

# Computer Algebra Independent Integration Tests

Summer 2024

4-Trig-functions/4.6-Cosecant/243-4.6.0

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# Contents

<b>1</b>	<b>Introduction</b>	<b>4</b>
1.1	Listing of CAS systems tested . . . . .	5
1.2	Results . . . . .	6
1.3	Time and leaf size Performance . . . . .	10
1.4	Performance based on number of rules Rubi used . . . . .	12
1.5	Performance based on number of steps Rubi used . . . . .	13
1.6	Solved integrals histogram based on leaf size of result . . . . .	14
1.7	Solved integrals histogram based on CPU time used . . . . .	15
1.8	Leaf size vs. CPU time used . . . . .	16
1.9	list of integrals with no known antiderivative . . . . .	17
1.10	List of integrals solved by CAS but has no known antiderivative . . . . .	17
1.11	list of integrals solved by CAS but failed verification . . . . .	17
1.12	Timing . . . . .	18
1.13	Verification . . . . .	18
1.14	Important notes about some of the results . . . . .	19
1.15	Current tree layout of integration tests . . . . .	22
1.16	Design of the test system . . . . .	23
<b>2</b>	<b>detailed summary tables of results</b>	<b>24</b>
2.1	List of integrals sorted by grade for each CAS . . . . .	25
2.2	Detailed conclusion table per each integral for all CAS systems . . . . .	29
2.3	Detailed conclusion table specific for Rubi results . . . . .	47
<b>3</b>	<b>Listing of integrals</b>	<b>50</b>
3.1	$\int \csc(a + bx) dx$ . . . . .	53
3.2	$\int \csc^2(a + bx) dx$ . . . . .	58
3.3	$\int \csc^3(a + bx) dx$ . . . . .	63
3.4	$\int \csc^4(a + bx) dx$ . . . . .	69
3.5	$\int \csc^5(a + bx) dx$ . . . . .	74
3.6	$\int \csc^6(a + bx) dx$ . . . . .	80
3.7	$\int \csc^7(a + bx) dx$ . . . . .	85

3.8	$\int \csc^8(a + bx) dx$	92
3.9	$\int \csc^{\frac{7}{2}}(a + bx) dx$	98
3.10	$\int \csc^{\frac{5}{2}}(a + bx) dx$	104
3.11	$\int \csc^{\frac{3}{2}}(a + bx) dx$	110
3.12	$\int \sqrt{\csc(a + bx)} dx$	116
3.13	$\int \frac{1}{\sqrt{\csc(a+bx)}} dx$	121
3.14	$\int \frac{1}{\csc^{\frac{3}{2}}(a+bx)} dx$	126
3.15	$\int \frac{1}{\csc^{\frac{5}{2}}(a+bx)} dx$	132
3.16	$\int \frac{1}{\csc^{\frac{7}{2}}(a+bx)} dx$	138
3.17	$\int (c \csc(a + bx))^{7/2} dx$	144
3.18	$\int (c \csc(a + bx))^{5/2} dx$	150
3.19	$\int (c \csc(a + bx))^{3/2} dx$	156
3.20	$\int \sqrt{c \csc(a + bx)} dx$	162
3.21	$\int \frac{1}{\sqrt{c \csc(a+bx)}} dx$	167
3.22	$\int \frac{1}{(c \csc(a+bx))^{3/2}} dx$	172
3.23	$\int \frac{1}{(c \csc(a+bx))^{5/2}} dx$	178
3.24	$\int \frac{1}{(c \csc(a+bx))^{7/2}} dx$	184
3.25	$\int \csc^{\frac{4}{3}}(a + bx) dx$	190
3.26	$\int \csc^{\frac{2}{3}}(a + bx) dx$	195
3.27	$\int \sqrt[3]{\csc(a + bx)} dx$	200
3.28	$\int \frac{1}{\sqrt[3]{\csc(a + bx)}} dx$	205
3.29	$\int \frac{1}{\csc^{\frac{2}{3}}(a+bx)} dx$	210
3.30	$\int \frac{1}{\csc^{\frac{4}{3}}(a+bx)} dx$	215
3.31	$\int (c \csc(a + bx))^{4/3} dx$	220
3.32	$\int (c \csc(a + bx))^{2/3} dx$	225
3.33	$\int \sqrt[3]{c \csc(a + bx)} dx$	230
3.34	$\int \frac{1}{\sqrt[3]{c \csc(a + bx)}} dx$	235
3.35	$\int \frac{1}{(c \csc(a+bx))^{2/3}} dx$	240
3.36	$\int \frac{1}{(c \csc(a+bx))^{4/3}} dx$	245
3.37	$\int \csc^n(a + bx) dx$	250
3.38	$\int (c \csc(a + bx))^n dx$	255
3.39	$\int \csc^2(x)^{7/2} dx$	260
3.40	$\int \csc^2(x)^{5/2} dx$	267
3.41	$\int \csc^2(x)^{3/2} dx$	273
3.42	$\int \sqrt{\csc^2(x)} dx$	279

3.43	$\int \frac{1}{\sqrt{\csc^2(x)}} dx$	284
3.44	$\int \frac{1}{\csc^2(x)^{3/2}} dx$	289
3.45	$\int \frac{1}{\csc^2(x)^{5/2}} dx$	295
3.46	$\int \frac{1}{\csc^2(x)^{7/2}} dx$	301
3.47	$\int (a \csc^2(x))^{7/2} dx$	307
3.48	$\int (a \csc^2(x))^{5/2} dx$	314
3.49	$\int (a \csc^2(x))^{3/2} dx$	321
3.50	$\int \sqrt{a \csc^2(x)} dx$	327
3.51	$\int \frac{1}{\sqrt{a \csc^2(x)}} dx$	332
3.52	$\int \frac{1}{(a \csc^2(x))^{3/2}} dx$	337
3.53	$\int \frac{1}{(a \csc^2(x))^{5/2}} dx$	342
3.54	$\int \frac{1}{(a \csc^2(x))^{7/2}} dx$	348
3.55	$\int (a \csc^3(x))^{5/2} dx$	355
3.56	$\int (a \csc^3(x))^{3/2} dx$	362
3.57	$\int \sqrt{a \csc^3(x)} dx$	369
3.58	$\int \frac{1}{\sqrt{a \csc^3(x)}} dx$	375
3.59	$\int \frac{1}{(a \csc^3(x))^{3/2}} dx$	381
3.60	$\int \frac{1}{(a \csc^3(x))^{5/2}} dx$	388
3.61	$\int (a \csc^4(x))^{7/2} dx$	395
3.62	$\int (a \csc^4(x))^{5/2} dx$	402
3.63	$\int (a \csc^4(x))^{3/2} dx$	408
3.64	$\int \sqrt{a \csc^4(x)} dx$	413
3.65	$\int \frac{1}{\sqrt{a \csc^4(x)}} dx$	418
3.66	$\int \frac{1}{(a \csc^4(x))^{3/2}} dx$	423
3.67	$\int \frac{1}{(a \csc^4(x))^{5/2}} dx$	429
3.68	$\int ((b \csc(c + dx))^p)^n dx$	436
3.69	$\int (a(b \csc(c + dx))^p)^n dx$	441
3.70	$\int (a \csc(e + fx))^m (b \csc(e + fx))^n dx$	446

**4 Appendix 451**

4.1	Listing of Grading functions	451
4.2	Links to plain text integration problems used in this report for each CAS469	

# CHAPTER 1

## INTRODUCTION

1.1	Listing of CAS systems tested . . . . .	5
1.2	Results . . . . .	6
1.3	Time and leaf size Performance . . . . .	10
1.4	Performance based on number of rules Rubi used . . . . .	12
1.5	Performance based on number of steps Rubi used . . . . .	13
1.6	Solved integrals histogram based on leaf size of result . . . . .	14
1.7	Solved integrals histogram based on CPU time used . . . . .	15
1.8	Leaf size vs. CPU time used . . . . .	16
1.9	list of integrals with no known antiderivative . . . . .	17
1.10	List of integrals solved by CAS but has no known antiderivative . . . . .	17
1.11	list of integrals solved by CAS but failed verification . . . . .	17
1.12	Timing . . . . .	18
1.13	Verification . . . . .	18
1.14	Important notes about some of the results . . . . .	19
1.15	Current tree layout of integration tests . . . . .	22
1.16	Design of the test system . . . . .	23

This report gives the result of running the computer algebra independent integration test. The download section in on the main webpage contains links to download the problems in plain text format used for all CAS systems. The number of integrals in this report is [ 70 ]. This is test number [ 243 ].

## 1.1 Listing of CAS systems tested

The following are the CAS systems tested:

1. Mathematica 14 (January 9, 2024) on windows 10 pro.
2. Rubi 4.17.3 (Sept 25, 2023) on Mathematica 14 on windows 10m pro.
3. Maple 2024 (March 1, 2024) on windows 10 pro.
4. Maxima 5.47 (June 1, 2023) using Lisp SBCL 2.4.0 on Linux Manjaro 23.1.2 KDE via sagemath 10.3.
5. FriCAS 1.3.10 built with sbcl 2.3.11 (January 10, 2024) on Linux Manjaro 23.1.2 KDE via sagemath 10.3.
6. Giac/Xcas 1.9.0-99 on Linux via sagemath 10.3.
7. Sympy 1.12 using Python 3.11.6 (Nov 14 2023, 09:36:21) [GCC 13.2.1 20230801] on Linux Manjaro 23.1.2 KDE.
8. Mupad using Matlab 2021a with Symbolic Math Toolbox Version 8.7 on windows 10.
9. Reduce CSL rev 6687 (January 9, 2024) on Linux Manjaro 23.1.2 KDE.

Maxima and Fricas and Giac are called using Sagemath. This was done using Sagemath `integrate` command by changing the name of the algorithm to use the different CAS systems.

Sympy was run directly in Python not via sagemath.

Reduce was called directly.

## 1.2 Results

Important note: A number of problems in this test suite have no antiderivative in closed form. This means the antiderivative of these integrals can not be expressed in terms of elementary, special functions or Hypergeometric2F1 functions. RootSum and RootOf are not allowed. If a CAS returns the above integral unevaluated within the time limit, then the result is counted as passed and assigned an A grade.

However, if CAS times out, then it is assigned an F grade even if the integral is not integrable, as this implies CAS could not determine that the integral is not integrable in the time limit.

If a CAS returns an antiderivative to such an integral, it is assigned an A grade automatically and this special result is listed in the introduction section of each individual test report to make it easy to identify as this can be important result to investigate.

The results given in in the table below reflects the above.

System	% solved	% Failed
Rubi	100.00 ( 70 )	0.00 ( 0 )
Mathematica	100.00 ( 70 )	0.00 ( 0 )
Maple	75.71 ( 53 )	24.29 ( 17 )
Fricas	75.71 ( 53 )	24.29 ( 17 )
Reduce	44.29 ( 31 )	55.71 ( 39 )
Giac	40.00 ( 28 )	60.00 ( 42 )
Maxima	40.00 ( 28 )	60.00 ( 42 )
Mupad	22.86 ( 16 )	77.14 ( 54 )
Sympy	12.86 ( 9 )	87.14 ( 61 )

Table 1.1: Percentage solved for each CAS

The table below gives additional break down of the grading of quality of the antiderivatives generated by each CAS. The grading is given using the letters A,B,C and F with A being the best quality. The grading is accomplished by comparing the antiderivative generated with the optimal antiderivatives included in the test suite. The following table describes the meaning of these grades.

grade	description
A	Integral was solved and antiderivative is optimal in quality and leaf size.
B	Integral was solved and antiderivative is optimal in quality but leaf size is larger than twice the optimal antiderivatives leaf size.
C	Integral was solved and antiderivative is non-optimal in quality. This can be due to one or more of the following reasons <ol style="list-style-type: none"> <li>1. antiderivative contains a hypergeometric function and the optimal antiderivative does not.</li> <li>2. antiderivative contains a special function and the optimal antiderivative does not.</li> <li>3. antiderivative contains the imaginary unit and the optimal antiderivative does not.</li> </ol>
F	Integral was not solved. Either the integral was returned unevaluated within the time limit, or it timed out, or CAS hanged or crashed or an exception was raised.

Table 1.2: Description of grading applied to integration result

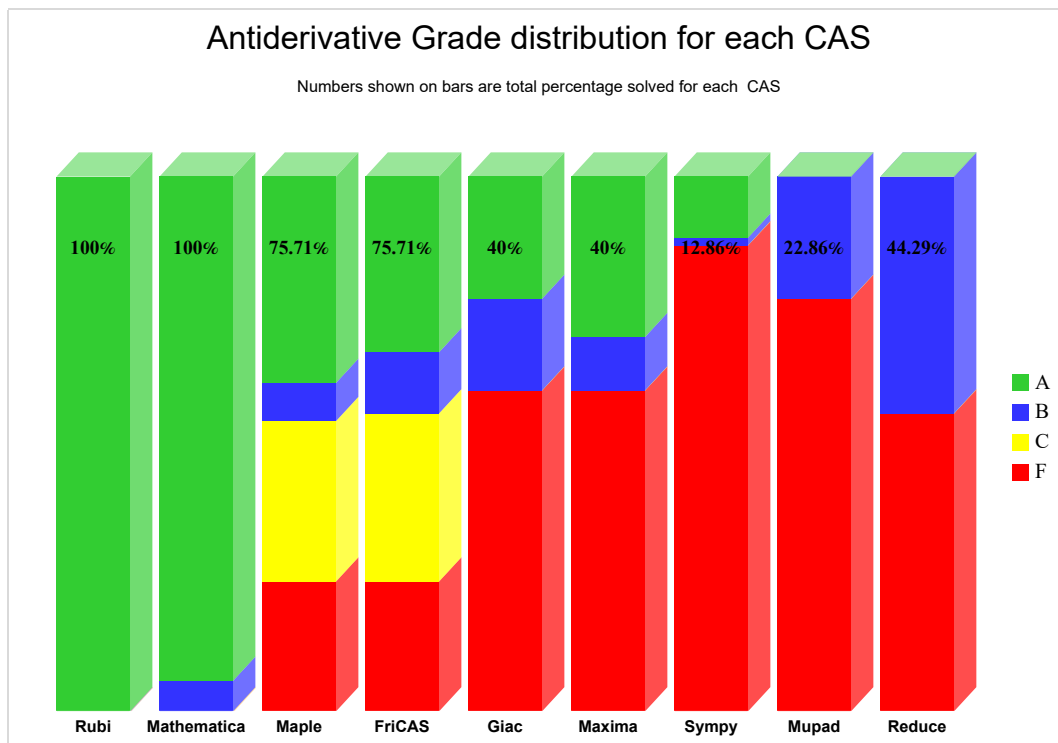
Grading is implemented for all CAS systems. Based on the above, the following table summarizes the grading for this test suite.

System	% A grade	% B grade	% C grade	% F grade
Rubi	100.000	0.000	0.000	0.000
Mathematica	94.286	5.714	0.000	0.000
Maple	38.571	7.143	30.000	24.286
Fricas	32.857	11.429	31.429	24.286
Maxima	30.000	10.000	0.000	60.000
Giac	22.857	17.143	0.000	60.000
Sympy	11.429	1.429	0.000	87.143
Mupad	0.000	22.857	0.000	77.143
Reduce	0.000	44.286	0.000	55.714

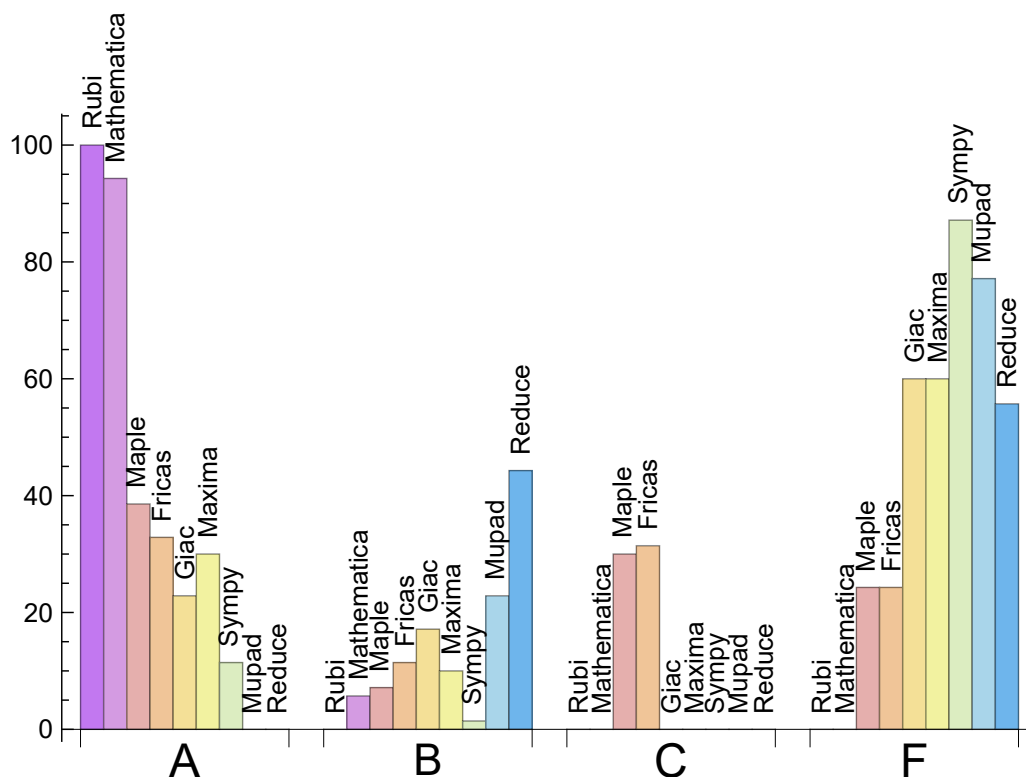
Table 1.3: Antiderivative Grade distribution of each CAS



The following is a Bar chart illustration of the data in the above table.



The figure below compares the grades of the CAS systems.



The following table shows the distribution of the different types of failures for each CAS. There are 3 types failures. The first is when CAS returns the input within the time limit, which means it could not solve it. This is the typical failure and given as **F**.

The second failure is due to time out. CAS could not solve the integral within the 3 minutes time limit which is assigned. This is assigned **F(-1)**.

The third is due to an exception generated, indicated as **F(-2)**. This most likely indicates an interface problem between sagemath and the CAS (applicable only to FriCAS, Maxima and Giac) or it could be an indication of an internal error in the CAS itself. This type of error requires more investigation to determine the cause.

System	Number failed	Percentage normal failure	Percentage time-out failure	Percentage exception failure
Rubi	0	0.00	0.00	0.00
Mathematica	0	0.00	0.00	0.00
Fricas	17	100.00	0.00	0.00
Maple	17	100.00	0.00	0.00
Reduce	39	100.00	0.00	0.00
Giac	42	92.86	0.00	7.14
Maxima	42	100.00	0.00	0.00
Mupad	54	0.00	100.00	0.00
Sympy	61	91.80	8.20	0.00

Table 1.4: Failure statistics for each CAS

### 1.3 Time and leaf size Performance

The table below summarizes the performance of each CAS system in terms of time used and leaf size of results.

Mean size is the average leaf size produced by the CAS (before any normalization). The Normalized mean is relative to the mean size of the optimal anti-derivative given in the input files.

For example, if CAS has **Normalized mean** of 3, then the mean size of its leaf size is 3 times as large as the mean size of the optimal leaf size.

Median size is value of leaf size where half the values are larger than this and half are smaller (before any normalization). i.e. The Middle value.

Similarly the **Normalized median** is relative to the median leaf size of the optimal.

For example, if a CAS has Normalized median of 1.2, then its median is 1.2 as large as the median leaf size of the optimal.

System	Mean time (sec)
Mathematica	0.07
Fricas	0.08
Giac	0.15
Reduce	0.18
Maxima	0.18
Maple	0.22
Rubi	0.29
Sympy	2.98
Mupad	12.11

Table 1.5: Time performance for each CAS

System	Mean size	Normalized mean	Median size	Normalized median
Reduce	29.87	0.71	30.00	0.54
Sympy	38.89	1.22	37.00	1.00
Mathematica	51.99	0.98	52.50	0.89
Rubi	59.96	1.03	56.00	1.00
Giac	63.14	1.50	56.50	1.39
Fricas	69.28	1.30	68.00	1.28
Mupad	76.38	1.15	39.50	1.01
Maple	80.68	1.41	50.00	1.12
Maxima	259.07	5.18	40.00	0.92

Table 1.6: Leaf size performance for each CAS

# 1.4 Performance based on number of rules Rubi used

This section shows how each CAS performed based on the number of rules Rubi needed to solve the same integral. One diagram is given for each CAS.

On the  $y$  axis is the percentage solved which Rubi itself needed the number of rules given the  $x$  axis. These plots show that as more rules are needed then most CAS system percentage of solving decreases which indicates the integral is becoming more complicated to solve.

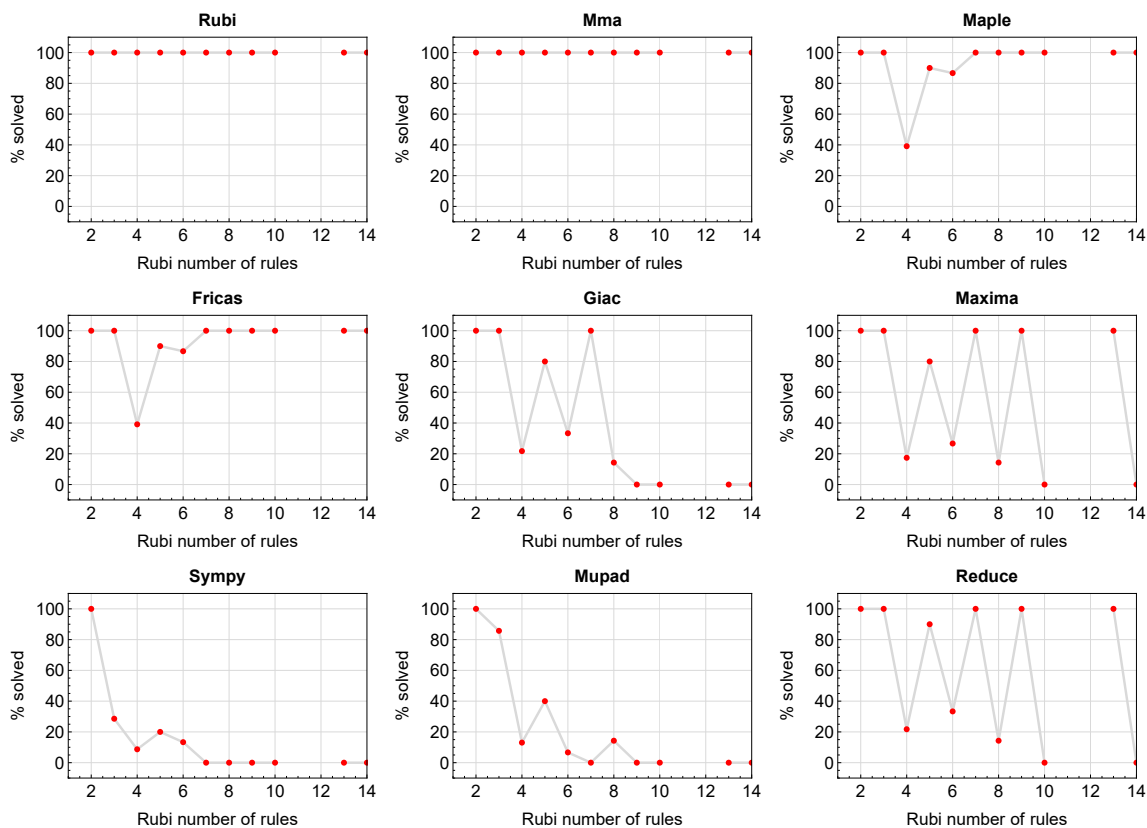


Figure 1.1: Solving statistics per number of Rubi rules used

## 1.5 Performance based on number of steps Rubi used

This section shows how each CAS performed based on the number of steps Rubi needed to solve the same integral. Note that the number of steps Rubi needed can be much higher than the number of rules, as the same rule could be used more than once.

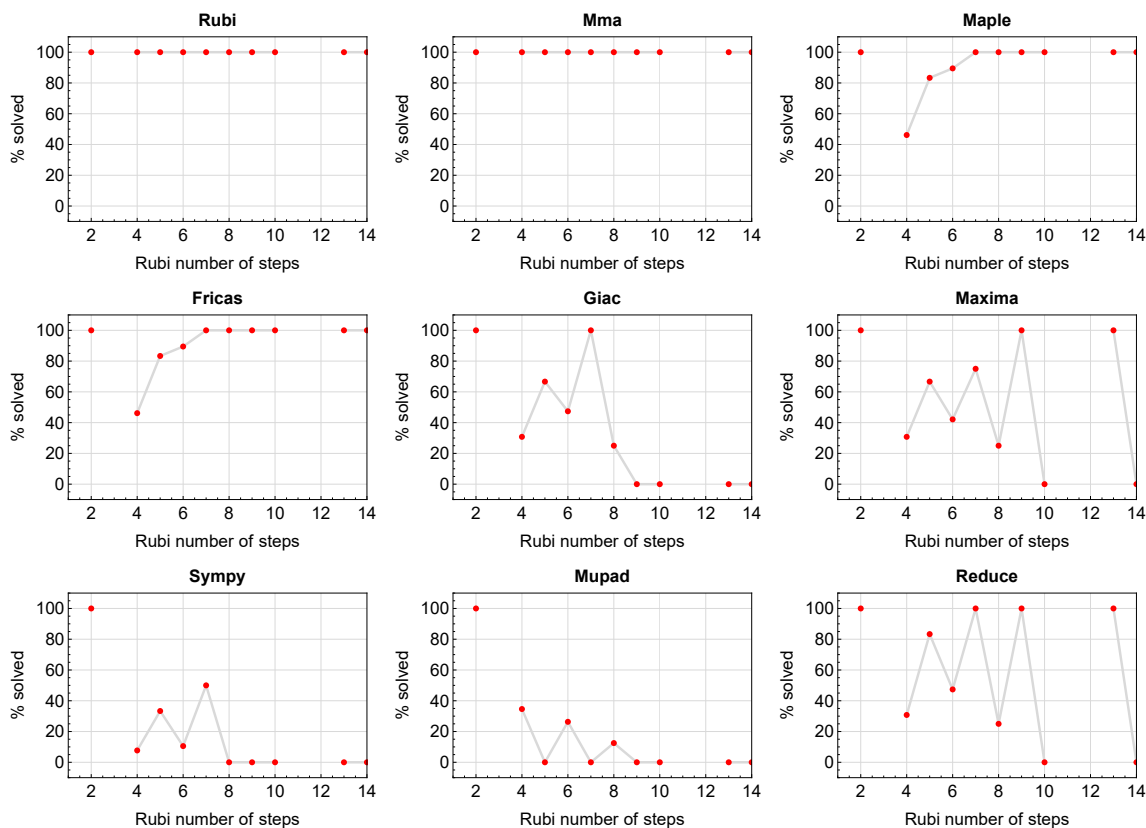


Figure 1.2: Solving statistics per number of Rubi steps used

The above diagram shows that the percentage of solved integrals decreases for most CAS systems as the number of steps increases. As expected, for integrals that required less steps by Rubi, CAS systems had more success which indicates the integral was not as hard to solve. As Rubi needed more steps to solve the integral, the solved percentage decreased for most CAS systems which indicates the integral is becoming harder to solve.

## 1.6 Solved integrals histogram based on leaf size of result

The following shows the distribution of solved integrals for each CAS system based on leaf size of the antiderivatives produced by each CAS. It shows that most integrals solved produced leaf size less than about 100 to 150. The bin size used is 40.

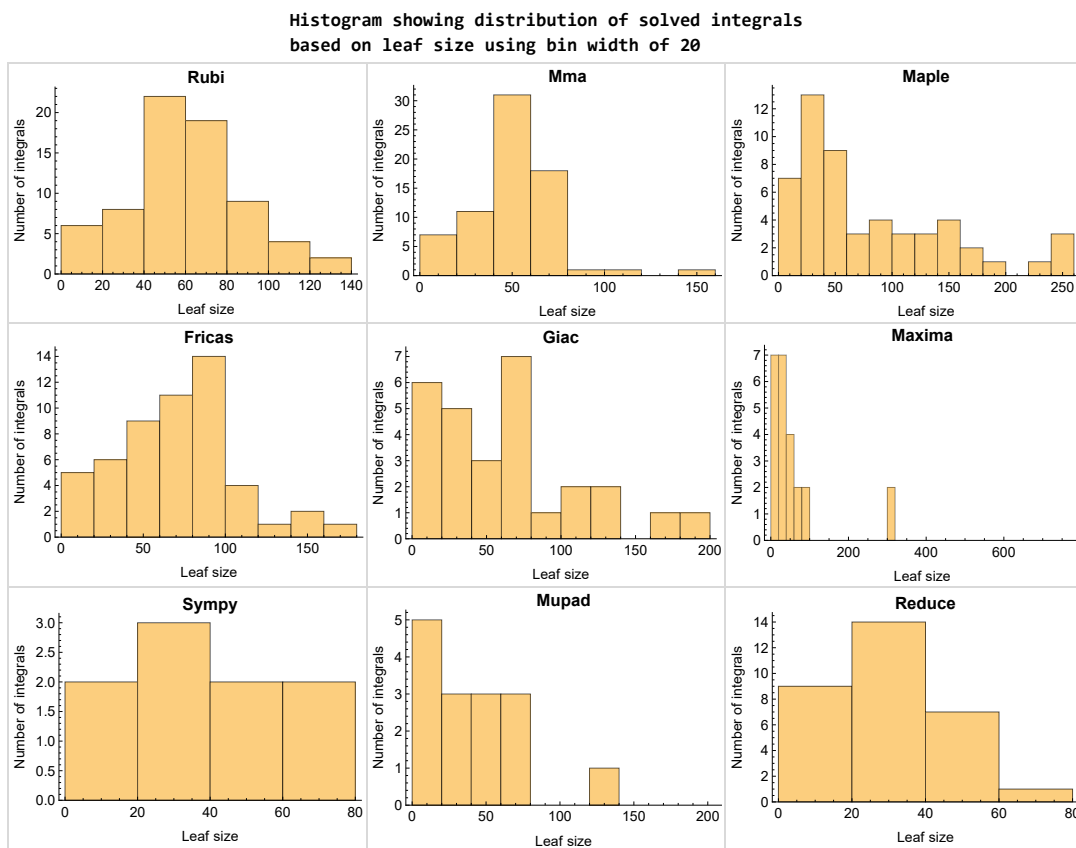


Figure 1.3: Solved integrals based on leaf size distribution

## 1.7 Solved integrals histogram based on CPU time used

The following shows the distribution of solved integrals for each CAS system based on CPU time used in seconds. The bin size used is 0.1 second.

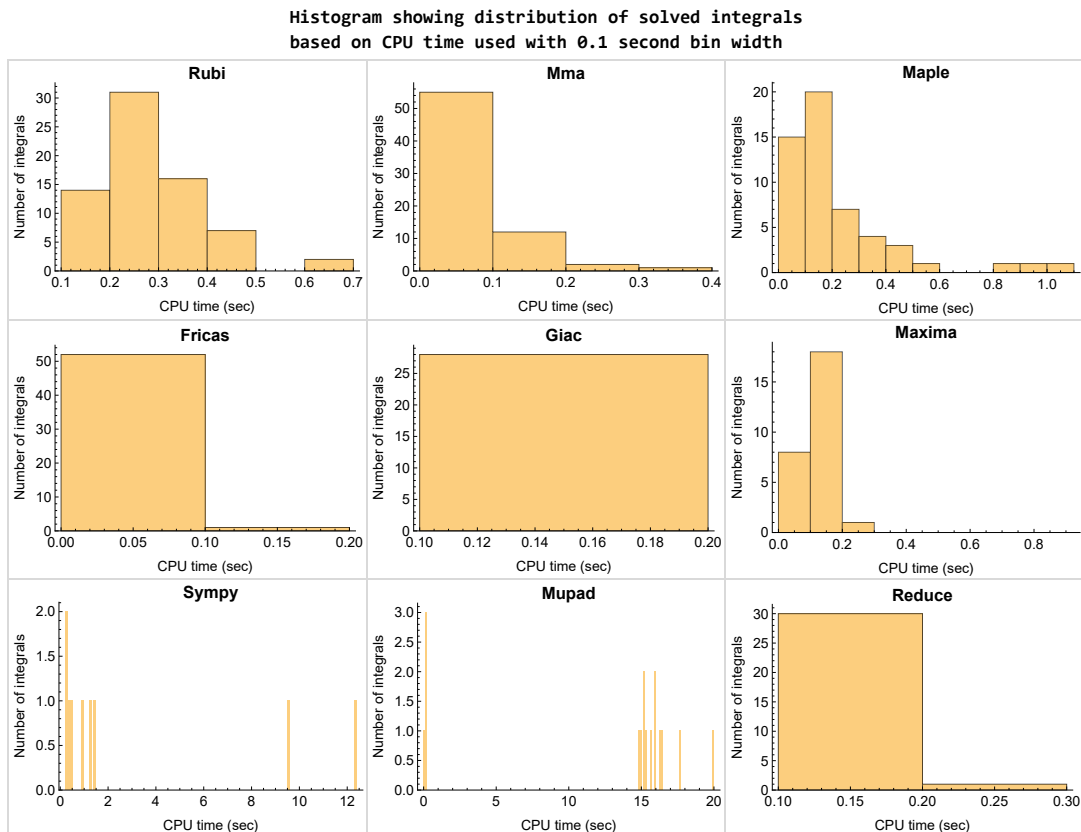


Figure 1.4: Solved integrals histogram based on CPU time used



## 1.8 Leaf size vs. CPU time used

The following shows the relation between the CPU time used to solve an integral and the leaf size of the antiderivative.

The result for Fracas, Maxima and Giac is shifted more to the right than the other CAS system due to the use of sagemath to call them, which causes an initial slight delay in the timing to start the integration due to overhead of starting a new process each time. This should also be taken into account when looking at the timing of these three CAS systems. Direct calls not using sagemath would result in faster timings, but current implementation uses sagemath as this makes testing much easier to do.

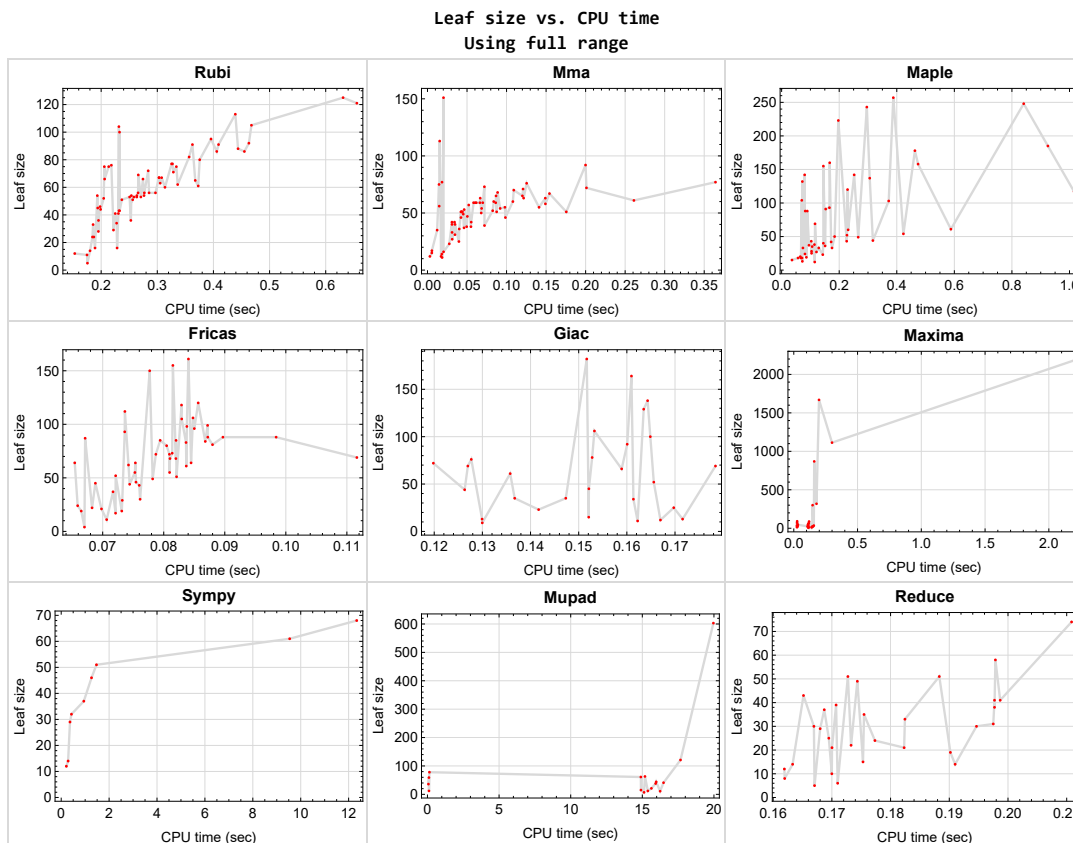


Figure 1.5: Leaf size vs. CPU time. Full range

## 1.9 list of integrals with no known antiderivative

{}

## 1.10 List of integrals solved by CAS but has no known antiderivative

Rubi {}

Mathematica {}

Maple {}

Maxima {}

Fricas {}

Sympy {}

Giac {}

Reduce {}

Mupad {}

## 1.11 list of integrals solved by CAS but failed verification

The following are integrals solved by CAS but the verification phase failed to verify the anti-derivative produced is correct. This does not necessarily mean that the anti-derivative is wrong as additional methods of verification might be needed, or more time is needed (3 minutes time limit was used). These integrals are listed here to make it possible to do further investigation to determine why the result could not be verified.

Rubi {}

Mathematica {}

Maple {39, 40, 41, 43, 44, 45, 46}

**Maxima** Verification phase not currently implemented.

**Fricas** Verification phase not currently implemented.

**Sympy** Verification phase not currently implemented.

**Giac** Verification phase not currently implemented.

**Reduce** Verification phase not currently implemented.

**Mupad** Verification phase not currently implemented.

## 1.12 Timing

The command `AbsoluteTiming[]` was used in Mathematica to obtain the elapsed time for each integrate call. In Maple, the command `Usage` was used as in the following example

```
cpu_time := Usage(assign ('result_of_int',int(expr,x)),output='realtime')
```

For all other CAS systems, the elapsed time to complete each integral was found by taking the difference between the time after the call completed from the time before the call was made. This was done using Python's `time.time()` call.

All elapsed times shown are in seconds. A time limit of 3 CPU minutes was used for each integral. If the integrate command did not complete within this time limit, the integral was aborted and considered to have failed and assigned an F grade. The time used by failed integrals due to time out was not counted in the final statistics.

## 1.13 Verification

A verification phase was applied on the result of integration for **Rubi** and **Mathematica**.

Future version of this report will implement verification for the other CAS systems. For the integrals whose result was not run through a verification phase, it is assumed that the antiderivative was correct.

Verification phase also had 3 minutes time out. An integral whose result was not verified could still be correct, but further investigation is needed on those integrals. These integrals were marked in the summary table below and also in each integral separate section so they are easy to identify and locate.

## 1.14 Important notes about some of the results

### Important note about Maxima results

Since tests were run in a batch mode, and using an automated script, then any integral where Maxima needed an interactive response from the user to answer a question during the evaluation of the integral will fail.

The exception raised is `ValueError`. Therefore Maxima results is lower than what would result if Maxima was run directly and each question was answered correctly.

The percentage of such failures were not counted for each test file, but for an example, for the `Timofeev` test file, there were about 14 such integrals out of total 705, or about 2 percent. This percentage can be higher or lower depending on the specific input test file.

Such integrals can be identified by looking at the output of the integration in each section for Maxima. The exception message will indicate the cause of error.

Maxima `integrate` was run using SageMath with the following settings set by default

```
'besselexpand : true'  
'display2d : false'  
'domain : complex'  
'keepfloat : true'  
'load(to_poly_solve)'  
'load(simplify_sum)'  
'load(abs_integrate)' 'load(diag)'
```

SageMath automatic loading of Maxima `abs_integrate` was found to cause some problems. So the following code was added to disable this effect.

```
from sage.interfaces.maxima_lib import maxima_lib  
maxima_lib.set('extra_definite_integration_methods', '[]')  
maxima_lib.set('extra_integration_methods', '[]')
```

See <https://ask.sagemath.org/question/43088/integrate-results-that-are-different-from-using-maxima/> for reference.

## Important note about FriCAS result

There were few integrals which failed due to SageMath interface and not because FriCAS system could not do the integration.

These will fail With error `Exception raised: NotImplementedError`.

The number of such cases seems to be very small. About 1 or 2 percent of all integrals. These can be identified by looking at the exception message given in the result.

## Important note about finding leaf size of antiderivative

For Mathematica, Rubi, and Maple, the builtin system function `LeafSize` was used to find the leaf size of each antiderivative.

The other CAS systems (SageMath and Sympy) do not have special builtin function for this purpose at this time. Therefore the leaf size for Fricas and Sympy antiderivative was determined using the following function, thanks to user `slelievre` at [https://ask.sagemath.org/question/57123/could-we-have-a-leaf\\_count-function-in-base-sagemath/](https://ask.sagemath.org/question/57123/could-we-have-a-leaf_count-function-in-base-sagemath/)

```
def tree_size(expr):
    r"""
    Return the tree size of this expression.
    """
    if expr not in SR:
        # deal with lists, tuples, vectors
        return 1 + sum(tree_size(a) for a in expr)
    expr = SR(expr)
    x, aa = expr.operator(), expr.operands()
    if x is None:
        return 1
    else:
        return 1 + sum(tree_size(a) for a in aa)
```

For Sympy, which was called directly from Python, the following code was used to obtain the leafsize of its result

```
try:  
    # 1.7 is a fudge factor since it is low side from actual leaf count  
    leafCount = round(1.7*count_ops(anti))  
  
except Exception as ee:  
    leafCount = 1
```

### Important note about Mupad results

Matlab's symbolic toolbox does not have a leaf count function to measure the size of the antiderivative. Maple was used to determine the leaf size of Mupad output by post processing Mupad result.

Currently no grading of the antiderivative for Mupad is implemented. If it can integrate the problem, it was assigned a B grade automatically as a placeholder. In the future, when grading function is implemented for Mupad, the tests will be rerun again.

The following is an example of using Matlab's symbolic toolbox (Mupad) to solve an integral

```
integrand = evalin(symengine, 'cos(x)*sin(x)')  
the_variable = evalin(symengine, 'x')  
anti = int(integrand, the_variable)
```

Which gives  $\sin(x)^2/2$

# 1.15 Current tree layout of integration tests

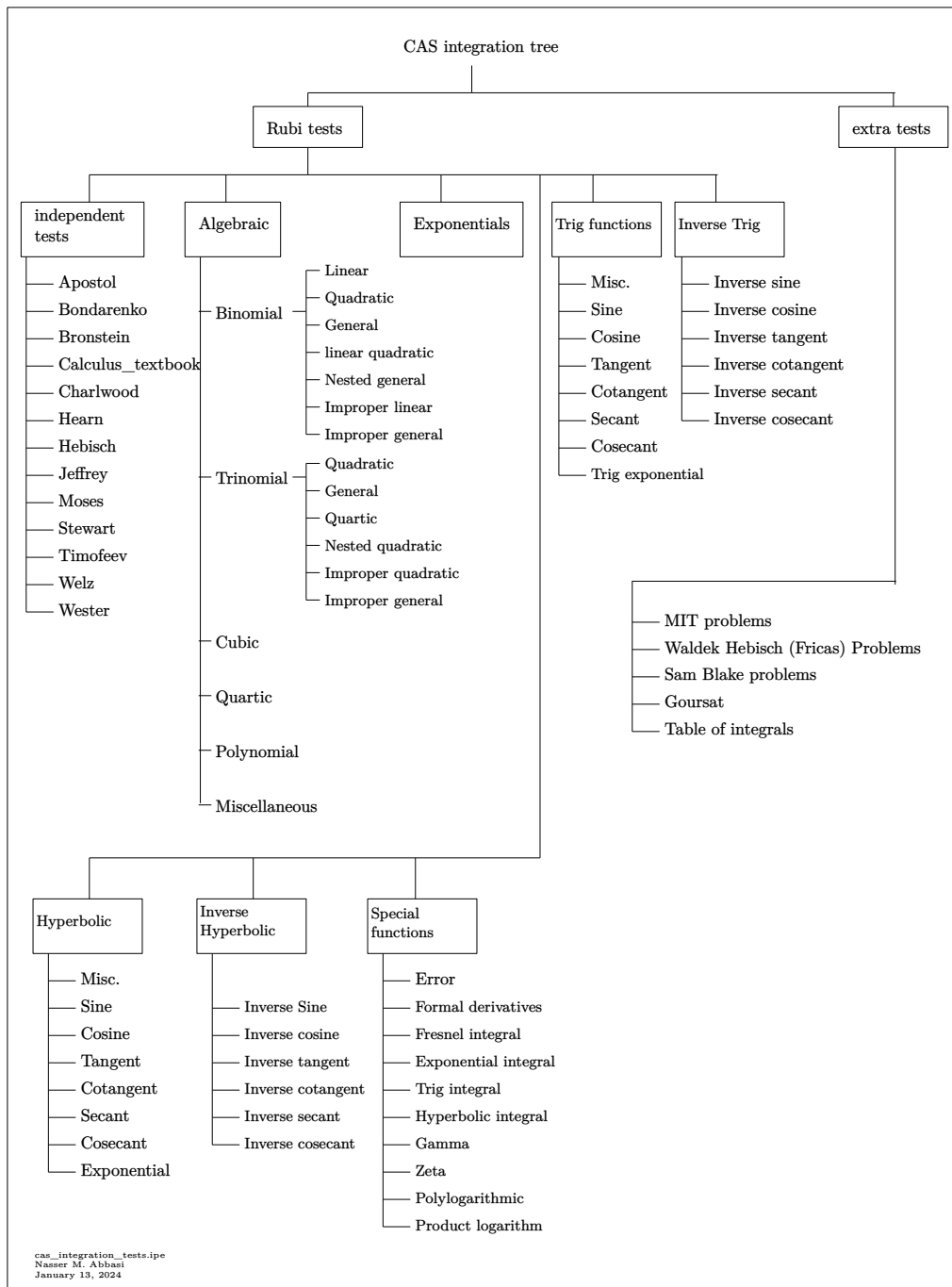
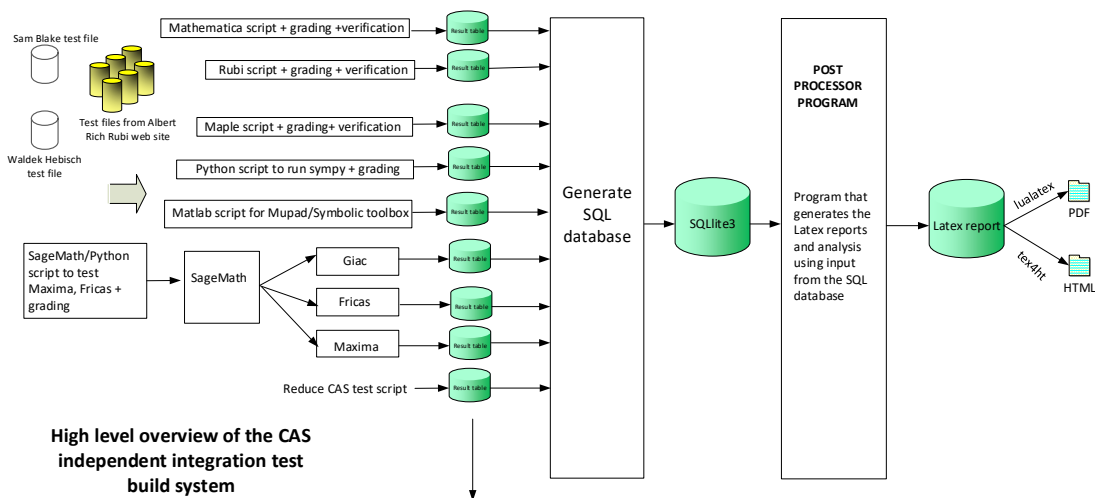


Figure 1.6: CAS integration tests tree

# 1.16 Design of the test system

The following diagram gives a high level view of the current test build system.



**High level overview of the CAS independent integration test build system**

One record (line) per one integral result. The line is CSV comma separated. This is description of each record

1. integer, the problem number.
2. integer. 0 for failed, 1 for passed, -1 for timeout, -2 for CAS specific exception. (this is not the grade field)
3. integer. Leaf size of result.
4. integer. Leaf size of the optimal antiderivative.
5. number. CPU time used to solve this integral. 0 if failed.
6. string. The integral in Latex format
7. string. The input used in CAS own syntax.
8. string. The result (antiderivative) produced by CAS in Latex format
9. string. The optimal antiderivative in Latex format.
10. integer. 0 or 1. Indicates if problem has known antiderivative or not
11. String. The result (antiderivative) in CAS own syntax.
12. String. The grade of the antiderivative. Can be "A", "B", "C", or "E"
13. String. Small string description of why the grade was given.
14. integer. 1 if result was verified or 0 if not verified. (For mma, rubi and maple only)

*The following fields are present only in Rubi Table file*

15. integer. Number of steps used.
16. integer. Number of rules used.
17. integer. Integrand leaf size.
18. real number. Ratio. Field 16 over field 17
19. String of form "{n,n,...}" which is list of the rules used by Rubi
20. String. The optimal antiderivative in Mathematica syntax

Nasser M. Abbasi  
January 13, 2024  
Design note



# CHAPTER 2

## DETAILED SUMMARY TABLES OF RESULTS

2.1	List of integrals sorted by grade for each CAS . . . . .	25
2.2	Detailed conclusion table per each integral for all CAS systems . . . . .	29
2.3	Detailed conclusion table specific for Rubi results . . . . .	47

## 2.1 List of integrals sorted by grade for each CAS

Rubi . . . . .	25
Mma . . . . .	25
Maple . . . . .	26
Fricas . . . . .	26
Maxima . . . . .	26
Giac . . . . .	27
Mupad . . . . .	27
Sympy . . . . .	27
Reduce . . . . .	28

### Rubi

**A grade** { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70 }

**B grade** { }

**C grade** { }

**F normal fail** { }

**F(-1) timedout fail** { }

**F(-2) exception fail** { }

### Mma

**A grade** { 1, 2, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70 }

**B grade** { 3, 5, 41, 42 }

**C grade** { }

**F normal fail** { }

**F(-1) timedout fail** { }

**F(-2) exception fail { }**

## **Maple**

**A grade { 1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 14, 16, 47, 48, 49, 50, 51, 52, 53, 54, 61, 62, 63, 64, 65, 66, 67 }**

**B grade { 9, 11, 13, 15, 42 }**

**C grade { 17, 18, 19, 20, 21, 22, 23, 24, 39, 40, 41, 43, 44, 45, 46, 55, 56, 57, 58, 59, 60 }**

**F normal fail { 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 68, 69, 70 }**

**F(-1) timedout fail { }**

**F(-2) exception fail { }**

## **Fricas**

**A grade { 2, 4, 6, 8, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 61, 62, 63, 64, 65, 66, 67 }**

**B grade { 1, 3, 5, 7, 39, 40, 41, 42 }**

**C grade { 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 55, 56, 57, 58, 59, 60 }**

**F normal fail { 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 68, 69, 70 }**

**F(-1) timedout fail { }**

**F(-2) exception fail { }**

## **Maxima**

**A grade { 1, 2, 3, 4, 5, 6, 7, 8, 43, 44, 45, 46, 50, 51, 61, 62, 63, 64, 65, 66, 67 }**

**B grade { 39, 40, 41, 42, 47, 48, 49 }**

**C grade { }**

**F normal fail { 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 52, 53, 54, 55, 56, 57, 58, 59, 60, 68, 69, 70 }**

**F(-1) timedout fail { }**

**F(-2) exception fail { }**

## Giac

**A grade** { 1, 2, 4, 6, 8, 43, 45, 46, 50, 52, 53, 54, 61, 62, 63, 64 }

**B grade** { 3, 5, 7, 39, 40, 41, 42, 44, 47, 48, 49, 51 }

**C grade** { }

**F normal fail** { 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 55, 56, 57, 58, 59, 60, 68, 69, 70 }

**F(-1) timedout fail** { }

**F(-2) exception fail** { 65, 66, 67 }

## Mupad

**A grade** { }

**B grade** { 1, 2, 3, 4, 5, 6, 7, 8, 12, 20, 43, 51, 61, 62, 63, 64 }

**C grade** { }

**F normal fail** { }

**F(-1) timedout fail** { 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 44, 45, 46, 47, 48, 49, 50, 52, 53, 54, 55, 56, 57, 58, 59, 60, 65, 66, 67, 68, 69, 70 }

**F(-2) exception fail** { }

## Sympy

**A grade** { 43, 44, 45, 46, 51, 52, 53, 54 }

**B grade** { 1 }

**C grade** { }

**F normal fail** { 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 40, 41, 42, 48, 49, 50, 55, 56, 57, 58, 59, 60, 62, 63, 64, 65, 66, 67, 68, 69, 70 }

**F(-1) timedout fail** { 9, 17, 39, 47, 61 }

**F(-2) exception fail** { }

## Reduce

**A grade** { }

**B grade** { 1, 2, 3, 4, 5, 6, 7, 8, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 61, 62, 63, 64, 65, 66, 67 }

**C grade** { }

**F normal fail** { 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 55, 56, 57, 58, 59, 60, 68, 69, 70 }

**F(-1) timedout fail** { }

**F(-2) exception fail** { }

## 2.2 Detailed conclusion table per each integral for all CAS systems

Detailed conclusion table per each integral is given by the table below. The elapsed time is in seconds. For failed result it is given as **F(-1)** if the failure was due to timeout. It is given as **F(-2)** if the failure was due to an exception being raised, which could indicate a bug in the system. If the failure was due to integral not being evaluated within the time limit, then it is given as **F**.

In this table, the column **N.S.** means **normalized size** and is defined as  $\frac{\text{antiderivative leaf size}}{\text{optimal antiderivative leaf size}}$ . To make the table fit the page, the name **Mathematica** was abbreviated to **MMA**.

