := Simp[(-b^2)*Cos[e + f*x]*((c + d*Sin[e + f*x])^(n + 1)/(a*f*(b*c - a*d)*(a + b*Sin[e + f*x]))], x] + Simp[d/(a*(b*c - a*d)) Int[(c + d*Sin[e + f*x])^n*(a^n - b*(n + 1)*Sin[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0] && LtQ[n, 0] && (IntegerQ[2*n] || EqQ[c, 0])`

rule 3284

```
Int[1/(((a_.) + (b_.)*sin[(e_.) + (f_.)*(x_)])*Sqrt[(c_.) + (d_.)*sin[(e_.) + (f_.)*(x_)]]), x_Symbol] := Simp[(2/(f*(a + b)*Sqrt[c + d]))*EllipticPi[2*(b/(a + b)), (1/2)*(e - Pi/2 + f*x), 2*(d/(c + d))], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0] && GtQ[c + d, 0]
```

rule 3286

```
Int[1/(((a_.) + (b_.)*sin[(e_.) + (f_.)*(x_)])*Sqrt[(c_.) + (d_.)*sin[(e_.) + (f_.)*(x_)]]), x_Symbol] := Simp[Sqrt[(c + d*Sin[e + f*x])]/(c + d)]/Sqrt[c + d*Sin[e + f*x]] Int[1/((a + b*Sin[e + f*x])*Sqrt[c/(c + d) + (d/(c + d))*Sin[e + f*x]]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0] && !GtQ[c + d, 0]
```

rule 3420

```
Int[1/(sin[(e_.) + (f_.)*(x_)])*Sqrt[(a_.) + (b_.)*sin[(e_.) + (f_.)*(x_)]]*(c_.) + (d_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] := Simp[1/c Int[1/(Sin[e + f*x]*Sqrt[a + b*Sin[e + f*x]]), x], x] - Simp[d/c Int[1/(Sqrt[a + b*Sin[e + f*x]]*(c + d*Sin[e + f*x])), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0]
```

Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 583 vs. 2(241) = 482.

Time = 1.15 (sec) , antiderivative size = 584, normalized size of antiderivative = 2.37

method	result
default	$\frac{\sqrt{-(-a-b\sin(fx+e))\cos(fx+e)^2} \left(-\frac{2\left(\frac{a}{b}-1\right)\sqrt{\frac{a+b\sin(fx+e)}{a-b}}\sqrt{\frac{(1-\sin(fx+e))b}{a+b}}\sqrt{-\frac{b(1+\sin(fx+e))}{a-b}} b \operatorname{EllipticPi}\left(\sqrt{\frac{a+b\sin(fx+e)}{a-b}}, -\frac{b}{a}\right)}{\sqrt{-(-a-b\sin(fx+e))\cos(fx+e)^2} a} \right)}{\sqrt{-(-a-b\sin(fx+e))\cos(fx+e)^2}}$

input

```
int(csc(f*x+e)/(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x,method=_RETURNVERBOSE)
```

output

```
(-(-a-b*sin(f*x+e))*cos(f*x+e)^2)^(1/2)/c*(-2*(1/b*a-1)*((a+b*sin(f*x+e))/
(a-b))^(1/2)*(1/(a+b)*(1-sin(f*x+e))*b)^(1/2)*(-b*(1+sin(f*x+e))/(a-b))^(1
/2)/(-(-a-b*sin(f*x+e))*cos(f*x+e)^2)^(1/2)*b/a*EllipticPi(((a+b*sin(f*x+e)
))/(a-b))^(1/2),-(-1/b*a+1)*b/a,((a-b)/(a+b))^(1/2))+(-sin(f*x+e)^2*b-a*si
n(f*x+e)+b*sin(f*x+e)+a)/(a-b)/((1+sin(f*x+e))*(sin(f*x+e)-1)*(-a-b*sin(f*
x+e)))^(1/2)+2*b/(2*a-2*b)*(1/b*a-1)*((a+b*sin(f*x+e))/(a-b))^(1/2)*(1/(a+
b)*(1-sin(f*x+e))*b)^(1/2)*(-b*(1+sin(f*x+e))/(a-b))^(1/2)/(-(-a-b*sin(f*x
+e))*cos(f*x+e)^2)^(1/2)*EllipticF(((a+b*sin(f*x+e))/(a-b))^(1/2),((a-b)/(
a+b))^(1/2))+b/(a-b)*(1/b*a-1)*((a+b*sin(f*x+e))/(a-b))^(1/2)*(1/(a+b)*(1-
sin(f*x+e))*b)^(1/2)*(-b*(1+sin(f*x+e))/(a-b))^(1/2)/(-(-a-b*sin(f*x+e))*c
os(f*x+e)^2)^(1/2)*((-1/b*a-1)*EllipticE(((a+b*sin(f*x+e))/(a-b))^(1/2),((
a-b)/(a+b))^(1/2))+EllipticF(((a+b*sin(f*x+e))/(a-b))^(1/2),((a-b)/(a+b))^(
1/2))))/cos(f*x+e)/(a+b*sin(f*x+e))^(1/2)/f
```

Fricas [F(-1)]

Timed out.

$$\int \frac{\csc(e + fx)}{\sqrt{a + b \sin(e + fx)}(c + c \sin(e + fx))} dx = \text{Timed out}$$

input

```
integrate(csc(f*x+e)/(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x, algorithm=
"fricas")
```

output

Timed out

Sympy [F]

$$\int \frac{\csc(e + fx)}{\sqrt{a + b \sin(e + fx)}(c + c \sin(e + fx))} dx = \frac{\int \frac{\csc(e+fx)}{\sqrt{a+b \sin(e+fx)} \sin(e+fx) + \sqrt{a+b \sin(e+fx)}} dx}{c}$$

input

```
integrate(csc(f*x+e)/(a+b*sin(f*x+e))**(1/2)/(c+c*sin(f*x+e)),x)
```

output

```
Integral(csc(e + f*x)/(sqrt(a + b*sin(e + f*x))*sin(e + f*x) + sqrt(a + b*
sin(e + f*x))), x)/c
```

Maxima [F]

$$\int \frac{\csc(e + fx)}{\sqrt{a + b \sin(e + fx)}(c + c \sin(e + fx))} dx$$

$$= \int \frac{\csc(fx + e)}{\sqrt{b \sin(fx + e) + a}(c \sin(fx + e) + c)} dx$$

input

```
integrate(csc(f*x+e)/(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x, algorithm=
"maxima")
```

output

```
integrate(csc(f*x + e)/(sqrt(b*sin(f*x + e) + a)*(c*sin(f*x + e) + c)), x)
```

Giac [F]

$$\int \frac{\csc(e + fx)}{\sqrt{a + b \sin(e + fx)}(c + c \sin(e + fx))} dx$$

$$= \int \frac{\csc(fx + e)}{\sqrt{b \sin(fx + e) + a}(c \sin(fx + e) + c)} dx$$

input

```
integrate(csc(f*x+e)/(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x, algorithm=
"giac")
```

output

```
integrate(csc(f*x + e)/(sqrt(b*sin(f*x + e) + a)*(c*sin(f*x + e) + c)), x)
```

Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e + fx)}{\sqrt{a + b \sin(e + fx)}(c + c \sin(e + fx))} dx$$

$$= \int \frac{1}{\sin(e + fx) \sqrt{a + b \sin(e + fx)} (c + c \sin(e + fx))} dx$$

input `int(1/(sin(e + f*x)*(a + b*sin(e + f*x))^(1/2)*(c + c*sin(e + f*x))),x)`

output `int(1/(sin(e + f*x)*(a + b*sin(e + f*x))^(1/2)*(c + c*sin(e + f*x))), x)`

Reduce [F]

$$\int \frac{\csc(e + fx)}{\sqrt{a + b \sin(e + fx)}(c + c \sin(e + fx))} dx = \int \frac{\frac{\sqrt{\sin(fx+e)b+a} \csc(fx+e)}{\sin(fx+e)^2 b+a \sin(fx+e)+\sin(fx+e)b+a} dx}{c}$$

input `int(csc(f*x+e)/(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x)`

output `int((sqrt(sin(e + f*x)*b + a)*csc(e + f*x))/(sin(e + f*x)**2*b + sin(e + f*x)*a + sin(e + f*x)*b + a),x)/c`

3.31
$$\int \frac{\sqrt{g \sin(e+fx)} \sqrt{a+b \sin(e+fx)}}{c+c \sin(e+fx)} dx$$

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Optimal result

Integrand size = 39, antiderivative size = 267

$$\int \frac{\sqrt{g \sin(e+fx)} \sqrt{a+b \sin(e+fx)}}{c+c \sin(e+fx)} dx$$

$$= \frac{2\sqrt{g} \operatorname{EllipticPi}\left(\frac{b}{a+b}, \arcsin\left(\frac{\sqrt{a+b}\sqrt{g \sin(e+fx)}}{\sqrt{g}\sqrt{a+b \sin(e+fx)}}\right), -\frac{a-b}{a+b}\right) \sec(e+fx) \sqrt{\frac{a(1-\sin(e+fx))}{a+b \sin(e+fx)}} \sqrt{\frac{a(1+\sin(e+fx))}{a+b \sin(e+fx)}} (a+b \sin(e+fx))}{\sqrt{a+bcf}}$$

$$+ \frac{gE\left(\arcsin\left(\frac{\cos(e+fx)}{1+\sin(e+fx)}\right) \middle| -\frac{a-b}{a+b}\right) \sqrt{\frac{\sin(e+fx)}{1+\sin(e+fx)}} \sqrt{a+b \sin(e+fx)}}{cf \sqrt{g \sin(e+fx)} \sqrt{\frac{a+b \sin(e+fx)}{(a+b)(1+\sin(e+fx))}}}$$

output

```
2*g^(1/2)*EllipticPi((a+b)^(1/2)*(g*sin(f*x+e))^(1/2)/g^(1/2)/(a+b*sin(f*x+e))^(1/2),b/(a+b),(-(a-b)/(a+b))^(1/2))*sec(f*x+e)*(a*(1-sin(f*x+e))/(a+b*sin(f*x+e)))^(1/2)*(a*(1+sin(f*x+e))/(a+b*sin(f*x+e)))^(1/2)*(a+b*sin(f*x+e))/(a+b)^(1/2)/c/f+g*EllipticE(cos(f*x+e)/(1+sin(f*x+e)),(-(a-b)/(a+b))^(1/2))*(sin(f*x+e)/(1+sin(f*x+e)))^(1/2)*(a+b*sin(f*x+e))^(1/2)/c/f/(g*sin(f*x+e))^(1/2)/((a+b*sin(f*x+e))/(a+b)/(1+sin(f*x+e)))^(1/2)
```

Mathematica [C] (warning: unable to verify)

Result contains complex when optimal does not.

Time = 36.98 (sec) , antiderivative size = 13199, normalized size of antiderivative = 49.43

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}}{c + c \sin(e + fx)} dx = \text{Result too large to show}$$

input

```
Integrate[(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]/(c + c*Sin[e + f*x]),x]
```

output

Result too large to show

Rubi [A] (verified)

Time = 0.96 (sec) , antiderivative size = 267, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.128$, Rules used = {3042, 3407, 3042, 3290, 3411}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}}{c \sin(e + fx) + c} dx \\ & \quad \downarrow \text{3042} \\ & \int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}}{c \sin(e + fx) + c} dx \\ & \quad \downarrow \text{3407} \\ & \frac{g \int \frac{\sqrt{a + b \sin(e + fx)}}{\sqrt{g \sin(e + fx)}} dx}{c} - g \int \frac{\sqrt{a + b \sin(e + fx)}}{\sqrt{g \sin(e + fx)} (\sin(e + fx) c + c)} dx \\ & \quad \downarrow \text{3042} \end{aligned}$$

$$\frac{g \int \frac{\sqrt{a+b \sin(e+fx)}}{\sqrt{g \sin(e+fx)}} dx}{c} - g \int \frac{\sqrt{a+b \sin(e+fx)}}{\sqrt{g \sin(e+fx)}(\sin(e+fx)c+c)} dx$$

↓ 3290

$$\frac{2\sqrt{g} \sec(e+fx) \sqrt{\frac{a(1-\sin(e+fx))}{a+b \sin(e+fx)}} \sqrt{\frac{a(\sin(e+fx)+1)}{a+b \sin(e+fx)}} (a+b \sin(e+fx)) \operatorname{EllipticPi}\left(\frac{b}{a+b}, \arcsin\left(\frac{\sqrt{a+b} \sqrt{g \sin(e+fx)}}{\sqrt{g} \sqrt{a+b \sin(e+fx)}}\right)\right), -}{cf\sqrt{a+b}}$$

$$g \int \frac{\sqrt{a+b \sin(e+fx)}}{\sqrt{g \sin(e+fx)}(\sin(e+fx)c+c)} dx$$

↓ 3411

$$\frac{g \sqrt{\frac{\sin(e+fx)}{\sin(e+fx)+1}} \sqrt{a+b \sin(e+fx)} E\left(\arcsin\left(\frac{\cos(e+fx)}{\sin(e+fx)+1}\right) \mid -\frac{a-b}{a+b}\right) +}{cf\sqrt{g \sin(e+fx)} \sqrt{\frac{a+b \sin(e+fx)}{(a+b)(\sin(e+fx)+1)}}} +$$

$$\frac{2\sqrt{g} \sec(e+fx) \sqrt{\frac{a(1-\sin(e+fx))}{a+b \sin(e+fx)}} \sqrt{\frac{a(\sin(e+fx)+1)}{a+b \sin(e+fx)}} (a+b \sin(e+fx)) \operatorname{EllipticPi}\left(\frac{b}{a+b}, \arcsin\left(\frac{\sqrt{a+b} \sqrt{g \sin(e+fx)}}{\sqrt{g} \sqrt{a+b \sin(e+fx)}}\right)\right), -}{cf\sqrt{a+b}}$$

input

```
Int[(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]])/(c + c*Sin[e + f*x]),x
]
```

output

```
(2*Sqrt[g]*EllipticPi[b/(a + b), ArcSin[(Sqrt[a + b]*Sqrt[g*Sin[e + f*x]])
/(Sqrt[g]*Sqrt[a + b*Sin[e + f*x]])], -((a - b)/(a + b))]*Sec[e + f*x]*Sqr
t[(a*(1 - Sin[e + f*x]))/(a + b*Sin[e + f*x])*Sqrt[(a*(1 + Sin[e + f*x]))
/(a + b*Sin[e + f*x])]*(a + b*Sin[e + f*x])]/(Sqrt[a + b]*c*f) + (g*Ellipt
icE[ArcSin[Cos[e + f*x]/(1 + Sin[e + f*x])], -((a - b)/(a + b))]*Sqrt[Sin[
e + f*x]/(1 + Sin[e + f*x])*Sqrt[a + b*Sin[e + f*x]])/(c*f*Sqrt[g*Sin[e +
f*x]]*Sqrt[(a + b*Sin[e + f*x])/((a + b)*(1 + Sin[e + f*x]))])
```

Defintions of rubi rules used

rule 3042

```
Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3290

```
Int[Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)]]/Sqrt[(c_) + (d_)*sin[(e_)
+ (f_)*(x_)]], x_Symbol] := Simp[2*((a + b*Sin[e + f*x])/(d*f*Rt[(a + b)/
(c + d), 2]*Cos[e + f*x]))*Sqrt[(b*c - a*d)*((1 + Sin[e + f*x])/((c - d)*(a
+ b*Sin[e + f*x])))]*Sqrt[(-(b*c - a*d))*((1 - Sin[e + f*x])/((c + d)*(a +
b*Sin[e + f*x])))]*EllipticPi[b*((c + d)/(d*(a + b))), ArcSin[Rt[(a + b)/(
c + d), 2]*(Sqrt[c + d*Sin[e + f*x]]/Sqrt[a + b*Sin[e + f*x]])], (a - b)*((
c + d)/((a + b)*(c - d))), x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c -
a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0] && PosQ[(a + b)/(c + d)]
```

rule 3407

```
Int[(Sqrt[(g_)*sin[(e_) + (f_)*(x_)]]*Sqrt[(a_) + (b_)*sin[(e_) + (f_
)*(x_)])]/((c_) + (d_)*sin[(e_) + (f_)*(x_)]), x_Symbol] := Simp[g/d I
nt[Sqrt[a + b*Sin[e + f*x]]/Sqrt[g*Sin[e + f*x]], x], x] - Simp[c*(g/d) I
nt[Sqrt[a + b*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c + d*Sin[e + f*x])), x]
, x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && (EqQ[a^2 -
b^2, 0] || EqQ[c^2 - d^2, 0])
```

rule 3411

```
Int[Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)]]/(Sqrt[(g_)*sin[(e_) + (f_
)*(x_)]]*((c_) + (d_)*sin[(e_) + (f_)*(x_)])), x_Symbol] := Simp[(-Sqrt[
a + b*Sin[e + f*x]]*(Sqrt[d*(Sin[e + f*x]/(c + d*Sin[e + f*x]))]/(d*f*Sqrt
[g*Sin[e + f*x]]*Sqrt[c^2*((a + b*Sin[e + f*x])/((a*c + b*d)*(c + d*Sin[e +
f*x])))])))*EllipticE[ArcSin[c*(Cos[e + f*x]/(c + d*Sin[e + f*x]))], (b*c -
a*d)/(b*c + a*d)], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d,
0] && NeQ[a^2 - b^2, 0] && EqQ[c^2 - d^2, 0]
```

Maple [C] (warning: unable to verify)

Result contains complex when optimal does not.

Time = 4.61 (sec) , antiderivative size = 5285, normalized size of antiderivative = 19.79

method	result	size
default	Expression too large to display	5285

input

```
int((g*sin(f*x+e))^(1/2)*(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x,method=
_RETURNVERBOSE)
```


output result too large to display

Fricas [F]

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}}{c + c \sin(e + fx)} dx = \int \frac{\sqrt{b \sin(fx + e) + a} \sqrt{g \sin(fx + e)}}{c \sin(fx + e) + c} dx$$

input `integrate((g*sin(f*x+e))^(1/2)*(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x,
algorithm="fricas")`

output `integral(sqrt(b*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))/(c*sin(f*x + e) + c
, x)`

Sympy [F]

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}}{c + c \sin(e + fx)} dx = \frac{\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}}{\sin(e + fx) + 1} dx}{c}$$

input `integrate((g*sin(f*x+e))**(1/2)*(a+b*sin(f*x+e))**(1/2)/(c+c*sin(f*x+e)),x
)`

output `Integral(sqrt(g*sin(e + f*x))*sqrt(a + b*sin(e + f*x))/(sin(e + f*x) + 1),
x)/c`

Maxima [F]

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}}{c + c \sin(e + fx)} dx = \int \frac{\sqrt{b \sin(fx + e) + a} \sqrt{g \sin(fx + e)}}{c \sin(fx + e) + c} dx$$

input `integrate((g*sin(f*x+e))^(1/2)*(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x,
algorithm="maxima")`

output `integrate(sqrt(b*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))/(c*sin(f*x + e) +
c), x)`

Giac [F]

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}}{c + c \sin(e + fx)} dx = \int \frac{\sqrt{b \sin(fx + e) + a} \sqrt{g \sin(fx + e)}}{c \sin(fx + e) + c} dx$$

input `integrate((g*sin(f*x+e))^(1/2)*(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x,
algorithm="giac")`

output `integrate(sqrt(b*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))/(c*sin(f*x + e) +
c), x)`

Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}}{c + c \sin(e + fx)} dx = \int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}}{c + c \sin(e + fx)} dx$$

input `int(((g*sin(e + f*x))^(1/2)*(a + b*sin(e + f*x))^(1/2))/(c + c*sin(e + f*x
)),x)`

output `int(((g*sin(e + f*x))^(1/2)*(a + b*sin(e + f*x))^(1/2))/(c + c*sin(e + f*x))), x)`

Reduce [F]

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}}{c + c \sin(e + fx)} dx = \frac{\sqrt{g} \left(\int \frac{\sqrt{\sin(fx+e)} \sqrt{\sin(fx+e)b+a}}{\sin(fx+e)+1} dx \right)}{c}$$

input `int((g*sin(f*x+e))^(1/2)*(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x)`

output `(sqrt(g)*int((sqrt(sin(e + f*x))*sqrt(sin(e + f*x)*b + a))/(sin(e + f*x) + 1),x))/c`

3.32
$$\int \frac{\sqrt{a+b \sin(e+fx)}}{\sqrt{g \sin(e+fx)}(c+c \sin(e+fx))} dx$$

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Optimal result

Integrand size = 39, antiderivative size = 116

$$\int \frac{\sqrt{a+b \sin(e+fx)}}{\sqrt{g \sin(e+fx)}(c+c \sin(e+fx))} dx$$

$$= -\frac{E\left(\arcsin\left(\frac{\cos(e+fx)}{1+\sin(e+fx)}\right) \mid -\frac{a-b}{a+b}\right) \sqrt{\frac{\sin(e+fx)}{1+\sin(e+fx)}} \sqrt{a+b \sin(e+fx)}}{cf \sqrt{g \sin(e+fx)} \sqrt{\frac{a+b \sin(e+fx)}{(a+b)(1+\sin(e+fx))}}}$$

output

```
-EllipticE(cos(f*x+e)/(1+sin(f*x+e)),(-(a-b)/(a+b))^(1/2))*(sin(f*x+e)/(1+sin(f*x+e)))^(1/2)*(a+b*sin(f*x+e))^(1/2)/c/f/(g*sin(f*x+e))^(1/2)/((a+b*sin(f*x+e))/(a+b)/(1+sin(f*x+e)))^(1/2)
```

Mathematica [B] (warning: unable to verify)

Leaf count is larger than twice the leaf count of optimal. 4679 vs. 2(116) = 232.

Time = 33.08 (sec) , antiderivative size = 4679, normalized size of antiderivative = 40.34

$$\int \frac{\sqrt{a+b \sin(e+fx)}}{\sqrt{g \sin(e+fx)}(c+c \sin(e+fx))} dx = \text{Result too large to show}$$

input

```
Integrate[Sqrt[a + b*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c + c*Sin[e + f*x])),x]
```

output

```
(-2*Sin[(e + f*x)/2]*(Cos[(e + f*x)/2] + Sin[(e + f*x)/2])*Sin[e + f*x]*Sqrt[a + b*Sin[e + f*x]]/(f*Sqrt[g*Sin[e + f*x]]*(c + c*Sin[e + f*x])) + ((Cos[(e + f*x)/2] + Sin[(e + f*x)/2])^2*Sqrt[Sin[e + f*x]]*((a*Sqrt[Sin[e + f*x]])/(2*Sqrt[a + b*Sin[e + f*x]]) - (b*Sqrt[Sin[e + f*x]])/(2*Sqrt[a + b*Sin[e + f*x]]) + (a*Cot[(e + f*x)/2]*Sqrt[Sin[e + f*x]])/(2*Sqrt[a + b*Sin[e + f*x]]) + (b*Cot[(e + f*x)/2]*Sqrt[Sin[e + f*x]])/(2*Sqrt[a + b*Sin[e + f*x]]) - (b*Cos[(3*(e + f*x))/2]*Csc[(e + f*x)/2]*Sqrt[Sin[e + f*x]])/(2*Sqrt[a + b*Sin[e + f*x]]) + (b*Csc[(e + f*x)/2]*Sqrt[Sin[e + f*x]]*Sin[(3*(e + f*x))/2])/(2*Sqrt[a + b*Sin[e + f*x]])*Sqrt[(a + 2*b*Tan[(e + f*x)/2] + a*Tan[(e + f*x)/2]^2)/(1 + Tan[(e + f*x)/2]^2)]*((Tan[(e + f*x)/2]*(1 + Tan[(e + f*x)/2]))/(1 + Tan[(e + f*x)/2]^2) + (Sqrt[-a^2 + b^2]*Sqrt[(a*(a + 2*b*Tan[(e + f*x)/2] + a*Tan[(e + f*x)/2]^2))/(a^2 - b^2)]*(EllipticE[ArcSin[Sqrt[(-b + Sqrt[-a^2 + b^2] - a*Tan[(e + f*x)/2])/Sqrt[-a^2 + b^2]]/Sqrt[2]], (2*Sqrt[-a^2 + b^2])/(-b + Sqrt[-a^2 + b^2])]*Tan[(e + f*x)/2] + EllipticF[ArcSin[Sqrt[(b + Sqrt[-a^2 + b^2] + a*Tan[(e + f*x)/2])/Sqrt[-a^2 + b^2]]/Sqrt[2]], (2*Sqrt[-a^2 + b^2])/(b + Sqrt[-a^2 + b^2])]*Sqrt[(a*Tan[(e + f*x)/2])/(-b + Sqrt[-a^2 + b^2])]*Sqrt[-((a*Tan[(e + f*x)/2])/(b + Sqrt[-a^2 + b^2]))])/(Sqrt[(a*Tan[(e + f*x)/2])/(-b + Sqrt[-a^2 + b^2])]*(a + 2*b*Tan[(e + f*x)/2] + a*Tan[(e + f*x)/2]^2)))/(f*Sqrt[g*Sin[e + f*x]]*(c + c*Sin[e + f*x])*Sqrt[Tan[(e + f*x)/2]/(2 + 2*Tan[(e + f*...
```

Rubi [A] (verified)

Time = 0.41 (sec) , antiderivative size = 116, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}} = 0.051$, Rules used = {3042, 3411}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sqrt{a + b \sin(e + fx)}}{(c \sin(e + fx) + c) \sqrt{g \sin(e + fx)}} dx$$

↓ 3042

$$\int \frac{\sqrt{a + b \sin(e + fx)}}{(c \sin(e + fx) + c) \sqrt{g \sin(e + fx)}} dx$$

↓ 3411

$$\frac{\sqrt{\frac{\sin(e+fx)}{\sin(e+fx)+1}} \sqrt{a + b \sin(e + fx)} E\left(\arcsin\left(\frac{\cos(e+fx)}{\sin(e+fx)+1}\right) \mid -\frac{a-b}{a+b}\right)}{cf \sqrt{g \sin(e + fx)} \sqrt{\frac{a+b \sin(e+fx)}{(a+b)(\sin(e+fx)+1)}}$$

input `Int[Sqrt[a + b*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c + c*Sin[e + f*x])),x]`

output `-((EllipticE[ArcSin[Cos[e + f*x]/(1 + Sin[e + f*x])], -(a - b)/(a + b)]*Sqrt[Sin[e + f*x]/(1 + Sin[e + f*x])] * Sqrt[a + b*Sin[e + f*x]]/(c*f*Sqrt[g*Sin[e + f*x]]*Sqrt[(a + b*Sin[e + f*x])/((a + b)*(1 + Sin[e + f*x]))]))`

Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3411 `Int[Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)]]/(Sqrt[(g_)*sin[(e_) + (f_)*(x_)])*(x_)])*((c_) + (d_)*sin[(e_) + (f_)*(x_)])], x_Symbol] := Simp[(-Sqrt[a + b*Sin[e + f*x]]*(Sqrt[d*(Sin[e + f*x]/(c + d*Sin[e + f*x]))]/(d*f*Sqrt[g*Sin[e + f*x]]*Sqrt[c^2*((a + b*Sin[e + f*x])/((a*c + b*d)*(c + d*Sin[e + f*x]))])))*EllipticE[ArcSin[c*(Cos[e + f*x]/(c + d*Sin[e + f*x])], (b*c - a*d)/(b*c + a*d)], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && EqQ[c^2 - d^2, 0]`

Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 1578 vs. $2(109) = 218$.

Time = 3.48 (sec) , antiderivative size = 1579, normalized size of antiderivative = 13.61

method	result	size
default	Expression too large to display	1579

input `int((a+b*sin(f*x+e))^(1/2)/(g*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x,method=_RETURNVERBOSE)`

output
$$\begin{aligned} & \frac{1}{2} \frac{c}{f} (a+b \sin(fx+e))^{1/2} \left((2 \cos(fx+e)+2) \sin(fx+e) + 2(1+\cos(fx+e))^2 \right) \\ & (-a^2+b^2)^{1/2} \left(\frac{1}{(-a^2+b^2)^{1/2}} (a \cot(fx+e) + (-a^2+b^2)^{1/2} - b - a \csc(fx+e)) \right)^{1/2} \\ & \left(\frac{1}{b+(-a^2+b^2)^{1/2}} a (-\csc(fx+e) + \cot(fx+e)) \right)^{1/2} \operatorname{EllipticE} \left(\frac{(-a \cot(fx+e) + (-a^2+b^2)^{1/2} + b + a \csc(fx+e))}{b+(-a^2+b^2)^{1/2}} \right)^{1/2}, \\ & \frac{1}{2} 2^{1/2} \left(\frac{b+(-a^2+b^2)^{1/2}}{(-a^2+b^2)^{1/2}} \right)^{1/2} \left(\frac{(-a \cot(fx+e) + (-a^2+b^2)^{1/2} + b + a \csc(fx+e))}{b+(-a^2+b^2)^{1/2}} \right)^{1/2} \\ & b \left((-2 \cos(fx+e) - 2) \sin(fx+e) - 2(1+\cos(fx+e))^2 \right) \frac{1}{(-a^2+b^2)^{1/2}} \\ & \left(\frac{1}{b+(-a^2+b^2)^{1/2}} (a \cot(fx+e) + (-a^2+b^2)^{1/2} - b - a \csc(fx+e)) \right)^{1/2} \left(\frac{1}{b+(-a^2+b^2)^{1/2}} \right)^{1/2} \\ & a \left(-\csc(fx+e) + \cot(fx+e) \right)^{1/2} \operatorname{EllipticE} \left(\frac{(-a \cot(fx+e) + (-a^2+b^2)^{1/2} + b + a \csc(fx+e))}{b+(-a^2+b^2)^{1/2}} \right)^{1/2}, \\ & \frac{1}{2} 2^{1/2} \left(\frac{b+(-a^2+b^2)^{1/2}}{(-a^2+b^2)^{1/2}} \right)^{1/2} \left(\frac{(-a \cot(fx+e) + (-a^2+b^2)^{1/2} + b + a \csc(fx+e))}{b+(-a^2+b^2)^{1/2}} \right)^{1/2} \\ & a^2 \left((2 \cos(fx+e) + 2) \sin(fx+e) + 2(1+\cos(fx+e))^2 \right) \frac{1}{(-a^2+b^2)^{1/2}} \\ & \left(\frac{1}{b+(-a^2+b^2)^{1/2}} (a \cot(fx+e) + (-a^2+b^2)^{1/2} - b - a \csc(fx+e)) \right)^{1/2} \left(\frac{1}{b+(-a^2+b^2)^{1/2}} \right)^{1/2} \\ & a \left(-\csc(fx+e) + \cot(fx+e) \right)^{1/2} \operatorname{EllipticE} \left(\frac{(-a \cot(fx+e) + (-a^2+b^2)^{1/2} + b + a \csc(fx+e))}{b+(-a^2+b^2)^{1/2}} \right)^{1/2}, \\ & \frac{1}{2} 2^{1/2} \left(\frac{b+(-a^2+b^2)^{1/2}}{(-a^2+b^2)^{1/2}} \right)^{1/2} \left(\frac{(-a \cot(fx+e) + (-a^2+b^2)^{1/2} + b + a \csc(fx+e))}{b+(-a^2+b^2)^{1/2}} \right)^{1/2} \\ & b^2 \left((1+\cos(fx+e)) \sin(fx+e) + (1+\cos(fx+e))^2 \right) (-a^2+b^2)^{1/2} \\ & \frac{1}{(-a^2+b^2)^{1/2}} \left(\frac{1}{b+(-a^2+b^2)^{1/2}} (a \cot(fx+e) + (-a^2+b^2)^{1/2} - b - a \csc(fx+e)) \right)^{1/2} \\ & \left(\frac{1}{b+(-a^2+b^2)^{1/2}} a \left(-\csc(fx+e) + \cot(fx+e) \right) \right)^{1/2} \operatorname{EllipticF}(\dots \end{aligned}$$

Fricas [F]

$$\int \frac{\sqrt{a + b \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c + c \sin(e + fx))} dx = \int \frac{\sqrt{b \sin(fx + e) + a}}{(c \sin(fx + e) + c) \sqrt{g \sin(fx + e)}} dx$$

input `integrate((a+b*sin(f*x+e))^(1/2)/(g*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x,
algorithm="fricas")`

output `integral(-sqrt(b*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))/(c*g*cos(f*x + e)^
2 - c*g*sin(f*x + e) - c*g), x)`

Sympy [F]

$$\int \frac{\sqrt{a + b \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c + c \sin(e + fx))} dx = \frac{\int \frac{\sqrt{a + b \sin(e + fx)}}{\sqrt{g \sin(e + fx)} \sin(e + fx) + \sqrt{g \sin(e + fx)}} dx}{c}$$

input `integrate((a+b*sin(f*x+e))**(1/2)/(g*sin(f*x+e))**(1/2)/(c+c*sin(f*x+e)),x
)`

output `Integral(sqrt(a + b*sin(e + f*x))/(sqrt(g*sin(e + f*x))*sin(e + f*x) + sqr
t(g*sin(e + f*x))), x)/c`

Maxima [F]

$$\int \frac{\sqrt{a + b \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c + c \sin(e + fx))} dx = \int \frac{\sqrt{b \sin(fx + e) + a}}{(c \sin(fx + e) + c) \sqrt{g \sin(fx + e)}} dx$$

input `integrate((a+b*sin(f*x+e))^(1/2)/(g*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x,
algorithm="maxima")`

output `integrate(sqrt(b*sin(f*x + e) + a)/((c*sin(f*x + e) + c)*sqrt(g*sin(f*x + e))), x)`

Giac [F]

$$\int \frac{\sqrt{a + b \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c + c \sin(e + fx))} dx = \int \frac{\sqrt{b \sin(fx + e) + a}}{(c \sin(fx + e) + c)\sqrt{g \sin(fx + e)}} dx$$

input `integrate((a+b*sin(f*x+e))^(1/2)/(g*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x, algorithm="giac")`

output `integrate(sqrt(b*sin(f*x + e) + a)/((c*sin(f*x + e) + c)*sqrt(g*sin(f*x + e))), x)`

Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{a + b \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c + c \sin(e + fx))} dx = \int \frac{\sqrt{a + b \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c + c \sin(e + fx))} dx$$

input `int((a + b*sin(e + f*x))^(1/2)/((g*sin(e + f*x))^(1/2)*(c + c*sin(e + f*x))),x)`

output `int((a + b*sin(e + f*x))^(1/2)/((g*sin(e + f*x))^(1/2)*(c + c*sin(e + f*x))), x)`

Reduce [F]

$$\int \frac{\sqrt{a + b \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c + c \sin(e + fx))} dx = \frac{\sqrt{g} \left(\int \frac{\sqrt{\sin(fx+e)} \sqrt{\sin(fx+e)b+a}}{\sin(fx+e)^2 + \sin(fx+e)} dx \right)}{cg}$$

input `int((a+b*sin(f*x+e))^(1/2)/(g*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x)`

output `(sqrt(g)*int((sqrt(sin(e + f*x))*sqrt(sin(e + f*x)*b + a))/(sin(e + f*x)**2 + sin(e + f*x)),x))/(c*g)`

$$3.33 \quad \int \frac{\sqrt{g \sin(e+fx)}}{\sqrt{a+b \sin(e+fx)}(c+c \sin(e+fx))} dx$$

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Optimal result

Integrand size = 39, antiderivative size = 252

$$\int \frac{\sqrt{g \sin(e+fx)}}{\sqrt{a+b \sin(e+fx)}(c+c \sin(e+fx))} dx$$

$$= \frac{gE\left(\arcsin\left(\frac{\cos(e+fx)}{1+\sin(e+fx)}\right) \mid -\frac{a-b}{a+b}\right) \sqrt{\frac{\sin(e+fx)}{1+\sin(e+fx)}} \sqrt{a+b \sin(e+fx)}}{(a-b)cf \sqrt{g \sin(e+fx)} \sqrt{\frac{a+b \sin(e+fx)}{(a+b)(1+\sin(e+fx))}}}$$

$$- \frac{2\sqrt{a+b} \sqrt{g} \sqrt{\frac{a(1-\csc(e+fx))}{a+b}} \sqrt{\frac{a(1+\csc(e+fx))}{a-b}} \operatorname{EllipticF}\left(\arcsin\left(\frac{\sqrt{g} \sqrt{a+b \sin(e+fx)}}{\sqrt{a+b} \sqrt{g \sin(e+fx)}}\right), -\frac{a+b}{a-b}\right) \tan(e+fx)}{(a-b)cf}$$

output

```
g*EllipticE(cos(f*x+e)/(1+sin(f*x+e)),(-a-b)/(a+b))^(1/2)*(sin(f*x+e)/(1+sin(f*x+e)))^(1/2)*(a+b*sin(f*x+e))^(1/2)/(a-b)/c/f/(g*sin(f*x+e))^(1/2)/((a+b*sin(f*x+e))/(a+b)/(1+sin(f*x+e)))^(1/2)-2*(a+b)^(1/2)*g^(1/2)*(a*(1-csc(f*x+e))/(a+b))^(1/2)*(a*(1+csc(f*x+e))/(a-b))^(1/2)*EllipticF(g^(1/2)*(a+b*sin(f*x+e))^(1/2)/(a+b)^(1/2)/(g*sin(f*x+e))^(1/2),(-a+b)/(a-b))^(1/2))*tan(f*x+e)/(a-b)/c/f
```

Mathematica [B] (warning: unable to verify)

Leaf count is larger than twice the leaf count of optimal. 5708 vs. $2(252) = 504$.

Time = 33.87 (sec) , antiderivative size = 5708, normalized size of antiderivative = 22.65

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)}(c + c \sin(e + fx))} dx = \text{Result too large to show}$$

input

```
Integrate[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + b*Sin[e + f*x]]*(c + c*Sin[e + f*x])),x]
```

output

Result too large to show

Rubi [A] (verified)

Time = 0.97 (sec) , antiderivative size = 252, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.128$, Rules used = {3042, 3415, 3042, 3295, 3411}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \frac{\sqrt{g \sin(e + fx)}}{(c \sin(e + fx) + c) \sqrt{a + b \sin(e + fx)}} dx \\ & \quad \downarrow \text{3042} \\ & \int \frac{\sqrt{g \sin(e + fx)}}{(c \sin(e + fx) + c) \sqrt{a + b \sin(e + fx)}} dx \\ & \quad \downarrow \text{3415} \\ & \frac{ag \int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}} dx}{c(a - b)} - \frac{g \int \frac{\sqrt{a + b \sin(e + fx)}}{\sqrt{g \sin(e + fx)} (\sin(e + fx)c + c)} dx}{a - b} \\ & \quad \downarrow \text{3042} \end{aligned}$$

$$\begin{aligned}
& \frac{ag \int \frac{1}{\sqrt{g \sin(e+fx)} \sqrt{a+b \sin(e+fx)}} dx}{c(a-b)} - \frac{g \int \frac{\sqrt{a+b \sin(e+fx)}}{\sqrt{g \sin(e+fx)} (\sin(e+fx)c+c)} dx}{a-b} \\
& \quad \downarrow \text{3295} \\
& \frac{g \int \frac{\sqrt{a+b \sin(e+fx)}}{\sqrt{g \sin(e+fx)} (\sin(e+fx)c+c)} dx}{a-b} - \\
& \frac{2\sqrt{g}\sqrt{a+b} \tan(e+fx) \sqrt{\frac{a(1-\csc(e+fx))}{a+b}} \sqrt{\frac{a(\csc(e+fx)+1)}{a-b}} \text{EllipticF}\left(\arcsin\left(\frac{\sqrt{g}\sqrt{a+b \sin(e+fx)}}{\sqrt{a+b}\sqrt{g \sin(e+fx)}}\right), -\frac{a+b}{a-b}\right)}{cf(a-b)} \\
& \quad \downarrow \text{3411} \\
& \frac{g \sqrt{\frac{\sin(e+fx)}{\sin(e+fx)+1}} \sqrt{a+b \sin(e+fx)} E\left(\arcsin\left(\frac{\cos(e+fx)}{\sin(e+fx)+1}\right) \mid -\frac{a-b}{a+b}\right)}{cf(a-b) \sqrt{g \sin(e+fx)} \sqrt{\frac{a+b \sin(e+fx)}{(a+b)(\sin(e+fx)+1)}}} - \\
& \frac{2\sqrt{g}\sqrt{a+b} \tan(e+fx) \sqrt{\frac{a(1-\csc(e+fx))}{a+b}} \sqrt{\frac{a(\csc(e+fx)+1)}{a-b}} \text{EllipticF}\left(\arcsin\left(\frac{\sqrt{g}\sqrt{a+b \sin(e+fx)}}{\sqrt{a+b}\sqrt{g \sin(e+fx)}}\right), -\frac{a+b}{a-b}\right)}{cf(a-b)}
\end{aligned}$$

input

```
Int[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + b*Sin[e + f*x]]*(c + c*Sin[e + f*x])),x
]
```

output

```
(g*EllipticE[ArcSin[Cos[e + f*x]/(1 + Sin[e + f*x])], -((a - b)/(a + b))]*
Sqrt[Sin[e + f*x]/(1 + Sin[e + f*x])]*Sqrt[a + b*Sin[e + f*x]]/((a - b)*c
*f*Sqrt[g*Sin[e + f*x]]*Sqrt[(a + b*Sin[e + f*x])/((a + b)*(1 + Sin[e + f
x]))]) - (2*Sqrt[a + b]*Sqrt[g]*Sqrt[(a*(1 - Csc[e + f*x]))/(a + b)]*Sqrt[
(a*(1 + Csc[e + f*x]))/(a - b)]*EllipticF[ArcSin[(Sqrt[g]*Sqrt[a + b*Sin[e
+ f*x]])/(Sqrt[a + b]*Sqrt[g*Sin[e + f*x]])], -((a + b)/(a - b))]*Tan[e +
f*x])/((a - b)*c*f)
```

Definitions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3295 `Int[1/(Sqrt[(d_)*sin[(e_) + (f_)*(x_)])*Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)])], x_Symbol] := Simp[-2*(Tan[e + f*x]/(a*f))*Rt[(a + b)/d, 2]*Sqrt[a*((1 - Csc[e + f*x])/(a + b))]*Sqrt[a*((1 + Csc[e + f*x])/(a - b))]*EllipticF[ArcSin[Sqrt[a + b*Sin[e + f*x]]/Sqrt[d*Sin[e + f*x]]/Rt[(a + b)/d, 2], -(a + b)/(a - b)], x] /; FreeQ[{a, b, d, e, f}, x] && NeQ[a^2 - b^2, 0] && PosQ[(a + b)/d]`

rule 3411 `Int[Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)]]/(Sqrt[(g_)*sin[(e_) + (f_)*(x_)])*((c_) + (d_)*sin[(e_) + (f_)*(x_)])], x_Symbol] := Simp[(-Sqrt[a + b*Sin[e + f*x]])*(Sqrt[d*(Sin[e + f*x])/(c + d*Sin[e + f*x])])/(d*f*Sqrt[g*Sin[e + f*x]]*Sqrt[c^2*((a + b*Sin[e + f*x])/(a*c + b*d)*(c + d*Sin[e + f*x]))]))*EllipticE[ArcSin[c*(Cos[e + f*x])/(c + d*Sin[e + f*x])], (b*c - a*d)/(b*c + a*d)], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && EqQ[c^2 - d^2, 0]`

rule 3415 `Int[Sqrt[(g_)*sin[(e_) + (f_)*(x_)]]/(Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)])*((c_) + (d_)*sin[(e_) + (f_)*(x_)])], x_Symbol] := Simp[(-a)*(g/(b*c - a*d)) Int[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]), x], x] + Simp[c*(g/(b*c - a*d)) Int[Sqrt[a + b*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c + d*Sin[e + f*x])), x], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && (EqQ[a^2 - b^2, 0] || EqQ[c^2 - d^2, 0])`

Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 1665 vs. $2(230) = 460$.

Time = 4.10 (sec) , antiderivative size = 1666, normalized size of antiderivative = 6.61

method	result	size
default	Expression too large to display	1666

input `int((g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x,method=_RETURNVERBOSE)`

output `-1/2/c/f*(g*sin(f*x+e))^(1/2)*(a+b*sin(f*x+e))^(1/2)*(((2*cos(f*x+e)-2)*sin(f*x+e)-2*(1+cos(f*x+e))^2)*(-a^2+b^2)^(1/2)*(-(a*cot(f*x+e)-(-a^2+b^2)^(1/2)-b-a*csc(f*x+e))/(b+(-a^2+b^2)^(1/2)))^(1/2)*(1/(-a^2+b^2)^(1/2)*(a*cot(f*x+e)+(-a^2+b^2)^(1/2)-b-a*csc(f*x+e)))^(1/2)*(1/(b+(-a^2+b^2)^(1/2))*a*(-csc(f*x+e)+cot(f*x+e)))^(1/2)*EllipticE((-a*cot(f*x+e)-(-a^2+b^2)^(1/2)-b-a*csc(f*x+e))/(b+(-a^2+b^2)^(1/2)))^(1/2),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2))*b+((2*cos(f*x+e)+2)*sin(f*x+e)+2*(1+cos(f*x+e))^2)*(-(a*cot(f*x+e)-(-a^2+b^2)^(1/2)-b-a*csc(f*x+e))/(b+(-a^2+b^2)^(1/2)))^(1/2)*(1/(-a^2+b^2)^(1/2)*(a*cot(f*x+e)+(-a^2+b^2)^(1/2)-b-a*csc(f*x+e)))^(1/2)*(1/(b+(-a^2+b^2)^(1/2))*a*(-csc(f*x+e)+cot(f*x+e)))^(1/2)*EllipticE((-a*cot(f*x+e)-(-a^2+b^2)^(1/2)-b-a*csc(f*x+e))/(b+(-a^2+b^2)^(1/2)))^(1/2),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2))*a^2+((-2*cos(f*x+e)-2)*sin(f*x+e)-2*(1+cos(f*x+e))^2)*(-(a*cot(f*x+e)-(-a^2+b^2)^(1/2)-b-a*csc(f*x+e))/(b+(-a^2+b^2)^(1/2)))^(1/2)*(1/(-a^2+b^2)^(1/2)*(a*cot(f*x+e)+(-a^2+b^2)^(1/2)-b-a*csc(f*x+e)))^(1/2)*(1/(b+(-a^2+b^2)^(1/2))*a*(-csc(f*x+e)+cot(f*x+e)))^(1/2)*EllipticE((-a*cot(f*x+e)-(-a^2+b^2)^(1/2)-b-a*csc(f*x+e))/(b+(-a^2+b^2)^(1/2)))^(1/2),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2))*b^2+((1+cos(f*x+e))*sin(f*x+e)+(1+cos(f*x+e))^2)*(-a^2+b^2)^(1/2)*(-(a*cot(f*x+e)-(-a^2+b^2)^(1/2)-b-a*csc(f*x+e))/(b+(-a^2+b^2)^(1/2)))^(1/2)*(1/(-a^2+b^2)^(1/2)*(a*cot(f*x+e)+(-a^2+b^2)...`

Fricas [F]

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)}(c + c \sin(e + fx))} dx$$

$$= \int \frac{\sqrt{g \sin(fx + e)}}{\sqrt{b \sin(fx + e) + a}(c \sin(fx + e) + c)} dx$$

input `integrate((g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x,algorithm="fricas")`

output

```
integral(-sqrt(b*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))/(b*c*cos(f*x + e)^
2 - (a + b)*c*sin(f*x + e) - (a + b)*c), x)
```

Sympy [F]

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)}(c + c \sin(e + fx))} dx = \frac{\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)} \sin(e + fx) + \sqrt{a + b \sin(e + fx)}} dx}{c}$$

input

```
integrate((g*sin(f*x+e))**(1/2)/(a+b*sin(f*x+e))**(1/2)/(c+c*sin(f*x+e)),x
)
```

output

```
Integral(sqrt(g*sin(e + f*x))/(sqrt(a + b*sin(e + f*x))*sin(e + f*x) + sqrt
(a + b*sin(e + f*x))), x)/c
```

Maxima [F]

$$\begin{aligned} & \int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)}(c + c \sin(e + fx))} dx \\ &= \int \frac{\sqrt{g \sin(fx + e)}}{\sqrt{b \sin(fx + e) + a}(c \sin(fx + e) + c)} dx \end{aligned}$$

input

```
integrate((g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x,
algorithm="maxima")
```

output

```
integrate(sqrt(g*sin(f*x + e))/(sqrt(b*sin(f*x + e) + a)*(c*sin(f*x + e) +
c)), x)
```


Giac [F]

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)}(c + c \sin(e + fx))} dx$$

$$= \int \frac{\sqrt{g \sin(fx + e)}}{\sqrt{b \sin(fx + e) + a}(c \sin(fx + e) + c)} dx$$

input `integrate((g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x,
algorithm="giac")`

output `integrate(sqrt(g*sin(f*x + e))/(sqrt(b*sin(f*x + e) + a)*(c*sin(f*x + e) +
c)), x)`

Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)}(c + c \sin(e + fx))} dx$$

$$= \int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)}(c + c \sin(e + fx))} dx$$

input `int((g*sin(e + f*x))^(1/2)/((a + b*sin(e + f*x))^(1/2)*(c + c*sin(e + f*x)
)),x)`

output `int((g*sin(e + f*x))^(1/2)/((a + b*sin(e + f*x))^(1/2)*(c + c*sin(e + f*x)
)), x)`

Reduce [F]

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)}(c + c \sin(e + fx))} dx$$

$$= \frac{\sqrt{g} \left(\int \frac{\sqrt{\sin(fx+e)} \sqrt{\sin(fx+e)b+a}}{\sin(fx+e)^2 b+a \sin(fx+e)+\sin(fx+e)b+a} dx \right)}{c}$$

input

```
int((g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x)
```

output

```
(sqrt(g)*int((sqrt(sin(e + f*x))*sqrt(sin(e + f*x)*b + a))/(sin(e + f*x)**
2*b + sin(e + f*x)*a + sin(e + f*x)*b + a),x))/c
```

$$3.34 \quad \int \frac{1}{\sqrt{g \sin(e+fx)} \sqrt{a+b \sin(e+fx)} (c+c \sin(e+fx))} dx$$

Optimal result	298
Mathematica [C] (warning: unable to verify)	299
Rubi [A] (verified)	300
Maple [B] (warning: unable to verify)	302
Fricas [F]	303
Sympy [F]	304
Maxima [F]	304
Giac [F]	305
Mupad [F(-1)]	305
Reduce [F]	306

Optimal result

Integrand size = 39, antiderivative size = 256

$$\int \frac{1}{\sqrt{g \sin(e+fx)} \sqrt{a+b \sin(e+fx)} (c+c \sin(e+fx))} dx$$

$$= -\frac{E\left(\arcsin\left(\frac{\cos(e+fx)}{1+\sin(e+fx)}\right) \mid -\frac{a-b}{a+b}\right) \sqrt{\frac{\sin(e+fx)}{1+\sin(e+fx)}} \sqrt{a+b \sin(e+fx)}}{(a-b)cf \sqrt{g \sin(e+fx)} \sqrt{\frac{a+b \sin(e+fx)}{(a+b)(1+\sin(e+fx))}}}$$

$$+ \frac{2b\sqrt{a+b} \sqrt{\frac{a(1-\csc(e+fx))}{a+b}} \sqrt{\frac{a(1+\csc(e+fx))}{a-b}} \operatorname{EllipticF}\left(\arcsin\left(\frac{\sqrt{g}\sqrt{a+b \sin(e+fx)}}{\sqrt{a+b}\sqrt{g \sin(e+fx)}}\right), -\frac{a+b}{a-b}\right) \tan(e+fx)}{a(a-b)cf\sqrt{g}}$$

output

```
-EllipticE(cos(f*x+e)/(1+sin(f*x+e)),(-(a-b)/(a+b))^(1/2))*(sin(f*x+e)/(1+sin(f*x+e)))^(1/2)*(a+b*sin(f*x+e))^(1/2)/(a-b)/c/f/(g*sin(f*x+e))^(1/2)/((a+b*sin(f*x+e))/(a+b)/(1+sin(f*x+e)))^(1/2)+2*b*(a+b)^(1/2)*(a*(1-csc(f*x+e))/(a+b))^(1/2)*(a*(1+csc(f*x+e))/(a-b))^(1/2)*EllipticF(g^(1/2)*(a+b*sin(f*x+e))^(1/2)/(a+b)^(1/2)/(g*sin(f*x+e))^(1/2),(-(a+b)/(a-b))^(1/2))*tan(f*x+e)/a/(a-b)/c/f/g^(1/2)
```

Mathematica [C] (warning: unable to verify)

Result contains complex when optimal does not.

Time = 21.57 (sec) , antiderivative size = 1667, normalized size of antiderivative = 6.51

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)} (c + c \sin(e + fx))} dx = \text{Too large to display}$$

input

```
Integrate[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]*(c + c*Sin[e + f*x])),x]
```

output

```
(-2*Sin[(e + f*x)/2]*(Cos[(e + f*x)/2] + Sin[(e + f*x)/2])*Sin[e + f*x]*Sqrt[a + b*Sin[e + f*x]]/((a - b)*f*Sqrt[g*Sin[e + f*x]]*(c + c*Sin[e + f*x])) + ((Cos[(e + f*x)/2] + Sin[(e + f*x)/2])^2*Sqrt[Sin[e + f*x]]*((4*a*(a - b)*Sqrt[((a + b)*Cot[(-e + Pi/2 - f*x)/2]^2)/(-a + b)]*EllipticF[ArcSin[Sqrt[(Csc[(-e + Pi/2 - f*x)/2]^2*(a + b*Sin[e + f*x]))/a]/Sqrt[2]], (-2*a)/(-a + b)]*Sec[e + f*x]*Sin[(-e + Pi/2 - f*x)/2]^4*Sqrt[-((a + b)*Csc[(-e + Pi/2 - f*x)/2]^2*Sin[e + f*x])/a])*Sqrt[(Csc[(-e + Pi/2 - f*x)/2]^2*(a + b*Sin[e + f*x])/a])/((a + b)*Sqrt[Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]) + (2*a*ArcTanh[(Sqrt[b]*Sqrt[Sin[e + f*x]])/Sqrt[a + b*Sin[e + f*x]])*Cos[e + f*x]^2/(Sqrt[b]*(1 - Sin[e + f*x]^2)) + 4*a^2*((Sqrt[((a + b)*Cot[(-e + Pi/2 - f*x)/2]^2)/(-a + b)]*EllipticF[ArcSin[Sqrt[(Csc[(-e + Pi/2 - f*x)/2]^2*(a + b*Sin[e + f*x]))/a]/Sqrt[2]], (-2*a)/(-a + b)]*Sec[e + f*x]*Sin[(-e + Pi/2 - f*x)/2]^4*Sqrt[-((a + b)*Csc[(-e + Pi/2 - f*x)/2]^2*Sin[e + f*x])/a])*Sqrt[(Csc[(-e + Pi/2 - f*x)/2]^2*(a + b*Sin[e + f*x])/a])/((a + b)*Sqrt[Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]) - (Sqrt[((a + b)*Cot[(-e + Pi/2 - f*x)/2]^2)/(-a + b)]*EllipticPi[-(a/b), ArcSin[Sqrt[(Csc[(-e + Pi/2 - f*x)/2]^2*(a + b*Sin[e + f*x]))/a]/Sqrt[2]], (-2*a)/(-a + b)]*Sec[e + f*x]*Sin[(-e + Pi/2 - f*x)/2]^4*Sqrt[-((a + b)*Csc[(-e + Pi/2 - f*x)/2]^2*Sin[e + f*x])/a])*Sqrt[(Csc[(-e + Pi/2 - f*x)/2]^2*(a + b*Sin[e + f*x])/a])/((b*Sqrt[Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]) - 2*b*((Cos...
```

Rubi [A] (verified)

Time = 0.99 (sec) , antiderivative size = 256, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.128$, Rules used = {3042, 3417, 3042, 3295, 3411}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{1}{(c \sin(e + fx) + c) \sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{1}{(c \sin(e + fx) + c) \sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}} dx \\
 & \quad \downarrow \text{3417} \\
 & \frac{\int \frac{\sqrt{a + b \sin(e + fx)}}{\sqrt{g \sin(e + fx)} (\sin(e + fx) c + c)} dx}{a - b} - \frac{b \int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}} dx}{c(a - b)} \\
 & \quad \downarrow \text{3042} \\
 & \frac{\int \frac{\sqrt{a + b \sin(e + fx)}}{\sqrt{g \sin(e + fx)} (\sin(e + fx) c + c)} dx}{a - b} - \frac{b \int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}} dx}{c(a - b)} \\
 & \quad \downarrow \text{3295} \\
 & \frac{\int \frac{\sqrt{a + b \sin(e + fx)}}{\sqrt{g \sin(e + fx)} (\sin(e + fx) c + c)} dx}{a - b} + \\
 & \frac{2b\sqrt{a + b} \tan(e + fx) \sqrt{\frac{a(1 - \csc(e + fx))}{a + b}} \sqrt{\frac{a(\csc(e + fx) + 1)}{a - b}} \text{EllipticF}\left(\arcsin\left(\frac{\sqrt{g} \sqrt{a + b \sin(e + fx)}}{\sqrt{a + b} \sqrt{g \sin(e + fx)}}\right), -\frac{a + b}{a - b}\right)}{acf\sqrt{g}(a - b)} \\
 & \quad \downarrow \text{3411} \\
 & \frac{2b\sqrt{a + b} \tan(e + fx) \sqrt{\frac{a(1 - \csc(e + fx))}{a + b}} \sqrt{\frac{a(\csc(e + fx) + 1)}{a - b}} \text{EllipticF}\left(\arcsin\left(\frac{\sqrt{g} \sqrt{a + b \sin(e + fx)}}{\sqrt{a + b} \sqrt{g \sin(e + fx)}}\right), -\frac{a + b}{a - b}\right)}{acf\sqrt{g}(a - b)} \\
 & \frac{\sqrt{\frac{\sin(e + fx)}{\sin(e + fx) + 1}} \sqrt{a + b \sin(e + fx)} E\left(\arcsin\left(\frac{\cos(e + fx)}{\sin(e + fx) + 1}\right) \mid -\frac{a - b}{a + b}\right)}{cf(a - b) \sqrt{g \sin(e + fx)} \sqrt{\frac{a + b \sin(e + fx)}{(a + b)(\sin(e + fx) + 1)}}}
 \end{aligned}$$

input `Int[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]*(c + c*Sin[e + f*x])),x]`

output `-((EllipticE[ArcSin[Cos[e + f*x]/(1 + Sin[e + f*x])], -((a - b)/(a + b))] * Sqrt[Sin[e + f*x]/(1 + Sin[e + f*x])] * Sqrt[a + b*Sin[e + f*x]])/((a - b)*c * f*Sqrt[g*Sin[e + f*x]]*Sqrt[(a + b*Sin[e + f*x])/((a + b)*(1 + Sin[e + f*x]))]) + (2*b*Sqrt[a + b]*Sqrt[(a*(1 - Csc[e + f*x]))/(a + b)]*Sqrt[(a*(1 + Csc[e + f*x]))/(a - b)]*EllipticF[ArcSin[(Sqrt[g]*Sqrt[a + b*Sin[e + f*x]])/(Sqrt[a + b]*Sqrt[g*Sin[e + f*x])], -((a + b)/(a - b))] * Tan[e + f*x])/(a*(a - b)*c*f*Sqrt[g])`

Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3295 `Int[1/(Sqrt[(d_)*sin[(e_) + (f_)*(x_)])*Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)]]), x_Symbol] := Simp[-2*(Tan[e + f*x]/(a*f))*Rt[(a + b)/d, 2]*Sqrt[a*((1 - Csc[e + f*x])/(a + b))]*Sqrt[a*((1 + Csc[e + f*x])/(a - b))]*EllipticF[ArcSin[Sqrt[a + b*Sin[e + f*x]]/Sqrt[d*Sin[e + f*x]]/Rt[(a + b)/d, 2], -(a + b)/(a - b)], x] /; FreeQ[{a, b, d, e, f}, x] && NeQ[a^2 - b^2, 0] && PosQ[(a + b)/d]`

rule 3411 `Int[Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)]]/(Sqrt[(g_)*sin[(e_) + (f_)*(x_)]]*((c_) + (d_)*sin[(e_) + (f_)*(x_)])), x_Symbol] := Simp[(-Sqrt[a + b*Sin[e + f*x]])*(Sqrt[d*(Sin[e + f*x]/(c + d*Sin[e + f*x]))]/(d*f*Sqrt[g*Sin[e + f*x]]*Sqrt[c^2*((a + b*Sin[e + f*x])/((a*c + b*d)*(c + d*Sin[e + f*x]))])))*EllipticE[ArcSin[c*(Cos[e + f*x]/(c + d*Sin[e + f*x])], (b*c - a*d)/(b*c + a*d)], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && EqQ[c^2 - d^2, 0]`

rule 3417

```
Int[1/(Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)]]*Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]])*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] :> Simp[b/(b*c - a*d) Int[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]), x], x] - Simp[d/(b*c - a*d) Int[Sqrt[a + b*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c + d*Sin[e + f*x])), x], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && (EqQ[a^2 - b^2, 0] || EqQ[c^2 - d^2, 0])
```

Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 2117 vs. $2(234) = 468$.

Time = 4.04 (sec) , antiderivative size = 2118, normalized size of antiderivative = 8.27

method	result	size
default	Expression too large to display	2118

input

```
int(1/(g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x,method=_RETURNVERBOSE)
```

output

```

1/2/c/f*(a+b*sin(f*x+e))^(1/2)*(((2*cos(f*x+e)-2)*sin(f*x+e)-2*(1+cos(f*x
+e))^2)*(-a^2+b^2)^(1/2)*(-(-a*cot(f*x+e)-(-a^2+b^2)^(1/2)+b+a*csc(f*x+e))
/(-a^2+b^2)^(1/2))^(1/2)*(-(csc(f*x+e)-cot(f*x+e))/(b+(-a^2+b^2)^(1/2))*a)
^(1/2)*EllipticE((-a*cot(f*x+e)+(-a^2+b^2)^(1/2)+b+a*csc(f*x+e))/(b+(-a^2
+b^2)^(1/2)))^(1/2),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1
/2))*((-a*cot(f*x+e)+(-a^2+b^2)^(1/2)+b+a*csc(f*x+e))/(b+(-a^2+b^2)^(1/2))
)^(1/2)*b+((2*cos(f*x+e)+2)*sin(f*x+e)+2*(1+cos(f*x+e))^2)*(-(-a*cot(f*x+e
)-(-a^2+b^2)^(1/2)+b+a*csc(f*x+e))/(-a^2+b^2)^(1/2))^(1/2)*(-(csc(f*x+e)-c
ot(f*x+e))/(b+(-a^2+b^2)^(1/2))*a)^(1/2)*EllipticE((-a*cot(f*x+e)+(-a^2+b
^2)^(1/2)+b+a*csc(f*x+e))/(b+(-a^2+b^2)^(1/2)))^(1/2),1/2*2^(1/2)*((b+(-a
^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2))*((-a*cot(f*x+e)+(-a^2+b^2)^(1/2)+b
+a*csc(f*x+e))/(b+(-a^2+b^2)^(1/2)))^(1/2)*a^2+((-2*cos(f*x+e)-2)*sin(f*x+e
)-2*(1+cos(f*x+e))^2)*(-(-a*cot(f*x+e)-(-a^2+b^2)^(1/2)+b+a*csc(f*x+e))/(-
a^2+b^2)^(1/2))^(1/2)*(-(csc(f*x+e)-cot(f*x+e))/(b+(-a^2+b^2)^(1/2))*a)^(1
/2)*EllipticE((-a*cot(f*x+e)+(-a^2+b^2)^(1/2)+b+a*csc(f*x+e))/(b+(-a^2+b
^2)^(1/2)))^(1/2),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2)
)*((-a*cot(f*x+e)+(-a^2+b^2)^(1/2)+b+a*csc(f*x+e))/(b+(-a^2+b^2)^(1/2)))^(
1/2)*b^2+((-1-cos(f*x+e))*sin(f*x+e)-(1+cos(f*x+e))^2)*(-a^2+b^2)^(1/2)*(-
(-a*cot(f*x+e)-(-a^2+b^2)^(1/2)+b+a*csc(f*x+e))/(-a^2+b^2)^(1/2))^(1/2)*(-
(csc(f*x+e)-cot(f*x+e))/(b+(-a^2+b^2)^(1/2))*a)^(1/2)*EllipticF((-a*co...

```

Fricas [F]

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)} (c + c \sin(e + fx))} dx$$

$$= \int \frac{1}{\sqrt{b \sin(fx + e) + a} (c \sin(fx + e) + c) \sqrt{g \sin(fx + e)}} dx$$

input

```

integrate(1/(g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x
, algorithm="fricas")

```

output

```

integral(-sqrt(b*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))/((a + b)*c*g*cos(f
*x + e)^2 - (a + b)*c*g + (b*c*g*cos(f*x + e)^2 - (a + b)*c*g)*sin(f*x + e
)), x)

```


Sympy [F]

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)} (c + c \sin(e + fx))} dx$$

$$= \frac{\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)} \sin(e + fx) + \sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}} dx}{c}$$

input `integrate(1/(g*sin(f*x+e))**(1/2)/(a+b*sin(f*x+e))**(1/2)/(c+c*sin(f*x+e)),x)`

output `Integral(1/(sqrt(g*sin(e + f*x))*sqrt(a + b*sin(e + f*x))*sin(e + f*x) + sqrt(g*sin(e + f*x))*sqrt(a + b*sin(e + f*x))), x)/c`

Maxima [F]

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)} (c + c \sin(e + fx))} dx$$

$$= \int \frac{1}{\sqrt{b \sin(fx + e) + a} (c \sin(fx + e) + c) \sqrt{g \sin(fx + e)}} dx$$

input `integrate(1/(g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x, algorithm="maxima")`

output `integrate(1/(sqrt(b*sin(f*x + e) + a)*(c*sin(f*x + e) + c)*sqrt(g*sin(f*x + e))), x)`

Giac [F]

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)} (c + c \sin(e + fx))} dx$$

$$= \int \frac{1}{\sqrt{b \sin(fx + e) + a} (c \sin(fx + e) + c) \sqrt{g \sin(fx + e)}} dx$$

input `integrate(1/(g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x, algorithm="giac")`

output `integrate(1/(sqrt(b*sin(f*x + e) + a)*(c*sin(f*x + e) + c)*sqrt(g*sin(f*x + e))), x)`

Mupad [F(-1)]

Timed out.

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)} (c + c \sin(e + fx))} dx$$

$$= \int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)} (c + c \sin(e + fx))} dx$$

input `int(1/((g*sin(e + f*x))^(1/2)*(a + b*sin(e + f*x))^(1/2)*(c + c*sin(e + f*x))),x)`

output `int(1/((g*sin(e + f*x))^(1/2)*(a + b*sin(e + f*x))^(1/2)*(c + c*sin(e + f*x))), x)`

Reduce [F]

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)} (c + c \sin(e + fx))} dx$$

$$= \frac{\sqrt{g} \left(\int \frac{\sqrt{\sin(fx+e)} \sqrt{\sin(fx+e)b+a}}{\sin(fx+e)^3 b + \sin(fx+e)^2 a + \sin(fx+e)^2 b + a \sin(fx+e)} dx \right)}{cg}$$

input `int(1/(g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2)/(c+c*sin(f*x+e)),x)`

output `(sqrt(g)*int((sqrt(sin(e + f*x))*sqrt(sin(e + f*x)*b + a))/(sin(e + f*x)**3*b + sin(e + f*x)**2*a + sin(e + f*x)**2*b + sin(e + f*x)*a),x))/(c*g)`

3.35 $\int \csc(e+fx) \sqrt{a+a \sin(e+fx)} \sqrt{c+d \sin(e+fx)} dx$

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Optimal result

Integrand size = 35, antiderivative size = 123

$$\int \csc(e+fx) \sqrt{a+a \sin(e+fx)} \sqrt{c+d \sin(e+fx)} dx$$

$$= -\frac{2\sqrt{a}\sqrt{d} \arctan\left(\frac{\sqrt{a}\sqrt{d} \cos(e+fx)}{\sqrt{a+a \sin(e+fx)} \sqrt{c+d \sin(e+fx)}}\right)}{f}$$

$$- \frac{2\sqrt{a}\sqrt{c} \operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{c} \cos(e+fx)}{\sqrt{a+a \sin(e+fx)} \sqrt{c+d \sin(e+fx)}}\right)}{f}$$

output

```
-2*a^(1/2)*d^(1/2)*arctan(a^(1/2)*d^(1/2)*cos(f*x+e)/(a+a*sin(f*x+e))^(1/2)
)/(c+d*sin(f*x+e))^(1/2))/f-2*a^(1/2)*c^(1/2)*arctanh(a^(1/2)*c^(1/2)*cos(
f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2))/f
```

Mathematica [C] (warning: unable to verify)

Result contains complex when optimal does not.

Time = 5.13 (sec) , antiderivative size = 567, normalized size of antiderivative = 4.61

$$\int \csc(e + fx) \sqrt{a + a \sin(e + fx)} \sqrt{c + d \sin(e + fx)} dx =$$

$$\left(\sqrt{c} \log \left(\frac{\left(\frac{1}{2} + \frac{i}{2}\right) e^{-\frac{ie}{2}} \left(-\sqrt{2c}(-1 + e^{i(e+fx)}) - i\sqrt{2d}(1 + e^{i(e+fx)}) + 2i\sqrt{c} \sqrt{2ce^{i(e+fx)} - id(-1 + e^{2i(e+fx)})} \right) f}{c^{3/2}(1 + e^{i(e+fx)})} \right) + \sqrt{c} \log \left(\frac{\left(\frac{1}{2} + \frac{i}{2}\right) e^{-\frac{ie}{2}} \left(-\sqrt{2c}(-1 + e^{i(e+fx)}) - i\sqrt{2d}(1 + e^{i(e+fx)}) + 2i\sqrt{c} \sqrt{2ce^{i(e+fx)} - id(-1 + e^{2i(e+fx)})} \right) f}{c^{3/2}(1 + e^{i(e+fx)})} \right) \right)$$

input

```
Integrate[Csc[e + f*x]*Sqrt[a + a*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]],x]
```

output

```
-(((Sqrt[c]*Log[((1/2 + I/2)*(-Sqrt[2]*c*(-1 + E^(I*(e + f*x)))) - I*Sqrt[2]*d*(1 + E^(I*(e + f*x))) + (2*I)*Sqrt[c]*Sqrt[2*c*E^(I*(e + f*x)) - I*d*(-1 + E^((2*I)*(e + f*x)))]*f)/(c^(3/2)*E^((I/2)*e)*(1 + E^(I*(e + f*x))))] + Sqrt[c]*Log[((1/2 + I/2)*((-I)*Sqrt[2]*d*(-1 + E^(I*(e + f*x))) + Sqrt[2]*c*(1 + E^(I*(e + f*x)))) + 2*Sqrt[c]*Sqrt[2*c*E^(I*(e + f*x)) - I*d*(-1 + E^((2*I)*(e + f*x)))]*f)/(c^(3/2)*E^((I/2)*e)*(-1 + E^(I*(e + f*x))))] - I*Sqrt[d]*(Log[(2*((-1)^(3/4)*d + (-1)^(1/4)*c*E^(I*(e + f*x))) + I*Sqrt[d]*Sqrt[2*c*E^(I*(e + f*x)) - I*d*(-1 + E^((2*I)*(e + f*x)))]*f)/(d^(3/2)*E^((I/2)*(e + 2*f*x)))] - Log[((1 + I)*Sqrt[2]*(c - I*d*Cos[e + f*x] + d*Sin[e + f*x] + (1 - I)*Sqrt[d]*Sqrt[(Cos[e + f*x] + I*Sin[e + f*x])*(c + d*Sin[e + f*x]])]/Sqrt[d]))*(Cos[(e + f*x)/2] + I*Sin[(e + f*x)/2])*Sqrt[a*(1 + Sin[e + f*x])*Sqrt[c + d*Sin[e + f*x]])/(f*(Cos[(e + f*x)/2] + Sin[(e + f*x)/2])*Sqrt[(Cos[e + f*x] + I*Sin[e + f*x])*(c + d*Sin[e + f*x])]))
```

Rubi [A] (verified)

Time = 0.89 (sec) , antiderivative size = 123, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}} = 0.200$, Rules used = {3042, 3428, 3042, 3254, 218, 3422, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
& \int \csc(e+fx) \sqrt{a \sin(e+fx) + a} \sqrt{c+d \sin(e+fx)} dx \\
& \quad \downarrow \text{3042} \\
& \int \frac{\sqrt{a \sin(e+fx) + a} \sqrt{c+d \sin(e+fx)}}{\sin(e+fx)} dx \\
& \quad \downarrow \text{3428} \\
& d \int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{c+d \sin(e+fx)}} dx + c \int \frac{\csc(e+fx) \sqrt{\sin(e+fx)a+a}}{\sqrt{c+d \sin(e+fx)}} dx \\
& \quad \downarrow \text{3042} \\
& d \int \frac{\sqrt{\sin(e+fx)a+a}}{\sqrt{c+d \sin(e+fx)}} dx + c \int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx) \sqrt{c+d \sin(e+fx)}} dx \\
& \quad \downarrow \text{3254} \\
& \frac{c \int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx) \sqrt{c+d \sin(e+fx)}} dx - 2ad \int \frac{1}{\frac{a^2 d \cos^2(e+fx)}{(\sin(e+fx)a+a)(c+d \sin(e+fx))} + a} d \frac{a \cos(e+fx)}{\sqrt{\sin(e+fx)a+a} \sqrt{c+d \sin(e+fx)}}}{f} \\
& \quad \downarrow \text{218} \\
& c \int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx) \sqrt{c+d \sin(e+fx)}} dx - \frac{2\sqrt{a}\sqrt{d} \arctan\left(\frac{\sqrt{a}\sqrt{d} \cos(e+fx)}{\sqrt{a \sin(e+fx)+a} \sqrt{c+d \sin(e+fx)}}\right)}{f} \\
& \quad \downarrow \text{3422} \\
& \frac{2ac \int \frac{1}{1 - \frac{ac \cos^2(e+fx)}{(\sin(e+fx)a+a)(c+d \sin(e+fx))}} d \frac{\cos(e+fx)}{\sqrt{\sin(e+fx)a+a} \sqrt{c+d \sin(e+fx)}}}{f} \\
& \quad \frac{2\sqrt{a}\sqrt{d} \arctan\left(\frac{\sqrt{a}\sqrt{d} \cos(e+fx)}{\sqrt{a \sin(e+fx)+a} \sqrt{c+d \sin(e+fx)}}\right)}{f} \\
& \quad \downarrow \text{219} \\
& \frac{2\sqrt{a}\sqrt{d} \arctan\left(\frac{\sqrt{a}\sqrt{d} \cos(e+fx)}{\sqrt{a \sin(e+fx)+a} \sqrt{c+d \sin(e+fx)}}\right)}{f} \\
& \quad \frac{2\sqrt{a}\sqrt{c} \operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{c} \cos(e+fx)}{\sqrt{a \sin(e+fx)+a} \sqrt{c+d \sin(e+fx)}}\right)}{f}
\end{aligned}$$

input `Int[Csc[e + f*x]*Sqrt[a + a*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]],x]`

output `(-2*Sqrt[a]*Sqrt[d]*ArcTan[(Sqrt[a]*Sqrt[d]*Cos[e + f*x])/(Sqrt[a + a*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]])]/f - (2*Sqrt[a]*Sqrt[c]*ArcTanh[(Sqrt[a]*Sqrt[c]*Cos[e + f*x])/(Sqrt[a + a*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]])])/f`

Defintions of rubi rules used

rule 218 `Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(Rt[a/b, 2]/a)*ArcTan[x/Rt[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]`

rule 219 `Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (GtQ[a, 0] || LtQ[b, 0])`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3254 `Int[Sqrt[(a_) + (b_.)*sin[(e_) + (f_.)*(x_)]]/Sqrt[(c_) + (d_.)*sin[(e_) + (f_.)*(x_)]], x_Symbol] := Simp[-2*(b/f) Subst[Int[1/(b + d*x^2), x], x, b*(Cos[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]])], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]`

rule 3422 `Int[Sqrt[(a_) + (b_.)*sin[(e_) + (f_.)*(x_)]]/(sin[(e_) + (f_.)*(x_)])*Sqrt[(c_) + (d_.)*sin[(e_) + (f_.)*(x_)]], x_Symbol] := Simp[-2*(a/f) Subst[Int[1/(1 - a*c*x^2), x], x, Cos[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]])], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0] && NeQ[b*c + a*d, 0]`

rule 3428

```
Int[(Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)])*Sqrt[(c_) + (d_)*sin[(e_)
+ (f_)*(x_)]])/sin[(e_) + (f_)*(x_)], x_Symbol] := Simp[d Int[Sqrt[a
+ b*Sin[e + f*x]]/Sqrt[c + d*Sin[e + f*x]], x], x] + Simp[c Int[Sqrt[a +
b*Sin[e + f*x]]/(Sin[e + f*x]*Sqrt[c + d*Sin[e + f*x]]), x], x] /; FreeQ[{a
, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && (NeQ[a^2 - b^2, 0] || NeQ[c^2
- d^2, 0])
```

Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 4966 vs. $2(99) = 198$.

Time = 10.51 (sec) , antiderivative size = 4967, normalized size of antiderivative = 40.38

output too large to display

input

```
int(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)*(c+d*sin(f*x+e))^(1/2),x)
```

output

```
-1/f/(-(d^2/c^2)^(1/2)*c)^(1/2)/(d^2/c^2)^(1/2)/(c-d)^2/c*(arctan(((d^2/c^
2)^(1/2)*c^2-d^2)*c*((d^2/c^2)^(1/2)-1)/(((d^2/c^2)^(1/2)*c^4+6*(d^2/c^2)^(
1/2)*d^2*c^2+d^4*(d^2/c^2)^(1/2)-4*c^2*d^2-4*d^4)*c)^(1/2)*((d^2/c^2)^(1/
2)*c*sin(1/2*f*x+1/2*e)-d*cos(1/2*f*x+1/2*e))/((d^2/c^2)^(1/2)*c*sin(1/2*f
*x+1/2*e)+d*cos(1/2*f*x+1/2*e))/(d*(c+2*d*sin(1/2*f*x+1/2*e)*cos(1/2*f*x+1
/2*e))/(2*(d^2/c^2)^(1/2)*c*sin(1/2*f*x+1/2*e)*cos(1/2*f*x+1/2*e)+d))^(1/2
))*cos(1/2*f*x+1/2*e)*(-(d^2/c^2)^(1/2)*c)^(1/2)*(d*(c+2*d*sin(1/2*f*x+1/2
*e)*cos(1/2*f*x+1/2*e))/(2*(d^2/c^2)^(1/2)*c*sin(1/2*f*x+1/2*e)*cos(1/2*f*
x+1/2*e)+d))^(1/2)*(((d^2/c^2)^(1/2)*c^4+6*(d^2/c^2)^(1/2)*d^2*c^2+d^4*(d^
2/c^2)^(1/2)-4*c^2*d^2-4*d^4)*c)^(1/2)*(d^2/c^2)^(1/2)*c+2*(-(d^2/c^2)^(1/
2)*c)^(1/2)*c^(5/2)*(d^2/c^2)^(1/2)*ln(-(c*cot(1/2*f*x+1/2*e)*cos(1/2*f*x+
1/2*e)-2*c*cot(1/2*f*x+1/2*e)+2*c^(1/2)*((c+2*d*sin(1/2*f*x+1/2*e)*cos(1/2
*f*x+1/2*e)))/(cos(1/2*f*x+1/2*e)+1)^2)^(1/2)*sin(1/2*f*x+1/2*e)+c*csc(1/2*
f*x+1/2*e)-sin(1/2*f*x+1/2*e)*c+2*d*cos(1/2*f*x+1/2*e)-2*d)/c^(1/2)/(-1+co
s(1/2*f*x+1/2*e))*c*csc(1/2*f*x+1/2*e)^2+2*d*cot(1/2*f*x+1/2*e)^(1/2)*d
*sin(1/2*f*x+1/2*e)+2*(-(d^2/c^2)^(1/2)*c)^(1/2)*c^(5/2)*(d^2/c^2)^(1/2)*l
n(-c^(1/2)*((c+2*d*sin(1/2*f*x+1/2*e)*cos(1/2*f*x+1/2*e))/(cos(1/2*f*x+1/2
*e)+1)^2)^(1/2)-c^(1/2)*((c+2*d*sin(1/2*f*x+1/2*e)*cos(1/2*f*x+1/2*e))/(co
s(1/2*f*x+1/2*e)+1)^2)^(1/2)*sec(1/2*f*x+1/2*e)-c*tan(1/2*f*x+1/2*e)-d)*(c
*csc(1/2*f*x+1/2*e)^2+2*d*cot(1/2*f*x+1/2*e)^(1/2)*d*sin(1/2*f*x+1/2*e)...
```


Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 333 vs. $2(99) = 198$.

Time = 0.78 (sec) , antiderivative size = 3539, normalized size of antiderivative = 28.77

$$\int \csc(e + fx) \sqrt{a + a \sin(e + fx)} \sqrt{c + d \sin(e + fx)} dx = \text{Too large to display}$$

input `integrate(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)*(c+d*sin(f*x+e))^(1/2),x, algorithm="fricas")`

output Too large to include

Sympy [F]

$$\begin{aligned} & \int \csc(e + fx) \sqrt{a + a \sin(e + fx)} \sqrt{c + d \sin(e + fx)} dx \\ &= \int \sqrt{a(\sin(e + fx) + 1)} \sqrt{c + d \sin(e + fx)} \csc(e + fx) dx \end{aligned}$$

input `integrate(csc(f*x+e)*(a+a*sin(f*x+e))**(1/2)*(c+d*sin(f*x+e))**(1/2),x)`

output `Integral(sqrt(a*(sin(e + f*x) + 1))*sqrt(c + d*sin(e + f*x))*csc(e + f*x), x)`

Maxima [F]

$$\begin{aligned} & \int \csc(e + fx) \sqrt{a + a \sin(e + fx)} \sqrt{c + d \sin(e + fx)} dx \\ &= \int \sqrt{a \sin(fx + e) + a} \sqrt{d \sin(fx + e) + c} \csc(fx + e) dx \end{aligned}$$

input `integrate(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)*(c+d*sin(f*x+e))^(1/2),x, algorithm="maxima")`

output

```
integrate(sqrt(a*sin(f*x + e) + a)*sqrt(d*sin(f*x + e) + c)*csc(f*x + e),
x)
```

Giac [F(-1)]

Timed out.

$$\int \csc(e + fx) \sqrt{a + a \sin(e + fx)} \sqrt{c + d \sin(e + fx)} dx = \text{Timed out}$$

input

```
integrate(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)*(c+d*sin(f*x+e))^(1/2),x, algo
rithm="giac")
```

output

Timed out

Mupad [F(-1)]

Timed out.

$$\begin{aligned} & \int \csc(e + fx) \sqrt{a + a \sin(e + fx)} \sqrt{c + d \sin(e + fx)} dx \\ &= \int \frac{\sqrt{a + a \sin(e + fx)} \sqrt{c + d \sin(e + fx)}}{\sin(e + fx)} dx \end{aligned}$$

input

```
int(((a + a*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))^(1/2))/sin(e + f*x),x
)
```

output

```
int(((a + a*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))^(1/2))/sin(e + f*x),
x)
```

Reduce [F]

$$\begin{aligned} & \int \csc(e + fx) \sqrt{a + a \sin(e + fx)} \sqrt{c + d \sin(e + fx)} dx \\ &= \sqrt{a} \left(\int \sqrt{\sin(fx + e) d + c} \sqrt{\sin(fx + e) + 1} \csc(fx + e) dx \right) \end{aligned}$$

input `int(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)*(c+d*sin(f*x+e))^(1/2),x)`

output `sqrt(a)*int(sqrt(sin(e + f*x)*d + c)*sqrt(sin(e + f*x) + 1)*csc(e + f*x),x)`
`)`

3.36
$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}} dx$$

Optimal result	315
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Maxima [F]	320
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Mupad [F(-1)]	320
Reduce [F]	321

Optimal result

Integrand size = 35, antiderivative size = 61

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}} dx = -\frac{2\sqrt{a}\operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{c}\cos(e+fx)}{\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\right)}{\sqrt{c}f}$$

output