Problem 1	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	B	B	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	12	12	12	15	19	30	37	15	14	12
N.S.	1	1.00	1.00	1.25	1.58	2.50	3.08	1.25	1.17	1.00
time (sec)	N/A	0.153	0.003	0.036	0.026	0.076	0.944	0.152	0.191	0.105

Problem 2	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	<b>F</b>	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	11	11	11	12	13	19	0	13	19	11
N.S.	1	1.00	1.00	1.09	1.18	1.73	0.00	1.18	1.73	1.00
time (sec)	N/A	0.175	0.018	0.115	0.025	0.066	0.000	0.130	0.190	16.243

Problem 3	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	B	A	A	B	<b>F</b>	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	34	34	75	38	46	72	0	92	41	36
N.S.	1	1.00	2.21	1.12	1.35	2.12	0.00	2.71	1.21	1.06
time (sec)	N/A	0.228	0.015	0.114	0.032	0.079	0.000	0.160	0.199	0.069

Problem 4	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	<b>F</b>	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	27	24	35	23	25	45	0	25	31	21
N.S.	1	0.89	1.30	0.85	0.93	1.67	0.00	0.93	1.15	0.78
time (sec)	N/A	0.187	0.012	0.143	0.029	0.069	0.000	0.170	0.197	15.635

Problem 5	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	B	A	A	B	<b>F</b>	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	55	60	113	50	71	112	0	138	58	59
N.S.	1	1.09	2.05	0.91	1.29	2.04	0.00	2.51	1.05	1.07
time (sec)	N/A	0.314	0.015	0.184	0.026	0.074	0.000	0.164	0.198	0.102

Problem 6	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	<b>F</b>	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	42	36	56	33	35	64	0	35	41	38
N.S.	1	0.86	1.33	0.79	0.83	1.52	0.00	0.83	0.98	0.90
time (sec)	N/A	0.195	0.015	0.175	0.025	0.065	0.000	0.137	0.198	15.929

Problem 7	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	B	<b>F</b>	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	76	86	151	60	91	155	0	182	74	78
N.S.	1	1.13	1.99	0.79	1.20	2.04	0.00	2.39	0.97	1.03
time (sec)	N/A	0.406	0.020	0.231	0.026	0.082	0.000	0.152	0.211	0.138

Problem 8	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	<b>F</b>	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	55	44	77	43	45	87	0	45	51	41
N.S.	1	0.80	1.40	0.78	0.82	1.58	0.00	0.82	0.93	0.75
time (sec)	N/A	0.199	0.018	0.226	0.031	0.067	0.000	0.152	0.188	16.481

Problem 9	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	B	<b>F</b>	C	<b>F(-1)</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	90	91	63	160	0	120	0	0	18	0
N.S.	1	1.01	0.70	1.78	0.00	1.33	0.00	0.00	0.20	0.00
time (sec)	N/A	0.409	0.149	0.167	0.000	0.086	0.000	0.000	0.205	0.000

Problem 10	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	<b>F</b>	C	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	67	67	50	88	0	88	0	0	18	0
N.S.	1	1.00	0.75	1.31	0.00	1.31	0.00	0.00	0.27	0.00
time (sec)	N/A	0.303	0.068	0.082	0.000	0.090	0.000	0.000	0.193	0.000

Problem 11	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	B	<b>F</b>	C	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	63	63	49	132	0	72	0	0	16	0
N.S.	1	1.00	0.78	2.10	0.00	1.14	0.00	0.00	0.25	0.00
time (sec)	N/A	0.305	0.045	0.072	0.000	0.081	0.000	0.000	0.191	0.000



Problem 12	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	<b>F</b>	C	<b>F</b>	<b>F</b>	<b>F</b>	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	41	41	40	69	0	51	0	0	9	61
N.S.	1	1.00	0.98	1.68	0.00	1.24	0.00	0.00	0.22	1.49
time (sec)	N/A	0.225	0.031	0.116	0.000	0.082	0.000	0.000	0.191	14.897

Problem 13	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	B	<b>F</b>	C	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	41	41	40	91	0	55	0	0	18	0
N.S.	1	1.00	0.98	2.22	0.00	1.34	0.00	0.00	0.44	0.00
time (sec)	N/A	0.231	0.035	0.153	0.000	0.081	0.000	0.000	0.195	0.000

Problem 14	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	<b>F</b>	C	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	67	67	53	88	0	68	0	0	18	0
N.S.	1	1.00	0.79	1.31	0.00	1.01	0.00	0.00	0.27	0.00
time (sec)	N/A	0.308	0.046	0.089	0.000	0.082	0.000	0.000	0.197	0.000

Problem 15	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	B	<b>F</b>	C	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	67	67	60	142	0	85	0	0	18	0
N.S.	1	1.00	0.90	2.12	0.00	1.27	0.00	0.00	0.27	0.00
time (sec)	N/A	0.303	0.084	0.080	0.000	0.082	0.000	0.000	0.194	0.000

Problem 16	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	<b>F</b>	C	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	90	95	65	104	0	81	0	0	18	0
N.S.	1	1.06	0.72	1.16	0.00	0.90	0.00	0.00	0.20	0.00
time (sec)	N/A	0.396	0.120	0.070	0.000	0.088	0.000	0.000	0.194	0.000

Problem 17	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	<b>F</b>	C	<b>F(-1)</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	103	105	67	248	0	150	0	0	24	0
N.S.	1	1.02	0.65	2.41	0.00	1.46	0.00	0.00	0.23	0.00
time (sec)	N/A	0.468	0.154	0.841	0.000	0.078	0.000	0.000	0.197	0.000

Problem 18	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	<b>F</b>	C	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	75	75	55	137	0	105	0	0	24	0
N.S.	1	1.00	0.73	1.83	0.00	1.40	0.00	0.00	0.32	0.00
time (sec)	N/A	0.334	0.141	0.306	0.000	0.083	0.000	0.000	0.198	0.000

Problem 19	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	<b>F</b>	C	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	71	71	54	223	0	83	0	0	20	0
N.S.	1	1.00	0.76	3.14	0.00	1.17	0.00	0.00	0.28	0.00
time (sec)	N/A	0.329	0.092	0.197	0.000	0.084	0.000	0.000	0.197	0.000

Problem 20	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	<b>F</b>	C	<b>F</b>	<b>F</b>	<b>F</b>	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	43	43	42	120	0	55	0	0	12	63
N.S.	1	1.00	0.98	2.79	0.00	1.28	0.00	0.00	0.28	1.47
time (sec)	N/A	0.233	0.031	0.229	0.000	0.075	0.000	0.000	0.191	15.184

Problem 21	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	<b>F</b>	C	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	43	43	42	243	0	62	0	0	24	0
N.S.	1	1.00	0.98	5.65	0.00	1.44	0.00	0.00	0.56	0.00
time (sec)	N/A	0.232	0.034	0.296	0.000	0.074	0.000	0.000	0.196	0.000

Problem 22	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	<b>F</b>	C	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	77	77	63	142	0	85	0	0	24	0
N.S.	1	1.00	0.82	1.84	0.00	1.10	0.00	0.00	0.31	0.00
time (sec)	N/A	0.325	0.067	0.252	0.000	0.079	0.000	0.000	0.182	0.000

Problem 23	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	<b>F</b>	C	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	77	77	60	257	0	96	0	0	24	0
N.S.	1	1.00	0.78	3.34	0.00	1.25	0.00	0.00	0.31	0.00
time (sec)	N/A	0.327	0.108	0.388	0.000	0.085	0.000	0.000	0.184	0.000

Problem 24	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	<b>F</b>	C	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	105	113	70	158	0	98	0	0	24	0
N.S.	1	1.08	0.67	1.50	0.00	0.93	0.00	0.00	0.23	0.00
time (sec)	N/A	0.439	0.109	0.474	0.000	0.084	0.000	0.000	0.177	0.000

Problem 25	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	51	51	54	0	0	0	0	0	10	0
N.S.	1	1.00	1.06	0.00	0.00	0.00	0.00	0.00	0.20	0.00
time (sec)	N/A	0.237	0.068	0.000	0.000	0.000	0.000	0.000	0.181	0.000

Problem 26	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	51	51	51	0	0	0	0	0	10	0
N.S.	1	1.00	1.00	0.00	0.00	0.00	0.00	0.00	0.20	0.00
time (sec)	N/A	0.256	0.044	0.000	0.000	0.000	0.000	0.000	0.190	0.000

Problem 27	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	53	53	51	0	0	0	0	0	10	0
N.S.	1	1.00	0.96	0.00	0.00	0.00	0.00	0.00	0.19	0.00
time (sec)	N/A	0.251	0.043	0.000	0.000	0.000	0.000	0.000	0.197	0.000

Problem 28	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	53	53	51	0	0	0	0	0	10	0
N.S.	1	1.00	0.96	0.00	0.00	0.00	0.00	0.00	0.19	0.00
time (sec)	N/A	0.263	0.043	0.000	0.000	0.000	0.000	0.000	0.192	0.000

Problem 29	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	53	53	51	0	0	0	0	0	10	0
N.S.	1	1.00	0.96	0.00	0.00	0.00	0.00	0.00	0.19	0.00
time (sec)	N/A	0.258	0.042	0.000	0.000	0.000	0.000	0.000	0.188	0.000

Problem 30	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	53	53	68	0	0	0	0	0	10	0
N.S.	1	1.00	1.28	0.00	0.00	0.00	0.00	0.00	0.19	0.00
time (sec)	N/A	0.271	0.089	0.000	0.000	0.000	0.000	0.000	0.195	0.000

Problem 31	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	54	54	57	0	0	0	0	0	14	0
N.S.	1	1.00	1.06	0.00	0.00	0.00	0.00	0.00	0.26	0.00
time (sec)	N/A	0.254	0.052	0.000	0.000	0.000	0.000	0.000	0.185	0.000

Problem 32	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	54	54	59	0	0	0	0	0	14	0
N.S.	1	1.00	1.09	0.00	0.00	0.00	0.00	0.00	0.26	0.00
time (sec)	N/A	0.276	0.066	0.000	0.000	0.000	0.000	0.000	0.192	0.000

Problem 33	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	56	56	59	0	0	0	0	0	14	0
N.S.	1	1.00	1.05	0.00	0.00	0.00	0.00	0.00	0.25	0.00
time (sec)	N/A	0.266	0.058	0.000	0.000	0.000	0.000	0.000	0.196	0.000

Problem 34	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	56	56	59	0	0	0	0	0	14	0
N.S.	1	1.00	1.05	0.00	0.00	0.00	0.00	0.00	0.25	0.00
time (sec)	N/A	0.277	0.060	0.000	0.000	0.000	0.000	0.000	0.196	0.000

Problem 35	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	56	56	59	0	0	0	0	0	14	0
N.S.	1	1.00	1.05	0.00	0.00	0.00	0.00	0.00	0.25	0.00
time (sec)	N/A	0.285	0.062	0.000	0.000	0.000	0.000	0.000	0.206	0.000

Problem 36	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	56	56	76	0	0	0	0	0	14	0
N.S.	1	1.00	1.36	0.00	0.00	0.00	0.00	0.00	0.25	0.00
time (sec)	N/A	0.297	0.125	0.000	0.000	0.000	0.000	0.000	0.212	0.000

Problem 37	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	69	69	59	0	0	0	0	0	10	0
N.S.	1	1.00	0.86	0.00	0.00	0.00	0.00	0.00	0.14	0.00
time (sec)	N/A	0.266	0.086	0.000	0.000	0.000	0.000	0.000	0.188	0.000

Problem 38	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	72	72	65	0	0	0	0	0	14	0
N.S.	1	1.00	0.90	0.00	0.00	0.00	0.00	0.00	0.19	0.00
time (sec)	N/A	0.284	0.087	0.000	0.000	0.000	0.000	0.000	0.202	0.000

Problem 39	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	B	B	<b>F(-1)</b>	B	B	<b>F(-1)</b>
verified	N/A	Yes	Yes	<b>No</b>	TBD	TBD	TBD	TBD	TBD	TBD
size	50	66	92	44	1669	93	0	129	38	0
N.S.	1	1.32	1.84	0.88	33.38	1.86	0.00	2.58	0.76	0.00
time (sec)	N/A	0.206	0.200	0.317	0.198	0.074	0.000	0.164	0.198	0.000

Problem 40	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	B	B	<b>F</b>	B	B	<b>F(-1)</b>
verified	N/A	Yes	Yes	<b>No</b>	TBD	TBD	TBD	TBD	TBD	TBD
size	36	45	72	36	869	69	0	100	30	0
N.S.	1	1.25	2.00	1.00	24.14	1.92	0.00	2.78	0.83	0.00
time (sec)	N/A	0.194	0.201	0.151	0.160	0.112	0.000	0.165	0.195	0.000

Problem 41	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	B	C	B	B	<b>F</b>	B	B	<b>F(-1)</b>
verified	N/A	Yes	Yes	<b>No</b>	TBD	TBD	TBD	TBD	TBD	TBD
size	22	24	51	25	300	44	0	69	21	0
N.S.	1	1.09	2.32	1.14	13.64	2.00	0.00	3.14	0.95	0.00
time (sec)	N/A	0.185	0.087	0.104	0.147	0.074	0.000	0.127	0.182	0.000

Problem 42	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	B	B	B	B	<b>F</b>	B	B	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	5	5	15	18	35	19	0	12	5	0
N.S.	1	1.00	3.00	3.60	7.00	3.80	0.00	2.40	1.00	0.00
time (sec)	N/A	0.176	0.005	0.071	0.157	0.073	0.000	0.167	0.167	0.000

Problem 43	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	A	A	A	A	B	B
verified	N/A	Yes	Yes	<b>No</b>	TBD	TBD	TBD	TBD	TBD	TBD
size	12	14	12	13	10	4	12	11	6	12
N.S.	1	1.17	1.00	1.08	0.83	0.33	1.00	0.92	0.50	1.00
time (sec)	N/A	0.180	0.017	0.072	0.117	0.067	0.216	0.162	0.171	15.385



Problem 44	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	A	A	A	B	B	<b>F(-1)</b>
verified	N/A	Yes	Yes	<b>No</b>	TBD	TBD	TBD	TBD	TBD	TBD
size	29	33	23	19	11	11	29	44	14	0
N.S.	1	1.14	0.79	0.66	0.38	0.38	1.00	1.52	0.48	0.00
time (sec)	N/A	0.186	0.028	0.086	0.140	0.071	0.367	0.126	0.163	0.000

Problem 45	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	A	A	A	A	B	<b>F(-1)</b>
verified	N/A	Yes	Yes	<b>No</b>	TBD	TBD	TBD	TBD	TBD	TBD
size	43	54	31	27	17	17	46	61	22	0
N.S.	1	1.26	0.72	0.63	0.40	0.40	1.07	1.42	0.51	0.00
time (sec)	N/A	0.193	0.035	0.121	0.142	0.072	1.266	0.136	0.173	0.000

Problem 46	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	A	A	A	A	B	<b>F(-1)</b>
verified	N/A	Yes	Yes	<b>No</b>	TBD	TBD	TBD	TBD	TBD	TBD
size	57	75	37	33	23	21	61	78	30	0
N.S.	1	1.32	0.65	0.58	0.40	0.37	1.07	1.37	0.53	0.00
time (sec)	N/A	0.206	0.046	0.129	0.147	0.070	9.539	0.153	0.167	0.000

Problem 47	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	B	A	<b>F(-1)</b>	B	B	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	84	100	61	52	2183	106	0	164	43	0
N.S.	1	1.19	0.73	0.62	25.99	1.26	0.00	1.95	0.51	0.00
time (sec)	N/A	0.233	0.261	0.227	2.199	0.085	0.000	0.161	0.165	0.000

Problem 48	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	B	A	<b>F</b>	B	B	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	65	76	51	40	1113	80	0	106	35	0
N.S.	1	1.17	0.78	0.62	17.12	1.23	0.00	1.63	0.54	0.00
time (sec)	N/A	0.218	0.176	0.145	0.300	0.080	0.000	0.153	0.175	0.000

Problem 49	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	B	A	<b>F</b>	B	B	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	46	52	39	28	318	49	0	72	24	0
N.S.	1	1.13	0.85	0.61	6.91	1.07	0.00	1.57	0.52	0.00
time (sec)	N/A	0.205	0.072	0.104	0.178	0.078	0.000	0.120	0.177	0.000

Problem 50	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	<b>F</b>	A	B	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	26	28	17	20	24	64	0	13	8	0
N.S.	1	1.08	0.65	0.77	0.92	2.46	0.00	0.50	0.31	0.00
time (sec)	N/A	0.195	0.006	0.065	0.152	0.084	0.000	0.172	0.162	0.000

Problem 51	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	A	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	14	16	14	18	13	22	14	34	12	15
N.S.	1	1.14	1.00	1.29	0.93	1.57	1.00	2.43	0.86	1.07
time (sec)	N/A	0.189	0.018	0.057	0.112	0.068	0.288	0.161	0.162	14.904

Problem 52	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	<b>F</b>	A	A	A	B	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	36	46	27	33	0	29	32	52	21	0
N.S.	1	1.28	0.75	0.92	0.00	0.81	0.89	1.44	0.58	0.00
time (sec)	N/A	0.198	0.031	0.074	0.000	0.073	0.428	0.166	0.170	0.000

Problem 53	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	<b>F</b>	A	A	A	B	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	55	75	36	37	0	37	51	66	29	0
N.S.	1	1.36	0.65	0.67	0.00	0.67	0.93	1.20	0.53	0.00
time (sec)	N/A	0.213	0.040	0.095	0.000	0.072	1.473	0.159	0.168	0.000

Problem 54	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	<b>F</b>	A	A	A	B	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	74	104	42	43	0	43	68	76	37	0
N.S.	1	1.41	0.57	0.58	0.00	0.58	0.92	1.03	0.50	0.00
time (sec)	N/A	0.232	0.055	0.103	0.000	0.076	12.337	0.128	0.169	0.000

Problem 55	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	<b>F</b>	C	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	123	121	58	185	0	161	0	0	16	0
N.S.	1	0.98	0.47	1.50	0.00	1.31	0.00	0.00	0.13	0.00
time (sec)	N/A	0.655	0.149	0.925	0.000	0.084	0.000	0.000	0.160	0.000

Problem 56	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	<b>F</b>	C	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	71	86	46	103	0	99	0	0	14	0
N.S.	1	1.21	0.65	1.45	0.00	1.39	0.00	0.00	0.20	0.00
time (sec)	N/A	0.455	0.099	0.372	0.000	0.087	0.000	0.000	0.163	0.000

Problem 57	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	<b>F</b>	C	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	48	61	46	155	0	64	0	0	11	0
N.S.	1	1.27	0.96	3.23	0.00	1.33	0.00	0.00	0.23	0.00
time (sec)	N/A	0.373	0.042	0.145	0.000	0.075	0.000	0.000	0.172	0.000

Problem 58	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	<b>F</b>	C	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	50	65	38	93	0	68	0	0	16	0
N.S.	1	1.30	0.76	1.86	0.00	1.36	0.00	0.00	0.32	0.00
time (sec)	N/A	0.368	0.050	0.166	0.000	0.081	0.000	0.000	0.165	0.000

Problem 59	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	<b>F</b>	C	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	79	88	52	178	0	84	0	0	16	0
N.S.	1	1.11	0.66	2.25	0.00	1.06	0.00	0.00	0.20	0.00
time (sec)	N/A	0.444	0.082	0.463	0.000	0.087	0.000	0.000	0.166	0.000

Problem 60	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	<b>F</b>	C	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	123	125	63	117	0	88	0	0	16	0
N.S.	1	1.02	0.51	0.95	0.00	0.72	0.00	0.00	0.13	0.00
time (sec)	N/A	0.631	0.122	1.016	0.000	0.098	0.000	0.000	0.161	0.000

Problem 61	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	<b>F(-1)</b>	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	164	66	59	61	66	118	0	69	51	603
N.S.	1	0.40	0.36	0.37	0.40	0.72	0.00	0.42	0.31	3.68
time (sec)	N/A	0.275	0.070	0.588	0.117	0.083	0.000	0.178	0.173	19.962

Problem 62	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	<b>F</b>	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	118	54	47	49	48	88	0	35	39	121
N.S.	1	0.46	0.40	0.42	0.41	0.75	0.00	0.30	0.33	1.03
time (sec)	N/A	0.264	0.050	0.266	0.111	0.087	0.000	0.147	0.171	17.665

Problem 63	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	<b>F</b>	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	62	36	33	35	30	52	0	23	25	44
N.S.	1	0.58	0.53	0.56	0.48	0.84	0.00	0.37	0.40	0.71
time (sec)	N/A	0.253	0.031	0.106	0.115	0.072	0.000	0.142	0.169	15.969

Problem 64	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	<b>F</b>	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	16	16	16	18	9	24	0	9	10	7
N.S.	1	1.00	1.00	1.12	0.56	1.50	0.00	0.56	0.62	0.44
time (sec)	N/A	0.228	0.020	0.067	0.114	0.066	0.000	0.130	0.170	15.119

Problem 65	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	<b>F</b>	<b>F(-2)</b>	B	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	36	29	25	24	25	46	0	0	15	0
N.S.	1	0.81	0.69	0.67	0.69	1.28	0.00	0.00	0.42	0.00
time (sec)	N/A	0.222	0.040	0.081	0.108	0.075	0.000	0.000	0.175	0.000

Problem 66	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	<b>F</b>	<b>F(-2)</b>	B	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	86	62	38	42	58	61	0	0	33	0
N.S.	1	0.72	0.44	0.49	0.67	0.71	0.00	0.00	0.38	0.00
time (sec)	N/A	0.336	0.055	0.172	0.115	0.084	0.000	0.000	0.182	0.000

Problem 67	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	<b>F</b>	<b>F(-2)</b>	B	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	132	92	55	54	88	73	0	0	49	0
N.S.	1	0.70	0.42	0.41	0.67	0.55	0.00	0.00	0.37	0.00
time (sec)	N/A	0.463	0.098	0.423	0.120	0.081	0.000	0.000	0.174	0.000

Problem 68	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	80	80	71	0	0	0	0	0	18	0
N.S.	1	1.00	0.89	0.00	0.00	0.00	0.00	0.00	0.22	0.00
time (sec)	N/A	0.376	0.121	0.000	0.000	0.000	0.000	0.000	0.182	0.000

Problem 69	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	82	82	73	0	0	0	0	0	21	0
N.S.	1	1.00	0.89	0.00	0.00	0.00	0.00	0.00	0.26	0.00
time (sec)	N/A	0.357	0.072	0.000	0.000	0.000	0.000	0.000	0.181	0.000

Problem 70	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	91	91	77	0	0	0	0	0	19	0
N.S.	1	1.00	0.85	0.00	0.00	0.00	0.00	0.00	0.21	0.00
time (sec)	N/A	0.363	0.364	0.000	0.000	0.000	0.000	0.000	0.195	0.000

## 2.3 Detailed conclusion table specific for Rubi results

The following table is specific to Rubi only. It gives additional statistics for each integral. the column **steps** is the number of steps used by Rubi to obtain the antiderivative. The **rules** column is the number of unique rules used. The **integrand size** column is the leaf size of the integrand. Finally the ratio  $\frac{\text{number of rules}}{\text{integrand size}}$  is also given. The larger this ratio is, the harder the integral is to solve. In this test file, problem number [55] had the largest ratio of [1.39999999999999991]

Table 2.1: Rubi specific breakdown of results for each integral

#	grade	number of steps used	number of unique rules	normalized antiderivative leaf size	integrand leaf size	$\frac{\text{number of rules}}{\text{integrand leaf size}}$
1	A	2	2	1.00	6	0.333
2	A	4	3	1.00	8	0.375
3	A	4	4	1.00	8	0.500
4	A	4	3	0.89	8	0.375
5	A	6	6	1.09	8	0.750
6	A	4	3	0.86	8	0.375
7	A	8	8	1.13	8	1.000
8	A	4	3	0.80	8	0.375
9	A	8	8	1.01	10	0.800
10	A	6	6	1.00	10	0.600
11	A	6	6	1.00	10	0.600
12	A	4	4	1.00	10	0.400
13	A	4	4	1.00	10	0.400
14	A	6	6	1.00	10	0.600
15	A	6	6	1.00	10	0.600
16	A	8	8	1.06	10	0.800
17	A	8	8	1.02	12	0.667
18	A	6	6	1.00	12	0.500
19	A	6	6	1.00	12	0.500
20	A	4	4	1.00	12	0.333
21	A	4	4	1.00	12	0.333

Continued on next page



Table 2.1 – continued from previous page

#	grade	number of steps used	number of unique rules	normalized antiderivative leaf size	integrand leaf size	$\frac{\text{number of rules}}{\text{integrand leaf size}}$
22	A	6	6	1.00	12	0.500
23	A	6	6	1.00	12	0.500
24	A	8	8	1.08	12	0.667
25	A	4	4	1.00	10	0.400
26	A	4	4	1.00	10	0.400
27	A	4	4	1.00	10	0.400
28	A	4	4	1.00	10	0.400
29	A	4	4	1.00	10	0.400
30	A	4	4	1.00	10	0.400
31	A	4	4	1.00	12	0.333
32	A	4	4	1.00	12	0.333
33	A	4	4	1.00	12	0.333
34	A	4	4	1.00	12	0.333
35	A	4	4	1.00	12	0.333
36	A	4	4	1.00	12	0.333
37	A	4	4	1.00	8	0.500
38	A	4	4	1.00	10	0.400
39	A	7	6	1.32	8	0.750
40	A	6	5	1.25	8	0.625
41	A	5	4	1.09	8	0.500
42	A	4	3	1.00	8	0.375
43	A	4	3	1.17	8	0.375
44	A	5	4	1.14	8	0.500
45	A	6	5	1.26	8	0.625
46	A	7	6	1.32	8	0.750
47	A	8	7	1.19	10	0.700
48	A	7	6	1.17	10	0.600
49	A	6	5	1.13	10	0.500
50	A	5	4	1.08	10	0.400
51	A	4	3	1.14	10	0.300
52	A	5	4	1.28	10	0.400
53	A	6	5	1.36	10	0.500

Continued on next page

Table 2.1 – continued from previous page

#	grade	number of steps used	number of unique rules	normalized antiderivative leaf size	integrand leaf size	$\frac{\text{number of rules}}{\text{integrand leaf size}}$
54	A	7	6	1.41	10	0.600
55	A	14	14	0.98	10	1.400
56	A	10	10	1.21	10	1.000
57	A	8	8	1.27	10	0.800
58	A	8	8	1.30	10	0.800
59	A	10	10	1.11	10	1.000
60	A	14	14	1.02	10	1.400
61	A	6	5	0.40	10	0.500
62	A	6	5	0.46	10	0.500
63	A	6	5	0.58	10	0.500
64	A	6	5	1.00	10	0.500
65	A	5	5	0.81	10	0.500
66	A	9	9	0.72	10	0.900
67	A	13	13	0.70	10	1.300
68	A	6	6	1.00	12	0.500
69	A	6	6	1.00	14	0.429
70	A	5	5	1.00	21	0.238

# CHAPTER 3

## LISTING OF INTEGRALS

3.1	$\int \csc(a + bx) dx$ . . . . .	53
3.2	$\int \csc^2(a + bx) dx$ . . . . .	58
3.3	$\int \csc^3(a + bx) dx$ . . . . .	63
3.4	$\int \csc^4(a + bx) dx$ . . . . .	69
3.5	$\int \csc^5(a + bx) dx$ . . . . .	74
3.6	$\int \csc^6(a + bx) dx$ . . . . .	80
3.7	$\int \csc^7(a + bx) dx$ . . . . .	85
3.8	$\int \csc^8(a + bx) dx$ . . . . .	92
3.9	$\int \csc^{\frac{7}{2}}(a + bx) dx$ . . . . .	98
3.10	$\int \csc^{\frac{5}{2}}(a + bx) dx$ . . . . .	104
3.11	$\int \csc^{\frac{3}{2}}(a + bx) dx$ . . . . .	110
3.12	$\int \sqrt{\csc(a + bx)} dx$ . . . . .	116
3.13	$\int \frac{1}{\sqrt{\csc(a+bx)}} dx$ . . . . .	121
3.14	$\int \frac{1}{\csc^{\frac{3}{2}}(a+bx)} dx$ . . . . .	126
3.15	$\int \frac{1}{\csc^{\frac{5}{2}}(a+bx)} dx$ . . . . .	132
3.16	$\int \frac{1}{\csc^{\frac{7}{2}}(a+bx)} dx$ . . . . .	138
3.17	$\int (c \csc(a + bx))^{\frac{7}{2}} dx$ . . . . .	144
3.18	$\int (c \csc(a + bx))^{\frac{5}{2}} dx$ . . . . .	150
3.19	$\int (c \csc(a + bx))^{\frac{3}{2}} dx$ . . . . .	156
3.20	$\int \sqrt{c \csc(a + bx)} dx$ . . . . .	162
3.21	$\int \frac{1}{\sqrt{c \csc(a+bx)}} dx$ . . . . .	167
3.22	$\int \frac{1}{(c \csc(a+bx))^{\frac{3}{2}}} dx$ . . . . .	172
3.23	$\int \frac{1}{(c \csc(a+bx))^{\frac{5}{2}}} dx$ . . . . .	178
3.24	$\int \frac{1}{(c \csc(a+bx))^{\frac{7}{2}}} dx$ . . . . .	184
3.25	$\int \csc^{\frac{4}{3}}(a + bx) dx$ . . . . .	190

3.26	$\int \csc^{\frac{2}{3}}(a + bx) dx$	195
3.27	$\int \sqrt[3]{\csc(a + bx)} dx$	200
3.28	$\int \frac{1}{\sqrt[3]{\csc(a + bx)}} dx$	205
3.29	$\int \frac{1}{\csc^{\frac{2}{3}}(a+bx)} dx$	210
3.30	$\int \frac{1}{\csc^{\frac{4}{3}}(a+bx)} dx$	215
3.31	$\int (c \csc(a + bx))^{\frac{4}{3}} dx$	220
3.32	$\int (c \csc(a + bx))^{\frac{2}{3}} dx$	225
3.33	$\int \sqrt[3]{c \csc(a + bx)} dx$	230
3.34	$\int \frac{1}{\sqrt[3]{c \csc(a + bx)}} dx$	235
3.35	$\int \frac{1}{(c \csc(a+bx))^{\frac{2}{3}}} dx$	240
3.36	$\int \frac{1}{(c \csc(a+bx))^{\frac{4}{3}}} dx$	245
3.37	$\int \csc^n(a + bx) dx$	250
3.38	$\int (c \csc(a + bx))^n dx$	255
3.39	$\int \csc^2(x)^{7/2} dx$	260
3.40	$\int \csc^2(x)^{5/2} dx$	267
3.41	$\int \csc^2(x)^{3/2} dx$	273
3.42	$\int \sqrt{\csc^2(x)} dx$	279
3.43	$\int \frac{1}{\sqrt{\csc^2(x)}} dx$	284
3.44	$\int \frac{1}{\csc^2(x)^{3/2}} dx$	289
3.45	$\int \frac{1}{\csc^2(x)^{5/2}} dx$	295
3.46	$\int \frac{1}{\csc^2(x)^{7/2}} dx$	301
3.47	$\int (a \csc^2(x))^{7/2} dx$	307
3.48	$\int (a \csc^2(x))^{5/2} dx$	314
3.49	$\int (a \csc^2(x))^{3/2} dx$	321
3.50	$\int \sqrt{a \csc^2(x)} dx$	327
3.51	$\int \frac{1}{\sqrt{a \csc^2(x)}} dx$	332
3.52	$\int \frac{1}{(a \csc^2(x))^{3/2}} dx$	337
3.53	$\int \frac{1}{(a \csc^2(x))^{5/2}} dx$	342
3.54	$\int \frac{1}{(a \csc^2(x))^{7/2}} dx$	348
3.55	$\int (a \csc^3(x))^{5/2} dx$	355
3.56	$\int (a \csc^3(x))^{3/2} dx$	362
3.57	$\int \sqrt{a \csc^3(x)} dx$	369
3.58	$\int \frac{1}{\sqrt{a \csc^3(x)}} dx$	375
3.59	$\int \frac{1}{(a \csc^3(x))^{3/2}} dx$	381
3.60	$\int \frac{1}{(a \csc^3(x))^{5/2}} dx$	388

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3.61	$\int (a \csc^4(x))^{7/2} dx$	395
3.62	$\int (a \csc^4(x))^{5/2} dx$	402
3.63	$\int (a \csc^4(x))^{3/2} dx$	408
3.64	$\int \sqrt{a \csc^4(x)} dx$	413
3.65	$\int \frac{1}{\sqrt{a \csc^4(x)}} dx$	418
3.66	$\int \frac{1}{(a \csc^4(x))^{3/2}} dx$	423
3.67	$\int \frac{1}{(a \csc^4(x))^{5/2}} dx$	429
3.68	$\int ((b \csc(c + dx))^p)^n dx$	436
3.69	$\int (a(b \csc(c + dx))^p)^n dx$	441
3.70	$\int (a \csc(e + fx))^m (b \csc(e + fx))^n dx$	446

### 3.1 $\int \csc(a + bx) dx$

Optimal result . . . . .	53
Mathematica [A] (verified) . . . . .	53
Rubi [A] (verified) . . . . .	54
Maple [A] (verified) . . . . .	55
Fricas [B] (verification not implemented) . . . . .	55
Sympy [B] (verification not implemented) . . . . .	56
Maxima [A] (verification not implemented) . . . . .	56
Giac [A] (verification not implemented) . . . . .	56
Mupad [B] (verification not implemented) . . . . .	57
Reduce [B] (verification not implemented) . . . . .	57

#### Optimal result

Integrand size = 6, antiderivative size = 12

$$\int \csc(a + bx) dx = -\frac{\operatorname{arctanh}(\cos(a + bx))}{b}$$

output

```
-arctanh(cos(b*x+a))/b
```

#### Mathematica [A] (verified)

Time = 0.00 (sec) , antiderivative size = 12, normalized size of antiderivative = 1.00

$$\int \csc(a + bx) dx = -\frac{\operatorname{arctanh}(\cos(a + bx))}{b}$$

input

```
Integrate[Csc[a + b*x],x]
```

output

```
-(ArcTanh[Cos[a + b*x]]/b)
```

**Rubi [A] (verified)**

Time = 0.15 (sec) , antiderivative size = 12, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.333$ , Rules used = {3042, 4257}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \csc(a + bx) dx$$

$$\downarrow 3042$$

$$\int \csc(a + bx) dx$$

$$\downarrow 4257$$

$$-\frac{\operatorname{arctanh}(\cos(a + bx))}{b}$$

input `Int[Csc[a + b*x],x]`

output `-(ArcTanh[Cos[a + b*x]]/b)`

**Defintions of rubi rules used**

rule 3042 `Int[u_, x_Symbol] :=> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear Q[u, x]`

rule 4257 `Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] :=> Simp[-ArcTanh[Cos[c + d*x]]/d, x] /; FreeQ[{c, d}, x]`

**Maple [A] (verified)**

Time = 0.04 (sec) , antiderivative size = 15, normalized size of antiderivative = 1.25

method	result	size
norman	$\frac{\ln\left(\tan\left(\frac{bx}{2} + \frac{a}{2}\right)\right)}{b}$	15
parallelrisc	$\frac{\ln\left(\tan\left(\frac{bx}{2} + \frac{a}{2}\right)\right)}{b}$	15
derivativedivides	$-\frac{\ln(\csc(bx+a)+\cot(bx+a))}{b}$	20
default	$-\frac{\ln(\csc(bx+a)+\cot(bx+a))}{b}$	20
risc	$-\frac{\ln(e^{i(bx+a)}+1)}{b} + \frac{\ln(e^{i(bx+a)}-1)}{b}$	35

input `int(csc(b*x+a),x,method=_RETURNVERBOSE)`

output `1/b*ln(tan(1/2*b*x+1/2*a))`

**Fricas [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 30 vs. 2(12) = 24.

Time = 0.08 (sec) , antiderivative size = 30, normalized size of antiderivative = 2.50

$$\int \csc(a + bx) dx = -\frac{\log\left(\frac{1}{2} \cos(bx + a) + \frac{1}{2}\right) - \log\left(-\frac{1}{2} \cos(bx + a) + \frac{1}{2}\right)}{2b}$$

input `integrate(csc(b*x+a),x, algorithm="fricas")`

output `-1/2*(log(1/2*cos(b*x + a) + 1/2) - log(-1/2*cos(b*x + a) + 1/2))/b`



**Sympy [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 37 vs.  $2(10) = 20$ .

Time = 0.94 (sec) , antiderivative size = 37, normalized size of antiderivative = 3.08

$$\int \csc(a + bx) dx = \begin{cases} -\frac{\log(\cot(a+bx) + \csc(a+bx))}{b} & \text{for } b \neq 0 \\ \frac{x(\cot(a)\csc(a) + \csc^2(a))}{\cot(a) + \csc(a)} & \text{otherwise} \end{cases}$$

input `integrate(csc(b*x+a), x)`

output `Piecewise((-log(cot(a + b*x) + csc(a + b*x))/b, Ne(b, 0)), (x*(cot(a)*csc(a) + csc(a)**2)/(cot(a) + csc(a)), True))`

**Maxima [A] (verification not implemented)**

Time = 0.03 (sec) , antiderivative size = 19, normalized size of antiderivative = 1.58

$$\int \csc(a + bx) dx = -\frac{\log(\cot(bx + a) + \csc(bx + a))}{b}$$

input `integrate(csc(b*x+a), x, algorithm="maxima")`

output `-log(cot(b*x + a) + csc(b*x + a))/b`

**Giac [A] (verification not implemented)**

Time = 0.15 (sec) , antiderivative size = 15, normalized size of antiderivative = 1.25

$$\int \csc(a + bx) dx = \frac{\log\left(\left|\tan\left(\frac{1}{2}bx + \frac{1}{2}a\right)\right|\right)}{b}$$

input `integrate(csc(b*x+a), x, algorithm="giac")`

output `log(abs(tan(1/2*b*x + 1/2*a)))/b`

**Mupad [B] (verification not implemented)**

Time = 0.11 (sec) , antiderivative size = 12, normalized size of antiderivative = 1.00

$$\int \csc(a + bx) dx = -\frac{\operatorname{atanh}(\cos(a + bx))}{b}$$

input `int(1/sin(a + b*x),x)`

output `-atanh(cos(a + b*x))/b`

**Reduce [B] (verification not implemented)**

Time = 0.19 (sec) , antiderivative size = 14, normalized size of antiderivative = 1.17

$$\int \csc(a + bx) dx = \frac{\log\left(\tan\left(\frac{bx}{2} + \frac{a}{2}\right)\right)}{b}$$

input `int(csc(b*x+a),x)`

output `log(tan((a + b*x)/2))/b`

## 3.2 $\int \csc^2(a + bx) dx$

Optimal result . . . . .	58
Mathematica [A] (verified) . . . . .	58
Rubi [A] (verified) . . . . .	59
Maple [A] (verified) . . . . .	60
Fricas [A] (verification not implemented) . . . . .	60
Sympy [F] . . . . .	61
Maxima [A] (verification not implemented) . . . . .	61
Giac [A] (verification not implemented) . . . . .	61
Mupad [B] (verification not implemented) . . . . .	62
Reduce [B] (verification not implemented) . . . . .	62

### Optimal result

Integrand size = 8, antiderivative size = 11

$$\int \csc^2(a + bx) dx = -\frac{\cot(a + bx)}{b}$$

output `-cot(b*x+a)/b`

### Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 11, normalized size of antiderivative = 1.00

$$\int \csc^2(a + bx) dx = -\frac{\cot(a + bx)}{b}$$

input `Integrate[Csc[a + b*x]^2,x]`

output `-(Cot[a + b*x]/b)`

**Rubi [A] (verified)**

Time = 0.17 (sec) , antiderivative size = 11, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 3,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.375$ , Rules used = {3042, 4254, 24}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \csc^2(a + bx) dx \\ & \quad \downarrow \text{3042} \\ & \int \csc(a + bx)^2 dx \\ & \quad \downarrow \text{4254} \\ & -\frac{\int 1 d \cot(a + bx)}{b} \\ & \quad \downarrow \text{24} \\ & -\frac{\cot(a + bx)}{b} \end{aligned}$$

input `Int[Csc[a + b*x]^2,x]`

output `-(Cot[a + b*x]/b)`

**Defintions of rubi rules used**

rule 24 `Int[a_, x_Symbol] := Simp[a*x, x] /; FreeQ[a, x]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 4254

```
Int[csc[(c_.) + (d_.)*(x_)]^(n_), x_Symbol] := Simp[-d^(-1) Subst[Int[Exp
andIntegrand[(1 + x^2)^(n/2 - 1), x], x], x, Cot[c + d*x]], x] /; FreeQ[{c,
d}, x] && IGtQ[n/2, 0]
```

**Maple [A] (verified)**

Time = 0.12 (sec) , antiderivative size = 12, normalized size of antiderivative = 1.09

method	result	size
derivativedivides	$-\frac{\cot(bx+a)}{b}$	12
default	$-\frac{\cot(bx+a)}{b}$	12
risch	$-\frac{2i}{b(e^{2i(bx+a)}-1)}$	20
parallelrisch	$-\frac{\cot\left(\frac{bx}{2} + \frac{a}{2}\right) + \tan\left(\frac{bx}{2} + \frac{a}{2}\right)}{2b}$	27
norman	$-\frac{\frac{1}{2b} + \frac{\tan\left(\frac{bx}{2} + \frac{a}{2}\right)^2}{2b}}{\tan\left(\frac{bx}{2} + \frac{a}{2}\right)}$	35

input

```
int(csc(b*x+a)^2,x,method=_RETURNVERBOSE)
```

output

```
-cot(b*x+a)/b
```

**Fricas [A] (verification not implemented)**

Time = 0.07 (sec) , antiderivative size = 19, normalized size of antiderivative = 1.73

$$\int \csc^2(a + bx) dx = -\frac{\cos(bx + a)}{b \sin(bx + a)}$$

input

```
integrate(csc(b*x+a)^2,x, algorithm="fricas")
```

output

```
-cos(b*x + a)/(b*sin(b*x + a))
```

**Sympy [F]**

$$\int \csc^2(a + bx) dx = \int \csc^2(a + bx) dx$$

input `integrate(csc(b*x+a)**2,x)`

output `Integral(csc(a + b*x)**2, x)`

**Maxima [A] (verification not implemented)**

Time = 0.03 (sec) , antiderivative size = 13, normalized size of antiderivative = 1.18

$$\int \csc^2(a + bx) dx = -\frac{1}{b \tan(bx + a)}$$

input `integrate(csc(b*x+a)^2,x, algorithm="maxima")`

output `-1/(b*tan(b*x + a))`

**Giac [A] (verification not implemented)**

Time = 0.13 (sec) , antiderivative size = 13, normalized size of antiderivative = 1.18

$$\int \csc^2(a + bx) dx = -\frac{1}{b \tan(bx + a)}$$

input `integrate(csc(b*x+a)^2,x, algorithm="giac")`

output `-1/(b*tan(b*x + a))`

**Mupad [B] (verification not implemented)**

Time = 16.24 (sec) , antiderivative size = 11, normalized size of antiderivative = 1.00

$$\int \csc^2(a + bx) dx = -\frac{\cot(a + bx)}{b}$$

input `int(1/sin(a + b*x)^2,x)`

output `-cot(a + b*x)/b`

**Reduce [B] (verification not implemented)**

Time = 0.19 (sec) , antiderivative size = 19, normalized size of antiderivative = 1.73

$$\int \csc^2(a + bx) dx = -\frac{\cos(bx + a)}{\sin(bx + a)b}$$

input `int(csc(b*x+a)^2,x)`

output `( - cos(a + b*x))/(sin(a + b*x)*b)`

### 3.3 $\int \csc^3(a + bx) dx$

Optimal result . . . . .	63
Mathematica [B] (verified) . . . . .	63
Rubi [A] (verified) . . . . .	64
Maple [A] (verified) . . . . .	65
Fricas [B] (verification not implemented) . . . . .	66
Sympy [F] . . . . .	66
Maxima [A] (verification not implemented) . . . . .	66
Giac [B] (verification not implemented) . . . . .	67
Mupad [B] (verification not implemented) . . . . .	67
Reduce [B] (verification not implemented) . . . . .	68

#### Optimal result

Integrand size = 8, antiderivative size = 34

$$\int \csc^3(a + bx) dx = -\frac{\operatorname{arctanh}(\cos(a + bx))}{2b} - \frac{\cot(a + bx) \csc(a + bx)}{2b}$$

output `-1/2*arctanh(cos(b*x+a))/b-1/2*cot(b*x+a)*csc(b*x+a)/b`

#### Mathematica [B] (verified)

Leaf count is larger than twice the leaf count of optimal. 75 vs. 2(34) = 68.

Time = 0.01 (sec) , antiderivative size = 75, normalized size of antiderivative = 2.21

$$\int \csc^3(a + bx) dx = -\frac{\csc^2\left(\frac{1}{2}(a + bx)\right)}{8b} - \frac{\log\left(\cos\left(\frac{1}{2}(a + bx)\right)\right)}{2b} + \frac{\log\left(\sin\left(\frac{1}{2}(a + bx)\right)\right)}{2b} + \frac{\sec^2\left(\frac{1}{2}(a + bx)\right)}{8b}$$

input `Integrate[Csc[a + b*x]^3,x]`

output `-1/8*Csc[(a + b*x)/2]^2/b - Log[Cos[(a + b*x)/2]]/(2*b) + Log[Sin[(a + b*x)/2]]/(2*b) + Sec[(a + b*x)/2]^2/(8*b)`



**Rubi [A] (verified)**

Time = 0.23 (sec) , antiderivative size = 34, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {3042, 4255, 3042, 4257}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \csc^3(a + bx) dx \\
 & \quad \downarrow \text{3042} \\
 & \int \csc(a + bx)^3 dx \\
 & \quad \downarrow \text{4255} \\
 & \frac{1}{2} \int \csc(a + bx) dx - \frac{\cot(a + bx) \csc(a + bx)}{2b} \\
 & \quad \downarrow \text{3042} \\
 & \frac{1}{2} \int \csc(a + bx) dx - \frac{\cot(a + bx) \csc(a + bx)}{2b} \\
 & \quad \downarrow \text{4257} \\
 & -\frac{\operatorname{arctanh}(\cos(a + bx))}{2b} - \frac{\cot(a + bx) \csc(a + bx)}{2b}
 \end{aligned}$$

input

`Int [Csc[a + b*x]^3,x]`

output

`-1/2*ArcTanh[Cos[a + b*x]]/b - (Cot[a + b*x]*Csc[a + b*x])/(2*b)`

## Definitions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 4255 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^n, x_Symbol] := Simp[(-b)*Cos[c + d*x]*((b*Csc[c + d*x])^(n - 1)/(d*(n - 1))), x] + Simp[b^2*((n - 2)/(n - 1)) Int[(b*Csc[c + d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[2*n]`

rule 4257 `Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] := Simp[-ArcTanh[Cos[c + d*x]]/d, x] /; FreeQ[{c, d}, x]`

## Maple [A] (verified)

Time = 0.11 (sec) , antiderivative size = 38, normalized size of antiderivative = 1.12

method	result	size
derivativedivides	$\frac{-\frac{\csc(bx+a)\cot(bx+a)}{2} + \frac{\ln(\csc(bx+a)-\cot(bx+a))}{2}}{b}$	38
default	$\frac{-\frac{\csc(bx+a)\cot(bx+a)}{2} + \frac{\ln(\csc(bx+a)-\cot(bx+a))}{2}}{b}$	38
parallelrisc	$\frac{\tan\left(\frac{bx}{2} + \frac{a}{2}\right)^2 - \cot\left(\frac{bx}{2} + \frac{a}{2}\right)^2 + 4 \ln\left(\tan\left(\frac{bx}{2} + \frac{a}{2}\right)\right)}{8b}$	43
norman	$\frac{-\frac{1}{8b} + \frac{\tan\left(\frac{bx}{2} + \frac{a}{2}\right)^4}{8b}}{\tan\left(\frac{bx}{2} + \frac{a}{2}\right)^2} + \frac{\ln\left(\tan\left(\frac{bx}{2} + \frac{a}{2}\right)\right)}{2b}$	51
risc	$\frac{e^{3i(bx+a)} + e^{i(bx+a)}}{b(e^{2i(bx+a)} - 1)^2} - \frac{\ln(e^{i(bx+a)} + 1)}{2b} + \frac{\ln(e^{i(bx+a)} - 1)}{2b}$	72

input `int(csc(b*x+a)^3,x,method=_RETURNVERBOSE)`

output `1/b*(-1/2*csc(b*x+a)*cot(b*x+a)+1/2*ln(csc(b*x+a)-cot(b*x+a)))`

**Fricas [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 72 vs.  $2(30) = 60$ .

Time = 0.08 (sec) , antiderivative size = 72, normalized size of antiderivative = 2.12

$$\int \csc^3(a + bx) dx = \frac{(\cos(bx + a)^2 - 1) \log\left(\frac{1}{2} \cos(bx + a) + \frac{1}{2}\right) - (\cos(bx + a)^2 - 1) \log\left(-\frac{1}{2} \cos(bx + a) + \frac{1}{2}\right) - 2 \cos(bx + a)}{4(b \cos(bx + a)^2 - b)}$$

input `integrate(csc(b*x+a)^3,x, algorithm="fricas")`

output 
$$-1/4*((\cos(b*x + a)^2 - 1)*\log(1/2*\cos(b*x + a) + 1/2) - (\cos(b*x + a)^2 - 1)*\log(-1/2*\cos(b*x + a) + 1/2) - 2*\cos(b*x + a))/(b*\cos(b*x + a)^2 - b)$$

**Sympy [F]**

$$\int \csc^3(a + bx) dx = \int \csc^3(a + bx) dx$$

input `integrate(csc(b*x+a)**3,x)`

output `Integral(csc(a + b*x)**3, x)`

**Maxima [A] (verification not implemented)**

Time = 0.03 (sec) , antiderivative size = 46, normalized size of antiderivative = 1.35

$$\int \csc^3(a + bx) dx = \frac{\frac{2 \cos(bx+a)}{\cos(bx+a)^2-1} - \log(\cos(bx + a) + 1) + \log(\cos(bx + a) - 1)}{4b}$$

input `integrate(csc(b*x+a)^3,x, algorithm="maxima")`

output  $1/4*(2*\cos(b*x + a)/(\cos(b*x + a)^2 - 1) - \log(\cos(b*x + a) + 1) + \log(\cos(b*x + a) - 1))/b$

### Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 92 vs.  $2(30) = 60$ .

Time = 0.16 (sec) , antiderivative size = 92, normalized size of antiderivative = 2.71

$$\int \csc^3(a + bx) dx = -\frac{\left(\frac{2(\cos(bx+a)-1)}{\cos(bx+a)+1} - 1\right)(\cos(bx+a)+1)}{\cos(bx+a)-1} + \frac{\cos(bx+a)-1}{\cos(bx+a)+1} - 2 \log\left(\frac{|-\cos(bx+a)+1|}{|\cos(bx+a)+1|}\right)$$

input `integrate(csc(b*x+a)^3,x, algorithm="giac")`

output  $-1/8*((2*(\cos(b*x + a) - 1)/(\cos(b*x + a) + 1) - 1)*(\cos(b*x + a) + 1)/(\cos(b*x + a) - 1) + (\cos(b*x + a) - 1)/(\cos(b*x + a) + 1) - 2*\log(\text{abs}(-\cos(b*x + a) + 1)/\text{abs}(\cos(b*x + a) + 1)))/b$

### Mupad [B] (verification not implemented)

Time = 0.07 (sec) , antiderivative size = 36, normalized size of antiderivative = 1.06

$$\int \csc^3(a + bx) dx = \frac{\cos(a + bx)}{2b(\cos(a + bx)^2 - 1)} - \frac{\text{atanh}(\cos(a + bx))}{2b}$$

input `int(1/sin(a + b*x)^3,x)`

output  $\cos(a + b*x)/(2*b*(\cos(a + b*x)^2 - 1)) - \text{atanh}(\cos(a + b*x))/(2*b)$

**Reduce [B] (verification not implemented)**

Time = 0.20 (sec) , antiderivative size = 41, normalized size of antiderivative = 1.21

$$\int \csc^3(a + bx) dx = \frac{-\cos(bx + a) + \log\left(\tan\left(\frac{bx}{2} + \frac{a}{2}\right)\right) \sin(bx + a)^2}{2 \sin(bx + a)^2 b}$$

input `int(csc(b*x+a)^3,x)`

output `( - cos(a + b*x) + log(tan((a + b*x)/2))*sin(a + b*x)**2)/(2*sin(a + b*x)*  
*2*b)`

### 3.4 $\int \csc^4(a + bx) dx$

Optimal result	69
Mathematica [A] (verified)	69
Rubi [A] (verified)	70
Maple [A] (verified)	71
Fricas [A] (verification not implemented)	71
Sympy [F]	72
Maxima [A] (verification not implemented)	72
Giac [A] (verification not implemented)	72
Mupad [B] (verification not implemented)	73
Reduce [B] (verification not implemented)	73

#### Optimal result

Integrand size = 8, antiderivative size = 27

$$\int \csc^4(a + bx) dx = -\frac{\cot(a + bx)}{b} - \frac{\cot^3(a + bx)}{3b}$$

output

```
-cot(b*x+a)/b-1/3*cot(b*x+a)^3/b
```

#### Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 35, normalized size of antiderivative = 1.30

$$\int \csc^4(a + bx) dx = -\frac{2 \cot(a + bx)}{3b} - \frac{\cot(a + bx) \csc^2(a + bx)}{3b}$$

input

```
Integrate[Csc[a + b*x]^4,x]
```

output

```
(-2*Cot[a + b*x])/(3*b) - (Cot[a + b*x]*Csc[a + b*x]^2)/(3*b)
```

**Rubi [A] (verified)**

Time = 0.19 (sec) , antiderivative size = 24, normalized size of antiderivative = 0.89, number of steps used = 4, number of rules used = 3,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.375$ , Rules used = {3042, 4254, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \csc^4(a + bx) dx \\ & \quad \downarrow \text{3042} \\ & \int \csc(a + bx)^4 dx \\ & \quad \downarrow \text{4254} \\ & - \frac{\int (\cot^2(a + bx) + 1) d \cot(a + bx)}{b} \\ & \quad \downarrow \text{2009} \\ & - \frac{\frac{1}{3} \cot^3(a + bx) + \cot(a + bx)}{b} \end{aligned}$$

input `Int[Csc[a + b*x]^4,x]`

output `-((Cot[a + b*x] + Cot[a + b*x]^3/3)/b)`

**Defintions of rubi rules used**

rule 2009 `Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 4254

```
Int[csc[(c_.) + (d_.)*(x_)]^(n_), x_Symbol] := Simp[-d^(-1) Subst[Int[Exp
andIntegrand[(1 + x^2)^(n/2 - 1), x], x], x, Cot[c + d*x]], x] /; FreeQ[{c,
d}, x] && IGtQ[n/2, 0]
```

**Maple [A] (verified)**

Time = 0.14 (sec) , antiderivative size = 23, normalized size of antiderivative = 0.85

method	result	size
derivativedivides	$\frac{\left(-\frac{2}{3} - \frac{\csc(bx+a)^2}{3}\right) \cot(bx+a)}{b}$	23
default	$\frac{\left(-\frac{2}{3} - \frac{\csc(bx+a)^2}{3}\right) \cot(bx+a)}{b}$	23
risch	$\frac{4i(3e^{2i(bx+a)} - 1)}{3b(e^{2i(bx+a)} - 1)^3}$	33
parallelrisch	$\frac{\tan\left(\frac{bx}{2} + \frac{a}{2}\right)^3 - \cot\left(\frac{bx}{2} + \frac{a}{2}\right)^3 + 9 \tan\left(\frac{bx}{2} + \frac{a}{2}\right) - 9 \cot\left(\frac{bx}{2} + \frac{a}{2}\right)}{24b}$	53
norman	$\frac{-\frac{1}{24b} - \frac{3 \tan\left(\frac{bx}{2} + \frac{a}{2}\right)^2}{8b} + \frac{3 \tan\left(\frac{bx}{2} + \frac{a}{2}\right)^4}{8b} + \frac{\tan\left(\frac{bx}{2} + \frac{a}{2}\right)^6}{24b}}{\tan\left(\frac{bx}{2} + \frac{a}{2}\right)^3}$	67

input

```
int(csc(b*x+a)^4,x,method=_RETURNVERBOSE)
```

output

```
1/b*(-2/3-1/3*csc(b*x+a)^2)*cot(b*x+a)
```

**Fricas [A] (verification not implemented)**

Time = 0.07 (sec) , antiderivative size = 45, normalized size of antiderivative = 1.67

$$\int \csc^4(a + bx) dx = -\frac{2 \cos(bx + a)^3 - 3 \cos(bx + a)}{3(b \cos(bx + a)^2 - b) \sin(bx + a)}$$

input

```
integrate(csc(b*x+a)^4,x, algorithm="fricas")
```



output

```
-1/3*(2*cos(b*x + a)^3 - 3*cos(b*x + a))/((b*cos(b*x + a)^2 - b)*sin(b*x + a))
```

**Sympy [F]**

$$\int \csc^4(a + bx) dx = \int \csc^4(a + bx) dx$$

input

```
integrate(csc(b*x+a)**4,x)
```

output

```
Integral(csc(a + b*x)**4, x)
```

**Maxima [A] (verification not implemented)**

Time = 0.03 (sec) , antiderivative size = 25, normalized size of antiderivative = 0.93

$$\int \csc^4(a + bx) dx = -\frac{3 \tan^2(bx + a) + 1}{3 b \tan^3(bx + a)}$$

input

```
integrate(csc(b*x+a)^4,x, algorithm="maxima")
```

output

```
-1/3*(3*tan(b*x + a)^2 + 1)/(b*tan(b*x + a)^3)
```

**Giac [A] (verification not implemented)**

Time = 0.17 (sec) , antiderivative size = 25, normalized size of antiderivative = 0.93

$$\int \csc^4(a + bx) dx = -\frac{3 \tan^2(bx + a) + 1}{3 b \tan^3(bx + a)}$$

input

```
integrate(csc(b*x+a)^4,x, algorithm="giac")
```

output  $-1/3*(3*\tan(b*x + a)^2 + 1)/(b*\tan(b*x + a)^3)$

### Mupad [B] (verification not implemented)

Time = 15.64 (sec) , antiderivative size = 21, normalized size of antiderivative = 0.78

$$\int \csc^4(a + bx) dx = -\frac{\cot(a + bx) (\cot(a + bx)^2 + 3)}{3b}$$

input  $\text{int}(1/\sin(a + b*x)^4, x)$

output  $-(\cot(a + b*x)*(\cot(a + b*x)^2 + 3))/(3*b)$

### Reduce [B] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 31, normalized size of antiderivative = 1.15

$$\int \csc^4(a + bx) dx = \frac{\cos(bx + a) (-2 \sin(bx + a)^2 - 1)}{3 \sin(bx + a)^3 b}$$

input  $\text{int}(\csc(b*x+a)^4, x)$

output  $(\cos(a + b*x)*(-2*\sin(a + b*x)**2 - 1))/(3*\sin(a + b*x)**3*b)$

### 3.5 $\int \csc^5(a + bx) dx$

Optimal result . . . . .	74
Mathematica [B] (verified) . . . . .	74
Rubi [A] (verified) . . . . .	75
Maple [A] (verified) . . . . .	76
Fricas [B] (verification not implemented) . . . . .	77
Sympy [F] . . . . .	77
Maxima [A] (verification not implemented) . . . . .	78
Giac [B] (verification not implemented) . . . . .	78
Mupad [B] (verification not implemented) . . . . .	79
Reduce [B] (verification not implemented) . . . . .	79

#### Optimal result

Integrand size = 8, antiderivative size = 55

$$\int \csc^5(a + bx) dx = -\frac{3\operatorname{arctanh}(\cos(a + bx))}{8b} - \frac{3 \cot(a + bx) \csc(a + bx)}{8b} - \frac{\cot(a + bx) \csc^3(a + bx)}{4b}$$

output `-3/8*arctanh(cos(b*x+a))/b-3/8*cot(b*x+a)*csc(b*x+a)/b-1/4*cot(b*x+a)*csc(b*x+a)^3/b`

#### Mathematica [B] (verified)

Leaf count is larger than twice the leaf count of optimal. 113 vs. 2(55) = 110.

Time = 0.02 (sec) , antiderivative size = 113, normalized size of antiderivative = 2.05

$$\int \csc^5(a + bx) dx = -\frac{3 \csc^2\left(\frac{1}{2}(a + bx)\right)}{32b} - \frac{\csc^4\left(\frac{1}{2}(a + bx)\right)}{64b} - \frac{3 \log\left(\cos\left(\frac{1}{2}(a + bx)\right)\right)}{8b} + \frac{3 \log\left(\sin\left(\frac{1}{2}(a + bx)\right)\right)}{8b} + \frac{3 \sec^2\left(\frac{1}{2}(a + bx)\right)}{32b} + \frac{\sec^4\left(\frac{1}{2}(a + bx)\right)}{64b}$$

input `Integrate[Csc[a + b*x]^5,x]`

output

$$\begin{aligned} & (-3*\text{Csc}[(a + b*x)/2]^2)/(32*b) - \text{Csc}[(a + b*x)/2]^4/(64*b) - (3*\text{Log}[\text{Cos}[(a \\ & + b*x)/2]])/(8*b) + (3*\text{Log}[\text{Sin}[(a + b*x)/2]])/(8*b) + (3*\text{Sec}[(a + b*x)/2] \\ & ^2)/(32*b) + \text{Sec}[(a + b*x)/2]^4/(64*b) \end{aligned}$$

**Rubi [A] (verified)**

Time = 0.31 (sec) , antiderivative size = 60, normalized size of antiderivative = 1.09, number of steps used = 6, number of rules used = 6,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.750$ , Rules used = {3042, 4255, 3042, 4255, 3042, 4257}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \csc^5(a + bx) dx \\ & \quad \downarrow \text{3042} \\ & \int \csc(a + bx)^5 dx \\ & \quad \downarrow \text{4255} \\ & \frac{3}{4} \int \csc^3(a + bx) dx - \frac{\cot(a + bx) \csc^3(a + bx)}{4b} \\ & \quad \downarrow \text{3042} \\ & \frac{3}{4} \int \csc(a + bx)^3 dx - \frac{\cot(a + bx) \csc^3(a + bx)}{4b} \\ & \quad \downarrow \text{4255} \\ & \frac{3}{4} \left( \frac{1}{2} \int \csc(a + bx) dx - \frac{\cot(a + bx) \csc(a + bx)}{2b} \right) - \frac{\cot(a + bx) \csc^3(a + bx)}{4b} \\ & \quad \downarrow \text{3042} \\ & \frac{3}{4} \left( \frac{1}{2} \int \csc(a + bx) dx - \frac{\cot(a + bx) \csc(a + bx)}{2b} \right) - \frac{\cot(a + bx) \csc^3(a + bx)}{4b} \\ & \quad \downarrow \text{4257} \\ & \frac{3}{4} \left( -\frac{\text{arctanh}(\cos(a + bx))}{2b} - \frac{\cot(a + bx) \csc(a + bx)}{2b} \right) - \frac{\cot(a + bx) \csc^3(a + bx)}{4b} \end{aligned}$$

input `Int[Csc[a + b*x]^5,x]`

output 
$$-1/4*(\text{Cot}[a + b*x]*\text{Csc}[a + b*x]^3)/b + (3*(-1/2*\text{ArcTanh}[\text{Cos}[a + b*x]]/b - (\text{Cot}[a + b*x]*\text{Csc}[a + b*x])/(2*b)))/4$$

**Defintions of rubi rules used**

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear Q[u, x]`

rule 4255 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^n_], x_Symbol] := Simp[(-b)*Cos[c + d*x]*(b*Csc[c + d*x]^(n - 1)/(d*(n - 1))), x] + Simp[b^2*(n - 2)/(n - 1) Int[(b*Csc[c + d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[2*n]`

rule 4257 `Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] := Simp[-ArcTanh[Cos[c + d*x]]/d, x] /; FreeQ[{c, d}, x]`

**Maple [A] (verified)**

Time = 0.18 (sec) , antiderivative size = 50, normalized size of antiderivative = 0.91

method	result	size
derivativedivides	$\frac{\left(-\frac{\csc(bx+a)^3}{4} - \frac{3 \csc(bx+a)}{8}\right) \cot(bx+a) + \frac{3 \ln(\csc(bx+a) - \cot(bx+a))}{8}}{b}$	50
default	$\frac{\left(-\frac{\csc(bx+a)^3}{4} - \frac{3 \csc(bx+a)}{8}\right) \cot(bx+a) + \frac{3 \ln(\csc(bx+a) - \cot(bx+a))}{8}}{b}$	50
parallelrisc	$\frac{-\cot\left(\frac{bx}{2} + \frac{a}{2}\right)^4 + \tan\left(\frac{bx}{2} + \frac{a}{2}\right)^4 - 8 \cot\left(\frac{bx}{2} + \frac{a}{2}\right)^2 + 8 \tan\left(\frac{bx}{2} + \frac{a}{2}\right)^2 + 24 \ln\left(\tan\left(\frac{bx}{2} + \frac{a}{2}\right)\right)}{64b}$	69
norman	$-\frac{1}{64b} - \frac{\tan\left(\frac{bx}{2} + \frac{a}{2}\right)^2}{8b} + \frac{\tan\left(\frac{bx}{2} + \frac{a}{2}\right)^6}{8b} + \frac{\tan\left(\frac{bx}{2} + \frac{a}{2}\right)^8}{64b} + \frac{3 \ln\left(\tan\left(\frac{bx}{2} + \frac{a}{2}\right)\right)}{8b}$	83
risc	$\frac{3e^{7i(bx+a)} - 11e^{5i(bx+a)} - 11e^{3i(bx+a)} + 3e^{i(bx+a)}}{4b(e^{2i(bx+a)} - 1)^4} - \frac{3 \ln(e^{i(bx+a)} + 1)}{8b} + \frac{3 \ln(e^{i(bx+a)} - 1)}{8b}$	99

input `int(csc(b*x+a)^5,x,method=_RETURNVERBOSE)`

output `1/b*((-1/4*csc(b*x+a)^3-3/8*csc(b*x+a))*cot(b*x+a)+3/8*ln(csc(b*x+a)-cot(b*x+a)))`

### Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 112 vs.  $2(49) = 98$ .

Time = 0.07 (sec) , antiderivative size = 112, normalized size of antiderivative = 2.04

$$\int \csc^5(a + bx) dx = \frac{6 \cos^3(bx + a) - 3(\cos^4(bx + a) - 2 \cos^2(bx + a) + 1) \log\left(\frac{1}{2} \cos(bx + a) + \frac{1}{2}\right) + 3(\cos^4(bx + a) - 2 \cos^2(bx + a) + 1) \log\left(\frac{1}{2} \cos(bx + a) - \frac{1}{2}\right)}{16(b \cos^4(bx + a) - 2b \cos^2(bx + a) + b)}$$

input `integrate(csc(b*x+a)^5,x, algorithm="fricas")`

output `1/16*(6*cos(b*x + a)^3 - 3*(cos(b*x + a)^4 - 2*cos(b*x + a)^2 + 1)*log(1/2*cos(b*x + a) + 1/2) + 3*(cos(b*x + a)^4 - 2*cos(b*x + a)^2 + 1)*log(-1/2*cos(b*x + a) + 1/2) - 10*cos(b*x + a))/(b*cos(b*x + a)^4 - 2*b*cos(b*x + a)^2 + b)`

### Sympy [F]

$$\int \csc^5(a + bx) dx = \int \csc^5(a + bx) dx$$

input `integrate(csc(b*x+a)**5,x)`

output `Integral(csc(a + b*x)**5, x)`

**Maxima [A] (verification not implemented)**

Time = 0.03 (sec) , antiderivative size = 71, normalized size of antiderivative = 1.29

$$\int \csc^5(a + bx) dx$$

$$= \frac{2 \left( 3 \cos(bx+a)^3 - 5 \cos(bx+a) \right)}{\cos(bx+a)^4 - 2 \cos(bx+a)^2 + 1} - 3 \log(\cos(bx+a) + 1) + 3 \log(\cos(bx+a) - 1)}{16b}$$

input `integrate(csc(b*x+a)^5,x, algorithm="maxima")`

output  $\frac{1}{16} * (2 * (3 * \cos(b * x + a)^3 - 5 * \cos(b * x + a)) / (\cos(b * x + a)^4 - 2 * \cos(b * x + a)^2 + 1) - 3 * \log(\cos(b * x + a) + 1) + 3 * \log(\cos(b * x + a) - 1)) / b$

**Giac [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 138 vs.  $2(49) = 98$ .

Time = 0.16 (sec) , antiderivative size = 138, normalized size of antiderivative = 2.51

$$\int \csc^5(a + bx) dx$$

$$= \frac{\left( \frac{8(\cos(bx+a)-1)}{\cos(bx+a)+1} - \frac{18(\cos(bx+a)-1)^2}{(\cos(bx+a)+1)^2} - 1 \right) (\cos(bx+a)+1)^2}{(\cos(bx+a)-1)^2} - \frac{8(\cos(bx+a)-1)}{\cos(bx+a)+1} + \frac{(\cos(bx+a)-1)^2}{(\cos(bx+a)+1)^2} + 12 \log\left(\frac{|\cos(bx+a)+1|}{|\cos(bx+a)-1|}\right)}{64b}$$

input `integrate(csc(b*x+a)^5,x, algorithm="giac")`

output  $\frac{1}{64} * ((8 * (\cos(b * x + a) - 1) / (\cos(b * x + a) + 1) - 18 * (\cos(b * x + a) - 1)^2 / (\cos(b * x + a) + 1)^2 - 1) * (\cos(b * x + a) + 1)^2 / (\cos(b * x + a) - 1)^2 - 8 * (\cos(b * x + a) - 1) / (\cos(b * x + a) + 1) + (\cos(b * x + a) - 1)^2 / (\cos(b * x + a) + 1)^2 + 12 * \log(\text{abs}(-\cos(b * x + a) + 1) / \text{abs}(\cos(b * x + a) + 1))) / b$

**Mupad [B] (verification not implemented)**

Time = 0.10 (sec) , antiderivative size = 59, normalized size of antiderivative = 1.07

$$\int \csc^5(a + bx) dx = -\frac{3 \operatorname{atanh}(\cos(a + bx))}{8b} - \frac{\frac{5 \cos(a+bx)}{8} - \frac{3 \cos(a+bx)^3}{8}}{b (\cos(a + bx)^4 - 2 \cos(a + bx)^2 + 1)}$$

input `int(1/sin(a + b*x)^5,x)`output `-(3*atanh(cos(a + b*x)))/(8*b) - ((5*cos(a + b*x))/8 - (3*cos(a + b*x)^3)/8)/(b*(cos(a + b*x)^4 - 2*cos(a + b*x)^2 + 1))`**Reduce [B] (verification not implemented)**

Time = 0.20 (sec) , antiderivative size = 58, normalized size of antiderivative = 1.05

$$\int \csc^5(a + bx) dx = \frac{-3 \cos(bx + a) \sin(bx + a)^2 - 2 \cos(bx + a) + 3 \log\left(\tan\left(\frac{bx}{2} + \frac{a}{2}\right)\right) \sin(bx + a)^4}{8 \sin(bx + a)^4 b}$$

input `int(csc(b*x+a)^5,x)`output `(-3*cos(a + b*x)*sin(a + b*x)**2 - 2*cos(a + b*x) + 3*log(tan((a + b*x)/2))*sin(a + b*x)**4)/(8*sin(a + b*x)**4*b)`



### 3.6 $\int \csc^6(a + bx) dx$

Optimal result . . . . .	80
Mathematica [A] (verified) . . . . .	80
Rubi [A] (verified) . . . . .	81
Maple [A] (verified) . . . . .	82
Fricas [A] (verification not implemented) . . . . .	82
Sympy [F] . . . . .	83
Maxima [A] (verification not implemented) . . . . .	83
Giac [A] (verification not implemented) . . . . .	83
Mupad [B] (verification not implemented) . . . . .	84
Reduce [B] (verification not implemented) . . . . .	84

#### Optimal result

Integrand size = 8, antiderivative size = 42

$$\int \csc^6(a + bx) dx = -\frac{\cot(a + bx)}{b} - \frac{2 \cot^3(a + bx)}{3b} - \frac{\cot^5(a + bx)}{5b}$$

output

```
-cot(b*x+a)/b-2/3*cot(b*x+a)^3/b-1/5*cot(b*x+a)^5/b
```

#### Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 56, normalized size of antiderivative = 1.33

$$\int \csc^6(a + bx) dx = -\frac{8 \cot(a + bx)}{15b} - \frac{4 \cot(a + bx) \csc^2(a + bx)}{15b} - \frac{\cot(a + bx) \csc^4(a + bx)}{5b}$$

input

```
Integrate[Csc[a + b*x]^6,x]
```

output

```
(-8*Cot[a + b*x])/(15*b) - (4*Cot[a + b*x]*Csc[a + b*x]^2)/(15*b) - (Cot[a + b*x]*Csc[a + b*x]^4)/(5*b)
```

**Rubi [A] (verified)**

Time = 0.20 (sec) , antiderivative size = 36, normalized size of antiderivative = 0.86, number of steps used = 4, number of rules used = 3,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.375$ , Rules used = {3042, 4254, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \csc^6(a + bx) dx \\ & \quad \downarrow \text{3042} \\ & \int \csc(a + bx)^6 dx \\ & \quad \downarrow \text{4254} \\ & \frac{\int (\cot^4(a + bx) + 2 \cot^2(a + bx) + 1) d \cot(a + bx)}{b} \\ & \quad \downarrow \text{2009} \\ & \frac{\frac{1}{5} \cot^5(a + bx) + \frac{2}{3} \cot^3(a + bx) + \cot(a + bx)}{b} \end{aligned}$$

input `Int[Csc[a + b*x]^6,x]`

output `-((Cot[a + b*x] + (2*Cot[a + b*x]^3)/3 + Cot[a + b*x]^5/5)/b)`

**Defintions of rubi rules used**

rule 2009 `Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 4254

```
Int[csc[(c_.) + (d_.)*(x_)]^(n_), x_Symbol] := Simp[-d^(-1) Subst[Int[Exp
andIntegrand[(1 + x^2)^(n/2 - 1), x], x], x, Cot[c + d*x]], x] /; FreeQ[{c,
d}, x] && IGtQ[n/2, 0]
```

**Maple [A] (verified)**

Time = 0.18 (sec) , antiderivative size = 33, normalized size of antiderivative = 0.79

method	result	size
derivativedivides	$\frac{\left(-\frac{8}{15} - \frac{\csc(bx+a)^4}{5} - \frac{4 \csc(bx+a)^2}{15}\right) \cot(bx+a)}{b}$	33
default	$\frac{\left(-\frac{8}{15} - \frac{\csc(bx+a)^4}{5} - \frac{4 \csc(bx+a)^2}{15}\right) \cot(bx+a)}{b}$	33
risch	$-\frac{16i(10e^{4i(bx+a)} - 5e^{2i(bx+a)} + 1)}{15b(e^{2i(bx+a)} - 1)^5}$	44
parallelrisch	$\frac{3 \tan\left(\frac{bx}{2} + \frac{a}{2}\right)^5 - 3 \cot\left(\frac{bx}{2} + \frac{a}{2}\right)^5 + 25 \tan\left(\frac{bx}{2} + \frac{a}{2}\right)^3 - 25 \cot\left(\frac{bx}{2} + \frac{a}{2}\right)^3 + 150 \tan\left(\frac{bx}{2} + \frac{a}{2}\right) - 150 \cot\left(\frac{bx}{2} + \frac{a}{2}\right)}{480b}$	81
norman	$-\frac{\frac{1}{160b} - \frac{5 \tan\left(\frac{bx}{2} + \frac{a}{2}\right)^2}{96b} - \frac{5 \tan\left(\frac{bx}{2} + \frac{a}{2}\right)^4}{16b} + \frac{5 \tan\left(\frac{bx}{2} + \frac{a}{2}\right)^6}{16b} + \frac{5 \tan\left(\frac{bx}{2} + \frac{a}{2}\right)^8}{96b} + \frac{\tan\left(\frac{bx}{2} + \frac{a}{2}\right)^{10}}{160b}}{\tan\left(\frac{bx}{2} + \frac{a}{2}\right)^5}$	99

input

```
int(csc(b*x+a)^6,x,method=_RETURNVERBOSE)
```

output

```
1/b*(-8/15-1/5*csc(b*x+a)^4-4/15*csc(b*x+a)^2)*cot(b*x+a)
```

**Fricas [A] (verification not implemented)**

Time = 0.07 (sec) , antiderivative size = 64, normalized size of antiderivative = 1.52

$$\int \csc^6(a + bx) dx = -\frac{8 \cos(bx + a)^5 - 20 \cos(bx + a)^3 + 15 \cos(bx + a)}{15 (b \cos(bx + a)^4 - 2b \cos(bx + a)^2 + b) \sin(bx + a)}$$

input

```
integrate(csc(b*x+a)^6,x, algorithm="fricas")
```

output 
$$-1/15*(8*\cos(b*x + a)^5 - 20*\cos(b*x + a)^3 + 15*\cos(b*x + a))/((b*\cos(b*x + a)^4 - 2*b*\cos(b*x + a)^2 + b)*\sin(b*x + a))$$

### Sympy [F]

$$\int \csc^6(a + bx) dx = \int \csc^6(a + bx) dx$$

input `integrate(csc(b*x+a)**6,x)`

output `Integral(csc(a + b*x)**6, x)`

### Maxima [A] (verification not implemented)

Time = 0.02 (sec) , antiderivative size = 35, normalized size of antiderivative = 0.83

$$\int \csc^6(a + bx) dx = -\frac{15 \tan^4(bx + a) + 10 \tan^2(bx + a) + 3}{15 b \tan^5(bx + a)}$$

input `integrate(csc(b*x+a)^6,x, algorithm="maxima")`

output 
$$-1/15*(15*\tan(b*x + a)^4 + 10*\tan(b*x + a)^2 + 3)/(b*\tan(b*x + a)^5)$$

### Giac [A] (verification not implemented)

Time = 0.14 (sec) , antiderivative size = 35, normalized size of antiderivative = 0.83

$$\int \csc^6(a + bx) dx = -\frac{15 \tan^4(bx + a) + 10 \tan^2(bx + a) + 3}{15 b \tan^5(bx + a)}$$

input `integrate(csc(b*x+a)^6,x, algorithm="giac")`

output  $-1/15*(15*\tan(b*x + a)^4 + 10*\tan(b*x + a)^2 + 3)/(b*\tan(b*x + a)^5)$

### Mupad [B] (verification not implemented)

Time = 15.93 (sec) , antiderivative size = 38, normalized size of antiderivative = 0.90

$$\int \csc^6(a + bx) dx = -\frac{\cot(a + bx)}{b} - \frac{2 \cot(a + bx)^3}{3b} - \frac{\cot(a + bx)^5}{5b}$$

input  $\text{int}(1/\sin(a + b*x)^6, x)$

output  $-\cot(a + b*x)/b - (2*\cot(a + b*x)^3)/(3*b) - \cot(a + b*x)^5/(5*b)$

### Reduce [B] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 41, normalized size of antiderivative = 0.98

$$\int \csc^6(a + bx) dx = \frac{\cos(bx + a) (-8 \sin(bx + a)^4 - 4 \sin(bx + a)^2 - 3)}{15 \sin(bx + a)^5 b}$$

input  $\text{int}(\csc(b*x+a)^6, x)$

output  $(\cos(a + b*x)*(-8*\sin(a + b*x)**4 - 4*\sin(a + b*x)**2 - 3))/(15*\sin(a + b*x)**5*b)$

### 3.7 $\int \csc^7(a + bx) dx$

Optimal result . . . . .	85
Mathematica [A] (verified) . . . . .	85
Rubi [A] (verified) . . . . .	86
Maple [A] (verified) . . . . .	88
Fricas [B] (verification not implemented) . . . . .	88
Sympy [F] . . . . .	89
Maxima [A] (verification not implemented) . . . . .	89
Giac [B] (verification not implemented) . . . . .	90
Mupad [B] (verification not implemented) . . . . .	90
Reduce [B] (verification not implemented) . . . . .	91

#### Optimal result

Integrand size = 8, antiderivative size = 76

$$\int \csc^7(a + bx) dx = -\frac{5\operatorname{arctanh}(\cos(a + bx))}{16b} - \frac{5 \cot(a + bx) \csc(a + bx)}{16b} - \frac{5 \cot(a + bx) \csc^3(a + bx)}{24b} - \frac{\cot(a + bx) \csc^5(a + bx)}{6b}$$

output `-5/16*arctanh(cos(b*x+a))/b-5/16*cot(b*x+a)*csc(b*x+a)/b-5/24*cot(b*x+a)*sc(b*x+a)^3/b-1/6*cot(b*x+a)*csc(b*x+a)^5/b`

#### Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 151, normalized size of antiderivative = 1.99

$$\int \csc^7(a + bx) dx = -\frac{5 \csc^2\left(\frac{1}{2}(a + bx)\right)}{64b} - \frac{\csc^4\left(\frac{1}{2}(a + bx)\right)}{64b} - \frac{\csc^6\left(\frac{1}{2}(a + bx)\right)}{384b} - \frac{5 \log\left(\cos\left(\frac{1}{2}(a + bx)\right)\right)}{16b} + \frac{5 \log\left(\sin\left(\frac{1}{2}(a + bx)\right)\right)}{16b} + \frac{5 \sec^2\left(\frac{1}{2}(a + bx)\right)}{64b} + \frac{\sec^4\left(\frac{1}{2}(a + bx)\right)}{64b} + \frac{\sec^6\left(\frac{1}{2}(a + bx)\right)}{384b}$$

input `Integrate[Csc[a + b*x]^7,x]`

output

```
(-5*Csc[(a + b*x)/2]^2)/(64*b) - Csc[(a + b*x)/2]^4/(64*b) - Csc[(a + b*x)/2]^6/(384*b) - (5*Log[Cos[(a + b*x)/2]])/(16*b) + (5*Log[Sin[(a + b*x)/2]])/(16*b) + (5*Sec[(a + b*x)/2]^2)/(64*b) + Sec[(a + b*x)/2]^4/(64*b) + Sec[(a + b*x)/2]^6/(384*b)
```

**Rubi [A] (verified)**

Time = 0.41 (sec) , antiderivative size = 86, normalized size of antiderivative = 1.13, number of steps used = 8, number of rules used = 8,  $\frac{\text{number of rules}}{\text{integrand size}} = 1.000$ , Rules used = {3042, 4255, 3042, 4255, 3042, 4255, 3042, 4257}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \csc^7(a + bx) dx \\
 & \quad \downarrow \text{3042} \\
 & \int \csc(a + bx)^7 dx \\
 & \quad \downarrow \text{4255} \\
 & \frac{5}{6} \int \csc^5(a + bx) dx - \frac{\cot(a + bx) \csc^5(a + bx)}{6b} \\
 & \quad \downarrow \text{3042} \\
 & \frac{5}{6} \int \csc(a + bx)^5 dx - \frac{\cot(a + bx) \csc^5(a + bx)}{6b} \\
 & \quad \downarrow \text{4255} \\
 & \frac{5}{6} \left( \frac{3}{4} \int \csc^3(a + bx) dx - \frac{\cot(a + bx) \csc^3(a + bx)}{4b} \right) - \frac{\cot(a + bx) \csc^5(a + bx)}{6b} \\
 & \quad \downarrow \text{3042} \\
 & \frac{5}{6} \left( \frac{3}{4} \int \csc(a + bx)^3 dx - \frac{\cot(a + bx) \csc^3(a + bx)}{4b} \right) - \frac{\cot(a + bx) \csc^5(a + bx)}{6b} \\
 & \quad \downarrow \text{4255}
 \end{aligned}$$

$$\frac{5}{6} \left( \frac{3}{4} \left( \frac{1}{2} \int \csc(a+bx) dx - \frac{\cot(a+bx) \csc(a+bx)}{2b} \right) - \frac{\cot(a+bx) \csc^3(a+bx)}{4b} \right) - \frac{\cot(a+bx) \csc^5(a+bx)}{6b}$$

↓ 3042

$$\frac{5}{6} \left( \frac{3}{4} \left( \frac{1}{2} \int \csc(a+bx) dx - \frac{\cot(a+bx) \csc(a+bx)}{2b} \right) - \frac{\cot(a+bx) \csc^3(a+bx)}{4b} \right) - \frac{\cot(a+bx) \csc^5(a+bx)}{6b}$$

↓ 4257

$$\frac{5}{6} \left( \frac{3}{4} \left( -\frac{\operatorname{arctanh}(\cos(a+bx))}{2b} - \frac{\cot(a+bx) \csc(a+bx)}{2b} \right) - \frac{\cot(a+bx) \csc^3(a+bx)}{4b} \right) - \frac{\cot(a+bx) \csc^5(a+bx)}{6b}$$

input `Int[Csc[a + b*x]^7,x]`

output `-1/6*(Cot[a + b*x]*Csc[a + b*x]^5)/b + (5*(-1/4*(Cot[a + b*x]*Csc[a + b*x]^3)/b + (3*(-1/2*ArcTanh[Cos[a + b*x]]/b - (Cot[a + b*x]*Csc[a + b*x])/(2*b))))/4)/6`

### Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 4255 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^n], x_Symbol] := Simp[(-b)*Cos[c + d*x]*((b*Csc[c + d*x])^(n-1)/(d*(n-1))), x] + Simp[b^2*((n-2)/(n-1)) Int[(b*Csc[c + d*x])^(n-2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[2*n]`

rule 4257 `Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] := Simp[-ArcTanh[Cos[c + d*x]]/d, x] /; FreeQ[{c, d}, x]`



### Maple [A] (verified)

Time = 0.23 (sec) , antiderivative size = 60, normalized size of antiderivative = 0.79

method	result
derivativedivides	$\frac{\left(-\frac{\csc(bx+a)^5}{6}-\frac{5 \csc(bx+a)^3}{24}-\frac{5 \csc(bx+a)}{16}\right) \cot(bx+a)+\frac{5 \ln(\csc(bx+a)-\cot(bx+a))}{16}}{b}$
default	$\frac{\left(-\frac{\csc(bx+a)^5}{6}-\frac{5 \csc(bx+a)^3}{24}-\frac{5 \csc(bx+a)}{16}\right) \cot(bx+a)+\frac{5 \ln(\csc(bx+a)-\cot(bx+a))}{16}}{b}$
parallelrisc	$\frac{\tan\left(\frac{bx}{2}+\frac{a}{2}\right)^6-\cot\left(\frac{bx}{2}+\frac{a}{2}\right)^6+9 \tan\left(\frac{bx}{2}+\frac{a}{2}\right)^4-9 \cot\left(\frac{bx}{2}+\frac{a}{2}\right)^4+45 \tan\left(\frac{bx}{2}+\frac{a}{2}\right)^2-45 \cot\left(\frac{bx}{2}+\frac{a}{2}\right)^2+120 \ln\left(\tan\left(\frac{bx}{2}+\frac{a}{2}\right)\right)}{384b}$
norman	$-\frac{1}{384b}-\frac{3 \tan\left(\frac{bx}{2}+\frac{a}{2}\right)^2}{128b}-\frac{15 \tan\left(\frac{bx}{2}+\frac{a}{2}\right)^4}{128b}+\frac{15 \tan\left(\frac{bx}{2}+\frac{a}{2}\right)^8}{128b}+\frac{3 \tan\left(\frac{bx}{2}+\frac{a}{2}\right)^{10}}{128b}+\frac{\tan\left(\frac{bx}{2}+\frac{a}{2}\right)^{12}}{384b}+\frac{5 \ln\left(\tan\left(\frac{bx}{2}+\frac{a}{2}\right)\right)}{16b}$
risc	$\frac{15 e^{11i(bx+a)}-85 e^{9i(bx+a)}+198 e^{7i(bx+a)}+198 e^{5i(bx+a)}-85 e^{3i(bx+a)}+15 e^{i(bx+a)}}{24b\left(e^{2i(bx+a)}-1\right)^6}-\frac{5 \ln\left(e^{i(bx+a)}+1\right)}{16b}+\frac{5 \ln\left(e^{i(bx+a)}-1\right)}{16b}$

input `int(csc(b*x+a)^7,x,method=_RETURNVERBOSE)`

output `1/b*((-1/6*csc(b*x+a)^5-5/24*csc(b*x+a)^3-5/16*csc(b*x+a))*cot(b*x+a)+5/16*ln(csc(b*x+a)-cot(b*x+a)))`

### Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 155 vs. 2(68) = 136.

Time = 0.08 (sec) , antiderivative size = 155, normalized size of antiderivative = 2.04

$$\int \csc^7(a+bx) dx = \frac{30 \cos(bx+a)^5 - 80 \cos(bx+a)^3 - 15(\cos(bx+a)^6 - 3 \cos(bx+a)^4 + 3 \cos(bx+a)^2 - 1) \log\left(\frac{1}{2} \csc(bx+a) + \frac{1}{2} \cot(bx+a)\right)}{96(b \cos(bx+a))^6 - 3}$$

input `integrate(csc(b*x+a)^7,x, algorithm="fricas")`

output

```
1/96*(30*cos(b*x + a)^5 - 80*cos(b*x + a)^3 - 15*(cos(b*x + a)^6 - 3*cos(b
*x + a)^4 + 3*cos(b*x + a)^2 - 1)*log(1/2*cos(b*x + a) + 1/2) + 15*(cos(b*
x + a)^6 - 3*cos(b*x + a)^4 + 3*cos(b*x + a)^2 - 1)*log(-1/2*cos(b*x + a)
+ 1/2) + 66*cos(b*x + a))/(b*cos(b*x + a)^6 - 3*b*cos(b*x + a)^4 + 3*b*cos
(b*x + a)^2 - b)
```

**Sympy [F]**

$$\int \csc^7(a + bx) dx = \int \csc^7(a + bx) dx$$

input

```
integrate(csc(b*x+a)**7,x)
```

output

```
Integral(csc(a + b*x)**7, x)
```

**Maxima [A] (verification not implemented)**

Time = 0.03 (sec) , antiderivative size = 91, normalized size of antiderivative = 1.20

$$\int \csc^7(a + bx) dx$$

$$= \frac{2 \left( 15 \cos(bx+a)^5 - 40 \cos(bx+a)^3 + 33 \cos(bx+a) \right)}{\cos(bx+a)^6 - 3 \cos(bx+a)^4 + 3 \cos(bx+a)^2 - 1} - \frac{15 \log(\cos(bx+a) + 1) + 15 \log(\cos(bx+a) - 1)}{96b}$$

input

```
integrate(csc(b*x+a)^7,x, algorithm="maxima")
```

output

```
1/96*(2*(15*cos(b*x + a)^5 - 40*cos(b*x + a)^3 + 33*cos(b*x + a))/(cos(b*x
+ a)^6 - 3*cos(b*x + a)^4 + 3*cos(b*x + a)^2 - 1) - 15*log(cos(b*x + a) +
1) + 15*log(cos(b*x + a) - 1))/b
```

**Giac [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 182 vs.  $2(68) = 136$ .

Time = 0.15 (sec) , antiderivative size = 182, normalized size of antiderivative = 2.39

$$\int \csc^7(a + bx) dx = \frac{\left( \frac{9(\cos(bx+a)-1)}{\cos(bx+a)+1} - \frac{45(\cos(bx+a)-1)^2}{(\cos(bx+a)+1)^2} + \frac{110(\cos(bx+a)-1)^3}{(\cos(bx+a)+1)^3} - 1 \right) (\cos(bx+a)+1)^3}{(\cos(bx+a)-1)^3} + \frac{45(\cos(bx+a)-1)}{\cos(bx+a)+1} - \frac{9(\cos(bx+a)-1)^2}{(\cos(bx+a)+1)^2} + \frac{(\cos(bx+a)-1)}{\cos(bx+a)+1}$$

$384b$

input `integrate(csc(b*x+a)^7,x, algorithm="giac")`

output 
$$\frac{-1/384 * ((9 * (\cos(b*x + a) - 1) / (\cos(b*x + a) + 1) - 45 * (\cos(b*x + a) - 1)^2 / (\cos(b*x + a) + 1)^2 + 110 * (\cos(b*x + a) - 1)^3 / (\cos(b*x + a) + 1)^3 - 1) * (\cos(b*x + a) + 1)^3 / (\cos(b*x + a) - 1)^3 + 45 * (\cos(b*x + a) - 1) / (\cos(b*x + a) + 1) - 9 * (\cos(b*x + a) - 1)^2 / (\cos(b*x + a) + 1)^2 + (\cos(b*x + a) - 1) / (\cos(b*x + a) + 1) - 60 * \log(\text{abs}(-\cos(b*x + a) + 1) / \text{abs}(\cos(b*x + a) + 1))) / b}{384b}$$

**Mupad [B] (verification not implemented)**

Time = 0.14 (sec) , antiderivative size = 78, normalized size of antiderivative = 1.03

$$\int \csc^7(a + bx) dx = \frac{\frac{5 \cos(a+bx)^5}{16} - \frac{5 \cos(a+bx)^3}{6} + \frac{11 \cos(a+bx)}{16}}{b (\cos(a + bx)^6 - 3 \cos(a + bx)^4 + 3 \cos(a + bx)^2 - 1)} - \frac{5 \operatorname{atanh}(\cos(a + bx))}{16b}$$

input `int(1/sin(a + b*x)^7,x)`

output 
$$\frac{((11 * \cos(a + b*x)) / 16 - (5 * \cos(a + b*x)^3) / 6 + (5 * \cos(a + b*x)^5) / 16) / (b * (3 * \cos(a + b*x)^2 - 3 * \cos(a + b*x)^4 + \cos(a + b*x)^6 - 1)) - (5 * \operatorname{atanh}(\cos(a + b*x))) / (16 * b)}{16b}$$

**Reduce [B] (verification not implemented)**

Time = 0.21 (sec) , antiderivative size = 74, normalized size of antiderivative = 0.97

$$\int \csc^7(a + bx) dx$$

$$= \frac{-15 \cos(bx + a) \sin(bx + a)^4 - 10 \cos(bx + a) \sin(bx + a)^2 - 8 \cos(bx + a) + 15 \log(\tan(\frac{bx}{2} + \frac{a}{2})) \sin(bx + a)^6}{48 \sin(bx + a)^6 b}$$

input `int(csc(b*x+a)^7,x)`output `( - 15*cos(a + b*x)*sin(a + b*x)**4 - 10*cos(a + b*x)*sin(a + b*x)**2 - 8*cos(a + b*x) + 15*log(tan((a + b*x)/2))*sin(a + b*x)**6)/(48*sin(a + b*x)**6*b)`

### 3.8 $\int \csc^8(a + bx) dx$

Optimal result	92
Mathematica [A] (verified)	92
Rubi [A] (verified)	93
Maple [A] (verified)	94
Fricas [A] (verification not implemented)	94
Sympy [F]	95
Maxima [A] (verification not implemented)	95
Giac [A] (verification not implemented)	96
Mupad [B] (verification not implemented)	96
Reduce [B] (verification not implemented)	96

#### Optimal result

Integrand size = 8, antiderivative size = 55

$$\int \csc^8(a + bx) dx = -\frac{\cot(a + bx)}{b} - \frac{\cot^3(a + bx)}{b} - \frac{3 \cot^5(a + bx)}{5b} - \frac{\cot^7(a + bx)}{7b}$$

output

```
-cot(b*x+a)/b-cot(b*x+a)^3/b-3/5*cot(b*x+a)^5/b-1/7*cot(b*x+a)^7/b
```

#### Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 77, normalized size of antiderivative = 1.40

$$\int \csc^8(a + bx) dx = -\frac{16 \cot(a + bx)}{35b} - \frac{8 \cot(a + bx) \csc^2(a + bx)}{35b} - \frac{6 \cot(a + bx) \csc^4(a + bx)}{35b} - \frac{\cot(a + bx) \csc^6(a + bx)}{7b}$$

input

```
Integrate[Csc[a + b*x]^8,x]
```

output

```
(-16*Cot[a + b*x])/(35*b) - (8*Cot[a + b*x]*Csc[a + b*x]^2)/(35*b) - (6*Cot[a + b*x]*Csc[a + b*x]^4)/(35*b) - (Cot[a + b*x]*Csc[a + b*x]^6)/(7*b)
```

**Rubi [A] (verified)**

Time = 0.20 (sec) , antiderivative size = 44, normalized size of antiderivative = 0.80, number of steps used = 4, number of rules used = 3,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.375$ , Rules used = {3042, 4254, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{array}{c}
 \int \csc^8(a + bx) dx \\
 \downarrow 3042 \\
 \int \csc(a + bx)^8 dx \\
 \downarrow 4254 \\
 - \frac{\int (\cot^6(a + bx) + 3 \cot^4(a + bx) + 3 \cot^2(a + bx) + 1) d \cot(a + bx)}{b} \\
 \downarrow 2009 \\
 - \frac{\frac{1}{7} \cot^7(a + bx) + \frac{3}{5} \cot^5(a + bx) + \cot^3(a + bx) + \cot(a + bx)}{b}
 \end{array}$$

input `Int[Csc[a + b*x]^8,x]`

output `-((Cot[a + b*x] + Cot[a + b*x]^3 + (3*Cot[a + b*x]^5)/5 + Cot[a + b*x]^7/7)/b)`

**Defintions of rubi rules used**

rule 2009 `Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 4254

```
Int[csc[(c_.) + (d_.)*(x_)]^(n_), x_Symbol] := Simp[-d^(-1) Subst[Int[Exp
andIntegrand[(1 + x^2)^(n/2 - 1), x], x], x, Cot[c + d*x]], x] /; FreeQ[{c,
d}, x] && IGtQ[n/2, 0]
```

**Maple [A] (verified)**

Time = 0.23 (sec) , antiderivative size = 43, normalized size of antiderivative = 0.78

method	result
derivativdivides	$\frac{\left(-\frac{16}{35} - \frac{\csc(bx+a)^6}{7} - \frac{6 \csc(bx+a)^4}{35} - \frac{8 \csc(bx+a)^2}{35}\right) \cot(bx+a)}{b}$
default	$\frac{\left(-\frac{16}{35} - \frac{\csc(bx+a)^6}{7} - \frac{6 \csc(bx+a)^4}{35} - \frac{8 \csc(bx+a)^2}{35}\right) \cot(bx+a)}{b}$
risch	$\frac{32i(35 e^{6i(bx+a)} - 21 e^{4i(bx+a)} + 7 e^{2i(bx+a)} - 1)}{35b(e^{2i(bx+a)} - 1)^7}$
parallelrisch	$\frac{5 \tan\left(\frac{bx}{2} + \frac{a}{2}\right)^7 - 5 \cot\left(\frac{bx}{2} + \frac{a}{2}\right)^7 + 49 \tan\left(\frac{bx}{2} + \frac{a}{2}\right)^5 - 49 \cot\left(\frac{bx}{2} + \frac{a}{2}\right)^5 + 245 \tan\left(\frac{bx}{2} + \frac{a}{2}\right)^3 - 245 \cot\left(\frac{bx}{2} + \frac{a}{2}\right)^3 + 1225 \tan\left(\frac{bx}{2} + \frac{a}{2}\right)}{4480b}$
norman	$\frac{-\frac{1}{896b} - \frac{7 \tan\left(\frac{bx}{2} + \frac{a}{2}\right)^2}{640b} - \frac{7 \tan\left(\frac{bx}{2} + \frac{a}{2}\right)^4}{128b} - \frac{35 \tan\left(\frac{bx}{2} + \frac{a}{2}\right)^6}{128b} + \frac{35 \tan\left(\frac{bx}{2} + \frac{a}{2}\right)^8}{128b} + \frac{7 \tan\left(\frac{bx}{2} + \frac{a}{2}\right)^{10}}{128b} + \frac{7 \tan\left(\frac{bx}{2} + \frac{a}{2}\right)^{12}}{640b} + \frac{\tan\left(\frac{bx}{2} + \frac{a}{2}\right)}{896b}}{\tan\left(\frac{bx}{2} + \frac{a}{2}\right)^7}$

input

```
int(csc(b*x+a)^8,x,method=_RETURNVERBOSE)
```

output

```
1/b*(-16/35-1/7*csc(b*x+a)^6-6/35*csc(b*x+a)^4-8/35*csc(b*x+a)^2)*cot(b*x+a)
```

**Fricas [A] (verification not implemented)**

Time = 0.07 (sec) , antiderivative size = 87, normalized size of antiderivative = 1.58

$$\int \csc^8(a + bx) dx$$

$$= -\frac{16 \cos(bx + a)^7 - 56 \cos(bx + a)^5 + 70 \cos(bx + a)^3 - 35 \cos(bx + a)}{35 (b \cos(bx + a)^6 - 3b \cos(bx + a)^4 + 3b \cos(bx + a)^2 - b) \sin(bx + a)}$$

input

```
integrate(csc(b*x+a)^8,x, algorithm="fricas")
```

output

```
-1/35*(16*cos(b*x + a)^7 - 56*cos(b*x + a)^5 + 70*cos(b*x + a)^3 - 35*cos(
b*x + a))/((b*cos(b*x + a)^6 - 3*b*cos(b*x + a)^4 + 3*b*cos(b*x + a)^2 - b
)*sin(b*x + a))
```

**Sympy [F]**

$$\int \csc^8(a + bx) dx = \int \csc^8(a + bx) dx$$

input

```
integrate(csc(b*x+a)**8,x)
```

output

```
Integral(csc(a + b*x)**8, x)
```

**Maxima [A] (verification not implemented)**

Time = 0.03 (sec) , antiderivative size = 45, normalized size of antiderivative = 0.82

$$\int \csc^8(a + bx) dx = -\frac{35 \tan^6(bx + a) + 35 \tan^4(bx + a) + 21 \tan^2(bx + a) + 5}{35 b \tan^7(bx + a)}$$

input

```
integrate(csc(b*x+a)^8,x, algorithm="maxima")
```

output

```
-1/35*(35*tan(b*x + a)^6 + 35*tan(b*x + a)^4 + 21*tan(b*x + a)^2 + 5)/(b*t
an(b*x + a)^7)
```



**Giac [A] (verification not implemented)**

Time = 0.15 (sec) , antiderivative size = 45, normalized size of antiderivative = 0.82

$$\int \csc^8(a + bx) dx = -\frac{35 \tan(bx + a)^6 + 35 \tan(bx + a)^4 + 21 \tan(bx + a)^2 + 5}{35 b \tan(bx + a)^7}$$

input `integrate(csc(b*x+a)^8,x, algorithm="giac")`output `-1/35*(35*tan(b*x + a)^6 + 35*tan(b*x + a)^4 + 21*tan(b*x + a)^2 + 5)/(b*tan(b*x + a)^7)`**Mupad [B] (verification not implemented)**

Time = 16.48 (sec) , antiderivative size = 41, normalized size of antiderivative = 0.75

$$\int \csc^8(a + bx) dx = -\frac{\tan(a + bx)^6 + \tan(a + bx)^4 + \frac{3 \tan(a + bx)^2}{5} + \frac{1}{7}}{b \tan(a + bx)^7}$$

input `int(1/sin(a + b*x)^8,x)`output `-((3*tan(a + b*x)^2)/5 + tan(a + b*x)^4 + tan(a + b*x)^6 + 1/7)/(b*tan(a + b*x)^7)`**Reduce [B] (verification not implemented)**

Time = 0.19 (sec) , antiderivative size = 51, normalized size of antiderivative = 0.93

$$\begin{aligned} & \int \csc^8(a + bx) dx \\ &= \frac{\cos(bx + a) (-16 \sin(bx + a)^6 - 8 \sin(bx + a)^4 - 6 \sin(bx + a)^2 - 5)}{35 \sin(bx + a)^7 b} \end{aligned}$$

input `int(csc(b*x+a)^8,x)`

output  $(\cos(a + b*x) * (-16*\sin(a + b*x)**6 - 8*\sin(a + b*x)**4 - 6*\sin(a + b*x)*$   
 $*2 - 5)) / (35*\sin(a + b*x)**7*b)$

### 3.9 $\int \csc^{\frac{7}{2}}(a + bx) dx$

Optimal result . . . . .	98
Mathematica [A] (verified) . . . . .	98
Rubi [A] (verified) . . . . .	99
Maple [B] (verified) . . . . .	101
Fricas [C] (verification not implemented) . . . . .	101
Sympy [F(-1)] . . . . .	102
Maxima [F] . . . . .	102
Giac [F] . . . . .	103
Mupad [F(-1)] . . . . .	103
Reduce [F] . . . . .	103

#### Optimal result

Integrand size = 10, antiderivative size = 90

$$\int \csc^{\frac{7}{2}}(a + bx) dx = -\frac{6 \cos(a + bx) \sqrt{\csc(a + bx)}}{5b} - \frac{2 \cos(a + bx) \csc^{\frac{5}{2}}(a + bx)}{5b} - \frac{6 \sqrt{\csc(a + bx)} E\left(\frac{1}{2}(a - \frac{\pi}{2} + bx) \mid 2\right) \sqrt{\sin(a + bx)}}{5b}$$

output

```
-6/5*cos(b*x+a)*csc(b*x+a)^(1/2)/b-2/5*cos(b*x+a)*csc(b*x+a)^(5/2)/b+6/5*csc(b*x+a)^(1/2)*EllipticE(cos(1/2*a+1/4*Pi+1/2*b*x),2^(1/2))*sin(b*x+a)^(1/2)/b
```

#### Mathematica [A] (verified)

Time = 0.15 (sec) , antiderivative size = 63, normalized size of antiderivative = 0.70

$$\int \csc^{\frac{7}{2}}(a + bx) dx = \frac{\csc^{\frac{5}{2}}(a + bx) \left( -7 \cos(a + bx) + 3 \cos(3(a + bx)) \right) + 12 E\left(\frac{1}{4}(-2a + \pi - 2bx) \mid 2\right) \sin^{\frac{5}{2}}(a + bx)}{10b}$$

input

```
Integrate[Csc[a + b*x]^(7/2),x]
```

output

```
(Csc[a + b*x]^(5/2)*(-7*Cos[a + b*x] + 3*Cos[3*(a + b*x)] + 12*EllipticE[(-2*a + Pi - 2*b*x)/4, 2]*Sin[a + b*x]^(5/2)))/(10*b)
```

**Rubi [A] (verified)**

Time = 0.41 (sec) , antiderivative size = 91, normalized size of antiderivative = 1.01, number of steps used = 8, number of rules used = 8,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.800$ , Rules used = {3042, 4255, 3042, 4255, 3042, 4258, 3042, 3119}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \csc^{7/2}(a + bx) dx \\
 & \quad \downarrow \text{3042} \\
 & \int \csc(a + bx)^{7/2} dx \\
 & \quad \downarrow \text{4255} \\
 & \frac{3}{5} \int \csc^{3/2}(a + bx) dx - \frac{2 \cos(a + bx) \csc^{5/2}(a + bx)}{5b} \\
 & \quad \downarrow \text{3042} \\
 & \frac{3}{5} \int \csc(a + bx)^{3/2} dx - \frac{2 \cos(a + bx) \csc^{5/2}(a + bx)}{5b} \\
 & \quad \downarrow \text{4255} \\
 & \frac{3}{5} \left( - \int \frac{1}{\sqrt{\csc(a + bx)}} dx - \frac{2 \cos(a + bx) \sqrt{\csc(a + bx)}}{b} \right) - \frac{2 \cos(a + bx) \csc^{5/2}(a + bx)}{5b} \\
 & \quad \downarrow \text{3042} \\
 & \frac{3}{5} \left( - \int \frac{1}{\sqrt{\csc(a + bx)}} dx - \frac{2 \cos(a + bx) \sqrt{\csc(a + bx)}}{b} \right) - \frac{2 \cos(a + bx) \csc^{5/2}(a + bx)}{5b} \\
 & \quad \downarrow \text{4258}
 \end{aligned}$$

$$\frac{3}{5} \left( -\sqrt{\sin(a+bx)} \sqrt{\csc(a+bx)} \int \sqrt{\sin(a+bx)} dx - \frac{2 \cos(a+bx) \sqrt{\csc(a+bx)}}{b} \right) - \frac{2 \cos(a+bx) \csc^{\frac{5}{2}}(a+bx)}{5b}$$

↓ 3042

$$\frac{3}{5} \left( -\sqrt{\sin(a+bx)} \sqrt{\csc(a+bx)} \int \sqrt{\sin(a+bx)} dx - \frac{2 \cos(a+bx) \sqrt{\csc(a+bx)}}{b} \right) - \frac{2 \cos(a+bx) \csc^{\frac{5}{2}}(a+bx)}{5b}$$

↓ 3119

$$\frac{3}{5} \left( -\frac{2 \cos(a+bx) \sqrt{\csc(a+bx)}}{b} - \frac{2 \sqrt{\sin(a+bx)} \sqrt{\csc(a+bx)} E\left(\frac{1}{2}(a+bx - \frac{\pi}{2}) \mid 2\right)}{b} \right) - \frac{2 \cos(a+bx) \csc^{\frac{5}{2}}(a+bx)}{5b}$$

input `Int[Csc[a + b*x]^(7/2), x]`

output `(-2*Cos[a + b*x]*Csc[a + b*x]^(5/2))/(5*b) + (3*((-2*Cos[a + b*x]*Sqrt[Csc[a + b*x]])/b - (2*Sqrt[Csc[a + b*x]]*EllipticE[(a - Pi/2 + b*x)/2, 2]*Sqrt[Sin[a + b*x]]/b))/5`

### Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] :=> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3119 `Int[Sqrt[sin[(c_.) + (d_.)*(x_)]], x_Symbol] :=> Simp[(2/d)*EllipticE[(1/2)*(c - Pi/2 + d*x), 2], x] /; FreeQ[{c, d}, x]`

rule 4255 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] :=> Simp[(-b)*Cos[c + d*x]*((b*Csc[c + d*x])^(n - 1)/(d*(n - 1))), x] + Simp[b^2*((n - 2)/(n - 1)) Int[(b*Csc[c + d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[2*n]`

rule 4258

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] := Simp[(b*Csc[c + d*x]
)^(n)*Sin[c + d*x]^n Int[1/Sin[c + d*x]^n, x], x] /; FreeQ[{b, c, d}, x] &&
EqQ[n^2, 1/4]
```

### Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 159 vs.  $2(76) = 152$ .

Time = 0.17 (sec) , antiderivative size = 160, normalized size of antiderivative = 1.78

method	result
default	$\frac{6\sqrt{\sin(bx+a)+1}\sqrt{-2\sin(bx+a)+2}\sqrt{-\sin(bx+a)}\sin(bx+a)^2\text{EllipticE}\left(\sqrt{\sin(bx+a)+1},\frac{\sqrt{2}}{2}\right)-3\sqrt{\sin(bx+a)+1}\sqrt{-2\sin(bx+a)}}{5\sin(bx+a)^{\frac{5}{2}}\cos(bx+a)b}$

input

```
int(csc(b*x+a)^(7/2),x,method=_RETURNVERBOSE)
```

output

```
1/5/sin(b*x+a)^(5/2)*(6*(sin(b*x+a)+1)^(1/2)*(-2*sin(b*x+a)+2)^(1/2)*(-sin
(b*x+a))^(1/2)*sin(b*x+a)^2*EllipticE((sin(b*x+a)+1)^(1/2),1/2*2^(1/2))-3*
(sin(b*x+a)+1)^(1/2)*(-2*sin(b*x+a)+2)^(1/2)*(-sin(b*x+a))^(1/2)*sin(b*x+a
)^2*EllipticF((sin(b*x+a)+1)^(1/2),1/2*2^(1/2))+6*sin(b*x+a)^4-4*sin(b*x+a
)^2-2)/cos(b*x+a)/b
```

### Fricas [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 0.09 (sec) , antiderivative size = 120, normalized size of antiderivative = 1.33

$$\int \csc^{\frac{7}{2}}(a + bx) dx =$$

$$\frac{3\sqrt{2i}(\cos(bx+a)^2 - 1)\text{weierstrassZeta}(4, 0, \text{weierstrassPInverse}(4, 0, \cos(bx+a) + i \sin(bx+a)))}{-}$$

input

```
integrate(csc(b*x+a)^(7/2),x, algorithm="fricas")
```

output

```
-1/5*(3*sqrt(2*I)*(cos(b*x + a)^2 - 1)*weierstrassZeta(4, 0, weierstrassPI
nverse(4, 0, cos(b*x + a) + I*sin(b*x + a))) + 3*sqrt(-2*I)*(cos(b*x + a)^
2 - 1)*weierstrassZeta(4, 0, weierstrassPInverse(4, 0, cos(b*x + a) - I*si
n(b*x + a))) + 2*(3*cos(b*x + a)^3 - 4*cos(b*x + a))/sqrt(sin(b*x + a)))/(
b*cos(b*x + a)^2 - b)
```

**Sympy [F(-1)]**

Timed out.

$$\int \csc^{\frac{7}{2}}(a + bx) dx = \text{Timed out}$$

input

```
integrate(csc(b*x+a)**(7/2),x)
```

output

Timed out

**Maxima [F]**

$$\int \csc^{\frac{7}{2}}(a + bx) dx = \int \csc(bx + a)^{\frac{7}{2}} dx$$

input

```
integrate(csc(b*x+a)^(7/2),x, algorithm="maxima")
```

output

```
integrate(csc(b*x + a)^(7/2), x)
```

**Giac [F]**

$$\int \csc^{\frac{7}{2}}(a + bx) dx = \int \csc(bx + a)^{\frac{7}{2}} dx$$

input `integrate(csc(b*x+a)^(7/2),x, algorithm="giac")`

output `integrate(csc(b*x + a)^(7/2), x)`

**Mupad [F(-1)]**

Timed out.

$$\int \csc^{\frac{7}{2}}(a + bx) dx = \int \left( \frac{1}{\sin(a + bx)} \right)^{7/2} dx$$

input `int((1/sin(a + b*x))^(7/2),x)`

output `int((1/sin(a + b*x))^(7/2), x)`

**Reduce [F]**

$$\int \csc^{\frac{7}{2}}(a + bx) dx = \int \sqrt{\csc(bx + a)} \csc(bx + a)^3 dx$$

input `int(csc(b*x+a)^(7/2),x)`

output `int(sqrt(csc(a + b*x))*csc(a + b*x)**3,x)`



### 3.10 $\int \csc^{\frac{5}{2}}(a + bx) dx$

Optimal result . . . . .	104
Mathematica [A] (verified) . . . . .	104
Rubi [A] (verified) . . . . .	105
Maple [A] (verified) . . . . .	106
Fricas [C] (verification not implemented) . . . . .	107
Sympy [F] . . . . .	107
Maxima [F] . . . . .	108
Giac [F] . . . . .	108
Mupad [F(-1)] . . . . .	108
Reduce [F] . . . . .	109

#### Optimal result

Integrand size = 10, antiderivative size = 67

$$\int \csc^{\frac{5}{2}}(a + bx) dx = -\frac{2 \cos(a + bx) \csc^{\frac{3}{2}}(a + bx)}{3b} + \frac{2\sqrt{\csc(a + bx)} \operatorname{EllipticF}\left(\frac{1}{2}\left(a - \frac{\pi}{2} + bx\right), 2\right) \sqrt{\sin(a + bx)}}{3b}$$

output