```
-2*a^(1/2)*arctanh(a^(1/2)*c^(1/2)*cos(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c+d*
sin(f*x+e))^(1/2))/c^(1/2)/f
```

Mathematica [C] (warning: unable to verify)

Result contains complex when optimal does not.

Time = 3.64 (sec) , antiderivative size = 367, normalized size of antiderivative = 6.02

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}} dx = \frac{\log\left(\frac{(1+i)e^{\frac{ie}{2}}\left(\sqrt{2c}(-1+e^{i(e+fx)})+i\sqrt{2d}(1+e^{i(e+fx)})-2i\sqrt{c}\sqrt{2ce^{i(e+fx)}-id(-1+e^{2i(e+fx)})}\right)f}{\sqrt{c}(1+e^{i(e+fx)})}\right)}{\dots} + \log\left(\frac{(1+i)e^{\frac{ie}{2}}(-i\sqrt{2c}}{\dots}\right)$$

input `Integrate[(Csc[e + f*x]*Sqrt[a + a*Sin[e + f*x]])/Sqrt[c + d*Sin[e + f*x]],x]`

output
$$-\left(\left(\text{Log}\left[\left(-1 - I\right)E^{\left(\frac{I}{2}\right)e}\left(\text{Sqrt}[2]*c\left(-1 + E^{I\left(e + f*x\right)}\right)\right) + I*\text{Sqrt}[2]*d\left(1 + E^{I\left(e + f*x\right)}\right)\right) - \left(2*I\right)*\text{Sqrt}[c]*\text{Sqrt}[2*c*E^{I\left(e + f*x\right)} - I*d\left(-1 + E^{\left(2*I\right)\left(e + f*x\right)}\right)]*f\right)/\left(\text{Sqrt}[c]*\left(1 + E^{I\left(e + f*x\right)}\right)\right)\right) + \text{Log}\left[\left(\left(1 + I\right)E^{\left(\frac{I}{2}\right)e}\left(-I\right)*\text{Sqrt}[2]*d\left(-1 + E^{I\left(e + f*x\right)}\right) + \text{Sqrt}[2]*c\left(1 + E^{I\left(e + f*x\right)}\right) + 2*\text{Sqrt}[c]*\text{Sqrt}[2*c*E^{I\left(e + f*x\right)} - I*d\left(-1 + E^{\left(2*I\right)\left(e + f*x\right)}\right)]*f\right)/\left(\text{Sqrt}[c]*\left(-1 + E^{I\left(e + f*x\right)}\right)\right)\right]*\left(\text{Cos}\left[\left(e + f*x\right)/2\right] - I*\text{Sin}\left[\left(e + f*x\right)/2\right]*\text{Sqrt}[a*\left(1 + \text{Sin}\left[e + f*x\right]\right)]*\text{Sqrt}\left[\left(\text{Cos}\left[e + f*x\right] + I*\text{Sin}\left[e + f*x\right]\right)*\left(c + d*\text{Sin}\left[e + f*x\right]\right)\right]\right)/\left(\text{Sqrt}[c]*f*\left(\text{Cos}\left[\left(e + f*x\right)/2\right] + \text{Sin}\left[\left(e + f*x\right)/2\right]*\text{Sqrt}[c + d*\text{Sin}\left[e + f*x\right]]\right)\right)\right)$$

Rubi [A] (verified)

Time = 0.37 (sec) , antiderivative size = 61, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.086$, Rules used = {3042, 3422, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e + fx)\sqrt{a \sin(e + fx) + a}}{\sqrt{c + d \sin(e + fx)}} dx$$

↓ 3042

$$\int \frac{\sqrt{a \sin(e + fx) + a}}{\sin(e + fx)\sqrt{c + d \sin(e + fx)}} dx$$

↓ 3422

$$\frac{2a \int \frac{1}{1 - \frac{ac \cos^2(e + fx)}{(\sin(e + fx)a + a)(c + d \sin(e + fx))}} d \frac{\cos(e + fx)}{\sqrt{\sin(e + fx)a + a}\sqrt{c + d \sin(e + fx)}}}{f}$$

↓ 219

$$\frac{2\sqrt{a} \operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{c} \cos(e + fx)}{\sqrt{a \sin(e + fx) + a}\sqrt{c + d \sin(e + fx)}}\right)}{\sqrt{c}f}$$

input `Int[(Csc[e + f*x]*Sqrt[a + a*Sin[e + f*x]])/Sqrt[c + d*Sin[e + f*x]],x]`

output `(-2*Sqrt[a]*ArcTanh[(Sqrt[a]*Sqrt[c]*Cos[e + f*x])/(Sqrt[a + a*Sin[e + f*x]])*Sqrt[c + d*Sin[e + f*x]])/(Sqrt[c]*f)`

Defintions of rubi rules used

rule 219 `Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (GtQ[a, 0] || LtQ[b, 0])`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3422 `Int[Sqrt[(a_) + (b_.)*sin[(e_) + (f_.)*(x_)]]/(sin[(e_) + (f_.)*(x_)])*Sqrt[(c_) + (d_.)*sin[(e_) + (f_.)*(x_)]], x_Symbol] := Simp[-2*(a/f) Subst[Int[1/(1 - a*c*x^2), x], x, Cos[e + f*x]/(Sqrt[a + b*Sin[e + f*x])*Sqrt[c + d*Sin[e + f*x]]], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0] && NeQ[b*c + a*d, 0]`

Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 394 vs. 2(49) = 98.

Time = 7.11 (sec) , antiderivative size = 395, normalized size of antiderivative = 6.48

method	result
default	$2\sqrt{\left(2\sin\left(\frac{fx}{2} + \frac{e}{2}\right)\cos\left(\frac{fx}{2} + \frac{e}{2}\right) + 1\right)a} \sqrt{\frac{c + 2d\sin\left(\frac{fx}{2} + \frac{e}{2}\right)\cos\left(\frac{fx}{2} + \frac{e}{2}\right)}{\left(\cos\left(\frac{fx}{2} + \frac{e}{2}\right) + 1\right)^2}} \left(\ln \left(\frac{c\left(1 - \cos\left(\frac{fx}{2} + \frac{e}{2}\right)\right)^2 \csc\left(\frac{fx}{2} + \frac{e}{2}\right) + 2\sqrt{c} \sqrt{\frac{c + 2d\sin\left(\frac{fx}{2} + \frac{e}{2}\right)\cos\left(\frac{fx}{2} + \frac{e}{2}\right)}{\left(\cos\left(\frac{fx}{2} + \frac{e}{2}\right) + 1\right)}}}{\sqrt{c}\left(1 - \cos\left(\frac{fx}{2} + \frac{e}{2}\right)\right)} \right) \right)$

input `int(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x,method=_RETURNVERBOSE)`

output `-2/f/c^(1/2)*((2*sin(1/2*f*x+1/2*e)*cos(1/2*f*x+1/2*e)+1)*a)^(1/2)/((1-cos(1/2*f*x+1/2*e))^2*csc(1/2*f*x+1/2*e)^2-2*csc(1/2*f*x+1/2*e)+2*cot(1/2*f*x+1/2*e)-1)*((c+2*d*sin(1/2*f*x+1/2*e)*cos(1/2*f*x+1/2*e))/(cos(1/2*f*x+1/2*e)+1)^2)^(1/2)*(ln(1/c^(1/2)/(1-cos(1/2*f*x+1/2*e)))*(c*(1-cos(1/2*f*x+1/2*e))^2*csc(1/2*f*x+1/2*e)+2*c^(1/2)*((c+2*d*sin(1/2*f*x+1/2*e)*cos(1/2*f*x+1/2*e))/(cos(1/2*f*x+1/2*e)+1)^2)^(1/2)*sin(1/2*f*x+1/2*e)-2*d*(1-cos(1/2*f*x+1/2*e))-sin(1/2*f*x+1/2*e)*c))+ln(-4*c^(1/2)*((c+2*d*sin(1/2*f*x+1/2*e)*cos(1/2*f*x+1/2*e))/(cos(1/2*f*x+1/2*e)+1)^2)^(1/2)-4*c^(1/2)*((c+2*d*sin(1/2*f*x+1/2*e)*cos(1/2*f*x+1/2*e))/(cos(1/2*f*x+1/2*e)+1)^2)^(1/2)*sec(1/2*f*x+1/2*e)-4*c*tan(1/2*f*x+1/2*e)-4*d)/(c+2*d*sin(1/2*f*x+1/2*e)*cos(1/2*f*x+1/2*e))^(1/2)`

Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 171 vs. $2(49) = 98$.

Time = 0.27 (sec) , antiderivative size = 1044, normalized size of antiderivative = 17.11

$$\int \frac{\csc(e + fx) \sqrt{a + a \sin(e + fx)}}{\sqrt{c + d \sin(e + fx)}} dx = \text{Too large to display}$$

input `integrate(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x, algorithm="fricas")`

output

```
[1/4*sqrt(a/c)*log(((a*c^4 - 28*a*c^3*d + 70*a*c^2*d^2 - 28*a*c*d^3 + a*d^4)*cos(f*x + e)^5 + a*c^4 + 4*a*c^3*d + 6*a*c^2*d^2 + 4*a*c*d^3 + a*d^4 - (31*a*c^4 - 196*a*c^3*d + 154*a*c^2*d^2 - 4*a*c*d^3 - a*d^4)*cos(f*x + e)^4 - 2*(81*a*c^4 - 252*a*c^3*d + 150*a*c^2*d^2 - 28*a*c*d^3 + a*d^4)*cos(f*x + e)^3 + 2*(79*a*c^4 - 100*a*c^3*d + 74*a*c^2*d^2 - 4*a*c*d^3 - a*d^4)*cos(f*x + e)^2 - 8*((c^4 - 7*c^3*d + 7*c^2*d^2 - c*d^3)*cos(f*x + e)^4 + 51*c^4 - 59*c^3*d + 17*c^2*d^2 - c*d^3 - 2*(5*c^4 - 14*c^3*d + 5*c^2*d^2)*cos(f*x + e)^3 - 2*(18*c^4 - 33*c^3*d + 12*c^2*d^2 - c*d^3)*cos(f*x + e)^2 + 2*(13*c^4 - 14*c^3*d + 5*c^2*d^2)*cos(f*x + e) - (51*c^4 - 59*c^3*d + 17*c^2*d^2 - c*d^3 - (c^4 - 7*c^3*d + 7*c^2*d^2 - c*d^3)*cos(f*x + e)^3 - (11*c^4 - 35*c^3*d + 17*c^2*d^2 - c*d^3)*cos(f*x + e)^2 + (25*c^4 - 31*c^3*d + 7*c^2*d^2 - c*d^3)*cos(f*x + e))*sin(f*x + e))*sqrt(a*sin(f*x + e) + a)*sqrt(d*sin(f*x + e) + c)*sqrt(a/c) + (289*a*c^4 - 476*a*c^3*d + 230*a*c^2*d^2 - 28*a*c*d^3 + a*d^4)*cos(f*x + e) + (a*c^4 + 4*a*c^3*d + 6*a*c^2*d^2 + 4*a*c*d^3 + a*d^4 + (a*c^4 - 28*a*c^3*d + 70*a*c^2*d^2 - 28*a*c*d^3 + a*d^4)*cos(f*x + e)^4 + 32*(a*c^4 - 7*a*c^3*d + 7*a*c^2*d^2 - a*c*d^3)*cos(f*x + e)^3 - 2*(65*a*c^4 - 140*a*c^3*d + 38*a*c^2*d^2 - 12*a*c*d^3 + a*d^4)*cos(f*x + e)^2 - 32*(9*a*c^4 - 15*a*c^3*d + 7*a*c^2*d^2 - a*c*d^3)*cos(f*x + e))*sin(f*x + e))/(cos(f*x + e)^5 + cos(f*x + e)^4 - 2*cos(f*x + e)^3 - 2*cos(f*x + e)^2 + (cos(f*x + e)^4 - 2*cos(f*x + e)^2 + 1)*sin(f*x + ...
```

Sympy [F]

$$\int \frac{\csc(e + fx) \sqrt{a + a \sin(e + fx)}}{\sqrt{c + d \sin(e + fx)}} dx = \int \frac{\sqrt{a(\sin(e + fx) + 1)} \csc(e + fx)}{\sqrt{c + d \sin(e + fx)}} dx$$

input

```
integrate(csc(f*x+e)*(a+a*sin(f*x+e))**(1/2)/(c+d*sin(f*x+e))**(1/2),x)
```

output

```
Integral(sqrt(a*(sin(e + f*x) + 1))*csc(e + f*x)/sqrt(c + d*sin(e + f*x)), x)
```


Maxima [F]

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}} dx = \int \frac{\sqrt{a\sin(fx+e)+a}\csc(fx+e)}{\sqrt{d\sin(fx+e)+c}} dx$$

input `integrate(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x, algorith="maxima")`

output `integrate(sqrt(a*sin(f*x + e) + a)*csc(f*x + e)/sqrt(d*sin(f*x + e) + c), x)`

Giac [F]

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}} dx = \int \frac{\sqrt{a\sin(fx+e)+a}\csc(fx+e)}{\sqrt{d\sin(fx+e)+c}} dx$$

input `integrate(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x, algorith="giac")`

output `integrate(sqrt(a*sin(f*x + e) + a)*csc(f*x + e)/sqrt(d*sin(f*x + e) + c), x)`

Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)\sqrt{a+a\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}} dx = \int \frac{\sqrt{a+a\sin(e+fx)}}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}} dx$$

input `int((a + a*sin(e + f*x))^(1/2)/(sin(e + f*x)*(c + d*sin(e + f*x))^(1/2)),x)`

output `int((a + a*sin(e + f*x))^(1/2)/(sin(e + f*x)*(c + d*sin(e + f*x))^(1/2)), x)`

Reduce [F]

$$\int \frac{\csc(e + fx) \sqrt{a + a \sin(e + fx)}}{\sqrt{c + d \sin(e + fx)}} dx$$

$$= \sqrt{a} \left(\int \frac{\sqrt{\sin(fx + e)d + c} \sqrt{\sin(fx + e) + 1} \csc(fx + e)}{\sin(fx + e)d + c} dx \right)$$

input `int(csc(f*x+e)*(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x)`

output `sqrt(a)*int((sqrt(sin(e + f*x)*d + c)*sqrt(sin(e + f*x) + 1)*csc(e + f*x))/(sin(e + f*x)*d + c),x)`

3.37 $\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}} dx$

Optimal result	322
Mathematica [B] (warning: unable to verify)	323
Rubi [A] (verified)	324
Maple [B] (warning: unable to verify)	326
Fricas [B] (verification not implemented)	327
Sympy [F]	328
Maxima [F]	329
Giac [F(-1)]	329
Mupad [F(-1)]	329
Reduce [F]	330

Optimal result

Integrand size = 35, antiderivative size = 140

$$\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}} dx$$

$$= -\frac{2\sqrt{c}\operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{c}\cos(e+fx)}{\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\right)}{\sqrt{a}f}$$

$$+ \frac{\sqrt{2}\sqrt{c-d}\operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{c-d}\cos(e+fx)}{\sqrt{2}\sqrt{a+a\sin(e+fx)}\sqrt{c+d\sin(e+fx)}}\right)}{\sqrt{a}f}$$

output

```
-2*c^(1/2)*arctanh(a^(1/2)*c^(1/2)*cos(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c+d*
sin(f*x+e))^(1/2))/a^(1/2)/f+2^(1/2)*(c-d)^(1/2)*arctanh(1/2*a^(1/2)*(c-d)
^(1/2)*cos(f*x+e)*2^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2))/a
^(1/2)/f
```

Mathematica [B] (warning: unable to verify)

Leaf count is larger than twice the leaf count of optimal. 885 vs. $2(140) = 280$.

Time = 16.35 (sec) , antiderivative size = 885, normalized size of antiderivative = 6.32

$$\int \frac{\csc(e + fx)\sqrt{c + d\sin(e + fx)}}{\sqrt{a + a\sin(e + fx)}} dx = \text{Too large to display}$$

input `Integrate[(Csc[e + f*x]*Sqrt[c + d*Sin[e + f*x]])/Sqrt[a + a*Sin[e + f*x]],x]`

output `(Csc[e + f*x]*(Sqrt[c]*Log[Tan[(e + f*x)/2]] - Sqrt[2]*Sqrt[c - d]*Log[1 + Tan[(e + f*x)/2]] + Sqrt[c]*Log[d + Sqrt[2]*Sqrt[c]*Sqrt[(1 + Cos[e + f*x])^(-1)]*Sqrt[c + d*Sin[e + f*x]] + c*Tan[(e + f*x)/2]] - Sqrt[c]*Log[c + Sqrt[2]*Sqrt[c]*Sqrt[(1 + Cos[e + f*x])^(-1)]*Sqrt[c + d*Sin[e + f*x]] + d*Tan[(e + f*x)/2]] + Sqrt[2]*Sqrt[c - d]*Log[c - d + 2*Sqrt[c - d]*Sqrt[(1 + Cos[e + f*x])^(-1)]*Sqrt[c + d*Sin[e + f*x]] + (-c + d)*Tan[(e + f*x)/2]])*Sqrt[c + d*Sin[e + f*x]])/(f*Sqrt[a*(1 + Sin[e + f*x])]*((Sqrt[c]*Csc[(e + f*x)/2]*Sec[(e + f*x)/2])/2 - (Sqrt[c - d]*Sec[(e + f*x)/2]^2)/(Sqrt[2]*(1 + Tan[(e + f*x)/2])) + (Sqrt[c]*((c*Sec[(e + f*x)/2]^2)/2 + (Sqrt[c]*d*Cos[e + f*x]*Sqrt[(1 + Cos[e + f*x])^(-1)])/(Sqrt[2]*Sqrt[c + d*Sin[e + f*x]])) + (Sqrt[c]*((1 + Cos[e + f*x])^(-1))^3/2)*Sin[e + f*x]*Sqrt[c + d*Sin[e + f*x]])/Sqrt[2]))/(d + Sqrt[2]*Sqrt[c]*Sqrt[(1 + Cos[e + f*x])^(-1)]*Sqrt[c + d*Sin[e + f*x]] + c*Tan[(e + f*x)/2]) - (Sqrt[c]*((d*Sec[(e + f*x)/2]^2)/2 + (Sqrt[c]*d*Cos[e + f*x]*Sqrt[(1 + Cos[e + f*x])^(-1)])/(Sqrt[2]*Sqrt[c + d*Sin[e + f*x]])) + (Sqrt[c]*((1 + Cos[e + f*x])^(-1))^3/2)*Sin[e + f*x]*Sqrt[c + d*Sin[e + f*x]])/Sqrt[2]))/(c + Sqrt[2]*Sqrt[c]*Sqrt[(1 + Cos[e + f*x])^(-1)]*Sqrt[c + d*Sin[e + f*x]] + d*Tan[(e + f*x)/2]) + (Sqrt[2]*Sqrt[c - d]*((-c + d)*Sec[(e + f*x)/2]^2)/2 + (Sqrt[c - d]*d*Cos[e + f*x]*Sqrt[(1 + Cos[e + f*x])^(-1)])/Sqrt[c + d*Sin[e + f*x]] + Sqrt[c - d]*((1 + Cos[e + f*x])^(-1))^3/2)*Sin[e + f*x]*Sqrt[c + d*Sin[e + ...`

Rubi [A] (verified)

Time = 0.91 (sec) , antiderivative size = 140, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}} = 0.200$, Rules used = {3042, 3423, 3042, 3261, 221, 3422, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{\sqrt{a\sin(e+fx)+a}} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{\sqrt{c+d\sin(e+fx)}}{\sin(e+fx)\sqrt{a\sin(e+fx)+a}} dx \\
 & \quad \downarrow \text{3423} \\
 & \frac{c \int \frac{\csc(e+fx)\sqrt{\sin(e+fx)a+a}}{\sqrt{c+d\sin(e+fx)}} dx}{a} - (c-d) \int \frac{1}{\sqrt{\sin(e+fx)a+a}\sqrt{c+d\sin(e+fx)}} dx \\
 & \quad \downarrow \text{3042} \\
 & \frac{c \int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}} dx}{a} - (c-d) \int \frac{1}{\sqrt{\sin(e+fx)a+a}\sqrt{c+d\sin(e+fx)}} dx \\
 & \quad \downarrow \text{3261} \\
 & \frac{2a(c-d) \int \frac{1}{2a^2 - \frac{a^3(c-d)\cos^2(e+fx)}{(\sin(e+fx)a+a)(c+d\sin(e+fx))}} d \frac{a \cos(e+fx)}{\sqrt{\sin(e+fx)a+a}\sqrt{c+d\sin(e+fx)}}}{f} + \\
 & \quad \frac{c \int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}} dx}{a} \\
 & \quad \downarrow \text{221} \\
 & \frac{c \int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}} dx}{a} + \frac{\sqrt{2}\sqrt{c-d} \operatorname{darctanh}\left(\frac{\sqrt{a}\sqrt{c-d}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}\sqrt{c+d\sin(e+fx)}}\right)}{\sqrt{a}f} \\
 & \quad \downarrow \text{3422}
 \end{aligned}$$

$$\frac{\sqrt{2}\sqrt{c-d}\operatorname{darctanh}\left(\frac{\sqrt{a}\sqrt{c-d}\cos(e+fx)}{\sqrt{2}\sqrt{a}\sin(e+fx)+a\sqrt{c+d}\sin(e+fx)}\right)}{\sqrt{a}f} - \frac{2c \int \frac{1}{1-\frac{ac\cos^2(e+fx)}{(\sin(e+fx)a+a)(c+d)\sin(e+fx)}} d \frac{\cos(e+fx)}{\sqrt{\sin(e+fx)a+a\sqrt{c+d}\sin(e+fx)}}}{f}$$

↓ 219

$$\frac{\sqrt{2}\sqrt{c-d}\operatorname{darctanh}\left(\frac{\sqrt{a}\sqrt{c-d}\cos(e+fx)}{\sqrt{2}\sqrt{a}\sin(e+fx)+a\sqrt{c+d}\sin(e+fx)}\right)}{\sqrt{a}f} - \frac{2\sqrt{c}\operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{c}\cos(e+fx)}{\sqrt{a}\sin(e+fx)+a\sqrt{c+d}\sin(e+fx)}\right)}{\sqrt{a}f}$$

input `Int[(Csc[e + f*x]*Sqrt[c + d*Sin[e + f*x]])/Sqrt[a + a*Sin[e + f*x]],x]`

output `(-2*Sqrt[c]*ArcTanh[(Sqrt[a]*Sqrt[c]*Cos[e + f*x])/(Sqrt[a + a*Sin[e + f*x]])*Sqrt[c + d*Sin[e + f*x]])]/(Sqrt[a]*f) + (Sqrt[2]*Sqrt[c - d]*ArcTanh[(Sqrt[a]*Sqrt[c - d]*Cos[e + f*x])/(Sqrt[2]*Sqrt[a + a*Sin[e + f*x]])*Sqrt[c + d*Sin[e + f*x]])]/(Sqrt[a]*f)`

Defintions of rubi rules used

rule 219 `Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (GtQ[a, 0] || LtQ[b, 0])`

rule 221 `Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(Rt[-a/b, 2]/a)*ArcTanh[x/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3261 `Int[1/(Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)])*Sqrt[(c_) + (d_)*sin[(e_) + (f_)*(x_)])], x_Symbol] := Simp[-2*(a/f) Subst[Int[1/(2*b^2 - (a*c - b*d)*x^2), x], x, b*(Cos[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]])], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]`

rule 3422 `Int[Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)]]/(sin[(e_) + (f_)*(x_)])*Sqrt[(c_) + (d_)*sin[(e_) + (f_)*(x_)])], x_Symbol] := Simp[-2*(a/f) Subst[Int[1/(1 - a*c*x^2), x], x, Cos[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]])], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && EqQ[a^2 - b^2, 0] && NeQ[b*c + a*d, 0]`

rule 3423 `Int[Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)]]/(sin[(e_) + (f_)*(x_)])*Sqrt[(c_) + (d_)*sin[(e_) + (f_)*(x_)])], x_Symbol] := Simp[(b*c - a*d)/c Int[1/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]])], x], x] + Simp[a/c Int[Sqrt[c + d*Sin[e + f*x]]/(Sin[e + f*x]*Sqrt[a + b*Sin[e + f*x]])], x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && EqQ[c^2 - d^2, 0]`

Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 790 vs. $2(113) = 226$.

Time = 8.82 (sec) , antiderivative size = 791, normalized size of antiderivative = 5.65

method	result	size
default	Expression too large to display	791

input `int(csc(f*x+e)*(c+d*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2),x,method=_RETURNVERBOSE)`

output

```

-1/2/f/c^(1/2)/(2*c-2*d)^(1/2)/((2*sin(1/2*f*x+1/2*e)*cos(1/2*f*x+1/2*e)+1
)*a)^(1/2)*((1-cos(1/2*f*x+1/2*e))^2*csc(1/2*f*x+1/2*e)^2-2*csc(1/2*f*x+1/
2*e)+2*cot(1/2*f*x+1/2*e)-1)*(c+2*d*sin(1/2*f*x+1/2*e)*cos(1/2*f*x+1/2*e))
^(1/2)*(c*ln(-4*c^(1/2)*((c+2*d*sin(1/2*f*x+1/2*e)*cos(1/2*f*x+1/2*e))/(co
s(1/2*f*x+1/2*e)+1)^2)^(1/2)-4*c^(1/2)*((c+2*d*sin(1/2*f*x+1/2*e)*cos(1/2*
f*x+1/2*e))/(cos(1/2*f*x+1/2*e)+1)^2)^(1/2)*sec(1/2*f*x+1/2*e)-4*c*tan(1/2
*f*x+1/2*e)-4*d)*(2*c-2*d)^(1/2)+ln(1/c^(1/2)/(1-cos(1/2*f*x+1/2*e))*(c*(1
-cos(1/2*f*x+1/2*e))^2*csc(1/2*f*x+1/2*e)+2*c^(1/2)*((c+2*d*sin(1/2*f*x+1/
2*e)*cos(1/2*f*x+1/2*e))/(cos(1/2*f*x+1/2*e)+1)^2)^(1/2)*sin(1/2*f*x+1/2*e
)-2*d*(1-cos(1/2*f*x+1/2*e))-sin(1/2*f*x+1/2*e)*c))*(2*c-2*d)^(1/2)*c-2*ln
(-4*((2*c-2*d)^(1/2)*((c+2*d*sin(1/2*f*x+1/2*e)*cos(1/2*f*x+1/2*e))/(cos(1
/2*f*x+1/2*e)+1)^2)^(1/2)*cos(1/2*f*x+1/2*e)+(2*c-2*d)^(1/2)*((c+2*d*sin(1
/2*f*x+1/2*e)*cos(1/2*f*x+1/2*e))/(cos(1/2*f*x+1/2*e)+1)^2)^(1/2)-c*cos(1/
2*f*x+1/2*e)+sin(1/2*f*x+1/2*e)*c+d*cos(1/2*f*x+1/2*e)-d*sin(1/2*f*x+1/2*e
)))/(cos(1/2*f*x+1/2*e)+sin(1/2*f*x+1/2*e))*c^(3/2)+2*ln(-4*((2*c-2*d)^(1/
2)*((c+2*d*sin(1/2*f*x+1/2*e)*cos(1/2*f*x+1/2*e))/(cos(1/2*f*x+1/2*e)+1)^2
)^(1/2)*cos(1/2*f*x+1/2*e)+(2*c-2*d)^(1/2)*((c+2*d*sin(1/2*f*x+1/2*e)*cos(
1/2*f*x+1/2*e))/(cos(1/2*f*x+1/2*e)+1)^2)^(1/2)-c*cos(1/2*f*x+1/2*e)+sin(1
/2*f*x+1/2*e)*c+d*cos(1/2*f*x+1/2*e)-d*sin(1/2*f*x+1/2*e))/(cos(1/2*f*x+1/
2*e)+sin(1/2*f*x+1/2*e))*c^(1/2)*d)/((c+2*d*sin(1/2*f*x+1/2*e)*cos(1/2...

```

Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 288 vs. $2(113) = 226$.

Time = 0.41 (sec) , antiderivative size = 2791, normalized size of antiderivative = 19.94

$$\int \frac{\csc(e + fx) \sqrt{c + d \sin(e + fx)}}{\sqrt{a + a \sin(e + fx)}} dx = \text{Too large to display}$$

input

```

integrate(csc(f*x+e)*(c+d*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2),x, algo
rithm="fricas")

```


output

```
[1/4*(sqrt(2)*sqrt((c - d)/a)*log(((c^2 - 14*c*d + 17*d^2)*cos(f*x + e)^3
+ 4*sqrt(2)*((c - 3*d)*cos(f*x + e)^2 - (3*c - d)*cos(f*x + e) + ((c - 3*d
)*cos(f*x + e) + 4*c - 4*d)*sin(f*x + e) - 4*c + 4*d)*sqrt(a*sin(f*x + e)
+ a)*sqrt(d*sin(f*x + e) + c)*sqrt((c - d)/a) - (13*c^2 - 22*c*d - 3*d^2)*
cos(f*x + e)^2 - 4*c^2 - 8*c*d - 4*d^2 - 2*(9*c^2 - 14*c*d + 9*d^2)*cos(f*
x + e) + ((c^2 - 14*c*d + 17*d^2)*cos(f*x + e)^2 - 4*c^2 - 8*c*d - 4*d^2 +
2*(7*c^2 - 18*c*d + 7*d^2)*cos(f*x + e))*sin(f*x + e))/(cos(f*x + e)^3 +
3*cos(f*x + e)^2 + (cos(f*x + e)^2 - 2*cos(f*x + e) - 4)*sin(f*x + e) - 2*
cos(f*x + e) - 4)) + sqrt(c/a)*log(((c^4 - 28*c^3*d + 70*c^2*d^2 - 28*c*d^
3 + d^4)*cos(f*x + e)^5 - (31*c^4 - 196*c^3*d + 154*c^2*d^2 - 4*c*d^3 - d^
4)*cos(f*x + e)^4 + c^4 + 4*c^3*d + 6*c^2*d^2 + 4*c*d^3 + d^4 - 2*(81*c^4
- 252*c^3*d + 150*c^2*d^2 - 28*c*d^3 + d^4)*cos(f*x + e)^3 + 2*(79*c^4 - 1
00*c^3*d + 74*c^2*d^2 - 4*c*d^3 - d^4)*cos(f*x + e)^2 - 8*((c^3 - 7*c^2*d
+ 7*c*d^2 - d^3)*cos(f*x + e)^4 - 2*(5*c^3 - 14*c^2*d + 5*c*d^2)*cos(f*x
+ e)^3 + 51*c^3 - 59*c^2*d + 17*c*d^2 - d^3 - 2*(18*c^3 - 33*c^2*d + 12*c*d
^2 - d^3)*cos(f*x + e)^2 + 2*(13*c^3 - 14*c^2*d + 5*c*d^2)*cos(f*x + e) +
((c^3 - 7*c^2*d + 7*c*d^2 - d^3)*cos(f*x + e)^3 - 51*c^3 + 59*c^2*d - 17*c
*d^2 + d^3 + (11*c^3 - 35*c^2*d + 17*c*d^2 - d^3)*cos(f*x + e)^2 - (25*c^3
- 31*c^2*d + 7*c*d^2 - d^3)*cos(f*x + e))*sin(f*x + e))*sqrt(a*sin(f*x +
e) + a)*sqrt(d*sin(f*x + e) + c)*sqrt(c/a) + (289*c^4 - 476*c^3*d + 230...
```

Sympy [F]

$$\int \frac{\csc(e + fx) \sqrt{c + d \sin(e + fx)}}{\sqrt{a + a \sin(e + fx)}} dx = \int \frac{\sqrt{c + d \sin(e + fx)} \csc(e + fx)}{\sqrt{a} (\sin(e + fx) + 1)} dx$$

input

```
integrate(csc(f*x+e)*(c+d*sin(f*x+e))**(1/2)/(a+a*sin(f*x+e))**(1/2),x)
```

output

```
Integral(sqrt(c + d*sin(e + f*x))*csc(e + f*x)/sqrt(a*(sin(e + f*x) + 1)),
x)
```

Maxima [F]

$$\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}} dx = \int \frac{\sqrt{d\sin(fx+e)+c}\csc(fx+e)}{\sqrt{a\sin(fx+e)+a}} dx$$

input `integrate(csc(f*x+e)*(c+d*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2),x, algorith="maxima")`

output `integrate(sqrt(d*sin(f*x + e) + c)*csc(f*x + e)/sqrt(a*sin(f*x + e) + a), x)`

Giac [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}} dx = \text{Timed out}$$

input `integrate(csc(f*x+e)*(c+d*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2),x, algorith="giac")`

output `Timed out`

Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{\sqrt{a+a\sin(e+fx)}} dx = \int \frac{\sqrt{c+d\sin(e+fx)}}{\sin(e+fx)\sqrt{a+a\sin(e+fx)}} dx$$

input `int((c + d*sin(e + f*x))^(1/2)/(sin(e + f*x)*(a + a*sin(e + f*x))^(1/2)),x)`

output

```
int((c + d*sin(e + f*x))^(1/2)/(sin(e + f*x)*(a + a*sin(e + f*x))^(1/2)),
x)
```

Reduce [F]

$$\int \frac{\csc(e + fx) \sqrt{c + d \sin(e + fx)}}{\sqrt{a + a \sin(e + fx)}} dx = \frac{\sqrt{a} \left(\int \frac{\sqrt{\sin(fx+e)d+c} \sqrt{\sin(fx+e)+1} \csc(fx+e)}{\sin(fx+e)+1} dx \right)}{a}$$

input

```
int(csc(f*x+e)*(c+d*sin(f*x+e))^(1/2)/(a+a*sin(f*x+e))^(1/2),x)
```

output

```
(sqrt(a)*int((sqrt(sin(e + f*x)*d + c)*sqrt(sin(e + f*x) + 1)*csc(e + f*x)
)/(sin(e + f*x) + 1),x))/a
```

3.38 $\int \frac{\csc(e+fx)}{\sqrt{a+a \sin(e+fx)}\sqrt{c+d \sin(e+fx)}} dx$

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Optimal result

Integrand size = 35, antiderivative size = 140

$$\int \frac{\csc(e+fx)}{\sqrt{a+a \sin(e+fx)}\sqrt{c+d \sin(e+fx)}} dx$$

$$= -\frac{2\arctanh\left(\frac{\sqrt{a}\sqrt{c}\cos(e+fx)}{\sqrt{a+a \sin(e+fx)}\sqrt{c+d \sin(e+fx)}}\right)}{\sqrt{a}\sqrt{c}f} + \frac{\sqrt{2}\arctanh\left(\frac{\sqrt{a}\sqrt{c-d}\cos(e+fx)}{\sqrt{2}\sqrt{a+a \sin(e+fx)}\sqrt{c+d \sin(e+fx)}}\right)}{\sqrt{a}\sqrt{c-d}f}$$

output

```
-2*arctanh(a^(1/2)*c^(1/2)*cos(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2))/a^(1/2)/c^(1/2)/f+2^(1/2)*arctanh(1/2*a^(1/2)*(c-d)^(1/2)*cos(f*x+e)*2^(1/2)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2))/a^(1/2)/(c-d)^(1/2)/f
```

Mathematica [B] (warning: unable to verify)

Leaf count is larger than twice the leaf count of optimal. 885 vs. 2(140) = 280.

Time = 16.65 (sec) , antiderivative size = 885, normalized size of antiderivative = 6.32

$$\int \frac{\csc(e+fx)}{\sqrt{a+a \sin(e+fx)}\sqrt{c+d \sin(e+fx)}} dx = \text{Too large to display}$$

input `Integrate[Csc[e + f*x]/(Sqrt[a + a*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]]),x]`

output
$$\begin{aligned} & (\text{Csc}[e + f*x] * (\text{Log}[\text{Tan}[(e + f*x)/2]] / \text{Sqrt}[c] - (\text{Sqrt}[2] * \text{Log}[1 + \text{Tan}[(e + f*x)/2]])) / \text{Sqrt}[c - d] + \text{Log}[d + \text{Sqrt}[2] * \text{Sqrt}[c] * \text{Sqrt}[(1 + \text{Cos}[e + f*x])^{-1}] * \text{Sqrt}[c + d * \text{Sin}[e + f*x]] + c * \text{Tan}[(e + f*x)/2]] / \text{Sqrt}[c] - \text{Log}[c + \text{Sqrt}[2] * \text{Sqrt}[c] * \text{Sqrt}[(1 + \text{Cos}[e + f*x])^{-1}] * \text{Sqrt}[c + d * \text{Sin}[e + f*x]] + d * \text{Tan}[(e + f*x)/2]] / \text{Sqrt}[c] + (\text{Sqrt}[2] * \text{Log}[c - d + 2 * \text{Sqrt}[c - d] * \text{Sqrt}[(1 + \text{Cos}[e + f*x])^{-1}] * \text{Sqrt}[c + d * \text{Sin}[e + f*x]] + (-c + d) * \text{Tan}[(e + f*x)/2]]) / \text{Sqrt}[c - d]) / (f * \text{Sqrt}[a * (1 + \text{Sin}[e + f*x])] * \text{Sqrt}[c + d * \text{Sin}[e + f*x]] * ((\text{Csc}[(e + f*x)/2] * \text{Sec}[(e + f*x)/2]) / (2 * \text{Sqrt}[c]) - \text{Sec}[(e + f*x)/2]^2 / (\text{Sqrt}[2] * \text{Sqrt}[c - d] * (1 + \text{Tan}[(e + f*x)/2]))) + ((c * \text{Sec}[(e + f*x)/2]^2) / 2 + (\text{Sqrt}[c] * d * \text{Cos}[e + f*x] * \text{Sqrt}[(1 + \text{Cos}[e + f*x])^{-1}]) / (\text{Sqrt}[2] * \text{Sqrt}[c + d * \text{Sin}[e + f*x]]) + (\text{Sqrt}[c] * ((1 + \text{Cos}[e + f*x])^{-1})^{3/2} * \text{Sin}[e + f*x] * \text{Sqrt}[c + d * \text{Sin}[e + f*x]]) / \text{Sqrt}[2]) / (\text{Sqrt}[c] * (d + \text{Sqrt}[2] * \text{Sqrt}[c] * \text{Sqrt}[(1 + \text{Cos}[e + f*x])^{-1}] * \text{Sqrt}[c + d * \text{Sin}[e + f*x]] + c * \text{Tan}[(e + f*x)/2])) - ((d * \text{Sec}[(e + f*x)/2]^2) / 2 + (\text{Sqrt}[c] * d * \text{Cos}[e + f*x] * \text{Sqrt}[(1 + \text{Cos}[e + f*x])^{-1}]) / (\text{Sqrt}[2] * \text{Sqrt}[c + d * \text{Sin}[e + f*x]]) + (\text{Sqrt}[c] * ((1 + \text{Cos}[e + f*x])^{-1})^{3/2} * \text{Sin}[e + f*x] * \text{Sqrt}[c + d * \text{Sin}[e + f*x]]) / \text{Sqrt}[2]) / (\text{Sqrt}[c] * (c + \text{Sqrt}[2] * \text{Sqrt}[c] * \text{Sqrt}[(1 + \text{Cos}[e + f*x])^{-1}] * \text{Sqrt}[c + d * \text{Sin}[e + f*x]] + d * \text{Tan}[(e + f*x)/2])) + (\text{Sqrt}[2] * (((-c + d) * \text{Sec}[(e + f*x)/2]^2) / 2 + (\text{Sqrt}[c - d] * d * \text{Cos}[e + f*x] * \text{Sqrt}[(1 + \text{Cos}[e + f*x])^{-1}]) / \text{Sqrt}[c + d * \text{Sin}[e + f*x]] + \text{Sqrt}[c - d] * ((1 + \text{Cos}[e + f*x])^{-1})^{3/2} * \text{Sin}[e + f*x] * \text{Sqrt}[c + d * \text{Sin}[e + f*x]])) / \dots \end{aligned}$$

Rubi [A] (verified)

Time = 0.87 (sec) , antiderivative size = 140, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}} = 0.200$, Rules used = {3042, 3426, 3042, 3261, 221, 3422, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e + fx)}{\sqrt{a \sin(e + fx) + a} \sqrt{c + d \sin(e + fx)}} dx$$

↓ 3042

$$\begin{aligned}
& \int \frac{1}{\sin(e+fx)\sqrt{a\sin(e+fx)+a}\sqrt{c+d\sin(e+fx)}} dx \\
& \quad \downarrow \text{3426} \\
& \frac{\int \frac{\csc(e+fx)\sqrt{\sin(e+fx)a+a}}{\sqrt{c+d\sin(e+fx)}} dx}{a} - \int \frac{1}{\sqrt{\sin(e+fx)a+a}\sqrt{c+d\sin(e+fx)}} dx \\
& \quad \downarrow \text{3042} \\
& \frac{\int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}} dx}{a} - \int \frac{1}{\sqrt{\sin(e+fx)a+a}\sqrt{c+d\sin(e+fx)}} dx \\
& \quad \downarrow \text{3261} \\
& \frac{2a \int \frac{1}{2a^2 - \frac{a^3(c-d)\cos^2(e+fx)}{(\sin(e+fx)a+a)(c+d\sin(e+fx))}} dx}{f} - \frac{d \int \frac{a \cos(e+fx)}{\sqrt{\sin(e+fx)a+a}\sqrt{c+d\sin(e+fx)}} dx}{f} + \frac{\int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}} dx}{a} \\
& \quad \downarrow \text{221} \\
& \frac{\int \frac{\sqrt{\sin(e+fx)a+a}}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}} dx}{a} + \frac{\sqrt{2} \operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{c-d}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}\sqrt{c+d\sin(e+fx)}}\right)}{\sqrt{a}f\sqrt{c-d}} \\
& \quad \downarrow \text{3422} \\
& \frac{\sqrt{2} \operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{c-d}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}\sqrt{c+d\sin(e+fx)}}\right)}{\sqrt{a}f\sqrt{c-d}} - \\
& \frac{2 \int \frac{1}{1 - \frac{ac \cos^2(e+fx)}{(\sin(e+fx)a+a)(c+d\sin(e+fx))}} dx}{f} - \frac{d \int \frac{\cos(e+fx)}{\sqrt{\sin(e+fx)a+a}\sqrt{c+d\sin(e+fx)}} dx}{f} \\
& \quad \downarrow \text{219} \\
& \frac{\sqrt{2} \operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{c-d}\cos(e+fx)}{\sqrt{2}\sqrt{a\sin(e+fx)+a}\sqrt{c+d\sin(e+fx)}}\right)}{\sqrt{a}f\sqrt{c-d}} - \frac{2 \operatorname{arctanh}\left(\frac{\sqrt{a}\sqrt{c}\cos(e+fx)}{\sqrt{a\sin(e+fx)+a}\sqrt{c+d\sin(e+fx)}}\right)}{\sqrt{a}\sqrt{c}f}
\end{aligned}$$

input

```
Int[Csc[e + f*x]/(Sqrt[a + a*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]]),x]
```

output

$$\frac{(-2 \operatorname{ArcTanh}[\frac{\sqrt{a} \sqrt{c} \cos[e + fx]}{(\sqrt{a + a \sin[e + fx]} \sqrt{c + d \sin[e + fx]})}] / (\sqrt{a} \sqrt{c} f) + \sqrt{2} \operatorname{ArcTanh}[\frac{\sqrt{a} \sqrt{c - d} \cos[e + fx]}{(\sqrt{2} \sqrt{a + a \sin[e + fx]} \sqrt{c + d \sin[e + fx]})}]}{(\sqrt{a} \sqrt{c - d} f)}$$

Defintions of rubi rules used

rule 219

$$\operatorname{Int}[(a + (b \cdot x)^2)^{-1}, x_{\text{Symbol}}] \rightarrow \operatorname{Simp}[(1 / (\operatorname{Rt}[a, 2] \operatorname{Rt}[-b, 2])) * \operatorname{ArcTanh}[\operatorname{Rt}[-b, 2] * (x / \operatorname{Rt}[a, 2])], x] /; \operatorname{FreeQ}\{a, b\}, x \ \&\& \operatorname{NegQ}[a/b] \ \&\& (\operatorname{GtQ}[a, 0] \ || \ \operatorname{LtQ}[b, 0])$$

rule 221

$$\operatorname{Int}[(a + (b \cdot x)^2)^{-1}, x_{\text{Symbol}}] \rightarrow \operatorname{Simp}[(\operatorname{Rt}[-a/b, 2] / a) * \operatorname{ArcTanh}[x / \operatorname{Rt}[-a/b, 2]], x] /; \operatorname{FreeQ}\{a, b\}, x \ \&\& \operatorname{NegQ}[a/b]$$

rule 3042

$$\operatorname{Int}[u, x_{\text{Symbol}}] \rightarrow \operatorname{Int}[\operatorname{DeactivateTrig}[u, x], x] /; \operatorname{FunctionOfTrigOfLinearQ}[u, x]$$

rule 3261

$$\operatorname{Int}[1 / (\sqrt{(a + (b \cdot \sin[e + fx]) + (f \cdot x)}) * \sqrt{(c + (d \cdot \sin[e + fx]) + (f \cdot x))}), x_{\text{Symbol}}] \rightarrow \operatorname{Simp}[-2 * (a/f) \operatorname{Subst}[\operatorname{Int}[1 / (2 * b^2 - (a * c - b * d) * x^2), x], x, b * (\cos[e + fx] / (\sqrt{a + b \sin[e + fx]} * \sqrt{c + d \sin[e + fx]})]], x] /; \operatorname{FreeQ}\{a, b, c, d, e, f\}, x \ \&\& \operatorname{NeQ}[b * c - a * d, 0] \ \&\& \operatorname{EqQ}[a^2 - b^2, 0] \ \&\& \operatorname{NeQ}[c^2 - d^2, 0]$$

rule 3422

$$\operatorname{Int}[\sqrt{(a + (b \cdot \sin[e + fx]) + (f \cdot x))} / (\sin[e + fx] * \sqrt{(c + (d \cdot \sin[e + fx]) + (f \cdot x))}), x_{\text{Symbol}}] \rightarrow \operatorname{Simp}[-2 * (a/f) \operatorname{Subst}[\operatorname{Int}[1 / (1 - a * c * x^2), x], x, \cos[e + fx] / (\sqrt{a + b \sin[e + fx]} * \sqrt{c + d \sin[e + fx]})], x] /; \operatorname{FreeQ}\{a, b, c, d, e, f\}, x \ \&\& \operatorname{NeQ}[b * c - a * d, 0] \ \&\& \operatorname{EqQ}[a^2 - b^2, 0] \ \&\& \operatorname{NeQ}[b * c + a * d, 0]$$

rule 3426