```
-2/3*cos(b*x+a)*csc(b*x+a)^(3/2)/b+2/3*csc(b*x+a)^(1/2)*InverseJacobiAM(1/2*a-1/4*Pi+1/2*b*x,2^(1/2))*sin(b*x+a)^(1/2)/b
```

#### Mathematica [A] (verified)

Time = 0.07 (sec) , antiderivative size = 50, normalized size of antiderivative = 0.75

$$\int \csc^{\frac{5}{2}}(a + bx) dx = -\frac{2 \csc^{\frac{3}{2}}(a + bx) \left( \cos(a + bx) + \operatorname{EllipticF}\left(\frac{1}{4}(-2a + \pi - 2bx), 2\right) \sin^{\frac{3}{2}}(a + bx) \right)}{3b}$$

input

```
Integrate[Csc[a + b*x]^(5/2), x]
```

output

$$\frac{(-2*\text{Csc}[a + b*x]^{(3/2)}*(\text{Cos}[a + b*x] + \text{EllipticF}[(-2*a + \text{Pi} - 2*b*x)/4, 2] * \text{Sin}[a + b*x]^{(3/2)}))/(3*b)}$$
**Rubi [A] (verified)**

Time = 0.30 (sec) , antiderivative size = 67, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 6,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.600$ , Rules used = {3042, 4255, 3042, 4258, 3042, 3120}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \csc^{\frac{5}{2}}(a + bx) dx \\ & \quad \downarrow \text{3042} \\ & \int \csc(a + bx)^{5/2} dx \\ & \quad \downarrow \text{4255} \\ & \frac{1}{3} \int \sqrt{\csc(a + bx)} dx - \frac{2 \cos(a + bx) \csc^{\frac{3}{2}}(a + bx)}{3b} \\ & \quad \downarrow \text{3042} \\ & \frac{1}{3} \int \sqrt{\csc(a + bx)} dx - \frac{2 \cos(a + bx) \csc^{\frac{3}{2}}(a + bx)}{3b} \\ & \quad \downarrow \text{4258} \\ & \frac{1}{3} \sqrt{\sin(a + bx)} \sqrt{\csc(a + bx)} \int \frac{1}{\sqrt{\sin(a + bx)}} dx - \frac{2 \cos(a + bx) \csc^{\frac{3}{2}}(a + bx)}{3b} \\ & \quad \downarrow \text{3042} \\ & \frac{1}{3} \sqrt{\sin(a + bx)} \sqrt{\csc(a + bx)} \int \frac{1}{\sqrt{\sin(a + bx)}} dx - \frac{2 \cos(a + bx) \csc^{\frac{3}{2}}(a + bx)}{3b} \\ & \quad \downarrow \text{3120} \\ & \frac{2 \sqrt{\sin(a + bx)} \sqrt{\csc(a + bx)} \text{EllipticF}\left(\frac{1}{2}(a + bx - \frac{\pi}{2}), 2\right)}{3b} - \frac{2 \cos(a + bx) \csc^{\frac{3}{2}}(a + bx)}{3b} \end{aligned}$$

input `Int[Csc[a + b*x]^(5/2), x]`

output 
$$\frac{-2\cos[a + b*x]*\text{Csc}[a + b*x]^{3/2}}{3*b} + \frac{2*\sqrt{\text{Csc}[a + b*x]}*\text{EllipticF}\left[\frac{a - \pi/2 + b*x}{2}, 2\right]*\sqrt{\sin[a + b*x]}}{3*b}$$

### Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] :=> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3120 `Int[1/Sqrt[sin[(c_.) + (d_.)*(x_)]], x_Symbol] :=> Simp[(2/d)*EllipticF[(1/2)*(c - Pi/2 + d*x), 2], x] /; FreeQ[{c, d}, x]`

rule 4255 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] :=> Simp[(-b)*Cos[c + d*x]*((b*Csc[c + d*x])^(n - 1)/(d*(n - 1))), x] + Simp[b^2*((n - 2)/(n - 1)) * Int[(b*Csc[c + d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[2*n]`

rule 4258 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] :=> Simp[(b*Csc[c + d*x])^n * Sin[c + d*x]^n * Int[1/Sin[c + d*x]^n, x], x] /; FreeQ[{b, c, d}, x] && EqQ[n^2, 1/4]`

### Maple [A] (verified)

Time = 0.08 (sec) , antiderivative size = 88, normalized size of antiderivative = 1.31

method	result	size
default	$\frac{\sqrt{\sin(bx+a)+1} \sqrt{-2\sin(bx+a)+2} \sqrt{-\sin(bx+a)} \text{EllipticF}\left(\sqrt{\sin(bx+a)+1}, \frac{\sqrt{2}}{2}\right) \sin(bx+a) - 2 \cos(bx+a)^2}{3 \sin(bx+a)^{\frac{3}{2}} \cos(bx+a)b}$	88

input `int(csc(b*x+a)^(5/2), x, method=_RETURNVERBOSE)`

output

```
1/3/sin(b*x+a)^(3/2)*((sin(b*x+a)+1)^(1/2)*(-2*sin(b*x+a)+2)^(1/2)*(-sin(b
*x+a))^(1/2)*EllipticF((sin(b*x+a)+1)^(1/2),1/2*2^(1/2))*sin(b*x+a)-2*cos(
b*x+a)^2)/cos(b*x+a)/b
```

### Fricas [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 0.09 (sec) , antiderivative size = 88, normalized size of antiderivative = 1.31

$$\int \csc^{\frac{5}{2}}(a + bx) dx$$

$$= \frac{-i\sqrt{2i}\sin(bx+a)\operatorname{weierstrassPInverse}(4,0,\cos(bx+a)+i\sin(bx+a))+i\sqrt{-2i}\sin(bx+a)\operatorname{weierstrassPInverse}(4,0,\cos(bx+a)-i\sin(bx+a))}{3b\sin(bx+a)}$$

input

```
integrate(csc(b*x+a)^(5/2),x, algorithm="fricas")
```

output

```
1/3*(-I*sqrt(2*I)*sin(b*x + a)*weierstrassPInverse(4, 0, cos(b*x + a) + I*
sin(b*x + a)) + I*sqrt(-2*I)*sin(b*x + a)*weierstrassPInverse(4, 0, cos(b*
x + a) - I*sin(b*x + a)) - 2*cos(b*x + a)/sqrt(sin(b*x + a)))/(b*sin(b*x +
a))
```

### Sympy [F]

$$\int \csc^{\frac{5}{2}}(a + bx) dx = \int \csc^{\frac{5}{2}}(a + bx) dx$$

input

```
integrate(csc(b*x+a)**(5/2),x)
```

output

```
Integral(csc(a + b*x)**(5/2), x)
```

**Maxima [F]**

$$\int \csc^{\frac{5}{2}}(a + bx) dx = \int \csc (bx + a)^{\frac{5}{2}} dx$$

input `integrate(csc(b*x+a)^(5/2),x, algorithm="maxima")`

output `integrate(csc(b*x + a)^(5/2), x)`

**Giac [F]**

$$\int \csc^{\frac{5}{2}}(a + bx) dx = \int \csc (bx + a)^{\frac{5}{2}} dx$$

input `integrate(csc(b*x+a)^(5/2),x, algorithm="giac")`

output `integrate(csc(b*x + a)^(5/2), x)`

**Mupad [F(-1)]**

Timed out.

$$\int \csc^{\frac{5}{2}}(a + bx) dx = \int \left( \frac{1}{\sin(a + bx)} \right)^{\frac{5}{2}} dx$$

input `int((1/sin(a + b*x))^(5/2),x)`

output `int((1/sin(a + b*x))^(5/2), x)`

**Reduce [F]**

$$\int \csc^{\frac{5}{2}}(a + bx) dx = \int \sqrt{\csc(bx + a)} \csc(bx + a)^2 dx$$

input `int(csc(b*x+a)^(5/2),x)`

output `int(sqrt(csc(a + b*x))*csc(a + b*x)**2,x)`

### 3.11 $\int \csc^{\frac{3}{2}}(a + bx) dx$

Optimal result . . . . .	110
Mathematica [A] (verified) . . . . .	110
Rubi [A] (verified) . . . . .	111
Maple [B] (verified) . . . . .	112
Fricas [C] (verification not implemented) . . . . .	113
Sympy [F] . . . . .	113
Maxima [F] . . . . .	114
Giac [F] . . . . .	114
Mupad [F(-1)] . . . . .	114
Reduce [F] . . . . .	115

#### Optimal result

Integrand size = 10, antiderivative size = 63

$$\int \csc^{\frac{3}{2}}(a + bx) dx = -\frac{2 \cos(a + bx) \sqrt{\csc(a + bx)}}{b} - \frac{2 \sqrt{\csc(a + bx)} E\left(\frac{1}{2}\left(a - \frac{\pi}{2} + bx\right) \middle| 2\right) \sqrt{\sin(a + bx)}}{b}$$

output

```
-2*cos(b*x+a)*csc(b*x+a)^(1/2)/b+2*csc(b*x+a)^(1/2)*EllipticE(cos(1/2*a+1/4*Pi+1/2*b*x),2^(1/2))*sin(b*x+a)^(1/2)/b
```

#### Mathematica [A] (verified)

Time = 0.05 (sec) , antiderivative size = 49, normalized size of antiderivative = 0.78

$$\int \csc^{\frac{3}{2}}(a + bx) dx = -\frac{2 \sqrt{\csc(a + bx)} \left( \cos(a + bx) - E\left(\frac{1}{4}(-2a + \pi - 2bx) \middle| 2\right) \sqrt{\sin(a + bx)} \right)}{b}$$

input

```
Integrate[Csc[a + b*x]^(3/2),x]
```

output

```
(-2*sqrt[Csc[a + b*x]]*(Cos[a + b*x] - EllipticE[(-2*a + Pi - 2*b*x)/4, 2]
*sqrt[Sin[a + b*x]]))/b
```

**Rubi [A] (verified)**

Time = 0.31 (sec) , antiderivative size = 63, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 6,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.600$ , Rules used = {3042, 4255, 3042, 4258, 3042, 3119}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \csc^{\frac{3}{2}}(a + bx) dx \\
 & \quad \downarrow \text{3042} \\
 & \int \csc(a + bx)^{3/2} dx \\
 & \quad \downarrow \text{4255} \\
 & - \int \frac{1}{\sqrt{\csc(a + bx)}} dx - \frac{2 \cos(a + bx) \sqrt{\csc(a + bx)}}{b} \\
 & \quad \downarrow \text{3042} \\
 & - \int \frac{1}{\sqrt{\csc(a + bx)}} dx - \frac{2 \cos(a + bx) \sqrt{\csc(a + bx)}}{b} \\
 & \quad \downarrow \text{4258} \\
 & - \sqrt{\sin(a + bx)} \sqrt{\csc(a + bx)} \int \sqrt{\sin(a + bx)} dx - \frac{2 \cos(a + bx) \sqrt{\csc(a + bx)}}{b} \\
 & \quad \downarrow \text{3042} \\
 & - \sqrt{\sin(a + bx)} \sqrt{\csc(a + bx)} \int \sqrt{\sin(a + bx)} dx - \frac{2 \cos(a + bx) \sqrt{\csc(a + bx)}}{b} \\
 & \quad \downarrow \text{3119} \\
 & \frac{2 \cos(a + bx) \sqrt{\csc(a + bx)}}{b} - \frac{2 \sqrt{\sin(a + bx)} \sqrt{\csc(a + bx)} E\left(\frac{1}{2}(a + bx - \frac{\pi}{2}) \mid 2\right)}{b}
 \end{aligned}$$



input `Int[Csc[a + b*x]^(3/2), x]`

output 
$$\frac{(-2\cos[a + bx]\sqrt{\csc[a + bx]})}{b} - \frac{(2\sqrt{\csc[a + bx]}\operatorname{EllipticE}[(a - \pi/2 + bx)/2, 2]\sqrt{\sin[a + bx]})}{b}$$

### Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3119 `Int[Sqrt[sin[(c_.) + (d_.)*(x_)]], x_Symbol] := Simp[(2/d)*EllipticE[(1/2)*(c - Pi/2 + d*x), 2], x] /; FreeQ[{c, d}, x]`

rule 4255 `Int[(csc[(c_.) + (d_.)*(x_)])*(b_.))^(n_), x_Symbol] := Simp[(-b)*Cos[c + d*x]*((b*Csc[c + d*x])^(n - 1)/(d*(n - 1))), x] + Simp[b^2*(n - 2)/(n - 1) Int[(b*Csc[c + d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[2*n]`

rule 4258 `Int[(csc[(c_.) + (d_.)*(x_)])*(b_.))^(n_), x_Symbol] := Simp[(b*Csc[c + d*x])^n*Sin[c + d*x]^n Int[1/Sin[c + d*x]^n, x], x] /; FreeQ[{b, c, d}, x] && EqQ[n^2, 1/4]`

### Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 131 vs.  $2(57) = 114$ .

Time = 0.07 (sec) , antiderivative size = 132, normalized size of antiderivative = 2.10

method	result
default	$\frac{2\sqrt{\sin(bx+a)+1}\sqrt{-2\sin(bx+a)+2}\sqrt{-\sin(bx+a)}\operatorname{EllipticE}\left(\sqrt{\sin(bx+a)+1}, \frac{\sqrt{2}}{2}\right) - \sqrt{\sin(bx+a)+1}\sqrt{-2\sin(bx+a)+2}\sqrt{-\sin(bx+a)}}{\cos(bx+a)\sqrt{\sin(bx+a)}b}$

input `int(csc(b*x+a)^(3/2), x, method=_RETURNVERBOSE)`

output

```
(2*(sin(b*x+a)+1)^(1/2)*(-2*sin(b*x+a)+2)^(1/2)*(-sin(b*x+a))^(1/2)*EllipticE((sin(b*x+a)+1)^(1/2),1/2*2^(1/2))-sin(b*x+a)+1)^(1/2)*(-2*sin(b*x+a)+2)^(1/2)*(-sin(b*x+a))^(1/2)*EllipticF((sin(b*x+a)+1)^(1/2),1/2*2^(1/2))-2*cos(b*x+a)^2)/cos(b*x+a)/sin(b*x+a)^(1/2)/b
```

### Fricas [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 0.08 (sec) , antiderivative size = 72, normalized size of antiderivative = 1.14

$$\int \csc^{\frac{3}{2}}(a + bx) dx =$$

$$\frac{\sqrt{2i}\operatorname{weierstrassZeta}(4, 0, \operatorname{weierstrassPInverse}(4, 0, \cos(bx + a) + i \sin(bx + a))) + \sqrt{-2i}\operatorname{weierstrassZeta}(4, 0, \operatorname{weierstrassPInverse}(4, 0, \cos(bx + a) - i \sin(bx + a)))}{b}$$

input

```
integrate(csc(b*x+a)^(3/2),x, algorithm="fricas")
```

output

```
-(sqrt(2*I)*weierstrassZeta(4, 0, weierstrassPInverse(4, 0, cos(b*x + a) + I*sin(b*x + a))) + sqrt(-2*I)*weierstrassZeta(4, 0, weierstrassPInverse(4, 0, cos(b*x + a) - I*sin(b*x + a)))) + 2*cos(b*x + a)/sqrt(sin(b*x + a)))/b
```

### Sympy [F]

$$\int \csc^{\frac{3}{2}}(a + bx) dx = \int \csc^{\frac{3}{2}}(a + bx) dx$$

input

```
integrate(csc(b*x+a)**(3/2),x)
```

output

```
Integral(csc(a + b*x)**(3/2), x)
```

**Maxima [F]**

$$\int \csc^{\frac{3}{2}}(a + bx) dx = \int \csc (bx + a)^{\frac{3}{2}} dx$$

input `integrate(csc(b*x+a)^(3/2),x, algorithm="maxima")`

output `integrate(csc(b*x + a)^(3/2), x)`

**Giac [F]**

$$\int \csc^{\frac{3}{2}}(a + bx) dx = \int \csc (bx + a)^{\frac{3}{2}} dx$$

input `integrate(csc(b*x+a)^(3/2),x, algorithm="giac")`

output `integrate(csc(b*x + a)^(3/2), x)`

**Mupad [F(-1)]**

Timed out.

$$\int \csc^{\frac{3}{2}}(a + bx) dx = \int \left( \frac{1}{\sin(a + bx)} \right)^{\frac{3}{2}} dx$$

input `int((1/sin(a + b*x))^(3/2),x)`

output `int((1/sin(a + b*x))^(3/2), x)`

**Reduce [F]**

$$\int \csc^{\frac{3}{2}}(a + bx) dx = \int \sqrt{\csc(bx + a)} \csc(bx + a) dx$$

input `int(csc(b*x+a)^(3/2),x)`

output `int(sqrt(csc(a + b*x))*csc(a + b*x),x)`

### 3.12 $\int \sqrt{\csc(a + bx)} dx$

Optimal result	116
Mathematica [A] (verified)	116
Rubi [A] (verified)	117
Maple [A] (verified)	118
Fricas [C] (verification not implemented)	118
Sympy [F]	119
Maxima [F]	119
Giac [F]	120
Mupad [B] (verification not implemented)	120
Reduce [F]	120

#### Optimal result

Integrand size = 10, antiderivative size = 41

$$\int \sqrt{\csc(a + bx)} dx = \frac{2\sqrt{\csc(a + bx)} \operatorname{EllipticF}\left(\frac{1}{2}(a - \frac{\pi}{2} + bx), 2\right) \sqrt{\sin(a + bx)}}{b}$$

output

```
2*csc(b*x+a)^(1/2)*InverseJacobiAM(1/2*a-1/4*Pi+1/2*b*x,2^(1/2))*sin(b*x+a)^(1/2)/b
```

#### Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 40, normalized size of antiderivative = 0.98

$$\int \sqrt{\csc(a + bx)} dx = -\frac{2\sqrt{\csc(a + bx)} \operatorname{EllipticF}\left(\frac{1}{4}(-2a + \pi - 2bx), 2\right) \sqrt{\sin(a + bx)}}{b}$$

input

```
Integrate[Sqrt[Csc[a + b*x]],x]
```

output

```
(-2*Sqrt[Csc[a + b*x]]*EllipticF[(-2*a + Pi - 2*b*x)/4, 2]*Sqrt[Sin[a + b*x]])/b
```

**Rubi [A] (verified)**

Time = 0.23 (sec) , antiderivative size = 41, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.400$ , Rules used = {3042, 4258, 3042, 3120}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \sqrt{\csc(a + bx)} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \sqrt{\csc(a + bx)} dx \\
 & \quad \downarrow \text{4258} \\
 & \sqrt{\sin(a + bx)} \sqrt{\csc(a + bx)} \int \frac{1}{\sqrt{\sin(a + bx)}} dx \\
 & \quad \downarrow \text{3042} \\
 & \sqrt{\sin(a + bx)} \sqrt{\csc(a + bx)} \int \frac{1}{\sqrt{\sin(a + bx)}} dx \\
 & \quad \downarrow \text{3120} \\
 & \frac{2\sqrt{\sin(a + bx)}\sqrt{\csc(a + bx)} \text{EllipticF}\left(\frac{1}{2}(a + bx - \frac{\pi}{2}), 2\right)}{b}
 \end{aligned}$$

input `Int[Sqrt[Csc[a + b*x]],x]`

output `(2*Sqrt[Csc[a + b*x]]*EllipticF[(a - Pi/2 + b*x)/2, 2]*Sqrt[Sin[a + b*x]])/b`

**Defintions of rubi rules used**

rule 3042 `Int[u_, x_Symbol] :=> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3120 `Int[1/Sqrt[sin[(c_.) + (d_.)*(x_)]], x_Symbol] :=> Simp[(2/d)*EllipticF[(1/2)*(c - Pi/2 + d*x), 2], x] /; FreeQ[{c, d}, x]`

rule 4258 `Int[(csc[(c_.) + (d_.)*(x_)])*(b_.))^(n_), x_Symbol] :=> Simp[(b*Csc[c + d*x])^n*Sin[c + d*x]^n Int[1/Sin[c + d*x]^n, x], x] /; FreeQ[{b, c, d}, x] && EqQ[n^2, 1/4]`

**Maple [A] (verified)**

Time = 0.12 (sec) , antiderivative size = 69, normalized size of antiderivative = 1.68

method	result	size
default	$\frac{\sqrt{\sin(bx+a)+1} \sqrt{-2\sin(bx+a)+2} \sqrt{-\sin(bx+a)} \operatorname{EllipticF}\left(\sqrt{\sin(bx+a)+1}, \frac{\sqrt{2}}{2}\right)}{\cos(bx+a)\sqrt{\sin(bx+a)} b}$	69

input `int(csc(b*x+a)^(1/2), x, method=_RETURNVERBOSE)`

output `((sin(b*x+a)+1)^(1/2)*(-2*sin(b*x+a)+2)^(1/2)*(-sin(b*x+a))^(1/2)*EllipticF((sin(b*x+a)+1)^(1/2), 1/2*2^(1/2))/cos(b*x+a)/sin(b*x+a)^(1/2)/b`

**Fricas [C] (verification not implemented)**

Result contains complex when optimal does not.

Time = 0.08 (sec) , antiderivative size = 51, normalized size of antiderivative = 1.24

$$\int \sqrt{\csc(a + bx)} dx = \frac{-i \sqrt{2i} \operatorname{weierstrassPInverse}(4, 0, \cos(bx + a) + i \sin(bx + a)) + i \sqrt{-2i} \operatorname{weierstrassPInverse}(4, 0, \cos(bx + a) + i \sin(bx + a))}{b}$$

input `integrate(csc(b*x+a)^(1/2),x, algorithm="fricas")`

output `(-I*sqrt(2*I)*weierstrassPInverse(4, 0, cos(b*x + a) + I*sin(b*x + a)) + I*sqrt(-2*I)*weierstrassPInverse(4, 0, cos(b*x + a) - I*sin(b*x + a)))/b`

### Sympy [F]

$$\int \sqrt{\csc(a + bx)} dx = \int \sqrt{\csc(a + bx)} dx$$

input `integrate(csc(b*x+a)**(1/2),x)`

output `Integral(sqrt(csc(a + b*x)), x)`

### Maxima [F]

$$\int \sqrt{\csc(a + bx)} dx = \int \sqrt{\csc(bx + a)} dx$$

input `integrate(csc(b*x+a)^(1/2),x, algorithm="maxima")`

output `integrate(sqrt(csc(b*x + a)), x)`



**Giac [F]**

$$\int \sqrt{\csc(a + bx)} dx = \int \sqrt{\csc(bx + a)} dx$$

input `integrate(csc(b*x+a)^(1/2),x, algorithm="giac")`

output `integrate(sqrt(csc(b*x + a)), x)`

**Mupad [B] (verification not implemented)**

Time = 14.90 (sec) , antiderivative size = 61, normalized size of antiderivative = 1.49

$$\int \sqrt{\csc(a + bx)} dx$$

$$= -\frac{2\sqrt{\sin(a + bx)} F\left(\operatorname{asin}\left(\frac{\sqrt{2}\sqrt{1-\sin(a+bx)}}{2}\right) \middle| 2\right) \sqrt{\cos(a + bx)^2} \sqrt{\frac{1}{\sin(a+bx)}}}{b \cos(a + bx)}$$

input `int((1/sin(a + b*x))^(1/2),x)`

output `-(2*sin(a + b*x)^(1/2)*ellipticF(asin((2^(1/2)*(1 - sin(a + b*x))^(1/2))/2), 2)*(cos(a + b*x)^2)^(1/2)*(1/sin(a + b*x))^(1/2))/(b*cos(a + b*x))`

**Reduce [F]**

$$\int \sqrt{\csc(a + bx)} dx = \int \sqrt{\csc(bx + a)} dx$$

input `int(csc(b*x+a)^(1/2),x)`

output `int(sqrt(csc(a + b*x)),x)`

### 3.13 $\int \frac{1}{\sqrt{\csc(a+bx)}} dx$

Optimal result	121
Mathematica [A] (verified)	121
Rubi [A] (verified)	122
Maple [B] (verified)	123
Fricas [C] (verification not implemented)	124
Sympy [F]	124
Maxima [F]	124
Giac [F]	125
Mupad [F(-1)]	125
Reduce [F]	125

#### Optimal result

Integrand size = 10, antiderivative size = 41

$$\int \frac{1}{\sqrt{\csc(a+bx)}} dx = \frac{2\sqrt{\csc(a+bx)}E\left(\frac{1}{2}\left(a - \frac{\pi}{2} + bx\right) \middle| 2\right) \sqrt{\sin(a+bx)}}{b}$$

output `-2*csc(b*x+a)^(1/2)*EllipticE(cos(1/2*a+1/4*Pi+1/2*b*x),2^(1/2))*sin(b*x+a)^(1/2)/b`

#### Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 40, normalized size of antiderivative = 0.98

$$\int \frac{1}{\sqrt{\csc(a+bx)}} dx = -\frac{2\sqrt{\csc(a+bx)}E\left(\frac{1}{4}(-2a + \pi - 2bx) \middle| 2\right) \sqrt{\sin(a+bx)}}{b}$$

input `Integrate[1/Sqrt[Csc[a + b*x]],x]`

output `(-2*Sqrt[Csc[a + b*x]]*EllipticE[(-2*a + Pi - 2*b*x)/4, 2]*Sqrt[Sin[a + b*x]])/b`

**Rubi [A] (verified)**

Time = 0.23 (sec) , antiderivative size = 41, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.400$ , Rules used = {3042, 4258, 3042, 3119}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{1}{\sqrt{\csc(a+bx)}} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{1}{\sqrt{\csc(a+bx)}} dx \\
 & \quad \downarrow \text{4258} \\
 & \sqrt{\sin(a+bx)} \sqrt{\csc(a+bx)} \int \sqrt{\sin(a+bx)} dx \\
 & \quad \downarrow \text{3042} \\
 & \sqrt{\sin(a+bx)} \sqrt{\csc(a+bx)} \int \sqrt{\sin(a+bx)} dx \\
 & \quad \downarrow \text{3119} \\
 & \frac{2\sqrt{\sin(a+bx)} \sqrt{\csc(a+bx)} E\left(\frac{1}{2}(a+bx - \frac{\pi}{2}) \mid 2\right)}{b}
 \end{aligned}$$

input `Int[1/Sqrt[Csc[a + b*x]],x]`

output `(2*Sqrt[Csc[a + b*x]]*EllipticE[(a - Pi/2 + b*x)/2, 2]*Sqrt[Sin[a + b*x]])/b`

## Definitions of rubi rules used

rule 3042 `Int[u_, x_Symbol] :=> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3119 `Int[Sqrt[sin[(c_.) + (d_.)*(x_)]], x_Symbol] :=> Simp[(2/d)*EllipticE[(1/2)*(c - Pi/2 + d*x), 2], x] /; FreeQ[{c, d}, x]`

rule 4258 `Int[(csc[(c_.) + (d_.)*(x_)])*(b_.))^(n_), x_Symbol] :=> Simp[(b*Csc[c + d*x])^n*Sin[c + d*x]^n Int[1/Sin[c + d*x]^n, x], x] /; FreeQ[{b, c, d}, x] && EqQ[n^2, 1/4]`

## Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 90 vs.  $2(37) = 74$ .

Time = 0.15 (sec) , antiderivative size = 91, normalized size of antiderivative = 2.22

method	result
default	$-\frac{\sqrt{\sin(bx+a)+1} \sqrt{-2 \sin(bx+a)+2} \sqrt{-\sin(bx+a)} \left( 2 \operatorname{EllipticE}\left(\sqrt{\sin(bx+a)+1}, \frac{\sqrt{2}}{2}\right) - \operatorname{EllipticF}\left(\sqrt{\sin(bx+a)+1}, \frac{\sqrt{2}}{2}\right) \right)}{\cos(bx+a) \sqrt{\sin(bx+a)} b}$
risch	$-\frac{i\sqrt{2}}{b \sqrt{\frac{ie^{i(bx+a)}}{e^{2i(bx+a)}-1}}} + i \left( -\frac{2i(-i+ie^{2i(bx+a)})}{\sqrt{e^{i(bx+a)}(-i+ie^{2i(bx+a)})}} - \frac{\sqrt{e^{i(bx+a)+1} \sqrt{-2e^{i(bx+a)+2} \sqrt{-e^{i(bx+a)}}} \left( -2 \operatorname{EllipticE}\left(\sqrt{e^{i(bx+a)+1}, \frac{\sqrt{2}}{2}\right) \right)}{\sqrt{ie^{3i(bx+a)}-ie^{i(bx+a)}}} \right)}{b \sqrt{\frac{ie^{i(bx+a)}}{e^{2i(bx+a)}-1}} (e^{2i(bx+a)}-1)} \right)$

input `int(1/csc(b*x+a)^(1/2), x, method=_RETURNVERBOSE)`

output `-(sin(b*x+a)+1)^(1/2)*(-2*sin(b*x+a)+2)^(1/2)*(-sin(b*x+a))^(1/2)*(2*EllipticE((sin(b*x+a)+1)^(1/2), 1/2*2^(1/2))-EllipticF((sin(b*x+a)+1)^(1/2), 1/2*2^(1/2)))/cos(b*x+a)/sin(b*x+a)^(1/2)/b`

**Fricas [C] (verification not implemented)**

Result contains complex when optimal does not.

Time = 0.08 (sec) , antiderivative size = 55, normalized size of antiderivative = 1.34

$$\int \frac{1}{\sqrt{\csc(a+bx)}} dx$$

$$= \frac{\sqrt{2i}\operatorname{weierstrassZeta}(4, 0, \operatorname{weierstrassPInverse}(4, 0, \cos(bx+a) + i \sin(bx+a))) + \sqrt{-2i}\operatorname{weierstrassZeta}(4, 0, \operatorname{weierstrassPInverse}(4, 0, \cos(bx+a) - i \sin(bx+a)))}{b}$$

input `integrate(1/csc(b*x+a)^(1/2),x, algorithm="fricas")`

output `(sqrt(2*I)*weierstrassZeta(4, 0, weierstrassPInverse(4, 0, cos(b*x + a) + I*sin(b*x + a))) + sqrt(-2*I)*weierstrassZeta(4, 0, weierstrassPInverse(4, 0, cos(b*x + a) - I*sin(b*x + a))))/b`

**Sympy [F]**

$$\int \frac{1}{\sqrt{\csc(a+bx)}} dx = \int \frac{1}{\sqrt{\csc(a+bx)}} dx$$

input `integrate(1/csc(b*x+a)**(1/2),x)`

output `Integral(1/sqrt(csc(a + b*x)), x)`

**Maxima [F]**

$$\int \frac{1}{\sqrt{\csc(a+bx)}} dx = \int \frac{1}{\sqrt{\csc(bx+a)}} dx$$

input `integrate(1/csc(b*x+a)^(1/2),x, algorithm="maxima")`

output `integrate(1/sqrt(csc(b*x + a)), x)`

### Giac [F]

$$\int \frac{1}{\sqrt{\csc(a + bx)}} dx = \int \frac{1}{\sqrt{\csc(bx + a)}} dx$$

input `integrate(1/csc(b*x+a)^(1/2),x, algorithm="giac")`

output `integrate(1/sqrt(csc(b*x + a)), x)`

### Mupad [F(-1)]

Timed out.

$$\int \frac{1}{\sqrt{\csc(a + bx)}} dx = \int \frac{1}{\sqrt{\frac{1}{\sin(a+bx)}}} dx$$

input `int(1/(1/sin(a + b*x))^(1/2),x)`

output `int(1/(1/sin(a + b*x))^(1/2), x)`

### Reduce [F]

$$\int \frac{1}{\sqrt{\csc(a + bx)}} dx = \int \frac{\sqrt{\csc(bx + a)}}{\csc(bx + a)} dx$$

input `int(1/csc(b*x+a)^(1/2),x)`

output `int(sqrt(csc(a + b*x))/csc(a + b*x),x)`

### 3.14 $\int \frac{1}{\csc^{\frac{3}{2}}(a+bx)} dx$

Optimal result	126
Mathematica [A] (verified)	126
Rubi [A] (verified)	127
Maple [A] (verified)	128
Fricas [C] (verification not implemented)	129
Sympy [F]	129
Maxima [F]	130
Giac [F]	130
Mupad [F(-1)]	130
Reduce [F]	131

#### Optimal result

Integrand size = 10, antiderivative size = 67

$$\int \frac{1}{\csc^{\frac{3}{2}}(a+bx)} dx = -\frac{2 \cos(a+bx)}{3b\sqrt{\csc(a+bx)}} + \frac{2\sqrt{\csc(a+bx)} \operatorname{EllipticF}\left(\frac{1}{2}(a - \frac{\pi}{2} + bx), 2\right) \sqrt{\sin(a+bx)}}{3b}$$

output

```
-2/3*cos(b*x+a)/b/csc(b*x+a)^(1/2)+2/3*csc(b*x+a)^(1/2)*InverseJacobiAM(1/2*a-1/4*Pi+1/2*b*x,2^(1/2))*sin(b*x+a)^(1/2)/b
```

#### Mathematica [A] (verified)

Time = 0.05 (sec) , antiderivative size = 53, normalized size of antiderivative = 0.79

$$\int \frac{1}{\csc^{\frac{3}{2}}(a+bx)} dx = -\frac{\sqrt{\csc(a+bx)} \left( 2 \operatorname{EllipticF}\left(\frac{1}{4}(-2a + \pi - 2bx), 2\right) \sqrt{\sin(a+bx)} + \sin(2(a+bx)) \right)}{3b}$$

input

```
Integrate[Csc[a + b*x]^(-3/2), x]
```

output

```
-1/3*(Sqrt[Csc[a + b*x]]*(2*EllipticF[(-2*a + Pi - 2*b*x)/4, 2]*Sqrt[Sin[a + b*x]] + Sin[2*(a + b*x)]))/b
```

**Rubi [A] (verified)**

Time = 0.31 (sec) , antiderivative size = 67, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 6,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.600$ , Rules used = {3042, 4256, 3042, 4258, 3042, 3120}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{1}{\csc^{\frac{3}{2}}(a + bx)} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{1}{\csc(a + bx)^{3/2}} dx \\
 & \quad \downarrow \text{4256} \\
 & \frac{1}{3} \int \sqrt{\csc(a + bx)} dx - \frac{2 \cos(a + bx)}{3b \sqrt{\csc(a + bx)}} \\
 & \quad \downarrow \text{3042} \\
 & \frac{1}{3} \int \sqrt{\csc(a + bx)} dx - \frac{2 \cos(a + bx)}{3b \sqrt{\csc(a + bx)}} \\
 & \quad \downarrow \text{4258} \\
 & \frac{1}{3} \sqrt{\sin(a + bx)} \sqrt{\csc(a + bx)} \int \frac{1}{\sqrt{\sin(a + bx)}} dx - \frac{2 \cos(a + bx)}{3b \sqrt{\csc(a + bx)}} \\
 & \quad \downarrow \text{3042} \\
 & \frac{1}{3} \sqrt{\sin(a + bx)} \sqrt{\csc(a + bx)} \int \frac{1}{\sqrt{\sin(a + bx)}} dx - \frac{2 \cos(a + bx)}{3b \sqrt{\csc(a + bx)}} \\
 & \quad \downarrow \text{3120} \\
 & \frac{2 \sqrt{\sin(a + bx)} \sqrt{\csc(a + bx)} \text{EllipticF}\left(\frac{1}{2}(a + bx - \frac{\pi}{2}), 2\right)}{3b} - \frac{2 \cos(a + bx)}{3b \sqrt{\csc(a + bx)}}
 \end{aligned}$$



input `Int[Csc[a + b*x]^(-3/2),x]`

output 
$$\frac{(-2*\cos[a + b*x])/(3*b*\sqrt{\csc[a + b*x]}) + (2*\sqrt{\csc[a + b*x]}*\text{EllipticF}[(a - \pi/2 + b*x)/2, 2]*\sqrt{\sin[a + b*x]})/(3*b)}$$

### Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3120 `Int[1/Sqrt[sin[(c_.) + (d_.)*(x_)]], x_Symbol] := Simp[(2/d)*EllipticF[(1/2)*(c - Pi/2 + d*x), 2], x] /; FreeQ[{c, d}, x]`

rule 4256 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^n, x_Symbol] := Simp[Cos[c + d*x]*((b*Csc[c + d*x])^(n + 1)/(b*d^n)), x] + Simp[(n + 1)/(b^2*n) Int[(b*Csc[c + d*x])^(n + 2), x], x] /; FreeQ[{b, c, d}, x] && LtQ[n, -1] && IntegerQ[2*n]`

rule 4258 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^n, x_Symbol] := Simp[(b*Csc[c + d*x])^n*Sin[c + d*x]^n Int[1/Sin[c + d*x]^n, x], x] /; FreeQ[{b, c, d}, x] && EqQ[n^2, 1/4]`

### Maple [A] (verified)

Time = 0.09 (sec) , antiderivative size = 88, normalized size of antiderivative = 1.31

method	result	size
default	$\frac{\sqrt{\sin(bx+a)+1} \sqrt{-2 \sin(bx+a)+2} \sqrt{-\sin(bx+a)} \text{EllipticF}\left(\sqrt{\sin(bx+a)+1}, \frac{\sqrt{2}}{2}\right) - \frac{2 \cos(bx+a)^2 \sin(bx+a)}{3}}{\cos(bx+a) \sqrt{\sin(bx+a)} b}$	88

input `int(1/csc(b*x+a)^(3/2),x,method=_RETURNVERBOSE)`

output

```
(1/3*(sin(b*x+a)+1)^(1/2)*(-2*sin(b*x+a)+2)^(1/2)*(-sin(b*x+a))^(1/2)*EllipticF((sin(b*x+a)+1)^(1/2),1/2*2^(1/2))-2/3*cos(b*x+a)^2*sin(b*x+a))/cos(b*x+a)/sin(b*x+a)^(1/2)/b
```

### Fricas [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 0.08 (sec) , antiderivative size = 68, normalized size of antiderivative = 1.01

$$\int \frac{1}{\csc^{\frac{3}{2}}(a+bx)} dx = \frac{-2 \cos(bx+a) \sqrt{\sin(bx+a)} + i \sqrt{2i} \operatorname{weierstrassPInverse}(4, 0, \cos(bx+a) + i \sin(bx+a)) - i \sqrt{-2i}}{3b}$$

input

```
integrate(1/csc(b*x+a)^(3/2),x, algorithm="fricas")
```

output

```
-1/3*(2*cos(b*x + a)*sqrt(sin(b*x + a)) + I*sqrt(2*I)*weierstrassPInverse(4, 0, cos(b*x + a) + I*sin(b*x + a)) - I*sqrt(-2*I)*weierstrassPInverse(4, 0, cos(b*x + a) - I*sin(b*x + a)))/b
```

### Sympy [F]

$$\int \frac{1}{\csc^{\frac{3}{2}}(a+bx)} dx = \int \frac{1}{\csc^{\frac{3}{2}}(a+bx)} dx$$

input

```
integrate(1/csc(b*x+a)**(3/2),x)
```

output

```
Integral(csc(a + b*x)**(-3/2), x)
```

**Maxima [F]**

$$\int \frac{1}{\csc^{\frac{3}{2}}(a + bx)} dx = \int \frac{1}{\csc(bx + a)^{\frac{3}{2}}} dx$$

input `integrate(1/csc(b*x+a)^(3/2),x, algorithm="maxima")`

output `integrate(csc(b*x + a)^(-3/2), x)`

**Giac [F]**

$$\int \frac{1}{\csc^{\frac{3}{2}}(a + bx)} dx = \int \frac{1}{\csc(bx + a)^{\frac{3}{2}}} dx$$

input `integrate(1/csc(b*x+a)^(3/2),x, algorithm="giac")`

output `integrate(csc(b*x + a)^(-3/2), x)`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{1}{\csc^{\frac{3}{2}}(a + bx)} dx = \int \frac{1}{\left(\frac{1}{\sin(a+bx)}\right)^{3/2}} dx$$

input `int(1/(1/sin(a + b*x))^(3/2),x)`

output `int(1/(1/sin(a + b*x))^(3/2), x)`

**Reduce [F]**

$$\int \frac{1}{\csc^{\frac{3}{2}}(a + bx)} dx = \int \frac{\sqrt{\csc(bx + a)}}{\csc(bx + a)^2} dx$$

input `int(1/csc(b*x+a)^(3/2),x)`

output `int(sqrt(csc(a + b*x))/csc(a + b*x)**2,x)`

**3.15**  $\int \frac{1}{\csc^{\frac{5}{2}}(a+bx)} dx$

Optimal result	132
Mathematica [A] (verified)	132
Rubi [A] (verified)	133
Maple [B] (verified)	134
Fricas [C] (verification not implemented)	135
Sympy [F]	135
Maxima [F]	136
Giac [F]	136
Mupad [F(-1)]	136
Reduce [F]	137

**Optimal result**

Integrand size = 10, antiderivative size = 67

$$\int \frac{1}{\csc^{\frac{5}{2}}(a+bx)} dx = -\frac{2 \cos(a+bx)}{5b \csc^{\frac{3}{2}}(a+bx)} + \frac{6\sqrt{\csc(a+bx)}E\left(\frac{1}{2}\left(a-\frac{\pi}{2}+bx\right)\middle|2\right)\sqrt{\sin(a+bx)}}{5b}$$

output

```
-2/5*cos(b*x+a)/b/csc(b*x+a)^(3/2)-6/5*csc(b*x+a)^(1/2)*EllipticE(cos(1/2*a+1/4*Pi+1/2*b*x),2^(1/2))*sin(b*x+a)^(1/2)/b
```

**Mathematica [A] (verified)**

Time = 0.08 (sec) , antiderivative size = 60, normalized size of antiderivative = 0.90

$$\int \frac{1}{\csc^{\frac{5}{2}}(a+bx)} dx = \frac{2\sqrt{\csc(a+bx)}\left(3E\left(\frac{1}{4}(-2a+\pi-2bx)\middle|2\right)\sqrt{\sin(a+bx)}+\cos(a+bx)\sin^2(a+bx)\right)}{5b}$$

input

```
Integrate[Csc[a + b*x]^(-5/2),x]
```

output

```
(-2*sqrt[Csc[a + b*x]]*(3*EllipticE[(-2*a + Pi - 2*b*x)/4, 2]*sqrt[Sin[a +
b*x]] + Cos[a + b*x]*Sin[a + b*x]^2))/(5*b)
```

**Rubi [A] (verified)**

Time = 0.30 (sec) , antiderivative size = 67, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 6,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.600$ , Rules used = {3042, 4256, 3042, 4258, 3042, 3119}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{1}{\csc^{\frac{5}{2}}(a+bx)} dx$$

$$\downarrow 3042$$

$$\int \frac{1}{\csc(a+bx)^{5/2}} dx$$

$$\downarrow 4256$$

$$\frac{3}{5} \int \frac{1}{\sqrt{\csc(a+bx)}} dx - \frac{2 \cos(a+bx)}{5b \csc^{\frac{3}{2}}(a+bx)}$$

$$\downarrow 3042$$

$$\frac{3}{5} \int \frac{1}{\sqrt{\csc(a+bx)}} dx - \frac{2 \cos(a+bx)}{5b \csc^{\frac{3}{2}}(a+bx)}$$

$$\downarrow 4258$$

$$\frac{3}{5} \sqrt{\sin(a+bx)} \sqrt{\csc(a+bx)} \int \sqrt{\sin(a+bx)} dx - \frac{2 \cos(a+bx)}{5b \csc^{\frac{3}{2}}(a+bx)}$$

$$\downarrow 3042$$

$$\frac{3}{5} \sqrt{\sin(a+bx)} \sqrt{\csc(a+bx)} \int \sqrt{\sin(a+bx)} dx - \frac{2 \cos(a+bx)}{5b \csc^{\frac{3}{2}}(a+bx)}$$

$$\downarrow 3119$$

$$\frac{6 \sqrt{\sin(a+bx)} \sqrt{\csc(a+bx)} E\left(\frac{1}{2}(a+bx - \frac{\pi}{2}) \mid 2\right)}{5b} - \frac{2 \cos(a+bx)}{5b \csc^{\frac{3}{2}}(a+bx)}$$

input `Int[Csc[a + b*x]^(-5/2),x]`

output `(-2*Cos[a + b*x])/(5*b*Csc[a + b*x]^(3/2)) + (6*Sqrt[Csc[a + b*x]]*EllipticE[(a - Pi/2 + b*x)/2, 2]*Sqrt[Sin[a + b*x]])/(5*b)`

### Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] :=> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3119 `Int[Sqrt[sin[(c_.) + (d_.)*(x_)]], x_Symbol] :=> Simp[(2/d)*EllipticE[(1/2)*(c - Pi/2 + d*x), 2], x] /; FreeQ[{c, d}, x]`

rule 4256 `Int[(csc[(c_.) + (d_.)*(x_)])*(b_.))^n, x_Symbol] :=> Simp[Cos[c + d*x]*((b*Csc[c + d*x])^(n + 1)/(b*d^n)), x] + Simp[(n + 1)/(b^2*n) Int[(b*Csc[c + d*x])^(n + 2), x], x] /; FreeQ[{b, c, d}, x] && LtQ[n, -1] && IntegerQ[2*n]`

rule 4258 `Int[(csc[(c_.) + (d_.)*(x_)])*(b_.))^n, x_Symbol] :=> Simp[(b*Csc[c + d*x])^n*Sin[c + d*x]^n Int[1/Sin[c + d*x]^n, x], x] /; FreeQ[{b, c, d}, x] && EqQ[n^2, 1/4]`

### Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 141 vs.  $2(57) = 114$ .

Time = 0.08 (sec) , antiderivative size = 142, normalized size of antiderivative = 2.12

method	result
default	$\frac{\frac{2 \sin^4(bx+a)}{5} - \frac{2 \sin^2(bx+a)}{5} - \frac{6 \sqrt{\sin(bx+a)+1} \sqrt{-2 \sin(bx+a)+2} \sqrt{-\sin(bx+a)}}{5} \operatorname{EllipticE}\left(\frac{\sqrt{\sin(bx+a)+1}, \frac{\sqrt{2}}{2}\right)}{\cos(bx+a) \sqrt{\sin(bx+a)} b} + \frac{3 \sqrt{\sin(bx+a)+1} \sqrt{-2 \sin(bx+a)+2}}{\cos(bx+a) \sqrt{\sin(bx+a)} b}$

input `int(1/csc(b*x+a)^(5/2),x,method=_RETURNVERBOSE)`

output

```
(2/5*sin(b*x+a)^4-2/5*sin(b*x+a)^2-6/5*(sin(b*x+a)+1)^(1/2)*(-2*sin(b*x+a)+2)^(1/2)*(-sin(b*x+a))^(1/2)*EllipticE((sin(b*x+a)+1)^(1/2),1/2*2^(1/2))+3/5*(sin(b*x+a)+1)^(1/2)*(-2*sin(b*x+a)+2)^(1/2)*(-sin(b*x+a))^(1/2)*EllipticF((sin(b*x+a)+1)^(1/2),1/2*2^(1/2)))/cos(b*x+a)/sin(b*x+a)^(1/2)/b
```

### Fricas [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 0.08 (sec) , antiderivative size = 85, normalized size of antiderivative = 1.27

$$\int \frac{1}{\csc^{\frac{5}{2}}(a+bx)} dx$$

$$= \frac{3\sqrt{2i}\text{weierstrassZeta}(4,0,\text{weierstrassPInverse}(4,0,\cos(bx+a)+i\sin(bx+a))) + 3\sqrt{-2i}\text{weierstrassZeta}(4,0,\text{weierstrassPInverse}(4,0,\cos(bx+a)-i\sin(bx+a))) + 2*(\cos(bx+a)^3 - \cos(bx+a))/\sqrt{\sin(bx+a)}}{5b}$$

input

```
integrate(1/csc(b*x+a)^(5/2),x, algorithm="fricas")
```

output

```
1/5*(3*sqrt(2*I)*weierstrassZeta(4, 0, weierstrassPInverse(4, 0, cos(b*x + a) + I*sin(b*x + a))) + 3*sqrt(-2*I)*weierstrassZeta(4, 0, weierstrassPInverse(4, 0, cos(b*x + a) - I*sin(b*x + a))) + 2*(cos(b*x + a)^3 - cos(b*x + a))/sqrt(sin(b*x + a)))/b
```

### Sympy [F]

$$\int \frac{1}{\csc^{\frac{5}{2}}(a+bx)} dx = \int \frac{1}{\csc^{\frac{5}{2}}(a+bx)} dx$$

input

```
integrate(1/csc(b*x+a)**(5/2),x)
```

output

```
Integral(csc(a + b*x)**(-5/2), x)
```



**Maxima [F]**

$$\int \frac{1}{\csc^{\frac{5}{2}}(a + bx)} dx = \int \frac{1}{\csc^{\frac{5}{2}}(bx + a)} dx$$

input `integrate(1/csc(b*x+a)^(5/2),x, algorithm="maxima")`

output `integrate(csc(b*x + a)^(-5/2), x)`

**Giac [F]**

$$\int \frac{1}{\csc^{\frac{5}{2}}(a + bx)} dx = \int \frac{1}{\csc^{\frac{5}{2}}(bx + a)} dx$$

input `integrate(1/csc(b*x+a)^(5/2),x, algorithm="giac")`

output `integrate(csc(b*x + a)^(-5/2), x)`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{1}{\csc^{\frac{5}{2}}(a + bx)} dx = \int \frac{1}{\left(\frac{1}{\sin(a+bx)}\right)^{\frac{5}{2}}} dx$$

input `int(1/(1/sin(a + b*x))^(5/2),x)`

output `int(1/(1/sin(a + b*x))^(5/2), x)`

**Reduce [F]**

$$\int \frac{1}{\csc^{\frac{5}{2}}(a + bx)} dx = \int \frac{\sqrt{\csc(bx + a)}}{\csc(bx + a)^3} dx$$

input `int(1/csc(b*x+a)^(5/2),x)`

output `int(sqrt(csc(a + b*x))/csc(a + b*x)**3,x)`

### 3.16 $\int \frac{1}{\csc^{\frac{7}{2}}(a+bx)} dx$

Optimal result	138
Mathematica [A] (verified)	138
Rubi [A] (verified)	139
Maple [A] (verified)	141
Fricas [C] (verification not implemented)	141
Sympy [F]	142
Maxima [F]	142
Giac [F]	142
Mupad [F(-1)]	143
Reduce [F]	143

#### Optimal result

Integrand size = 10, antiderivative size = 90

$$\int \frac{1}{\csc^{\frac{7}{2}}(a+bx)} dx = -\frac{2 \cos(a+bx)}{7b \csc^{\frac{5}{2}}(a+bx)} - \frac{10 \cos(a+bx)}{21b \sqrt{\csc(a+bx)}} + \frac{10 \sqrt{\csc(a+bx)} \operatorname{EllipticF}\left(\frac{1}{2}(a - \frac{\pi}{2} + bx), 2\right) \sqrt{\sin(a+bx)}}{21b}$$

output

```
-2/7*cos(b*x+a)/b/csc(b*x+a)^(5/2)-10/21*cos(b*x+a)/b/csc(b*x+a)^(1/2)+10/21*csc(b*x+a)^(1/2)*InverseJacobiAM(1/2*a-1/4*Pi+1/2*b*x,2^(1/2))*sin(b*x+a)^(1/2)/b
```

#### Mathematica [A] (verified)

Time = 0.12 (sec) , antiderivative size = 65, normalized size of antiderivative = 0.72

$$\int \frac{1}{\csc^{\frac{7}{2}}(a+bx)} dx = \frac{\sqrt{\csc(a+bx)} \left( 40 \operatorname{EllipticF}\left(\frac{1}{4}(-2a + \pi - 2bx), 2\right) \sqrt{\sin(a+bx)} + 26 \sin(2(a+bx)) - 3 \sin(4(a+bx)) \right)}{84b}$$

input

```
Integrate[Csc[a + b*x]^(-7/2), x]
```

output

```
-1/84*(Sqrt[Csc[a + b*x]]*(40*EllipticF[(-2*a + Pi - 2*b*x)/4, 2]*Sqrt[Sin
[a + b*x]] + 26*Sin[2*(a + b*x)] - 3*Sin[4*(a + b*x)]))/b
```

**Rubi [A] (verified)**

Time = 0.40 (sec) , antiderivative size = 95, normalized size of antiderivative = 1.06, number of steps used = 8, number of rules used = 8,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.800$ , Rules used = {3042, 4256, 3042, 4256, 3042, 4258, 3042, 3120}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{1}{\csc^{\frac{7}{2}}(a+bx)} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{1}{\csc(a+bx)^{7/2}} dx \\
 & \quad \downarrow \text{4256} \\
 & \frac{5}{7} \int \frac{1}{\csc^{\frac{3}{2}}(a+bx)} dx - \frac{2 \cos(a+bx)}{7b \csc^{\frac{5}{2}}(a+bx)} \\
 & \quad \downarrow \text{3042} \\
 & \frac{5}{7} \int \frac{1}{\csc(a+bx)^{3/2}} dx - \frac{2 \cos(a+bx)}{7b \csc^{\frac{5}{2}}(a+bx)} \\
 & \quad \downarrow \text{4256} \\
 & \frac{5}{7} \left( \frac{1}{3} \int \sqrt{\csc(a+bx)} dx - \frac{2 \cos(a+bx)}{3b \sqrt{\csc(a+bx)}} \right) - \frac{2 \cos(a+bx)}{7b \csc^{\frac{5}{2}}(a+bx)} \\
 & \quad \downarrow \text{3042} \\
 & \frac{5}{7} \left( \frac{1}{3} \int \sqrt{\csc(a+bx)} dx - \frac{2 \cos(a+bx)}{3b \sqrt{\csc(a+bx)}} \right) - \frac{2 \cos(a+bx)}{7b \csc^{\frac{5}{2}}(a+bx)} \\
 & \quad \downarrow \text{4258} \\
 & \frac{5}{7} \left( \frac{1}{3} \sqrt{\sin(a+bx)} \sqrt{\csc(a+bx)} \int \frac{1}{\sqrt{\sin(a+bx)}} dx - \frac{2 \cos(a+bx)}{3b \sqrt{\csc(a+bx)}} \right) - \frac{2 \cos(a+bx)}{7b \csc^{\frac{5}{2}}(a+bx)}
 \end{aligned}$$

$$\begin{aligned} & \downarrow 3042 \\ & \frac{5}{7} \left( \frac{1}{3} \sqrt{\sin(a+bx)} \sqrt{\csc(a+bx)} \int \frac{1}{\sqrt{\sin(a+bx)}} dx - \frac{2 \cos(a+bx)}{3b \sqrt{\csc(a+bx)}} \right) - \frac{2 \cos(a+bx)}{7b \csc^{\frac{5}{2}}(a+bx)} \\ & \downarrow 3120 \\ & \frac{5}{7} \left( \frac{2 \sqrt{\sin(a+bx)} \sqrt{\csc(a+bx)} \operatorname{EllipticF}\left(\frac{1}{2}(a+bx - \frac{\pi}{2}), 2\right)}{3b} - \frac{2 \cos(a+bx)}{3b \sqrt{\csc(a+bx)}} \right) - \\ & \quad \frac{2 \cos(a+bx)}{7b \csc^{\frac{5}{2}}(a+bx)} \end{aligned}$$

input `Int[Csc[a + b*x]^(-7/2), x]`

output `(-2*Cos[a + b*x])/(7*b*Csc[a + b*x]^(5/2)) + (5*((-2*Cos[a + b*x])/(3*b*Sqrt[Csc[a + b*x]]) + (2*Sqrt[Csc[a + b*x]]*EllipticF[(a - Pi/2 + b*x)/2, 2]*Sqrt[Sin[a + b*x]])/(3*b)))/7`

### Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3120 `Int[1/Sqrt[sin[(c_.) + (d_.)*(x_)]], x_Symbol] := Simp[(2/d)*EllipticF[(1/2)*(c - Pi/2 + d*x), 2], x] /; FreeQ[{c, d}, x]`

rule 4256 `Int[(csc[(c_.) + (d_.)*(x_)])*(b_.))^(n_), x_Symbol] := Simp[Cos[c + d*x]*((b*Csc[c + d*x])^(n + 1)/(b*d^n)), x] + Simp[(n + 1)/(b^2*n) Int[(b*Csc[c + d*x])^(n + 2), x], x] /; FreeQ[{b, c, d}, x] && LtQ[n, -1] && IntegerQ[2*n]`

rule 4258 `Int[(csc[(c_.) + (d_.)*(x_)])*(b_.))^(n_), x_Symbol] := Simp[(b*Csc[c + d*x])^n*Sin[c + d*x]^n Int[1/Sin[c + d*x]^n, x], x] /; FreeQ[{b, c, d}, x] && EqQ[n^2, 1/4]`

**Maple [A] (verified)**

Time = 0.07 (sec) , antiderivative size = 104, normalized size of antiderivative = 1.16

method	result	size
default	$\frac{\frac{2 \cos(bx+a)^4 \sin(bx+a)}{7} + \frac{5 \sqrt{\sin(bx+a)+1} \sqrt{-2 \sin(bx+a)+2} \sqrt{-\sin(bx+a)} \operatorname{EllipticF}\left(\sqrt{\sin(bx+a)+1}, \frac{\sqrt{2}}{2}\right) - \frac{16 \cos(bx+a)^2 \sin(bx+a)}{21}}{21 \cos(bx+a) \sqrt{\sin(bx+a)} b}$	104

input `int(1/csc(b*x+a)^(7/2),x,method=_RETURNVERBOSE)`output 
$$\frac{(2/7 \cos(bx+a)^4 \sin(bx+a) + 5/21 (\sin(bx+a)+1)^{1/2} (-2 \sin(bx+a)+2)^{1/2} (-\sin(bx+a))^{1/2} \operatorname{EllipticF}(\sqrt{\sin(bx+a)+1}, 1/2 \sqrt{2}) - 16/21 \cos(bx+a)^2 \sin(bx+a)) / \cos(bx+a) / \sin(bx+a)^{1/2}}{21 b}$$
**Fricas [C] (verification not implemented)**

Result contains complex when optimal does not.

Time = 0.09 (sec) , antiderivative size = 81, normalized size of antiderivative = 0.90

$$\int \frac{1}{\csc^{7/2}(a+bx)} dx$$

$$= \frac{2(3 \cos(bx+a)^3 - 8 \cos(bx+a)) \sqrt{\sin(bx+a)} - 5i \sqrt{2i} \operatorname{weierstrassPInverse}(4, 0, \cos(bx+a) + i \sin(bx+a)) + 5i \sqrt{2i} \operatorname{weierstrassPInverse}(4, 0, \cos(bx+a) - i \sin(bx+a))}{21 b}$$

input `integrate(1/csc(b*x+a)^(7/2),x, algorithm="fricas")`output 
$$\frac{1/21 * (2 * (3 * \cos(b*x + a)^3 - 8 * \cos(b*x + a)) * \sqrt{\sin(b*x + a)} - 5 * I * \sqrt{2 * I} * \operatorname{weierstrassPInverse}(4, 0, \cos(b*x + a) + I * \sin(b*x + a)) + 5 * I * \sqrt{2 * I} * \operatorname{weierstrassPInverse}(4, 0, \cos(b*x + a) - I * \sin(b*x + a)))}{21 b}$$

**Sympy [F]**

$$\int \frac{1}{\csc^{\frac{7}{2}}(a + bx)} dx = \int \frac{1}{\csc^{\frac{7}{2}}(a + bx)} dx$$

input `integrate(1/csc(b*x+a)**(7/2),x)`

output `Integral(csc(a + b*x)**(-7/2), x)`

**Maxima [F]**

$$\int \frac{1}{\csc^{\frac{7}{2}}(a + bx)} dx = \int \frac{1}{\csc^{\frac{7}{2}}(bx + a)} dx$$

input `integrate(1/csc(b*x+a)^(7/2),x, algorithm="maxima")`

output `integrate(csc(b*x + a)^(-7/2), x)`

**Giac [F]**

$$\int \frac{1}{\csc^{\frac{7}{2}}(a + bx)} dx = \int \frac{1}{\csc^{\frac{7}{2}}(bx + a)} dx$$

input `integrate(1/csc(b*x+a)^(7/2),x, algorithm="giac")`

output `integrate(csc(b*x + a)^(-7/2), x)`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{1}{\csc^{\frac{7}{2}}(a + bx)} dx = \int \frac{1}{\left(\frac{1}{\sin(a+bx)}\right)^{\frac{7}{2}}} dx$$

input `int(1/(1/sin(a + b*x))^(7/2),x)`output `int(1/(1/sin(a + b*x))^(7/2), x)`**Reduce [F]**

$$\int \frac{1}{\csc^{\frac{7}{2}}(a + bx)} dx = \int \frac{\sqrt{\csc(bx + a)}}{\csc(bx + a)^4} dx$$

input `int(1/csc(b*x+a)^(7/2),x)`output `int(sqrt(csc(a + b*x))/csc(a + b*x)**4,x)`



### 3.17 $\int (c \csc(a + bx))^{7/2} dx$

Optimal result	144
Mathematica [A] (verified)	144
Rubi [A] (verified)	145
Maple [C] (verified)	147
Fricas [C] (verification not implemented)	147
Sympy [F(-1)]	148
Maxima [F]	148
Giac [F]	149
Mupad [F(-1)]	149
Reduce [F]	149

#### Optimal result

Integrand size = 12, antiderivative size = 103

$$\int (c \csc(a + bx))^{7/2} dx = -\frac{6c^3 \cos(a + bx) \sqrt{c \csc(a + bx)}}{5b} - \frac{2c \cos(a + bx) (c \csc(a + bx))^{5/2}}{5b} - \frac{6c^4 E\left(\frac{1}{2}(a - \frac{\pi}{2} + bx) \mid 2\right)}{5b \sqrt{c \csc(a + bx)} \sqrt{\sin(a + bx)}}$$

output

```
-6/5*c^3*cos(b*x+a)*(c*csc(b*x+a))^(1/2)/b-2/5*c*cos(b*x+a)*(c*csc(b*x+a))^(5/2)/b+6/5*c^4*EllipticE(cos(1/2*a+1/4*Pi+1/2*b*x),2^(1/2))/b/(c*csc(b*x+a))^(1/2)/sin(b*x+a)^(1/2)
```

#### Mathematica [A] (verified)

Time = 0.15 (sec) , antiderivative size = 67, normalized size of antiderivative = 0.65

$$\int (c \csc(a + bx))^{7/2} dx = \frac{(c \csc(a + bx))^{7/2} \left( 24E\left(\frac{1}{4}(-2a + \pi - 2bx) \mid 2\right) \sin^{\frac{7}{2}}(a + bx) - 10 \sin(2(a + bx)) + 3 \sin(4(a + bx)) \right)}{20b}$$

input

```
Integrate[(c*Csc[a + b*x])^(7/2),x]
```

output

```
((c*Csc[a + b*x])^(7/2)*(24*EllipticE[(-2*a + Pi - 2*b*x)/4, 2]*Sin[a + b*x]^(7/2) - 10*Sin[2*(a + b*x)] + 3*Sin[4*(a + b*x)]))/(20*b)
```

**Rubi [A] (verified)**

Time = 0.47 (sec) , antiderivative size = 105, normalized size of antiderivative = 1.02, number of steps used = 8, number of rules used = 8,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.667$ , Rules used = {3042, 4255, 3042, 4255, 3042, 4258, 3042, 3119}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int (c \csc(a + bx))^{7/2} dx \\
 & \quad \downarrow \text{3042} \\
 & \int (c \csc(a + bx))^{7/2} dx \\
 & \quad \downarrow \text{4255} \\
 & \frac{3}{5} c^2 \int (c \csc(a + bx))^{3/2} dx - \frac{2c \cos(a + bx) (c \csc(a + bx))^{5/2}}{5b} \\
 & \quad \downarrow \text{3042} \\
 & \frac{3}{5} c^2 \int (c \csc(a + bx))^{3/2} dx - \frac{2c \cos(a + bx) (c \csc(a + bx))^{5/2}}{5b} \\
 & \quad \downarrow \text{4255} \\
 & \frac{3}{5} c^2 \left( c^2 \left( - \int \frac{1}{\sqrt{c \csc(a + bx)}} dx \right) - \frac{2c \cos(a + bx) \sqrt{c \csc(a + bx)}}{b} \right) - \\
 & \quad \frac{2c \cos(a + bx) (c \csc(a + bx))^{5/2}}{5b} \\
 & \quad \downarrow \text{3042} \\
 & \frac{3}{5} c^2 \left( c^2 \left( - \int \frac{1}{\sqrt{c \csc(a + bx)}} dx \right) - \frac{2c \cos(a + bx) \sqrt{c \csc(a + bx)}}{b} \right) - \\
 & \quad \frac{2c \cos(a + bx) (c \csc(a + bx))^{5/2}}{5b} \\
 & \quad \downarrow \text{4258}
 \end{aligned}$$

$$\frac{3}{5}c^2 \left( -\frac{c^2 \int \sqrt{\sin(a+bx)} dx}{\sqrt{\sin(a+bx)} \sqrt{c \csc(a+bx)}} - \frac{2c \cos(a+bx) \sqrt{c \csc(a+bx)}}{b} \right) - \frac{2c \cos(a+bx) (c \csc(a+bx))^{5/2}}{5b}$$

↓ 3042

$$\frac{3}{5}c^2 \left( -\frac{c^2 \int \sqrt{\sin(a+bx)} dx}{\sqrt{\sin(a+bx)} \sqrt{c \csc(a+bx)}} - \frac{2c \cos(a+bx) \sqrt{c \csc(a+bx)}}{b} \right) - \frac{2c \cos(a+bx) (c \csc(a+bx))^{5/2}}{5b}$$

↓ 3119

$$\frac{3}{5}c^2 \left( -\frac{2c^2 E\left(\frac{1}{2}(a+bx - \frac{\pi}{2}) \mid 2\right)}{b \sqrt{\sin(a+bx)} \sqrt{c \csc(a+bx)}} - \frac{2c \cos(a+bx) \sqrt{c \csc(a+bx)}}{b} \right) - \frac{2c \cos(a+bx) (c \csc(a+bx))^{5/2}}{5b}$$

input `Int[(c*Csc[a + b*x])^(7/2),x]`

output `(-2*c*Cos[a + b*x]*(c*Csc[a + b*x])^(5/2))/(5*b) + (3*c^2*((-2*c*Cos[a + b*x]*Sqrt[c*Csc[a + b*x]])/b - (2*c^2*EllipticE[(a - Pi/2 + b*x)/2, 2])/(b*Sqrt[c*Csc[a + b*x]]*Sqrt[Sin[a + b*x]])))/5`

### Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3119 `Int[Sqrt[sin[(c_.) + (d_.)*(x_)]], x_Symbol] := Simp[(2/d)*EllipticE[(1/2)*(c - Pi/2 + d*x), 2], x] /; FreeQ[{c, d}, x]`

rule 4255 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] := Simp[(-b)*Cos[c + d*x]*((b*Csc[c + d*x])^(n - 1)/(d*(n - 1))), x] + Simp[b^2*((n - 2)/(n - 1)) * Int[(b*Csc[c + d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[2*n]`

rule 4258

```
Int[(csc[(c_.) + (d_.)*(x_.)]*(b_.))^(n_), x_Symbol] := Simp[(b*Csc[c + d*x]
)^n*Sin[c + d*x]^n Int[1/Sin[c + d*x]^n, x], x] /; FreeQ[{b, c, d}, x] &&
EqQ[n^2, 1/4]
```

### Maple [C] (verified)

Result contains complex when optimal does not.

Time = 0.84 (sec) , antiderivative size = 248, normalized size of antiderivative = 2.41

method	result
default	$-\frac{\sqrt{2} \sqrt{c \csc(bx+a)} c^3 \left( (-6 \cos(bx+a) - 6) \sqrt{1+i \cot(bx+a) - i \csc(bx+a)} \sqrt{i(\csc(bx+a) - \cot(bx+a))} \operatorname{EllipticE}\left(\sqrt{1+i \cot(bx+a)}\right) \right)}{1}$

input

```
int((c*csc(b*x+a))^(7/2),x,method=_RETURNVERBOSE)
```

output

```
-1/5/b*2^(1/2)*(c*csc(b*x+a))^(1/2)*c^3*((-6*cos(b*x+a)-6)*(1+I*cot(b*x+a)
-I*csc(b*x+a))^(1/2)*(I*(csc(b*x+a)-cot(b*x+a)))^(1/2)*EllipticE((1+I*cot(
b*x+a)-I*csc(b*x+a))^(1/2),1/2*2^(1/2))*(1-I*cot(b*x+a)+I*csc(b*x+a))^(1/2
)+(3*cos(b*x+a)+3)*(1+I*cot(b*x+a)-I*csc(b*x+a))^(1/2)*(I*(csc(b*x+a)-cot(
b*x+a)))^(1/2)*(1-I*cot(b*x+a)+I*csc(b*x+a))^(1/2)*EllipticF((1+I*cot(b*x+
a)-I*csc(b*x+a))^(1/2),1/2*2^(1/2))+2^(1/2)*(3+csc(b*x+a)*cot(b*x+a)))
```

### Fricas [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 0.08 (sec) , antiderivative size = 150, normalized size of antiderivative = 1.46

$$\int (c \csc(a + bx))^{7/2} dx =$$

$$\frac{3(c^3 \cos(bx + a)^2 - c^3) \sqrt{2i} \operatorname{weierstrassZeta}(4, 0, \operatorname{weierstrassPInverse}(4, 0, \cos(bx + a) + i \sin(bx + a)))}{1}$$

input

```
integrate((c*csc(b*x+a))^(7/2),x, algorithm="fricas")
```

output

```
-1/5*(3*(c^3*cos(b*x + a)^2 - c^3)*sqrt(2*I*c)*weierstrassZeta(4, 0, weierstrassPInverse(4, 0, cos(b*x + a) + I*sin(b*x + a))) + 3*(c^3*cos(b*x + a)^2 - c^3)*sqrt(-2*I*c)*weierstrassZeta(4, 0, weierstrassPInverse(4, 0, cos(b*x + a) - I*sin(b*x + a))) + 2*(3*c^3*cos(b*x + a)^3 - 4*c^3*cos(b*x + a))*sqrt(c/sin(b*x + a))/(b*cos(b*x + a)^2 - b)
```

**Sympy [F(-1)]**

Timed out.

$$\int (c \csc(a + bx))^{7/2} dx = \text{Timed out}$$

input

```
integrate((c*csc(b*x+a))**(7/2),x)
```

output

Timed out

**Maxima [F]**

$$\int (c \csc(a + bx))^{7/2} dx = \int (c \csc(bx + a))^{7/2} dx$$

input

```
integrate((c*csc(b*x+a))^(7/2),x, algorithm="maxima")
```

output

```
integrate((c*csc(b*x + a))^(7/2), x)
```

**Giac [F]**

$$\int (c \csc(a + bx))^{7/2} dx = \int (c \csc(bx + a))^{\frac{7}{2}} dx$$

input `integrate((c*csc(b*x+a))^(7/2),x, algorithm="giac")`

output `integrate((c*csc(b*x + a))^(7/2), x)`

**Mupad [F(-1)]**

Timed out.

$$\int (c \csc(a + bx))^{7/2} dx = \int \left( \frac{c}{\sin(a + bx)} \right)^{7/2} dx$$

input `int((c/sin(a + b*x))^(7/2),x)`

output `int((c/sin(a + b*x))^(7/2), x)`

**Reduce [F]**

$$\int (c \csc(a + bx))^{7/2} dx = \sqrt{c} \left( \int \sqrt{\csc(bx + a)} \csc(bx + a)^3 dx \right) c^3$$

input `int((c*csc(b*x+a))^(7/2),x)`

output `sqrt(c)*int(sqrt(csc(a + b*x))*csc(a + b*x)**3,x)*c**3`

### 3.18 $\int (c \csc(a + bx))^{5/2} dx$

Optimal result	150
Mathematica [A] (verified)	150
Rubi [A] (verified)	151
Maple [C] (verified)	152
Fricas [C] (verification not implemented)	153
Sympy [F]	153
Maxima [F]	154
Giac [F]	154
Mupad [F(-1)]	154
Reduce [F]	155

#### Optimal result

Integrand size = 12, antiderivative size = 75

$$\int (c \csc(a + bx))^{5/2} dx = -\frac{2c \cos(a + bx)(c \csc(a + bx))^{3/2}}{3b} + \frac{2c^2 \sqrt{c \csc(a + bx)} \operatorname{EllipticF}\left(\frac{1}{2}(a - \frac{\pi}{2} + bx), 2\right) \sqrt{\sin(a + bx)}}{3b}$$

output

```
-2/3*c*cos(b*x+a)*(c*csc(b*x+a))^(3/2)/b+2/3*c^2*(c*csc(b*x+a))^(1/2)*InverseJacobiAM(1/2*a-1/4*Pi+1/2*b*x,2^(1/2))*sin(b*x+a)^(1/2)/b
```

#### Mathematica [A] (verified)

Time = 0.14 (sec) , antiderivative size = 55, normalized size of antiderivative = 0.73

$$\int (c \csc(a + bx))^{5/2} dx = \frac{(c \csc(a + bx))^{5/2} \left( 2 \operatorname{EllipticF}\left(\frac{1}{4}(-2a + \pi - 2bx), 2\right) \sin^{\frac{5}{2}}(a + bx) + \sin(2(a + bx)) \right)}{3b}$$

input

```
Integrate[(c*Csc[a + b*x])^(5/2),x]
```

output

$$-1/3*((c*\text{Csc}[a + b*x])^{5/2}*(2*\text{EllipticF}[(-2*a + \text{Pi} - 2*b*x)/4, 2]*\text{Sin}[a + b*x])^{5/2} + \text{Sin}[2*(a + b*x)]))/b$$
**Rubi [A] (verified)**

Time = 0.33 (sec) , antiderivative size = 75, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 6,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {3042, 4255, 3042, 4258, 3042, 3120}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int (c \csc(a + bx))^{5/2} dx \\
 & \quad \downarrow \text{3042} \\
 & \int (c \csc(a + bx))^{5/2} dx \\
 & \quad \downarrow \text{4255} \\
 & \frac{1}{3}c^2 \int \sqrt{c \csc(a + bx)} dx - \frac{2c \cos(a + bx)(c \csc(a + bx))^{3/2}}{3b} \\
 & \quad \downarrow \text{3042} \\
 & \frac{1}{3}c^2 \int \sqrt{c \csc(a + bx)} dx - \frac{2c \cos(a + bx)(c \csc(a + bx))^{3/2}}{3b} \\
 & \quad \downarrow \text{4258} \\
 & \frac{1}{3}c^2 \sqrt{\sin(a + bx)} \sqrt{c \csc(a + bx)} \int \frac{1}{\sqrt{\sin(a + bx)}} dx - \frac{2c \cos(a + bx)(c \csc(a + bx))^{3/2}}{3b} \\
 & \quad \downarrow \text{3042} \\
 & \frac{1}{3}c^2 \sqrt{\sin(a + bx)} \sqrt{c \csc(a + bx)} \int \frac{1}{\sqrt{\sin(a + bx)}} dx - \frac{2c \cos(a + bx)(c \csc(a + bx))^{3/2}}{3b} \\
 & \quad \downarrow \text{3120} \\
 & \frac{2c^2 \sqrt{\sin(a + bx)} \text{EllipticF}\left(\frac{1}{2}(a + bx - \frac{\pi}{2}), 2\right) \sqrt{c \csc(a + bx)}}{3b} - \frac{2c \cos(a + bx)(c \csc(a + bx))^{3/2}}{3b}
 \end{aligned}$$



input `Int[(c*Csc[a + b*x])^(5/2),x]`

output  $(-2*c*\cos[a + b*x]*(c*\operatorname{Csc}[a + b*x])^{3/2})/(3*b) + (2*c^2*\sqrt{c*\operatorname{Csc}[a + b*x]})*\operatorname{EllipticF}[(a - \pi/2 + b*x)/2, 2]*\sqrt{\sin[a + b*x]}/(3*b)$

### Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] :=> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3120 `Int[1/Sqrt[sin[(c_.) + (d_.)*(x_)]], x_Symbol] :=> Simp[(2/d)*EllipticF[(1/2)*(c - Pi/2 + d*x), 2], x] /; FreeQ[{c, d}, x]`

rule 4255 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] :=> Simp[(-b)*Cos[c + d*x]*((b*Csc[c + d*x])^(n - 1)/(d*(n - 1))), x] + Simp[b^2*((n - 2)/(n - 1)) Int[(b*Csc[c + d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[2*n]`

rule 4258 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] :=> Simp[(b*Csc[c + d*x])^n*Sin[c + d*x]^n Int[1/Sin[c + d*x]^n, x], x] /; FreeQ[{b, c, d}, x] && EqQ[n^2, 1/4]`

### Maple [C] (verified)

Result contains complex when optimal does not.

Time = 0.31 (sec) , antiderivative size = 137, normalized size of antiderivative = 1.83

method	result
default	$\frac{\sqrt{2}c^2\sqrt{c\operatorname{csc}(bx+a)}\left(i(\cos(bx+a)+1)\sqrt{1+i\cot(bx+a)-i\operatorname{csc}(bx+a)}\sqrt{1-i\cot(bx+a)+i\operatorname{csc}(bx+a)}\sqrt{i(\operatorname{csc}(bx+a)-\cot(bx+a))}\operatorname{EllipticF}\left[\frac{1}{2}\left(\frac{c-\pi/2+bx}{d}\right), 2\right]\sqrt{\sin(bx+a)}\right)}{3b}$

input `int((c*csc(b*x+a))^(5/2),x,method=_RETURNVERBOSE)`

output

```
1/3/b*2^(1/2)*c^2*(c*csc(b*x+a))^(1/2)*(I*(cos(b*x+a)+1)*(1+I*cot(b*x+a)-I
*csc(b*x+a))^(1/2)*(1-I*cot(b*x+a)+I*csc(b*x+a))^(1/2)*(I*(csc(b*x+a)-cot(
b*x+a)))^(1/2)*EllipticF((1+I*cot(b*x+a)-I*csc(b*x+a))^(1/2),1/2*2^(1/2))-
2^(1/2)*cot(b*x+a))
```

### Fricas [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 0.08 (sec) , antiderivative size = 105, normalized size of antiderivative = 1.40

$$\int (c \csc(a + bx))^{5/2} dx = \frac{-i \sqrt{2i} c^2 \sin(bx + a) \operatorname{weierstrassPInverse}(4, 0, \cos(bx + a) + i \sin(bx + a)) + i \sqrt{-2i} c^2 \sin(bx + a) \operatorname{weierstrassPInverse}(4, 0, \cos(bx + a) - i \sin(bx + a)) - 2c^2 \sqrt{c/\sin(bx + a)} \cos(bx + a)}{3b \sin(bx + a)}$$

input

```
integrate((c*csc(b*x+a))^(5/2),x, algorithm="fricas")
```

output

```
1/3*(-I*sqrt(2*I*c)*c^2*sin(b*x + a)*weierstrassPInverse(4, 0, cos(b*x + a)
) + I*sin(b*x + a)) + I*sqrt(-2*I*c)*c^2*sin(b*x + a)*weierstrassPInverse(
4, 0, cos(b*x + a) - I*sin(b*x + a)) - 2*c^2*sqrt(c/sin(b*x + a))*cos(b*x
+ a))/(b*sin(b*x + a))
```

### Sympy [F]

$$\int (c \csc(a + bx))^{5/2} dx = \int (c \csc(a + bx))^{\frac{5}{2}} dx$$

input

```
integrate((c*csc(b*x+a))**(5/2),x)
```

output

```
Integral((c*csc(a + b*x))**(5/2), x)
```

**Maxima [F]**

$$\int (c \csc(a + bx))^{5/2} dx = \int (c \csc(bx + a))^{\frac{5}{2}} dx$$

input `integrate((c*csc(b*x+a))^(5/2),x, algorithm="maxima")`

output `integrate((c*csc(b*x + a))^(5/2), x)`

**Giac [F]**

$$\int (c \csc(a + bx))^{5/2} dx = \int (c \csc(bx + a))^{\frac{5}{2}} dx$$

input `integrate((c*csc(b*x+a))^(5/2),x, algorithm="giac")`

output `integrate((c*csc(b*x + a))^(5/2), x)`

**Mupad [F(-1)]**

Timed out.

$$\int (c \csc(a + bx))^{5/2} dx = \int \left( \frac{c}{\sin(a + bx)} \right)^{5/2} dx$$

input `int((c/sin(a + b*x))^(5/2),x)`

output `int((c/sin(a + b*x))^(5/2), x)`

**Reduce [F]**

$$\int (c \csc(a + bx))^{5/2} dx = \sqrt{c} \left( \int \sqrt{\csc(bx + a)} \csc(bx + a)^2 dx \right) c^2$$

input `int((c*csc(b*x+a))^(5/2),x)`

output `sqrt(c)*int(sqrt(csc(a + b*x))*csc(a + b*x)**2,x)*c**2`

### 3.19 $\int (c \csc(a + bx))^{3/2} dx$

Optimal result . . . . .	156
Mathematica [A] (verified) . . . . .	156
Rubi [A] (verified) . . . . .	157
Maple [C] (verified) . . . . .	158
Fricas [C] (verification not implemented) . . . . .	159
Sympy [F] . . . . .	159
Maxima [F] . . . . .	160
Giac [F] . . . . .	160
Mupad [F(-1)] . . . . .	160
Reduce [F] . . . . .	161

#### Optimal result

Integrand size = 12, antiderivative size = 71

$$\int (c \csc(a + bx))^{3/2} dx = -\frac{2c \cos(a + bx) \sqrt{c \csc(a + bx)}}{b} - \frac{2c^2 E\left(\frac{1}{2}(a - \frac{\pi}{2} + bx) \mid 2\right)}{b \sqrt{c \csc(a + bx)} \sqrt{\sin(a + bx)}}$$

output

```
-2*c*cos(b*x+a)*(c*csc(b*x+a))^(1/2)/b+2*c^2*EllipticE(cos(1/2*a+1/4*Pi+1/2*b*x),2^(1/2))/b/(c*csc(b*x+a))^(1/2)/sin(b*x+a)^(1/2)
```

#### Mathematica [A] (verified)

Time = 0.09 (sec) , antiderivative size = 54, normalized size of antiderivative = 0.76

$$\int (c \csc(a + bx))^{3/2} dx = \frac{(c \csc(a + bx))^{3/2} \left( 2E\left(\frac{1}{4}(-2a + \pi - 2bx) \mid 2\right) \sin^{\frac{3}{2}}(a + bx) - \sin(2(a + bx)) \right)}{b}$$

input

```
Integrate[(c*Csc[a + b*x])^(3/2),x]
```

output

```
((c*Csc[a + b*x])^(3/2)*(2*EllipticE[(-2*a + Pi - 2*b*x)/4, 2]*Sin[a + b*x]^(3/2) - Sin[2*(a + b*x)]))/b
```

**Rubi [A] (verified)**

Time = 0.33 (sec) , antiderivative size = 71, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 6,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {3042, 4255, 3042, 4258, 3042, 3119}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int (c \csc(a + bx))^{3/2} dx \\
 & \quad \downarrow \text{3042} \\
 & \int (c \csc(a + bx))^{3/2} dx \\
 & \quad \downarrow \text{4255} \\
 & c^2 \left( - \int \frac{1}{\sqrt{c \csc(a + bx)}} dx \right) - \frac{2c \cos(a + bx) \sqrt{c \csc(a + bx)}}{b} \\
 & \quad \downarrow \text{3042} \\
 & c^2 \left( - \int \frac{1}{\sqrt{c \csc(a + bx)}} dx \right) - \frac{2c \cos(a + bx) \sqrt{c \csc(a + bx)}}{b} \\
 & \quad \downarrow \text{4258} \\
 & - \frac{c^2 \int \sqrt{\sin(a + bx)} dx}{\sqrt{\sin(a + bx)} \sqrt{c \csc(a + bx)}} - \frac{2c \cos(a + bx) \sqrt{c \csc(a + bx)}}{b} \\
 & \quad \downarrow \text{3042} \\
 & - \frac{c^2 \int \sqrt{\sin(a + bx)} dx}{\sqrt{\sin(a + bx)} \sqrt{c \csc(a + bx)}} - \frac{2c \cos(a + bx) \sqrt{c \csc(a + bx)}}{b} \\
 & \quad \downarrow \text{3119} \\
 & - \frac{2c^2 E\left(\frac{1}{2}(a + bx - \frac{\pi}{2}) \middle| 2\right)}{b \sqrt{\sin(a + bx)} \sqrt{c \csc(a + bx)}} - \frac{2c \cos(a + bx) \sqrt{c \csc(a + bx)}}{b}
 \end{aligned}$$

input `Int[(c*Csc[a + b*x])^(3/2),x]`

output  $(-2*c*\text{Cos}[a + b*x]*\text{Sqrt}[c*\text{Csc}[a + b*x]])/b - (2*c^2*\text{EllipticE}[(a - \text{Pi}/2 + b*x)/2, 2])/(b*\text{Sqrt}[c*\text{Csc}[a + b*x]]*\text{Sqrt}[\text{Sin}[a + b*x]])$

### Defintions of rubi rules used

rule 3042  $\text{Int}[u_, x\_Symbol] \rightarrow \text{Int}[\text{DeactivateTrig}[u, x], x] /; \text{FunctionOfTrigOfLinearQ}[u, x]$

rule 3119  $\text{Int}[\text{Sqrt}[\text{sin}[(c_.) + (d_.)*(x_.)]], x\_Symbol] \rightarrow \text{Simp}[(2/d)*\text{EllipticE}[(1/2)*(c - \text{Pi}/2 + d*x), 2], x] /; \text{FreeQ}[\{c, d\}, x]$

rule 4255  $\text{Int}[(\text{csc}[(c_.) + (d_.)*(x_.)]*(b_.))^n, x\_Symbol] \rightarrow \text{Simp}[(-b)*\text{Cos}[c + d*x]*((b*\text{Csc}[c + d*x])^{n-1}/(d*(n-1))), x] + \text{Simp}[b^2*((n-2)/(n-1))\text{Int}[(b*\text{Csc}[c + d*x])^{n-2}, x], x] /; \text{FreeQ}[\{b, c, d\}, x] \&\& \text{GtQ}[n, 1] \&\& \text{IntegerQ}[2*n]$

rule 4258  $\text{Int}[(\text{csc}[(c_.) + (d_.)*(x_.)]*(b_.))^n, x\_Symbol] \rightarrow \text{Simp}[(b*\text{Csc}[c + d*x])^n*\text{Sin}[c + d*x]^n \text{Int}[1/\text{Sin}[c + d*x]^n, x], x] /; \text{FreeQ}[\{b, c, d\}, x] \&\& \text{EqQ}[n^2, 1/4]$

### Maple [C] (verified)

Result contains complex when optimal does not.

Time = 0.20 (sec) , antiderivative size = 223, normalized size of antiderivative = 3.14

method	result
default	$-\frac{\sqrt{2}(\sqrt{2}-\sqrt{1+i(-\csc(bx+a)+\cot(bx+a))}\sqrt{1-i(-\csc(bx+a)+\cot(bx+a))}\sqrt{-i(-\csc(bx+a)+\cot(bx+a))})}{2}\text{EllipticE}(\sqrt{1+i(-\csc(bx+a)+\cot(bx+a))})$

input  $\text{int}((c*\text{csc}(b*x+a))^{3/2}, x, \text{method}=\_RETURNVERBOSE)$

output

```
-1/b*2^(1/2)*(2^(1/2)-(1+I*(-csc(b*x+a)+cot(b*x+a)))^(1/2)*(1-I*(-csc(b*x+a)+cot(b*x+a)))^(1/2)*(-I*(-csc(b*x+a)+cot(b*x+a)))^(1/2)*(2*EllipticE((1+I*cot(b*x+a)-I*csc(b*x+a))^(1/2),1/2*2^(1/2))*cos(b*x+a)-EllipticF((1+I*cot(b*x+a)-I*csc(b*x+a))^(1/2),1/2*2^(1/2))*cos(b*x+a)+2*EllipticE((1+I*cot(b*x+a)-I*csc(b*x+a))^(1/2),1/2*2^(1/2))-EllipticF((1+I*cot(b*x+a)-I*csc(b*x+a))^(1/2),1/2*2^(1/2))))*c*(c*csc(b*x+a))^(1/2)
```

### Fricas [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 0.08 (sec) , antiderivative size = 83, normalized size of antiderivative = 1.17

$$\int (c \csc(a + bx))^{3/2} dx =$$

$$2c \sqrt{\frac{c}{\sin(bx+a)}} \cos(bx+a) + \sqrt{2i} c \text{weierstrassZeta}(4, 0, \text{weierstrassPInverse}(4, 0, \cos(bx+a) + i \sin(bx+a)))$$


---

$b$

input

```
integrate((c*csc(b*x+a))^(3/2),x, algorithm="fricas")
```

output

```
-(2*c*sqrt(c/sin(b*x + a))*cos(b*x + a) + sqrt(2*I*c)*c*weierstrassZeta(4, 0, weierstrassPInverse(4, 0, cos(b*x + a) + I*sin(b*x + a))) + sqrt(-2*I*c)*c*weierstrassZeta(4, 0, weierstrassPInverse(4, 0, cos(b*x + a) - I*sin(b*x + a))))/b
```

### Sympy [F]

$$\int (c \csc(a + bx))^{3/2} dx = \int (c \csc(a + bx))^{\frac{3}{2}} dx$$

input

```
integrate((c*csc(b*x+a))**(3/2),x)
```

output

```
Integral((c*csc(a + b*x))**(3/2), x)
```



**Maxima [F]**

$$\int (c \csc(a + bx))^{3/2} dx = \int (c \csc(bx + a))^{\frac{3}{2}} dx$$

input `integrate((c*csc(b*x+a))^(3/2),x, algorithm="maxima")`

output `integrate((c*csc(b*x + a))^(3/2), x)`

**Giac [F]**

$$\int (c \csc(a + bx))^{3/2} dx = \int (c \csc(bx + a))^{\frac{3}{2}} dx$$

input `integrate((c*csc(b*x+a))^(3/2),x, algorithm="giac")`

output `integrate((c*csc(b*x + a))^(3/2), x)`

**Mupad [F(-1)]**

Timed out.

$$\int (c \csc(a + bx))^{3/2} dx = \int \left( \frac{c}{\sin(a + bx)} \right)^{3/2} dx$$

input `int((c/sin(a + b*x))^(3/2),x)`

output `int((c/sin(a + b*x))^(3/2), x)`

**Reduce [F]**

$$\int (c \csc(a + bx))^{3/2} dx = \sqrt{c} \left( \int \sqrt{\csc(bx + a)} \csc(bx + a) dx \right) c$$

input `int((c*csc(b*x+a))^(3/2),x)`

output `sqrt(c)*int(sqrt(csc(a + b*x))*csc(a + b*x),x)*c`

### 3.20 $\int \sqrt{c \csc(a + bx)} dx$

Optimal result	162
Mathematica [A] (verified)	162
Rubi [A] (verified)	163
Maple [C] (verified)	164
Fricas [C] (verification not implemented)	165
Sympy [F]	165
Maxima [F]	165
Giac [F]	166
Mupad [B] (verification not implemented)	166
Reduce [F]	166

#### Optimal result

Integrand size = 12, antiderivative size = 43

$$\int \sqrt{c \csc(a + bx)} dx = \frac{2\sqrt{c \csc(a + bx)} \operatorname{EllipticF}\left(\frac{1}{2}\left(a - \frac{\pi}{2} + bx\right), 2\right) \sqrt{\sin(a + bx)}}{b}$$

output

$2*(c*\csc(b*x+a))^{(1/2)}*InverseJacobiAM(1/2*a-1/4*Pi+1/2*b*x,2^{(1/2)})*\sin(b*x+a)^{(1/2)}/b$

#### Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 42, normalized size of antiderivative = 0.98

$$\int \sqrt{c \csc(a + bx)} dx = -\frac{2\sqrt{c \csc(a + bx)} \operatorname{EllipticF}\left(\frac{1}{4}(-2a + \pi - 2bx), 2\right) \sqrt{\sin(a + bx)}}{b}$$

input

`Integrate[Sqrt[c*Csc[a + b*x]],x]`

output

$(-2*\sqrt{c*\csc[a + b*x]}*\operatorname{EllipticF}[(-2*a + \pi - 2*b*x)/4, 2]*\sqrt{\sin[a + b*x]})/b$

**Rubi [A] (verified)**

Time = 0.23 (sec) , antiderivative size = 43, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.333$ , Rules used = {3042, 4258, 3042, 3120}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \sqrt{c \csc(a + bx)} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \sqrt{c \csc(a + bx)} dx \\
 & \quad \downarrow \text{4258} \\
 & \sqrt{\sin(a + bx)} \sqrt{c \csc(a + bx)} \int \frac{1}{\sqrt{\sin(a + bx)}} dx \\
 & \quad \downarrow \text{3042} \\
 & \sqrt{\sin(a + bx)} \sqrt{c \csc(a + bx)} \int \frac{1}{\sqrt{\sin(a + bx)}} dx \\
 & \quad \downarrow \text{3120} \\
 & \frac{2\sqrt{\sin(a + bx)} \operatorname{EllipticF}\left(\frac{1}{2}(a + bx - \frac{\pi}{2}), 2\right) \sqrt{c \csc(a + bx)}}{b}
 \end{aligned}$$

input `Int[Sqrt[c*Csc[a + b*x]],x]`

output `(2*Sqrt[c*Csc[a + b*x]]*EllipticF[(a - Pi/2 + b*x)/2, 2]*Sqrt[Sin[a + b*x]])/b`

### Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] :=> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3120 `Int[1/Sqrt[sin[(c_.) + (d_.)*(x_)]], x_Symbol] :=> Simp[(2/d)*EllipticF[(1/2)*(c - Pi/2 + d*x), 2], x] /; FreeQ[{c, d}, x]`

rule 4258 `Int[(csc[(c_.) + (d_.)*(x_)*(b_.)]^(n_), x_Symbol] :=> Simp[(b*Csc[c + d*x])^n*Sin[c + d*x]^n Int[1/Sin[c + d*x]^n, x], x] /; FreeQ[{b, c, d}, x] && EqQ[n^2, 1/4]`

### Maple [C] (verified)

Result contains complex when optimal does not.

Time = 0.23 (sec) , antiderivative size = 120, normalized size of antiderivative = 2.79

method	result
default	$\frac{i(\cos(bx+a)+1)\sqrt{c \csc(bx+a)} \sqrt{1+i \cot(bx+a)-i \csc(bx+a)} \sqrt{1-i \cot(bx+a)+i \csc(bx+a)} \sqrt{i(\csc(bx+a)-\cot(bx+a))}}{b} \text{EllipticF}(\dots)$

input `int((c*csc(b*x+a))^(1/2),x,method=_RETURNVERBOSE)`

output `I/b*(cos(b*x+a)+1)*(c*csc(b*x+a))^(1/2)*(1+I*cot(b*x+a)-I*csc(b*x+a))^(1/2)*(1-I*cot(b*x+a)+I*csc(b*x+a))^(1/2)*(I*(csc(b*x+a)-cot(b*x+a)))^(1/2)*EllipticF((1+I*cot(b*x+a)-I*csc(b*x+a))^(1/2),1/2*2^(1/2))*2^(1/2)`

**Fricas [C] (verification not implemented)**

Result contains complex when optimal does not.

Time = 0.08 (sec) , antiderivative size = 55, normalized size of antiderivative = 1.28

$$\int \sqrt{c \csc(a + bx)} dx$$

$$= \frac{-i \sqrt{2i} \operatorname{cweierstrassPInverse}(4, 0, \cos(bx + a) + i \sin(bx + a)) + i \sqrt{-2i} \operatorname{cweierstrassPInverse}(4, 0, \cos(bx + a) - i \sin(bx + a))}{b}$$

input `integrate((c*csc(b*x+a))^(1/2),x, algorithm="fricas")`

output `(-I*sqrt(2*I*c)*weierstrassPInverse(4, 0, cos(b*x + a) + I*sin(b*x + a)) + I*sqrt(-2*I*c)*weierstrassPInverse(4, 0, cos(b*x + a) - I*sin(b*x + a)))/b`

**Sympy [F]**

$$\int \sqrt{c \csc(a + bx)} dx = \int \sqrt{c \csc(a + bx)} dx$$

input `integrate((c*csc(b*x+a))**(1/2),x)`

output `Integral(sqrt(c*csc(a + b*x)), x)`

**Maxima [F]**

$$\int \sqrt{c \csc(a + bx)} dx = \int \sqrt{c \csc(bx + a)} dx$$

input `integrate((c*csc(b*x+a))^(1/2),x, algorithm="maxima")`

output `integrate(sqrt(c*csc(b*x + a)), x)`

**Giac [F]**

$$\int \sqrt{c \csc(a + bx)} dx = \int \sqrt{c \csc(bx + a)} dx$$

input `integrate((c*csc(b*x+a))^(1/2),x, algorithm="giac")`

output `integrate(sqrt(c*csc(b*x + a)), x)`

**Mupad [B] (verification not implemented)**

Time = 15.18 (sec) , antiderivative size = 63, normalized size of antiderivative = 1.47

$$\int \sqrt{c \csc(a + bx)} dx$$

$$= -\frac{2 \sqrt{\sin(a + bx)} F\left(\operatorname{asin}\left(\frac{\sqrt{2} \sqrt{1 - \sin(a + bx)}}{2}\right) \middle| 2\right) \sqrt{\frac{c}{\sin(a + bx)}} \sqrt{\cos(a + bx)^2}}{b \cos(a + bx)}$$

input `int((c/sin(a + b*x))^(1/2),x)`

output `-(2*sin(a + b*x)^(1/2)*ellipticF(asin((2^(1/2)*(1 - sin(a + b*x))^(1/2))/2), 2)*(c/sin(a + b*x))^(1/2)*(cos(a + b*x)^2)^(1/2))/(b*cos(a + b*x))`

**Reduce [F]**

$$\int \sqrt{c \csc(a + bx)} dx = \sqrt{c} \left( \int \sqrt{\csc(bx + a)} dx \right)$$

input `int((c*csc(b*x+a))^(1/2),x)`

output `sqrt(c)*int(sqrt(csc(a + b*x)),x)`

### 3.21 $\int \frac{1}{\sqrt{c \csc(a+bx)}} dx$

Optimal result	167
Mathematica [A] (verified)	167
Rubi [A] (verified)	168
Maple [C] (verified)	169
Fricas [C] (verification not implemented)	170
Sympy [F]	170
Maxima [F]	170
Giac [F]	171
Mupad [F(-1)]	171
Reduce [F]	171

#### Optimal result

Integrand size = 12, antiderivative size = 43

$$\int \frac{1}{\sqrt{c \csc(a+bx)}} dx = \frac{2E\left(\frac{1}{2}\left(a - \frac{\pi}{2} + bx\right) \middle| 2\right)}{b\sqrt{c \csc(a+bx)}\sqrt{\sin(a+bx)}}$$

output `-2*EllipticE(cos(1/2*a+1/4*Pi+1/2*b*x),2^(1/2))/b/(c*csc(b*x+a))^(1/2)/sin(b*x+a)^(1/2)`

#### Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 42, normalized size of antiderivative = 0.98

$$\int \frac{1}{\sqrt{c \csc(a+bx)}} dx = -\frac{2E\left(\frac{1}{4}(-2a + \pi - 2bx) \middle| 2\right)}{b\sqrt{c \csc(a+bx)}\sqrt{\sin(a+bx)}}$$

input `Integrate[1/Sqrt[c*Csc[a + b*x]],x]`

output `(-2*EllipticE[(-2*a + Pi - 2*b*x)/4, 2])/(b*Sqrt[c*Csc[a + b*x]]*Sqrt[Sin[a + b*x]])`



**Rubi [A] (verified)**

Time = 0.23 (sec) , antiderivative size = 43, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.333$ , Rules used = {3042, 4258, 3042, 3119}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{1}{\sqrt{c \csc(a + bx)}} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{1}{\sqrt{c \csc(a + bx)}} dx \\
 & \quad \downarrow \text{4258} \\
 & \frac{\int \sqrt{\sin(a + bx)} dx}{\sqrt{\sin(a + bx)} \sqrt{c \csc(a + bx)}} \\
 & \quad \downarrow \text{3042} \\
 & \frac{\int \sqrt{\sin(a + bx)} dx}{\sqrt{\sin(a + bx)} \sqrt{c \csc(a + bx)}} \\
 & \quad \downarrow \text{3119} \\
 & \frac{2E\left(\frac{1}{2}(a + bx - \frac{\pi}{2}) \mid 2\right)}{b\sqrt{\sin(a + bx)} \sqrt{c \csc(a + bx)}}
 \end{aligned}$$

input `Int[1/Sqrt[c*Csc[a + b*x]],x]`

output `(2*EllipticE[(a - Pi/2 + b*x)/2, 2])/(b*Sqrt[c*Csc[a + b*x]]*Sqrt[Sin[a + b*x]])`

**Defintions of rubi rules used**

rule 3042 `Int[u_, x_Symbol] :=> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear Q[u, x]`

rule 3119 `Int[Sqrt[sin[(c_.) + (d_.)*(x_)]], x_Symbol] :=> Simp[(2/d)*EllipticE[(1/2)*(c - Pi/2 + d*x), 2], x] /; FreeQ[{c, d}, x]`

rule 4258 `Int[(csc[(c_.) + (d_.)*(x_)*(b_.)]^(n_), x_Symbol] :=> Simp[(b*Csc[c + d*x])^n*Sin[c + d*x]^n Int[1/Sin[c + d*x]^n, x], x] /; FreeQ[{b, c, d}, x] && EqQ[n^2, 1/4]`

**Maple [C] (verified)**

Result contains complex when optimal does not.

Time = 0.30 (sec) , antiderivative size = 243, normalized size of antiderivative = 5.65

method	result
default	$-\frac{\sqrt{2} \left( \sqrt{1+i(-\csc(bx+a)+\cot(bx+a))} \sqrt{-i(-\csc(bx+a)+\cot(bx+a))} \sqrt{1-i(-\csc(bx+a)+\cot(bx+a))} (-\cos(bx+a)-1) \operatorname{EllipticE} \left( \sqrt{e^{i(bx+a)}+1}, \frac{\sqrt{2}}{2} \right) \right)}{\dots}$
risch	$-\frac{i\sqrt{2}}{b\sqrt{\frac{ic e^{i(bx+a)}}{e^{2i(bx+a)}-1}}} + i \left( \frac{2i(ic e^{2i(bx+a)}-ic)}{c\sqrt{e^{i(bx+a)}}(ic e^{2i(bx+a)}-ic)} - \frac{\sqrt{e^{i(bx+a)}+1} \sqrt{-2e^{i(bx+a)}+2} \sqrt{-e^{i(bx+a)}} (-2 \operatorname{EllipticE}(\sqrt{e^{i(bx+a)}+1}, \frac{\sqrt{2}}{2}))}{\sqrt{ic e^{3i(bx+a)}-ic e^{i(bx+a)}}} \right) \frac{1}{b\sqrt{\frac{ic e^{i(bx+a)}}{e^{2i(bx+a)}-1}} (e^{2i(bx+a)}-1)}$

input `int(1/(c*csc(b*x+a))^(1/2),x,method=_RETURNVERBOSE)`

output `-1/b*2^(1/2)*((1+I*(-csc(b*x+a)+cot(b*x+a)))^(1/2)*(-I*(-csc(b*x+a)+cot(b*x+a)))^(1/2)*(1-I*(-csc(b*x+a)+cot(b*x+a)))^(1/2)*(-cos(b*x+a)-1)*EllipticF((1+I*cot(b*x+a)-I*csc(b*x+a))^(1/2),1/2*2^(1/2))+2*(1+I*(-csc(b*x+a)+cot(b*x+a)))^(1/2)*(-I*(-csc(b*x+a)+cot(b*x+a)))^(1/2)*(1-I*(-csc(b*x+a)+cot(b*x+a)))^(1/2)*(cos(b*x+a)+1)*EllipticE((1+I*cot(b*x+a)-I*csc(b*x+a))^(1/2),1/2*2^(1/2))+2^(1/2)*(cos(b*x+a)-1))/(c*csc(b*x+a))^(1/2)*csc(b*x+a)`

**Fricas [C] (verification not implemented)**

Result contains complex when optimal does not.

Time = 0.07 (sec) , antiderivative size = 62, normalized size of antiderivative = 1.44

$$\int \frac{1}{\sqrt{c \csc(a + bx)}} dx$$

$$= \frac{\sqrt{2i} c \operatorname{weierstrassZeta}(4, 0, \operatorname{weierstrassPInverse}(4, 0, \cos(bx + a) + i \sin(bx + a))) + \sqrt{-2i} c \operatorname{weierstrassZeta}(4, 0, \operatorname{weierstrassPInverse}(4, 0, \cos(bx + a) - i \sin(bx + a)))}{bc}$$

input `integrate(1/(c*csc(b*x+a))^(1/2),x, algorithm="fricas")`

output `(sqrt(2*I*c)*weierstrassZeta(4, 0, weierstrassPInverse(4, 0, cos(b*x + a) + I*sin(b*x + a))) + sqrt(-2*I*c)*weierstrassZeta(4, 0, weierstrassPInverse(4, 0, cos(b*x + a) - I*sin(b*x + a))))/(b*c)`

**Sympy [F]**

$$\int \frac{1}{\sqrt{c \csc(a + bx)}} dx = \int \frac{1}{\sqrt{c \csc(a + bx)}} dx$$

input `integrate(1/(c*csc(b*x+a))**(1/2),x)`

output `Integral(1/sqrt(c*csc(a + b*x)), x)`

**Maxima [F]**

$$\int \frac{1}{\sqrt{c \csc(a + bx)}} dx = \int \frac{1}{\sqrt{c \csc(bx + a)}} dx$$

input `integrate(1/(c*csc(b*x+a))^(1/2),x, algorithm="maxima")`

output `integrate(1/sqrt(c*csc(b*x + a)), x)`

### Giac [F]

$$\int \frac{1}{\sqrt{c \csc(a + bx)}} dx = \int \frac{1}{\sqrt{c \csc(bx + a)}} dx$$

input `integrate(1/(c*csc(b*x+a))^(1/2),x, algorithm="giac")`

output `integrate(1/sqrt(c*csc(b*x + a)), x)`

### Mupad [F(-1)]

Timed out.

$$\int \frac{1}{\sqrt{c \csc(a + bx)}} dx = \int \frac{1}{\sqrt{\frac{c}{\sin(a+bx)}}} dx$$

input `int(1/(c/sin(a + b*x))^(1/2),x)`

output `int(1/(c/sin(a + b*x))^(1/2), x)`

### Reduce [F]

$$\int \frac{1}{\sqrt{c \csc(a + bx)}} dx = \frac{\sqrt{c} \left( \int \frac{\sqrt{\csc(bx+a)}}{\csc(bx+a)} dx \right)}{c}$$

input `int(1/(c*csc(b*x+a))^(1/2),x)`

output `(sqrt(c)*int(sqrt(csc(a + b*x))/csc(a + b*x),x))/c`

### 3.22 $\int \frac{1}{(c \csc(a+bx))^{3/2}} dx$

Optimal result	172
Mathematica [A] (verified)	172
Rubi [A] (verified)	173
Maple [C] (verified)	174
Fricas [C] (verification not implemented)	175
Sympy [F]	175
Maxima [F]	176
Giac [F]	176
Mupad [F(-1)]	176
Reduce [F]	177

#### Optimal result

Integrand size = 12, antiderivative size = 77

$$\int \frac{1}{(c \csc(a + bx))^{3/2}} dx = -\frac{2 \cos(a + bx)}{3bc\sqrt{c \csc(a + bx)}} + \frac{2\sqrt{c \csc(a + bx)} \operatorname{EllipticF}\left(\frac{1}{2}\left(a - \frac{\pi}{2} + bx\right), 2\right) \sqrt{\sin(a + bx)}}{3bc^2}$$

output

```
-2/3*cos(b*x+a)/b/c/(c*csc(b*x+a))^(1/2)+2/3*(c*csc(b*x+a))^(1/2)*InverseJ
acobiAM(1/2*a-1/4*Pi+1/2*b*x,2^(1/2))*sin(b*x+a)^(1/2)/b/c^2
```

#### Mathematica [A] (verified)

Time = 0.07 (sec) , antiderivative size = 63, normalized size of antiderivative = 0.82

$$\int \frac{1}{(c \csc(a + bx))^{3/2}} dx = \frac{\csc^2(a + bx) \left( 2 \operatorname{EllipticF}\left(\frac{1}{4}(-2a + \pi - 2bx), 2\right) \sqrt{\sin(a + bx)} + \sin(2(a + bx)) \right)}{3b(c \csc(a + bx))^{3/2}}$$

input

```
Integrate[(c*Csc[a + b*x])^(-3/2),x]
```

output

$$-1/3*(\text{Csc}[a + b*x]^2*(2*\text{EllipticF}[(-2*a + \text{Pi} - 2*b*x)/4, 2]*\text{Sqrt}[\text{Sin}[a + b*x]] + \text{Sin}[2*(a + b*x)]))/(b*(c*\text{Csc}[a + b*x])^(3/2))$$
**Rubi [A] (verified)**

Time = 0.33 (sec) , antiderivative size = 77, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 6,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {3042, 4256, 3042, 4258, 3042, 3120}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \frac{1}{(c \csc(a + bx))^{3/2}} dx \\ & \quad \downarrow 3042 \\ & \int \frac{1}{(c \csc(a + bx))^{3/2}} dx \\ & \quad \downarrow 4256 \\ & \frac{\int \sqrt{c \csc(a + bx)} dx}{3c^2} - \frac{2 \cos(a + bx)}{3bc \sqrt{c \csc(a + bx)}} \\ & \quad \downarrow 3042 \\ & \frac{\int \sqrt{c \csc(a + bx)} dx}{3c^2} - \frac{2 \cos(a + bx)}{3bc \sqrt{c \csc(a + bx)}} \\ & \quad \downarrow 4258 \\ & \frac{\sqrt{\sin(a + bx)} \sqrt{c \csc(a + bx)} \int \frac{1}{\sqrt{\sin(a + bx)}} dx}{3c^2} - \frac{2 \cos(a + bx)}{3bc \sqrt{c \csc(a + bx)}} \\ & \quad \downarrow 3042 \\ & \frac{\sqrt{\sin(a + bx)} \sqrt{c \csc(a + bx)} \int \frac{1}{\sqrt{\sin(a + bx)}} dx}{3c^2} - \frac{2 \cos(a + bx)}{3bc \sqrt{c \csc(a + bx)}} \\ & \quad \downarrow 3120 \\ & \frac{2 \sqrt{\sin(a + bx)} \text{EllipticF}\left(\frac{1}{2}(a + bx - \frac{\pi}{2}), 2\right) \sqrt{c \csc(a + bx)}}{3bc^2} - \frac{2 \cos(a + bx)}{3bc \sqrt{c \csc(a + bx)}} \end{aligned}$$

input `Int[(c*Csc[a + b*x])^(-3/2),x]`

output `(-2*Cos[a + b*x])/(3*b*c*Sqrt[c*Csc[a + b*x]]) + (2*Sqrt[c*Csc[a + b*x]]*EllipticF[(a - Pi/2 + b*x)/2, 2]*Sqrt[Sin[a + b*x]])/(3*b*c^2)`

### Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3120 `Int[1/Sqrt[sin[(c_) + (d_)*(x_)]], x_Symbol] := Simp[(2/d)*EllipticF[(1/2)*(c - Pi/2 + d*x), 2], x] /; FreeQ[{c, d}, x]`

rule 4256 `Int[(csc[(c_) + (d_)*(x_)]*(b_))^(n_), x_Symbol] := Simp[Cos[c + d*x]*((b*Csc[c + d*x])^(n + 1)/(b*d^n)), x] + Simp[(n + 1)/(b^2*n) Int[(b*Csc[c + d*x])^(n + 2), x], x] /; FreeQ[{b, c, d}, x] && LtQ[n, -1] && IntegerQ[2*n]`

rule 4258 `Int[(csc[(c_) + (d_)*(x_)]*(b_))^(n_), x_Symbol] := Simp[(b*Csc[c + d*x])^n*Sin[c + d*x]^n Int[1/Sin[c + d*x]^n, x], x] /; FreeQ[{b, c, d}, x] && EqQ[n^2, 1/4]`

### Maple [C] (verified)

Result contains complex when optimal does not.

Time = 0.25 (sec) , antiderivative size = 142, normalized size of antiderivative = 1.84

method	result
default	$\frac{\sqrt{2} \left( i \sqrt{1+i \cot (bx+a)-i \csc (bx+a)} \sqrt{1-i \cot (bx+a)+i \csc (bx+a)} \sqrt{i(\csc (bx+a)-\cot (bx+a))} \operatorname{EllipticF}\left(\sqrt{1+i \cot (bx+a)-i \csc (bx+a)}, 2\right) \right)}{3 b \sqrt{c \csc (bx+a)} c}$

input `int(1/(c*csc(b*x+a))^(3/2),x,method=_RETURNVERBOSE)`

output