```
Int[1/(sin[(e_.) + (f_.)*(x_)]*Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]*S
qrt[(c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]], x_Symbol] := Simp[-b/a Int[1/
(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]]), x], x] + Simp[1/a In
t[Sqrt[a + b*Sin[e + f*x]]/(Sin[e + f*x]*Sqrt[c + d*Sin[e + f*x]]), x], x]
/; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && (NeQ[a^2 - b^2, 0]
|| NeQ[c^2 - d^2, 0])
```

Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 597 vs. $2(113) = 226$.

Time = 9.60 (sec) , antiderivative size = 598, normalized size of antiderivative = 4.27

method	result
default	$\sqrt{\frac{c+2d\sin\left(\frac{fx}{2}+\frac{e}{2}\right)\cos\left(\frac{fx}{2}+\frac{e}{2}\right)}{\left(\cos\left(\frac{fx}{2}+\frac{e}{2}\right)+1\right)^2}} \left(\cos\left(\frac{fx}{2}+\frac{e}{2}\right)+1\right) \left(\cos\left(\frac{fx}{2}+\frac{e}{2}\right)+\sin\left(\frac{fx}{2}+\frac{e}{2}\right)\right) \ln \left(-\frac{c \cot\left(\frac{fx}{2}+\frac{e}{2}\right) \cos\left(\frac{fx}{2}+\frac{e}{2}\right) - 2c \cot\left(\frac{fx}{2}+\frac{e}{2}\right) + 2d \cot\left(\frac{fx}{2}+\frac{e}{2}\right) \sin\left(\frac{fx}{2}+\frac{e}{2}\right)}{\left(\cos\left(\frac{fx}{2}+\frac{e}{2}\right)+1\right)^2} \right)$

input

```
int(csc(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x,method=_RET
URNVERBOSE)
```


output

```

1/f/c^(1/2)/(2*c-2*d)^(1/2)*((c+2*d*sin(1/2*f*x+1/2*e))*cos(1/2*f*x+1/2*e))
/(cos(1/2*f*x+1/2*e)+1)^2^(1/2)*(cos(1/2*f*x+1/2*e)+1)*(cos(1/2*f*x+1/2*
e))+sin(1/2*f*x+1/2*e))*(ln(-(c*cot(1/2*f*x+1/2*e))*cos(1/2*f*x+1/2*e)-2*c*co
t(1/2*f*x+1/2*e)+2*c^(1/2)*((c+2*d*sin(1/2*f*x+1/2*e))*cos(1/2*f*x+1/2*e)))/
(cos(1/2*f*x+1/2*e)+1)^2^(1/2)*sin(1/2*f*x+1/2*e)+c*csc(1/2*f*x+1/2*e)-si
n(1/2*f*x+1/2*e)*c+2*d*cos(1/2*f*x+1/2*e)-2*d)/c^(1/2)/(-1+cos(1/2*f*x+1/2
*e)))*(2*c-2*d)^(1/2)+ln(-4*c^(1/2)*((c+2*d*sin(1/2*f*x+1/2*e))*cos(1/2*f*x
+1/2*e)))/(cos(1/2*f*x+1/2*e)+1)^2^(1/2)-4*c^(1/2)*((c+2*d*sin(1/2*f*x+1/2
*e))*cos(1/2*f*x+1/2*e))/(cos(1/2*f*x+1/2*e)+1)^2^(1/2)*sec(1/2*f*x+1/2*e)
-4*c*tan(1/2*f*x+1/2*e)-4*d)*(2*c-2*d)^(1/2)-2*ln(-4*((2*c-2*d)^(1/2)*((c+
2*d*sin(1/2*f*x+1/2*e))*cos(1/2*f*x+1/2*e))/(cos(1/2*f*x+1/2*e)+1)^2^(1/2)
*cos(1/2*f*x+1/2*e)+(2*c-2*d)^(1/2)*((c+2*d*sin(1/2*f*x+1/2*e))*cos(1/2*f*x
+1/2*e))/(cos(1/2*f*x+1/2*e)+1)^2^(1/2)-c*cos(1/2*f*x+1/2*e)+sin(1/2*f*x+
1/2*e)*c+d*cos(1/2*f*x+1/2*e)-d*sin(1/2*f*x+1/2*e))/(cos(1/2*f*x+1/2*e)+si
n(1/2*f*x+1/2*e))*c^(1/2))/((2*sin(1/2*f*x+1/2*e))*cos(1/2*f*x+1/2*e)+1)*a
)^(1/2)/(c+2*d*sin(1/2*f*x+1/2*e))*cos(1/2*f*x+1/2*e))^(1/2)

```

Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 290 vs. $2(113) = 226$.

Time = 0.48 (sec) , antiderivative size = 3005, normalized size of antiderivative = 21.46

$$\int \frac{\csc(e + fx)}{\sqrt{a + a \sin(e + fx)} \sqrt{c + d \sin(e + fx)}} dx = \text{Too large to display}$$

input

```

integrate(csc(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x, algo
rithm="fricas")

```

output

Too large to include

Sympy [F]

$$\int \frac{\csc(e + fx)}{\sqrt{a + a \sin(e + fx)} \sqrt{c + d \sin(e + fx)}} dx$$

$$= \int \frac{\csc(e + fx)}{\sqrt{a(\sin(e + fx) + 1)} \sqrt{c + d \sin(e + fx)}} dx$$

input `integrate(csc(f*x+e)/(a+a*sin(f*x+e))**(1/2)/(c+d*sin(f*x+e))**(1/2),x)`

output `Integral(csc(e + f*x)/(sqrt(a*(sin(e + f*x) + 1))*sqrt(c + d*sin(e + f*x))), x)`

Maxima [F]

$$\int \frac{\csc(e + fx)}{\sqrt{a + a \sin(e + fx)} \sqrt{c + d \sin(e + fx)}} dx$$

$$= \int \frac{\csc(fx + e)}{\sqrt{a \sin(fx + e) + a} \sqrt{d \sin(fx + e) + c}} dx$$

input `integrate(csc(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x, algorith="maxima")`

output `integrate(csc(f*x + e)/(sqrt(a*sin(f*x + e) + a)*sqrt(d*sin(f*x + e) + c)), x)`

Giac [F(-1)]

Timed out.

$$\int \frac{\csc(e + fx)}{\sqrt{a + a \sin(e + fx)} \sqrt{c + d \sin(e + fx)}} dx = \text{Timed out}$$

input `integrate(csc(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x, algorithm="giac")`

output `Timed out`

Mupad [F(-1)]

Timed out.

$$\begin{aligned} & \int \frac{\csc(e + fx)}{\sqrt{a + a \sin(e + fx)} \sqrt{c + d \sin(e + fx)}} dx \\ &= \int \frac{1}{\sin(e + fx) \sqrt{a + a \sin(e + fx)} \sqrt{c + d \sin(e + fx)}} dx \end{aligned}$$

input `int(1/(sin(e + f*x)*(a + a*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))^(1/2)),x)`

output `int(1/(sin(e + f*x)*(a + a*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))^(1/2)), x)`

Reduce [F]

$$\begin{aligned} & \int \frac{\csc(e + fx)}{\sqrt{a + a \sin(e + fx)} \sqrt{c + d \sin(e + fx)}} dx \\ &= \frac{\sqrt{a} \left(\int \frac{\sqrt{\sin(fx+e)d+c} \sqrt{\sin(fx+e)+1} \csc(fx+e)}{\sin(fx+e)^2 d + \sin(fx+e)c + \sin(fx+e)d+c} dx \right)}{a} \end{aligned}$$

input `int(csc(f*x+e)/(a+a*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x)`

output `(sqrt(a)*int((sqrt(sin(e + f*x)*d + c)*sqrt(sin(e + f*x) + 1)*csc(e + f*x))/
(sin(e + f*x)**2*d + sin(e + f*x)*c + sin(e + f*x)*d + c),x))/a`

3.39
$$\int \frac{\sin^2(e+fx)}{(a+b \sin(e+fx))^2(c+d \sin(e+fx))} dx$$

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Optimal result

Integrand size = 33, antiderivative size = 181

$$\int \frac{\sin^2(e+fx)}{(a+b \sin(e+fx))^2(c+d \sin(e+fx))} dx$$

$$= -\frac{2a(a^2c - 2b^2c + abd) \arctan\left(\frac{b+a \tan(\frac{1}{2}(e+fx))}{\sqrt{a^2-b^2}}\right)}{(a^2 - b^2)^{3/2} (bc - ad)^2 f}$$

$$+ \frac{2c^2 \arctan\left(\frac{d+c \tan(\frac{1}{2}(e+fx))}{\sqrt{c^2-d^2}}\right)}{(bc - ad)^2 \sqrt{c^2 - d^2} f} + \frac{a^2 \cos(e+fx)}{(a^2 - b^2) (bc - ad) f (a + b \sin(e+fx))}$$

output

```
-2*a*(a^2*c+a*b*d-2*b^2*c)*arctan((b+a*tan(1/2*f*x+1/2*e))/(a^2-b^2)^(1/2))
/(a^2-b^2)^(3/2)/(-a*d+b*c)^2/f+2*c^2*arctan((d+c*tan(1/2*f*x+1/2*e))/(c^2-d^2)^(1/2))
/(-a*d+b*c)^2/(c^2-d^2)^(1/2)/f+a^2*cos(f*x+e)/(a^2-b^2)/(-a*d+b*c)/f/(a+b*sin(f*x+e))
```

Mathematica [A] (verified)

Time = 2.23 (sec) , antiderivative size = 178, normalized size of antiderivative = 0.98

$$\int \frac{\sin^2(e + fx)}{(a + b \sin(e + fx))^2 (c + d \sin(e + fx))} dx$$

$$= \frac{2a(a^2c - 2b^2c + abd) \arctan\left(\frac{b + a \tan\left(\frac{1}{2}(e + fx)\right)}{\sqrt{a^2 - b^2}}\right)}{(a^2 - b^2)^{3/2} (bc - ad)^2} + \frac{2c^2 \arctan\left(\frac{d + c \tan\left(\frac{1}{2}(e + fx)\right)}{\sqrt{c^2 - d^2}}\right)}{(bc - ad)^2 \sqrt{c^2 - d^2}} - \frac{a^2 \cos(e + fx)}{(a - b)(a + b)(-bc + ad)(a + b \sin(e + fx))}$$

input

```
Integrate[Sin[e + f*x]^2/((a + b*SIN[e + f*x])^2*(c + d*SIN[e + f*x])),x]
```

output

```
((-2*a*(a^2*c - 2*b^2*c + a*b*d)*ArcTan[(b + a*Tan[(e + f*x)/2])/Sqrt[a^2 - b^2]])/((a^2 - b^2)^(3/2)*(b*c - a*d)^2) + (2*c^2*ArcTan[(d + c*Tan[(e + f*x)/2])/Sqrt[c^2 - d^2]])/((b*c - a*d)^2*Sqrt[c^2 - d^2]) - (a^2*Cos[e + f*x])/((a - b)*(a + b)*(-b*c) + a*d)*(a + b*SIN[e + f*x]))/f
```

Rubi [A] (verified)Time = 0.86 (sec) , antiderivative size = 225, normalized size of antiderivative = 1.24, number of steps used = 10, number of rules used = 9, $\frac{\text{number of rules}}{\text{integrand size}} = 0.273$, Rules used = {3042, 3535, 25, 3042, 3480, 3042, 3139, 1083, 217}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sin^2(e + fx)}{(a + b \sin(e + fx))^2 (c + d \sin(e + fx))} dx$$

$$\downarrow \text{3042}$$

$$\int \frac{\sin(e + fx)^2}{(a + b \sin(e + fx))^2 (c + d \sin(e + fx))} dx$$

$$\downarrow \text{3535}$$

$$\frac{a^2 \cos(e + fx)}{f (a^2 - b^2) (bc - ad) (a + b \sin(e + fx))} - \frac{\int -\frac{abc + (ca^2 + bda - b^2c) \sin(e + fx)}{(a + b \sin(e + fx))(c + d \sin(e + fx))} dx}{(a^2 - b^2) (bc - ad)}$$

$$\begin{aligned}
& \int \frac{abc+(ca^2+bd a-b^2c)\sin(e+fx)}{(a+b\sin(e+fx))(c+d\sin(e+fx))} dx + \frac{a^2\cos(e+fx)}{f(a^2-b^2)(bc-ad)(a+b\sin(e+fx))} \\
& \quad \downarrow 25 \\
& \int \frac{abc+(ca^2+bd a-b^2c)\sin(e+fx)}{(a^2-b^2)(bc-ad)} dx + \frac{a^2\cos(e+fx)}{f(a^2-b^2)(bc-ad)(a+b\sin(e+fx))} \\
& \quad \downarrow 3042 \\
& \frac{c^2(a^2-b^2)\int\frac{1}{c+d\sin(e+fx)}dx}{bc-ad} - \frac{a(a^2c+abd-2b^2c)\int\frac{1}{a+b\sin(e+fx)}dx}{bc-ad} + \\
& \quad \frac{(a^2-b^2)(bc-ad)}{a^2\cos(e+fx)} \\
& \quad \frac{f(a^2-b^2)(bc-ad)(a+b\sin(e+fx))} \\
& \quad \downarrow 3042 \\
& \frac{c^2(a^2-b^2)\int\frac{1}{c+d\sin(e+fx)}dx}{bc-ad} - \frac{a(a^2c+abd-2b^2c)\int\frac{1}{a+b\sin(e+fx)}dx}{bc-ad} + \\
& \quad \frac{(a^2-b^2)(bc-ad)}{a^2\cos(e+fx)} \\
& \quad \frac{f(a^2-b^2)(bc-ad)(a+b\sin(e+fx))} \\
& \quad \downarrow 3139 \\
& \frac{2c^2(a^2-b^2)\int\frac{1}{c\tan^2(\frac{1}{2}(e+fx))+2d\tan(\frac{1}{2}(e+fx))+c}d\tan(\frac{1}{2}(e+fx))}{f(bc-ad)} - \frac{2a(a^2c+abd-2b^2c)\int\frac{1}{a\tan^2(\frac{1}{2}(e+fx))+2b\tan(\frac{1}{2}(e+fx))+a}d\tan(\frac{1}{2}(e+fx))}{f(bc-ad)} \\
& \quad \frac{(a^2-b^2)(bc-ad)}{a^2\cos(e+fx)} \\
& \quad \frac{f(a^2-b^2)(bc-ad)(a+b\sin(e+fx))} \\
& \quad \downarrow 1083 \\
& \frac{4a(a^2c+abd-2b^2c)\int\frac{1}{-(2b+2a\tan(\frac{1}{2}(e+fx)))^2-4(a^2-b^2)}d(2b+2a\tan(\frac{1}{2}(e+fx)))}{f(bc-ad)} - \frac{4c^2(a^2-b^2)\int\frac{1}{-(2d+2c\tan(\frac{1}{2}(e+fx)))^2-4(c^2-d^2)}d(2d+2c\tan(\frac{1}{2}(e+fx)))}{f(bc-ad)} \\
& \quad \frac{(a^2-b^2)(bc-ad)}{a^2\cos(e+fx)} \\
& \quad \frac{f(a^2-b^2)(bc-ad)(a+b\sin(e+fx))} \\
& \quad \downarrow 217
\end{aligned}$$

$$\frac{2c^2(a^2-b^2) \arctan\left(\frac{2c \tan\left(\frac{1}{2}(e+fx)\right)+2d}{2\sqrt{c^2-d^2}}\right)}{f\sqrt{c^2-d^2}(bc-ad)} - \frac{2a(a^2c+abd-2b^2c) \arctan\left(\frac{2a \tan\left(\frac{1}{2}(e+fx)\right)+2b}{2\sqrt{a^2-b^2}}\right)}{f\sqrt{a^2-b^2}(bc-ad)} + \frac{(a^2-b^2)(bc-ad)}{a^2 \cos(e+fx)} \frac{1}{f(a^2-b^2)(bc-ad)(a+b \sin(e+fx))}$$

input `Int[Sin[e + f*x]^2/((a + b*Sin[e + f*x])^2*(c + d*Sin[e + f*x])),x]`

output `((-2*a*(a^2*c - 2*b^2*c + a*b*d)*ArcTan[(2*b + 2*a*Tan[(e + f*x)/2])/(2*sqrt[a^2 - b^2])])/(sqrt[a^2 - b^2]*(b*c - a*d)*f) + (2*(a^2 - b^2)*c^2*ArcTan[(2*d + 2*c*Tan[(e + f*x)/2])/(2*sqrt[c^2 - d^2])])/(b*c - a*d)*sqrt[c^2 - d^2]*f)/((a^2 - b^2)*(b*c - a*d)) + (a^2*cos[e + f*x])/((a^2 - b^2)*(b*c - a*d)*f*(a + b*Sin[e + f*x]))`

Definitions of rubi rules used

rule 25 `Int[-(Fx_), x_Symbol] := Simp[Identity[-1] Int[Fx, x], x]`

rule 217 `Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] && (LtQ[a, 0] || LtQ[b, 0])`

rule 1083 `Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] := Simp[-2 Subst[Int[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c}, x]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3139 `Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] := With[{e = FreeFactors[Tan[(c + d*x)/2], x]}, Simp[2*(e/d) Subst[Int[1/(a + 2*b*e*x + a*e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x] /; FreeQ[{a, b, c, d}, x] && NeQ[a^2 - b^2, 0]`

rule 3480

```
Int[((A_.) + (B_.)*sin[(e_.) + (f_.)*(x_.)])/(((a_.) + (b_.)*sin[(e_.) + (f_.)*(x_.)])*((c_.) + (d_.)*sin[(e_.) + (f_.)*(x_.)])), x_Symbol] := Simp[(A*b - a*B)/(b*c - a*d) Int[1/(a + b*Sin[e + f*x]), x], x] + Simp[(B*c - A*d)/(b*c - a*d) Int[1/(c + d*Sin[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f, A, B}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]
```

rule 3535

```
Int[((a_.) + (b_.)*sin[(e_.) + (f_.)*(x_.)])^(m_)*((c_.) + (d_.)*sin[(e_.) + (f_.)*(x_.)])^(n_)*((A_.) + (C_.)*sin[(e_.) + (f_.)*(x_.)]^2), x_Symbol] := Simp[(-A*b^2 + a^2*C)*Cos[e + f*x]*(a + b*Sin[e + f*x])^(m + 1)*((c + d*Sin[e + f*x])^(n + 1)/(f*(m + 1)*(b*c - a*d)*(a^2 - b^2))), x] + Simp[1/((m + 1)*(b*c - a*d)*(a^2 - b^2)) Int[(a + b*Sin[e + f*x])^(m + 1)*(c + d*Sin[e + f*x])^n*Simp[a*(m + 1)*(b*c - a*d)*(A + C) + d*(A*b^2 + a^2*C)*(m + n + 2) - (c*(A*b^2 + a^2*C) + b*(m + 1)*(b*c - a*d)*(A + C))*Sin[e + f*x] - d*(A*b^2 + a^2*C)*(m + n + 3)*Sin[e + f*x]^2, x], x], x] /; FreeQ[{a, b, c, d, e, f, A, C, n}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0] && LtQ[m, -1] && ((EqQ[a, 0] && IntegerQ[m] && !IntegerQ[n]) || !(IntegerQ[2*n] && LtQ[n, -1] && ((IntegerQ[n] && !IntegerQ[m]) || EqQ[a, 0])))
```

Maple [A] (verified)

Time = 1.05 (sec) , antiderivative size = 244, normalized size of antiderivative = 1.35

method	result
derivativedivides	$-\frac{2a \left(\frac{b(ad-bc) \tan\left(\frac{fx}{2} + \frac{e}{2}\right) + \frac{a(ad-bc)}{a^2-b^2} + \frac{(a^2c+abd-2b^2c) \arctan\left(\frac{2a \tan\left(\frac{fx}{2} + \frac{e}{2}\right) + 2b}{2\sqrt{a^2-b^2}}\right)}{(a^2-b^2)^{\frac{3}{2}}} \right)}{a^2d^2-2abcd+b^2c^2} + \frac{8c^2 \arctan\left(\frac{2c \tan\left(\frac{fx}{2} + \frac{e}{2}\right) + 2d}{2\sqrt{c^2-d^2}}\right)}{(4a^2d^2-8abcd+4b^2c^2)\sqrt{c^2-d^2}}$
default	$-\frac{2a \left(\frac{b(ad-bc) \tan\left(\frac{fx}{2} + \frac{e}{2}\right) + \frac{a(ad-bc)}{a^2-b^2} + \frac{(a^2c+abd-2b^2c) \arctan\left(\frac{2a \tan\left(\frac{fx}{2} + \frac{e}{2}\right) + 2b}{2\sqrt{a^2-b^2}}\right)}{(a^2-b^2)^{\frac{3}{2}}} \right)}{a^2d^2-2abcd+b^2c^2} + \frac{8c^2 \arctan\left(\frac{2c \tan\left(\frac{fx}{2} + \frac{e}{2}\right) + 2d}{2\sqrt{c^2-d^2}}\right)}{(4a^2d^2-8abcd+4b^2c^2)\sqrt{c^2-d^2}}$
risch	$\frac{2ia^2(ib+ae^{i(fx+e)})}{b(a^2-b^2)(ad-bc)f(-ibe^{2i(fx+e)}+ib+2ae^{i(fx+e)})} + \frac{c^2 \ln\left(e^{i(fx+e)} + \frac{ic\sqrt{-c^2+d^2+c^2-d^2}}{\sqrt{-c^2+d^2}d}\right)}{\sqrt{-c^2+d^2}(ad-bc)^2f} - \frac{c^2 \ln\left(e^{i(fx+e)} + \frac{ic}{\sqrt{-c^2+d^2}}\right)}{\sqrt{-c^2+d^2}}$

input `int(sin(f*x+e)^2/(a+b*sin(f*x+e))^2/(c+d*sin(f*x+e)),x,method=_RETURNVERBOSE)`

output `1/f*(-2*a/(a^2*d^2-2*a*b*c*d+b^2*c^2)*((b*(a*d-b*c)/(a^2-b^2)*tan(1/2*f*x+1/2*e)+a*(a*d-b*c)/(a^2-b^2))/(a*tan(1/2*f*x+1/2*e)^2+2*b*tan(1/2*f*x+1/2*e)+a)+(a^2*c+a*b*d-2*b^2*c)/(a^2-b^2)^(3/2)*arctan(1/2*(2*a*tan(1/2*f*x+1/2*e)+2*b)/(a^2-b^2)^(1/2)))+8*c^2/(4*a^2*d^2-8*a*b*c*d+4*b^2*c^2)/(c^2-d^2)^(1/2)*arctan(1/2*(2*c*tan(1/2*f*x+1/2*e)+2*d)/(c^2-d^2)^(1/2))`

Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 625 vs. 2(171) = 342.

Time = 59.94 (sec) , antiderivative size = 2837, normalized size of antiderivative = 15.67

$$\int \frac{\sin^2(e + fx)}{(a + b \sin(e + fx))^2 (c + d \sin(e + fx))} dx = \text{Too large to display}$$

input `integrate(sin(f*x+e)^2/(a+b*sin(f*x+e))^2/(c+d*sin(f*x+e)),x, algorithm="fricas")`

output

```
[1/2*((a^3*b*c^2*d - a^3*b*d^3 + (a^4 - 2*a^2*b^2)*c^3 - (a^4 - 2*a^2*b^2)
*c*d^2 + (a^2*b^2*c^2*d - a^2*b^2*d^3 + (a^3*b - 2*a*b^3)*c^3 - (a^3*b - 2
*a*b^3)*c*d^2)*sin(f*x + e))*sqrt(-a^2 + b^2)*log(((2*a^2 - b^2)*cos(f*x +
e)^2 - 2*a*b*sin(f*x + e) - a^2 - b^2 + 2*(a*cos(f*x + e)*sin(f*x + e) +
b*cos(f*x + e))*sqrt(-a^2 + b^2))/(b^2*cos(f*x + e)^2 - 2*a*b*sin(f*x + e)
- a^2 - b^2)) - ((a^4*b - 2*a^2*b^3 + b^5)*c^2*sin(f*x + e) + (a^5 - 2*a^
3*b^2 + a*b^4)*c^2)*sqrt(-c^2 + d^2)*log(((2*c^2 - d^2)*cos(f*x + e)^2 - 2
*c*d*sin(f*x + e) - c^2 - d^2 + 2*(c*cos(f*x + e)*sin(f*x + e) + d*cos(f*x
+ e))*sqrt(-c^2 + d^2))/(d^2*cos(f*x + e)^2 - 2*c*d*sin(f*x + e) - c^2 -
d^2)) + 2*((a^4*b - a^2*b^3)*c^3 - (a^5 - a^3*b^2)*c^2*d - (a^4*b - a^2*b^
3)*c*d^2 + (a^5 - a^3*b^2)*d^3)*cos(f*x + e))/(((a^4*b^3 - 2*a^2*b^5 + b^7
)*c^4 - 2*(a^5*b^2 - 2*a^3*b^4 + a*b^6)*c^3*d + (a^6*b - 3*a^4*b^3 + 3*a^2
*b^5 - b^7)*c^2*d^2 + 2*(a^5*b^2 - 2*a^3*b^4 + a*b^6)*c*d^3 - (a^6*b - 2*a
^4*b^3 + a^2*b^5)*d^4)*f*sin(f*x + e) + ((a^5*b^2 - 2*a^3*b^4 + a*b^6)*c^4
- 2*(a^6*b - 2*a^4*b^3 + a^2*b^5)*c^3*d + (a^7 - 3*a^5*b^2 + 3*a^3*b^4 -
a*b^6)*c^2*d^2 + 2*(a^6*b - 2*a^4*b^3 + a^2*b^5)*c*d^3 - (a^7 - 2*a^5*b^2
+ a^3*b^4)*d^4)*f), -1/2*(2*((a^4*b - 2*a^2*b^3 + b^5)*c^2*sin(f*x + e) +
(a^5 - 2*a^3*b^2 + a*b^4)*c^2)*sqrt(c^2 - d^2)*arctan(-(c*sin(f*x + e) + d
)/(sqrt(c^2 - d^2)*cos(f*x + e))) - (a^3*b*c^2*d - a^3*b*d^3 + (a^4 - 2*a^
2*b^2)*c^3 - (a^4 - 2*a^2*b^2)*c*d^2 + (a^2*b^2*c^2*d - a^2*b^2*d^3 + (...
```

Sympy [F(-1)]

Timed out.

$$\int \frac{\sin^2(e + fx)}{(a + b \sin(e + fx))^2 (c + d \sin(e + fx))} dx = \text{Timed out}$$

input

```
integrate(sin(f*x+e)**2/(a+b*sin(f*x+e))**2/(c+d*sin(f*x+e)),x)
```

output

Timed out

Maxima [F(-2)]

Exception generated.

$$\int \frac{\sin^2(e + fx)}{(a + b \sin(e + fx))^2 (c + d \sin(e + fx))} dx = \text{Exception raised: ValueError}$$

input `integrate(sin(f*x+e)^2/(a+b*sin(f*x+e))^2/(c+d*sin(f*x+e)),x, algorithm="maxima")`

output Exception raised: ValueError >> Computation failed since Maxima requested additional constraints; using the 'assume' command before evaluation *may* help (example of legal syntax is 'assume(4*d^2-4*c^2>0)', see `assume?` f or more de

Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 292, normalized size of antiderivative = 1.61

$$\int \frac{\sin^2(e + fx)}{(a + b \sin(e + fx))^2 (c + d \sin(e + fx))} dx$$

$$= \frac{2 \left(\left(\pi \left\lfloor \frac{fx+e}{2\pi} + \frac{1}{2} \right\rfloor \operatorname{sgn}(c) + \arctan \left(\frac{c \tan \left(\frac{1}{2} fx + \frac{1}{2} e \right) + d}{\sqrt{c^2 - d^2}} \right) \right) c^2}{(b^2 c^2 - 2abcd + a^2 d^2) \sqrt{c^2 - d^2}} - \frac{(a^3 c - 2ab^2 c + a^2 b d) \left(\pi \left\lfloor \frac{fx+e}{2\pi} + \frac{1}{2} \right\rfloor \operatorname{sgn}(a) + \arctan \left(\frac{a \tan \left(\frac{1}{2} fx + \frac{1}{2} e \right) + b}{\sqrt{a^2 - b^2}} \right) \right)}{(a^2 b^2 c^2 - b^4 c^2 - 2a^3 bcd + 2ab^3 cd + a^4 d^2 - a^2 b^2 d^2) \sqrt{a^2 - b^2}}}{f}$$

input `integrate(sin(f*x+e)^2/(a+b*sin(f*x+e))^2/(c+d*sin(f*x+e)),x, algorithm="giac")`

output `2*((pi*floor(1/2*(f*x + e)/pi + 1/2)*sgn(c) + arctan((c*tan(1/2*f*x + 1/2*e) + d)/sqrt(c^2 - d^2)))*c^2/((b^2*c^2 - 2*a*b*c*d + a^2*d^2)*sqrt(c^2 - d^2)) - (a^3*c - 2*a*b^2*c + a^2*b*d)*(pi*floor(1/2*(f*x + e)/pi + 1/2)*sgn(a) + arctan((a*tan(1/2*f*x + 1/2*e) + b)/sqrt(a^2 - b^2)))/((a^2*b^2*c^2 - b^4*c^2 - 2*a^3*b*c*d + 2*a*b^3*c*d + a^4*d^2 - a^2*b^2*d^2)*sqrt(a^2 - b^2)) + (a*b*tan(1/2*f*x + 1/2*e) + a^2)/((a^2*b*c - b^3*c - a^3*d + a*b^2*d)*(a*tan(1/2*f*x + 1/2*e)^2 + 2*b*tan(1/2*f*x + 1/2*e) + a))/f`

Mupad [B] (verification not implemented)

Time = 51.16 (sec) , antiderivative size = 23933, normalized size of antiderivative = 132.23

$$\int \frac{\sin^2(e + fx)}{(a + b \sin(e + fx))^2 (c + d \sin(e + fx))} dx = \text{Too large to display}$$

input `int(sin(e + f*x)^2/((a + b*sin(e + f*x))^2*(c + d*sin(e + f*x))),x)`

output

```
- ((2*a^2)/((a^2 - b^2)*(a*d - b*c)) + (2*a*b*tan(e/2 + (f*x)/2))/((a^2 -
b^2)*(a*d - b*c)))/(f*(a + 2*b*tan(e/2 + (f*x)/2) + a*tan(e/2 + (f*x)/2)^2
)) - (c^2*atan(((c^2*(d^2 - c^2)^(1/2))*((32*(2*a^4*b^4*c^6 - a^2*b^6*c^6 -
a^6*b^2*c^6 + a^8*c^4*d^2 - 3*a^3*b^5*c^5*d + 2*a^5*b^3*c^5*d + 2*a^7*b*c
^3*d^3 + 8*a^4*b^4*c^4*d^2 - 5*a^5*b^3*c^3*d^3 + a^6*b^2*c^2*d^4 - 6*a^6*b
^2*c^4*d^2)))/(a^7*d^3 - b^7*c^3 + 2*a^2*b^5*c^3 - a^4*b^3*c^3 + a^3*b^4*d^
3 - 2*a^5*b^2*d^3 - 3*a^2*b^5*c*d^2 - 6*a^3*b^4*c^2*d + 6*a^4*b^3*c*d^2 +
3*a^5*b^2*c^2*d + 3*a*b^6*c^2*d - 3*a^6*b*c*d^2) + (32*tan(e/2 + (f*x)/2)*
(2*a^7*b*c^6 - 2*a*b^7*c^6 - 2*a^8*c^5*d + 9*a^3*b^5*c^6 - 8*a^5*b^3*c^6 +
2*a^8*c^3*d^3 + 2*a^2*b^6*c^5*d - 13*a^4*b^4*c^5*d + 2*a^6*b^2*c*d^5 + 10
*a^6*b^2*c^5*d + 4*a^7*b*c^2*d^4 - 4*a^7*b*c^4*d^2 - 8*a^3*b^5*c^4*d^2 + 1
6*a^4*b^4*c^3*d^3 - 10*a^5*b^3*c^2*d^4 + 13*a^5*b^3*c^4*d^2 - 13*a^6*b^2*c
^3*d^3)))/(a^7*d^3 - b^7*c^3 + 2*a^2*b^5*c^3 - a^4*b^3*c^3 + a^3*b^4*d^3 -
2*a^5*b^2*d^3 - 3*a^2*b^5*c*d^2 - 6*a^3*b^4*c^2*d + 6*a^4*b^3*c*d^2 + 3*a^
5*b^2*c^2*d + 3*a*b^6*c^2*d - 3*a^6*b*c*d^2) + (c^2*(d^2 - c^2)^(1/2))*((32
*(a*b^9*c^7 - a^3*b^7*c^7 + a^10*c^2*d^5 - 3*a^2*b^8*c^6*d + 2*a^4*b^6*c^6
*d + a^6*b^4*c^6*d - a^7*b^3*c*d^6 - 4*a^9*b*c^3*d^4 + a^3*b^7*c^5*d^2 + 6
*a^4*b^6*c^4*d^3 - 9*a^5*b^5*c^3*d^4 + 3*a^5*b^5*c^5*d^2 + 5*a^6*b^4*c^2*d
^5 - 12*a^6*b^4*c^4*d^3 + 13*a^7*b^3*c^3*d^4 - 4*a^7*b^3*c^5*d^2 - 6*a^8*b
^2*c^2*d^5 + 6*a^8*b^2*c^4*d^3 + a^9*b*c*d^6))/(a^7*d^3 - b^7*c^3 + 2*a...
```

Reduce [B] (verification not implemented)

Time = 0.21 (sec) , antiderivative size = 1480, normalized size of antiderivative = 8.18

$$\int \frac{\sin^2(e + fx)}{(a + b \sin(e + fx))^2 (c + d \sin(e + fx))} dx = \text{Too large to display}$$

input `int(sin(f*x+e)^2/(a+b*sin(f*x+e))^2/(c+d*sin(f*x+e)),x)`

output

```
( - 2*sqrt(a**2 - b**2)*atan((tan((e + f*x)/2)*a + b)/sqrt(a**2 - b**2))*sin(e + f*x)*a**3*b*c**3 + 2*sqrt(a**2 - b**2)*atan((tan((e + f*x)/2)*a + b)/sqrt(a**2 - b**2))*sin(e + f*x)*a**3*b*c*d**2 - 2*sqrt(a**2 - b**2)*atan((tan((e + f*x)/2)*a + b)/sqrt(a**2 - b**2))*sin(e + f*x)*a**2*b**2*c**2*d + 2*sqrt(a**2 - b**2)*atan((tan((e + f*x)/2)*a + b)/sqrt(a**2 - b**2))*sin(e + f*x)*a**2*b**2*d**3 + 4*sqrt(a**2 - b**2)*atan((tan((e + f*x)/2)*a + b)/sqrt(a**2 - b**2))*sin(e + f*x)*a*b**3*c**3 - 4*sqrt(a**2 - b**2)*atan((tan((e + f*x)/2)*a + b)/sqrt(a**2 - b**2))*sin(e + f*x)*a*b**3*c*d**2 - 2*sqrt(a**2 - b**2)*atan((tan((e + f*x)/2)*a + b)/sqrt(a**2 - b**2))*a**4*c**3 + 2*sqrt(a**2 - b**2)*atan((tan((e + f*x)/2)*a + b)/sqrt(a**2 - b**2))*a**4*c*d**2 - 2*sqrt(a**2 - b**2)*atan((tan((e + f*x)/2)*a + b)/sqrt(a**2 - b**2))*a**3*b*c**2*d + 2*sqrt(a**2 - b**2)*atan((tan((e + f*x)/2)*a + b)/sqrt(a**2 - b**2))*a**3*b*d**3 + 4*sqrt(a**2 - b**2)*atan((tan((e + f*x)/2)*a + b)/sqrt(a**2 - b**2))*a**2*b**2*c**3 - 4*sqrt(a**2 - b**2)*atan((tan((e + f*x)/2)*a + b)/sqrt(a**2 - b**2))*a**2*b**2*c*d**2 + 2*sqrt(c**2 - d**2)*atan((tan((e + f*x)/2)*c + d)/sqrt(c**2 - d**2))*sin(e + f*x)*a**4*b*c**2 - 4*sqrt(c**2 - d**2)*atan((tan((e + f*x)/2)*c + d)/sqrt(c**2 - d**2))*sin(e + f*x)*a**2*b**3*c**2 + 2*sqrt(c**2 - d**2)*atan((tan((e + f*x)/2)*c + d)/sqrt(c**2 - d**2))*sin(e + f*x)*b**5*c**2 + 2*sqrt(c**2 - d**2)*atan((tan((e + f*x)/2)*c + d)/sqrt(c**2 - d**2))*a**5*c**2 - 4*sqrt(c...
```

3.40
$$\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{a+b\sin(e+fx)} dx$$

Optimal result	350
Mathematica [C] (verified)	351
Rubi [A] (verified)	351
Maple [A] (verified)	353
Fricas [F(-1)]	354
Sympy [F]	354
Maxima [F]	355
Giac [F]	355
Mupad [F(-1)]	355
Reduce [F]	356

Optimal result

Integrand size = 33, antiderivative size = 154

$$\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{a+b\sin(e+fx)} dx$$

$$= \frac{2c \operatorname{EllipticPi}\left(2, \frac{1}{2}\left(e - \frac{\pi}{2} + fx\right), \frac{2d}{c+d}\right) \sqrt{\frac{c+d\sin(e+fx)}{c+d}}}{af\sqrt{c+d\sin(e+fx)}}$$

$$- \frac{2(bc-ad) \operatorname{EllipticPi}\left(\frac{2b}{a+b}, \frac{1}{2}\left(e - \frac{\pi}{2} + fx\right), \frac{2d}{c+d}\right) \sqrt{\frac{c+d\sin(e+fx)}{c+d}}}{a(a+b)f\sqrt{c+d\sin(e+fx)}}$$

output

```
-2*c*EllipticPi(cos(1/2*e+1/4*Pi+1/2*f*x), 2, 2^(1/2)*(d/(c+d))^(1/2))*((c+d
*sin(f*x+e))/(c+d))^(1/2)/a/f/(c+d*sin(f*x+e))^(1/2)+2*(-a*d+b*c)*Elliptic
Pi(cos(1/2*e+1/4*Pi+1/2*f*x), 2*b/(a+b), 2^(1/2)*(d/(c+d))^(1/2))*((c+d*sin(
f*x+e))/(c+d))^(1/2)/a/(a+b)/f/(c+d*sin(f*x+e))^(1/2)
```

Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 21.61 (sec) , antiderivative size = 179, normalized size of antiderivative = 1.16

$$\int \frac{\csc(e + fx)\sqrt{c + d\sin(e + fx)}}{a + b\sin(e + fx)} dx$$

$$= \frac{2i \left(\text{EllipticPi} \left(\frac{c+d}{c}, \text{iarcsinh} \left(\sqrt{-\frac{1}{c+d}} \sqrt{c + d\sin(e + fx)} \right), \frac{c+d}{c-d} \right) - \text{EllipticPi} \left(\frac{b(c+d)}{bc-ad}, \text{iarcsinh} \left(\sqrt{-\frac{1}{c+d}} \sqrt{c + d\sin(e + fx)} \right) \right) \right)}{a\sqrt{-\frac{1}{c+d}}f}$$

input

```
Integrate[(Csc[e + f*x]*Sqrt[c + d*Sin[e + f*x]])/(a + b*Sin[e + f*x]),x]
```

output

```
((2*I)*(EllipticPi[(c + d)/c, I*ArcSinh[Sqrt[-(c + d)^(-1)]*Sqrt[c + d*Sin[e + f*x]]], (c + d)/(c - d)] - EllipticPi[(b*(c + d))/(b*c - a*d), I*ArcSinh[Sqrt[-(c + d)^(-1)]*Sqrt[c + d*Sin[e + f*x]]], (c + d)/(c - d)])*Sec[e + f*x]*Sqrt[-((d*(-1 + Sin[e + f*x]))/(c + d))]*Sqrt[(d*(1 + Sin[e + f*x]))/(-c + d)]/(a*Sqrt[-(c + d)^(-1)]*f)
```

Rubi [A] (verified)

Time = 0.86 (sec) , antiderivative size = 154, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}} = 0.182$, Rules used = {3042, 3414, 3042, 3286, 3042, 3284}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e + fx)\sqrt{c + d\sin(e + fx)}}{a + b\sin(e + fx)} dx$$

$$\downarrow \text{3042}$$

$$\int \frac{\sqrt{c + d\sin(e + fx)}}{\sin(e + fx)(a + b\sin(e + fx))} dx$$

$$\downarrow \text{3414}$$

$$\begin{aligned}
& \frac{c \int \frac{\csc(e+fx)}{\sqrt{c+d \sin(e+fx)}} dx}{a} - \frac{(bc-ad) \int \frac{1}{(a+b \sin(e+fx))\sqrt{c+d \sin(e+fx)}} dx}{a} \\
& \quad \downarrow 3042 \\
& \frac{c \int \frac{1}{\sin(e+fx)\sqrt{c+d \sin(e+fx)}} dx}{a} - \frac{(bc-ad) \int \frac{1}{(a+b \sin(e+fx))\sqrt{c+d \sin(e+fx)}} dx}{a} \\
& \quad \downarrow 3286 \\
& \frac{c \sqrt{\frac{c+d \sin(e+fx)}{c+d}} \int \frac{\csc(e+fx)}{\sqrt{\frac{c}{c+d} + \frac{d \sin(e+fx)}{c+d}}} dx}{a \sqrt{c+d \sin(e+fx)}} - \\
& \frac{(bc-ad) \sqrt{\frac{c+d \sin(e+fx)}{c+d}} \int \frac{1}{(a+b \sin(e+fx))\sqrt{\frac{c}{c+d} + \frac{d \sin(e+fx)}{c+d}}} dx}{a \sqrt{c+d \sin(e+fx)}} \\
& \quad \downarrow 3042 \\
& \frac{c \sqrt{\frac{c+d \sin(e+fx)}{c+d}} \int \frac{1}{\sin(e+fx)\sqrt{\frac{c}{c+d} + \frac{d \sin(e+fx)}{c+d}}} dx}{a \sqrt{c+d \sin(e+fx)}} - \\
& \frac{(bc-ad) \sqrt{\frac{c+d \sin(e+fx)}{c+d}} \int \frac{1}{(a+b \sin(e+fx))\sqrt{\frac{c}{c+d} + \frac{d \sin(e+fx)}{c+d}}} dx}{a \sqrt{c+d \sin(e+fx)}} \\
& \quad \downarrow 3284 \\
& \frac{2c \sqrt{\frac{c+d \sin(e+fx)}{c+d}} \operatorname{EllipticPi}\left(2, \frac{1}{2}(e+fx - \frac{\pi}{2}), \frac{2d}{c+d}\right)}{af \sqrt{c+d \sin(e+fx)}} - \\
& \frac{2(bc-ad) \sqrt{\frac{c+d \sin(e+fx)}{c+d}} \operatorname{EllipticPi}\left(\frac{2b}{a+b}, \frac{1}{2}(e+fx - \frac{\pi}{2}), \frac{2d}{c+d}\right)}{af(a+b) \sqrt{c+d \sin(e+fx)}}
\end{aligned}$$

input `Int[(Csc[e + f*x]*Sqrt[c + d*Sin[e + f*x]])/(a + b*Sin[e + f*x]),x]`

output `(2*c*EllipticPi[2, (e - Pi/2 + f*x)/2, (2*d)/(c + d)]*Sqrt[(c + d*Sin[e + f*x])/(c + d)]/(a*f*Sqrt[c + d*Sin[e + f*x]]) - (2*(b*c - a*d)*EllipticPi[(2*b)/(a + b), (e - Pi/2 + f*x)/2, (2*d)/(c + d)]*Sqrt[(c + d*Sin[e + f*x])/(c + d)]/(a*(a + b)*f*Sqrt[c + d*Sin[e + f*x]])`

Definitions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3284 `Int[1/(((a_.) + (b_.)*sin[(e_.) + (f_.)*(x_)])*Sqrt[(c_.) + (d_.)*sin[(e_.) + (f_.)*(x_)]]), x_Symbol] := Simp[(2/(f*(a + b)*Sqrt[c + d]))*EllipticPi[2*(b/(a + b)), (1/2)*(e - Pi/2 + f*x), 2*(d/(c + d))], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0] && GtQ[c + d, 0]`

rule 3286 `Int[1/(((a_.) + (b_.)*sin[(e_.) + (f_.)*(x_)])*Sqrt[(c_.) + (d_.)*sin[(e_.) + (f_.)*(x_)]]), x_Symbol] := Simp[Sqrt[(c + d*Sin[e + f*x])]/(c + d)]/Sqrt[c + d*Sin[e + f*x]] Int[1/((a + b*Sin[e + f*x])*Sqrt[c/(c + d) + (d/(c + d))*Sin[e + f*x]]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0] && !GtQ[c + d, 0]`

rule 3414 `Int[Sqrt[(a_.) + (b_.)*sin[(e_.) + (f_.)*(x_)]]/(sin[(e_.) + (f_.)*(x_)]*((c_.) + (d_.)*sin[(e_.) + (f_.)*(x_)]]), x_Symbol] := Simp[a/c Int[1/(Sin[e + f*x]*Sqrt[a + b*Sin[e + f*x]]), x], x] + Simp[(b*c - a*d)/c Int[1/(Sqrt[a + b*Sin[e + f*x]]*(c + d*Sin[e + f*x])), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0]`

Maple [A] (verified)

Time = 0.88 (sec) , antiderivative size = 190, normalized size of antiderivative = 1.23

method	result
default	$-\frac{2\left(\operatorname{EllipticPi}\left(\sqrt{\frac{c+d\sin(fx+e)}{c-d}}, \frac{c-d}{c}, \sqrt{\frac{c-d}{c+d}}\right) - \operatorname{EllipticPi}\left(\sqrt{\frac{c+d\sin(fx+e)}{c-d}}, -\frac{(c-d)b}{ad-bc}, \sqrt{\frac{c-d}{c+d}}\right)\right) \sqrt{-\frac{d(1+\sin(fx+e))}{c-d}} \sqrt{-\frac{(\sin(fx+e))}{c+d}}}{a \cos(fx+e) \sqrt{c+d\sin(fx+e)} f}$

input `int(csc(f*x+e)*(c+d*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e)),x,method=_RETURNVERBOSE)`

output