```
1/3/b*2^(1/2)/(c*csc(b*x+a))^(1/2)/c*(I*(1+I*cot(b*x+a)-I*csc(b*x+a))^(1/2)
)*(1-I*cot(b*x+a)+I*csc(b*x+a))^(1/2)*(I*(csc(b*x+a)-cot(b*x+a)))^(1/2)*Eli
pticF((1+I*cot(b*x+a)-I*csc(b*x+a))^(1/2),1/2*2^(1/2))*(csc(b*x+a)+cot(b
*x+a))-2^(1/2)*cos(b*x+a))
```

**Fricas [C] (verification not implemented)**

Result contains complex when optimal does not.

Time = 0.08 (sec) , antiderivative size = 85, normalized size of antiderivative = 1.10

$$\int \frac{1}{(c \csc(a + bx))^{3/2}} dx = \frac{2 \sqrt{\frac{c}{\sin(bx+a)}} \cos(bx+a) \sin(bx+a) + i \sqrt{2i} \operatorname{weierstrassPInverse}(4, 0, \cos(bx+a) + i \sin(bx+a)) - i \operatorname{weierstrassPInverse}(4, 0, \cos(bx+a) - i \sin(bx+a))}{3bc^2}$$

input

```
integrate(1/(c*csc(b*x+a))^(3/2),x, algorithm="fricas")
```

output

```
-1/3*(2*sqrt(c/sin(b*x + a))*cos(b*x + a)*sin(b*x + a) + I*sqrt(2*I*c)*wei
erstrassPInverse(4, 0, cos(b*x + a) + I*sin(b*x + a)) - I*sqrt(-2*I*c)*wei
erstrassPInverse(4, 0, cos(b*x + a) - I*sin(b*x + a)))/(b*c^2)
```

**Sympy [F]**

$$\int \frac{1}{(c \csc(a + bx))^{3/2}} dx = \int \frac{1}{(c \csc(a + bx))^{\frac{3}{2}}} dx$$

input

```
integrate(1/(c*csc(b*x+a))**(3/2),x)
```

output

```
Integral((c*csc(a + b*x))**(-3/2), x)
```



**Maxima [F]**

$$\int \frac{1}{(c \csc(a + bx))^{3/2}} dx = \int \frac{1}{(c \csc(bx + a))^{\frac{3}{2}}} dx$$

input `integrate(1/(c*csc(b*x+a))^(3/2),x, algorithm="maxima")`

output `integrate((c*csc(b*x + a))^(-3/2), x)`

**Giac [F]**

$$\int \frac{1}{(c \csc(a + bx))^{3/2}} dx = \int \frac{1}{(c \csc(bx + a))^{\frac{3}{2}}} dx$$

input `integrate(1/(c*csc(b*x+a))^(3/2),x, algorithm="giac")`

output `integrate((c*csc(b*x + a))^(-3/2), x)`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{1}{(c \csc(a + bx))^{3/2}} dx = \int \frac{1}{\left(\frac{c}{\sin(a+bx)}\right)^{3/2}} dx$$

input `int(1/(c/sin(a + b*x))^(3/2),x)`

output `int(1/(c/sin(a + b*x))^(3/2), x)`

**Reduce [F]**

$$\int \frac{1}{(c \csc(a + bx))^{3/2}} dx = \frac{\sqrt{c} \left( \int \frac{\sqrt{\csc(bx+a)}}{\csc(bx+a)^2} dx \right)}{c^2}$$

input `int(1/(c*csc(b*x+a))^(3/2),x)`

output `(sqrt(c)*int(sqrt(csc(a + b*x))/csc(a + b*x)**2,x))/c**2`

### 3.23 $\int \frac{1}{(c \csc(a+bx))^{5/2}} dx$

Optimal result	178
Mathematica [A] (verified)	178
Rubi [A] (verified)	179
Maple [C] (verified)	180
Fricas [C] (verification not implemented)	181
Sympy [F]	181
Maxima [F]	182
Giac [F]	182
Mupad [F(-1)]	182
Reduce [F]	183

#### Optimal result

Integrand size = 12, antiderivative size = 77

$$\int \frac{1}{(c \csc(a + bx))^{5/2}} dx = -\frac{2 \cos(a + bx)}{5bc(c \csc(a + bx))^{3/2}} + \frac{6E\left(\frac{1}{2}\left(a - \frac{\pi}{2} + bx\right) \middle| 2\right)}{5bc^2 \sqrt{c \csc(a + bx)} \sqrt{\sin(a + bx)}}$$

```
output -2/5*cos(b*x+a)/b/c/(c*csc(b*x+a))^(3/2)-6/5*EllipticE(cos(1/2*a+1/4*Pi+1/2*b*x),2^(1/2))/b/c^2/(c*csc(b*x+a))^(1/2)/sin(b*x+a)^(1/2)
```

#### Mathematica [A] (verified)

Time = 0.11 (sec) , antiderivative size = 60, normalized size of antiderivative = 0.78

$$\int \frac{1}{(c \csc(a + bx))^{5/2}} dx = \frac{-\frac{12E\left(\frac{1}{4}(-2a + \pi - 2bx) \middle| 2\right)}{\sqrt{\sin(a + bx)}} - 2 \sin(2(a + bx))}{10bc^2 \sqrt{c \csc(a + bx)}}$$

```
input Integrate[(c*Csc[a + b*x])^(-5/2),x]
```

```
output ((-12*EllipticE[(-2*a + Pi - 2*b*x)/4, 2])/Sqrt[Sin[a + b*x]] - 2*Sin[2*(a + b*x)])/(10*b*c^2*Sqrt[c*Csc[a + b*x]])
```

**Rubi [A] (verified)**

Time = 0.33 (sec) , antiderivative size = 77, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 6,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {3042, 4256, 3042, 4258, 3042, 3119}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{1}{(c \csc(a + bx))^{5/2}} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{1}{(c \csc(a + bx))^{5/2}} dx \\
 & \quad \downarrow \text{4256} \\
 & \frac{3 \int \frac{1}{\sqrt{c \csc(a+bx)}} dx}{5c^2} - \frac{2 \cos(a + bx)}{5bc(c \csc(a + bx))^{3/2}} \\
 & \quad \downarrow \text{3042} \\
 & \frac{3 \int \frac{1}{\sqrt{c \csc(a+bx)}} dx}{5c^2} - \frac{2 \cos(a + bx)}{5bc(c \csc(a + bx))^{3/2}} \\
 & \quad \downarrow \text{4258} \\
 & \frac{3 \int \sqrt{\sin(a + bx)} dx}{5c^2 \sqrt{\sin(a + bx)} \sqrt{c \csc(a + bx)}} - \frac{2 \cos(a + bx)}{5bc(c \csc(a + bx))^{3/2}} \\
 & \quad \downarrow \text{3042} \\
 & \frac{3 \int \sqrt{\sin(a + bx)} dx}{5c^2 \sqrt{\sin(a + bx)} \sqrt{c \csc(a + bx)}} - \frac{2 \cos(a + bx)}{5bc(c \csc(a + bx))^{3/2}} \\
 & \quad \downarrow \text{3119} \\
 & \frac{6E\left(\frac{1}{2}(a + bx - \frac{\pi}{2}) \mid 2\right)}{5bc^2 \sqrt{\sin(a + bx)} \sqrt{c \csc(a + bx)}} - \frac{2 \cos(a + bx)}{5bc(c \csc(a + bx))^{3/2}}
 \end{aligned}$$

input `Int[(c*Csc[a + b*x])^(-5/2),x]`

output 
$$\frac{(-2\cos[a + b*x])/(5*b*c*(c*\text{Csc}[a + b*x])^{3/2}) + (6*\text{EllipticE}[(a - \text{Pi}/2 + b*x)/2, 2])/(5*b*c^2*\text{Sqrt}[c*\text{Csc}[a + b*x]]*\text{Sqrt}[\text{Sin}[a + b*x]])}{1}$$

### Defintions of rubi rules used

rule 3042 
$$\text{Int}[u_, x\_Symbol] \text{ :> Int}[\text{DeactivateTrig}[u, x], x] \text{ /; FunctionOfTrigOfLinearQ}[u, x]$$

rule 3119 
$$\text{Int}[\text{Sqrt}[\text{sin}[(c_.) + (d_.)*(x_.)]], x\_Symbol] \text{ :> Simp}[(2/d)*\text{EllipticE}[(1/2)*(c - \text{Pi}/2 + d*x), 2], x] \text{ /; FreeQ}\{c, d\}, x]$$

rule 4256 
$$\text{Int}[(\text{csc}[(c_.) + (d_.)*(x_.)]*(b_.))^{(n_.)}, x\_Symbol] \text{ :> Simp}[\text{Cos}[c + d*x]*((b*\text{Csc}[c + d*x])^{(n + 1)}/(b*d^n)), x] + \text{Simp}[(n + 1)/(b^2*n) \text{ Int}[(b*\text{Csc}[c + d*x])^{(n + 2)}, x], x] \text{ /; FreeQ}\{b, c, d\}, x] \ \&\& \ \text{LtQ}[n, -1] \ \&\& \ \text{IntegerQ}[2*n]$$

rule 4258 
$$\text{Int}[(\text{csc}[(c_.) + (d_.)*(x_.)]*(b_.))^{(n_.)}, x\_Symbol] \text{ :> Simp}[(b*\text{Csc}[c + d*x])^n*\text{Sin}[c + d*x]^n \text{ Int}[1/\text{Sin}[c + d*x]^n, x], x] \text{ /; FreeQ}\{b, c, d\}, x] \ \&\& \ \text{EqQ}[n^2, 1/4]$$

### Maple [C] (verified)

Result contains complex when optimal does not.

Time = 0.39 (sec) , antiderivative size = 257, normalized size of antiderivative = 3.34

method	result
default	$\frac{\sqrt{2} \left( (-6 \cos(bx+a) - 6) \sqrt{1+i \cot(bx+a) - i \csc(bx+a)} \sqrt{i(\csc(bx+a) - \cot(bx+a))} \text{EllipticE}\left(\sqrt{1+i \cot(bx+a) - i \csc(bx+a)}, \frac{\sqrt{2}}{2}\right) \right)}{1}$

input 
$$\text{int}(1/(c*\text{csc}(b*x+a))^{5/2}, x, \text{method}=\_RETURNVERBOSE)$$

output

```
1/5/b*2^(1/2)*((-6*cos(b*x+a)-6)*(1+I*cot(b*x+a)-I*csc(b*x+a))^(1/2)*(I*(c
sc(b*x+a)-cot(b*x+a)))^(1/2)*EllipticE((1+I*cot(b*x+a)-I*csc(b*x+a))^(1/2)
,1/2*2^(1/2))*(1-I*cot(b*x+a)+I*csc(b*x+a))^(1/2)+(3*cos(b*x+a)+3)*(1+I*co
t(b*x+a)-I*csc(b*x+a))^(1/2)*(I*(csc(b*x+a)-cot(b*x+a)))^(1/2)*(1-I*cot(b*
x+a)+I*csc(b*x+a))^(1/2)*EllipticF((1+I*cot(b*x+a)-I*csc(b*x+a))^(1/2),1/2
*2^(1/2))+cos(b*x+a)^3-4*cos(b*x+a)+3)*2^(1/2))/(c*csc(b*x+a))^(1/2)/c^2*
csc(b*x+a)
```

### Fricas [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 0.09 (sec) , antiderivative size = 96, normalized size of antiderivative = 1.25

$$\int \frac{1}{(c \csc(a + bx))^{5/2}} dx = \frac{2(\cos(bx + a)^3 - \cos(bx + a))\sqrt{\frac{c}{\sin(bx + a)}} + 3\sqrt{2i} c \text{weierstrassZeta}(4, 0, \text{weierstrassPInverse}(4, 0, \cos(bx + a) + I \sin(bx + a))) + 3\sqrt{-2I} c \text{weierstrassZeta}(4, 0, \text{weierstrassPInverse}(4, 0, \cos(bx + a) - I \sin(bx + a)))}{(b \cdot c^3)}$$

input

```
integrate(1/(c*csc(b*x+a))^(5/2),x, algorithm="fricas")
```

output

```
1/5*(2*(cos(b*x + a)^3 - cos(b*x + a))*sqrt(c/sin(b*x + a)) + 3*sqrt(2*I*c
)*weierstrassZeta(4, 0, weierstrassPInverse(4, 0, cos(b*x + a) + I*sin(b*x
+ a))) + 3*sqrt(-2*I*c)*weierstrassZeta(4, 0, weierstrassPInverse(4, 0, c
os(b*x + a) - I*sin(b*x + a))))/(b*c^3)
```

### Sympy [F]

$$\int \frac{1}{(c \csc(a + bx))^{5/2}} dx = \int \frac{1}{(c \csc(a + bx))^{\frac{5}{2}}} dx$$

input

```
integrate(1/(c*csc(b*x+a))**(5/2),x)
```

output

```
Integral((c*csc(a + b*x))**(-5/2), x)
```

**Maxima [F]**

$$\int \frac{1}{(c \csc(a + bx))^{5/2}} dx = \int \frac{1}{(c \csc(bx + a))^{5/2}} dx$$

input `integrate(1/(c*csc(b*x+a))^(5/2),x, algorithm="maxima")`

output `integrate((c*csc(b*x + a))^(5/2), x)`

**Giac [F]**

$$\int \frac{1}{(c \csc(a + bx))^{5/2}} dx = \int \frac{1}{(c \csc(bx + a))^{5/2}} dx$$

input `integrate(1/(c*csc(b*x+a))^(5/2),x, algorithm="giac")`

output `integrate((c*csc(b*x + a))^(5/2), x)`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{1}{(c \csc(a + bx))^{5/2}} dx = \int \frac{1}{\left(\frac{c}{\sin(a+bx)}\right)^{5/2}} dx$$

input `int(1/(c/sin(a + b*x))^(5/2),x)`

output `int(1/(c/sin(a + b*x))^(5/2), x)`

**Reduce [F]**

$$\int \frac{1}{(c \csc(a + bx))^{5/2}} dx = \frac{\sqrt{c} \left( \int \frac{\sqrt{\csc(bx+a)}}{\csc(bx+a)^3} dx \right)}{c^3}$$

input `int(1/(c*csc(b*x+a))^(5/2),x)`

output `(sqrt(c)*int(sqrt(csc(a + b*x))/csc(a + b*x)**3,x))/c**3`



### 3.24 $\int \frac{1}{(c \csc(a+bx))^{7/2}} dx$

Optimal result	184
Mathematica [A] (verified)	184
Rubi [A] (verified)	185
Maple [C] (verified)	187
Fricas [C] (verification not implemented)	187
Sympy [F]	188
Maxima [F]	188
Giac [F]	188
Mupad [F(-1)]	189
Reduce [F]	189

#### Optimal result

Integrand size = 12, antiderivative size = 105

$$\int \frac{1}{(c \csc(a + bx))^{7/2}} dx = -\frac{2 \cos(a + bx)}{7bc(c \csc(a + bx))^{5/2}} - \frac{10 \cos(a + bx)}{21bc^3 \sqrt{c \csc(a + bx)}} + \frac{10 \sqrt{c \csc(a + bx)} \operatorname{EllipticF}\left(\frac{1}{2}\left(a - \frac{\pi}{2} + bx\right), 2\right) \sqrt{\sin(a + bx)}}{21bc^4}$$

output

```
-2/7*cos(b*x+a)/b/c/(c*csc(b*x+a))^(5/2)-10/21*cos(b*x+a)/b/c^3/(c*csc(b*x+a))^(1/2)+10/21*(c*csc(b*x+a))^(1/2)*InverseJacobiAM(1/2*a-1/4*Pi+1/2*b*x,2^(1/2))*sin(b*x+a)^(1/2)/b/c^4
```

#### Mathematica [A] (verified)

Time = 0.11 (sec) , antiderivative size = 70, normalized size of antiderivative = 0.67

$$\int \frac{1}{(c \csc(a + bx))^{7/2}} dx = \frac{\sqrt{c \csc(a + bx)} \left( 40 \operatorname{EllipticF}\left(\frac{1}{4}(-2a + \pi - 2bx), 2\right) \sqrt{\sin(a + bx)} + 26 \sin(2(a + bx)) - 3 \sin(4(a + bx)) \right)}{84bc^4}$$

input

```
Integrate[(c*Csc[a + b*x])^(-7/2),x]
```

output

```
-1/84*(Sqrt[c*Csc[a + b*x]]*(40*EllipticF[(-2*a + Pi - 2*b*x)/4, 2]*Sqrt[Sin[a + b*x]] + 26*Sin[2*(a + b*x)] - 3*Sin[4*(a + b*x)]))/(b*c^4)
```

**Rubi [A] (verified)**

Time = 0.44 (sec) , antiderivative size = 113, normalized size of antiderivative = 1.08, number of steps used = 8, number of rules used = 8,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.667$ , Rules used = {3042, 4256, 3042, 4256, 3042, 4258, 3042, 3120}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{1}{(c \csc(a + bx))^{7/2}} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{1}{(c \csc(a + bx))^{7/2}} dx \\
 & \quad \downarrow \text{4256} \\
 & \frac{5 \int \frac{1}{(c \csc(a + bx))^{3/2}} dx}{7c^2} - \frac{2 \cos(a + bx)}{7bc(c \csc(a + bx))^{5/2}} \\
 & \quad \downarrow \text{3042} \\
 & \frac{5 \int \frac{1}{(c \csc(a + bx))^{3/2}} dx}{7c^2} - \frac{2 \cos(a + bx)}{7bc(c \csc(a + bx))^{5/2}} \\
 & \quad \downarrow \text{4256} \\
 & \frac{5 \left( \frac{\int \sqrt{c \csc(a + bx)} dx}{3c^2} - \frac{2 \cos(a + bx)}{3bc \sqrt{c \csc(a + bx)}} \right)}{7c^2} - \frac{2 \cos(a + bx)}{7bc(c \csc(a + bx))^{5/2}} \\
 & \quad \downarrow \text{3042} \\
 & \frac{5 \left( \frac{\int \sqrt{c \csc(a + bx)} dx}{3c^2} - \frac{2 \cos(a + bx)}{3bc \sqrt{c \csc(a + bx)}} \right)}{7c^2} - \frac{2 \cos(a + bx)}{7bc(c \csc(a + bx))^{5/2}} \\
 & \quad \downarrow \text{4258}
 \end{aligned}$$

$$\frac{5 \left( \frac{\sqrt{\sin(a+bx)} \sqrt{c \csc(a+bx)} \int \frac{1}{\sqrt{\sin(a+bx)}} dx}{3c^2} - \frac{2 \cos(a+bx)}{3bc \sqrt{c \csc(a+bx)}} \right)}{7c^2} - \frac{2 \cos(a+bx)}{7bc (c \csc(a+bx))^{5/2}}$$

↓ 3042

$$\frac{5 \left( \frac{\sqrt{\sin(a+bx)} \sqrt{c \csc(a+bx)} \int \frac{1}{\sqrt{\sin(a+bx)}} dx}{3c^2} - \frac{2 \cos(a+bx)}{3bc \sqrt{c \csc(a+bx)}} \right)}{7c^2} - \frac{2 \cos(a+bx)}{7bc (c \csc(a+bx))^{5/2}}$$

↓ 3120

$$\frac{5 \left( \frac{2 \sqrt{\sin(a+bx)} \operatorname{EllipticF}\left(\frac{1}{2}(a+bx-\frac{\pi}{2}), 2\right) \sqrt{c \csc(a+bx)}}{3bc^2} - \frac{2 \cos(a+bx)}{3bc \sqrt{c \csc(a+bx)}} \right)}{7c^2} - \frac{2 \cos(a+bx)}{7bc (c \csc(a+bx))^{5/2}}$$

input `Int[(c*Csc[a + b*x])^(-7/2),x]`

output `(-2*Cos[a + b*x])/(7*b*c*(c*Csc[a + b*x])^(5/2)) + (5*((-2*Cos[a + b*x])/(3*b*c*Sqrt[c*Csc[a + b*x]]) + (2*Sqrt[c*Csc[a + b*x]]*EllipticF[(a - Pi/2 + b*x)/2, 2]*Sqrt[Sin[a + b*x]])/(3*b*c^2)))/(7*c^2)`

### Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3120 `Int[1/Sqrt[sin[(c_.) + (d_.)*(x_)]], x_Symbol] := Simp[(2/d)*EllipticF[(1/2)*(c - Pi/2 + d*x), 2], x] /; FreeQ[{c, d}, x]`

rule 4256 `Int[(csc[(c_.) + (d_.)*(x_)])*(b_.)^(n_), x_Symbol] := Simp[Cos[c + d*x]*((b*Csc[c + d*x])^(n + 1)/(b*d^n)), x] + Simp[(n + 1)/(b^2*n) Int[(b*Csc[c + d*x])^(n + 2), x], x] /; FreeQ[{b, c, d}, x] && LtQ[n, -1] && IntegerQ[2*n]`

rule 4258

```
Int[(csc[(c_.) + (d_.)*(x_.)]*(b_.))^(n_), x_Symbol] := Simp[(b*Csc[c + d*x]
)^(n)*Sin[c + d*x]^n Int[1/Sin[c + d*x]^n, x], x] /; FreeQ[{b, c, d}, x] &&
EqQ[n^2, 1/4]
```

**Maple [C] (verified)**

Result contains complex when optimal does not.

Time = 0.47 (sec) , antiderivative size = 158, normalized size of antiderivative = 1.50

method	result
default	$\frac{\sqrt{2} \left( \sqrt{2} \left( 3 \cos(bx+a)^3 - 8 \cos(bx+a) \right) + i \sqrt{1+i \cot(bx+a) - i \csc(bx+a)} \operatorname{EllipticF} \left( \sqrt{1+i \cot(bx+a) - i \csc(bx+a)}, \frac{\sqrt{2}}{2} \right) \sqrt{1-i \cot(bx+a)} \right)}{21 b c^3 \sqrt{c \csc(bx+a)}}$

input

```
int(1/(c*csc(b*x+a))^(7/2),x,method=_RETURNVERBOSE)
```

output

```
1/21/b*2^(1/2)/c^3/(c*csc(b*x+a))^(1/2)*(2^(1/2)*(3*cos(b*x+a)^3-8*cos(b*x
+a))+I*(1+I*cot(b*x+a)-I*csc(b*x+a))^(1/2)*EllipticF((1+I*cot(b*x+a)-I*csc
(b*x+a))^(1/2),1/2*2^(1/2))*(1-I*cot(b*x+a)+I*csc(b*x+a))^(1/2)*(I*(csc(b*
x+a)-cot(b*x+a)))^(1/2)*(5*cot(b*x+a)+5*csc(b*x+a)))
```

**Fricas [C] (verification not implemented)**

Result contains complex when optimal does not.

Time = 0.08 (sec) , antiderivative size = 98, normalized size of antiderivative = 0.93

$$\int \frac{1}{(c \csc(a + bx))^{7/2}} dx = \frac{2(3 \cos(bx + a)^3 - 8 \cos(bx + a)) \sqrt{\frac{c}{\sin(bx + a)}} \sin(bx + a) - 5i \sqrt{2i} \operatorname{cweierstrass}}{(c \csc(a + bx))^{7/2}}$$

input

```
integrate(1/(c*csc(b*x+a))^(7/2),x, algorithm="fricas")
```

output

```
1/21*(2*(3*cos(b*x + a)^3 - 8*cos(b*x + a))*sqrt(c/sin(b*x + a))*sin(b*x +
a) - 5*I*sqrt(2*I*c)*weierstrassPInverse(4, 0, cos(b*x + a) + I*sin(b*x +
a)) + 5*I*sqrt(-2*I*c)*weierstrassPInverse(4, 0, cos(b*x + a) - I*sin(b*x
+ a)))/(b*c^4)
```

**Sympy [F]**

$$\int \frac{1}{(c \csc(a + bx))^{7/2}} dx = \int \frac{1}{(c \csc(a + bx))^{\frac{7}{2}}} dx$$

input `integrate(1/(c*csc(b*x+a))**(7/2), x)`

output `Integral((c*csc(a + b*x))**(-7/2), x)`

**Maxima [F]**

$$\int \frac{1}{(c \csc(a + bx))^{7/2}} dx = \int \frac{1}{(c \csc(bx + a))^{\frac{7}{2}}} dx$$

input `integrate(1/(c*csc(b*x+a))^(7/2), x, algorithm="maxima")`

output `integrate((c*csc(b*x + a))^(7/2), x)`

**Giac [F]**

$$\int \frac{1}{(c \csc(a + bx))^{7/2}} dx = \int \frac{1}{(c \csc(bx + a))^{\frac{7}{2}}} dx$$

input `integrate(1/(c*csc(b*x+a))^(7/2), x, algorithm="giac")`

output `integrate((c*csc(b*x + a))^(7/2), x)`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{1}{(c \csc(a + bx))^{7/2}} dx = \int \frac{1}{\left(\frac{c}{\sin(a+bx)}\right)^{7/2}} dx$$

input `int(1/(c/sin(a + b*x))^(7/2),x)`output `int(1/(c/sin(a + b*x))^(7/2), x)`**Reduce [F]**

$$\int \frac{1}{(c \csc(a + bx))^{7/2}} dx = \frac{\sqrt{c} \left( \int \frac{\sqrt{\csc(bx+a)}}{\csc(bx+a)^4} dx \right)}{c^4}$$

input `int(1/(c*csc(b*x+a))^(7/2),x)`output `(sqrt(c)*int(sqrt(csc(a + b*x))/csc(a + b*x)**4,x))/c**4`

### 3.25 $\int \csc^{\frac{4}{3}}(a + bx) dx$

Optimal result	190
Mathematica [A] (verified)	190
Rubi [A] (verified)	191
Maple [F]	192
Fricas [F]	192
Sympy [F]	193
Maxima [F]	193
Giac [F]	193
Mupad [F(-1)]	194
Reduce [F]	194

#### Optimal result

Integrand size = 10, antiderivative size = 51

$$\int \csc^{\frac{4}{3}}(a + bx) dx = -\frac{3 \cos(a + bx) \sqrt[3]{\csc(a + bx)} \operatorname{Hypergeometric2F1}\left(-\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \sin^2(a + bx)\right)}{b \sqrt{\cos^2(a + bx)}}$$

output `-3*cos(b*x+a)*csc(b*x+a)^(1/3)*hypergeom([-1/6, 1/2],[5/6],sin(b*x+a)^2)/b/(cos(b*x+a)^2)^(1/2)`

#### Mathematica [A] (verified)

Time = 0.07 (sec) , antiderivative size = 54, normalized size of antiderivative = 1.06

$$\int \csc^{\frac{4}{3}}(a + bx) dx = \frac{\cos(a + bx) \sqrt[3]{\csc(a + bx)} \left(-3 + 2 \operatorname{Hypergeometric2F1}\left(\frac{1}{6}, \frac{1}{2}, \frac{3}{2}, \cos^2(a + bx)\right) \sqrt[6]{\sin^2(a + bx)}\right)}{b}$$

input `Integrate[Csc[a + b*x]^(4/3),x]`

output

```
(Cos[a + b*x]*Csc[a + b*x]^(1/3)*(-3 + 2*Hypergeometric2F1[1/6, 1/2, 3/2,
Cos[a + b*x]^2]*(Sin[a + b*x]^2)^(1/6)))/b
```

**Rubi [A] (verified)**

Time = 0.24 (sec) , antiderivative size = 51, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.400$ , Rules used = {3042, 4259, 3042, 3122}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \csc^{\frac{4}{3}}(a + bx) dx \\
 & \quad \downarrow \text{3042} \\
 & \int \csc(a + bx)^{\frac{4}{3}} dx \\
 & \quad \downarrow \text{4259} \\
 & \sqrt[3]{\sin(a + bx)} \sqrt[3]{\csc(a + bx)} \int \frac{1}{\sin^{\frac{4}{3}}(a + bx)} dx \\
 & \quad \downarrow \text{3042} \\
 & \sqrt[3]{\sin(a + bx)} \sqrt[3]{\csc(a + bx)} \int \frac{1}{\sin(a + bx)^{\frac{4}{3}}} dx \\
 & \quad \downarrow \text{3122} \\
 & -\frac{3 \cos(a + bx) \sqrt[3]{\csc(a + bx)} \operatorname{Hypergeometric2F1}\left(-\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \sin^2(a + bx)\right)}{b \sqrt{\cos^2(a + bx)}}
 \end{aligned}$$

input

```
Int[Csc[a + b*x]^(4/3),x]
```

output

```
(-3*Cos[a + b*x]*Csc[a + b*x]^(1/3)*Hypergeometric2F1[-1/6, 1/2, 5/6, Sin[
a + b*x]^2])/(b*Sqrt[Cos[a + b*x]^2])
```



**Defintions of rubi rules used**

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3122 `Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] := Simp[Cos[c + d*x]*((b*Sin[c + d*x])^(n + 1)/(b*d*(n + 1)*Sqrt[Cos[c + d*x]^2]))*Hypergeometric2F1[1/2, (n + 1)/2, (n + 3)/2, Sin[c + d*x]^2], x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[2*n]`

rule 4259 `Int[(csc[(c_.) + (d_.)*(x_)])*(b_.)]^(n_), x_Symbol] := Simp[(b*Csc[c + d*x])^(n - 1)*((Sin[c + d*x]/b)^(n - 1) Int[1/(Sin[c + d*x]/b)^n, x]), x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[n]`

**Maple [F]**

$$\int \csc (bx + a)^{\frac{4}{3}} dx$$

input `int(csc(b*x+a)^(4/3),x)`

output `int(csc(b*x+a)^(4/3),x)`

**Fricas [F]**

$$\int \csc^{\frac{4}{3}}(a + bx) dx = \int \csc (bx + a)^{\frac{4}{3}} dx$$

input `integrate(csc(b*x+a)^(4/3),x, algorithm="fricas")`

output `integral(csc(b*x + a)^(4/3), x)`

**Sympy [F]**

$$\int \csc^{\frac{4}{3}}(a + bx) dx = \int \csc^{\frac{4}{3}}(a + bx) dx$$

input `integrate(csc(b*x+a)**(4/3),x)`

output `Integral(csc(a + b*x)**(4/3), x)`

**Maxima [F]**

$$\int \csc^{\frac{4}{3}}(a + bx) dx = \int \csc^{\frac{4}{3}}(bx + a) dx$$

input `integrate(csc(b*x+a)^(4/3),x, algorithm="maxima")`

output `integrate(csc(b*x + a)^(4/3), x)`

**Giac [F]**

$$\int \csc^{\frac{4}{3}}(a + bx) dx = \int \csc^{\frac{4}{3}}(bx + a) dx$$

input `integrate(csc(b*x+a)^(4/3),x, algorithm="giac")`

output `integrate(csc(b*x + a)^(4/3), x)`

**Mupad [F(-1)]**

Timed out.

$$\int \csc^{\frac{4}{3}}(a + bx) dx = \int \left( \frac{1}{\sin(a + bx)} \right)^{\frac{4}{3}} dx$$

input `int((1/sin(a + b*x))^(4/3),x)`output `int((1/sin(a + b*x))^(4/3), x)`**Reduce [F]**

$$\int \csc^{\frac{4}{3}}(a + bx) dx = \int \csc(bx + a)^{\frac{4}{3}} dx$$

input `int(csc(b*x+a)^(4/3),x)`output `int(csc(a + b*x)**(1/3)*csc(a + b*x),x)`

### 3.26 $\int \csc^{\frac{2}{3}}(a + bx) dx$

Optimal result	195
Mathematica [A] (verified)	195
Rubi [A] (verified)	196
Maple [F]	197
Fricas [F]	197
Sympy [F]	198
Maxima [F]	198
Giac [F]	198
Mupad [F(-1)]	199
Reduce [F]	199

#### Optimal result

Integrand size = 10, antiderivative size = 51

$$\int \csc^{\frac{2}{3}}(a + bx) dx = \frac{3 \cos(a + bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{6}, \frac{1}{2}, \frac{7}{6}, \sin^2(a + bx)\right)}{b \sqrt{\cos^2(a + bx)} \sqrt[3]{\csc(a + bx)}}$$

output `3*cos(b*x+a)*hypergeom([1/6, 1/2],[7/6],sin(b*x+a)^2)/b/(cos(b*x+a)^2)^(1/2)/csc(b*x+a)^(1/3)`

#### Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 51, normalized size of antiderivative = 1.00

$$\int \csc^{\frac{2}{3}}(a + bx) dx = -\frac{\cos(a + bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{5}{6}, \frac{3}{2}, \cos^2(a + bx)\right)}{b \sqrt[3]{\csc(a + bx)} \sqrt[6]{\sin^2(a + bx)}}$$

input `Integrate[Csc[a + b*x]^(2/3),x]`

output `-((Cos[a + b*x]*Hypergeometric2F1[1/2, 5/6, 3/2, Cos[a + b*x]^2])/(b*Csc[a + b*x]^(1/3)*(Sin[a + b*x]^2)^(1/6)))`

**Rubi [A] (verified)**

Time = 0.26 (sec) , antiderivative size = 51, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.400$ , Rules used = {3042, 4259, 3042, 3122}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \csc^{\frac{2}{3}}(a + bx) dx \\
 & \quad \downarrow \text{3042} \\
 & \int \csc(a + bx)^{2/3} dx \\
 & \quad \downarrow \text{4259} \\
 & \sin^{\frac{2}{3}}(a + bx) \csc^{\frac{2}{3}}(a + bx) \int \frac{1}{\sin^{\frac{2}{3}}(a + bx)} dx \\
 & \quad \downarrow \text{3042} \\
 & \sin^{\frac{2}{3}}(a + bx) \csc^{\frac{2}{3}}(a + bx) \int \frac{1}{\sin(a + bx)^{2/3}} dx \\
 & \quad \downarrow \text{3122} \\
 & \frac{3 \cos(a + bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{6}, \frac{1}{2}, \frac{7}{6}, \sin^2(a + bx)\right)}{b \sqrt{\cos^2(a + bx)} \sqrt[3]{\csc(a + bx)}}
 \end{aligned}$$

input `Int[Csc[a + b*x]^(2/3), x]`

output `(3*Cos[a + b*x]*Hypergeometric2F1[1/6, 1/2, 7/6, Sin[a + b*x]^2])/(b*Sqrt[Cos[a + b*x]^2]*Csc[a + b*x]^(1/3))`

**Defintions of rubi rules used**

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3122 `Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] := Simp[Cos[c + d*x]*((b*Sin[c + d*x])^(n + 1)/(b*d*(n + 1)*Sqrt[Cos[c + d*x]^2]))*Hypergeometric2F1[1/2, (n + 1)/2, (n + 3)/2, Sin[c + d*x]^2], x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[2*n]`

rule 4259 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] := Simp[(b*Csc[c + d*x])^(n - 1)*((Sin[c + d*x]/b)^(n - 1) Int[1/(Sin[c + d*x]/b)^n, x]), x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[n]`

**Maple [F]**

$$\int \csc (bx + a)^{\frac{2}{3}} dx$$

input `int(csc(b*x+a)^(2/3),x)`

output `int(csc(b*x+a)^(2/3),x)`

**Fricas [F]**

$$\int \csc^{\frac{2}{3}}(a + bx) dx = \int \csc (bx + a)^{\frac{2}{3}} dx$$

input `integrate(csc(b*x+a)^(2/3),x, algorithm="fricas")`

output `integral(csc(b*x + a)^(2/3), x)`

**Sympy [F]**

$$\int \csc^{\frac{2}{3}}(a + bx) dx = \int \csc^{\frac{2}{3}}(a + bx) dx$$

input `integrate(csc(b*x+a)**(2/3),x)`

output `Integral(csc(a + b*x)**(2/3), x)`

**Maxima [F]**

$$\int \csc^{\frac{2}{3}}(a + bx) dx = \int \csc(bx + a)^{\frac{2}{3}} dx$$

input `integrate(csc(b*x+a)^(2/3),x, algorithm="maxima")`

output `integrate(csc(b*x + a)^(2/3), x)`

**Giac [F]**

$$\int \csc^{\frac{2}{3}}(a + bx) dx = \int \csc(bx + a)^{\frac{2}{3}} dx$$

input `integrate(csc(b*x+a)^(2/3),x, algorithm="giac")`

output `integrate(csc(b*x + a)^(2/3), x)`

**Mupad [F(-1)]**

Timed out.

$$\int \csc^{\frac{2}{3}}(a + bx) dx = \int \left( \frac{1}{\sin(a + bx)} \right)^{\frac{2}{3}} dx$$

input `int((1/sin(a + b*x))^(2/3),x)`output `int((1/sin(a + b*x))^(2/3), x)`**Reduce [F]**

$$\int \csc^{\frac{2}{3}}(a + bx) dx = \int \csc(bx + a)^{\frac{2}{3}} dx$$

input `int(csc(b*x+a)^(2/3),x)`output `int(csc(a + b*x)**(2/3),x)`



### 3.27 $\int \sqrt[3]{\csc(a + bx)} dx$

Optimal result	200
Mathematica [A] (verified)	200
Rubi [A] (verified)	201
Maple [F]	202
Fricas [F]	202
Sympy [F]	203
Maxima [F]	203
Giac [F]	203
Mupad [F(-1)]	204
Reduce [F]	204

#### Optimal result

Integrand size = 10, antiderivative size = 53

$$\int \sqrt[3]{\csc(a + bx)} dx = \frac{3 \cos(a + bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{3}, \frac{1}{2}, \frac{4}{3}, \sin^2(a + bx)\right)}{2b \sqrt{\cos^2(a + bx)} \csc^{\frac{2}{3}}(a + bx)}$$

output  $\frac{3/2 \cdot \cos(b \cdot x + a) \cdot \operatorname{hypergeom}([1/3, 1/2], [4/3], \sin(b \cdot x + a)^2)}{b \cdot (\cos(b \cdot x + a)^2)^{1/2} / \csc(b \cdot x + a)^{2/3}}$

#### Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 51, normalized size of antiderivative = 0.96

$$\int \sqrt[3]{\csc(a + bx)} dx = -\frac{\cos(a + bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{2}, \cos^2(a + bx)\right)}{b \csc^{\frac{2}{3}}(a + bx) \sqrt[3]{\sin^2(a + bx)}}$$

input `Integrate[Csc[a + b*x]^(1/3), x]`

output  $-\left(\frac{\cos[a + b \cdot x] \cdot \operatorname{Hypergeometric2F1}[1/2, 2/3, 3/2, \cos[a + b \cdot x]^2]}{b \cdot \csc[a + b \cdot x]^{2/3} \cdot (\sin[a + b \cdot x]^2)^{1/3}}\right)$

**Rubi [A] (verified)**

Time = 0.25 (sec) , antiderivative size = 53, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.400$ , Rules used = {3042, 4259, 3042, 3122}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \sqrt[3]{\csc(a+bx)} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \sqrt[3]{\csc(a+bx)} dx \\
 & \quad \downarrow \text{4259} \\
 & \sqrt[3]{\sin(a+bx)} \sqrt[3]{\csc(a+bx)} \int \frac{1}{\sqrt[3]{\sin(a+bx)}} dx \\
 & \quad \downarrow \text{3042} \\
 & \sqrt[3]{\sin(a+bx)} \sqrt[3]{\csc(a+bx)} \int \frac{1}{\sqrt[3]{\sin(a+bx)}} dx \\
 & \quad \downarrow \text{3122} \\
 & \frac{3 \cos(a+bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{3}, \frac{1}{2}, \frac{4}{3}, \sin^2(a+bx)\right)}{2b \sqrt{\cos^2(a+bx)} \csc^{\frac{2}{3}}(a+bx)}
 \end{aligned}$$

input `Int[Csc[a + b*x]^(1/3),x]`

output `(3*Cos[a + b*x]*Hypergeometric2F1[1/3, 1/2, 4/3, Sin[a + b*x]^2])/(2*b*Sqr  
t[Cos[a + b*x]^2]*Csc[a + b*x]^(2/3))`

**Defintions of rubi rules used**

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3122 `Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] := Simp[Cos[c + d*x]*((b*Sin[c + d*x])^(n + 1)/(b*d*(n + 1)*Sqrt[Cos[c + d*x]^2]))*Hypergeometric2F1[1/2, (n + 1)/2, (n + 3)/2, Sin[c + d*x]^2], x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[2*n]`

rule 4259 `Int[(csc[(c_.) + (d_.)*(x_)])*(b_.)^(n_), x_Symbol] := Simp[(b*Csc[c + d*x])^(n - 1)*((Sin[c + d*x]/b)^(n - 1) Int[1/(Sin[c + d*x]/b)^n, x]), x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[n]`

**Maple [F]**

$$\int \csc(bx + a)^{\frac{1}{3}} dx$$

input `int(csc(b*x+a)^(1/3),x)`

output `int(csc(b*x+a)^(1/3),x)`

**Fricas [F]**

$$\int \sqrt[3]{\csc(a + bx)} dx = \int \csc(bx + a)^{\frac{1}{3}} dx$$

input `integrate(csc(b*x+a)^(1/3),x, algorithm="fricas")`

output `integral(csc(b*x + a)^(1/3), x)`

**Sympy [F]**

$$\int \sqrt[3]{\csc(a + bx)} dx = \int \sqrt[3]{\csc(a + bx)} dx$$

input `integrate(csc(b*x+a)**(1/3),x)`

output `Integral(csc(a + b*x)**(1/3), x)`

**Maxima [F]**

$$\int \sqrt[3]{\csc(a + bx)} dx = \int \csc(bx + a)^{\frac{1}{3}} dx$$

input `integrate(csc(b*x+a)^(1/3),x, algorithm="maxima")`

output `integrate(csc(b*x + a)^(1/3), x)`

**Giac [F]**

$$\int \sqrt[3]{\csc(a + bx)} dx = \int \csc(bx + a)^{\frac{1}{3}} dx$$

input `integrate(csc(b*x+a)^(1/3),x, algorithm="giac")`

output `integrate(csc(b*x + a)^(1/3), x)`

**Mupad [F(-1)]**

Timed out.

$$\int \sqrt[3]{\csc(a + bx)} dx = \int \left( \frac{1}{\sin(a + bx)} \right)^{1/3} dx$$

input `int((1/sin(a + b*x))^(1/3),x)`output `int((1/sin(a + b*x))^(1/3), x)`**Reduce [F]**

$$\int \sqrt[3]{\csc(a + bx)} dx = \int \csc(bx + a)^{\frac{1}{3}} dx$$

input `int(csc(b*x+a)^(1/3),x)`output `int(csc(a + b*x)**(1/3),x)`

### 3.28 $\int \frac{1}{\sqrt[3]{\csc(a + bx)}} dx$

Optimal result	205
Mathematica [A] (verified)	205
Rubi [A] (verified)	206
Maple [F]	207
Fricas [F]	207
Sympy [F]	208
Maxima [F]	208
Giac [F]	208
Mupad [F(-1)]	209
Reduce [F]	209

#### Optimal result

Integrand size = 10, antiderivative size = 53

$$\int \frac{1}{\sqrt[3]{\csc(a + bx)}} dx = \frac{3 \cos(a + bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{2}{3}, \frac{5}{3}, \sin^2(a + bx)\right)}{4b \sqrt{\cos^2(a + bx)} \csc^{\frac{4}{3}}(a + bx)}$$

output `3/4*cos(b*x+a)*hypergeom([1/2, 2/3],[5/3],sin(b*x+a)^2)/b/(cos(b*x+a)^2)^(1/2)/csc(b*x+a)^(4/3)`

#### Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 51, normalized size of antiderivative = 0.96

$$\int \frac{1}{\sqrt[3]{\csc(a + bx)}} dx = -\frac{\cos(a + bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{3}, \frac{1}{2}, \frac{3}{2}, \cos^2(a + bx)\right)}{b \csc^{\frac{4}{3}}(a + bx) \sin^2(a + bx)^{2/3}}$$

input `Integrate[Csc[a + b*x]^(-1/3),x]`

output `-((Cos[a + b*x]*Hypergeometric2F1[1/3, 1/2, 3/2, Cos[a + b*x]^2])/(b*Csc[a + b*x]^(4/3)*(Sin[a + b*x]^2)^(2/3)))`

**Rubi [A] (verified)**

Time = 0.26 (sec) , antiderivative size = 53, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.400$ , Rules used = {3042, 4259, 3042, 3122}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{1}{\sqrt[3]{\csc(a+bx)}} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{1}{\sqrt[3]{\csc(a+bx)}} dx \\
 & \quad \downarrow \text{4259} \\
 & \sin^{\frac{2}{3}}(a+bx) \csc^{\frac{2}{3}}(a+bx) \int \sqrt[3]{\sin(a+bx)} dx \\
 & \quad \downarrow \text{3042} \\
 & \sin^{\frac{2}{3}}(a+bx) \csc^{\frac{2}{3}}(a+bx) \int \sqrt[3]{\sin(a+bx)} dx \\
 & \quad \downarrow \text{3122} \\
 & \frac{3 \cos(a+bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{2}{3}, \frac{5}{3}, \sin^2(a+bx)\right)}{4b \sqrt{\cos^2(a+bx)} \csc^{\frac{4}{3}}(a+bx)}
 \end{aligned}$$

input `Int[Csc[a + b*x]^(-1/3),x]`

output `(3*Cos[a + b*x]*Hypergeometric2F1[1/2, 2/3, 5/3, Sin[a + b*x]^2])/(4*b*Sqr  
t[Cos[a + b*x]^2]*Csc[a + b*x]^(4/3))`

**Defintions of rubi rules used**

rule 3042 `Int[u_, x_Symbol] :=> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3122 `Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] :=> Simp[Cos[c + d*x]*((b*Sin[c + d*x])^(n + 1)/(b*d*(n + 1)*Sqrt[Cos[c + d*x]^2]))*Hypergeometric2F1[1/2, (n + 1)/2, (n + 3)/2, Sin[c + d*x]^2], x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[2*n]`

rule 4259 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] :=> Simp[(b*Csc[c + d*x])^(n - 1)*((Sin[c + d*x]/b)^(n - 1) Int[1/(Sin[c + d*x]/b)^n, x]), x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[n]`

**Maple [F]**

$$\int \frac{1}{\csc(bx + a)^{\frac{1}{3}}} dx$$

input `int(1/csc(b*x+a)^(1/3),x)`

output `int(1/csc(b*x+a)^(1/3),x)`

**Fricas [F]**

$$\int \frac{1}{\sqrt[3]{\csc(a + bx)}} dx = \int \frac{1}{\csc(bx + a)^{\frac{1}{3}}} dx$$

input `integrate(1/csc(b*x+a)^(1/3),x, algorithm="fricas")`

output `integral(csc(b*x + a)^(-1/3), x)`



**Sympy [F]**

$$\int \frac{1}{\sqrt[3]{\csc(a+bx)}} dx = \int \frac{1}{\sqrt[3]{\csc(a+bx)}} dx$$

input `integrate(1/csc(b*x+a)**(1/3),x)`

output `Integral(csc(a + b*x)**(-1/3), x)`

**Maxima [F]**

$$\int \frac{1}{\sqrt[3]{\csc(a+bx)}} dx = \int \frac{1}{\csc(bx+a)^{\frac{1}{3}}} dx$$

input `integrate(1/csc(b*x+a)^(1/3),x, algorithm="maxima")`

output `integrate(csc(b*x + a)^(-1/3), x)`

**Giac [F]**

$$\int \frac{1}{\sqrt[3]{\csc(a+bx)}} dx = \int \frac{1}{\csc(bx+a)^{\frac{1}{3}}} dx$$

input `integrate(1/csc(b*x+a)^(1/3),x, algorithm="giac")`

output `integrate(csc(b*x + a)^(-1/3), x)`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{1}{\sqrt[3]{\csc(a+bx)}} dx = \int \frac{1}{\left(\frac{1}{\sin(a+bx)}\right)^{1/3}} dx$$

input `int(1/(1/sin(a + b*x))^(1/3),x)`output `int(1/(1/sin(a + b*x))^(1/3), x)`**Reduce [F]**

$$\int \frac{1}{\sqrt[3]{\csc(a+bx)}} dx = \int \frac{1}{\csc(bx+a)^{1/3}} dx$$

input `int(1/csc(b*x+a)^(1/3),x)`output `int(1/csc(a + b*x)**(1/3),x)`

### 3.29 $\int \frac{1}{\csc^{\frac{2}{3}}(a+bx)} dx$

Optimal result	210
Mathematica [A] (verified)	210
Rubi [A] (verified)	211
Maple [F]	212
Fricas [F]	212
Sympy [F]	213
Maxima [F]	213
Giac [F]	213
Mupad [F(-1)]	214
Reduce [F]	214

#### Optimal result

Integrand size = 10, antiderivative size = 53

$$\int \frac{1}{\csc^{\frac{2}{3}}(a+bx)} dx = \frac{3 \cos(a+bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{5}{6}, \frac{11}{6}, \sin^2(a+bx)\right)}{5b \sqrt{\cos^2(a+bx)} \csc^{\frac{5}{3}}(a+bx)}$$

output

`3/5*cos(b*x+a)*hypergeom([1/2, 5/6],[11/6],sin(b*x+a)^2)/b/(cos(b*x+a)^2)^(1/2)/csc(b*x+a)^(5/3)`

#### Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 51, normalized size of antiderivative = 0.96

$$\int \frac{1}{\csc^{\frac{2}{3}}(a+bx)} dx = -\frac{\cos(a+bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{6}, \frac{1}{2}, \frac{3}{2}, \cos^2(a+bx)\right)}{b \csc^{\frac{5}{3}}(a+bx) \sin^2(a+bx)^{5/6}}$$

input

`Integrate[Csc[a + b*x]^(-2/3),x]`

output

`-((Cos[a + b*x]*Hypergeometric2F1[1/6, 1/2, 3/2, Cos[a + b*x]^2])/(b*Csc[a + b*x]^(5/3)*(Sin[a + b*x]^2)^(5/6)))`

**Rubi [A] (verified)**

Time = 0.26 (sec) , antiderivative size = 53, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.400$ , Rules used = {3042, 4259, 3042, 3122}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{1}{\csc^{\frac{2}{3}}(a+bx)} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{1}{\csc(a+bx)^{2/3}} dx \\
 & \quad \downarrow \text{4259} \\
 & \sqrt[3]{\sin(a+bx)} \sqrt[3]{\csc(a+bx)} \int \sin^{\frac{2}{3}}(a+bx) dx \\
 & \quad \downarrow \text{3042} \\
 & \sqrt[3]{\sin(a+bx)} \sqrt[3]{\csc(a+bx)} \int \sin(a+bx)^{2/3} dx \\
 & \quad \downarrow \text{3122} \\
 & \frac{3 \cos(a+bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{5}{6}, \frac{11}{6}, \sin^2(a+bx)\right)}{5b \sqrt{\cos^2(a+bx)} \csc^{\frac{5}{3}}(a+bx)}
 \end{aligned}$$

input `Int[Csc[a + b*x]^(-2/3),x]`

output `(3*Cos[a + b*x]*Hypergeometric2F1[1/2, 5/6, 11/6, Sin[a + b*x]^2])/(5*b*Sqrt[Cos[a + b*x]^2]*Csc[a + b*x]^(5/3))`

**Defintions of rubi rules used**

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3122 `Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] := Simp[Cos[c + d*x]*((b*Sin[c + d*x])^(n + 1)/(b*d*(n + 1)*Sqrt[Cos[c + d*x]^2]))*Hypergeometric2F1[1/2, (n + 1)/2, (n + 3)/2, Sin[c + d*x]^2], x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[2*n]`

rule 4259 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] := Simp[(b*Csc[c + d*x])^(n - 1)*((Sin[c + d*x]/b)^(n - 1) Int[1/(Sin[c + d*x]/b)^n, x]), x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[n]`

**Maple [F]**

$$\int \frac{1}{\csc(bx + a)^{\frac{2}{3}}} dx$$

input `int(1/csc(b*x+a)^(2/3), x)`

output `int(1/csc(b*x+a)^(2/3), x)`

**Fricas [F]**

$$\int \frac{1}{\csc^{\frac{2}{3}}(a + bx)} dx = \int \frac{1}{\csc(bx + a)^{\frac{2}{3}}} dx$$

input `integrate(1/csc(b*x+a)^(2/3), x, algorithm="fricas")`

output `integral(csc(b*x + a)^(-2/3), x)`

**Sympy [F]**

$$\int \frac{1}{\csc^{\frac{2}{3}}(a + bx)} dx = \int \frac{1}{\csc^{\frac{2}{3}}(a + bx)} dx$$

input `integrate(1/csc(b*x+a)**(2/3),x)`

output `Integral(csc(a + b*x)**(-2/3), x)`

**Maxima [F]**

$$\int \frac{1}{\csc^{\frac{2}{3}}(a + bx)} dx = \int \frac{1}{\csc^{\frac{2}{3}}(bx + a)} dx$$

input `integrate(1/csc(b*x+a)^(2/3),x, algorithm="maxima")`

output `integrate(csc(b*x + a)^(-2/3), x)`

**Giac [F]**

$$\int \frac{1}{\csc^{\frac{2}{3}}(a + bx)} dx = \int \frac{1}{\csc^{\frac{2}{3}}(bx + a)} dx$$

input `integrate(1/csc(b*x+a)^(2/3),x, algorithm="giac")`

output `integrate(csc(b*x + a)^(-2/3), x)`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{1}{\csc^{\frac{2}{3}}(a + bx)} dx = \int \frac{1}{\left(\frac{1}{\sin(a+bx)}\right)^{\frac{2}{3}}} dx$$

input `int(1/(1/sin(a + b*x))^(2/3),x)`output `int(1/(1/sin(a + b*x))^(2/3), x)`**Reduce [F]**

$$\int \frac{1}{\csc^{\frac{2}{3}}(a + bx)} dx = \int \frac{1}{\csc(bx + a)^{\frac{2}{3}}} dx$$

input `int(1/csc(b*x+a)^(2/3),x)`output `int(1/csc(a + b*x)**(2/3),x)`

### 3.30 $\int \frac{1}{\csc^{\frac{4}{3}}(a+bx)} dx$

Optimal result	215
Mathematica [A] (verified)	215
Rubi [A] (verified)	216
Maple [F]	217
Fricas [F]	217
Sympy [F]	218
Maxima [F]	218
Giac [F]	218
Mupad [F(-1)]	219
Reduce [F]	219

#### Optimal result

Integrand size = 10, antiderivative size = 53

$$\int \frac{1}{\csc^{\frac{4}{3}}(a+bx)} dx = \frac{3 \cos(a+bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{7}{6}, \frac{13}{6}, \sin^2(a+bx)\right)}{7b \sqrt{\cos^2(a+bx)} \csc^{\frac{7}{3}}(a+bx)}$$

output `3/7*cos(b*x+a)*hypergeom([1/2, 7/6],[13/6],sin(b*x+a)^2)/b/(cos(b*x+a)^2)^(1/2)/csc(b*x+a)^(7/3)`

#### Mathematica [A] (verified)

Time = 0.09 (sec) , antiderivative size = 68, normalized size of antiderivative = 1.28

$$\int \frac{1}{\csc^{\frac{4}{3}}(a+bx)} dx = -\frac{\cos(a+bx) \left( \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{5}{6}, \frac{3}{2}, \cos^2(a+bx)\right) + 3 \sqrt[6]{\sin^2(a+bx)} \right)}{4b \sqrt[3]{\csc(a+bx)} \sqrt[6]{\sin^2(a+bx)}}$$

input `Integrate[Csc[a + b*x]^(-4/3),x]`



output

```
-1/4*(Cos[a + b*x]*(Hypergeometric2F1[1/2, 5/6, 3/2, Cos[a + b*x]^2] + 3*(Sin[a + b*x]^2)^(1/6)))/(b*Csc[a + b*x]^(1/3)*(Sin[a + b*x]^2)^(1/6))
```

**Rubi [A] (verified)**

Time = 0.27 (sec) , antiderivative size = 53, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.400$ , Rules used = {3042, 4259, 3042, 3122}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \frac{1}{\csc^{\frac{4}{3}}(a+bx)} dx \\ & \quad \downarrow \text{3042} \\ & \int \frac{1}{\csc(a+bx)^{\frac{4}{3}}} dx \\ & \quad \downarrow \text{4259} \\ & \sin^{\frac{2}{3}}(a+bx) \csc^{\frac{2}{3}}(a+bx) \int \sin^{\frac{4}{3}}(a+bx) dx \\ & \quad \downarrow \text{3042} \\ & \sin^{\frac{2}{3}}(a+bx) \csc^{\frac{2}{3}}(a+bx) \int \sin(a+bx)^{\frac{4}{3}} dx \\ & \quad \downarrow \text{3122} \\ & \frac{3 \cos(a+bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{7}{6}, \frac{13}{6}, \sin^2(a+bx)\right)}{7b \sqrt{\cos^2(a+bx)} \csc^{\frac{7}{3}}(a+bx)} \end{aligned}$$

input