```
-2*(EllipticPi(((c+d*sin(f*x+e))/(c-d))^(1/2), (c-d)/c, ((c-d)/(c+d))^(1/2))
-EllipticPi(((c+d*sin(f*x+e))/(c-d))^(1/2), -(c-d)*b/(a*d-b*c), ((c-d)/(c+d)
)^(1/2)))*(-d*(1+sin(f*x+e))/(c-d))^(1/2)*(-(sin(f*x+e)-1)*d/(c+d))^(1/2)*
((c+d*sin(f*x+e))/(c-d))^(1/2)*(c-d)/a/cos(f*x+e)/(c+d*sin(f*x+e))^(1/2)/f
```

Fricas [F(-1)]

Timed out.

$$\int \frac{\csc(e + fx) \sqrt{c + d \sin(e + fx)}}{a + b \sin(e + fx)} dx = \text{Timed out}$$

input

```
integrate(csc(f*x+e)*(c+d*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e)),x, algorithm=
"fricas")
```

output

Timed out

Sympy [F]

$$\int \frac{\csc(e + fx) \sqrt{c + d \sin(e + fx)}}{a + b \sin(e + fx)} dx = \int \frac{\sqrt{c + d \sin(e + fx)} \csc(e + fx)}{a + b \sin(e + fx)} dx$$

input

```
integrate(csc(f*x+e)*(c+d*sin(f*x+e))**(1/2)/(a+b*sin(f*x+e)),x)
```

output

```
Integral(sqrt(c + d*sin(e + f*x))*csc(e + f*x)/(a + b*sin(e + f*x)), x)
```

Maxima [F]

$$\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{a+b\sin(e+fx)} dx = \int \frac{\sqrt{d\sin(fx+e)+c}\csc(fx+e)}{b\sin(fx+e)+a} dx$$

input `integrate(csc(f*x+e)*(c+d*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e)),x, algorithm="maxima")`

output `integrate(sqrt(d*sin(f*x + e) + c)*csc(f*x + e)/(b*sin(f*x + e) + a), x)`

Giac [F]

$$\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{a+b\sin(e+fx)} dx = \int \frac{\sqrt{d\sin(fx+e)+c}\csc(fx+e)}{b\sin(fx+e)+a} dx$$

input `integrate(csc(f*x+e)*(c+d*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e)),x, algorithm="giac")`

output `integrate(sqrt(d*sin(f*x + e) + c)*csc(f*x + e)/(b*sin(f*x + e) + a), x)`

Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)\sqrt{c+d\sin(e+fx)}}{a+b\sin(e+fx)} dx = \int \frac{\sqrt{c+d\sin(e+fx)}}{\sin(e+fx)(a+b\sin(e+fx))} dx$$

input `int((c + d*sin(e + f*x))^(1/2)/(sin(e + f*x)*(a + b*sin(e + f*x))),x)`

output `int((c + d*sin(e + f*x))^(1/2)/(sin(e + f*x)*(a + b*sin(e + f*x))), x)`

Reduce [F]

$$\int \frac{\csc(e + fx)\sqrt{c + d\sin(e + fx)}}{a + b\sin(e + fx)} dx = \int \frac{\sqrt{\sin(fx + e)d + c} \csc(fx + e)}{\sin(fx + e)b + a} dx$$

input `int(csc(f*x+e)*(c+d*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e)),x)`

output `int((sqrt(sin(e + f*x)*d + c)*csc(e + f*x))/(sin(e + f*x)*b + a),x)`

3.41
$$\int \frac{\csc(e+fx)}{(a+b \sin(e+fx))\sqrt{c+d \sin(e+fx)}} dx$$

Optimal result	357
Mathematica [C] (verified)	358
Rubi [A] (verified)	358
Maple [A] (verified)	360
Fricas [F(-1)]	361
Sympy [F]	361
Maxima [F]	362
Giac [F]	362
Mupad [F(-1)]	362
Reduce [F]	363

Optimal result

Integrand size = 33, antiderivative size = 146

$$\int \frac{\csc(e+fx)}{(a+b \sin(e+fx))\sqrt{c+d \sin(e+fx)}} dx$$

$$= \frac{2 \operatorname{EllipticPi}\left(2, \frac{1}{2}\left(e - \frac{\pi}{2} + fx\right), \frac{2d}{c+d}\right) \sqrt{\frac{c+d \sin(e+fx)}{c+d}}}{af \sqrt{c+d \sin(e+fx)}}$$

$$- \frac{2b \operatorname{EllipticPi}\left(\frac{2b}{a+b}, \frac{1}{2}\left(e - \frac{\pi}{2} + fx\right), \frac{2d}{c+d}\right) \sqrt{\frac{c+d \sin(e+fx)}{c+d}}}{a(a+b)f \sqrt{c+d \sin(e+fx)}}$$

output

```
-2*EllipticPi(cos(1/2*e+1/4*Pi+1/2*f*x),2,2^(1/2)*(d/(c+d))^(1/2))*((c+d*
sin(f*x+e))/(c+d))^(1/2)/a/f/(c+d*sin(f*x+e))^(1/2)+2*b*EllipticPi(cos(1/2*
e+1/4*Pi+1/2*f*x),2*b/(a+b),2^(1/2)*(d/(c+d))^(1/2))*((c+d*sin(f*x+e))/(c+
d))^(1/2)/a/(a+b)/f/(c+d*sin(f*x+e))^(1/2)
```

Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 23.03 (sec) , antiderivative size = 203, normalized size of antiderivative = 1.39

$$\int \frac{\csc(e + fx)}{(a + b \sin(e + fx))\sqrt{c + d \sin(e + fx)}} dx =$$

$$\frac{2i \left((-bc + ad) \operatorname{EllipticPi} \left(\frac{c+d}{c}, \operatorname{I} \operatorname{ArcSinh} \left(\sqrt{-\frac{1}{c+d}} \sqrt{c + d \sin(e + fx)} \right), \frac{c+d}{c-d} \right) + bc \operatorname{EllipticPi} \left(\frac{b(c+d)}{bc-ad}, \operatorname{I} \operatorname{ArcSinh} \left(\sqrt{-\frac{1}{c+d}} \sqrt{c + d \sin(e + fx)} \right), \frac{c+d}{c-d} \right) \right)}{ac \sqrt{-\frac{1}{c+d}} (bc - ad)}$$

input `Integrate[Csc[e + f*x]/((a + b*Sin[e + f*x])*Sqrt[c + d*Sin[e + f*x]]),x]`

output `((-2*I)*((-b*c) + a*d)*EllipticPi[(c + d)/c, I*ArcSinh[Sqrt[-(c + d)^(-1)]*Sqrt[c + d*Sin[e + f*x]]], (c + d)/(c - d)] + b*c*EllipticPi[(b*(c + d))/(b*c - a*d), I*ArcSinh[Sqrt[-(c + d)^(-1)]*Sqrt[c + d*Sin[e + f*x]]], (c + d)/(c - d)]*Sec[e + f*x]*Sqrt[-((d*(-1 + Sin[e + f*x]))/(c + d))*Sqrt[-((d*(1 + Sin[e + f*x]))/(c - d))]/(a*c*Sqrt[-(c + d)^(-1)]*(b*c - a*d)*f)`

Rubi [A] (verified)

Time = 0.85 (sec) , antiderivative size = 146, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}} = 0.182$, Rules used = {3042, 3420, 3042, 3286, 3042, 3284}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e + fx)}{(a + b \sin(e + fx))\sqrt{c + d \sin(e + fx)}} dx$$

$$\downarrow \text{3042}$$

$$\int \frac{1}{\sin(e + fx)(a + b \sin(e + fx))\sqrt{c + d \sin(e + fx)}} dx$$

$$\downarrow \text{3420}$$

$$\begin{aligned}
& \frac{\int \frac{\csc(e+fx)}{\sqrt{c+d\sin(e+fx)}} dx}{a} - \frac{b \int \frac{1}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}} dx}{a} \\
& \quad \downarrow 3042 \\
& \frac{\int \frac{1}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}} dx}{a} - \frac{b \int \frac{1}{(a+b\sin(e+fx))\sqrt{c+d\sin(e+fx)}} dx}{a} \\
& \quad \downarrow 3286 \\
& \frac{\sqrt{\frac{c+d\sin(e+fx)}{c+d}} \int \frac{\csc(e+fx)}{\sqrt{\frac{c}{c+d} + \frac{d\sin(e+fx)}{c+d}}} dx}{a\sqrt{c+d\sin(e+fx)}} - \frac{b\sqrt{\frac{c+d\sin(e+fx)}{c+d}} \int \frac{1}{(a+b\sin(e+fx))\sqrt{\frac{c}{c+d} + \frac{d\sin(e+fx)}{c+d}}} dx}{a\sqrt{c+d\sin(e+fx)}} \\
& \quad \downarrow 3042 \\
& \frac{\sqrt{\frac{c+d\sin(e+fx)}{c+d}} \int \frac{1}{\sin(e+fx)\sqrt{\frac{c}{c+d} + \frac{d\sin(e+fx)}{c+d}}} dx}{a\sqrt{c+d\sin(e+fx)}} - \frac{b\sqrt{\frac{c+d\sin(e+fx)}{c+d}} \int \frac{1}{(a+b\sin(e+fx))\sqrt{\frac{c}{c+d} + \frac{d\sin(e+fx)}{c+d}}} dx}{a\sqrt{c+d\sin(e+fx)}} \\
& \quad \downarrow 3284 \\
& \frac{2\sqrt{\frac{c+d\sin(e+fx)}{c+d}} \operatorname{EllipticPi}\left(2, \frac{1}{2}(e+fx - \frac{\pi}{2}), \frac{2d}{c+d}\right)}{af\sqrt{c+d\sin(e+fx)}} - \\
& \frac{2b\sqrt{\frac{c+d\sin(e+fx)}{c+d}} \operatorname{EllipticPi}\left(\frac{2b}{a+b}, \frac{1}{2}(e+fx - \frac{\pi}{2}), \frac{2d}{c+d}\right)}{af(a+b)\sqrt{c+d\sin(e+fx)}}
\end{aligned}$$

input `Int[Csc[e + f*x]/((a + b*Sin[e + f*x])*Sqrt[c + d*Sin[e + f*x]]),x]`

output `(2*EllipticPi[2, (e - Pi/2 + f*x)/2, (2*d)/(c + d)]*Sqrt[(c + d*Sin[e + f*x])/(c + d)]/(a*f*Sqrt[c + d*Sin[e + f*x]]) - (2*b*EllipticPi[(2*b)/(a + b), (e - Pi/2 + f*x)/2, (2*d)/(c + d)]*Sqrt[(c + d*Sin[e + f*x])/(c + d)]/(a*(a + b)*f*Sqrt[c + d*Sin[e + f*x]])`

Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3284 `Int[1/(((a_.) + (b_.)*sin[(e_.) + (f_.)*(x_)])*Sqrt[(c_.) + (d_.)*sin[(e_.) + (f_.)*(x_)]]), x_Symbol] := Simp[(2/(f*(a + b)*Sqrt[c + d]))*EllipticPi[2*(b/(a + b)), (1/2)*(e - Pi/2 + f*x), 2*(d/(c + d))], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0] && GtQ[c + d, 0]`

rule 3286 `Int[1/(((a_.) + (b_.)*sin[(e_.) + (f_.)*(x_)])*Sqrt[(c_.) + (d_.)*sin[(e_.) + (f_.)*(x_)]]), x_Symbol] := Simp[Sqrt[(c + d*Sin[e + f*x])]/(c + d)]/Sqrt[c + d*Sin[e + f*x]] Int[1/((a + b*Sin[e + f*x])*Sqrt[c/(c + d) + (d/(c + d))*Sin[e + f*x]]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0] && !GtQ[c + d, 0]`

rule 3420 `Int[1/(sin[(e_.) + (f_.)*(x_)])*Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.)*(x_)]]*(c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] := Simp[1/c Int[1/(Sin[e + f*x]*Sqrt[a + b*Sin[e + f*x]]), x], x] - Simp[d/c Int[1/(Sqrt[a + b*Sin[e + f*x]]*(c + d*Sin[e + f*x])), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0]`

Maple [A] (verified)

Time = 0.71 (sec) , antiderivative size = 254, normalized size of antiderivative = 1.74

method	result
default	$-\frac{2(c-d)\sqrt{\frac{c+d\sin(fx+e)}{c-d}}\sqrt{-\frac{(\sin(fx+e)-1)d}{c+d}}\sqrt{-\frac{d(1+\sin(fx+e))}{c-d}}}{ac(ad-bc)\cos(fx+e)\sqrt{c+d\sin(fx+e)}}\left(\text{EllipticPi}\left(\sqrt{\frac{c+d\sin(fx+e)}{c-d}}, \frac{c-d}{c}, \sqrt{\frac{c-d}{c+d}}\right)ad - \text{EllipticPi}\left(\sqrt{\frac{c+d\sin(fx+e)}{c-d}}, \frac{c-d}{c}, \sqrt{\frac{c-d}{c+d}}\right)\right)$

input `int(csc(f*x+e)/(a+b*sin(f*x+e))/(c+d*sin(f*x+e))^(1/2),x,method=_RETURNVERBOSE)`

output

```
-2*(c-d)*((c+d*sin(f*x+e))/(c-d))^(1/2)*(-(sin(f*x+e)-1)*d/(c+d))^(1/2)*(-
d*(1+sin(f*x+e))/(c-d))^(1/2)*(EllipticPi(((c+d*sin(f*x+e))/(c-d))^(1/2),(
c-d)/c,((c-d)/(c+d))^(1/2))*a*d-EllipticPi(((c+d*sin(f*x+e))/(c-d))^(1/2),
(c-d)/c,((c-d)/(c+d))^(1/2))*b*c+b*EllipticPi(((c+d*sin(f*x+e))/(c-d))^(1/
2),-(c-d)*b/(a*d-b*c),((c-d)/(c+d))^(1/2))*c)/a/c/(a*d-b*c)/cos(f*x+e)/(c+
d*sin(f*x+e))^(1/2)/f
```

Fricas [F(-1)]

Timed out.

$$\int \frac{\csc(e + fx)}{(a + b \sin(e + fx)) \sqrt{c + d \sin(e + fx)}} dx = \text{Timed out}$$

input

```
integrate(csc(f*x+e)/(a+b*sin(f*x+e))/(c+d*sin(f*x+e))^(1/2),x, algorithm=
"fricas")
```

output

Timed out

Sympy [F]

$$\begin{aligned} & \int \frac{\csc(e + fx)}{(a + b \sin(e + fx)) \sqrt{c + d \sin(e + fx)}} dx \\ &= \int \frac{\csc(e + fx)}{(a + b \sin(e + fx)) \sqrt{c + d \sin(e + fx)}} dx \end{aligned}$$

input

```
integrate(csc(f*x+e)/(a+b*sin(f*x+e))/(c+d*sin(f*x+e))**(1/2),x)
```

output

```
Integral(csc(e + f*x)/((a + b*sin(e + f*x))*sqrt(c + d*sin(e + f*x))), x)
```

Maxima [F]

$$\int \frac{\csc(e + fx)}{(a + b \sin(e + fx))\sqrt{c + d \sin(e + fx)}} dx = \int \frac{\csc(fx + e)}{(b \sin(fx + e) + a)\sqrt{d \sin(fx + e) + c}} dx$$

input `integrate(csc(f*x+e)/(a+b*sin(f*x+e))/(c+d*sin(f*x+e))^(1/2),x, algorithm="maxima")`

output `integrate(csc(f*x + e)/((b*sin(f*x + e) + a)*sqrt(d*sin(f*x + e) + c)), x)`

Giac [F]

$$\int \frac{\csc(e + fx)}{(a + b \sin(e + fx))\sqrt{c + d \sin(e + fx)}} dx = \int \frac{\csc(fx + e)}{(b \sin(fx + e) + a)\sqrt{d \sin(fx + e) + c}} dx$$

input `integrate(csc(f*x+e)/(a+b*sin(f*x+e))/(c+d*sin(f*x+e))^(1/2),x, algorithm="giac")`

output `integrate(csc(f*x + e)/((b*sin(f*x + e) + a)*sqrt(d*sin(f*x + e) + c)), x)`

Mupad [F(-1)]

Timed out.

$$\begin{aligned} & \int \frac{\csc(e + fx)}{(a + b \sin(e + fx))\sqrt{c + d \sin(e + fx)}} dx \\ &= \int \frac{1}{\sin(e + fx) (a + b \sin(e + fx)) \sqrt{c + d \sin(e + fx)}} dx \end{aligned}$$

input `int(1/(sin(e + f*x)*(a + b*sin(e + f*x))*(c + d*sin(e + f*x))^(1/2)),x)`

output `int(1/(sin(e + f*x)*(a + b*sin(e + f*x))*(c + d*sin(e + f*x))^(1/2)), x)`

Reduce [F]

$$\int \frac{\csc(e + fx)}{(a + b \sin(e + fx))\sqrt{c + d \sin(e + fx)}} dx$$

$$= \int \frac{\sqrt{\sin(fx + e)d + c} \csc(fx + e)}{\sin(fx + e)^2 bd + \sin(fx + e) ad + \sin(fx + e) bc + ac} dx$$

input `int(csc(f*x+e)/(a+b*sin(f*x+e))/(c+d*sin(f*x+e))^(1/2),x)`

output `int((sqrt(sin(e + f*x)*d + c)*csc(e + f*x))/(sin(e + f*x)**2*b*d + sin(e + f*x)*a*d + sin(e + f*x)*b*c + a*c),x)`

$$3.42 \quad \int \frac{\sqrt{g \sin(e+fx)} \sqrt{a+b \sin(e+fx)}}{c+d \sin(e+fx)} dx$$

Optimal result	364
Mathematica [F]	365
Rubi [A] (verified)	365
Maple [C] (warning: unable to verify)	367
Fricas [F(-1)]	368
Sympy [F]	368
Maxima [F]	368
Giac [F]	369
Mupad [F(-1)]	369
Reduce [F]	369

Optimal result

Integrand size = 39, antiderivative size = 254

$$\int \frac{\sqrt{g \sin(e+fx)} \sqrt{a+b \sin(e+fx)}}{c+d \sin(e+fx)} dx$$

$$= \frac{2\sqrt{a+b} \sqrt{g} \sqrt{\frac{a(1-\csc(e+fx))}{a+b}} \sqrt{\frac{a(1+\csc(e+fx))}{a-b}} \operatorname{EllipticPi}\left(\frac{a+b}{b}, \arcsin\left(\frac{\sqrt{g} \sqrt{a+b \sin(e+fx)}}{\sqrt{a+b} \sqrt{g \sin(e+fx)}}\right), -\frac{a+b}{a-b}\right) \tan(e+fx)}{df}$$

$$- \frac{2(bc-ad) \sqrt{-\cot^2(e+fx)} \sqrt{\frac{b+a \csc(e+fx)}{a+b}} \operatorname{EllipticPi}\left(\frac{2c}{c+d}, \arcsin\left(\frac{\sqrt{1-\csc(e+fx)}}{\sqrt{2}}\right), \frac{2a}{a+b}\right) \sqrt{g \sin(e+fx)}}{d(c+d)f \sqrt{a+b \sin(e+fx)}}$$

output

```
2*(a+b)^(1/2)*g^(1/2)*(a*(1-csc(f*x+e))/(a+b))^(1/2)*(a*(1+csc(f*x+e))/(a-b))^(1/2)*EllipticPi(g^(1/2)*(a+b*sin(f*x+e))^(1/2)/(a+b)^(1/2)/(g*sin(f*x+e))^(1/2), (a+b)/b, (-(a+b)/(a-b))^(1/2))*tan(f*x+e)/d/f-2*(-a*d+b*c)*(-cot(f*x+e)^2)^(1/2)*((b+a*csc(f*x+e))/(a+b))^(1/2)*EllipticPi(1/2*(1-csc(f*x+e))^(1/2)*2^(1/2), 2*c/(c+d), 2^(1/2)*(a/(a+b))^(1/2))*(g*sin(f*x+e))^(1/2)*tan(f*x+e)/d/(c+d)/f/(a+b*sin(f*x+e))^(1/2)
```

Mathematica [F]

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}}{c + d \sin(e + fx)} dx = \int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}}{c + d \sin(e + fx)} dx$$

input

```
Integrate[(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]/(c + d*Sin[e + f*x]), x]
```

output

```
Integrate[(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]/(c + d*Sin[e + f*x]), x]
```

Rubi [A] (verified)

Time = 0.95 (sec) , antiderivative size = 254, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.128$, Rules used = {3042, 3408, 3042, 3288, 3416}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}}{c + d \sin(e + fx)} dx \\ & \quad \downarrow \text{3042} \\ & \int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}}{c + d \sin(e + fx)} dx \\ & \quad \downarrow \text{3408} \\ & \frac{b \int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)}} dx}{d} - \frac{(bc - ad) \int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)}(c + d \sin(e + fx))} dx}{d} \\ & \quad \downarrow \text{3042} \\ & \frac{b \int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)}} dx}{d} - \frac{(bc - ad) \int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)}(c + d \sin(e + fx))} dx}{d} \\ & \quad \downarrow \text{3288} \end{aligned}$$

$$\frac{2\sqrt{g}\sqrt{a+b}\tan(e+fx)\sqrt{\frac{a(1-\csc(e+fx))}{a+b}}\sqrt{\frac{a(\csc(e+fx)+1)}{a-b}}\text{EllipticPi}\left(\frac{a+b}{b}, \arcsin\left(\frac{\sqrt{g}\sqrt{a+b}\sin(e+fx)}{\sqrt{a+b}\sqrt{g\sin(e+fx)}}\right), -\frac{a+b}{a-b}\right)}{(bc-ad)\int\frac{df}{\sqrt{a+b\sin(e+fx)}(c+d\sin(e+fx))}dx}$$

↓ 3416

$$\frac{2\sqrt{g}\sqrt{a+b}\tan(e+fx)\sqrt{\frac{a(1-\csc(e+fx))}{a+b}}\sqrt{\frac{a(\csc(e+fx)+1)}{a-b}}\text{EllipticPi}\left(\frac{a+b}{b}, \arcsin\left(\frac{\sqrt{g}\sqrt{a+b}\sin(e+fx)}{\sqrt{a+b}\sqrt{g\sin(e+fx)}}\right), -\frac{a+b}{a-b}\right)}{2(bc-ad)\tan(e+fx)\sqrt{-\cot^2(e+fx)}\sqrt{g\sin(e+fx)}\sqrt{\frac{a\csc(e+fx)+b}{a+b}}\text{EllipticPi}\left(\frac{2c}{c+d}, \arcsin\left(\frac{\sqrt{1-\csc(e+fx)}}{\sqrt{2}}\right), \frac{2}{a-b}\right)}{df(c+d)\sqrt{a+b\sin(e+fx)}}$$

input `Int[(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]])/(c + d*Sin[e + f*x]),x]`

output `(2*Sqrt[a + b]*Sqrt[g]*Sqrt[(a*(1 - Csc[e + f*x]))/(a + b)]*Sqrt[(a*(1 + Csc[e + f*x]))/(a - b)]*EllipticPi[(a + b)/b, ArcSin[(Sqrt[g]*Sqrt[a + b*Sin[e + f*x]])/(Sqrt[a + b]*Sqrt[g*Sin[e + f*x]])], -((a + b)/(a - b))]*Tan[e + f*x]/(d*f) - (2*(b*c - a*d)*Sqrt[-Cot[e + f*x]^2]*Sqrt[(b + a*Csc[e + f*x])/(a + b)]*EllipticPi[(2*c)/(c + d), ArcSin[Sqrt[1 - Csc[e + f*x]]/Sqrt[2]], (2*a)/(a + b)]*Sqrt[g*Sin[e + f*x]]*Tan[e + f*x]/(d*(c + d)*f*Sqrt[a + b*Sin[e + f*x]])`

Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3288 `Int[Sqrt[(b_.)*sin[(e_.) + (f_.)*(x_)]]/Sqrt[(c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]], x_Symbol] := Simp[2*b*(Tan[e + f*x]/(d*f))*Rt[(c + d)/b, 2]*Sqrt[c*((1 + Csc[e + f*x])/(c - d))]*Sqrt[c*((1 - Csc[e + f*x])/(c + d))]*EllipticPi[(c + d)/d, ArcSin[Sqrt[c + d*Sin[e + f*x]]/Sqrt[b*Sin[e + f*x]]/Rt[(c + d)/b, 2]], -(c + d)/(c - d), x] /; FreeQ[{b, c, d, e, f}, x] && NeQ[c^2 - d^2, 0] && PosQ[(c + d)/b]`

rule 3408

```
Int[(Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)])*Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.
)*(x_)])]/((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] :> Simp[b/d I
nt[Sqrt[g*Sin[e + f*x]]/Sqrt[a + b*Sin[e + f*x]], x], x] - Simp[(b*c - a*d)
/d Int[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + b*Sin[e + f*x]]*(c + d*Sin[e + f*x]
)), x], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && NeQ[a
^2 - b^2, 0] && NeQ[c^2 - d^2, 0]
```

rule 3416

```
Int[Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.
)*(x_)])*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[2*Sqrt[
-Cot[e + f*x]^2]*(Sqrt[g*Sin[e + f*x]]/(f*(c + d)*Cot[e + f*x]*Sqrt[a + b*S
in[e + f*x]]))*Sqrt[(b + a*Csc[e + f*x])/(a + b)]*EllipticPi[2*(c/(c + d)),
ArcSin[Sqrt[1 - Csc[e + f*x]]/Sqrt[2]], 2*(a/(a + b))], x] /; FreeQ[{a, b,
c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 -
d^2, 0]
```

Maple [C] (warning: unable to verify)

Result contains complex when optimal does not.

Time = 2.37 (sec) , antiderivative size = 5439, normalized size of antiderivative = 21.41

method	result	size
default	Expression too large to display	5439

input

```
int((g*sin(f*x+e))^(1/2)*(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,method=
_RETURNVERBOSE)
```

output

```
result too large to display
```


Fricas [F(-1)]

Timed out.

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}}{c + d \sin(e + fx)} dx = \text{Timed out}$$

input `integrate((g*sin(f*x+e))^(1/2)*(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,
algorithm="fricas")`

output Timed out

Sympy [F]

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}}{c + d \sin(e + fx)} dx = \int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}}{c + d \sin(e + fx)} dx$$

input `integrate((g*sin(f*x+e))**(1/2)*(a+b*sin(f*x+e))**(1/2)/(c+d*sin(f*x+e)),x
)`

output `Integral(sqrt(g*sin(e + f*x))*sqrt(a + b*sin(e + f*x))/(c + d*sin(e + f*x))
, x)`

Maxima [F]

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}}{c + d \sin(e + fx)} dx = \int \frac{\sqrt{b \sin(fx + e) + a} \sqrt{g \sin(fx + e)}}{d \sin(fx + e) + c} dx$$

input `integrate((g*sin(f*x+e))^(1/2)*(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,
algorithm="maxima")`

output `integrate(sqrt(b*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))/(d*sin(f*x + e) +
c), x)`

Giac [F]

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}}{c + d \sin(e + fx)} dx = \int \frac{\sqrt{b \sin(fx + e) + a} \sqrt{g \sin(fx + e)}}{d \sin(fx + e) + c} dx$$

input `integrate((g*sin(f*x+e))^(1/2)*(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,
algorithm="giac")`

output `integrate(sqrt(b*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))/(d*sin(f*x + e) +
c), x)`

Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}}{c + d \sin(e + fx)} dx = \int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}}{c + d \sin(e + fx)} dx$$

input `int(((g*sin(e + f*x))^(1/2)*(a + b*sin(e + f*x))^(1/2))/(c + d*sin(e + f*x
)),x)`

output `int(((g*sin(e + f*x))^(1/2)*(a + b*sin(e + f*x))^(1/2))/(c + d*sin(e + f*x
)), x)`

Reduce [F]

$$\begin{aligned} & \int \frac{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}}{c + d \sin(e + fx)} dx \\ &= \sqrt{g} \left(\int \frac{\sqrt{\sin(fx + e)} \sqrt{\sin(fx + e) b + a}}{\sin(fx + e) d + c} dx \right) \end{aligned}$$

input `int((g*sin(f*x+e))^(1/2)*(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x)`

output

```
sqrt(g)*int((sqrt(sin(e + f*x))*sqrt(sin(e + f*x)*b + a))/(sin(e + f*x)*d  
+ c),x)
```

$$3.43 \quad \int \frac{\sqrt{a+b \sin(e+fx)}}{\sqrt{g \sin(e+fx)}(c+d \sin(e+fx))} dx$$

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Optimal result

Integrand size = 39, antiderivative size = 250

$$\int \frac{\sqrt{a+b \sin(e+fx)}}{\sqrt{g \sin(e+fx)}(c+d \sin(e+fx))} dx =$$

$$\frac{2\sqrt{a+b} \sqrt{\frac{a(1-\csc(e+fx))}{a+b}} \sqrt{\frac{a(1+\csc(e+fx))}{a-b}} \operatorname{EllipticF}\left(\arcsin\left(\frac{\sqrt{g}\sqrt{a+b \sin(e+fx)}}{\sqrt{a+b}\sqrt{g \sin(e+fx)}}\right), -\frac{a+b}{a-b}\right) \tan(e+fx)}{cf\sqrt{g}}$$

$$+ \frac{2(bc-ad)\sqrt{-\cot^2(e+fx)} \sqrt{\frac{b+a \csc(e+fx)}{a+b}} \operatorname{EllipticPi}\left(\frac{2c}{c+d}, \arcsin\left(\frac{\sqrt{1-\csc(e+fx)}}{\sqrt{2}}\right), \frac{2a}{a+b}\right) \sqrt{g \sin(e+fx)}}{c(c+d)fg\sqrt{a+b \sin(e+fx)}}$$

output

```
-2*(a+b)^(1/2)*(a*(1-csc(f*x+e))/(a+b))^(1/2)*(a*(1+csc(f*x+e))/(a-b))^(1/2)*EllipticF(g^(1/2)*(a+b*sin(f*x+e))^(1/2)/(a+b)^(1/2)/(g*sin(f*x+e))^(1/2),(-(a+b)/(a-b))^(1/2))*tan(f*x+e)/c/f/g^(1/2)+2*(-a*d+b*c)*(-cot(f*x+e))^2^(1/2)*((b+a*csc(f*x+e))/(a+b))^(1/2)*EllipticPi(1/2*(1-csc(f*x+e))^(1/2)*2^(1/2),2*c/(c+d),2^(1/2)*(a/(a+b))^(1/2))*(g*sin(f*x+e))^(1/2)*tan(f*x+e)/c/(c+d)/f/g/(a+b*sin(f*x+e))^(1/2)
```

Mathematica [F]

$$\int \frac{\sqrt{a + b \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c + d \sin(e + fx))} dx = \int \frac{\sqrt{a + b \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c + d \sin(e + fx))} dx$$

input

```
Integrate[Sqrt[a + b*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c + d*Sin[e + f*
x])), x]
```

output

```
Integrate[Sqrt[a + b*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c + d*Sin[e + f*
x])), x]
```

Rubi [A] (verified)

Time = 0.95 (sec) , antiderivative size = 250, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.128$, Rules used = {3042, 3412, 3042, 3295, 3416}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \frac{\sqrt{a + b \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c + d \sin(e + fx))} dx \\ & \quad \downarrow \text{3042} \\ & \int \frac{\sqrt{a + b \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c + d \sin(e + fx))} dx \\ & \quad \downarrow \text{3412} \\ & \frac{(bc - ad) \int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)}(c + d \sin(e + fx))} dx}{cg} + \frac{a \int \frac{1}{\sqrt{g \sin(e + fx)}\sqrt{a + b \sin(e + fx)}} dx}{c} \\ & \quad \downarrow \text{3042} \\ & \frac{(bc - ad) \int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)}(c + d \sin(e + fx))} dx}{cg} + \frac{a \int \frac{1}{\sqrt{g \sin(e + fx)}\sqrt{a + b \sin(e + fx)}} dx}{c} \end{aligned}$$

$$\frac{(bc - ad) \int \frac{\sqrt{g \sin(e+fx)}}{\sqrt{a+b \sin(e+fx)}(c+d \sin(e+fx))} dx}{2\sqrt{a+b} \tan(e+fx) \sqrt{\frac{a(1-\csc(e+fx))}{a+b}} \sqrt{\frac{a(\csc(e+fx)+1)}{a-b}} \text{EllipticF}\left(\arcsin\left(\frac{\sqrt{g}\sqrt{a+b \sin(e+fx)}}{\sqrt{a+b}\sqrt{g \sin(e+fx)}}\right), -\frac{a+b}{a-b}\right)}{cf\sqrt{g}}$$

↓ 3295

$$\frac{2(bc - ad) \tan(e+fx) \sqrt{-\cot^2(e+fx)} \sqrt{g \sin(e+fx)} \sqrt{\frac{a \csc(e+fx)+b}{a+b}} \text{EllipticPi}\left(\frac{2c}{c+d}, \arcsin\left(\frac{\sqrt{1-\csc(e+fx)}}{\sqrt{2}}\right), \frac{2}{a}\right)}{2\sqrt{a+b} \tan(e+fx) \sqrt{\frac{a(1-\csc(e+fx))}{a+b}} \sqrt{\frac{a(\csc(e+fx)+1)}{a-b}} \text{EllipticF}\left(\arcsin\left(\frac{\sqrt{g}\sqrt{a+b \sin(e+fx)}}{\sqrt{a+b}\sqrt{g \sin(e+fx)}}\right), -\frac{a+b}{a-b}\right)}{cf\sqrt{g}}$$

↓ 3416

input `Int[Sqrt[a + b*Sin[e + f*x]]/(Sqrt[g*Sin[e + f*x]]*(c + d*Sin[e + f*x])),x]`

output `(-2*Sqrt[a + b]*Sqrt[(a*(1 - Csc[e + f*x]))/(a + b)]*Sqrt[(a*(1 + Csc[e + f*x]))/(a - b)]*EllipticF[ArcSin[(Sqrt[g]*Sqrt[a + b*Sin[e + f*x]])/(Sqrt[a + b]*Sqrt[g*Sin[e + f*x]])], -((a + b)/(a - b))*Tan[e + f*x]]/(c*f*Sqrt[g]) + (2*(b*c - a*d)*Sqrt[-Cot[e + f*x]^2]*Sqrt[(b + a*Csc[e + f*x])/(a + b)]*EllipticPi[(2*c)/(c + d), ArcSin[Sqrt[1 - Csc[e + f*x]]/Sqrt[2]], (2*a)/(a + b)]*Sqrt[g*Sin[e + f*x]]*Tan[e + f*x])/(c*(c + d)*f*g*Sqrt[a + b*Sin[e + f*x]])`

Definitions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3295 `Int[1/(Sqrt[(d_)*sin[(e_)] + (f_)*(x_)])*Sqrt[(a_)] + (b_)*sin[(e_)] + (f_)*(x_)], x_Symbol] := Simp[-2*(Tan[e + f*x]/(a*f))*Rt[(a + b)/d, 2]*Sqrt[a*((1 - Csc[e + f*x])/(a + b))]*Sqrt[a*((1 + Csc[e + f*x])/(a - b))]*EllipticF[ArcSin[Sqrt[a + b*Sin[e + f*x]]/Sqrt[d*Sin[e + f*x]]/Rt[(a + b)/d, 2], -(a + b)/(a - b)], x] /; FreeQ[{a, b, d, e, f}, x] && NeQ[a^2 - b^2, 0] && PosQ[(a + b)/d]`

rule 3412 `Int[Sqrt[(a_)] + (b_)*sin[(e_)] + (f_)*(x_)]/(Sqrt[(g_)*sin[(e_)] + (f_)*(x_)]*((c_)] + (d_)*sin[(e_)] + (f_)*(x_))), x_Symbol] := Simp[a/c Int[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]), x], x] + Simp[(b*c - a*d)/(c*g) Int[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + b*Sin[e + f*x]]*(c + d*Sin[e + f*x])), x], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]`

rule 3416 `Int[Sqrt[(g_)*sin[(e_)] + (f_)*(x_)]/(Sqrt[(a_)] + (b_)*sin[(e_)] + (f_)*(x_)]*((c_)] + (d_)*sin[(e_)] + (f_)*(x_))), x_Symbol] := Simp[2*Sqrt[-Cot[e + f*x]^2]*(Sqrt[g*Sin[e + f*x]]/(f*(c + d)*Cot[e + f*x]*Sqrt[a + b*Sin[e + f*x]]))*Sqrt[(b + a*Csc[e + f*x])/(a + b)]*EllipticPi[2*(c/(c + d)), ArcSin[Sqrt[1 - Csc[e + f*x]]/Sqrt[2]], 2*(a/(a + b))], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]`

Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 2892 vs. $2(228) = 456$.

Time = 1.70 (sec) , antiderivative size = 2893, normalized size of antiderivative = 11.57

method	result	size
default	Expression too large to display	2893

input `int((a+b*sin(f*x+e))^(1/2)/(g*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,method=_RETURNVERBOSE)`

output `-1/f/(-c^2+d^2)^(1/2)/(c*(-a^2+b^2)^(1/2)+a*(-c^2+d^2)^(1/2)-a*d+b*c)/(a*d+a*(-c^2+d^2)^(1/2)-b*c-c*(-a^2+b^2)^(1/2))*(a*d-b*c)*(2*(-c^2+d^2)^(1/2)*(-a^2+b^2)^(1/2)*EllipticPi((1/(b+(-a^2+b^2)^(1/2))*(-a*cot(f*x+e)+(-a^2+b^2)^(1/2)+b+a*csc(f*x+e)))^(1/2),-(b+(-a^2+b^2)^(1/2))*c/(a*d+a*(-c^2+d^2)^(1/2)-b*c-c*(-a^2+b^2)^(1/2)),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2)))^(1/2))*b-4*(-c^2+d^2)^(1/2)*(-a^2+b^2)^(1/2)*EllipticF((1/(b+(-a^2+b^2)^(1/2))*(-a*cot(f*x+e)+(-a^2+b^2)^(1/2)+b+a*csc(f*x+e)))^(1/2),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2)))^(1/2))*b+2*(-c^2+d^2)^(1/2)*(-a^2+b^2)^(1/2)*EllipticPi((1/(b+(-a^2+b^2)^(1/2))*(-a*cot(f*x+e)+(-a^2+b^2)^(1/2)+b+a*csc(f*x+e)))^(1/2),-(b+(-a^2+b^2)^(1/2))*c/(c*(-a^2+b^2)^(1/2)+a*(-c^2+d^2)^(1/2)-a*d+b*c),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2)))^(1/2))*b-(-c^2+d^2)^(1/2)*EllipticPi((1/(b+(-a^2+b^2)^(1/2))*(-a*cot(f*x+e)+(-a^2+b^2)^(1/2)+b+a*csc(f*x+e)))^(1/2),-(b+(-a^2+b^2)^(1/2))*c/(a*d+a*(-c^2+d^2)^(1/2)-b*c-c*(-a^2+b^2)^(1/2)),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2)))^(1/2))*a^2+2*(-c^2+d^2)^(1/2)*EllipticPi((1/(b+(-a^2+b^2)^(1/2))*(-a*cot(f*x+e)+(-a^2+b^2)^(1/2)+b+a*csc(f*x+e)))^(1/2),-(b+(-a^2+b^2)^(1/2))*c/(a*d+a*(-c^2+d^2)^(1/2)-b*c-c*(-a^2+b^2)^(1/2)),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2)))^(1/2))*b^2+2*(-c^2+d^2)^(1/2)*EllipticF((1/(b+(-a^2+b^2)^(1/2))*(-a*cot(f*x+e)+(-a^2+b^2)^(1/2)+b+a*csc(f*x+e)))^(1/2),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2)))^(1/2))...`

Fricas [F(-1)]

Timed out.

$$\int \frac{\sqrt{a + b \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c + d \sin(e + fx))} dx = \text{Timed out}$$

input `integrate((a+b*sin(f*x+e))^(1/2)/(g*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,algorithm="fricas")`

output `Timed out`

Sympy [F]

$$\int \frac{\sqrt{a + b \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c + d \sin(e + fx))} dx = \int \frac{\sqrt{a + b \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c + d \sin(e + fx))} dx$$

input `integrate((a+b*sin(f*x+e))**(1/2)/(g*sin(f*x+e))**(1/2)/(c+d*sin(f*x+e)),x)`

output `Integral(sqrt(a + b*sin(e + f*x))/(sqrt(g*sin(e + f*x))*(c + d*sin(e + f*x))), x)`

Maxima [F]

$$\int \frac{\sqrt{a + b \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c + d \sin(e + fx))} dx = \int \frac{\sqrt{b \sin(fx + e) + a}}{(d \sin(fx + e) + c)\sqrt{g \sin(fx + e)}} dx$$

input `integrate((a+b*sin(f*x+e))^(1/2)/(g*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x, algorithm="maxima")`

output `integrate(sqrt(b*sin(f*x + e) + a)/((d*sin(f*x + e) + c)*sqrt(g*sin(f*x + e))), x)`

Giac [F]

$$\int \frac{\sqrt{a + b \sin(e + fx)}}{\sqrt{g \sin(e + fx)}(c + d \sin(e + fx))} dx = \int \frac{\sqrt{b \sin(fx + e) + a}}{(d \sin(fx + e) + c)\sqrt{g \sin(fx + e)}} dx$$

input `integrate((a+b*sin(f*x+e))^(1/2)/(g*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x, algorithm="giac")`

output `integrate(sqrt(b*sin(f*x + e) + a)/((d*sin(f*x + e) + c)*sqrt(g*sin(f*x + e))), x)`

Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{a + b \sin(e + f x)}}{\sqrt{g \sin(e + f x)}(c + d \sin(e + f x))} dx = \int \frac{\sqrt{a + b \sin(e + f x)}}{\sqrt{g \sin(e + f x)}(c + d \sin(e + f x))} dx$$

input `int((a + b*sin(e + f*x))^(1/2)/((g*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))),x)`

output `int((a + b*sin(e + f*x))^(1/2)/((g*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))), x)`

Reduce [F]

$$\int \frac{\sqrt{a + b \sin(e + f x)}}{\sqrt{g \sin(e + f x)}(c + d \sin(e + f x))} dx = \frac{\sqrt{g} \left(\int \frac{\sqrt{\sin(fx+e)} \sqrt{\sin(fx+e)b+a}}{\sin(fx+e)^2 d + \sin(fx+e)c} dx \right)}{g}$$

input `int((a+b*sin(f*x+e))^(1/2)/(g*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x)`

output `(sqrt(g)*int((sqrt(sin(e + f*x))*sqrt(sin(e + f*x)*b + a))/(sin(e + f*x)**2*d + sin(e + f*x)*c),x))/g`

$$3.44 \quad \int \frac{\sqrt{g \sin(e+fx)}}{\sqrt{a+b \sin(e+fx)}(c+d \sin(e+fx))} dx$$

Optimal result	378
Mathematica [F]	378
Rubi [A] (verified)	379
Maple [B] (warning: unable to verify)	380
Fricas [F(-1)]	381
Sympy [F]	382
Maxima [F]	382
Giac [F]	382
Mupad [F(-1)]	383
Reduce [F]	383

Optimal result

Integrand size = 39, antiderivative size = 114

$$\int \frac{\sqrt{g \sin(e+fx)}}{\sqrt{a+b \sin(e+fx)}(c+d \sin(e+fx))} dx$$

$$= \frac{2\sqrt{-\cot^2(e+fx)}\sqrt{\frac{b+a \csc(e+fx)}{a+b}} \operatorname{EllipticPi}\left(\frac{2c}{c+d}, \arcsin\left(\frac{\sqrt{1-\csc(e+fx)}}{\sqrt{2}}\right), \frac{2a}{a+b}\right) \sqrt{g \sin(e+fx)} \tan(e+fx)}{(c+d)f\sqrt{a+b \sin(e+fx)}}$$

output `2*(-cot(f*x+e)^2)^(1/2)*((b+a*csc(f*x+e))/(a+b))^(1/2)*EllipticPi(1/2*(1-csc(f*x+e))^(1/2)*2^(1/2),2*c/(c+d),2^(1/2)*(a/(a+b))^(1/2))*(g*sin(f*x+e))^(1/2)*tan(f*x+e)/(c+d)/f/(a+b*sin(f*x+e))^(1/2)`

Mathematica [F]

$$\int \frac{\sqrt{g \sin(e+fx)}}{\sqrt{a+b \sin(e+fx)}(c+d \sin(e+fx))} dx$$

$$= \int \frac{\sqrt{g \sin(e+fx)}}{\sqrt{a+b \sin(e+fx)}(c+d \sin(e+fx))} dx$$

input `Integrate[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + b*Sin[e + f*x]]*(c + d*Sin[e + f*x])),x]`

output `Integrate[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + b*Sin[e + f*x]]*(c + d*Sin[e + f*x])), x]`

Rubi [A] (verified)

Time = 0.42 (sec) , antiderivative size = 114, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}} = 0.051$, Rules used = {3042, 3416}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)}(c + d \sin(e + fx))} dx$$

↓ 3042

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)}(c + d \sin(e + fx))} dx$$

↓ 3416

$$\frac{2 \tan(e + fx) \sqrt{-\cot^2(e + fx)} \sqrt{g \sin(e + fx)} \sqrt{\frac{a \csc(e + fx) + b}{a + b}} \text{EllipticPi}\left(\frac{2c}{c + d}, \arcsin\left(\frac{\sqrt{1 - \csc(e + fx)}}{\sqrt{2}}\right), \frac{2a}{a + b}\right)}{f(c + d) \sqrt{a + b \sin(e + fx)}}$$

input `Int[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + b*Sin[e + f*x]]*(c + d*Sin[e + f*x])),x]`

output `(2*Sqrt[-Cot[e + f*x]^2]*Sqrt[(b + a*Csc[e + f*x])/(a + b)]*EllipticPi[(2*c)/(c + d), ArcSin[Sqrt[1 - Csc[e + f*x]]/Sqrt[2]], (2*a)/(a + b)]*Sqrt[g*Sin[e + f*x]]*Tan[e + f*x])/((c + d)*f*Sqrt[a + b*Sin[e + f*x]])`

Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3416 `Int[Sqrt[(g_)*sin[(e_) + (f_)*(x_)]]/(Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)])*(x_)])*((c_) + (d_)*sin[(e_) + (f_)*(x_)])], x_Symbol] := Simp[2*Sqrt[-Cot[e + f*x]^2]*(Sqrt[g*Sin[e + f*x]]/(f*(c + d)*Cot[e + f*x]*Sqrt[a + b*Sin[e + f*x]])*Sqrt[(b + a*Csc[e + f*x])/(a + b)]*EllipticPi[2*(c/(c + d)), ArcSin[Sqrt[1 - Csc[e + f*x]]/Sqrt[2]], 2*(a/(a + b))], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]`

Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 2589 vs. $2(107) = 214$.

Time = 2.16 (sec) , antiderivative size = 2590, normalized size of antiderivative = 22.72

method	result	size
default	Expression too large to display	2590

input `int((g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,method=_RETURNVERBOSE)`

output