```
Int[Csc[a + b*x]^(-4/3),x]
```

output

```
(3*Cos[a + b*x]*Hypergeometric2F1[1/2, 7/6, 13/6, Sin[a + b*x]^2])/(7*b*Sqrt[Cos[a + b*x]^2]*Csc[a + b*x]^(7/3))
```

### Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] :=> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3122 `Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] :=> Simp[Cos[c + d*x]*((b*Sin[c + d*x])^(n + 1)/(b*d*(n + 1)*Sqrt[Cos[c + d*x]^2]))*Hypergeometric2F1[1/2, (n + 1)/2, (n + 3)/2, Sin[c + d*x]^2], x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[2*n]`

rule 4259 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] :=> Simp[(b*Csc[c + d*x])^(n - 1)*((Sin[c + d*x]/b)^(n - 1) Int[1/(Sin[c + d*x]/b)^n, x]), x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[n]`

### Maple [F]

$$\int \frac{1}{\csc^{\frac{4}{3}}(bx + a)} dx$$

input `int(1/csc(b*x+a)^(4/3),x)`

output `int(1/csc(b*x+a)^(4/3),x)`

### Fricas [F]

$$\int \frac{1}{\csc^{\frac{4}{3}}(a + bx)} dx = \int \frac{1}{\csc^{\frac{4}{3}}(bx + a)} dx$$

input `integrate(1/csc(b*x+a)^(4/3),x, algorithm="fricas")`

output `integral(csc(b*x + a)^(-4/3), x)`

**Sympy [F]**

$$\int \frac{1}{\csc^{\frac{4}{3}}(a + bx)} dx = \int \frac{1}{\csc^{\frac{4}{3}}(a + bx)} dx$$

input `integrate(1/csc(b*x+a)**(4/3),x)`

output `Integral(csc(a + b*x)**(-4/3), x)`

**Maxima [F]**

$$\int \frac{1}{\csc^{\frac{4}{3}}(a + bx)} dx = \int \frac{1}{\csc^{\frac{4}{3}}(bx + a)} dx$$

input `integrate(1/csc(b*x+a)^(4/3),x, algorithm="maxima")`

output `integrate(csc(b*x + a)^(-4/3), x)`

**Giac [F]**

$$\int \frac{1}{\csc^{\frac{4}{3}}(a + bx)} dx = \int \frac{1}{\csc^{\frac{4}{3}}(bx + a)} dx$$

input `integrate(1/csc(b*x+a)^(4/3),x, algorithm="giac")`

output `integrate(csc(b*x + a)^(-4/3), x)`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{1}{\csc^{\frac{4}{3}}(a + bx)} dx = \int \frac{1}{\left(\frac{1}{\sin(a+bx)}\right)^{\frac{4}{3}}} dx$$

input `int(1/(1/sin(a + b*x))^(4/3),x)`output `int(1/(1/sin(a + b*x))^(4/3), x)`**Reduce [F]**

$$\int \frac{1}{\csc^{\frac{4}{3}}(a + bx)} dx = \int \frac{1}{\csc^{\frac{4}{3}}(bx + a)} dx$$

input `int(1/csc(b*x+a)^(4/3),x)`output `int(1/(csc(a + b*x)**(1/3)*csc(a + b*x)),x)`

### 3.31 $\int (c \csc(a + bx))^{4/3} dx$

Optimal result	220
Mathematica [A] (verified)	220
Rubi [A] (verified)	221
Maple [F]	222
Fricas [F]	222
Sympy [F]	223
Maxima [F]	223
Giac [F]	223
Mupad [F(-1)]	224
Reduce [F]	224

#### Optimal result

Integrand size = 12, antiderivative size = 54

$$\int (c \csc(a + bx))^{4/3} dx = \frac{3c \cos(a + bx) \sqrt[3]{c \csc(a + bx)} \operatorname{Hypergeometric2F1}\left(-\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \sin^2(a + bx)\right)}{b \sqrt{\cos^2(a + bx)}}$$

output

```
-3*c*cos(b*x+a)*(c*csc(b*x+a))^(1/3)*hypergeom([-1/6, 1/2],[5/6],sin(b*x+a)^2)/b/(cos(b*x+a)^2)^(1/2)
```

#### Mathematica [A] (verified)

Time = 0.05 (sec) , antiderivative size = 57, normalized size of antiderivative = 1.06

$$\int (c \csc(a + bx))^{4/3} dx = \frac{c \cos(a + bx) \sqrt[3]{c \csc(a + bx)} \left(-3 + 2 \operatorname{Hypergeometric2F1}\left(\frac{1}{6}, \frac{1}{2}, \frac{3}{2}, \cos^2(a + bx)\right)\right) \sqrt[6]{\sin^2(a + bx)}}{b}$$

input

```
Integrate[(c*Csc[a + b*x])^(4/3),x]
```

output

```
(c*cos[a + b*x]*(c*csc[a + b*x])^(1/3)*(-3 + 2*Hypergeometric2F1[1/6, 1/2,
3/2, Cos[a + b*x]^2]*(Sin[a + b*x]^2)^(1/6)))/b
```

**Rubi [A] (verified)**

Time = 0.25 (sec) , antiderivative size = 54, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.333$ , Rules used = {3042, 4259, 3042, 3122}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int (c \csc(a + bx))^{4/3} dx \\
 & \quad \downarrow \text{3042} \\
 & \int (c \csc(a + bx))^{4/3} dx \\
 & \quad \downarrow \text{4259} \\
 & \sqrt[3]{\frac{\sin(a + bx)}{c}} \sqrt[3]{c \csc(a + bx)} \int \frac{1}{\left(\frac{\sin(a + bx)}{c}\right)^{4/3}} dx \\
 & \quad \downarrow \text{3042} \\
 & \sqrt[3]{\frac{\sin(a + bx)}{c}} \sqrt[3]{c \csc(a + bx)} \int \frac{1}{\left(\frac{\sin(a + bx)}{c}\right)^{4/3}} dx \\
 & \quad \downarrow \text{3122} \\
 & -\frac{3c \cos(a + bx) \sqrt[3]{c \csc(a + bx)} \operatorname{Hypergeometric2F1}\left(-\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \sin^2(a + bx)\right)}{b \sqrt{\cos^2(a + bx)}}
 \end{aligned}$$

input

```
Int[(c*Csc[a + b*x])^(4/3), x]
```

output

```
(-3*c*cos[a + b*x]*(c*csc[a + b*x])^(1/3)*Hypergeometric2F1[-1/6, 1/2, 5/6,
Sin[a + b*x]^2])/(b*Sqrt[Cos[a + b*x]^2])
```

**Defintions of rubi rules used**

rule 3042 `Int[u_, x_Symbol] :=> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3122 `Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] :=> Simp[Cos[c + d*x]*((b*Sin[c + d*x])^(n + 1)/(b*d*(n + 1)*Sqrt[Cos[c + d*x]^2]))*Hypergeometric2F1[1/2, (n + 1)/2, (n + 3)/2, Sin[c + d*x]^2], x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[2*n]`

rule 4259 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] :=> Simp[(b*Csc[c + d*x])^(n - 1)*((Sin[c + d*x]/b)^(n - 1) Int[1/(Sin[c + d*x]/b)^n, x]), x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[n]`

**Maple [F]**

$$\int (c \csc (bx + a))^{\frac{4}{3}} dx$$

input `int((c*csc(b*x+a))^(4/3),x)`

output `int((c*csc(b*x+a))^(4/3),x)`

**Fricas [F]**

$$\int (c \csc (a + bx))^{\frac{4}{3}} dx = \int (c \csc (bx + a))^{\frac{4}{3}} dx$$

input `integrate((c*csc(b*x+a))^(4/3),x, algorithm="fricas")`

output `integral((c*csc(b*x + a))^(1/3)*c*csc(b*x + a), x)`

**Sympy [F]**

$$\int (c \csc(a + bx))^{4/3} dx = \int (c \csc(a + bx))^{\frac{4}{3}} dx$$

input `integrate((c*csc(b*x+a))**(4/3),x)`

output `Integral((c*csc(a + b*x))**(4/3), x)`

**Maxima [F]**

$$\int (c \csc(a + bx))^{4/3} dx = \int (c \csc(bx + a))^{\frac{4}{3}} dx$$

input `integrate((c*csc(b*x+a))^(4/3),x, algorithm="maxima")`

output `integrate((c*csc(b*x + a))^(4/3), x)`

**Giac [F]**

$$\int (c \csc(a + bx))^{4/3} dx = \int (c \csc(bx + a))^{\frac{4}{3}} dx$$

input `integrate((c*csc(b*x+a))^(4/3),x, algorithm="giac")`

output `integrate((c*csc(b*x + a))^(4/3), x)`



**Mupad [F(-1)]**

Timed out.

$$\int (c \csc(a + bx))^{4/3} dx = \int \left( \frac{c}{\sin(a + bx)} \right)^{4/3} dx$$

input `int((c/sin(a + b*x))^(4/3),x)`output `int((c/sin(a + b*x))^(4/3), x)`**Reduce [F]**

$$\int (c \csc(a + bx))^{4/3} dx = c^{4/3} \left( \int \csc(bx + a)^{4/3} dx \right)$$

input `int((c*csc(b*x+a))^(4/3),x)`output `c**(1/3)*int(csc(a + b*x)**(1/3)*csc(a + b*x),x)*c`

### 3.32 $\int (c \csc(a + bx))^{2/3} dx$

Optimal result	225
Mathematica [A] (verified)	225
Rubi [A] (verified)	226
Maple [F]	227
Fricas [F]	227
Sympy [F]	228
Maxima [F]	228
Giac [F]	228
Mupad [F(-1)]	229
Reduce [F]	229

#### Optimal result

Integrand size = 12, antiderivative size = 54

$$\int (c \csc(a + bx))^{2/3} dx = \frac{3c \cos(a + bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{6}, \frac{1}{2}, \frac{7}{6}, \sin^2(a + bx)\right)}{b \sqrt{\cos^2(a + bx)} \sqrt[3]{c \csc(a + bx)}}$$

output `3*c*cos(b*x+a)*hypergeom([1/6, 1/2],[7/6],sin(b*x+a)^2)/b/(cos(b*x+a)^2)^(1/2)/(c*csc(b*x+a))^(1/3)`

#### Mathematica [A] (verified)

Time = 0.07 (sec) , antiderivative size = 59, normalized size of antiderivative = 1.09

$$\int (c \csc(a + bx))^{2/3} dx = \frac{\cos(a + bx) (c \csc(a + bx))^{2/3} \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{5}{6}, \frac{3}{2}, \cos^2(a + bx)\right) \sin(a + bx)}{b \sqrt[6]{\sin^2(a + bx)}}$$

input `Integrate[(c*Csc[a + b*x])^(2/3),x]`

output

```
-((Cos[a + b*x]*(c*Csc[a + b*x])^(2/3)*Hypergeometric2F1[1/2, 5/6, 3/2, Cos[a + b*x]^2]*Sin[a + b*x])/(b*(Sin[a + b*x]^2)^(1/6)))
```

**Rubi [A] (verified)**

Time = 0.28 (sec) , antiderivative size = 54, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.333$ , Rules used = {3042, 4259, 3042, 3122}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int (c \csc(a + bx))^{2/3} dx \\ & \quad \downarrow \text{3042} \\ & \int (c \csc(a + bx))^{2/3} dx \\ & \quad \downarrow \text{4259} \\ & \left(\frac{\sin(a + bx)}{c}\right)^{2/3} (c \csc(a + bx))^{2/3} \int \frac{1}{\left(\frac{\sin(a + bx)}{c}\right)^{2/3}} dx \\ & \quad \downarrow \text{3042} \\ & \left(\frac{\sin(a + bx)}{c}\right)^{2/3} (c \csc(a + bx))^{2/3} \int \frac{1}{\left(\frac{\sin(a + bx)}{c}\right)^{2/3}} dx \\ & \quad \downarrow \text{3122} \\ & \frac{3c \cos(a + bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{6}, \frac{1}{2}, \frac{7}{6}, \sin^2(a + bx)\right)}{b \sqrt{\cos^2(a + bx)} \sqrt[3]{c \csc(a + bx)}} \end{aligned}$$

input

```
Int[(c*Csc[a + b*x])^(2/3),x]
```

output

```
(3*c*cos[a + b*x]*Hypergeometric2F1[1/6, 1/2, 7/6, Sin[a + b*x]^2])/(b*Sqrt[Cos[a + b*x]^2]*(c*Csc[a + b*x])^(1/3))
```

**Defintions of rubi rules used**

rule 3042 `Int[u_, x_Symbol] :=> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3122 `Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] :=> Simp[Cos[c + d*x]*((b*Sin[c + d*x])^(n + 1)/(b*d*(n + 1)*Sqrt[Cos[c + d*x]^2]))*Hypergeometric2F1[1/2, (n + 1)/2, (n + 3)/2, Sin[c + d*x]^2], x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[2*n]`

rule 4259 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] :=> Simp[(b*Csc[c + d*x])^(n - 1)*((Sin[c + d*x]/b)^(n - 1) Int[1/(Sin[c + d*x]/b)^n, x]), x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[n]`

**Maple [F]**

$$\int (c \csc (bx + a))^{\frac{2}{3}} dx$$

input `int((c*csc(b*x+a))^(2/3),x)`

output `int((c*csc(b*x+a))^(2/3),x)`

**Fricas [F]**

$$\int (c \csc (a + bx))^{2/3} dx = \int (c \csc (bx + a))^{\frac{2}{3}} dx$$

input `integrate((c*csc(b*x+a))^(2/3),x, algorithm="fricas")`

output `integral((c*csc(b*x + a))^(2/3), x)`

**Sympy [F]**

$$\int (c \csc(a + bx))^{2/3} dx = \int (c \csc(a + bx))^{\frac{2}{3}} dx$$

input `integrate((c*csc(b*x+a))**(2/3),x)`

output `Integral((c*csc(a + b*x))**(2/3), x)`

**Maxima [F]**

$$\int (c \csc(a + bx))^{2/3} dx = \int (c \csc(bx + a))^{\frac{2}{3}} dx$$

input `integrate((c*csc(b*x+a))^(2/3),x, algorithm="maxima")`

output `integrate((c*csc(b*x + a))^(2/3), x)`

**Giac [F]**

$$\int (c \csc(a + bx))^{2/3} dx = \int (c \csc(bx + a))^{\frac{2}{3}} dx$$

input `integrate((c*csc(b*x+a))^(2/3),x, algorithm="giac")`

output `integrate((c*csc(b*x + a))^(2/3), x)`

**Mupad [F(-1)]**

Timed out.

$$\int (c \csc(a + bx))^{2/3} dx = \int \left( \frac{c}{\sin(a + bx)} \right)^{2/3} dx$$

input `int((c/sin(a + b*x))^(2/3),x)`output `int((c/sin(a + b*x))^(2/3), x)`**Reduce [F]**

$$\int (c \csc(a + bx))^{2/3} dx = c^{2/3} \left( \int \csc(bx + a)^{2/3} dx \right)$$

input `int((c*csc(b*x+a))^(2/3),x)`output `c**(2/3)*int(csc(a + b*x)**(2/3),x)`

### 3.33 $\int \sqrt[3]{c \csc(a + bx)} dx$

Optimal result	230
Mathematica [A] (verified)	230
Rubi [A] (verified)	231
Maple [F]	232
Fricas [F]	232
Sympy [F]	233
Maxima [F]	233
Giac [F]	233
Mupad [F(-1)]	234
Reduce [F]	234

#### Optimal result

Integrand size = 12, antiderivative size = 56

$$\int \sqrt[3]{c \csc(a + bx)} dx = \frac{3c \cos(a + bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{3}, \frac{1}{2}, \frac{4}{3}, \sin^2(a + bx)\right)}{2b \sqrt{\cos^2(a + bx)} (c \csc(a + bx))^{2/3}}$$

output  $3/2*c*cos(b*x+a)*\operatorname{hypergeom}([1/3, 1/2], [4/3], \sin(b*x+a)^2)/b/(\cos(b*x+a)^2)^{(1/2)/(c*\csc(b*x+a))^{(2/3)}}$

#### Mathematica [A] (verified)

Time = 0.06 (sec) , antiderivative size = 59, normalized size of antiderivative = 1.05

$$\int \sqrt[3]{c \csc(a + bx)} dx = -\frac{\cos(a + bx) \sqrt[3]{c \csc(a + bx)} \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{2}, \cos^2(a + bx)\right) \sin(a + bx)}{b \sqrt[3]{\sin^2(a + bx)}}$$

input  $\operatorname{Integrate}[(c*\operatorname{Csc}[a + b*x])^{(1/3)}, x]$

output

```

-((Cos[a + b*x]*(c*Csc[a + b*x])^(1/3)*Hypergeometric2F1[1/2, 2/3, 3/2, Cos[a + b*x]^2]*Sin[a + b*x])/(b*(Sin[a + b*x]^2)^(1/3)))

```

**Rubi [A] (verified)**

Time = 0.27 (sec) , antiderivative size = 56, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.333$ , Rules used = {3042, 4259, 3042, 3122}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
& \int \sqrt[3]{c \csc(a + bx)} dx \\
& \quad \downarrow \text{3042} \\
& \int \sqrt[3]{c \csc(a + bx)} dx \\
& \quad \downarrow \text{4259} \\
& \sqrt[3]{\frac{\sin(a + bx)}{c}} \sqrt[3]{c \csc(a + bx)} \int \frac{1}{\sqrt[3]{\frac{\sin(a + bx)}{c}}} dx \\
& \quad \downarrow \text{3042} \\
& \sqrt[3]{\frac{\sin(a + bx)}{c}} \sqrt[3]{c \csc(a + bx)} \int \frac{1}{\sqrt[3]{\frac{\sin(a + bx)}{c}}} dx \\
& \quad \downarrow \text{3122} \\
& \frac{3c \cos(a + bx) \text{Hypergeometric2F1}\left(\frac{1}{3}, \frac{1}{2}, \frac{4}{3}, \sin^2(a + bx)\right)}{2b \sqrt{\cos^2(a + bx)} (c \csc(a + bx))^{2/3}}
\end{aligned}$$

input

```

Int[(c*Csc[a + b*x])^(1/3),x]

```

output

```

(3*c*cos[a + b*x]*Hypergeometric2F1[1/3, 1/2, 4/3, Sin[a + b*x]^2])/(2*b*Sqrt[Cos[a + b*x]^2]*(c*Csc[a + b*x])^(2/3))

```



**Defintions of rubi rules used**

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3122 `Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] := Simp[Cos[c + d*x]*((b*Sin[c + d*x])^(n + 1)/(b*d*(n + 1)*Sqrt[Cos[c + d*x]^2]))*Hypergeometric2F1[1/2, (n + 1)/2, (n + 3)/2, Sin[c + d*x]^2], x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[2*n]`

rule 4259 `Int[(csc[(c_.) + (d_.)*(x_)])*(b_.)]^(n_), x_Symbol] := Simp[(b*Csc[c + d*x])^(n - 1)*((Sin[c + d*x]/b)^(n - 1) Int[1/(Sin[c + d*x]/b)^n, x]), x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[n]`

**Maple [F]**

$$\int (c \csc (bx + a))^{\frac{1}{3}} dx$$

input `int((c*csc(b*x+a))^(1/3),x)`

output `int((c*csc(b*x+a))^(1/3),x)`

**Fricas [F]**

$$\int \sqrt[3]{c \csc(a + bx)} dx = \int (c \csc (bx + a))^{\frac{1}{3}} dx$$

input `integrate((c*csc(b*x+a))^(1/3),x, algorithm="fricas")`

output `integral((c*csc(b*x + a))^(1/3), x)`

**Sympy [F]**

$$\int \sqrt[3]{c \csc(a + bx)} dx = \int \sqrt[3]{c \csc(a + bx)} dx$$

input `integrate((c*csc(b*x+a))**(1/3),x)`

output `Integral((c*csc(a + b*x))**(1/3), x)`

**Maxima [F]**

$$\int \sqrt[3]{c \csc(a + bx)} dx = \int (c \csc(bx + a))^{\frac{1}{3}} dx$$

input `integrate((c*csc(b*x+a))^(1/3),x, algorithm="maxima")`

output `integrate((c*csc(b*x + a))^(1/3), x)`

**Giac [F]**

$$\int \sqrt[3]{c \csc(a + bx)} dx = \int (c \csc(bx + a))^{\frac{1}{3}} dx$$

input `integrate((c*csc(b*x+a))^(1/3),x, algorithm="giac")`

output `integrate((c*csc(b*x + a))^(1/3), x)`

**Mupad [F(-1)]**

Timed out.

$$\int \sqrt[3]{c \csc(a + bx)} dx = \int \left( \frac{c}{\sin(a + bx)} \right)^{1/3} dx$$

input `int((c/sin(a + b*x))^(1/3),x)`output `int((c/sin(a + b*x))^(1/3), x)`**Reduce [F]**

$$\int \sqrt[3]{c \csc(a + bx)} dx = c^{1/3} \left( \int \csc(bx + a)^{1/3} dx \right)$$

input `int((c*csc(b*x+a))^(1/3),x)`output `c**(1/3)*int(csc(a + b*x)**(1/3),x)`

### 3.34 $\int \frac{1}{\sqrt[3]{c \csc(a + bx)}} dx$

Optimal result	235
Mathematica [A] (verified)	235
Rubi [A] (verified)	236
Maple [F]	237
Fricas [F]	237
Sympy [F]	238
Maxima [F]	238
Giac [F]	238
Mupad [F(-1)]	239
Reduce [F]	239

#### Optimal result

Integrand size = 12, antiderivative size = 56

$$\int \frac{1}{\sqrt[3]{c \csc(a + bx)}} dx = \frac{3c \cos(a + bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{2}{3}, \frac{5}{3}, \sin^2(a + bx)\right)}{4b \sqrt{\cos^2(a + bx)} (c \csc(a + bx))^{4/3}}$$

output

```
3/4*c*cos(b*x+a)*hypergeom([1/2, 2/3], [5/3], sin(b*x+a)^2)/b/(cos(b*x+a)^2)^(1/2)/(c*csc(b*x+a))^(4/3)
```

#### Mathematica [A] (verified)

Time = 0.06 (sec) , antiderivative size = 59, normalized size of antiderivative = 1.05

$$\int \frac{1}{\sqrt[3]{c \csc(a + bx)}} dx = -\frac{\cos(a + bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{3}, \frac{1}{2}, \frac{3}{2}, \cos^2(a + bx)\right) \sin(a + bx)}{b \sqrt[3]{c \csc(a + bx)} \sin^2(a + bx)^{2/3}}$$

input

```
Integrate[(c*Csc[a + b*x])^(-1/3), x]
```

output

```

-((Cos[a + b*x]*Hypergeometric2F1[1/3, 1/2, 3/2, Cos[a + b*x]^2]*Sin[a + b
*x])/((b*(c*Csc[a + b*x])^(1/3)*(Sin[a + b*x]^2)^(2/3)))

```

**Rubi [A] (verified)**

Time = 0.28 (sec) , antiderivative size = 56, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.333$ , Rules used = {3042, 4259, 3042, 3122}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
& \int \frac{1}{\sqrt[3]{c \csc(a + bx)}} dx \\
& \quad \downarrow \text{3042} \\
& \int \frac{1}{\sqrt[3]{c \csc(a + bx)}} dx \\
& \quad \downarrow \text{4259} \\
& \left(\frac{\sin(a + bx)}{c}\right)^{2/3} (c \csc(a + bx))^{2/3} \int \sqrt[3]{\frac{\sin(a + bx)}{c}} dx \\
& \quad \downarrow \text{3042} \\
& \left(\frac{\sin(a + bx)}{c}\right)^{2/3} (c \csc(a + bx))^{2/3} \int \sqrt[3]{\frac{\sin(a + bx)}{c}} dx \\
& \quad \downarrow \text{3122} \\
& \frac{3c \cos(a + bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{2}{3}, \frac{5}{3}, \sin^2(a + bx)\right)}{4b \sqrt{\cos^2(a + bx)} (c \csc(a + bx))^{4/3}}
\end{aligned}$$

input

```

Int[(c*Csc[a + b*x])^(-1/3),x]

```

output

```

(3*c*cos[a + b*x]*Hypergeometric2F1[1/2, 2/3, 5/3, Sin[a + b*x]^2])/((4*b*S
qrt[Cos[a + b*x]^2]*(c*Csc[a + b*x])^(4/3))

```

### Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] :=> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3122 `Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] :=> Simp[Cos[c + d*x]*((b*Sin[c + d*x])^(n + 1)/(b*d*(n + 1)*Sqrt[Cos[c + d*x]^2]))*Hypergeometric2F1[1/2, (n + 1)/2, (n + 3)/2, Sin[c + d*x]^2], x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[2*n]`

rule 4259 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] :=> Simp[(b*Csc[c + d*x])^(n - 1)*((Sin[c + d*x]/b)^(n - 1) Int[1/(Sin[c + d*x]/b)^n, x]), x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[n]`

### Maple [F]

$$\int \frac{1}{(c \csc(bx + a))^{\frac{1}{3}}} dx$$

input `int(1/(c*csc(b*x+a))^(1/3),x)`

output `int(1/(c*csc(b*x+a))^(1/3),x)`

### Fricas [F]

$$\int \frac{1}{\sqrt[3]{c \csc(a + bx)}} dx = \int \frac{1}{(c \csc(bx + a))^{\frac{1}{3}}} dx$$

input `integrate(1/(c*csc(b*x+a))^(1/3),x, algorithm="fricas")`

output `integral((c*csc(b*x + a))^(2/3)/(c*csc(b*x + a)), x)`

**Sympy [F]**

$$\int \frac{1}{\sqrt[3]{c \csc(a + bx)}} dx = \int \frac{1}{\sqrt[3]{c \csc(a + bx)}} dx$$

input `integrate(1/(c*csc(b*x+a))**(1/3),x)`

output `Integral((c*csc(a + b*x))**(-1/3), x)`

**Maxima [F]**

$$\int \frac{1}{\sqrt[3]{c \csc(a + bx)}} dx = \int \frac{1}{(c \csc(bx + a))^{\frac{1}{3}}} dx$$

input `integrate(1/(c*csc(b*x+a))^(1/3),x, algorithm="maxima")`

output `integrate((c*csc(b*x + a))^(1/3), x)`

**Giac [F]**

$$\int \frac{1}{\sqrt[3]{c \csc(a + bx)}} dx = \int \frac{1}{(c \csc(bx + a))^{\frac{1}{3}}} dx$$

input `integrate(1/(c*csc(b*x+a))^(1/3),x, algorithm="giac")`

output `integrate((c*csc(b*x + a))^(1/3), x)`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{1}{\sqrt[3]{c \csc(a + bx)}} dx = \int \frac{1}{\left(\frac{c}{\sin(a+bx)}\right)^{1/3}} dx$$

input `int(1/(c/sin(a + b*x))^(1/3),x)`output `int(1/(c/sin(a + b*x))^(1/3), x)`**Reduce [F]**

$$\int \frac{1}{\sqrt[3]{c \csc(a + bx)}} dx = \frac{\int \frac{1}{\csc(bx+a)^{1/3}} dx}{c^{1/3}}$$

input `int(1/(c*csc(b*x+a))^(1/3),x)`output `int(1/csc(a + b*x)**(1/3),x)/c**(1/3)`



### 3.35 $\int \frac{1}{(c \csc(a+bx))^{2/3}} dx$

Optimal result	240
Mathematica [A] (verified)	240
Rubi [A] (verified)	241
Maple [F]	242
Fricas [F]	242
Sympy [F]	243
Maxima [F]	243
Giac [F]	243
Mupad [F(-1)]	244
Reduce [F]	244

#### Optimal result

Integrand size = 12, antiderivative size = 56

$$\int \frac{1}{(c \csc(a+bx))^{2/3}} dx = \frac{3c \cos(a+bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{5}{6}, \frac{11}{6}, \sin^2(a+bx)\right)}{5b \sqrt{\cos^2(a+bx)} (c \csc(a+bx))^{5/3}}$$

output  $3/5*c*\cos(b*x+a)*\operatorname{hypergeom}([1/2, 5/6], [11/6], \sin(b*x+a)^2)/b/(\cos(b*x+a)^2)^{(1/2)}/(c*\csc(b*x+a))^{(5/3)}$

#### Mathematica [A] (verified)

Time = 0.06 (sec) , antiderivative size = 59, normalized size of antiderivative = 1.05

$$\int \frac{1}{(c \csc(a+bx))^{2/3}} dx = \frac{\cos(a+bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{6}, \frac{1}{2}, \frac{3}{2}, \cos^2(a+bx)\right) \sin(a+bx)}{b(c \csc(a+bx))^{2/3} \sin^2(a+bx)^{5/6}}$$

input  $\operatorname{Integrate}[(c*\operatorname{Csc}[a + b*x])^{(-2/3)}, x]$

output  $-((\operatorname{Cos}[a + b*x]*\operatorname{Hypergeometric2F1}[1/6, 1/2, 3/2, \operatorname{Cos}[a + b*x]^2]*\operatorname{Sin}[a + b*x])/b*(c*\operatorname{Csc}[a + b*x])^{(2/3)}*(\operatorname{Sin}[a + b*x]^2)^{(5/6))})$

**Rubi [A] (verified)**

Time = 0.29 (sec) , antiderivative size = 56, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.333$ , Rules used = {3042, 4259, 3042, 3122}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{1}{(c \csc(a + bx))^{2/3}} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{1}{(c \csc(a + bx))^{2/3}} dx \\
 & \quad \downarrow \text{4259} \\
 & \sqrt[3]{\frac{\sin(a + bx)}{c}} \sqrt[3]{c \csc(a + bx)} \int \left( \frac{\sin(a + bx)}{c} \right)^{2/3} dx \\
 & \quad \downarrow \text{3042} \\
 & \sqrt[3]{\frac{\sin(a + bx)}{c}} \sqrt[3]{c \csc(a + bx)} \int \left( \frac{\sin(a + bx)}{c} \right)^{2/3} dx \\
 & \quad \downarrow \text{3122} \\
 & \frac{3c \cos(a + bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{5}{6}, \frac{11}{6}, \sin^2(a + bx)\right)}{5b \sqrt{\cos^2(a + bx)} (c \csc(a + bx))^{5/3}}
 \end{aligned}$$

input `Int[(c*Csc[a + b*x])^(-2/3),x]`

output `(3*c*Cos[a + b*x]*Hypergeometric2F1[1/2, 5/6, 11/6, Sin[a + b*x]^2])/(5*b*Sqrt[Cos[a + b*x]^2]*(c*Csc[a + b*x])^(5/3))`

### Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3122 `Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] := Simp[Cos[c + d*x]*((b*Sin[c + d*x])^(n + 1)/(b*d*(n + 1)*Sqrt[Cos[c + d*x]^2]))*Hypergeometric2F1[1/2, (n + 1)/2, (n + 3)/2, Sin[c + d*x]^2], x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[2*n]`

rule 4259 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] := Simp[(b*Csc[c + d*x])^(n - 1)*((Sin[c + d*x]/b)^(n - 1) Int[1/(Sin[c + d*x]/b)^n, x]), x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[n]`

### Maple [F]

$$\int \frac{1}{(c \csc(bx + a))^{\frac{2}{3}}} dx$$

input `int(1/(c*csc(b*x+a))^(2/3),x)`

output `int(1/(c*csc(b*x+a))^(2/3),x)`

### Fricas [F]

$$\int \frac{1}{(c \csc(a + bx))^{2/3}} dx = \int \frac{1}{(c \csc(bx + a))^{\frac{2}{3}}} dx$$

input `integrate(1/(c*csc(b*x+a))^(2/3),x, algorithm="fricas")`

output `integral((c*csc(b*x + a))^(1/3)/(c*csc(b*x + a)), x)`

**Sympy [F]**

$$\int \frac{1}{(c \csc(a + bx))^{2/3}} dx = \int \frac{1}{(c \csc(a + bx))^{2/3}} dx$$

input `integrate(1/(c*csc(b*x+a))**(2/3), x)`

output `Integral((c*csc(a + b*x))**(-2/3), x)`

**Maxima [F]**

$$\int \frac{1}{(c \csc(a + bx))^{2/3}} dx = \int \frac{1}{(c \csc(bx + a))^{2/3}} dx$$

input `integrate(1/(c*csc(b*x+a))^(2/3), x, algorithm="maxima")`

output `integrate((c*csc(b*x + a))^(2/3), x)`

**Giac [F]**

$$\int \frac{1}{(c \csc(a + bx))^{2/3}} dx = \int \frac{1}{(c \csc(bx + a))^{2/3}} dx$$

input `integrate(1/(c*csc(b*x+a))^(2/3), x, algorithm="giac")`

output `integrate((c*csc(b*x + a))^(2/3), x)`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{1}{(c \csc(a + bx))^{2/3}} dx = \int \frac{1}{\left(\frac{c}{\sin(a+bx)}\right)^{2/3}} dx$$

input `int(1/(c/sin(a + b*x))^(2/3),x)`output `int(1/(c/sin(a + b*x))^(2/3), x)`**Reduce [F]**

$$\int \frac{1}{(c \csc(a + bx))^{2/3}} dx = \frac{\int \frac{1}{\csc(bx+a)^{2/3}} dx}{c^{2/3}}$$

input `int(1/(c*csc(b*x+a))^(2/3),x)`output `int(1/csc(a + b*x)**(2/3),x)/c**(2/3)`

### 3.36 $\int \frac{1}{(c \csc(a+bx))^{4/3}} dx$

Optimal result	245
Mathematica [A] (verified)	245
Rubi [A] (verified)	246
Maple [F]	247
Fricas [F]	247
Sympy [F]	248
Maxima [F]	248
Giac [F]	248
Mupad [F(-1)]	249
Reduce [F]	249

#### Optimal result

Integrand size = 12, antiderivative size = 56

$$\int \frac{1}{(c \csc(a + bx))^{4/3}} dx = \frac{3c \cos(a + bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{7}{6}, \frac{13}{6}, \sin^2(a + bx)\right)}{7b \sqrt{\cos^2(a + bx)} (c \csc(a + bx))^{7/3}}$$

output

```
3/7*c*cos(b*x+a)*hypergeom([1/2, 7/6],[13/6],sin(b*x+a)^2)/b/(cos(b*x+a)^2)^(1/2)/(c*csc(b*x+a))^(7/3)
```

#### Mathematica [A] (verified)

Time = 0.13 (sec) , antiderivative size = 76, normalized size of antiderivative = 1.36

$$\int \frac{1}{(c \csc(a + bx))^{4/3}} dx = \frac{\csc^2(a + bx) \left( 2 \cot(a + bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{5}{6}, \frac{3}{2}, \cos^2(a + bx)\right) \sin^2(a + bx)^{5/6} + 3 \sin(2(a + bx)) \right)}{8b(c \csc(a + bx))^{4/3}}$$

input

```
Integrate[(c*Csc[a + b*x])^(-4/3),x]
```

output

```
-1/8*(Csc[a + b*x]^2*(2*Cot[a + b*x]*Hypergeometric2F1[1/2, 5/6, 3/2, Cos[
a + b*x]^2]*(Sin[a + b*x]^2)^(5/6) + 3*Sin[2*(a + b*x)]))/(b*(c*Csc[a + b*
x])^(4/3))
```

**Rubi [A] (verified)**

Time = 0.30 (sec) , antiderivative size = 56, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.333$ , Rules used = {3042, 4259, 3042, 3122}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{1}{(c \csc(a + bx))^{4/3}} dx$$

↓ 3042

$$\int \frac{1}{(c \csc(a + bx))^{4/3}} dx$$

↓ 4259

$$\left(\frac{\sin(a + bx)}{c}\right)^{2/3} (c \csc(a + bx))^{2/3} \int \left(\frac{\sin(a + bx)}{c}\right)^{4/3} dx$$

↓ 3042

$$\left(\frac{\sin(a + bx)}{c}\right)^{2/3} (c \csc(a + bx))^{2/3} \int \left(\frac{\sin(a + bx)}{c}\right)^{4/3} dx$$

↓ 3122

$$\frac{3c \cos(a + bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{7}{6}, \frac{13}{6}, \sin^2(a + bx)\right)}{7b \sqrt{\cos^2(a + bx)} (c \csc(a + bx))^{7/3}}$$

input

```
Int[(c*Csc[a + b*x])^(-4/3),x]
```

output

```
(3*c*Cos[a + b*x]*Hypergeometric2F1[1/2, 7/6, 13/6, Sin[a + b*x]^2])/(7*b*
Sqrt[Cos[a + b*x]^2]*(c*Csc[a + b*x])^(7/3))
```

**Defintions of rubi rules used**

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3122 `Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] := Simp[Cos[c + d*x]*((b*Sin[c + d*x])^(n + 1)/(b*d*(n + 1)*Sqrt[Cos[c + d*x]^2]))*Hypergeometric2F1[1/2, (n + 1)/2, (n + 3)/2, Sin[c + d*x]^2], x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[2*n]`

rule 4259 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] := Simp[(b*Csc[c + d*x])^(n - 1)*((Sin[c + d*x]/b)^(n - 1) Int[1/(Sin[c + d*x]/b)^n, x]), x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[n]`

**Maple [F]**

$$\int \frac{1}{(c \csc(bx + a))^{\frac{4}{3}}} dx$$

input `int(1/(c*csc(b*x+a))^(4/3),x)`

output `int(1/(c*csc(b*x+a))^(4/3),x)`

**Fricas [F]**

$$\int \frac{1}{(c \csc(a + bx))^{4/3}} dx = \int \frac{1}{(c \csc(bx + a))^{\frac{4}{3}}} dx$$

input `integrate(1/(c*csc(b*x+a))^(4/3),x, algorithm="fricas")`

output `integral((c*csc(b*x + a))^(2/3)/(c^2*csc(b*x + a)^2), x)`



**Sympy [F]**

$$\int \frac{1}{(c \csc(a + bx))^{4/3}} dx = \int \frac{1}{(c \csc(a + bx))^{4/3}} dx$$

input `integrate(1/(c*csc(b*x+a))**(4/3), x)`

output `Integral((c*csc(a + b*x))**(-4/3), x)`

**Maxima [F]**

$$\int \frac{1}{(c \csc(a + bx))^{4/3}} dx = \int \frac{1}{(c \csc(bx + a))^{4/3}} dx$$

input `integrate(1/(c*csc(b*x+a))^(4/3), x, algorithm="maxima")`

output `integrate((c*csc(b*x + a))^(4/3), x)`

**Giac [F]**

$$\int \frac{1}{(c \csc(a + bx))^{4/3}} dx = \int \frac{1}{(c \csc(bx + a))^{4/3}} dx$$

input `integrate(1/(c*csc(b*x+a))^(4/3), x, algorithm="giac")`

output `integrate((c*csc(b*x + a))^(4/3), x)`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{1}{(c \csc(a + bx))^{4/3}} dx = \int \frac{1}{\left(\frac{c}{\sin(a+bx)}\right)^{4/3}} dx$$

input `int(1/(c/sin(a + b*x))^(4/3),x)`output `int(1/(c/sin(a + b*x))^(4/3), x)`**Reduce [F]**

$$\int \frac{1}{(c \csc(a + bx))^{4/3}} dx = \frac{\int \frac{1}{\csc(bx+a)^{4/3}} dx}{c^{4/3}}$$

input `int(1/(c*csc(b*x+a))^(4/3),x)`output `int(1/(csc(a + b*x)**(1/3)*csc(a + b*x)),x)/(c**(1/3)*c)`

### 3.37 $\int \csc^n(a + bx) dx$

Optimal result	250
Mathematica [A] (verified)	250
Rubi [A] (verified)	251
Maple [F]	252
Fricas [F]	252
Sympy [F]	253
Maxima [F]	253
Giac [F]	253
Mupad [F(-1)]	254
Reduce [F]	254

#### Optimal result

Integrand size = 8, antiderivative size = 69

$$\int \csc^n(a + bx) dx = \frac{\cos(a + bx) \csc^{-1+n}(a + bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{1-n}{2}, \frac{3-n}{2}, \sin^2(a + bx)\right)}{b(1-n)\sqrt{\cos^2(a + bx)}}$$

output

`cos(b*x+a)*csc(b*x+a)^(-1+n)*hypergeom([1/2, 1/2-1/2*n],[3/2-1/2*n],sin(b*x+a)^2)/b/(1-n)/(cos(b*x+a)^2)^(1/2)`

#### Mathematica [A] (verified)

Time = 0.09 (sec) , antiderivative size = 59, normalized size of antiderivative = 0.86

$$\int \csc^n(a + bx) dx = \frac{\cos(a + bx) \csc^{-1+n}(a + bx) \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{1+n}{2}, \frac{3}{2}, \cos^2(a + bx)\right) \sin^2(a + bx)^{\frac{1}{2}(-1+n)}}{b}$$

input

`Integrate[Csc[a + b*x]^n,x]`

output

```

-((Cos[a + b*x]*Csc[a + b*x]^(-1 + n)*Hypergeometric2F1[1/2, (1 + n)/2, 3/
2, Cos[a + b*x]^2]*(Sin[a + b*x]^2)^((-1 + n)/2))/b)

```

**Rubi [A] (verified)**

Time = 0.27 (sec) , antiderivative size = 69, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {3042, 4259, 3042, 3122}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
& \int \csc^n(a + bx) dx \\
& \quad \downarrow \text{3042} \\
& \int \csc(a + bx)^n dx \\
& \quad \downarrow \text{4259} \\
& \sin^n(a + bx) \csc^n(a + bx) \int \sin^{-n}(a + bx) dx \\
& \quad \downarrow \text{3042} \\
& \sin^n(a + bx) \csc^n(a + bx) \int \sin(a + bx)^{-n} dx \\
& \quad \downarrow \text{3122} \\
& \frac{\cos(a + bx) \csc^{n-1}(a + bx) \text{Hypergeometric2F1}\left(\frac{1}{2}, \frac{1-n}{2}, \frac{3-n}{2}, \sin^2(a + bx)\right)}{b(1-n)\sqrt{\cos^2(a + bx)}}
\end{aligned}$$

input

```
Int[Csc[a + b*x]^n,x]
```

output

```

(Cos[a + b*x]*Csc[a + b*x]^(-1 + n)*Hypergeometric2F1[1/2, (1 - n)/2, (3 -
n)/2, Sin[a + b*x]^2])/(b*(1 - n)*Sqrt[Cos[a + b*x]^2])

```

**Defintions of rubi rules used**

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3122 `Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] := Simp[Cos[c + d*x]*((b*Sin[c + d*x])^(n + 1)/(b*d*(n + 1)*Sqrt[Cos[c + d*x]^2]))*Hypergeometric2F1[1/2, (n + 1)/2, (n + 3)/2, Sin[c + d*x]^2], x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[2*n]`

rule 4259 `Int[(csc[(c_.) + (d_.)*(x_)])*(b_.)]^(n_), x_Symbol] := Simp[(b*Csc[c + d*x])^(n - 1)*((Sin[c + d*x]/b)^(n - 1) Int[1/(Sin[c + d*x]/b)^n, x]), x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[n]`

**Maple [F]**

$$\int \csc (bx + a)^n dx$$

input `int(csc(b*x+a)^n,x)`

output `int(csc(b*x+a)^n,x)`

**Fricas [F]**

$$\int \csc^n(a + bx) dx = \int \csc (bx + a)^n dx$$

input `integrate(csc(b*x+a)^n,x, algorithm="fricas")`

output `integral(csc(b*x + a)^n, x)`

**Sympy [F]**

$$\int \csc^n(a + bx) dx = \int \csc^n(a + bx) dx$$

input `integrate(csc(b*x+a)**n,x)`

output `Integral(csc(a + b*x)**n, x)`

**Maxima [F]**

$$\int \csc^n(a + bx) dx = \int \csc(bx + a)^n dx$$

input `integrate(csc(b*x+a)^n,x, algorithm="maxima")`

output `integrate(csc(b*x + a)^n, x)`

**Giac [F]**

$$\int \csc^n(a + bx) dx = \int \csc(bx + a)^n dx$$

input `integrate(csc(b*x+a)^n,x, algorithm="giac")`

output `integrate(csc(b*x + a)^n, x)`

**Mupad [F(-1)]**

Timed out.

$$\int \csc^n(a + bx) dx = \int \left( \frac{1}{\sin(a + bx)} \right)^n dx$$

input `int((1/sin(a + b*x))^n,x)`output `int((1/sin(a + b*x))^n, x)`**Reduce [F]**

$$\int \csc^n(a + bx) dx = \int \csc(bx + a)^n dx$$

input `int(csc(b*x+a)^n,x)`output `int(csc(a + b*x)**n,x)`

### 3.38 $\int (c \csc(a + bx))^n dx$

Optimal result	255
Mathematica [A] (verified)	255
Rubi [A] (verified)	256
Maple [F]	257
Fricas [F]	257
Sympy [F]	258
Maxima [F]	258
Giac [F]	258
Mupad [F(-1)]	259
Reduce [F]	259

#### Optimal result

Integrand size = 10, antiderivative size = 72

$$\int (c \csc(a + bx))^n dx = \frac{c \cos(a + bx)(c \csc(a + bx))^{-1+n} \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{1-n}{2}, \frac{3-n}{2}, \sin^2(a + bx)\right)}{b(1-n)\sqrt{\cos^2(a + bx)}}$$

output

```
c*cos(b*x+a)*(c*csc(b*x+a))^(n-1)*hypergeom([1/2, 1/2-1/2*n], [3/2-1/2*n], sin(b*x+a)^2)/b/(1-n)/(cos(b*x+a)^2)^(1/2)
```

#### Mathematica [A] (verified)

Time = 0.09 (sec) , antiderivative size = 65, normalized size of antiderivative = 0.90

$$\int (c \csc(a + bx))^n dx = \frac{\cos(a + bx)(c \csc(a + bx))^n \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{1+n}{2}, \frac{3}{2}, \cos^2(a + bx)\right) \sin(a + bx) \sin^2(a + bx)^{\frac{1}{2}(n-1)}}{b}$$

input

```
Integrate[(c*Csc[a + b*x])^n,x]
```



output

```

-((Cos[a + b*x]*(c*Csc[a + b*x])^n*Hypergeometric2F1[1/2, (1 + n)/2, 3/2,
Cos[a + b*x]^2]*Sin[a + b*x]*(Sin[a + b*x]^2)^((-1 + n)/2))/b)

```

**Rubi [A] (verified)**

Time = 0.28 (sec) , antiderivative size = 72, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.400$ , Rules used = {3042, 4259, 3042, 3122}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
& \int (c \csc(a + bx))^n dx \\
& \quad \downarrow \text{3042} \\
& \int (c \csc(a + bx))^n dx \\
& \quad \downarrow \text{4259} \\
& \left(\frac{\sin(a + bx)}{c}\right)^n (c \csc(a + bx))^n \int \left(\frac{\sin(a + bx)}{c}\right)^{-n} dx \\
& \quad \downarrow \text{3042} \\
& \left(\frac{\sin(a + bx)}{c}\right)^n (c \csc(a + bx))^n \int \left(\frac{\sin(a + bx)}{c}\right)^{-n} dx \\
& \quad \downarrow \text{3122} \\
& \frac{c \cos(a + bx)(c \csc(a + bx))^{n-1} \text{Hypergeometric2F1}\left(\frac{1}{2}, \frac{1-n}{2}, \frac{3-n}{2}, \sin^2(a + bx)\right)}{b(1-n)\sqrt{\cos^2(a + bx)}}
\end{aligned}$$

input

```

Int[(c*Csc[a + b*x])^n,x]

```

output

```

(c*Cos[a + b*x]*(c*Csc[a + b*x])^(-1 + n)*Hypergeometric2F1[1/2, (1 - n)/2,
(3 - n)/2, Sin[a + b*x]^2])/(b*(1 - n)*Sqrt[Cos[a + b*x]^2])

```

### Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] :=> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3122 `Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] :=> Simp[Cos[c + d*x]*((b*Sin[c + d*x])^(n + 1)/(b*d*(n + 1)*Sqrt[Cos[c + d*x]^2]))*Hypergeometric2F1[1/2, (n + 1)/2, (n + 3)/2, Sin[c + d*x]^2], x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[2*n]`

rule 4259 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] :=> Simp[(b*Csc[c + d*x])^(n - 1)*((Sin[c + d*x]/b)^(n - 1) Int[1/(Sin[c + d*x]/b)^n, x]), x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[n]`

### Maple [F]

$$\int (c \csc (bx + a))^n dx$$

input `int((c*csc(b*x+a))^n,x)`

output `int((c*csc(b*x+a))^n,x)`

### Fricas [F]

$$\int (c \csc (a + bx))^n dx = \int (c \csc (bx + a))^n dx$$

input `integrate((c*csc(b*x+a))^n,x, algorithm="fricas")`

output `integral((c*csc(b*x + a))^n, x)`

**Sympy [F]**

$$\int (c \csc(a + bx))^n dx = \int (c \csc(a + bx))^n dx$$

input `integrate((c*csc(b*x+a))**n,x)`

output `Integral((c*csc(a + b*x))**n, x)`

**Maxima [F]**

$$\int (c \csc(a + bx))^n dx = \int (c \csc(bx + a))^n dx$$

input `integrate((c*csc(b*x+a))^n,x, algorithm="maxima")`

output `integrate((c*csc(b*x + a))^n, x)`

**Giac [F]**

$$\int (c \csc(a + bx))^n dx = \int (c \csc(bx + a))^n dx$$

input `integrate((c*csc(b*x+a))^n,x, algorithm="giac")`

output `integrate((c*csc(b*x + a))^n, x)`

**Mupad [F(-1)]**

Timed out.

$$\int (c \csc(a + bx))^n dx = \int \left( \frac{c}{\sin(a + bx)} \right)^n dx$$

input `int((c/sin(a + b*x))^n,x)`output `int((c/sin(a + b*x))^n, x)`**Reduce [F]**

$$\int (c \csc(a + bx))^n dx = c^n \left( \int \csc(bx + a)^n dx \right)$$

input `int((c*csc(b*x+a))^n,x)`output `c**n*int(csc(a + b*x)**n,x)`

### 3.39 $\int \csc^2(x)^{7/2} dx$

Optimal result . . . . .	260
Mathematica [A] (verified) . . . . .	260
Rubi [A] (verified) . . . . .	261
Maple [C] (warning: unable to verify) . . . . .	263
Fricas [B] (verification not implemented) . . . . .	263
Sympy [F(-1)] . . . . .	264
Maxima [B] (verification not implemented) . . . . .	264
Giac [B] (verification not implemented) . . . . .	265
Mupad [F(-1)] . . . . .	266
Reduce [B] (verification not implemented) . . . . .	266

#### Optimal result

Integrand size = 8, antiderivative size = 50

$$\int \csc^2(x)^{7/2} dx = -\frac{5}{16} \operatorname{arcsinh}(\cot(x)) - \frac{5}{16} \cot(x) \sqrt{\csc^2(x)} - \frac{5}{24} \cot(x) \csc^2(x)^{3/2} - \frac{1}{6} \cot(x) \csc^2(x)^{5/2}$$

output `-5/16*arcsinh(cot(x))-5/16*cot(x)*(csc(x)^2)^(1/2)-5/24*cot(x)*(csc(x)^2)^(3/2)-1/6*cot(x)*(csc(x)^2)^(5/2)`

#### Mathematica [A] (verified)

Time = 0.20 (sec) , antiderivative size = 92, normalized size of antiderivative = 1.84

$$\int \csc^2(x)^{7/2} dx = \frac{1}{384} \sqrt{\csc^2(x)} \left( -30 \csc^2\left(\frac{x}{2}\right) - 6 \csc^4\left(\frac{x}{2}\right) - \csc^6\left(\frac{x}{2}\right) - 120 \left( \log\left(\cos\left(\frac{x}{2}\right)\right) - \log\left(\sin\left(\frac{x}{2}\right)\right) \right) + 30 \sec^2\left(\frac{x}{2}\right) + 6 \sec^4\left(\frac{x}{2}\right) + \sec^6\left(\frac{x}{2}\right) \right) \sin(x)$$

input `Integrate[(Csc[x]^2)^(7/2),x]`

output

```
(Sqrt[Csc[x]^2]*(-30*Csc[x/2]^2 - 6*Csc[x/2]^4 - Csc[x/2]^6 - 120*(Log[Cos[x/2]] - Log[Sin[x/2]])) + 30*Sec[x/2]^2 + 6*Sec[x/2]^4 + Sec[x/2]^6)*Sin[x])/384
```

**Rubi [A] (verified)**

Time = 0.21 (sec) , antiderivative size = 66, normalized size of antiderivative = 1.32, number of steps used = 7, number of rules used = 6,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.750$ , Rules used = {3042, 4610, 211, 211, 211, 222}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \csc^2(x)^{7/2} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \left( \sec \left( x + \frac{\pi}{2} \right)^2 \right)^{7/2} dx \\
 & \quad \downarrow \text{4610} \\
 & - \int (\cot^2(x) + 1)^{5/2} d \cot(x) \\
 & \quad \downarrow \text{211} \\
 & -\frac{5}{6} \int (\cot^2(x) + 1)^{3/2} d \cot(x) - \frac{1}{6} \cot(x) (\cot^2(x) + 1)^{5/2} \\
 & \quad \downarrow \text{211} \\
 & -\frac{5}{6} \left( \frac{3}{4} \int \sqrt{\cot^2(x) + 1} d \cot(x) + \frac{1}{4} \cot(x) (\cot^2(x) + 1)^{3/2} \right) - \frac{1}{6} \cot(x) (\cot^2(x) + 1)^{5/2} \\
 & \quad \downarrow \text{211} \\
 & -\frac{5}{6} \left( \frac{3}{4} \left( \frac{1}{2} \int \frac{1}{\sqrt{\cot^2(x) + 1}} d \cot(x) + \frac{1}{2} \sqrt{\cot^2(x) + 1} \cot(x) \right) + \frac{1}{4} \cot(x) (\cot^2(x) + 1)^{3/2} \right) - \\
 & \quad \frac{1}{6} \cot(x) (\cot^2(x) + 1)^{5/2} \\
 & \quad \downarrow \text{222}
 \end{aligned}$$

$$-\frac{5}{6} \left( \frac{3}{4} \left( \frac{1}{2} \operatorname{arcsinh}(\cot(x)) + \frac{1}{2} \cot(x) \sqrt{\cot^2(x) + 1} \right) + \frac{1}{4} \cot(x) (\cot^2(x) + 1)^{3/2} \right) - \frac{1}{6} \cot(x) (\cot^2(x) + 1)^{5/2}$$

input `Int[(Csc[x]^2)^(7/2), x]`

output `-1/6*(Cot[x]*(1 + Cot[x]^2)^(5/2)) - (5*((Cot[x]*(1 + Cot[x]^2)^(3/2))/4 + (3*(ArcSinh[Cot[x]]/2 + (Cot[x]*Sqrt[1 + Cot[x]^2])/2))/4))/6`

### Defintions of rubi rules used

rule 211 `Int[((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] := Simp[x*((a + b*x^2)^p/(2*p + 1)), x] + Simp[2*a*(p/(2*p + 1)) Int[(a + b*x^2)^(p - 1), x], x] /; FreeQ[{a, b}, x] && GtQ[p, 0] && (IntegerQ[4*p] || IntegerQ[6*p])`

rule 222 `Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] := Simp[ArcSinh[Rt[b, 2]*(x/Sqrt[a])]/Rt[b, 2], x] /; FreeQ[{a, b}, x] && GtQ[a, 0] && PosQ[b]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 4610 `Int[((b_.)*sec[(e_.) + (f_.)*(x_)]^2)^(p_), x_Symbol] := With[{ff = FreeFactors[Tan[e + f*x], x]}, Simp[b*(ff/f) Subst[Int[(b + b*ff^2*x^2)^(p - 1), x], x, Tan[e + f*x]/ff], x] /; FreeQ[{b, e, f, p}, x] && !IntegerQ[p]`

**Maple [C] (warning: unable to verify)**

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.32 (sec) , antiderivative size = 44, normalized size of antiderivative = 0.88

method	result
default	$-\frac{\sqrt{4}(-15 \ln(-\cot(x)+\csc(x)) \sin(x)^6+15 \cos(x)^5-40 \cos(x)^3+33 \cos(x)) \operatorname{csgn}(\csc(x)) \csc(x)^6}{96}$
risch	$-\frac{i \sqrt{-\frac{e^{2ix}}{(e^{2ix}-1)^2}} (15 e^{10ix}-85 e^{8ix}+198 e^{6ix}+198 e^{4ix}-85 e^{2ix}+15)}{24(e^{2ix}-1)^5} + \frac{5 \sqrt{-\frac{e^{2ix}}{(e^{2ix}-1)^2}} \ln(e^{ix}-1) \sin(x)}{8} - \frac{5 \sqrt{-\frac{e^{2ix}}{(e^{2ix}-1)^2}}}{8}$

input `int((csc(x)^2)^(7/2),x,method=_RETURNVERBOSE)`

output `-1/96*4^(1/2)*(-15*ln(-cot(x)+csc(x))*sin(x)^6+15*cos(x)^5-40*cos(x)^3+33*cos(x))*csgn(csc(x))*csc(x)^6`

**Fricas [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 93 vs. 2(36) = 72.

Time = 0.07 (sec) , antiderivative size = 93, normalized size of antiderivative = 1.86

$$\int \csc^2(x)^{7/2} dx = \frac{30 \cos(x)^5 - 80 \cos(x)^3 - 15 (\cos(x)^6 - 3 \cos(x)^4 + 3 \cos(x)^2 - 1) \log\left(\frac{1}{2} \cos(x) + \frac{1}{2}\right) - 15 (\cos(x)^6 - 3 \cos(x)^4 + 3 \cos(x)^2 - 1) \log\left(-\frac{1}{2} \cos(x) + \frac{1}{2}\right) + 66 \cos(x)}{96 (\cos(x)^6 - 3 \cos(x)^4 + 3 \cos(x)^2 - 1)}$$

input `integrate((csc(x)^2)^(7/2),x, algorithm="fricas")`

output `1/96*(30*cos(x)^5 - 80*cos(x)^3 - 15*(cos(x)^6 - 3*cos(x)^4 + 3*cos(x)^2 - 1)*log(1/2*cos(x) + 1/2) + 15*(cos(x)^6 - 3*cos(x)^4 + 3*cos(x)^2 - 1)*log(-1/2*cos(x) + 1/2) + 66*cos(x))/(cos(x)^6 - 3*cos(x)^4 + 3*cos(x)^2 - 1)`



**Sympy [F(-1)]**

Timed out.

$$\int \csc^2(x)^{7/2} dx = \text{Timed out}$$

input `integrate((csc(x)**2)**(7/2),x)`output `Timed out`**Maxima [B] (verification not implemented)**Leaf count of result is larger than twice the leaf count of optimal. 1669 vs.  $2(36) = 72$ .

Time = 0.20 (sec) , antiderivative size = 1669, normalized size of antiderivative = 33.38

$$\int \csc^2(x)^{7/2} dx = \text{Too large to display}$$

input `integrate((csc(x)^2)^(7/2),x, algorithm="maxima")`

output