```

1/f*c/(a*d+a*(-c^2+d^2)^(1/2)-b*c-c*(-a^2+b^2)^(1/2))/(c*(-a^2+b^2)^(1/2)+
a*(-c^2+d^2)^(1/2)-a*d+b*c)/(-c^2+d^2)^(1/2)*(2*(-c^2+d^2)^(1/2)*(-a^2+b^2)
)^(1/2)*EllipticPi((1/(b+(-a^2+b^2)^(1/2))*(-a*cot(f*x+e)+(-a^2+b^2)^(1/2)
+b+a*csc(f*x+e)))^(1/2),(b+(-a^2+b^2)^(1/2))*c/(c*(-a^2+b^2)^(1/2)+a*(-c^2
+d^2)^(1/2)-a*d+b*c),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(
1/2))*b+2*(-c^2+d^2)^(1/2)*(-a^2+b^2)^(1/2)*EllipticPi((1/(b+(-a^2+b^2)^(1
/2))*(-a*cot(f*x+e)+(-a^2+b^2)^(1/2)+b+a*csc(f*x+e)))^(1/2),-(b+(-a^2+b^2)
^(1/2))*c/(a*d+a*(-c^2+d^2)^(1/2)-b*c-c*(-a^2+b^2)^(1/2)),1/2*2^(1/2)*((b+
(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2))*b-(-c^2+d^2)^(1/2)*EllipticPi((
1/(b+(-a^2+b^2)^(1/2))*(-a*cot(f*x+e)+(-a^2+b^2)^(1/2)+b+a*csc(f*x+e)))^(1
/2),(b+(-a^2+b^2)^(1/2))*c/(c*(-a^2+b^2)^(1/2)+a*(-c^2+d^2)^(1/2)-a*d+b*c)
,1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2))*a^2+2*(-c^2+d^
2)^(1/2)*EllipticPi((1/(b+(-a^2+b^2)^(1/2))*(-a*cot(f*x+e)+(-a^2+b^2)^(1/2)
)+b+a*csc(f*x+e)))^(1/2),(b+(-a^2+b^2)^(1/2))*c/(c*(-a^2+b^2)^(1/2)+a*(-c^
2+d^2)^(1/2)-a*d+b*c),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(
1/2))*b^2-(-c^2+d^2)^(1/2)*EllipticPi((1/(b+(-a^2+b^2)^(1/2))*(-a*cot(f*x
+e)+(-a^2+b^2)^(1/2)+b+a*csc(f*x+e)))^(1/2),-(b+(-a^2+b^2)^(1/2))*c/(a*d+a
*(-c^2+d^2)^(1/2)-b*c-c*(-a^2+b^2)^(1/2)),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2)
)/(-a^2+b^2)^(1/2))^(1/2))*a^2+2*(-c^2+d^2)^(1/2)*EllipticPi((1/(b+(-a^2+b
^2)^(1/2))*(-a*cot(f*x+e)+(-a^2+b^2)^(1/2)+b+a*csc(f*x+e)))^(1/2),-(b+...

```

Fricas [F(-1)]

Timed out.

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)}(c + d \sin(e + fx))} dx = \text{Timed out}$$

input

```

integrate((g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,
algorithm="fricas")

```

output

Timed out

Sympy [F]

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)}(c + d \sin(e + fx))} dx$$

$$= \int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)}(c + d \sin(e + fx))} dx$$

input `integrate((g*sin(f*x+e))**(1/2)/(a+b*sin(f*x+e))**(1/2)/(c+d*sin(f*x+e)),x)`

output `Integral(sqrt(g*sin(e + f*x))/(sqrt(a + b*sin(e + f*x))*(c + d*sin(e + f*x))), x)`

Maxima [F]

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)}(c + d \sin(e + fx))} dx = \int \frac{\sqrt{g \sin(fx + e)}}{\sqrt{b \sin(fx + e) + a}(d \sin(fx + e) + c)} dx$$

input `integrate((g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,algorithm="maxima")`

output `integrate(sqrt(g*sin(f*x + e))/(sqrt(b*sin(f*x + e) + a)*(d*sin(f*x + e) + c)), x)`

Giac [F]

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)}(c + d \sin(e + fx))} dx = \int \frac{\sqrt{g \sin(fx + e)}}{\sqrt{b \sin(fx + e) + a}(d \sin(fx + e) + c)} dx$$

input `integrate((g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,algorithm="giac")`

output `integrate(sqrt(g*sin(f*x + e))/(sqrt(b*sin(f*x + e) + a)*(d*sin(f*x + e) + c)), x)`

Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)}(c + d \sin(e + fx))} dx$$

$$= \int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)}(c + d \sin(e + fx))} dx$$

input `int((g*sin(e + f*x))^(1/2)/((a + b*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))),x)`

output `int((g*sin(e + f*x))^(1/2)/((a + b*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))), x)`

Reduce [F]

$$\int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)}(c + d \sin(e + fx))} dx$$

$$= \sqrt{g} \left(\int \frac{\sqrt{\sin(fx + e)} \sqrt{\sin(fx + e) b + a}}{\sin(fx + e)^2 bd + \sin(fx + e) ad + \sin(fx + e) bc + ac} dx \right)$$

input `int((g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x)`

output `sqrt(g)*int((sqrt(sin(e + f*x))*sqrt(sin(e + f*x)*b + a))/(sin(e + f*x)**2 *b*d + sin(e + f*x)*a*d + sin(e + f*x)*b*c + a*c),x)`

3.45
$$\int \frac{1}{\sqrt{g \sin(e+fx)} \sqrt{a+b \sin(e+fx)} (c+d \sin(e+fx))} dx$$

Optimal result	384
Mathematica [B] (warning: unable to verify)	385
Rubi [A] (verified)	385
Maple [B] (warning: unable to verify)	387
Fricas [F]	388
Sympy [F]	389
Maxima [F]	389
Giac [F]	390
Mupad [F(-1)]	390
Reduce [F]	391

Optimal result

Integrand size = 39, antiderivative size = 246

$$\int \frac{1}{\sqrt{g \sin(e+fx)} \sqrt{a+b \sin(e+fx)} (c+d \sin(e+fx))} dx =$$

$$\frac{2\sqrt{a+b} \sqrt{\frac{a(1-\csc(e+fx))}{a+b}} \sqrt{\frac{a(1+\csc(e+fx))}{a-b}} \operatorname{EllipticF}\left(\arcsin\left(\frac{\sqrt{g}\sqrt{a+b \sin(e+fx)}}{\sqrt{a+b}\sqrt{g \sin(e+fx)}}\right), -\frac{a+b}{a-b}\right) \tan(e+fx)}{acf\sqrt{g}}$$

$$\frac{2d\sqrt{-\cot^2(e+fx)} \sqrt{\frac{b+a \csc(e+fx)}{a+b}} \operatorname{EllipticPi}\left(\frac{2c}{c+d}, \arcsin\left(\frac{\sqrt{1-\csc(e+fx)}}{\sqrt{2}}\right), \frac{2a}{a+b}\right) \sqrt{g \sin(e+fx)} \tan(e+fx)}{c(c+d)fg\sqrt{a+b \sin(e+fx)}}$$

output

```
-2*(a+b)^(1/2)*(a*(1-csc(f*x+e))/(a+b))^(1/2)*(a*(1+csc(f*x+e))/(a-b))^(1/2)*EllipticF(g^(1/2)*(a+b*sin(f*x+e))^(1/2)/(a+b)^(1/2)/(g*sin(f*x+e))^(1/2),(-(a+b)/(a-b))^(1/2))*tan(f*x+e)/a/c/f/g^(1/2)-2*d*(-cot(f*x+e)^2)^(1/2)*((b+a*csc(f*x+e))/(a+b))^(1/2)*EllipticPi(1/2*(1-csc(f*x+e))^(1/2)*2^(1/2),2*c/(c+d),2^(1/2)*(a/(a+b))^(1/2))*(g*sin(f*x+e))^(1/2)*tan(f*x+e)/c/(c+d)/f/g/(a+b*sin(f*x+e))^(1/2)
```

Mathematica [B] (warning: unable to verify)

Leaf count is larger than twice the leaf count of optimal. 5612 vs. $2(246) = 492$.

Time = 33.80 (sec) , antiderivative size = 5612, normalized size of antiderivative = 22.81

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)} (c + d \sin(e + fx))} dx = \text{Result too large to show}$$

input

```
Integrate[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]*(c + d*Sin[e + f*x])),x]
```

output

Result too large to show

Rubi [A] (verified)

Time = 0.99 (sec) , antiderivative size = 246, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.128$, Rules used = {3042, 3418, 3042, 3295, 3416}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{array}{c} \int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)} (c + d \sin(e + fx))} dx \\ \downarrow \text{3042} \\ \int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)} (c + d \sin(e + fx))} dx \\ \downarrow \text{3418} \\ \frac{\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)}} dx}{c} - \frac{d \int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{a + b \sin(e + fx)} (c + d \sin(e + fx))} dx}{cg} \\ \downarrow \text{3042} \end{array}$$

$$\begin{aligned}
& \frac{\int \frac{1}{\sqrt{g \sin(e+fx)} \sqrt{a+b \sin(e+fx)}} dx}{c} - \frac{d \int \frac{\sqrt{g \sin(e+fx)}}{\sqrt{a+b \sin(e+fx)}(c+d \sin(e+fx))} dx}{cg} \\
& \quad \downarrow \text{3295} \\
& \frac{d \int \frac{\sqrt{g \sin(e+fx)}}{\sqrt{a+b \sin(e+fx)}(c+d \sin(e+fx))} dx}{cg} - \\
& \frac{2\sqrt{a+b} \tan(e+fx) \sqrt{\frac{a(1-\csc(e+fx))}{a+b}} \sqrt{\frac{a(\csc(e+fx)+1)}{a-b}} \text{EllipticF}\left(\arcsin\left(\frac{\sqrt{g} \sqrt{a+b \sin(e+fx)}}{\sqrt{a+b} \sqrt{g \sin(e+fx)}}\right), -\frac{a+b}{a-b}\right)}{acf\sqrt{g}} \\
& \quad \downarrow \text{3416} \\
& \frac{2d \tan(e+fx) \sqrt{-\cot^2(e+fx)} \sqrt{g \sin(e+fx)} \sqrt{\frac{a \csc(e+fx)+b}{a+b}} \text{EllipticPi}\left(\frac{2c}{c+d}, \arcsin\left(\frac{\sqrt{1-\csc(e+fx)}}{\sqrt{2}}\right), \frac{2a}{a+b}\right)}{cf g(c+d) \sqrt{a+b \sin(e+fx)}} \\
& \frac{2\sqrt{a+b} \tan(e+fx) \sqrt{\frac{a(1-\csc(e+fx))}{a+b}} \sqrt{\frac{a(\csc(e+fx)+1)}{a-b}} \text{EllipticF}\left(\arcsin\left(\frac{\sqrt{g} \sqrt{a+b \sin(e+fx)}}{\sqrt{a+b} \sqrt{g \sin(e+fx)}}\right), -\frac{a+b}{a-b}\right)}{acf\sqrt{g}}
\end{aligned}$$

input `Int[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]*(c + d*Sin[e + f*x])),x]`

output `(-2*Sqrt[a + b]*Sqrt[(a*(1 - Csc[e + f*x]))/(a + b)]*Sqrt[(a*(1 + Csc[e + f*x]))/(a - b)]*EllipticF[ArcSin[(Sqrt[g]*Sqrt[a + b*Sin[e + f*x]])/(Sqrt[a + b]*Sqrt[g*Sin[e + f*x]])], -(a + b)/(a - b)]*Tan[e + f*x]/(a*c*f*Sqrt[g]) - (2*d*Sqrt[-Cot[e + f*x]^2]*Sqrt[(b + a*Csc[e + f*x])/(a + b)]*EllipticPi[(2*c)/(c + d), ArcSin[Sqrt[1 - Csc[e + f*x]]/Sqrt[2]], (2*a)/(a + b)]*Sqrt[g*Sin[e + f*x]]*Tan[e + f*x]/(c*(c + d)*f*g*Sqrt[a + b*Sin[e + f*x]]))`

Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3295

```
Int[1/(Sqrt[(d_)*sin[(e_)] + (f_)*(x_)])*Sqrt[(a_) + (b_)*sin[(e_)] + (f_)*(x_)]), x_Symbol] := Simp[-2*(Tan[e + f*x]/(a*f))*Rt[(a + b)/d, 2]*Sqrt[a*((1 - Csc[e + f*x])/(a + b))*Sqrt[a*((1 + Csc[e + f*x])/(a - b))]*EllipticF[ArcSin[Sqrt[a + b*Sin[e + f*x]]/Sqrt[d*Sin[e + f*x]]/Rt[(a + b)/d, 2]], -(a + b)/(a - b)], x] /; FreeQ[{a, b, d, e, f}, x] && NeQ[a^2 - b^2, 0] && PosQ[(a + b)/d]
```

rule 3416

```
Int[Sqrt[(g_)*sin[(e_)] + (f_)*(x_)]/(Sqrt[(a_) + (b_)*sin[(e_)] + (f_)*(x_)])*((c_) + (d_)*sin[(e_)] + (f_)*(x_))), x_Symbol] := Simp[2*Sqrt[-Cot[e + f*x]^2*(Sqrt[g*Sin[e + f*x]]/(f*(c + d)*Cot[e + f*x]*Sqrt[a + b*Sin[e + f*x]))]*Sqrt[(b + a*Csc[e + f*x])/(a + b)]*EllipticPi[2*(c/(c + d)), ArcSin[Sqrt[1 - Csc[e + f*x]]/Sqrt[2]], 2*(a/(a + b))], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]
```

rule 3418

```
Int[1/(Sqrt[(g_)*sin[(e_)] + (f_)*(x_)])*Sqrt[(a_) + (b_)*sin[(e_)] + (f_)*(x_)])*((c_) + (d_)*sin[(e_)] + (f_)*(x_))), x_Symbol] := Simp[1/c Int[1/(Sqrt[g*Sin[e + f*x]]*Sqrt[a + b*Sin[e + f*x]]), x], x] - Simp[d/(c*g) Int[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + b*Sin[e + f*x]]*(c + d*Sin[e + f*x])), x], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]
```

Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 3271 vs. $2(224) = 448$.

Time = 2.19 (sec) , antiderivative size = 3272, normalized size of antiderivative = 13.30

method	result	size
default	Expression too large to display	3272

input

```
int(1/(g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x,method=_RETURNVERBOSE)
```

output

```

2/f/a/(a*d+a*(-c^2+d^2)^(1/2)-b*c-c*(-a^2+b^2)^(1/2))/(c*(-a^2+b^2)^(1/2)+
a*(-c^2+d^2)^(1/2)-a*d+b*c)/(-c^2+d^2)^(1/2)*2^(1/2)/(g*sin(f*x+e))^(1/2)*
(a+b*sin(f*x+e))^(1/2)*(4*EllipticF((1/(b+(-a^2+b^2)^(1/2)))*(a*(-cot(f*x+e)
)+csc(f*x+e))+(-a^2+b^2)^(1/2)+b))^(1/2),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))
/(-a^2+b^2)^(1/2))^(1/2))*a*b*d*(-c^2+d^2)^(1/2)*(-a^2+b^2)^(1/2)-4*Ellipt
icF((1/(b+(-a^2+b^2)^(1/2)))*(a*(-cot(f*x+e)+csc(f*x+e))+(-a^2+b^2)^(1/2)+b
))^(1/2),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2))*b^2*c*
(-c^2+d^2)^(1/2)*(-a^2+b^2)^(1/2)-2*EllipticPi((1/(b+(-a^2+b^2)^(1/2)))*(a*
(-cot(f*x+e)+csc(f*x+e))+(-a^2+b^2)^(1/2)+b))^(1/2),(b+(-a^2+b^2)^(1/2))*c
/(c*(-a^2+b^2)^(1/2)+a*(-c^2+d^2)^(1/2)-a*d+b*c),1/2*2^(1/2)*((b+(-a^2+b^2
)^(1/2))/(-a^2+b^2)^(1/2))^(1/2))*a*b*d*(-c^2+d^2)^(1/2)*(-a^2+b^2)^(1/2)-
2*EllipticPi((1/(b+(-a^2+b^2)^(1/2)))*(a*(-cot(f*x+e)+csc(f*x+e))+(-a^2+b^2
)^(1/2)+b))^(1/2),-(b+(-a^2+b^2)^(1/2))*c/(a*d+a*(-c^2+d^2)^(1/2)-b*c-c*(-
a^2+b^2)^(1/2)),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2))
*a*b*d*(-c^2+d^2)^(1/2)*(-a^2+b^2)^(1/2)-2*EllipticF((1/(b+(-a^2+b^2)^(1/2
)))*(a*(-cot(f*x+e)+csc(f*x+e))+(-a^2+b^2)^(1/2)+b))^(1/2),1/2*2^(1/2)*((b+
(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2))*a^3*d*(-c^2+d^2)^(1/2)+2*Ellipt
icF((1/(b+(-a^2+b^2)^(1/2)))*(a*(-cot(f*x+e)+csc(f*x+e))+(-a^2+b^2)^(1/2)+b
))^(1/2),1/2*2^(1/2)*((b+(-a^2+b^2)^(1/2))/(-a^2+b^2)^(1/2))^(1/2))*a^2*b*
c*(-c^2+d^2)^(1/2)+4*EllipticF((1/(b+(-a^2+b^2)^(1/2)))*(a*(-cot(f*x+e)+...

```

Fricas [F]

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)} (c + d \sin(e + fx))} dx$$

$$= \int \frac{1}{\sqrt{b \sin(fx + e) + a(d \sin(fx + e) + c)} \sqrt{g \sin(fx + e)}} dx$$

input

```

integrate(1/(g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x
, algorithm="fricas")

```

output

```

integral(-sqrt(b*sin(f*x + e) + a)*sqrt(g*sin(f*x + e))/((b*c + a*d)*g*cos
(f*x + e)^2 - (b*c + a*d)*g + (b*d*g*cos(f*x + e)^2 - (a*c + b*d)*g)*sin(f
*x + e)), x)

```

Sympy [F]

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)} (c + d \sin(e + fx))} dx$$

$$= \int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)} (c + d \sin(e + fx))} dx$$

input `integrate(1/(g*sin(f*x+e))**(1/2)/(a+b*sin(f*x+e))**(1/2)/(c+d*sin(f*x+e)),x)`

output `Integral(1/(sqrt(g*sin(e + f*x))*sqrt(a + b*sin(e + f*x))*(c + d*sin(e + f*x))), x)`

Maxima [F]

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)} (c + d \sin(e + fx))} dx$$

$$= \int \frac{1}{\sqrt{b \sin(fx + e) + a} (d \sin(fx + e) + c) \sqrt{g \sin(fx + e)}} dx$$

input `integrate(1/(g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x, algorithm="maxima")`

output `integrate(1/(sqrt(b*sin(f*x + e) + a)*(d*sin(f*x + e) + c)*sqrt(g*sin(f*x + e))), x)`

Giac [F]

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)} (c + d \sin(e + fx))} dx$$

$$= \int \frac{1}{\sqrt{b \sin(fx + e) + a} (d \sin(fx + e) + c) \sqrt{g \sin(fx + e)}} dx$$

input `integrate(1/(g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x
, algorithm="giac")`

output `integrate(1/(sqrt(b*sin(f*x + e) + a)*(d*sin(f*x + e) + c)*sqrt(g*sin(f*x
+ e))), x)`

Mupad [F(-1)]

Timed out.

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)} (c + d \sin(e + fx))} dx$$

$$= \int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)} (c + d \sin(e + fx))} dx$$

input `int(1/((g*sin(e + f*x))^(1/2)*(a + b*sin(e + f*x))^(1/2)*(c + d*sin(e + f*
x))),x)`

output `int(1/((g*sin(e + f*x))^(1/2)*(a + b*sin(e + f*x))^(1/2)*(c + d*sin(e + f*
x))), x)`

Reduce [F]

$$\int \frac{1}{\sqrt{g \sin(e + fx)} \sqrt{a + b \sin(e + fx)} (c + d \sin(e + fx))} dx$$

$$= \frac{\sqrt{g} \left(\int \frac{\sqrt{\sin(fx+e)} \sqrt{\sin(fx+e)b+a}}{\sin(fx+e)^3 bd + \sin(fx+e)^2 ad + \sin(fx+e)^2 bc + \sin(fx+e)ac} dx \right)}{g}$$

input `int(1/(g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e)),x)`

output `(sqrt(g)*int((sqrt(sin(e + f*x))*sqrt(sin(e + f*x)*b + a))/(sin(e + f*x)**3*b*d + sin(e + f*x)**2*a*d + sin(e + f*x)**2*b*c + sin(e + f*x)*a*c),x))/g`

3.46
$$\int \frac{\sqrt{g \sin(e+fx)} \sqrt{c+d \sin(e+fx)}}{a+b \sin(e+fx)} dx$$

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Optimal result

Integrand size = 39, antiderivative size = 254

$$\int \frac{\sqrt{g \sin(e+fx)} \sqrt{c+d \sin(e+fx)}}{a+b \sin(e+fx)} dx$$

$$= \frac{2\sqrt{c+d}\sqrt{g}\sqrt{\frac{c(1-\csc(e+fx))}{c+d}}\sqrt{\frac{c(1+\csc(e+fx))}{c-d}} \operatorname{EllipticPi}\left(\frac{c+d}{d}, \arcsin\left(\frac{\sqrt{g}\sqrt{c+d \sin(e+fx)}}{\sqrt{c+d}\sqrt{g \sin(e+fx)}}\right), -\frac{c+d}{c-d}\right) \tan(e+fx)}{bf}$$

$$+ \frac{2(bc-ad)\sqrt{-\cot^2(e+fx)}\sqrt{\frac{d+c \csc(e+fx)}{c+d}} \operatorname{EllipticPi}\left(\frac{2a}{a+b}, \arcsin\left(\frac{\sqrt{1-\csc(e+fx)}}{\sqrt{2}}\right), \frac{2c}{c+d}\right) \sqrt{g \sin(e+fx)}}{b(a+b)f\sqrt{c+d \sin(e+fx)}}$$

output

```
2*(c+d)^(1/2)*g^(1/2)*(c*(1-csc(f*x+e))/(c+d))^(1/2)*(c*(1+csc(f*x+e))/(c-d))^(1/2)*EllipticPi(g^(1/2)*(c+d*sin(f*x+e))^(1/2)/(c+d)^(1/2)/(g*sin(f*x+e))^(1/2), (c+d)/d, (-(c+d)/(c-d))^(1/2))*tan(f*x+e)/b/f+2*(-a*d+b*c)*(-cot(f*x+e)^2)^(1/2)*((d+c*csc(f*x+e))/(c+d))^(1/2)*EllipticPi(1/2*(1-csc(f*x+e))^(1/2)*2^(1/2), 2*a/(a+b), 2^(1/2)*(c/(c+d))^(1/2))*(g*sin(f*x+e))^(1/2)*tan(f*x+e)/b/(a+b)/f/(c+d*sin(f*x+e))^(1/2)
```

Mathematica [F]

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{c + d \sin(e + fx)}}{a + b \sin(e + fx)} dx = \int \frac{\sqrt{g \sin(e + fx)} \sqrt{c + d \sin(e + fx)}}{a + b \sin(e + fx)} dx$$

input

```
Integrate[(Sqrt[g*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]]/(a + b*Sin[e + f*x]), x]
```

output

```
Integrate[(Sqrt[g*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]]/(a + b*Sin[e + f*x]), x]
```

Rubi [A] (verified)

Time = 0.96 (sec) , antiderivative size = 254, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.128$, Rules used = {3042, 3408, 3042, 3288, 3416}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \frac{\sqrt{g \sin(e + fx)} \sqrt{c + d \sin(e + fx)}}{a + b \sin(e + fx)} dx \\ & \quad \downarrow \text{3042} \\ & \int \frac{\sqrt{g \sin(e + fx)} \sqrt{c + d \sin(e + fx)}}{a + b \sin(e + fx)} dx \\ & \quad \downarrow \text{3408} \\ & \frac{(bc - ad) \int \frac{\sqrt{g \sin(e + fx)}}{(a + b \sin(e + fx)) \sqrt{c + d \sin(e + fx)}} dx}{b} + \frac{d \int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{c + d \sin(e + fx)}} dx}{b} \\ & \quad \downarrow \text{3042} \\ & \frac{(bc - ad) \int \frac{\sqrt{g \sin(e + fx)}}{(a + b \sin(e + fx)) \sqrt{c + d \sin(e + fx)}} dx}{b} + \frac{d \int \frac{\sqrt{g \sin(e + fx)}}{\sqrt{c + d \sin(e + fx)}} dx}{b} \\ & \quad \downarrow \text{3288} \end{aligned}$$

$$\frac{(bc - ad) \int \frac{\sqrt{g \sin(e+fx)}}{(a+b \sin(e+fx)) \sqrt{c+d \sin(e+fx)}} dx + 2\sqrt{g}\sqrt{c+d} \tan(e+fx) \sqrt{\frac{c(1-\csc(e+fx))}{c+d}} \sqrt{\frac{c(\csc(e+fx)+1)}{c-d}} \text{EllipticPi}\left(\frac{c+d}{d}, \arcsin\left(\frac{\sqrt{g}\sqrt{c+d \sin(e+fx)}}{\sqrt{c+d}\sqrt{g \sin(e+fx)}}\right), -\frac{c+d}{c-d}\right)}{bf}$$

↓ 3416

$$\frac{2(bc - ad) \tan(e+fx) \sqrt{-\cot^2(e+fx)} \sqrt{g \sin(e+fx)} \sqrt{\frac{c \csc(e+fx)+d}{c+d}} \text{EllipticPi}\left(\frac{2a}{a+b}, \arcsin\left(\frac{\sqrt{1-\csc(e+fx)}}{\sqrt{2}}\right), \frac{2}{c+d}\right) + 2\sqrt{g}\sqrt{c+d} \tan(e+fx) \sqrt{\frac{c(1-\csc(e+fx))}{c+d}} \sqrt{\frac{c(\csc(e+fx)+1)}{c-d}} \text{EllipticPi}\left(\frac{c+d}{d}, \arcsin\left(\frac{\sqrt{g}\sqrt{c+d \sin(e+fx)}}{\sqrt{c+d}\sqrt{g \sin(e+fx)}}\right), -\frac{c+d}{c-d}\right)}{bf(a+b)\sqrt{c+d \sin(e+fx)}}$$

input `Int[(Sqrt[g*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]])/(a + b*Sin[e + f*x]),x]`

output `(2*Sqrt[c + d]*Sqrt[g]*Sqrt[(c*(1 - Csc[e + f*x]))/(c + d)]*Sqrt[(c*(1 + Csc[e + f*x]))/(c - d)]*EllipticPi[(c + d)/d, ArcSin[(Sqrt[g]*Sqrt[c + d*Sin[e + f*x]])/(Sqrt[c + d]*Sqrt[g*Sin[e + f*x]])], -((c + d)/(c - d))]*Tan[e + f*x])/(b*f) + (2*(b*c - a*d)*Sqrt[-Cot[e + f*x]^2]*Sqrt[(d + c*Csc[e + f*x])/(c + d)]*EllipticPi[(2*a)/(a + b), ArcSin[Sqrt[1 - Csc[e + f*x]]/Sqrt[2]], (2*c)/(c + d)]*Sqrt[g*Sin[e + f*x]]*Tan[e + f*x])/(b*(a + b)*f*Sqrt[c + d*Sin[e + f*x]])`

Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3288 `Int[Sqrt[(b_.)*sin[(e_.) + (f_.)*(x_)]/Sqrt[(c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]], x_Symbol] := Simp[2*b*(Tan[e + f*x]/(d*f))*Rt[(c + d)/b, 2]*Sqrt[c*((1 + Csc[e + f*x])/(c - d))]*Sqrt[c*((1 - Csc[e + f*x])/(c + d))]*EllipticPi[(c + d)/d, ArcSin[Sqrt[c + d*Sin[e + f*x]]/Sqrt[b*Sin[e + f*x]]/Rt[(c + d)/b, 2]], -(c + d)/(c - d)], x] /; FreeQ[{b, c, d, e, f}, x] && NeQ[c^2 - d^2, 0] && PosQ[(c + d)/b]`

rule 3408

```
Int[(Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)])*Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.
)*(x_)])]/((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)]), x_Symbol] :> Simp[b/d I
nt[Sqrt[g*Sin[e + f*x]]/Sqrt[a + b*Sin[e + f*x]], x], x] - Simp[(b*c - a*d)
/d Int[Sqrt[g*Sin[e + f*x]]/(Sqrt[a + b*Sin[e + f*x]]*(c + d*Sin[e + f*x]
)), x], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && NeQ[a
^2 - b^2, 0] && NeQ[c^2 - d^2, 0]
```

rule 3416

```
Int[Sqrt[(g_.)*sin[(e_.) + (f_.)*(x_)]]/(Sqrt[(a_) + (b_.)*sin[(e_.) + (f_.
)*(x_)])*((c_) + (d_.)*sin[(e_.) + (f_.)*(x_)])), x_Symbol] :> Simp[2*Sqrt[
-Cot[e + f*x]^2]*(Sqrt[g*Sin[e + f*x]]/(f*(c + d)*Cot[e + f*x]*Sqrt[a + b*S
in[e + f*x]]))*Sqrt[(b + a*Csc[e + f*x])/(a + b)]*EllipticPi[2*(c/(c + d)),
ArcSin[Sqrt[1 - Csc[e + f*x]]/Sqrt[2]], 2*(a/(a + b))], x] /; FreeQ[{a, b,
c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 -
d^2, 0]
```

Maple [C] (warning: unable to verify)

Result contains complex when optimal does not.

Time = 2.12 (sec) , antiderivative size = 5439, normalized size of antiderivative = 21.41

method	result	size
default	Expression too large to display	5439

input

```
int((g*sin(f*x+e))^(1/2)*(c+d*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e)),x,method=
_RETURNVERBOSE)
```

output

```
result too large to display
```

Fricas [F(-1)]

Timed out.

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{c + d \sin(e + fx)}}{a + b \sin(e + fx)} dx = \text{Timed out}$$

input `integrate((g*sin(f*x+e))^(1/2)*(c+d*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e)),x,
algorithm="fricas")`

output Timed out

Sympy [F]

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{c + d \sin(e + fx)}}{a + b \sin(e + fx)} dx = \int \frac{\sqrt{g \sin(e + fx)} \sqrt{c + d \sin(e + fx)}}{a + b \sin(e + fx)} dx$$

input `integrate((g*sin(f*x+e))**(1/2)*(c+d*sin(f*x+e))**(1/2)/(a+b*sin(f*x+e)),x
)`

output `Integral(sqrt(g*sin(e + f*x))*sqrt(c + d*sin(e + f*x))/(a + b*sin(e + f*x))
, x)`

Maxima [F]

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{c + d \sin(e + fx)}}{a + b \sin(e + fx)} dx = \int \frac{\sqrt{d \sin(fx + e) + c} \sqrt{g \sin(fx + e)}}{b \sin(fx + e) + a} dx$$

input `integrate((g*sin(f*x+e))^(1/2)*(c+d*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e)),x,
algorithm="maxima")`

output `integrate(sqrt(d*sin(f*x + e) + c)*sqrt(g*sin(f*x + e))/(b*sin(f*x + e) +
a), x)`

Giac [F]

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{c + d \sin(e + fx)}}{a + b \sin(e + fx)} dx = \int \frac{\sqrt{d \sin(fx + e) + c} \sqrt{g \sin(fx + e)}}{b \sin(fx + e) + a} dx$$

input `integrate((g*sin(f*x+e))^(1/2)*(c+d*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e)),x,
algorithm="giac")`

output `integrate(sqrt(d*sin(f*x + e) + c)*sqrt(g*sin(f*x + e))/(b*sin(f*x + e) +
a), x)`

Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{g \sin(e + fx)} \sqrt{c + d \sin(e + fx)}}{a + b \sin(e + fx)} dx = \int \frac{\sqrt{g \sin(e + fx)} \sqrt{c + d \sin(e + fx)}}{a + b \sin(e + fx)} dx$$

input `int(((g*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))^(1/2))/(a + b*sin(e + f*x
)),x)`

output `int(((g*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))^(1/2))/(a + b*sin(e + f*x
)), x)`

Reduce [F]

$$\begin{aligned} & \int \frac{\sqrt{g \sin(e + fx)} \sqrt{c + d \sin(e + fx)}}{a + b \sin(e + fx)} dx \\ &= \sqrt{g} \left(\int \frac{\sqrt{\sin(fx + e)} \sqrt{\sin(fx + e) d + c}}{\sin(fx + e) b + a} dx \right) \end{aligned}$$

input `int((g*sin(f*x+e))^(1/2)*(c+d*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e)),x)`

output `sqrt(g)*int((sqrt(sin(e + f*x))*sqrt(sin(e + f*x)*d + c))/(sin(e + f*x)*b + a),x)`

$$3.47 \quad \int \frac{\sqrt{g \sin(e+fx)}}{(a+b \sin(e+fx))\sqrt{c+d \sin(e+fx)}} dx$$

Optimal result	399
Mathematica [F]	399
Rubi [A] (verified)	400
Maple [B] (warning: unable to verify)	401
Fricas [F(-1)]	402
Sympy [F]	403
Maxima [F]	403
Giac [F]	403
Mupad [F(-1)]	404
Reduce [F]	404

Optimal result

Integrand size = 39, antiderivative size = 114

$$\int \frac{\sqrt{g \sin(e+fx)}}{(a+b \sin(e+fx))\sqrt{c+d \sin(e+fx)}} dx$$

$$= \frac{2\sqrt{-\cot^2(e+fx)}\sqrt{\frac{d+c \csc(e+fx)}{c+d}} \operatorname{EllipticPi}\left(\frac{2a}{a+b}, \arcsin\left(\frac{\sqrt{1-\csc(e+fx)}}{\sqrt{2}}\right), \frac{2c}{c+d}\right) \sqrt{g \sin(e+fx)} \tan(e+fx)}{(a+b)f\sqrt{c+d \sin(e+fx)}}$$

output `2*(-cot(f*x+e)^2)^(1/2)*((d+c*csc(f*x+e))/(c+d))^(1/2)*EllipticPi(1/2*(1-csc(f*x+e))^(1/2)*2^(1/2),2*a/(a+b),2^(1/2)*(c/(c+d))^(1/2))*(g*sin(f*x+e))^(1/2)*tan(f*x+e)/(a+b)/f/(c+d*sin(f*x+e))^(1/2)`

Mathematica [F]

$$\int \frac{\sqrt{g \sin(e+fx)}}{(a+b \sin(e+fx))\sqrt{c+d \sin(e+fx)}} dx$$

$$= \int \frac{\sqrt{g \sin(e+fx)}}{(a+b \sin(e+fx))\sqrt{c+d \sin(e+fx)}} dx$$

input `Integrate[Sqrt[g*Sin[e + f*x]]/((a + b*Sin[e + f*x])*Sqrt[c + d*Sin[e + f*x]]),x]`

output `Integrate[Sqrt[g*Sin[e + f*x]]/((a + b*Sin[e + f*x])*Sqrt[c + d*Sin[e + f*x]]), x]`

Rubi [A] (verified)

Time = 0.42 (sec) , antiderivative size = 114, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}} = 0.051$, Rules used = {3042, 3416}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\sqrt{g \sin(e + fx)}}{(a + b \sin(e + fx)) \sqrt{c + d \sin(e + fx)}} dx$$

↓ 3042

$$\int \frac{\sqrt{g \sin(e + fx)}}{(a + b \sin(e + fx)) \sqrt{c + d \sin(e + fx)}} dx$$

↓ 3416

$$\frac{2 \tan(e + fx) \sqrt{-\cot^2(e + fx)} \sqrt{g \sin(e + fx)} \sqrt{\frac{c \csc(e + fx) + d}{c + d}} \text{EllipticPi}\left(\frac{2a}{a + b}, \arcsin\left(\frac{\sqrt{1 - \csc(e + fx)}}{\sqrt{2}}\right), \frac{2c}{c + d}\right)}{f(a + b) \sqrt{c + d \sin(e + fx)}}$$

input `Int[Sqrt[g*Sin[e + f*x]]/((a + b*Sin[e + f*x])*Sqrt[c + d*Sin[e + f*x]]),x]`

output `(2*Sqrt[-Cot[e + f*x]^2]*Sqrt[(d + c*Csc[e + f*x])/(c + d)]*EllipticPi[(2*a)/(a + b), ArcSin[Sqrt[1 - Csc[e + f*x]]/Sqrt[2]], (2*c)/(c + d)]*Sqrt[g*Sin[e + f*x]]*Tan[e + f*x])/((a + b)*f*Sqrt[c + d*Sin[e + f*x]])`

Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3416 `Int[Sqrt[(g_)*sin[(e_) + (f_)*(x_)]]/(Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)])*(x_)])*((c_) + (d_)*sin[(e_) + (f_)*(x_)]), x_Symbol] := Simp[2*Sqrt[-Cot[e + f*x]^2]*(Sqrt[g*Sin[e + f*x]]/(f*(c + d)*Cot[e + f*x]*Sqrt[a + b*Sin[e + f*x]])*Sqrt[(b + a*Csc[e + f*x])/(a + b)]*EllipticPi[2*(c/(c + d)), ArcSin[Sqrt[1 - Csc[e + f*x]]/Sqrt[2]], 2*(a/(a + b))], x] /; FreeQ[{a, b, c, d, e, f, g}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]`

Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 2581 vs. $2(107) = 214$.

Time = 2.13 (sec) , antiderivative size = 2582, normalized size of antiderivative = 22.65

method	result	size
default	Expression too large to display	2582

input `int((g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))/(c+d*sin(f*x+e))^(1/2),x,method=_RETURNVERBOSE)`

output

```

1/f*a/(c*(-a^2+b^2)^(1/2)-a*(-c^2+d^2)^(1/2)-a*d+b*c)/(a*(-c^2+d^2)^(1/2)+
c*(-a^2+b^2)^(1/2)+a*d-b*c)/(-a^2+b^2)^(1/2)*(EllipticPi((1/(d+(-c^2+d^2)^(
1/2)))*(c*csc(f*x+e)-cot(f*x+e)*c+(-c^2+d^2)^(1/2)+d))^(1/2),(d+(-c^2+d^2)
^(1/2))*a/(a*(-c^2+d^2)^(1/2)+c*(-a^2+b^2)^(1/2)+a*d-b*c),1/2*2^(1/2)*((d+
(-c^2+d^2)^(1/2))/(-c^2+d^2)^(1/2))^(1/2))*a*c*d+EllipticPi((1/(d+(-c^2+d^
2)^(1/2)))*(c*csc(f*x+e)-cot(f*x+e)*c+(-c^2+d^2)^(1/2)+d))^(1/2),(d+(-c^2+d
^2)^(1/2))*a/(a*(-c^2+d^2)^(1/2)+c*(-a^2+b^2)^(1/2)+a*d-b*c),1/2*2^(1/2)*
(d+(-c^2+d^2)^(1/2))/(-c^2+d^2)^(1/2))^(1/2))*a*c*(-c^2+d^2)^(1/2)+Ellipti
cPi((1/(d+(-c^2+d^2)^(1/2)))*(c*csc(f*x+e)-cot(f*x+e)*c+(-c^2+d^2)^(1/2)+d)
)^(1/2),(d+(-c^2+d^2)^(1/2))*a/(a*(-c^2+d^2)^(1/2)+c*(-a^2+b^2)^(1/2)+a*d-
b*c),1/2*2^(1/2)*((d+(-c^2+d^2)^(1/2))/(-c^2+d^2)^(1/2))^(1/2))*b*c^2-2*El
lipticPi((1/(d+(-c^2+d^2)^(1/2)))*(c*csc(f*x+e)-cot(f*x+e)*c+(-c^2+d^2)^(1/
2)+d))^(1/2),(d+(-c^2+d^2)^(1/2))*a/(a*(-c^2+d^2)^(1/2)+c*(-a^2+b^2)^(1/2)
+a*d-b*c),1/2*2^(1/2)*((d+(-c^2+d^2)^(1/2))/(-c^2+d^2)^(1/2))^(1/2))*b*d^2
-2*EllipticPi((1/(d+(-c^2+d^2)^(1/2)))*(c*csc(f*x+e)-cot(f*x+e)*c+(-c^2+d^2)
)^(1/2)+d))^(1/2),(d+(-c^2+d^2)^(1/2))*a/(a*(-c^2+d^2)^(1/2)+c*(-a^2+b^2)^(
1/2)+a*d-b*c),1/2*2^(1/2)*((d+(-c^2+d^2)^(1/2))/(-c^2+d^2)^(1/2))^(1/2))*
b*d*(-c^2+d^2)^(1/2)-EllipticPi((1/(d+(-c^2+d^2)^(1/2)))*(c*csc(f*x+e)-cot(
f*x+e)*c+(-c^2+d^2)^(1/2)+d))^(1/2),(d+(-c^2+d^2)^(1/2))*a/(a*(-c^2+d^2)^(
1/2)+c*(-a^2+b^2)^(1/2)+a*d-b*c),1/2*2^(1/2)*((d+(-c^2+d^2)^(1/2))/(-c^...

```

Fricas [F(-1)]

Timed out.

$$\int \frac{\sqrt{g \sin(e + fx)}}{(a + b \sin(e + fx)) \sqrt{c + d \sin(e + fx)}} dx = \text{Timed out}$$

input

```

integrate((g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))/(c+d*sin(f*x+e))^(1/2),x,
algorithm="fricas")

```

output

Timed out

Sympy [F]

$$\int \frac{\sqrt{g \sin(e + fx)}}{(a + b \sin(e + fx)) \sqrt{c + d \sin(e + fx)}} dx$$

$$= \int \frac{\sqrt{g \sin(e + fx)}}{(a + b \sin(e + fx)) \sqrt{c + d \sin(e + fx)}} dx$$

input `integrate((g*sin(f*x+e))**(1/2)/(a+b*sin(f*x+e))/(c+d*sin(f*x+e))**(1/2),x)`

output `Integral(sqrt(g*sin(e + f*x))/((a + b*sin(e + f*x))*sqrt(c + d*sin(e + f*x))), x)`

Maxima [F]

$$\int \frac{\sqrt{g \sin(e + fx)}}{(a + b \sin(e + fx)) \sqrt{c + d \sin(e + fx)}} dx = \int \frac{\sqrt{g \sin(fx + e)}}{(b \sin(fx + e) + a) \sqrt{d \sin(fx + e) + c}} dx$$

input `integrate((g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))/(c+d*sin(f*x+e))^(1/2),x,algorithm="maxima")`

output `integrate(sqrt(g*sin(f*x + e))/((b*sin(f*x + e) + a)*sqrt(d*sin(f*x + e) + c)), x)`

Giac [F]

$$\int \frac{\sqrt{g \sin(e + fx)}}{(a + b \sin(e + fx)) \sqrt{c + d \sin(e + fx)}} dx = \int \frac{\sqrt{g \sin(fx + e)}}{(b \sin(fx + e) + a) \sqrt{d \sin(fx + e) + c}} dx$$

input `integrate((g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))/(c+d*sin(f*x+e))^(1/2),x,algorithm="giac")`

output `integrate(sqrt(g*sin(f*x + e))/((b*sin(f*x + e) + a)*sqrt(d*sin(f*x + e) + c)), x)`

Mupad [F(-1)]

Timed out.

$$\int \frac{\sqrt{g \sin(e + fx)}}{(a + b \sin(e + fx)) \sqrt{c + d \sin(e + fx)}} dx$$

$$= \int \frac{\sqrt{g \sin(e + fx)}}{(a + b \sin(e + fx)) \sqrt{c + d \sin(e + fx)}} dx$$

input `int((g*sin(e + f*x))^(1/2)/((a + b*sin(e + f*x))*(c + d*sin(e + f*x))^(1/2)),x)`

output `int((g*sin(e + f*x))^(1/2)/((a + b*sin(e + f*x))*(c + d*sin(e + f*x))^(1/2)), x)`

Reduce [F]

$$\int \frac{\sqrt{g \sin(e + fx)}}{(a + b \sin(e + fx)) \sqrt{c + d \sin(e + fx)}} dx$$

$$= \sqrt{g} \left(\int \frac{\sqrt{\sin(fx + e)} \sqrt{\sin(fx + e) d + c}}{\sin(fx + e)^2 bd + \sin(fx + e) ad + \sin(fx + e) bc + ac} dx \right)$$

input `int((g*sin(f*x+e))^(1/2)/(a+b*sin(f*x+e))/(c+d*sin(f*x+e))^(1/2),x)`

output `sqrt(g)*int((sqrt(sin(e + f*x))*sqrt(sin(e + f*x)*d + c))/(sin(e + f*x)**2 *b*d + sin(e + f*x)*a*d + sin(e + f*x)*b*c + a*c),x)`

3.48 $\int \csc(e+fx) \sqrt{a+b \sin(e+fx)} \sqrt{c+d \sin(e+fx)} dx$

Optimal result	405
Mathematica [A] (verified)	406
Rubi [A] (verified)	406
Maple [C] (warning: unable to verify)	409
Fricas [F(-1)]	409
Sympy [F]	409
Maxima [F]	410
Giac [F]	410
Mupad [F(-1)]	411
Reduce [F]	411

Optimal result

Integrand size = 35, antiderivative size = 391

$$\int \csc(e+fx) \sqrt{a+b \sin(e+fx)} \sqrt{c+d \sin(e+fx)} dx =$$

$$\frac{2\sqrt{c+d} \operatorname{EllipticPi}\left(\frac{a(c+d)}{(a+b)c}, \arcsin\left(\frac{\sqrt{a+b}\sqrt{c+d \sin(e+fx)}}{\sqrt{c+d}\sqrt{a+b \sin(e+fx)}}\right), \frac{(a-b)(c+d)}{(a+b)(c-d)}\right) \sec(e+fx) \sqrt{-\frac{(bc-ad)(1-\sin(e+fx))}{(c+d)(a+b \sin(e+fx))}}}{\sqrt{a+bf}}$$

$$+ \frac{2\sqrt{c+d} \operatorname{EllipticPi}\left(\frac{b(c+d)}{(a+b)d}, \arcsin\left(\frac{\sqrt{a+b}\sqrt{c+d \sin(e+fx)}}{\sqrt{c+d}\sqrt{a+b \sin(e+fx)}}\right), \frac{(a-b)(c+d)}{(a+b)(c-d)}\right) \sec(e+fx) \sqrt{-\frac{(bc-ad)(1-\sin(e+fx))}{(c+d)(a+b \sin(e+fx))}}}{\sqrt{a+bf}}$$

output

```
-2*(c+d)^(1/2)*EllipticPi((a+b)^(1/2)*(c+d*sin(f*x+e))^(1/2)/(c+d)^(1/2)/(a+b*sin(f*x+e))^(1/2), a*(c+d)/(a+b)/c, ((a-b)*(c+d)/(a+b)/(c-d))^(1/2))*sec(f*x+e)*((-a*d+b*c)*(1-sin(f*x+e))/(c+d)/(a+b*sin(f*x+e)))^(1/2)*((-a*d+b*c)*(1+sin(f*x+e))/(c-d)/(a+b*sin(f*x+e)))^(1/2)*(a+b*sin(f*x+e))/(a+b)^(1/2)/f+2*(c+d)^(1/2)*EllipticPi((a+b)^(1/2)*(c+d*sin(f*x+e))^(1/2)/(c+d)^(1/2)/(a+b*sin(f*x+e))^(1/2), b*(c+d)/(a+b)/d, ((a-b)*(c+d)/(a+b)/(c-d))^(1/2))*sec(f*x+e)*((-a*d+b*c)*(1-sin(f*x+e))/(c+d)/(a+b*sin(f*x+e)))^(1/2)*((-a*d+b*c)*(1+sin(f*x+e))/(c-d)/(a+b*sin(f*x+e)))^(1/2)*(a+b*sin(f*x+e))/(a+b)^(1/2)/f
```

Mathematica [A] (verified)

Time = 0.33 (sec) , antiderivative size = 274, normalized size of antiderivative = 0.70

$$\int \csc(e + fx) \sqrt{a + b \sin(e + fx)} \sqrt{c + d \sin(e + fx)} dx =$$

$$\frac{2\sqrt{c+d} \left(\text{EllipticPi} \left(\frac{a(c+d)}{(a+b)c}, \arcsin \left(\frac{\sqrt{a+b} \sqrt{c+d \sin(e+fx)}}{\sqrt{c+d} \sqrt{a+b \sin(e+fx)}} \right), \frac{(a-b)(c+d)}{(a+b)(c-d)} \right) - \text{EllipticPi} \left(\frac{b(c+d)}{(a+b)d}, \arcsin \left(\frac{\sqrt{a+b} \sqrt{c+d \sin(e+fx)}}{\sqrt{c+d} \sqrt{a+b \sin(e+fx)}} \right) \right)}{\sqrt{a+b}}$$

input

```
Integrate[Csc[e + f*x]*Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]],x
]
```

output

```
(-2*Sqrt[c + d]*(EllipticPi[(a*(c + d))/((a + b)*c), ArcSin[(Sqrt[a + b]*S
qrt[c + d*Sin[e + f*x])]/(Sqrt[c + d]*Sqrt[a + b*Sin[e + f*x])]], ((a - b)
*(c + d))/((a + b)*(c - d))] - EllipticPi[(b*(c + d))/((a + b)*d), ArcSin[
(Sqrt[a + b]*Sqrt[c + d*Sin[e + f*x])]/(Sqrt[c + d]*Sqrt[a + b*Sin[e + f*x]
])]], ((a - b)*(c + d))/((a + b)*(c - d)))*Sec[e + f*x]*Sqrt[((b*c - a*d)
*(-1 + Sin[e + f*x]))/((c + d)*(a + b*Sin[e + f*x]))]*Sqrt[((b*c - a*d)*(1
+ Sin[e + f*x]))/((c - d)*(a + b*Sin[e + f*x]))]*(a + b*Sin[e + f*x]))/(S
qrt[a + b]*f)
```

Rubi [A] (verified)

Time = 1.11 (sec) , antiderivative size = 391, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.143$, Rules used = {3042, 3428, 3042, 3290, 3424}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \csc(e + fx) \sqrt{a + b \sin(e + fx)} \sqrt{c + d \sin(e + fx)} dx$$

$$\downarrow \text{3042}$$

$$\int \frac{\sqrt{a + b \sin(e + fx)} \sqrt{c + d \sin(e + fx)}}{\sin(e + fx)} dx$$

$$\downarrow \text{3428}$$

$$\begin{aligned}
& d \int \frac{\sqrt{a + b \sin(e + fx)}}{\sqrt{c + d \sin(e + fx)}} dx + c \int \frac{\csc(e + fx) \sqrt{a + b \sin(e + fx)}}{\sqrt{c + d \sin(e + fx)}} dx \\
& \quad \downarrow \text{3042} \\
& d \int \frac{\sqrt{a + b \sin(e + fx)}}{\sqrt{c + d \sin(e + fx)}} dx + c \int \frac{\sqrt{a + b \sin(e + fx)}}{\sin(e + fx) \sqrt{c + d \sin(e + fx)}} dx \\
& \quad \downarrow \text{3290} \\
& c \int \frac{\sqrt{a + b \sin(e + fx)}}{\sin(e + fx) \sqrt{c + d \sin(e + fx)}} dx + \\
& \frac{2\sqrt{c+d} \sec(e+fx)(a+b \sin(e+fx)) \sqrt{-\frac{(bc-ad)(1-\sin(e+fx))}{(c+d)(a+b \sin(e+fx))}} \sqrt{\frac{(bc-ad)(\sin(e+fx)+1)}{(c-d)(a+b \sin(e+fx))}} \text{EllipticPi}\left(\frac{b(c+d)}{(a+b)d}, \arcsin\left(\frac{\sqrt{a+b}}{\sqrt{c+d}}\right)\right)}{f\sqrt{a+b}} \\
& \quad \downarrow \text{3424} \\
& \frac{2\sqrt{c+d} \sec(e+fx)(a+b \sin(e+fx)) \sqrt{-\frac{(bc-ad)(1-\sin(e+fx))}{(c+d)(a+b \sin(e+fx))}} \sqrt{\frac{(bc-ad)(\sin(e+fx)+1)}{(c-d)(a+b \sin(e+fx))}} \text{EllipticPi}\left(\frac{b(c+d)}{(a+b)d}, \arcsin\left(\frac{\sqrt{a+b}}{\sqrt{c+d}}\right)\right)}{f\sqrt{a+b}} \\
& \frac{2\sqrt{c+d} \sec(e+fx)(a+b \sin(e+fx)) \sqrt{-\frac{(bc-ad)(1-\sin(e+fx))}{(c+d)(a+b \sin(e+fx))}} \sqrt{\frac{(bc-ad)(\sin(e+fx)+1)}{(c-d)(a+b \sin(e+fx))}} \text{EllipticPi}\left(\frac{a(c+d)}{(a+b)c}, \arcsin\left(\frac{\sqrt{a+b}}{\sqrt{c+d}}\right)\right)}{f\sqrt{a+b}}
\end{aligned}$$

input `Int[Csc[e + f*x]*Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]],x]`

output `(-2*Sqrt[c + d]*EllipticPi[(a*(c + d))/((a + b)*c), ArcSin[(Sqrt[a + b]*Sqrt[c + d*Sin[e + f*x]])/(Sqrt[c + d]*Sqrt[a + b*Sin[e + f*x]])], ((a - b)*(c + d))/((a + b)*(c - d))*Sec[e + f*x]*Sqrt[-(((b*c - a*d)*(1 - Sin[e + f*x]))/((c + d)*(a + b*Sin[e + f*x])))]*Sqrt[((b*c - a*d)*(1 + Sin[e + f*x]))/((c - d)*(a + b*Sin[e + f*x]))]*(a + b*Sin[e + f*x])/(Sqrt[a + b]*f) + (2*Sqrt[c + d]*EllipticPi[(b*(c + d))/((a + b)*d), ArcSin[(Sqrt[a + b]*Sqrt[c + d*Sin[e + f*x]])/(Sqrt[c + d]*Sqrt[a + b*Sin[e + f*x]])], ((a - b)*(c + d))/((a + b)*(c - d))*Sec[e + f*x]*Sqrt[-(((b*c - a*d)*(1 - Sin[e + f*x]))/((c + d)*(a + b*Sin[e + f*x])))]*Sqrt[((b*c - a*d)*(1 + Sin[e + f*x]))/((c - d)*(a + b*Sin[e + f*x]))]*(a + b*Sin[e + f*x])/(Sqrt[a + b]*f)`

Definitions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3290 `Int[Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)]]/Sqrt[(c_) + (d_)*sin[(e_) + (f_)*(x_)]], x_Symbol] := Simp[2*((a + b*Sin[e + f*x])/(d*f*Rt[(a + b)/(c + d), 2]*Cos[e + f*x]))*Sqrt[(b*c - a*d)*((1 + Sin[e + f*x])/((c - d)*(a + b*Sin[e + f*x])))]*Sqrt[(-(b*c - a*d))*((1 - Sin[e + f*x])/((c + d)*(a + b*Sin[e + f*x])))]*EllipticPi[b*((c + d)/(d*(a + b))), ArcSin[Rt[(a + b)/(c + d), 2]*(Sqrt[c + d*Sin[e + f*x]]/Sqrt[a + b*Sin[e + f*x]])], (a - b)*((c + d)/((a + b)*(c - d))), x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0] && PosQ[(a + b)/(c + d)]`

rule 3424 `Int[Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)]]/(sin[(e_) + (f_)*(x_)]*Sqrt[(c_) + (d_)*sin[(e_) + (f_)*(x_)]]), x_Symbol] := Simp[-2*((a + b*Sin[e + f*x])/(c*f*Rt[(a + b)/(c + d), 2]*Cos[e + f*x]))*Sqrt[(-(b*c - a*d))*((1 - Sin[e + f*x])/((c + d)*(a + b*Sin[e + f*x])))]*Sqrt[(b*c - a*d)*((1 + Sin[e + f*x])/((c - d)*(a + b*Sin[e + f*x])))]*EllipticPi[a*((c + d)/(c*(a + b))), ArcSin[Rt[(a + b)/(c + d), 2]*(Sqrt[c + d*Sin[e + f*x]]/Sqrt[a + b*Sin[e + f*x]])], (a - b)*((c + d)/((a + b)*(c - d))), x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]`

rule 3428 `Int[(Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)]]*Sqrt[(c_) + (d_)*sin[(e_) + (f_)*(x_)]])/sin[(e_) + (f_)*(x_)], x_Symbol] := Simp[d Int[Sqrt[a + b*Sin[e + f*x]]/Sqrt[c + d*Sin[e + f*x]], x], x] + Simp[c Int[Sqrt[a + b*Sin[e + f*x]]/(Sin[e + f*x]*Sqrt[c + d*Sin[e + f*x]]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && (NeQ[a^2 - b^2, 0] || NeQ[c^2 - d^2, 0])`

Maple [C] (warning: unable to verify)

Result contains complex when optimal does not.

Time = 8.97 (sec) , antiderivative size = 250622, normalized size of antiderivative = 640.98

method	result	size
default	Expression too large to display	250622

input `int(csc(f*x+e)*(a+b*sin(f*x+e))^(1/2)*(c+d*sin(f*x+e))^(1/2),x,method=_RETURNVERBOSE)`

output `result too large to display`

Fricas [F(-1)]

Timed out.

$$\int \csc(e + fx) \sqrt{a + b \sin(e + fx)} \sqrt{c + d \sin(e + fx)} dx = \text{Timed out}$$

input `integrate(csc(f*x+e)*(a+b*sin(f*x+e))^(1/2)*(c+d*sin(f*x+e))^(1/2),x, algorithm="fricas")`

output `Timed out`

Sympy [F]

$$\begin{aligned} & \int \csc(e + fx) \sqrt{a + b \sin(e + fx)} \sqrt{c + d \sin(e + fx)} dx \\ &= \int \sqrt{a + b \sin(e + fx)} \sqrt{c + d \sin(e + fx)} \csc(e + fx) dx \end{aligned}$$

input `integrate(csc(f*x+e)*(a+b*sin(f*x+e))**(1/2)*(c+d*sin(f*x+e))**(1/2),x)`

output `Integral(sqrt(a + b*sin(e + f*x))*sqrt(c + d*sin(e + f*x))*csc(e + f*x), x)`

Maxima [F]

$$\int \csc(e + fx) \sqrt{a + b \sin(e + fx)} \sqrt{c + d \sin(e + fx)} dx$$

$$= \int \sqrt{b \sin(fx + e) + a} \sqrt{d \sin(fx + e) + c} \csc(fx + e) dx$$

input `integrate(csc(f*x+e)*(a+b*sin(f*x+e))^(1/2)*(c+d*sin(f*x+e))^(1/2),x, algorith="maxima")`

output `integrate(sqrt(b*sin(f*x + e) + a)*sqrt(d*sin(f*x + e) + c)*csc(f*x + e), x)`

Giac [F]

$$\int \csc(e + fx) \sqrt{a + b \sin(e + fx)} \sqrt{c + d \sin(e + fx)} dx$$

$$= \int \sqrt{b \sin(fx + e) + a} \sqrt{d \sin(fx + e) + c} \csc(fx + e) dx$$

input `integrate(csc(f*x+e)*(a+b*sin(f*x+e))^(1/2)*(c+d*sin(f*x+e))^(1/2),x, algorith="giac")`

output `integrate(sqrt(b*sin(f*x + e) + a)*sqrt(d*sin(f*x + e) + c)*csc(f*x + e), x)`

Mupad [F(-1)]

Timed out.

$$\int \csc(e + fx) \sqrt{a + b \sin(e + fx)} \sqrt{c + d \sin(e + fx)} dx$$

$$= \int \frac{\sqrt{a + b \sin(e + fx)} \sqrt{c + d \sin(e + fx)}}{\sin(e + fx)} dx$$

input

```
int(((a + b*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))^(1/2))/sin(e + f*x),x)
```

output

```
int(((a + b*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))^(1/2))/sin(e + f*x),x)
```

Reduce [F]

$$\int \csc(e + fx) \sqrt{a + b \sin(e + fx)} \sqrt{c + d \sin(e + fx)} dx$$

$$= \int \sqrt{\sin(fx + e) d + c} \sqrt{\sin(fx + e) b + a} \csc(fx + e) dx$$

input

```
int(csc(f*x+e)*(a+b*sin(f*x+e))^(1/2)*(c+d*sin(f*x+e))^(1/2),x)
```

output

```
int(sqrt(sin(e + f*x)*d + c)*sqrt(sin(e + f*x)*b + a)*csc(e + f*x),x)
```

$$3.49 \quad \int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}} dx$$

Optimal result	412
Mathematica [A] (verified)	413
Rubi [A] (verified)	413
Maple [B] (warning: unable to verify)	415
Fricas [F(-1)]	415
Sympy [F]	415
Maxima [F]	416
Giac [F]	416
Mupad [F(-1)]	417
Reduce [F]	417

Optimal result

Integrand size = 35, antiderivative size = 198

$$\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}} dx = \frac{2\sqrt{c+d}\operatorname{EllipticPi}\left(\frac{a(c+d)}{(a+b)c}, \arcsin\left(\frac{\sqrt{a+b}\sqrt{c+d\sin(e+fx)}}{\sqrt{c+d}\sqrt{a+b\sin(e+fx)}}\right), \frac{(a-b)(c+d)}{(a+b)(c-d)}\right) \sec(e+fx) \sqrt{-\frac{(bc-ad)(1-\sin(e+fx))}{(c+d)(a+b\sin(e+fx))}}}{\sqrt{a+bcf}}$$

output

```
-2*(c+d)^(1/2)*EllipticPi((a+b)^(1/2)*(c+d*sin(f*x+e))^(1/2)/(c+d)^(1/2)/(a+b*sin(f*x+e))^(1/2), a*(c+d)/(a+b)/c, ((a-b)*(c+d)/(a+b)/(c-d))^(1/2))*sec(f*x+e)*(-(-a*d+b*c)*(1-sin(f*x+e))/(c+d)/(a+b*sin(f*x+e)))^(1/2)*((-a*d+b*c)*(1+sin(f*x+e))/(c-d)/(a+b*sin(f*x+e)))^(1/2)*(a+b*sin(f*x+e))/(a+b)^(1/2)/c/f
```

Mathematica [A] (verified)

Time = 0.20 (sec) , antiderivative size = 197, normalized size of antiderivative = 0.99

$$\int \frac{\csc(e + fx) \sqrt{a + b \sin(e + fx)}}{\sqrt{c + d \sin(e + fx)}} dx = \frac{2\sqrt{c + d} \operatorname{EllipticPi}\left(\frac{a(c+d)}{(a+b)c}, \arcsin\left(\frac{\sqrt{a+b}\sqrt{c+d\sin(e+fx)}}{\sqrt{c+d}\sqrt{a+b\sin(e+fx)}}\right), \frac{(a-b)(c+d)}{(a+b)(c-d)}\right) \sec(e + fx) \sqrt{\frac{(-bc+ad)(1-\sin(e+fx))}{(c+d)(a+b\sin(e+fx))}}}{\sqrt{a + bcf}}$$

input

```
Integrate[(Csc[e + f*x]*Sqrt[a + b*Sin[e + f*x]])/Sqrt[c + d*Sin[e + f*x]]
, x]
```

output

```
(-2*Sqrt[c + d]*EllipticPi[(a*(c + d))/((a + b)*c), ArcSin[(Sqrt[a + b]*Sqrt[c + d*Sin[e + f*x]])/(Sqrt[c + d]*Sqrt[a + b*Sin[e + f*x]])], ((a - b)*(c + d))/((a + b)*(c - d)))*Sec[e + f*x]*Sqrt[((-b*c) + a*d)*(1 - Sin[e + f*x])]/((c + d)*(a + b*Sin[e + f*x]))]*Sqrt[((b*c - a*d)*(1 + Sin[e + f*x]))]/((c - d)*(a + b*Sin[e + f*x]))]*(a + b*Sin[e + f*x])/(Sqrt[a + b]*c*f)
```

Rubi [A] (verified)

Time = 0.48 (sec) , antiderivative size = 198, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}} = 0.057$, Rules used = {3042, 3424}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e + fx) \sqrt{a + b \sin(e + fx)}}{\sqrt{c + d \sin(e + fx)}} dx$$

↓ 3042

$$\int \frac{\sqrt{a + b \sin(e + fx)}}{\sin(e + fx) \sqrt{c + d \sin(e + fx)}} dx$$

↓ 3424

$$\frac{2\sqrt{c+d}\sec(e+fx)(a+b\sin(e+fx))\sqrt{-\frac{(bc-ad)(1-\sin(e+fx))}{(c+d)(a+b\sin(e+fx))}}\sqrt{\frac{(bc-ad)(\sin(e+fx)+1)}{(c-d)(a+b\sin(e+fx))}}\operatorname{EllipticPi}\left(\frac{a(c+d)}{(a+b)c}, \arcsin\left(\frac{a+b\sin(e+fx)}{c}\right)\right)}{cf\sqrt{a+b}}$$

input

```
Int[(Csc[e + f*x]*Sqrt[a + b*Sin[e + f*x]])/Sqrt[c + d*Sin[e + f*x]],x]
```

output

```
(-2*Sqrt[c + d]*EllipticPi[(a*(c + d))/((a + b)*c), ArcSin[(Sqrt[a + b]*Sqrt[c + d*Sin[e + f*x]])/(Sqrt[c + d]*Sqrt[a + b*Sin[e + f*x]])], ((a - b)*(c + d))/((a + b)*(c - d)))*Sec[e + f*x]*Sqrt[-((b*c - a*d)*(1 - Sin[e + f*x]))/((c + d)*(a + b*Sin[e + f*x]))]*Sqrt[((b*c - a*d)*(1 + Sin[e + f*x]))/((c - d)*(a + b*Sin[e + f*x]))]*(a + b*Sin[e + f*x])/(Sqrt[a + b]*c*f)
```

Defintions of rubi rules used

rule 3042

```
Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]
```

rule 3424

```
Int[Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)]]/(sin[(e_) + (f_)*(x_)]*Sqrt[(c_) + (d_)*sin[(e_) + (f_)*(x_)]]), x_Symbol] := Simp[-2*((a + b*Sin[e + f*x])/(c*f*Rt[(a + b)/(c + d), 2]*Cos[e + f*x]))*Sqrt[(-(b*c - a*d))*((1 - Sin[e + f*x])/((c + d)*(a + b*Sin[e + f*x])))]*Sqrt[(b*c - a*d)*((1 + Sin[e + f*x])/((c - d)*(a + b*Sin[e + f*x])))]*EllipticPi[a*((c + d)/(c*(a + b))), ArcSin[Rt[(a + b)/(c + d), 2]*(Sqrt[c + d*Sin[e + f*x]])/Sqrt[a + b*Sin[e + f*x]]], (a - b)*((c + d)/((a + b)*(c - d))), x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]
```

Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 30788 vs. $2(183) = 366$.

Time = 7.35 (sec) , antiderivative size = 30789, normalized size of antiderivative = 155.50

method	result	size
default	Expression too large to display	30789

input `int(csc(f*x+e)*(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x,method=_RETURNVERBOSE)`

output `result too large to display`

Fricas [F(-1)]

Timed out.

$$\int \frac{\csc(e + fx) \sqrt{a + b \sin(e + fx)}}{\sqrt{c + d \sin(e + fx)}} dx = \text{Timed out}$$

input `integrate(csc(f*x+e)*(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x, algorithm="fricas")`

output `Timed out`

Sympy [F]

$$\int \frac{\csc(e + fx) \sqrt{a + b \sin(e + fx)}}{\sqrt{c + d \sin(e + fx)}} dx = \int \frac{\sqrt{a + b \sin(e + fx)} \csc(e + fx)}{\sqrt{c + d \sin(e + fx)}} dx$$

input `integrate(csc(f*x+e)*(a+b*sin(f*x+e))**(1/2)/(c+d*sin(f*x+e))**(1/2),x)`

output `Integral(sqrt(a + b*sin(e + f*x))*csc(e + f*x)/sqrt(c + d*sin(e + f*x)), x)`

Maxima [F]

$$\int \frac{\csc(e + fx)\sqrt{a + b\sin(e + fx)}}{\sqrt{c + d\sin(e + fx)}} dx = \int \frac{\sqrt{b\sin(fx + e) + a}\csc(fx + e)}{\sqrt{d\sin(fx + e) + c}} dx$$

input `integrate(csc(f*x+e)*(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x, algorith="maxima")`

output `integrate(sqrt(b*sin(f*x + e) + a)*csc(f*x + e)/sqrt(d*sin(f*x + e) + c), x)`

Giac [F]

$$\int \frac{\csc(e + fx)\sqrt{a + b\sin(e + fx)}}{\sqrt{c + d\sin(e + fx)}} dx = \int \frac{\sqrt{b\sin(fx + e) + a}\csc(fx + e)}{\sqrt{d\sin(fx + e) + c}} dx$$

input `integrate(csc(f*x+e)*(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x, algorith="giac")`

output `integrate(sqrt(b*sin(f*x + e) + a)*csc(f*x + e)/sqrt(d*sin(f*x + e) + c), x)`

Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}} dx = \int \frac{\sqrt{a+b\sin(e+fx)}}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}} dx$$

input `int((a + b*sin(e + f*x))^(1/2)/(sin(e + f*x)*(c + d*sin(e + f*x))^(1/2)),x)`

output `int((a + b*sin(e + f*x))^(1/2)/(sin(e + f*x)*(c + d*sin(e + f*x))^(1/2)),x)`

Reduce [F]

$$\begin{aligned} & \int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}} dx \\ &= \int \frac{\sqrt{\sin(fx+e)d+c}\sqrt{\sin(fx+e)b+a}\csc(fx+e)}{\sin(fx+e)d+c} dx \end{aligned}$$

input `int(csc(f*x+e)*(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x)`

output `int((sqrt(sin(e + f*x)*d + c)*sqrt(sin(e + f*x)*b + a)*csc(e + f*x))/(sin(e + f*x)*d + c),x)`

3.50
$$\int \frac{\csc(e+fx)}{\sqrt{a+b \sin(e+fx)} \sqrt{c+d \sin(e+fx)}} dx$$

Optimal result	418
Mathematica [A] (verified)	419
Rubi [A] (verified)	419
Maple [B] (warning: unable to verify)	422
Fricas [F(-1)]	422
Sympy [F]	422
Maxima [F]	423
Giac [F]	423
Mupad [F(-1)]	424
Reduce [F]	424

Optimal result

Integrand size = 35, antiderivative size = 398

$$\int \frac{\csc(e+fx)}{\sqrt{a+b \sin(e+fx)} \sqrt{c+d \sin(e+fx)}} dx =$$

$$\frac{2\sqrt{c+d} \operatorname{EllipticPi}\left(\frac{a(c+d)}{(a+b)c}, \arcsin\left(\frac{\sqrt{a+b}\sqrt{c+d \sin(e+fx)}}{\sqrt{c+d}\sqrt{a+b \sin(e+fx)}}\right), \frac{(a-b)(c+d)}{(a+b)(c-d)}\right) \sec(e+fx) \sqrt{-\frac{(bc-ad)(1-\sin(e+fx))}{(c+d)(a+b \sin(e+fx))}} \sqrt{a\sqrt{a+bc}f}}{2b\sqrt{a+b} \operatorname{EllipticF}\left(\arcsin\left(\frac{\sqrt{c+d}\sqrt{a+b \sin(e+fx)}}{\sqrt{a+b}\sqrt{c+d \sin(e+fx)}}\right), \frac{(a+b)(c-d)}{(a-b)(c+d)}\right) \sec(e+fx) \sqrt{\frac{(bc-ad)(1-\sin(e+fx))}{(a+b)(c+d \sin(e+fx))}} \sqrt{-\frac{(bc-ad)}{(a-b)(c+d)}} \sqrt{c+d}(bc-ad)f}$$

output

```
-2*(c+d)^(1/2)*EllipticPi((a+b)^(1/2)*(c+d*sin(f*x+e))^(1/2)/(c+d)^(1/2)/(a+b*sin(f*x+e))^(1/2), a*(c+d)/(a+b)/c, ((a-b)*(c+d)/(a+b)/(c-d))^(1/2))*sec(f*x+e)*((-a*d+b*c)*(1-sin(f*x+e))/(c+d)/(a+b*sin(f*x+e)))^(1/2)*((-a*d+b*c)*(1+sin(f*x+e))/(c-d)/(a+b*sin(f*x+e)))^(1/2)*(a+b*sin(f*x+e))/a/(a+b)^(1/2)/c/f-2*b*(a+b)^(1/2)*EllipticF((c+d)^(1/2)*(a+b*sin(f*x+e))^(1/2)/(a+b)^(1/2)/(c+d*sin(f*x+e))^(1/2), ((a+b)*(c-d)/(a-b)/(c+d))^(1/2))*sec(f*x+e)*((-a*d+b*c)*(1-sin(f*x+e))/(a+b)/(c+d*sin(f*x+e)))^(1/2)*((-a*d+b*c)*(1+sin(f*x+e))/(a-b)/(c+d*sin(f*x+e)))^(1/2)*(c+d*sin(f*x+e))/a/(c+d)^(1/2)/(-a*d+b*c)/f
```

Mathematica [A] (verified)

Time = 2.76 (sec) , antiderivative size = 374, normalized size of antiderivative = 0.94

$$\int \frac{\csc(e + fx)}{\sqrt{a + b \sin(e + fx)} \sqrt{c + d \sin(e + fx)}} dx$$

$$= 2 \sec(e + fx) \left(- \frac{(c+d) \operatorname{EllipticPi}\left(\frac{a(c+d)}{(a+b)c}, \arcsin\left(\frac{\sqrt{a+b} \sqrt{c+d \sin(e+fx)}}{\sqrt{c+d} \sqrt{a+b \sin(e+fx)}}\right), \frac{(a-b)(c+d)}{(a+b)(c-d)}\right) \sqrt{\frac{(bc-ad)(-1+\sin(e+fx))}{(c+d)(a+b \sin(e+fx))}} \sqrt{\frac{(bc-ad)(1+\sin(e+fx))}{(c-d)(a+b \sin(e+fx))}}}{c} \right)$$

input

```
Integrate[Csc[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]]),x]
```

output

```
(2*Sec[e + f*x]*(-(((c + d)*EllipticPi[(a*(c + d))/((a + b)*c), ArcSin[(Sqrt[a + b]*Sqrt[c + d*Sin[e + f*x]])/(Sqrt[c + d]*Sqrt[a + b*Sin[e + f*x]])], ((a - b)*(c + d))/((a + b)*(c - d))]*Sqrt[((b*c - a*d)*(-1 + Sin[e + f*x]))]/((c + d)*(a + b*Sin[e + f*x]))]*Sqrt[((b*c - a*d)*(1 + Sin[e + f*x]))]/((c - d)*(a + b*Sin[e + f*x]))]*(a + b*Sin[e + f*x])/c - (b*(a + b)*EllipticF[ArcSin[(Sqrt[c + d]*Sqrt[a + b*Sin[e + f*x]])/(Sqrt[a + b]*Sqrt[c + d*Sin[e + f*x]])], ((a + b)*(c - d))/((a - b)*(c + d))]*Sqrt[((-b*c) + a*d)*(-1 + Sin[e + f*x])/((a + b)*(c + d*Sin[e + f*x]))]*Sqrt[((-b*c) + a*d)*(1 + Sin[e + f*x])/((a - b)*(c + d*Sin[e + f*x]))]*(c + d*Sin[e + f*x])]/(b*c - a*d)))/(a*Sqrt[a + b]*Sqrt[c + d]*f)
```

Rubi [A] (verified)Time = 1.11 (sec) , antiderivative size = 398, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.143$, Rules used = {3042, 3426, 3042, 3297, 3424}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc(e + fx)}{\sqrt{a + b \sin(e + fx)} \sqrt{c + d \sin(e + fx)}} dx$$

↓ 3042

$$\begin{aligned}
 & \int \frac{1}{\sin(e+fx)\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}} dx \\
 & \quad \downarrow \text{3426} \\
 & \frac{\int \frac{\csc(e+fx)\sqrt{a+b\sin(e+fx)}}{\sqrt{c+d\sin(e+fx)}} dx}{a} - \frac{b \int \frac{1}{\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}} dx}{a} \\
 & \quad \downarrow \text{3042} \\
 & \frac{\int \frac{\sqrt{a+b\sin(e+fx)}}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}} dx}{a} - \frac{b \int \frac{1}{\sqrt{a+b\sin(e+fx)}\sqrt{c+d\sin(e+fx)}} dx}{a} \\
 & \quad \downarrow \text{3297} \\
 & \frac{\int \frac{\sqrt{a+b\sin(e+fx)}}{\sin(e+fx)\sqrt{c+d\sin(e+fx)}} dx}{a} - \\
 & \frac{2b\sqrt{a+b}\sec(e+fx)(c+d\sin(e+fx))\sqrt{\frac{(bc-ad)(1-\sin(e+fx))}{(a+b)(c+d\sin(e+fx))}}\sqrt{-\frac{(bc-ad)(\sin(e+fx)+1)}{(a-b)(c+d\sin(e+fx))}} \operatorname{EllipticF}\left(\arcsin\left(\frac{\sqrt{c+d}\sqrt{a+b}}{\sqrt{a+b}\sqrt{c+d}}\right)\right)}{af\sqrt{c+d}(bc-ad)} \\
 & \quad \downarrow \text{3424} \\
 & \frac{2b\sqrt{a+b}\sec(e+fx)(c+d\sin(e+fx))\sqrt{\frac{(bc-ad)(1-\sin(e+fx))}{(a+b)(c+d\sin(e+fx))}}\sqrt{-\frac{(bc-ad)(\sin(e+fx)+1)}{(a-b)(c+d\sin(e+fx))}} \operatorname{EllipticF}\left(\arcsin\left(\frac{\sqrt{c+d}\sqrt{a+b}}{\sqrt{a+b}\sqrt{c+d}}\right)\right)}{af\sqrt{c+d}(bc-ad)} \\
 & \frac{2\sqrt{c+d}\sec(e+fx)(a+b\sin(e+fx))\sqrt{-\frac{(bc-ad)(1-\sin(e+fx))}{(c+d)(a+b\sin(e+fx))}}\sqrt{\frac{(bc-ad)(\sin(e+fx)+1)}{(c-d)(a+b\sin(e+fx))}} \operatorname{EllipticPi}\left(\frac{a(c+d)}{(a+b)c}, \arcsin\left(\frac{\sqrt{c+d}}{\sqrt{a+b}}\right)\right)}{acf\sqrt{a+b}}
 \end{aligned}$$

input `Int[Csc[e + f*x]/(Sqrt[a + b*Sin[e + f*x]]*Sqrt[c + d*Sin[e + f*x]]),x]`

output `(-2*Sqrt[c + d]*EllipticPi[(a*(c + d))/((a + b)*c), ArcSin[(Sqrt[a + b]*Sqrt[c + d*Sin[e + f*x]])/(Sqrt[c + d]*Sqrt[a + b*Sin[e + f*x]])], ((a - b)*(c + d))/((a + b)*(c - d)))*Sec[e + f*x]*Sqrt[-(((b*c - a*d)*(1 - Sin[e + f*x]))/((c + d)*(a + b*Sin[e + f*x])))]*Sqrt[((b*c - a*d)*(1 + Sin[e + f*x]))/((c - d)*(a + b*Sin[e + f*x]))]*(a + b*Sin[e + f*x])/(a*Sqrt[a + b]*c*f) - (2*b*Sqrt[a + b]*EllipticF[ArcSin[(Sqrt[c + d]*Sqrt[a + b*Sin[e + f*x]])/(Sqrt[a + b]*Sqrt[c + d*Sin[e + f*x]])], ((a + b)*(c - d))/((a - b)*(c + d)))*Sec[e + f*x]*Sqrt[((b*c - a*d)*(1 - Sin[e + f*x]))/((a + b)*(c + d*Sin[e + f*x]))]*Sqrt[-(((b*c - a*d)*(1 + Sin[e + f*x]))/((a - b)*(c + d*Sin[e + f*x])))]*(c + d*Sin[e + f*x])/(a*Sqrt[c + d]*(b*c - a*d)*f)`

Definitions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3297 `Int[1/(Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)])*Sqrt[(c_) + (d_)*sin[(e_) + (f_)*(x_)])], x_Symbol] := Simp[2*((c + d*SIN[e + f*x])/(f*(b*c - a*d)*Rt[(c + d)/(a + b), 2]*Cos[e + f*x]))*Sqrt[(b*c - a*d)*((1 - Sin[e + f*x])/((a + b)*(c + d*SIN[e + f*x])))]*Sqrt[(-(b*c - a*d))*((1 + Sin[e + f*x])/((a - b)*(c + d*SIN[e + f*x])))]*EllipticF[ArcSin[Rt[(c + d)/(a + b), 2]*(Sqrt[a + b*SIN[e + f*x]]/Sqrt[c + d*SIN[e + f*x]])], (a + b)*((c - d)/((a - b)*(c + d)))]], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0] && PosQ[(c + d)/(a + b)]`

rule 3424 `Int[Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)]]/(sin[(e_) + (f_)*(x_)])*Sqrt[(c_) + (d_)*sin[(e_) + (f_)*(x_)])], x_Symbol] := Simp[-2*((a + b*SIN[e + f*x])/(c*f*Rt[(a + b)/(c + d), 2]*Cos[e + f*x]))*Sqrt[(-(b*c - a*d))*((1 - Sin[e + f*x])/((c + d)*(a + b*SIN[e + f*x])))]*Sqrt[(b*c - a*d)*((1 + Sin[e + f*x])/((c - d)*(a + b*SIN[e + f*x])))]*EllipticPi[a*((c + d)/(c*(a + b))), ArcSin[Rt[(a + b)/(c + d), 2]*(Sqrt[c + d*SIN[e + f*x]]/Sqrt[a + b*SIN[e + f*x]])], (a - b)*((c + d)/((a + b)*(c - d)))]], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && NeQ[a^2 - b^2, 0] && NeQ[c^2 - d^2, 0]`

rule 3426 `Int[1/(sin[(e_) + (f_)*(x_)])*Sqrt[(a_) + (b_)*sin[(e_) + (f_)*(x_)])*Sqrt[(c_) + (d_)*sin[(e_) + (f_)*(x_)])], x_Symbol] := Simp[-b/a Int[1/(Sqrt[a + b*SIN[e + f*x]]*Sqrt[c + d*SIN[e + f*x]]), x], x] + Simp[1/a Int[Sqrt[a + b*SIN[e + f*x]]/(Sin[e + f*x]*Sqrt[c + d*SIN[e + f*x]]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c - a*d, 0] && (NeQ[a^2 - b^2, 0] || NeQ[c^2 - d^2, 0])`

Maple [B] (warning: unable to verify)

Leaf count of result is larger than twice the leaf count of optimal. 25571 vs. $2(368) = 736$.

Time = 9.52 (sec) , antiderivative size = 25572, normalized size of antiderivative = 64.25

method	result	size
default	Expression too large to display	25572

input `int(csc(f*x+e)/(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x,method=_RETURNVERBOSE)`

output `result too large to display`

Fricas [F(-1)]

Timed out.

$$\int \frac{\csc(e + fx)}{\sqrt{a + b \sin(e + fx)} \sqrt{c + d \sin(e + fx)}} dx = \text{Timed out}$$

input `integrate(csc(f*x+e)/(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x, algorithm="fricas")`

output `Timed out`

Sympy [F]

$$\begin{aligned} & \int \frac{\csc(e + fx)}{\sqrt{a + b \sin(e + fx)} \sqrt{c + d \sin(e + fx)}} dx \\ &= \int \frac{\csc(e + fx)}{\sqrt{a + b \sin(e + fx)} \sqrt{c + d \sin(e + fx)}} dx \end{aligned}$$

input `integrate(csc(f*x+e)/(a+b*sin(f*x+e))**(1/2)/(c+d*sin(f*x+e))**(1/2),x)`

output

```
Integral(csc(e + f*x)/(sqrt(a + b*sin(e + f*x))*sqrt(c + d*sin(e + f*x))),
x)
```

Maxima [F]

$$\int \frac{\csc(e + fx)}{\sqrt{a + b \sin(e + fx)} \sqrt{c + d \sin(e + fx)}} dx$$

$$= \int \frac{\csc(fx + e)}{\sqrt{b \sin(fx + e) + a} \sqrt{d \sin(fx + e) + c}} dx$$

input

```
integrate(csc(f*x+e)/(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x, algo
rithm="maxima")
```

output

```
integrate(csc(f*x + e)/(sqrt(b*sin(f*x + e) + a)*sqrt(d*sin(f*x + e) + c))
, x)
```

Giac [F]

$$\int \frac{\csc(e + fx)}{\sqrt{a + b \sin(e + fx)} \sqrt{c + d \sin(e + fx)}} dx$$

$$= \int \frac{\csc(fx + e)}{\sqrt{b \sin(fx + e) + a} \sqrt{d \sin(fx + e) + c}} dx$$

input

```
integrate(csc(f*x+e)/(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x, algo
rithm="giac")
```

output

```
integrate(csc(f*x + e)/(sqrt(b*sin(f*x + e) + a)*sqrt(d*sin(f*x + e) + c))
, x)
```


Mupad [F(-1)]

Timed out.

$$\int \frac{\csc(e + fx)}{\sqrt{a + b \sin(e + fx)} \sqrt{c + d \sin(e + fx)}} dx$$

$$= \int \frac{1}{\sin(e + fx) \sqrt{a + b \sin(e + fx)} \sqrt{c + d \sin(e + fx)}} dx$$

input `int(1/(sin(e + f*x)*(a + b*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))^(1/2)),x)`

output `int(1/(sin(e + f*x)*(a + b*sin(e + f*x))^(1/2)*(c + d*sin(e + f*x))^(1/2)), x)`

Reduce [F]

$$\int \frac{\csc(e + fx)}{\sqrt{a + b \sin(e + fx)} \sqrt{c + d \sin(e + fx)}} dx$$

$$= \int \frac{\sqrt{\sin(fx + e)} d + c \sqrt{\sin(fx + e)} b + a \csc(fx + e)}{\sin(fx + e)^2 bd + \sin(fx + e) ad + \sin(fx + e) bc + ac} dx$$

input `int(csc(f*x+e)/(a+b*sin(f*x+e))^(1/2)/(c+d*sin(f*x+e))^(1/2),x)`

output `int((sqrt(sin(e + f*x)*d + c)*sqrt(sin(e + f*x)*b + a)*csc(e + f*x))/(sin(e + f*x)**2*b*d + sin(e + f*x)*a*d + sin(e + f*x)*b*c + a*c),x)`

3.51 $\int (a+a \sin(e+fx))^m (A+B \sin(e+fx))^p (c-c \sin(e+fx))^n dx$

Optimal result	425
Mathematica [F]	425
Rubi [A] (verified)	426
Maple [F]	428
Fricas [F]	428
Sympy [F(-1)]	429
Maxima [F]	429
Giac [F]	430
Mupad [F(-1)]	430
Reduce [F]	431

Optimal result

Integrand size = 38, antiderivative size = 157

$$\int (a + a \sin(e + fx))^m (A + B \sin(e + fx))^p (c - c \sin(e + fx))^n dx$$

$$= \frac{2^{\frac{1}{2}+n} \operatorname{AppellF1}\left(\frac{1}{2} + m, \frac{1}{2} - n, -p, \frac{3}{2} + m, \frac{1}{2}(1 + \sin(e + fx)), -\frac{B(1 + \sin(e + fx))}{A - B}\right) \sec(e + fx)(1 - \sin(e + fx))}{af(1 + 2m)}$$

output

```
2^(1/2+n)*AppellF1(1/2+m,-p,1/2-n,3/2+m,-B*(1+sin(f*x+e))/(A-B),1/2+1/2*si
n(f*x+e))*sec(f*x+e)*(1-sin(f*x+e))^(1/2-n)*(a+a*sin(f*x+e))^(1+m)*(A+B*si
n(f*x+e))^p*(c-c*sin(f*x+e))^n/a/f/(1+2*m)/(((A+B*sin(f*x+e))/(A-B))^p)
```

Mathematica [F]

$$\int (a + a \sin(e + fx))^m (A + B \sin(e + fx))^p (c - c \sin(e + fx))^n dx$$

$$= \int (a + a \sin(e + fx))^m (A + B \sin(e + fx))^p (c - c \sin(e + fx))^n dx$$

input

```
Integrate[(a + a*Sin[e + f*x])^m*(A + B*Sin[e + f*x])^p*(c - c*Sin[e + f*x])^n,x]
```

output

```
Integrate[(a + a*Sin[e + f*x])^m*(A + B*Sin[e + f*x])^p*(c - c*Sin[e + f*x])^n, x]
```

Rubi [A] (verified)

Time = 0.47 (sec) , antiderivative size = 157, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.132$, Rules used = {3042, 3487, 157, 156, 155}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int (a \sin(e + fx) + a)^m (c - c \sin(e + fx))^n (A + B \sin(e + fx))^p dx$$

↓ 3042

$$\int (a \sin(e + fx) + a)^m (c - c \sin(e + fx))^n (A + B \sin(e + fx))^p dx$$

↓ 3487

$$\frac{\sec(e + fx) \sqrt{a \sin(e + fx) + a} \sqrt{c - c \sin(e + fx)} \int (\sin(e + fx)a + a)^{m-\frac{1}{2}} (A + B \sin(e + fx))^p (c - c \sin(e + fx))^n dx}{f}$$

↓ 157

$$\frac{\sec(e + fx) \sqrt{a \sin(e + fx) + a} \sqrt{c - c \sin(e + fx)} (A + B \sin(e + fx))^p \left(\frac{A+B \sin(e+fx)}{A-B} \right)^{-p} \int (\sin(e + fx)a + a)^n dx}{f}$$

↓ 156

$$\frac{2^{n-\frac{1}{2}} \sec(e + fx) \sqrt{a \sin(e + fx) + a} (1 - \sin(e + fx))^{\frac{1}{2}-n} (c - c \sin(e + fx))^n (A + B \sin(e + fx))^p \left(\frac{A+B \sin(e+fx)}{A-B} \right)^{-p} dx}{f}$$

↓ 155

$$\frac{2^{n+\frac{1}{2}} \sec(e+fx)(1-\sin(e+fx))^{\frac{1}{2}-n}(a\sin(e+fx)+a)^{m+1}(c-c\sin(e+fx))^n(A+B\sin(e+fx))^p \left(\frac{A+B\sin(e+fx)}{A-\sin(e+fx)}\right)^p}{af(2m+1)}$$

input

```
Int[(a + a*Sin[e + f*x])^m*(A + B*Sin[e + f*x])^p*(c - c*Sin[e + f*x])^n,x
]
```

output

```
(2^(1/2 + n)*AppellF1[1/2 + m, 1/2 - n, -p, 3/2 + m, (1 + Sin[e + f*x])/2,
-((B*(1 + Sin[e + f*x]))/(A - B))] * Sec[e + f*x]*(1 - Sin[e + f*x])^(1/2 -
n)*(a + a*Sin[e + f*x])^(1 + m)*(A + B*Sin[e + f*x])^p*(c - c*Sin[e + f*x]
)^n)/(a*f*(1 + 2*m)*((A + B*Sin[e + f*x])/(A - B))^p)
```

Defintions of rubi rules used

rule 155

```
Int[((a_) + (b_)*(x_))^(m_)*((c_) + (d_)*(x_))^(n_)*((e_) + (f_)*(x_))
^(p_), x_] := Simp[((a + b*x)^(m + 1)/(b*(m + 1)*Simplify[b/(b*c - a*d)]^n*
Simplify[b/(b*e - a*f)]^p))*AppellF1[m + 1, -n, -p, m + 2, (-d)*((a + b*x)/
(b*c - a*d)), (-f)*((a + b*x)/(b*e - a*f))], x] /; FreeQ[{a, b, c, d, e, f,
m, n, p}, x] && !IntegerQ[m] && !IntegerQ[n] && !IntegerQ[p] && GtQ[Sim
plify[b/(b*c - a*d)], 0] && GtQ[Simplify[b/(b*e - a*f)], 0] && !(GtQ[Simpl
ify[d/(d*a - c*b)], 0] && GtQ[Simplify[d/(d*e - c*f)], 0] && SimplerQ[c + d
*x, a + b*x]) && !(GtQ[Simplify[f/(f*a - e*b)], 0] && GtQ[Simplify[f/(f*c
- e*d)], 0] && SimplerQ[e + f*x, a + b*x])
```

rule 156

```
Int[((a_) + (b_)*(x_))^(m_)*((c_) + (d_)*(x_))^(n_)*((e_) + (f_)*(x_))
^(p_), x_] := Simp[(e + f*x)^FracPart[p]/(Simplify[b/(b*e - a*f)]^IntPart[p]
*(b*((e + f*x)/(b*e - a*f)))^FracPart[p]) Int[(a + b*x)^m*(c + d*x)^n*Si
mp[b*(e/(b*e - a*f)) + b*f*(x/(b*e - a*f)), x]^p, x], x] /; FreeQ[{a, b, c,
d, e, f, m, n, p}, x] && !IntegerQ[m] && !IntegerQ[n] && !IntegerQ[p] &
& GtQ[Simplify[b/(b*c - a*d)], 0] && !GtQ[Simplify[b/(b*e - a*f)], 0]
```

rule 157

```
Int[((a_) + (b_)*(x_)^(m_))*((c_) + (d_)*(x_)^(n_))*((e_) + (f_)*(x_))
^(p_), x_] := Simp[(c + d*x)^FracPart[n]/(Simplify[b/(b*c - a*d)]^IntPart[n]
)*(b*((c + d*x)/(b*c - a*d)))^FracPart[n] Int[(a + b*x)^m*Simp[b*(c/(b*c
- a*d)) + b*d*(x/(b*c - a*d)), x]^n*(e + f*x)^p, x], x] /; FreeQ[{a, b, c,
d, e, f, m, n, p}, x] && !IntegerQ[m] && !IntegerQ[n] && !IntegerQ[p] &
& !GtQ[Simplify[b/(b*c - a*d)], 0] && !SimplerQ[c + d*x, a + b*x] && !Si
mplerQ[e + f*x, a + b*x]
```

rule 3042

```
Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 3487

```
Int[((a_) + (b_)*sin[(e_) + (f_)*(x_)])^(m_)*((A_) + (B_)*sin[(e_) +
(f_)*(x_)])^(p_)*((c_) + (d_)*sin[(e_) + (f_)*(x_)])^(n_), x_Symbol]
:= Simp[Sqrt[a + b*Sin[e + f*x]]*(Sqrt[c + d*Sin[e + f*x]]/(f*Cos[e + f*x])
) Subst[Int[(a + b*x)^(m - 1/2)*(c + d*x)^(n - 1/2)*(A + B*x)^p, x], x, S
in[e + f*x]], x] /; FreeQ[{a, b, c, d, e, f, A, B, m, n, p}, x] && EqQ[b*c
+ a*d, 0] && EqQ[a^2 - b^2, 0]
```

Maple [F]

$$\int (a + a \sin(fx + e))^m (A + B \sin(fx + e))^p (c - c \sin(fx + e))^n dx$$

input

```
int((a+a*sin(f*x+e))^m*(A+B*sin(f*x+e))^p*(c-c*sin(f*x+e))^n,x)
```

output

```
int((a+a*sin(f*x+e))^m*(A+B*sin(f*x+e))^p*(c-c*sin(f*x+e))^n,x)
```

Fricas [F]

$$\begin{aligned} & \int (a + a \sin(e + fx))^m (A + B \sin(e + fx))^p (c - c \sin(e + fx))^n dx \\ &= \int (B \sin(fx + e) + A)^p (a \sin(fx + e) + a)^m (-c \sin(fx + e) + c)^n dx \end{aligned}$$

input `integrate((a+a*sin(f*x+e))^m*(A+B*sin(f*x+e))^p*(c-c*sin(f*x+e))^n,x, algo
rithm="fricas")`

output `integral((B*sin(f*x + e) + A)^p*(a*sin(f*x + e) + a)^m*(-c*sin(f*x + e) +
c)^n, x)`

Sympy [F(-1)]

Timed out.

$$\int (a + a \sin(e + fx))^m (A + B \sin(e + fx))^p (c - c \sin(e + fx))^n dx = \text{Timed out}$$

input `integrate((a+a*sin(f*x+e))**m*(A+B*sin(f*x+e))**p*(c-c*sin(f*x+e))**n,x)`

output Timed out

Maxima [F]

$$\begin{aligned} & \int (a + a \sin(e + fx))^m (A + B \sin(e + fx))^p (c - c \sin(e + fx))^n dx \\ &= \int (B \sin(fx + e) + A)^p (a \sin(fx + e) + a)^m (-c \sin(fx + e) + c)^n dx \end{aligned}$$

input `integrate((a+a*sin(f*x+e))^m*(A+B*sin(f*x+e))^p*(c-c*sin(f*x+e))^n,x, algo
rithm="maxima")`

output `integrate((B*sin(f*x + e) + A)^p*(a*sin(f*x + e) + a)^m*(-c*sin(f*x + e) +
c)^n, x)`

Giac [F]

$$\int (a + a \sin(e + fx))^m (A + B \sin(e + fx))^p (c - c \sin(e + fx))^n dx$$

$$= \int (B \sin(fx + e) + A)^p (a \sin(fx + e) + a)^m (-c \sin(fx + e) + c)^n dx$$

input `integrate((a+a*sin(f*x+e))^m*(A+B*sin(f*x+e))^p*(c-c*sin(f*x+e))^n,x, algorithm="giac")`

output `integrate((B*sin(f*x + e) + A)^p*(a*sin(f*x + e) + a)^m*(-c*sin(f*x + e) + c)^n, x)`

Mupad [F(-1)]

Timed out.

$$\int (a + a \sin(e + fx))^m (A + B \sin(e + fx))^p (c - c \sin(e + fx))^n dx$$

$$= \int (A + B \sin(e + fx))^p (a + a \sin(e + fx))^m (c - c \sin(e + fx))^n dx$$

input `int((A + B*sin(e + f*x))^p*(a + a*sin(e + f*x))^m*(c - c*sin(e + f*x))^n,x)`

output `int((A + B*sin(e + f*x))^p*(a + a*sin(e + f*x))^m*(c - c*sin(e + f*x))^n,x)`

Reduce [F]

$$\int (a + a \sin(e + fx))^m (A + B \sin(e + fx))^p (c - c \sin(e + fx))^n dx$$

$$= \int (\sin(fx + e)b + a)^p (a + a \sin(fx + e))^m (-\sin(fx + e)c + c)^n dx$$

input `int((a+a*sin(f*x+e))^m*(A+B*sin(f*x+e))^p*(c-c*sin(f*x+e))^n,x)`

output `int((sin(e + f*x)*b + a)**p*(sin(e + f*x)*a + a)**m*(- sin(e + f*x)*c + c)**n,x)`

CHAPTER 4

APPENDIX

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4.1 Listing of Grading functions

The following are the current version of the grading functions used for grading the quality of the antiderivative with reference to the optimal antiderivative included in the test suite.

There is a version for Maple and for Mathematica/Rubi. There is a version for grading Sympy and version for use with Sagemath.

The following are links to the current source code.

The following are the listings of source code of the grading functions.

Mathematica and Rubi grading function

```
(* Original version thanks to Albert Rich emailed on 03/21/2017 *)
(* ::Package:: *)

(* Nasser: April 7, 2022. add second output which gives reason for the grade *)
(*                               Small rewrite of logic in main function to make it*)
(*                               match Maple's logic. No change in functionality otherwise*)

(* ::Subsection:: *)
(*GradeAntiderivative[result,optimal]*)
```

```

(* ::Text:: *)
(*If result and optimal are mathematical expressions, *)
(*      GradeAntiderivative[result,optimal] returns*)
(* "F" if the result fails to integrate an expression that*)
(*      is integrable*)
(* "C" if result involves higher level functions than necessary*)
(* "B" if result is more than twice the size of the optimal*)
(*      antiderivative*)
(* "A" if result can be considered optimal*)

GradeAntiderivative[result_,optimal_] := Module[{expnResult,expnOptimal,leafCountResult,leafCountOptimal},
  expnResult = ExpnType[result];
  expnOptimal = ExpnType[optimal];
  leafCountResult = LeafCount[result];
  leafCountOptimal = LeafCount[optimal];

  (*Print["expnResult=",expnResult," expnOptimal=",expnOptimal];*)
  If[expnResult<=expnOptimal,
    If[Not[FreeQ[result,Complex]], (*result contains complex*)
      If[Not[FreeQ[optimal,Complex]], (*optimal contains complex*)
        If[leafCountResult<=2*leafCountOptimal,
          finalresult={"A",""}
          ,(*ELSE*)
          finalresult={"B","Both result and optimal contain complex but leaf count
          ]
        ,(*ELSE*)
          finalresult={"C","Result contains complex when optimal does not."}
        ]
      ,(*ELSE*)(*result does not contains complex*)
        If[leafCountResult<=2*leafCountOptimal,
          finalresult={"A",""}
          ,(*ELSE*)
          finalresult={"B","Leaf count is larger than twice the leaf count of optimal.
          ]
        ]
      ,(*ELSE*)(*expnResult>expnOptimal*)
        If[FreeQ[result,Integrate] && FreeQ[result,Int],
          finalresult={"C","Result contains higher order function than in optimal. Order "
          ,

```

```

        finalresult={"F","Contains unresolved integral."}
    ]
];

finalresult
]

(* ::Text:: *)
(*The following summarizes the type number assigned an *)
(*expression based on the functions it involves*)
(*1 = rational function*)
(*2 = algebraic function*)
(*3 = elementary function*)
(*4 = special function*)
(*5 = hyperpergeometric function*)
(*6 = appell function*)
(*7 = rootsum function*)
(*8 = integrate function*)
(*9 = unknown function*)

ExpnType[expn_] :=
  If[AtomQ[expn],
    1,
  If[ListQ[expn],
    Max[Map[ExpnType,expn]],
  If[Head[expn]===Power,
    If[IntegerQ[expn[[2]]],
      ExpnType[expn[[1]]],
    If[Head[expn[[2]]]===Rational,
      If[IntegerQ[expn[[1]]] || Head[expn[[1]]]===Rational,
        1,
        Max[ExpnType[expn[[1]],2]],
      Max[ExpnType[expn[[1]],ExpnType[expn[[2]],3]],
    If[Head[expn]===Plus || Head[expn]===Times,
      Max[ExpnType[First[expn]],ExpnType[Rest[expn]]],
    If[ElementaryFunctionQ[Head[expn]],
      Max[3,ExpnType[expn[[1]]]],
    If[SpecialFunctionQ[Head[expn]],
      Apply[Max,Append[Map[ExpnType,Apply[List,expn]],4]],
    If[HypergeometricFunctionQ[Head[expn]],

```

```
Apply[Max, Append[Map[ExpnType, Apply[List, expn]], 5]],
If[AppellFunctionQ[Head[expn]],
Apply[Max, Append[Map[ExpnType, Apply[List, expn]], 6]],
If[Head[expn]===RootSum,
Apply[Max, Append[Map[ExpnType, Apply[List, expn]], 7]],
If[Head[expn]===Integrate || Head[expn]===Int,
Apply[Max, Append[Map[ExpnType, Apply[List, expn]], 8]],
9]]]]]]]]]]]]
```

```
ElementaryFunctionQ[func_] :=
MemberQ[{
Exp, Log,
Sin, Cos, Tan, Cot, Sec, Csc,
ArcSin, ArcCos, ArcTan, ArcCot, ArcSec, ArcCsc,
Sinh, Cosh, Tanh, Coth, Sech, Csch,
ArcSinh, ArcCosh, ArcTanh, ArcCoth, ArcSech, ArcCsch
}, func]
```

```
SpecialFunctionQ[func_] :=
MemberQ[{
Erf, Erfc, Erfi,
FresnelS, FresnelC,
ExpIntegralE, ExpIntegralEi, LogIntegral,
SinIntegral, CosIntegral, SinhIntegral, CoshIntegral,
Gamma, LogGamma, PolyGamma,
Zeta, PolyLog, ProductLog,
EllipticF, EllipticE, EllipticPi
}, func]
```

```
HypergeometricFunctionQ[func_] :=
MemberQ[{Hypergeometric1F1, Hypergeometric2F1, HypergeometricPFQ}, func]
```

```
AppellFunctionQ[func_] :=
MemberQ[{AppellF1}, func]
```

Maple grading function

```

# File: GradeAntiderivative.mpl
# Original version thanks to Albert Rich emailed on 03/21/2017

#Nasser 03/22/2017 Use Maple leaf count instead since buildin
#Nasser 03/23/2017 missing 'ln' for ElementaryFunctionQ added
#Nasser 03/24/2017 corrected the check for complex result
#Nasser 10/27/2017 check for leafsize and do not call ExpnType()
#
#                   if leaf size is "too large". Set at 500,000
#Nasser 12/22/2019 Added debug flag, added 'dilog' to special functions
#
#                   see problem 156, file Apostol_Problems
#Nasser 4/07/2022 add second output which gives reason for the grade

GradeAntiderivative := proc(result,optimal)
local leaf_count_result,
      leaf_count_optimal,
      ExpnType_result,
      ExpnType_optimal,
      debug:=false;

      leaf_count_result:=leafcount(result);
      #do NOT call ExpnType() if leaf size is too large. Recursion problem
      if leaf_count_result > 500000 then
          return "B","result has leaf size over 500,000. Avoiding possible recursion issue";
      fi;

      leaf_count_optimal := leafcount(optimal);
      ExpnType_result := ExpnType(result);
      ExpnType_optimal := ExpnType(optimal);

      if debug then
          print("ExpnType_result",ExpnType_result," ExpnType_optimal=",ExpnType_optimal);
      fi;

# If result and optimal are mathematical expressions,
# GradeAntiderivative[result,optimal] returns
# "F" if the result fails to integrate an expression that
# is integrable
# "C" if result involves higher level functions than necessary
# "B" if result is more than twice the size of the optimal

```

```

# antiderivative
# "A" if result can be considered optimal

#This check below actually is not needed, since I only
#call this grading only for passed integrals. i.e. I check
#for "F" before calling this. But no harm of keeping it here.
#just in case.

if not type(result,freeof('int')) then
    return "F","Result contains unresolved integral";
fi;

if ExpnType_result<=ExpnType_optimal then
    if debug then
        print("ExpnType_result<=ExpnType_optimal");
    fi;
    if is_contains_complex(result) then
        if is_contains_complex(optimal) then
            if debug then
                print("both result and optimal complex");
            fi;
            if leaf_count_result<=2*leaf_count_optimal then
                return "A"," ";
            else
                return "B",cat("Both result and optimal contain complex but leaf count of
                                convert(leaf_count_result,string)," vs. $2 (" ,
                                convert(leaf_count_optimal,string)," ) = ",convert(2*leaf

        end if
    else #result contains complex but optimal is not
        if debug then
            print("result contains complex but optimal is not");
        fi;
        return "C","Result contains complex when optimal does not.";
    fi;
else # result do not contain complex
    # this assumes optimal do not as well. No check is needed here.
    if debug then
        print("result do not contain complex, this assumes optimal do not as well
    fi;

```

```

        if leaf_count_result<=2*leaf_count_optimal then
            if debug then
                print("leaf_count_result<=2*leaf_count_optimal");
            fi;
            return "A"," ";
        else
            if debug then
                print("leaf_count_result>2*leaf_count_optimal");
            fi;
            return "B",cat("Leaf count of result is larger than twice the leaf count of
                            convert(leaf_count_result,string)," $ vs. $2(",
                            convert(leaf_count_optimal,string),")=",convert(2*leaf_co
            fi;
        fi;
    else #ExpnType(result) > ExpnType(optimal)
        if debug then
            print("ExpnType(result) > ExpnType(optimal)");
        fi;
        return "C",cat("Result contains higher order function than in optimal. Order ",
                        convert(ExpnType_result,string)," vs. order ",
                        convert(ExpnType_optimal,string),".");
    fi;
end proc:

#
# is_contains_complex(result)
# takes expressions and returns true if it contains "I" else false
#
#Nasser 032417
is_contains_complex:= proc(expression)
    return (has(expression,I));
end proc:

# The following summarizes the type number assigned an expression
# based on the functions it involves
# 1 = rational function
# 2 = algebraic function
# 3 = elementary function
# 4 = special function
# 5 = hyperpergeometric function

```

```

# 6 = appell function
# 7 = rootsum function
# 8 = integrate function
# 9 = unknown function

ExpnType := proc(expn)
  if type(expn,'atomic') then
    1
  elif type(expn,'list') then
    apply(max,map(ExpnType,expn))
  elif type(expn,'sqrt') then
    if type(op(1,expn),'rational') then
      1
    else
      max(2,ExpnType(op(1,expn)))
    end if
  elif type(expn,'^^') then
    if type(op(2,expn),'integer') then
      ExpnType(op(1,expn))
    elif type(op(2,expn),'rational') then
      if type(op(1,expn),'rational') then
        1
      else
        max(2,ExpnType(op(1,expn)))
      end if
    else
      max(3,ExpnType(op(1,expn)),ExpnType(op(2,expn)))
    end if
  elif type(expn,'+'') or type(expn,'*') then
    max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
  elif ElementaryFunctionQ(op(0,expn)) then
    max(3,ExpnType(op(1,expn)))
  elif SpecialFunctionQ(op(0,expn)) then
    max(4,apply(max,map(ExpnType,[op(expn)])))
  elif HypergeometricFunctionQ(op(0,expn)) then
    max(5,apply(max,map(ExpnType,[op(expn)])))
  elif AppellFunctionQ(op(0,expn)) then
    max(6,apply(max,map(ExpnType,[op(expn)])))
  elif op(0,expn)='int' then
    max(8,apply(max,map(ExpnType,[op(expn)]))) else
    9

```



```

    end if
end proc:

ElementaryFunctionQ := proc(func)
  member(func, [
    exp, log, ln,
    sin, cos, tan, cot, sec, csc,
    arcsin, arccos, arctan, arccot, arcsec, arccsc,
    sinh, cosh, tanh, coth, sech, csch,
    arcsinh, arccosh, arctanh, arccoth, arcsech, arccsch])
end proc:

SpecialFunctionQ := proc(func)
  member(func, [
    erf, erfc, erfi,
    FresnelS, FresnelC,
    Ei, Ei, Li, Si, Ci, Shi, Chi,
    GAMMA, lnGAMMA, Psi, Zeta, polylog, dilog, LambertW,
    EllipticF, EllipticE, EllipticPi])
end proc:

HypergeometricFunctionQ := proc(func)
  member(func, [Hypergeometric1F1, hypergeom, HypergeometricPFQ])
end proc:

AppellFunctionQ := proc(func)
  member(func, [AppellF1])
end proc:

# u is a sum or product. rest(u) returns all but the
# first term or factor of u.
rest := proc(u) local v;
  if nops(u)=2 then
    op(2,u)
  else
    apply(op(0,u), op(2..nops(u),u))
  end if
end proc:

#leafcount(u) returns the number of nodes in u.

```

```
#Nasser 3/23/17 Replaced by build-in leafCount from package in Maple
leafcount := proc(u)
  MmaTranslator[Mma][LeafCount](u);
end proc;
```

Sympy grading function

```
#Dec 24, 2019. Nasser M. Abbasi:
#      Port of original Maple grading function by
#      Albert Rich to use with Sympy/Python
#Dec 27, 2019 Nasser. Added `RootSum`. See problem 177, Timofeev file
#      added 'exp_polar'
from sympy import *

def leaf_count(expr):
  #sympy do not have leaf count function. This is approximation
  return round(1.7*count_ops(expr))

def is_sqrt(expr):
  if isinstance(expr,Pow):
    if expr.args[1] == Rational(1,2):
      return True
    else:
      return False
  else:
    return False

def is_elementary_function(func):
  return func in [exp,log,ln,sin,cos,tan,cot,sec,csc,
    asin,acos,atan,acot,asec,acsc,sinh,cosh,tanh,coth,sech,csch,
    asinh,acosh,atanh,acoth,asech,acsch
  ]

def is_special_function(func):
  return func in [ erf,erfc,erfi,
    fresnels,fresnelc,Ei,Ei,Li,Si,Ci,Shi,Chi,
    gamma,loggamma,digamma,zeta,polylog,LambertW,
    elliptic_f,elliptic_e,elliptic_pi,exp_polar
  ]
```

```

def is_hypergeometric_function(func):
    return func in [hyper]

def is_appell_function(func):
    return func in [appellf1]

def is_atom(expn):
    try:
        if expn.isAtom or isinstance(expn,int) or isinstance(expn,float):
            return True
        else:
            return False

    except AttributeError as error:
        return False

def expnType(expn):
    debug=False
    if debug:
        print("expn=",expn,"type(expn)=",type(expn))

    if is_atom(expn):
        return 1
    elif isinstance(expn,list):
        return max(map(expnType, expn)) #apply(max,map(ExpnType,expn))
    elif is_sqrt(expn):
        if isinstance(expn.args[0],Rational): #type(op(1,expn),'rational')
            return 1
        else:
            return max(2,expnType(expn.args[0])) #max(2,ExpnType(op(1,expn)))
    elif isinstance(expn,Pow): #type(expn,'^')
        if isinstance(expn.args[1],Integer): #type(op(2,expn),'integer')
            return expnType(expn.args[0]) #ExpnType(op(1,expn))
        elif isinstance(expn.args[1],Rational): #type(op(2,expn),'rational')
            if isinstance(expn.args[0],Rational): #type(op(1,expn),'rational')
                return 1
            else:
                return max(2,expnType(expn.args[0])) #max(2,ExpnType(op(1,expn)))
        else:
            return max(3,expnType(expn.args[0]),expnType(expn.args[1])) #max(3,ExpnType(op(1,expn)),ExpnType(op(2,expn)))
    elif isinstance(expn,Add) or isinstance(expn,Mul): #type(expn,'+') or type(expn,'*')

```

```

    m1 = expnType(expn.args[0])
    m2 = expnType(list(expn.args[1:]))
    return max(m1,m2) #max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
elif is_elementary_function(expn.func): #ElementaryFunctionQ(op(0,expn))
    return max(3,expnType(expn.args[0])) #max(3,ExpnType(op(1,expn)))
elif is_special_function(expn.func): #SpecialFunctionQ(op(0,expn))
    m1 = max(map(expnType, list(expn.args)))
    return max(4,m1) #max(4,apply(max,map(ExpnType,[op(expn)])))
elif is_hypergeometric_function(expn.func): #HypergeometricFunctionQ(op(0,expn))
    m1 = max(map(expnType, list(expn.args)))
    return max(5,m1) #max(5,apply(max,map(ExpnType,[op(expn)])))
elif is_appell_function(expn.func):
    m1 = max(map(expnType, list(expn.args)))
    return max(6,m1) #max(5,apply(max,map(ExpnType,[op(expn)])))
elif isinstance(expn,RootSum):
    m1 = max(map(expnType, list(expn.args))) #Apply[Max,Append[Map[ExpnType,Apply[List,expn]],7]],
    return max(7,m1)
elif str(expn).find("Integral") != -1:
    m1 = max(map(expnType, list(expn.args)))
    return max(8,m1) #max(5,apply(max,map(ExpnType,[op(expn)])))
else:
    return 9

#main function
def grade_antiderivative(result,optimal):

    #print("Enter grade_antiderivative for sagemath")
    #print("Enter grade_antiderivative, result=",result," optimal=",optimal)

    leaf_count_result = leaf_count(result)
    leaf_count_optimal = leaf_count(optimal)

    #print("leaf_count_result=",leaf_count_result)
    #print("leaf_count_optimal=",leaf_count_optimal)

    expnType_result = expnType(result)
    expnType_optimal = expnType(optimal)

    if str(result).find("Integral") != -1:
        grade = "F"
        grade_annotation = ""

```

```

else:
    if expnType_result <= expnType_optimal:
        if result.has(I):
            if optimal.has(I): #both result and optimal complex
                if leaf_count_result <= 2*leaf_count_optimal:
                    grade = "A"
                    grade_annotation = ""
                else:
                    grade = "B"
                    grade_annotation = "Both result and optimal contain complex but leaf count of result is lar
            else: #result contains complex but optimal is not
                grade = "C"
                grade_annotation = "Result contains complex when optimal does not."
        else: # result do not contain complex, this assumes optimal do not as well
            if leaf_count_result <= 2*leaf_count_optimal:
                grade = "A"
                grade_annotation = ""
            else:
                grade = "B"
                grade_annotation = "Leaf count of result is larger than twice the leaf count of optimal. "+str(
        else:
            grade = "C"
            grade_annotation = "Result contains higher order function than in optimal. Order "+str(ExpnType

#print("Before returning. grade=",grade, " grade_annotation=",grade_annotation)

return grade, grade_annotation

```

SageMath grading function

```

#Dec 24, 2019. Nasser: Ported original Maple grading function by
#      Albert Rich to use with Sagemath. This is used to
#      grade Fracas, Giac and Maxima results.
#Dec 24, 2019. Nasser: Added 'exp_integral_e' and 'sng', 'sin_integral'
#      'arctan2', 'floor', 'abs', 'log_integral'
#June 4, 2022 Made default grade_annotation "none" instead of "" due
#      issue later when reading the file.
#July 14, 2022. Added ellipticF. This is until they fix sagemath, then remove it.

```

```

from sage.all import *
from sage.symbolic.operators import add_vararg, mul_vararg

debug=False;

def tree_size(expr):
    r"""
    Return the tree size of this expression.
    """
    #print("Enter tree_size, expr is ",expr)

    if expr not in SR:
        # deal with lists, tuples, vectors
        return 1 + sum(tree_size(a) for a in expr)
    expr = SR(expr)
    x, aa = expr.operator(), expr.operands()
    if x is None:
        return 1
    else:
        return 1 + sum(tree_size(a) for a in aa)

def is_sqrt(expr):
    if expr.operator() == operator.pow: #isinstance(expr,Pow):
        if expr.operands()[1]==1/2: #expr.args[1] == Rational(1,2):
            if debug: print ("expr is sqrt")
            return True
        else:
            return False
    else:
        return False

def is_elementary_function(func):
    #debug=False
    m = func.name() in ['exp','log','ln',
        'sin','cos','tan','cot','sec','csc',
        'arcsin','arccos','arctan','arccot','arcsec','arccsc',
        'sinh','cosh','tanh','coth','sech','csch',
        'arcsinh','arccosh','arctanh','arcoth','arcsech','arcsch','sgn',
        'arctan2','floor','abs'
    ]
    if debug:

```

```

    if m:
        print ("func ", func , " is elementary_function")
    else:
        print ("func ", func , " is NOT elementary_function")

    return m

def is_special_function(func):
    #debug=False
    if debug:
        print ("type(func)=", type(func))

    m= func.name() in ['erf','erfc','erfi','fresnel_sin','fresnel_cos','Ei',
        'Ei','Li','Si','sin_integral','Ci','cos_integral','Shi','sinh_integral',
        'Chi','cosh_integral','gamma','log_gamma','psi','zeta',
        'polylog','lambert_w','elliptic_f','elliptic_e','ellipticF',
        'elliptic_pi','exp_integral_e','log_integral',
        'weierstrassPInverse','weierstrass','weierstrassP','weierstrassZeta',
        'weierstrassPPrime','weierstrassSigma']

    if debug:
        print ("m=",m)
    if m:
        print ("func ", func , " is special_function")
    else:
        print ("func ", func , " is NOT special_function")

    return m

def is_hypergeometric_function(func):
    return func.name() in ['hypergeometric','hypergeometric_M','hypergeometric_U']

def is_appell_function(func):
    return func.name() in ['hypergeometric'] #[appellf1] can't find this in sagemath

def is_atom(expn):

    #debug=False

```

```

if debug:
    print ("Enter is_atom, expn=",expn)

if not hasattr(expn, 'parent'):
    return False

#thanks to answer at https://ask.sagemath.org/question/49179/what-is-sagemath-equivalent-to-atomic
try:
    if expn.parent() is SR:
        return expn.operator() is None
    if expn.parent() in (ZZ, QQ, AA, QQbar):
        return expn in expn.parent() # Should always return True
    if hasattr(expn.parent(),"base_ring") and hasattr(expn.parent(),"gens"):
        return expn in expn.parent().base_ring() or expn in expn.parent().gens()

    return False

except AttributeError as error:
    print ("Exception,AttributeError in is_atom")
    print ("caught exception" , type(error).__name__ )
    return False

def expnType(expn):

    if debug:
        print (">>>>>Enter expnType, expn=", expn)
        print (">>>>>is_atom(expn)=", is_atom(expn))

    if is_atom(expn):
        return 1
    elif type(expn)==list: #instance(expn,list):
        return max(map(expnType, expn)) #apply(max,map(ExpnType,expn))
    elif is_sqrt(expn):
        if type(expn.operands()[0])==Rational: #type(instance(expn.args[0],Rational):
            return 1
        else:
            return max(2,expnType(expn.operands()[0])) #max(2,expnType(expn.args[0]))
    elif expn.operator() == operator.pow: #instance(expn,Pow)
        if type(expn.operands()[1])==Integer: #instance(expn.args[1],Integer)

```



```

    return expnType(expn.operands()[0]) #expnType(expn.args[0])
elif type(expn.operands()[1])==Rational: #isinstance(expn.args[1],Rational)
    if type(expn.operands()[0])==Rational: #isinstance(expn.args[0],Rational)
        return 1
    else:
        return max(2,expnType(expn.operands()[0])) #max(2,expnType(expn.args[0]))
else:
    return max(3,expnType(expn.operands()[0]),expnType(expn.operands()[1])) #max(3,expnType(expn
elif expn.operator() == add_vararg or expn.operator() == mul_vararg: #isinstance(expn,Add) or isins
    m1 = expnType(expn.operands()[0]) #expnType(expn.args[0])
    m2 = expnType(expn.operands()[1:]) #expnType(list(expn.args[1:]))
    return max(m1,m2) #max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
elif is_elementary_function(expn.operator()): #is_elementary_function(expn.func)
    return max(3,expnType(expn.operands()[0]))
elif is_special_function(expn.operator()): #is_special_function(expn.func)
    m1 = max(map(expnType, expn.operands())) #max(map(expnType, list(expn.args)))
    return max(4,m1) #max(4,m1)
elif is_hypergeometric_function(expn.operator()): #is_hypergeometric_function(expn.func)
    m1 = max(map(expnType, expn.operands())) #max(map(expnType, list(expn.args)))
    return max(5,m1) #max(5,m1)
elif is_appell_function(expn.operator()):
    m1 = max(map(expnType, expn.operands())) #max(map(expnType, list(expn.args)))
    return max(6,m1) #max(6,m1)
elif str(expn).find("Integral") != -1: #this will never happen, since it
    #is checked before calling the grading function that is passed.
    #but kept it here.
    m1 = max(map(expnType, expn.operands())) #max(map(expnType, list(expn.args)))
    return max(8,m1) #max(5,apply(max,map(ExpnType,[op(expn)])))
else:
    return 9

#main function
def grade_antiderivative(result,optimal):

if debug:
    print ("Enter grade_antiderivative for sagemath")
    print ("Enter grade_antiderivative, result=",result)
    print ("Enter grade_antiderivative, optimal=",optimal)
    print ("type(anti)=", type(result))
    print ("type(optimal)=", type(optimal))

```

```

leaf_count_result = tree_size(result) #leaf_count(result)
leaf_count_optimal = tree_size(optimal) #leaf_count(optimal)

#if debug: print ("leaf_count_result=", leaf_count_result, "leaf_count_optimal=",leaf_count_optimal)

expnType_result = expnType(result)
expnType_optimal = expnType(optimal)

if debug: print ("expnType_result=", expnType_result, "expnType_optimal=",expnType_optimal)

if expnType_result <= expnType_optimal:
    if result.has(I):
        if optimal.has(I): #both result and optimal complex
            if leaf_count_result <= 2*leaf_count_optimal:
                grade = "A"
                grade_annotation = "none"
            else:
                grade = "B"
                grade_annotation = "Both result and optimal contain complex but leaf count of result is larger"
        else: #result contains complex but optimal is not
            grade = "C"
            grade_annotation = "Result contains complex when optimal does not."
    else: # result do not contain complex, this assumes optimal do not as well
        if leaf_count_result <= 2*leaf_count_optimal:
            grade = "A"
            grade_annotation = "none"
        else:
            grade = "B"
            grade_annotation = "Leaf count of result is larger than twice the leaf count of optimal. "+str(leaf_count_result - 2*leaf_count_optimal)
    else:
        grade = "C"
        grade_annotation = "Result contains higher order function than in optimal. Order "+str(expnType_result - expnType_optimal)

print("Before returning. grade=",grade, " grade_annotation=",grade_annotation)

return grade, grade_annotation

```

4.2 Links to plain text integration problems used in this report for each CAS

1. Mathematica integration problems as .m file
2. Maple integration problems as .txt file
3. Sagemath integration problems as .sage file
4. Reduce integration problems as .txt file
5. Mupad integration problems as .txt file
6. Sympy integration problems as .py file