```

-1/96*(4*(15*cos(11*x) - 85*cos(9*x) + 198*cos(7*x) + 198*cos(5*x) - 85*cos(3*x) + 15*cos(x))*cos(12*x) - 60*(6*cos(10*x) - 15*cos(8*x) + 20*cos(6*x) - 15*cos(4*x) + 6*cos(2*x) - 1)*cos(11*x) + 24*(85*cos(9*x) - 198*cos(7*x) - 198*cos(5*x) + 85*cos(3*x) - 15*cos(x))*cos(10*x) - 340*(15*cos(8*x) - 20*cos(6*x) + 15*cos(4*x) - 6*cos(2*x) + 1)*cos(9*x) + 60*(198*cos(7*x) + 198*cos(5*x) - 85*cos(3*x) + 15*cos(x))*cos(8*x) - 792*(20*cos(6*x) - 15*cos(4*x) + 6*cos(2*x) - 1)*cos(7*x) - 80*(198*cos(5*x) - 85*cos(3*x) + 15*cos(x))*cos(6*x) + 792*(15*cos(4*x) - 6*cos(2*x) + 1)*cos(5*x) - 300*(17*cos(3*x) - 3*cos(x))*cos(4*x) + 340*(6*cos(2*x) - 1)*cos(3*x) - 360*cos(2*x)*cos(x) + 15*(2*(6*cos(10*x) - 15*cos(8*x) + 20*cos(6*x) - 15*cos(4*x) + 6*cos(2*x) - 1)*cos(12*x) - cos(12*x)^2 + 12*(15*cos(8*x) - 20*cos(6*x) + 15*cos(4*x) - 6*cos(2*x) + 1)*cos(10*x) - 36*cos(10*x)^2 + 30*(20*cos(6*x) - 15*cos(4*x) + 6*cos(2*x) - 1)*cos(8*x) - 225*cos(8*x)^2 + 40*(15*cos(4*x) - 6*cos(2*x) + 1)*cos(6*x) - 400*cos(6*x)^2 + 30*(6*cos(2*x) - 1)*cos(4*x) - 225*cos(4*x)^2 - 36*cos(2*x)^2 + 2*(6*sin(10*x) - 15*sin(8*x) + 20*sin(6*x) - 15*sin(4*x) + 6*sin(2*x))*sin(12*x) - sin(12*x)^2 + 12*(15*sin(8*x) - 20*sin(6*x) + 15*sin(4*x) - 6*sin(2*x))*sin(10*x) - 36*sin(10*x)^2 + 30*(20*sin(6*x) - 15*sin(4*x) + 6*sin(2*x))*sin(8*x) - 225*sin(8*x)^2 + 120*(5*sin(4*x) - 2*sin(2*x))*sin(6*x) - 400*sin(6*x)^2 - 225*sin(4*x)^2 + 180*sin(4*x)*sin(2*x) - 36*sin(2*x)^2 + 12*cos(2*x) - 1)*log(cos(x)^2 ...

```

### Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 129 vs.  $2(36) = 72$ .

Time = 0.16 (sec) , antiderivative size = 129, normalized size of antiderivative = 2.58

$$\int \csc^2(x)^{7/2} dx = -\frac{\frac{45(\cos(x)-1)}{\cos(x)+1} - \frac{9(\cos(x)-1)^2}{(\cos(x)+1)^2} + \frac{(\cos(x)-1)^3}{(\cos(x)+1)^3}}{384 \operatorname{sgn}(\sin(x))} - \frac{\left(\frac{9(\cos(x)-1)}{\cos(x)+1} - \frac{45(\cos(x)-1)^2}{(\cos(x)+1)^2} + \frac{110(\cos(x)-1)^3}{(\cos(x)+1)^3} - 1\right)(\cos(x)+1)^3}{384(\cos(x)-1)^3 \operatorname{sgn}(\sin(x))} + \frac{5 \log\left(-\frac{\cos(x)-1}{\cos(x)+1}\right)}{32 \operatorname{sgn}(\sin(x))}$$

input

```
integrate((csc(x)^2)^(7/2),x, algorithm="giac")
```

output

```
-1/384*(45*(cos(x) - 1)/(cos(x) + 1) - 9*(cos(x) - 1)^2/(cos(x) + 1)^2 + (
cos(x) - 1)^3/(cos(x) + 1)^3)/sgn(sin(x)) - 1/384*(9*(cos(x) - 1)/(cos(x)
+ 1) - 45*(cos(x) - 1)^2/(cos(x) + 1)^2 + 110*(cos(x) - 1)^3/(cos(x) + 1)^
3 - 1)*(cos(x) + 1)^3/((cos(x) - 1)^3*sgn(sin(x))) + 5/32*log(-(cos(x) - 1
)/(cos(x) + 1))/sgn(sin(x))
```

**Mupad [F(-1)]**

Timed out.

$$\int \csc^2(x)^{7/2} dx = \int \left( \frac{1}{\sin(x)^2} \right)^{7/2} dx$$

input

```
int((1/sin(x)^2)^(7/2),x)
```

output

```
int((1/sin(x)^2)^(7/2), x)
```

**Reduce [B] (verification not implemented)**

Time = 0.20 (sec) , antiderivative size = 38, normalized size of antiderivative = 0.76

$$\int \csc^2(x)^{7/2} dx = \frac{-15 \cos(x) \sin(x)^4 - 10 \cos(x) \sin(x)^2 - 8 \cos(x) + 15 \log\left(\tan\left(\frac{x}{2}\right)\right) \sin(x)^6}{48 \sin(x)^6}$$

input

```
int((csc(x)^2)^(7/2),x)
```

output

```
( - 15*cos(x)*sin(x)**4 - 10*cos(x)*sin(x)**2 - 8*cos(x) + 15*log(tan(x/2)
)*sin(x)**6)/(48*sin(x)**6)
```

### 3.40 $\int \csc^2(x)^{5/2} dx$

Optimal result	267
Mathematica [A] (verified)	267
Rubi [A] (verified)	268
Maple [C] (warning: unable to verify)	269
Fricas [B] (verification not implemented)	270
Sympy [F]	270
Maxima [B] (verification not implemented)	270
Giac [B] (verification not implemented)	271
Mupad [F(-1)]	272
Reduce [B] (verification not implemented)	272

#### Optimal result

Integrand size = 8, antiderivative size = 36

$$\int \csc^2(x)^{5/2} dx = -\frac{3}{8} \operatorname{arcsinh}(\cot(x)) - \frac{3}{8} \cot(x) \sqrt{\csc^2(x)} - \frac{1}{4} \cot(x) \csc^2(x)^{3/2}$$

output

$$-3/8*\operatorname{arcsinh}(\cot(x))-3/8*\cot(x)*(csc(x)^2)^{(1/2)}-1/4*\cot(x)*(csc(x)^2)^{(3/2)}$$

#### Mathematica [A] (verified)

Time = 0.20 (sec) , antiderivative size = 72, normalized size of antiderivative = 2.00

$$\int \csc^2(x)^{5/2} dx = \frac{1}{64} \sqrt{\csc^2(x)} \left( -6 \csc^2\left(\frac{x}{2}\right) - \csc^4\left(\frac{x}{2}\right) \right) + 24 \left( -\log\left(\cos\left(\frac{x}{2}\right)\right) + \log\left(\sin\left(\frac{x}{2}\right)\right) \right) + 6 \sec^2\left(\frac{x}{2}\right) + \sec^4\left(\frac{x}{2}\right) \sin(x)$$

input

$$\text{Integrate}[(\text{Csc}[x]^2)^{(5/2)}, x]$$

output

$$(\text{Sqrt}[\text{Csc}[x]^2]*(-6*\text{Csc}[x/2]^2 - \text{Csc}[x/2]^4 + 24*(-\text{Log}[\text{Cos}[x/2]] + \text{Log}[\text{Sin}[x/2]]) + 6*\text{Sec}[x/2]^2 + \text{Sec}[x/2]^4)*\text{Sin}[x])/64$$

**Rubi [A] (verified)**

Time = 0.19 (sec) , antiderivative size = 45, normalized size of antiderivative = 1.25, number of steps used = 6, number of rules used = 5,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.625$ , Rules used = {3042, 4610, 211, 211, 222}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \csc^2(x)^{5/2} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \left( \sec \left( x + \frac{\pi}{2} \right)^2 \right)^{5/2} dx \\
 & \quad \downarrow \text{4610} \\
 & - \int (\cot^2(x) + 1)^{3/2} d \cot(x) \\
 & \quad \downarrow \text{211} \\
 & -\frac{3}{4} \int \sqrt{\cot^2(x) + 1} d \cot(x) - \frac{1}{4} \cot(x) (\cot^2(x) + 1)^{3/2} \\
 & \quad \downarrow \text{211} \\
 & -\frac{3}{4} \left( \frac{1}{2} \int \frac{1}{\sqrt{\cot^2(x) + 1}} d \cot(x) + \frac{1}{2} \sqrt{\cot^2(x) + 1} \cot(x) \right) - \frac{1}{4} \cot(x) (\cot^2(x) + 1)^{3/2} \\
 & \quad \downarrow \text{222} \\
 & -\frac{3}{4} \left( \frac{1}{2} \operatorname{arcsinh}(\cot(x)) + \frac{1}{2} \cot(x) \sqrt{\cot^2(x) + 1} \right) - \frac{1}{4} \cot(x) (\cot^2(x) + 1)^{3/2}
 \end{aligned}$$

input

```
Int[(Csc[x]^2)^(5/2), x]
```

output

```
-1/4*(Cot[x]*(1 + Cot[x]^2)^(3/2)) - (3*(ArcSinh[Cot[x]]/2 + (Cot[x]*Sqrt[1 + Cot[x]^2])/2))/4
```

**Defintions of rubi rules used**

```
rule 211 Int[((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] := Simp[x*((a + b*x^2)^p/(2*p + 1)), x] + Simp[2*a*(p/(2*p + 1)) Int[(a + b*x^2)^(p - 1), x], x] /; FreeQ[{a, b}, x] && GtQ[p, 0] && (IntegerQ[4*p] || IntegerQ[6*p])
```

```
rule 222 Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] := Simp[ArcSinh[Rt[b, 2]*(x/Sqrt[a])]/Rt[b, 2], x] /; FreeQ[{a, b}, x] && GtQ[a, 0] && PosQ[b]
```

```
rule 3042 Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]
```

```
rule 4610 Int[((b_.)*sec[(e_.) + (f_.)*(x_)^2]^(p_), x_Symbol] := With[{ff = FreeFactors[Tan[e + f*x], x]}, Simp[b*(ff/f) Subst[Int[(b + b*ff^2*x^2)^(p - 1), x], x, Tan[e + f*x]/ff], x] /; FreeQ[{b, e, f, p}, x] && !IntegerQ[p]
```

**Maple [C] (warning: unable to verify)**

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.15 (sec) , antiderivative size = 36, normalized size of antiderivative = 1.00

method	result	size
default	$\frac{\sqrt{4} \operatorname{csgn}(\csc(x)) (3 \ln(-\cot(x) + \csc(x)) + 3 \cot(x)^3 \csc(x) - 5 \cot(x) \csc(x)^3)}{16}$	36
risch	$-\frac{i \sqrt{-\frac{e^{2ix}}{(e^{2ix}-1)^2}} (3e^{6ix} - 11e^{4ix} - 11e^{2ix} + 3)}{4(e^{2ix}-1)^3} - \frac{3 \sqrt{-\frac{e^{2ix}}{(e^{2ix}-1)^2}} \ln(e^{ix}+1) \sin(x)}{4} + \frac{3 \sqrt{-\frac{e^{2ix}}{(e^{2ix}-1)^2}} \ln(e^{ix}-1) \sin(x)}{4}$	11

```
input int((csc(x)^2)^(5/2), x, method=_RETURNVERBOSE)
```

```
output 1/16*4^(1/2)*csgn(csc(x))*(3*ln(-cot(x)+csc(x))+3*cot(x)^3*csc(x)-5*cot(x)*csc(x)^3)
```

**Fricas [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 69 vs.  $2(26) = 52$ .

Time = 0.11 (sec) , antiderivative size = 69, normalized size of antiderivative = 1.92

$$\int \csc^2(x)^{5/2} dx = \frac{6 \cos(x)^3 - 3(\cos(x)^4 - 2 \cos(x)^2 + 1) \log\left(\frac{1}{2} \cos(x) + \frac{1}{2}\right) + 3(\cos(x)^4 - 2 \cos(x)^2 + 1) \log\left(-\frac{1}{2} \cos(x) + \frac{1}{2}\right) - 10 \cos(x)}{16(\cos(x)^4 - 2 \cos(x)^2 + 1)}$$

input `integrate((csc(x)^2)^(5/2),x, algorithm="fricas")`

output `1/16*(6*cos(x)^3 - 3*(cos(x)^4 - 2*cos(x)^2 + 1)*log(1/2*cos(x) + 1/2) + 3*(cos(x)^4 - 2*cos(x)^2 + 1)*log(-1/2*cos(x) + 1/2) - 10*cos(x))/(cos(x)^4 - 2*cos(x)^2 + 1)`

**Sympy [F]**

$$\int \csc^2(x)^{5/2} dx = \int (\csc^2(x))^{5/2} dx$$

input `integrate((csc(x)**2)**(5/2),x)`

output `Integral((csc(x)**2)**(5/2), x)`

**Maxima [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 869 vs.  $2(26) = 52$ .

Time = 0.16 (sec) , antiderivative size = 869, normalized size of antiderivative = 24.14

$$\int \csc^2(x)^{5/2} dx = \text{Too large to display}$$

input `integrate((csc(x)^2)^(5/2),x, algorithm="maxima")`

output

```

-1/16*(4*(3*cos(7*x) - 11*cos(5*x) - 11*cos(3*x) + 3*cos(x))*cos(8*x) - 12
*(4*cos(6*x) - 6*cos(4*x) + 4*cos(2*x) - 1)*cos(7*x) + 16*(11*cos(5*x) + 1
1*cos(3*x) - 3*cos(x))*cos(6*x) - 44*(6*cos(4*x) - 4*cos(2*x) + 1)*cos(5*x
) - 24*(11*cos(3*x) - 3*cos(x))*cos(4*x) + 44*(4*cos(2*x) - 1)*cos(3*x) -
48*cos(2*x)*cos(x) + 3*(2*(4*cos(6*x) - 6*cos(4*x) + 4*cos(2*x) - 1)*cos(8
*x) - cos(8*x)^2 + 8*(6*cos(4*x) - 4*cos(2*x) + 1)*cos(6*x) - 16*cos(6*x)^
2 + 12*(4*cos(2*x) - 1)*cos(4*x) - 36*cos(4*x)^2 - 16*cos(2*x)^2 + 4*(2*si
n(6*x) - 3*sin(4*x) + 2*sin(2*x))*sin(8*x) - sin(8*x)^2 + 16*(3*sin(4*x) -
2*sin(2*x))*sin(6*x) - 16*sin(6*x)^2 - 36*sin(4*x)^2 + 48*sin(4*x)*sin(2*
x) - 16*sin(2*x)^2 + 8*cos(2*x) - 1)*log(cos(x)^2 + sin(x)^2 + 2*cos(x) +
1) - 3*(2*(4*cos(6*x) - 6*cos(4*x) + 4*cos(2*x) - 1)*cos(8*x) - cos(8*x)^2
+ 8*(6*cos(4*x) - 4*cos(2*x) + 1)*cos(6*x) - 16*cos(6*x)^2 + 12*(4*cos(2*
x) - 1)*cos(4*x) - 36*cos(4*x)^2 - 16*cos(2*x)^2 + 4*(2*sin(6*x) - 3*sin(4
*x) + 2*sin(2*x))*sin(8*x) - sin(8*x)^2 + 16*(3*sin(4*x) - 2*sin(2*x))*sin
(6*x) - 16*sin(6*x)^2 - 36*sin(4*x)^2 + 48*sin(4*x)*sin(2*x) - 16*sin(2*x)
^2 + 8*cos(2*x) - 1)*log(cos(x)^2 + sin(x)^2 - 2*cos(x) + 1) + 4*(3*sin(7*
x) - 11*sin(5*x) - 11*sin(3*x) + 3*sin(x))*sin(8*x) - 24*(2*sin(6*x) - 3*s
in(4*x) + 2*sin(2*x))*sin(7*x) + 16*(11*sin(5*x) + 11*sin(3*x) - 3*sin(x))
*sin(6*x) - 88*(3*sin(4*x) - 2*sin(2*x))*sin(5*x) - 24*(11*sin(3*x) - 3*si
n(x))*sin(4*x) + 176*sin(3*x)*sin(2*x) - 48*sin(2*x)*sin(x) + 12*cos(x)...

```

### Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 100 vs.  $2(26) = 52$ .

Time = 0.16 (sec) , antiderivative size = 100, normalized size of antiderivative = 2.78

$$\int \csc^2(x)^{5/2} dx = -\frac{(\cos(x) - 1)\operatorname{sgn}(\sin(x))}{8(\cos(x) + 1)} + \frac{(\cos(x) - 1)^2\operatorname{sgn}(\sin(x))}{64(\cos(x) + 1)^2} + \frac{\left(\frac{8(\cos(x)-1)}{\cos(x)+1} - \frac{18(\cos(x)-1)^2}{(\cos(x)+1)^2} - 1\right)(\cos(x) + 1)^2}{64(\cos(x) - 1)^2\operatorname{sgn}(\sin(x))} + \frac{3 \log\left(-\frac{\cos(x)-1}{\cos(x)+1}\right)}{16\operatorname{sgn}(\sin(x))}$$

input

```
integrate((csc(x)^2)^(5/2),x, algorithm="giac")
```



output

```
-1/8*(cos(x) - 1)*sgn(sin(x))/(cos(x) + 1) + 1/64*(cos(x) - 1)^2*sgn(sin(x))
)/(cos(x) + 1)^2 + 1/64*(8*(cos(x) - 1)/(cos(x) + 1) - 18*(cos(x) - 1)^2/
(cos(x) + 1)^2 - 1)*(cos(x) + 1)^2/((cos(x) - 1)^2*sgn(sin(x))) + 3/16*log
(-(cos(x) - 1)/(cos(x) + 1))/sgn(sin(x))
```

**Mupad [F(-1)]**

Timed out.

$$\int \csc^2(x)^{5/2} dx = \int \left( \frac{1}{\sin(x)^2} \right)^{5/2} dx$$

input

```
int((1/sin(x)^2)^(5/2), x)
```

output

```
int((1/sin(x)^2)^(5/2), x)
```

**Reduce [B] (verification not implemented)**

Time = 0.19 (sec) , antiderivative size = 30, normalized size of antiderivative = 0.83

$$\int \csc^2(x)^{5/2} dx = \frac{-3 \cos(x) \sin(x)^2 - 2 \cos(x) + 3 \log(\tan(\frac{x}{2})) \sin(x)^4}{8 \sin(x)^4}$$

input

```
int((csc(x)^2)^(5/2), x)
```

output

```
( - 3*cos(x)*sin(x)**2 - 2*cos(x) + 3*log(tan(x/2))*sin(x)**4)/(8*sin(x)**
4)
```

### 3.41 $\int \csc^2(x)^{3/2} dx$

Optimal result . . . . .	273
Mathematica [B] (verified) . . . . .	273
Rubi [A] (verified) . . . . .	274
Maple [C] (warning: unable to verify) . . . . .	275
Fricas [B] (verification not implemented) . . . . .	276
Sympy [F] . . . . .	276
Maxima [B] (verification not implemented) . . . . .	276
Giac [B] (verification not implemented) . . . . .	277
Mupad [F(-1)] . . . . .	278
Reduce [B] (verification not implemented) . . . . .	278

#### Optimal result

Integrand size = 8, antiderivative size = 22

$$\int \csc^2(x)^{3/2} dx = -\frac{1}{2} \operatorname{arcsinh}(\cot(x)) - \frac{1}{2} \cot(x) \sqrt{\csc^2(x)}$$

output `-1/2*arcsinh(cot(x))-1/2*cot(x)*(csc(x)^2)^(1/2)`

#### Mathematica [B] (verified)

Leaf count is larger than twice the leaf count of optimal. 51 vs. 2(22) = 44.

Time = 0.09 (sec) , antiderivative size = 51, normalized size of antiderivative = 2.32

$$\int \csc^2(x)^{3/2} dx = \frac{1}{8} \sqrt{\csc^2(x)} \left( -\csc^2\left(\frac{x}{2}\right) - 4 \log\left(\cos\left(\frac{x}{2}\right)\right) + 4 \log\left(\sin\left(\frac{x}{2}\right)\right) + \sec^2\left(\frac{x}{2}\right) \right) \sin(x)$$

input `Integrate[(Csc[x]^2)^(3/2),x]`

output `(Sqrt[Csc[x]^2]*(-Csc[x/2]^2 - 4*Log[Cos[x/2]] + 4*Log[Sin[x/2]] + Sec[x/2]^2)*Sin[x])/8`

**Rubi [A] (verified)**

Time = 0.18 (sec) , antiderivative size = 24, normalized size of antiderivative = 1.09, number of steps used = 5, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {3042, 4610, 211, 222}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \csc^2(x)^{3/2} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \left( \sec \left( x + \frac{\pi}{2} \right)^2 \right)^{3/2} dx \\
 & \quad \downarrow \text{4610} \\
 & - \int \sqrt{\cot^2(x) + 1} d \cot(x) \\
 & \quad \downarrow \text{211} \\
 & -\frac{1}{2} \int \frac{1}{\sqrt{\cot^2(x) + 1}} d \cot(x) - \frac{1}{2} \sqrt{\cot^2(x) + 1} \cot(x) \\
 & \quad \downarrow \text{222} \\
 & -\frac{1}{2} \operatorname{arcsinh}(\cot(x)) - \frac{1}{2} \cot(x) \sqrt{\cot^2(x) + 1}
 \end{aligned}$$

input `Int[(Csc[x]^2)^(3/2), x]`

output `-1/2*ArcSinh[Cot[x]] - (Cot[x]*Sqrt[1 + Cot[x]^2])/2`

## Definitions of rubi rules used

rule 211 `Int[((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] := Simp[x*((a + b*x^2)^p/(2*p + 1)), x] + Simp[2*a*(p/(2*p + 1)) Int[(a + b*x^2)^(p - 1), x], x] /; FreeQ[{a, b}, x] && GtQ[p, 0] && (IntegerQ[4*p] || IntegerQ[6*p])`

rule 222 `Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] := Simp[ArcSinh[Rt[b, 2]*(x/Sqrt[a])]/Rt[b, 2], x] /; FreeQ[{a, b}, x] && GtQ[a, 0] && PosQ[b]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 4610 `Int[((b_.)*sec[(e_.) + (f_.)*(x_)])^2)^(p_), x_Symbol] := With[{ff = FreeFactors[Tan[e + f*x], x]}, Simp[b*(ff/f) Subst[Int[(b + b*ff^2*x^2)^(p - 1), x], x, Tan[e + f*x]/ff], x] /; FreeQ[{b, e, f, p}, x] && !IntegerQ[p]`

## Maple [C] (warning: unable to verify)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.10 (sec) , antiderivative size = 25, normalized size of antiderivative = 1.14

method	result	size
default	$-\frac{\sqrt{4} \operatorname{csgn}(\csc(x))(-\ln(-\cot(x)+\csc(x))+\csc(x) \cot(x))}{4}$	25
risch	$-\frac{i \sqrt{-\frac{e^{2ix}}{(e^{2ix}-1)^2}} (e^{2ix}+1)}{e^{2ix}-1} + \sqrt{-\frac{e^{2ix}}{(e^{2ix}-1)^2}} \ln(e^{ix}-1) \sin(x) - \sqrt{-\frac{e^{2ix}}{(e^{2ix}-1)^2}} \ln(e^{ix}+1) \sin(x)$	98

input `int((csc(x)^2)^(3/2), x, method=_RETURNVERBOSE)`

output `-1/4*4^(1/2)*csgn(csc(x))*(-ln(-cot(x)+csc(x))+csc(x)*cot(x))`

**Fricas [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 44 vs.  $2(16) = 32$ .

Time = 0.07 (sec) , antiderivative size = 44, normalized size of antiderivative = 2.00

$$\int \csc^2(x)^{3/2} dx = \frac{(\cos(x)^2 - 1) \log\left(\frac{1}{2} \cos(x) + \frac{1}{2}\right) - (\cos(x)^2 - 1) \log\left(-\frac{1}{2} \cos(x) + \frac{1}{2}\right) - 2 \cos(x)}{4 (\cos(x)^2 - 1)}$$

input `integrate((csc(x)^2)^(3/2),x, algorithm="fricas")`

output `-1/4*((cos(x)^2 - 1)*log(1/2*cos(x) + 1/2) - (cos(x)^2 - 1)*log(-1/2*cos(x) + 1/2) - 2*cos(x))/(cos(x)^2 - 1)`

**Sympy [F]**

$$\int \csc^2(x)^{3/2} dx = \int (\csc^2(x))^{\frac{3}{2}} dx$$

input `integrate((csc(x)**2)**(3/2),x)`

output `Integral((csc(x)**2)**(3/2), x)`

**Maxima [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 300 vs.  $2(16) = 32$ .

Time = 0.15 (sec) , antiderivative size = 300, normalized size of antiderivative = 13.64

$$\int \csc^2(x)^{3/2} dx = \frac{4 (\cos(3x) + \cos(x)) \cos(4x) - 4 (2 \cos(2x) - 1) \cos(3x) - 8 \cos(2x) \cos(x) + (2 (2 \cos(2x) - 1) \cos(x))}{4 (\cos(x)^2 - 1)}$$

input `integrate((csc(x)^2)^(3/2),x, algorithm="maxima")`

output `-1/4*(4*(cos(3*x) + cos(x))*cos(4*x) - 4*(2*cos(2*x) - 1)*cos(3*x) - 8*cos(2*x)*cos(x) + (2*(2*cos(2*x) - 1)*cos(4*x) - cos(4*x)^2 - 4*cos(2*x)^2 - sin(4*x)^2 + 4*sin(4*x)*sin(2*x) - 4*sin(2*x)^2 + 4*cos(2*x) - 1)*log(cos(x)^2 + sin(x)^2 + 2*cos(x) + 1) - (2*(2*cos(2*x) - 1)*cos(4*x) - cos(4*x)^2 - 4*cos(2*x)^2 - sin(4*x)^2 + 4*sin(4*x)*sin(2*x) - 4*sin(2*x)^2 + 4*cos(2*x) - 1)*log(cos(x)^2 + sin(x)^2 - 2*cos(x) + 1) + 4*(sin(3*x) + sin(x))*sin(4*x) - 8*sin(3*x)*sin(2*x) - 8*sin(2*x)*sin(x) + 4*cos(x))/(2*(2*cos(2*x) - 1)*cos(4*x) - cos(4*x)^2 - 4*cos(2*x)^2 - sin(4*x)^2 + 4*sin(4*x)*sin(2*x) - 4*sin(2*x)^2 + 4*cos(2*x) - 1)`

### Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 69 vs.  $2(16) = 32$ .

Time = 0.13 (sec) , antiderivative size = 69, normalized size of antiderivative = 3.14

$$\int \csc^2(x)^{3/2} dx = -\frac{\left(\frac{2(\cos(x)-1)}{\cos(x)+1} - 1\right)(\cos(x) + 1)}{8(\cos(x) - 1)\operatorname{sgn}(\sin(x))} + \frac{\log\left(-\frac{\cos(x)-1}{\cos(x)+1}\right)}{4\operatorname{sgn}(\sin(x))} - \frac{\cos(x) - 1}{8(\cos(x) + 1)\operatorname{sgn}(\sin(x))}$$

input `integrate((csc(x)^2)^(3/2),x, algorithm="giac")`

output `-1/8*(2*(cos(x) - 1)/(cos(x) + 1) - 1)*(cos(x) + 1)/((cos(x) - 1)*sgn(sin(x))) + 1/4*log(-(cos(x) - 1)/(cos(x) + 1))/sgn(sin(x)) - 1/8*(cos(x) - 1)/((cos(x) + 1)*sgn(sin(x)))`

**Mupad [F(-1)]**

Timed out.

$$\int \csc^2(x)^{3/2} dx = \int \left( \frac{1}{\sin(x)^2} \right)^{3/2} dx$$

input `int((1/sin(x)^2)^(3/2),x)`output `int((1/sin(x)^2)^(3/2), x)`**Reduce [B] (verification not implemented)**

Time = 0.18 (sec) , antiderivative size = 21, normalized size of antiderivative = 0.95

$$\int \csc^2(x)^{3/2} dx = \frac{-\cos(x) + \log\left(\tan\left(\frac{x}{2}\right)\right) \sin(x)^2}{2 \sin(x)^2}$$

input `int((csc(x)^2)^(3/2),x)`output `( - cos(x) + log(tan(x/2))*sin(x)**2)/(2*sin(x)**2)`

### 3.42 $\int \sqrt{\csc^2(x)} dx$

Optimal result	279
Mathematica [B] (verified)	279
Rubi [A] (verified)	280
Maple [B] (verified)	281
Fricas [B] (verification not implemented)	281
Sympy [F]	282
Maxima [B] (verification not implemented)	282
Giac [B] (verification not implemented)	282
Mupad [F(-1)]	283
Reduce [B] (verification not implemented)	283

#### Optimal result

Integrand size = 8, antiderivative size = 5

$$\int \sqrt{\csc^2(x)} dx = -\operatorname{arcsinh}(\cot(x))$$

output `-arcsinh(cot(x))`

#### Mathematica [B] (verified)

Leaf count is larger than twice the leaf count of optimal. 15 vs.  $2(5) = 10$ .

Time = 0.01 (sec) , antiderivative size = 15, normalized size of antiderivative = 3.00

$$\int \sqrt{\csc^2(x)} dx = -\operatorname{arctanh}(\cos(x))\sqrt{\csc^2(x)}\sin(x)$$

input `Integrate[Sqrt[Csc[x]^2],x]`

output `-(ArcTanh[Cos[x]]*Sqrt[Csc[x]^2]*Sin[x])`



**Rubi [A] (verified)**

Time = 0.18 (sec) , antiderivative size = 5, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 3,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.375$ , Rules used = {3042, 4610, 222}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{array}{c}
 \int \sqrt{\csc^2(x)} dx \\
 \downarrow 3042 \\
 \int \sqrt{\sec\left(x + \frac{\pi}{2}\right)^2} dx \\
 \downarrow 4610 \\
 - \int \frac{1}{\sqrt{\cot^2(x) + 1}} d \cot(x) \\
 \downarrow 222 \\
 -\operatorname{arcsinh}(\cot(x))
 \end{array}$$

input `Int [Sqrt [Csc [x] ^2] , x]`

output `-ArcSinh [Cot [x]]`

**Defintions of rubi rules used**

rule 222 `Int [1/Sqrt [(a_) + (b_.)*(x_)^2], x_Symbol] :> Simp [ArcSinh [Rt [b, 2]*(x/Sqrt [a])]/Rt [b, 2], x] /; FreeQ [{a, b}, x] && GtQ [a, 0] && PosQ [b]`

rule 3042 `Int [u_, x_Symbol] :> Int [DeactivateTrig [u, x], x] /; FunctionOfTrigOfLinear Q [u, x]`

rule 4610

```
Int[((b_.)*sec[(e_.) + (f_.)*(x_)]^2)^(p_), x_Symbol] := With[{ff = FreeFac
tors[Tan[e + f*x], x]}, Simp[b*(ff/f) Subst[Int[(b + b*ff^2*x^2)^(p - 1),
x], x, Tan[e + f*x]/ff], x]] /; FreeQ[{b, e, f, p}, x] && !IntegerQ[p]
```

**Maple [B] (verified)**

Leaf count of result is larger than twice the leaf count of optimal. 17 vs.  $2(5) = 10$ .

Time = 0.07 (sec) , antiderivative size = 18, normalized size of antiderivative = 3.60

method	result	size
default	$\ln(-\cot(x) + \csc(x)) \sin(x) \sqrt{\csc(x)^2}$	18
risch	$2\sqrt{-\frac{e^{2ix}}{(e^{2ix}-1)^2}} \ln(e^{ix}-1) \sin(x) - 2\sqrt{-\frac{e^{2ix}}{(e^{2ix}-1)^2}} \ln(e^{ix}+1) \sin(x)$	62

input

```
int((csc(x)^2)^(1/2), x, method=_RETURNVERBOSE)
```

output

```
ln(-cot(x)+csc(x))*sin(x)*(csc(x)^2)^(1/2)
```

**Fricas [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 19 vs.  $2(5) = 10$ .

Time = 0.07 (sec) , antiderivative size = 19, normalized size of antiderivative = 3.80

$$\int \sqrt{\csc^2(x)} dx = -\frac{1}{2} \log\left(\frac{1}{2} \cos(x) + \frac{1}{2}\right) + \frac{1}{2} \log\left(-\frac{1}{2} \cos(x) + \frac{1}{2}\right)$$

input

```
integrate((csc(x)^2)^(1/2), x, algorithm="fricas")
```

output

```
-1/2*log(1/2*cos(x) + 1/2) + 1/2*log(-1/2*cos(x) + 1/2)
```

**Sympy [F]**

$$\int \sqrt{\csc^2(x)} dx = \int \sqrt{\csc^2(x)} dx$$

input `integrate((csc(x)**2)**(1/2),x)`

output `Integral(sqrt(csc(x)**2), x)`

**Maxima [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 35 vs.  $2(5) = 10$ .

Time = 0.16 (sec) , antiderivative size = 35, normalized size of antiderivative = 7.00

$$\int \sqrt{\csc^2(x)} dx = -\frac{1}{2} \log(\cos(x)^2 + \sin(x)^2 + 2 \cos(x) + 1) + \frac{1}{2} \log(\cos(x)^2 + \sin(x)^2 - 2 \cos(x) + 1)$$

input `integrate((csc(x)^2)^(1/2),x, algorithm="maxima")`

output `-1/2*log(cos(x)^2 + sin(x)^2 + 2*cos(x) + 1) + 1/2*log(cos(x)^2 + sin(x)^2 - 2*cos(x) + 1)`

**Giac [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 12 vs.  $2(5) = 10$ .

Time = 0.17 (sec) , antiderivative size = 12, normalized size of antiderivative = 2.40

$$\int \sqrt{\csc^2(x)} dx = \frac{\log(|\tan(\frac{1}{2}x)|)}{\operatorname{sgn}(\sin(x))}$$

input `integrate((csc(x)^2)^(1/2),x, algorithm="giac")`

output `log(abs(tan(1/2*x)))/sgn(sin(x))`

### Mupad [F(-1)]

Timed out.

$$\int \sqrt{\csc^2(x)} dx = \int \sqrt{\frac{1}{\sin(x)^2}} dx$$

input `int((1/sin(x)^2)^(1/2), x)`

output `int((1/sin(x)^2)^(1/2), x)`

### Reduce [B] (verification not implemented)

Time = 0.17 (sec) , antiderivative size = 5, normalized size of antiderivative = 1.00

$$\int \sqrt{\csc^2(x)} dx = \log\left(\tan\left(\frac{x}{2}\right)\right)$$

input `int((csc(x)^2)^(1/2), x)`

output `log(tan(x/2))`

### 3.43 $\int \frac{1}{\sqrt{\csc^2(x)}} dx$

Optimal result	284
Mathematica [A] (verified)	284
Rubi [A] (verified)	285
Maple [C] (warning: unable to verify)	286
Fricas [A] (verification not implemented)	286
Sympy [A] (verification not implemented)	287
Maxima [A] (verification not implemented)	287
Giac [A] (verification not implemented)	287
Mupad [B] (verification not implemented)	288
Reduce [B] (verification not implemented)	288

#### Optimal result

Integrand size = 8, antiderivative size = 12

$$\int \frac{1}{\sqrt{\csc^2(x)}} dx = -\frac{\cot(x)}{\sqrt{\csc^2(x)}}$$

output

```
-cot(x)/(csc(x)^2)^(1/2)
```

#### Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 12, normalized size of antiderivative = 1.00

$$\int \frac{1}{\sqrt{\csc^2(x)}} dx = -\frac{\cot(x)}{\sqrt{\csc^2(x)}}$$

input

```
Integrate[1/Sqrt[Csc[x]^2],x]
```

output

```
-(Cot[x]/Sqrt[Csc[x]^2])
```

**Rubi [A] (verified)**

Time = 0.18 (sec) , antiderivative size = 14, normalized size of antiderivative = 1.17, number of steps used = 4, number of rules used = 3,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.375$ , Rules used = {3042, 4610, 208}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \frac{1}{\sqrt{\csc^2(x)}} dx \\ & \quad \downarrow \text{3042} \\ & \int \frac{1}{\sqrt{\sec(x + \frac{\pi}{2})^2}} dx \\ & \quad \downarrow \text{4610} \\ & - \int \frac{1}{(\cot^2(x) + 1)^{3/2}} d \cot(x) \\ & \quad \downarrow \text{208} \\ & - \frac{\cot(x)}{\sqrt{\cot^2(x) + 1}} \end{aligned}$$

input `Int[1/Sqrt[Csc[x]^2], x]`

output `-(Cot[x]/Sqrt[1 + Cot[x]^2])`

**Defintions of rubi rules used**

rule 208 `Int[((a_) + (b_.)*(x_)^2)^(-3/2), x_Symbol] := Simp[x/(a*Sqrt[a + b*x^2]), x] /; FreeQ[{a, b}, x]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 4610

```
Int[((b_.)*sec[(e_.) + (f_.)*(x_)]^2)^(p_), x_Symbol] := With[{ff = FreeFac
tors[Tan[e + f*x], x]}, Simp[b*(ff/f) Subst[Int[(b + b*ff^2*x^2)^(p - 1),
x], x, Tan[e + f*x]/ff], x]] /; FreeQ[{b, e, f, p}, x] && !IntegerQ[p]
```

**Maple [C] (warning: unable to verify)**

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.07 (sec) , antiderivative size = 13, normalized size of antiderivative = 1.08

method	result	size
default	$-\frac{\sqrt{4}(\cos(x)+1)\operatorname{csgn}(\csc(x))}{2}$	13
risch	$-\frac{ie^{2ix}}{2\sqrt{-\frac{e^{2ix}}{(e^{2ix}-1)^2}}(e^{2ix}-1)} - \frac{i}{2(e^{2ix}-1)\sqrt{-\frac{e^{2ix}}{(e^{2ix}-1)^2}}}$	67

input

```
int(1/(csc(x)^2)^(1/2),x,method=_RETURNVERBOSE)
```

output

```
-1/2*4^(1/2)*(cos(x)+1)*csgn(csc(x))
```

**Fricas [A] (verification not implemented)**

Time = 0.07 (sec) , antiderivative size = 4, normalized size of antiderivative = 0.33

$$\int \frac{1}{\sqrt{\csc^2(x)}} dx = -\cos(x)$$

input

```
integrate(1/(csc(x)^2)^(1/2),x, algorithm="fricas")
```

output

```
-\cos(x)
```

**Sympy [A] (verification not implemented)**

Time = 0.22 (sec) , antiderivative size = 12, normalized size of antiderivative = 1.00

$$\int \frac{1}{\sqrt{\csc^2(x)}} dx = -\frac{\cot(x)}{\sqrt{\csc^2(x)}}$$

input `integrate(1/(csc(x)**2)**(1/2),x)`output `-cot(x)/sqrt(csc(x)**2)`**Maxima [A] (verification not implemented)**

Time = 0.12 (sec) , antiderivative size = 10, normalized size of antiderivative = 0.83

$$\int \frac{1}{\sqrt{\csc^2(x)}} dx = -\frac{1}{\sqrt{\tan(x)^2 + 1}}$$

input `integrate(1/(csc(x)^2)^(1/2),x, algorithm="maxima")`output `-1/sqrt(tan(x)^2 + 1)`**Giac [A] (verification not implemented)**

Time = 0.16 (sec) , antiderivative size = 11, normalized size of antiderivative = 0.92

$$\int \frac{1}{\sqrt{\csc^2(x)}} dx = -\cos(x) \operatorname{sgn}(\sin(x)) + \operatorname{sgn}(\sin(x))$$

input `integrate(1/(csc(x)^2)^(1/2),x, algorithm="giac")`output `-\cos(x)*sgn(sin(x)) + sgn(sin(x))`



**Mupad [B] (verification not implemented)**

Time = 15.39 (sec) , antiderivative size = 12, normalized size of antiderivative = 1.00

$$\int \frac{1}{\sqrt{\csc^2(x)}} dx = -\frac{\sin(2x)}{2\sqrt{\sin(x)^2}}$$

input `int(1/(1/sin(x)^2)^(1/2),x)`

output `-sin(2*x)/(2*(sin(x)^2)^(1/2))`

**Reduce [B] (verification not implemented)**

Time = 0.17 (sec) , antiderivative size = 6, normalized size of antiderivative = 0.50

$$\int \frac{1}{\sqrt{\csc^2(x)}} dx = -\cos(x) + 1$$

input `int(1/(csc(x)^2)^(1/2),x)`

output `- cos(x) + 1`

### 3.44 $\int \frac{1}{\csc^2(x)^{3/2}} dx$

Optimal result . . . . .	289
Mathematica [A] (verified) . . . . .	289
Rubi [A] (verified) . . . . .	290
Maple [C] (warning: unable to verify) . . . . .	291
Fricas [A] (verification not implemented) . . . . .	292
Sympy [A] (verification not implemented) . . . . .	292
Maxima [A] (verification not implemented) . . . . .	292
Giac [B] (verification not implemented) . . . . .	293
Mupad [F(-1)] . . . . .	293
Reduce [B] (verification not implemented) . . . . .	293

#### Optimal result

Integrand size = 8, antiderivative size = 29

$$\int \frac{1}{\csc^2(x)^{3/2}} dx = -\frac{\cot(x)}{3 \csc^2(x)^{3/2}} - \frac{2 \cot(x)}{3 \sqrt{\csc^2(x)}}$$

output `-1/3*cot(x)/(csc(x)^2)^(3/2)-2/3*cot(x)/(csc(x)^2)^(1/2)`

#### Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 23, normalized size of antiderivative = 0.79

$$\int \frac{1}{\csc^2(x)^{3/2}} dx = \frac{(-9 \cos(x) + \cos(3x)) \csc(x)}{12 \sqrt{\csc^2(x)}}$$

input `Integrate[(Csc[x]^2)^(-3/2),x]`

output `((-9*Cos[x] + Cos[3*x])*Csc[x])/(12*sqrt[Csc[x]^2])`

**Rubi [A] (verified)**

Time = 0.19 (sec) , antiderivative size = 33, normalized size of antiderivative = 1.14, number of steps used = 5, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {3042, 4610, 209, 208}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{1}{\csc^2(x)^{3/2}} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{1}{\left(\sec\left(x + \frac{\pi}{2}\right)\right)^{3/2}} dx \\
 & \quad \downarrow \text{4610} \\
 & - \int \frac{1}{(\cot^2(x) + 1)^{5/2}} d \cot(x) \\
 & \quad \downarrow \text{209} \\
 & -\frac{2}{3} \int \frac{1}{(\cot^2(x) + 1)^{3/2}} d \cot(x) - \frac{\cot(x)}{3(\cot^2(x) + 1)^{3/2}} \\
 & \quad \downarrow \text{208} \\
 & -\frac{2 \cot(x)}{3\sqrt{\cot^2(x) + 1}} - \frac{\cot(x)}{3(\cot^2(x) + 1)^{3/2}}
 \end{aligned}$$

input `Int[(Csc[x]^2)^(-3/2),x]`

output `-1/3*Cot[x]/(1 + Cot[x]^2)^(3/2) - (2*Cot[x])/(3*Sqrt[1 + Cot[x]^2])`

## Definitions of rubi rules used

rule 208  $\text{Int}[(a_ + (b_.)*(x_)^2)^{-3/2}, x\_Symbol] \rightarrow \text{Simp}[x/(a*\text{Sqrt}[a + b*x^2]), x] \text{ /; FreeQ}\{a, b\}, x]$

rule 209  $\text{Int}[(a_ + (b_.)*(x_)^2)^{p_}, x\_Symbol] \rightarrow \text{Simp}[(-x)*((a + b*x^2)^{p + 1} / (2*a*(p + 1))), x] + \text{Simp}[(2*p + 3)/(2*a*(p + 1)) \text{ Int}[(a + b*x^2)^{p + 1}], x] \text{ /; FreeQ}\{a, b\}, x] \ \&\& \ \text{ILtQ}[p + 3/2, 0]$

rule 3042  $\text{Int}[u_, x\_Symbol] \rightarrow \text{Int}[\text{DeactivateTrig}[u, x], x] \text{ /; FunctionOfTrigOfLinearQ}[u, x]$

rule 4610  $\text{Int}[(b_.)*\text{sec}[(e_.) + (f_.)*(x_)]^2)^{p_}, x\_Symbol] \rightarrow \text{With}\{\text{ff} = \text{FreeFactors}[\text{Tan}[e + f*x], x]\}, \text{Simp}[b*(\text{ff}/f) \text{ Subst}[\text{Int}[(b + b*\text{ff}^2*x^2)^{p - 1}], x], x, \text{Tan}[e + f*x]/\text{ff}], x] \text{ /; FreeQ}\{b, e, f, p\}, x] \ \&\& \ \text{!IntegerQ}[p]$

## Maple [C] (warning: unable to verify)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.09 (sec) , antiderivative size = 19, normalized size of antiderivative = 0.66

method	result	size
default	$\frac{\sqrt{4}(\cos(x)^3 - 3\cos(x) - 2)\text{csgn}(\csc(x))}{6}$	19
risch	$\frac{ie^{4ix}}{24(e^{2ix}-1)\sqrt{-\frac{e^{2ix}}{(e^{2ix}-1)^2}}} - \frac{3ie^{2ix}}{8\sqrt{-\frac{e^{2ix}}{(e^{2ix}-1)^2}}(e^{2ix}-1)} - \frac{3i}{8(e^{2ix}-1)\sqrt{-\frac{e^{2ix}}{(e^{2ix}-1)^2}}} + \frac{ie^{-2ix}}{24(e^{2ix}-1)\sqrt{-\frac{e^{2ix}}{(e^{2ix}-1)^2}}}$	137

input  $\text{int}(1/(\csc(x)^2)^{(3/2)}, x, \text{method}=\_RETURNVERBOSE)$

output  $1/6*4^{(1/2)}*(\cos(x)^3 - 3*\cos(x) - 2)*\text{csgn}(\csc(x))$

**Fricas [A] (verification not implemented)**

Time = 0.07 (sec) , antiderivative size = 11, normalized size of antiderivative = 0.38

$$\int \frac{1}{\csc^2(x)^{3/2}} dx = \frac{1}{3} \cos(x)^3 - \cos(x)$$

input `integrate(1/(csc(x)^2)^(3/2),x, algorithm="fricas")`output `1/3*cos(x)^3 - cos(x)`**Sympy [A] (verification not implemented)**

Time = 0.37 (sec) , antiderivative size = 29, normalized size of antiderivative = 1.00

$$\int \frac{1}{\csc^2(x)^{3/2}} dx = -\frac{2 \cot^3(x)}{3 (\csc^2(x))^{\frac{3}{2}}} - \frac{\cot(x)}{(\csc^2(x))^{\frac{3}{2}}}$$

input `integrate(1/(csc(x)**2)**(3/2),x)`output `-2*cot(x)**3/(3*(csc(x)**2)**(3/2)) - cot(x)/(csc(x)**2)**(3/2)`**Maxima [A] (verification not implemented)**

Time = 0.14 (sec) , antiderivative size = 11, normalized size of antiderivative = 0.38

$$\int \frac{1}{\csc^2(x)^{3/2}} dx = \frac{1}{12} \cos(3x) - \frac{3}{4} \cos(x)$$

input `integrate(1/(csc(x)^2)^(3/2),x, algorithm="maxima")`output `1/12*cos(3*x) - 3/4*cos(x)`

**Giac [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 44 vs.  $2(21) = 42$ .

Time = 0.13 (sec) , antiderivative size = 44, normalized size of antiderivative = 1.52

$$\int \frac{1}{\csc^2(x)^{3/2}} dx = -\frac{4 \left( \frac{3(\cos(x)-1)\operatorname{sgn}(\sin(x))}{\cos(x)+1} - \operatorname{sgn}(\sin(x)) \right)}{3 \left( \frac{\cos(x)-1}{\cos(x)+1} - 1 \right)^3} + \frac{4}{3} \operatorname{sgn}(\sin(x))$$

input `integrate(1/(csc(x)^2)^(3/2),x, algorithm="giac")`

output `-4/3*(3*(cos(x) - 1)*sgn(sin(x))/(cos(x) + 1) - sgn(sin(x)))/((cos(x) - 1)/(cos(x) + 1) - 1)^3 + 4/3*sgn(sin(x))`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{1}{\csc^2(x)^{3/2}} dx = \int \frac{1}{\left( \frac{1}{\sin(x)^2} \right)^{3/2}} dx$$

input `int(1/(1/sin(x)^2)^(3/2),x)`

output `int(1/(1/sin(x)^2)^(3/2), x)`

**Reduce [B] (verification not implemented)**

Time = 0.16 (sec) , antiderivative size = 14, normalized size of antiderivative = 0.48

$$\int \frac{1}{\csc^2(x)^{3/2}} dx = -\frac{\cos(x)\sin(x)^2}{3} - \frac{2\cos(x)}{3} + \frac{2}{3}$$

input `int(1/(csc(x)^2)^(3/2),x)`

output  $(-\cos(x)\sin(x)^2 - 2\cos(x) + 2)/3$

### 3.45 $\int \frac{1}{\csc^2(x)^{5/2}} dx$

Optimal result . . . . .	295
Mathematica [A] (verified) . . . . .	295
Rubi [A] (verified) . . . . .	296
Maple [C] (warning: unable to verify) . . . . .	297
Fricas [A] (verification not implemented) . . . . .	298
Sympy [A] (verification not implemented) . . . . .	298
Maxima [A] (verification not implemented) . . . . .	298
Giac [A] (verification not implemented) . . . . .	299
Mupad [F(-1)] . . . . .	299
Reduce [B] (verification not implemented) . . . . .	300

#### Optimal result

Integrand size = 8, antiderivative size = 43

$$\int \frac{1}{\csc^2(x)^{5/2}} dx = -\frac{\cot(x)}{5 \csc^2(x)^{5/2}} - \frac{4 \cot(x)}{15 \csc^2(x)^{3/2}} - \frac{8 \cot(x)}{15 \sqrt{\csc^2(x)}}$$

output `-1/5*cot(x)/(csc(x)^2)^(5/2)-4/15*cot(x)/(csc(x)^2)^(3/2)-8/15*cot(x)/(csc(x)^2)^(1/2)`

#### Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 31, normalized size of antiderivative = 0.72

$$\int \frac{1}{\csc^2(x)^{5/2}} dx = -\frac{(150 \cos(x) - 25 \cos(3x) + 3 \cos(5x)) \csc(x)}{240 \sqrt{\csc^2(x)}}$$

input `Integrate[(Csc[x]^2)^(-5/2), x]`

output `-1/240*((150*Cos[x] - 25*Cos[3*x] + 3*Cos[5*x])*Csc[x])/Sqrt[Csc[x]^2]`



**Rubi [A] (verified)**

Time = 0.19 (sec) , antiderivative size = 54, normalized size of antiderivative = 1.26, number of steps used = 6, number of rules used = 5,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.625$ , Rules used = {3042, 4610, 209, 209, 208}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{1}{\csc^2(x)^{5/2}} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{1}{\left(\sec\left(x + \frac{\pi}{2}\right)\right)^{5/2}} dx \\
 & \quad \downarrow \text{4610} \\
 & - \int \frac{1}{(\cot^2(x) + 1)^{7/2}} d \cot(x) \\
 & \quad \downarrow \text{209} \\
 & -\frac{4}{5} \int \frac{1}{(\cot^2(x) + 1)^{5/2}} d \cot(x) - \frac{\cot(x)}{5(\cot^2(x) + 1)^{5/2}} \\
 & \quad \downarrow \text{209} \\
 & -\frac{4}{5} \left( \frac{2}{3} \int \frac{1}{(\cot^2(x) + 1)^{3/2}} d \cot(x) + \frac{\cot(x)}{3(\cot^2(x) + 1)^{3/2}} \right) - \frac{\cot(x)}{5(\cot^2(x) + 1)^{5/2}} \\
 & \quad \downarrow \text{208} \\
 & -\frac{\cot(x)}{5(\cot^2(x) + 1)^{5/2}} - \frac{4}{5} \left( \frac{2 \cot(x)}{3\sqrt{\cot^2(x) + 1}} + \frac{\cot(x)}{3(\cot^2(x) + 1)^{3/2}} \right)
 \end{aligned}$$

input `Int[(Csc[x]^2)^(-5/2), x]`

output `-1/5*Cot[x]/(1 + Cot[x]^2)^(5/2) - (4*(Cot[x]/(3*(1 + Cot[x]^2)^(3/2))) + (2*Cot[x]))/(3*Sqrt[1 + Cot[x]^2]))/5`

## Definitions of rubi rules used

rule 208  $\text{Int}[(a_ + (b_ \cdot)(x_ )^2)^{-3/2}, x\_Symbol] \rightarrow \text{Simp}[x/(a\sqrt{a + b \cdot x^2}), x] \text{ /; FreeQ}\{a, b\}, x]$

rule 209  $\text{Int}[(a_ + (b_ \cdot)(x_ )^2)^{p_}, x\_Symbol] \rightarrow \text{Simp}[(-x) \cdot (a + b \cdot x^2)^{p+1} / (2 \cdot a \cdot (p+1)), x] + \text{Simp}[(2 \cdot p + 3) / (2 \cdot a \cdot (p+1)) \text{ Int}[(a + b \cdot x^2)^{p+1}], x], x] \text{ /; FreeQ}\{a, b\}, x \ \&\& \text{ ILtQ}[p + 3/2, 0]$

rule 3042  $\text{Int}[u_, x\_Symbol] \rightarrow \text{Int}[\text{DeactivateTrig}[u, x], x] \text{ /; FunctionOfTrigOfLinearQ}[u, x]$

rule 4610  $\text{Int}[(b_ \cdot) \sec[(e_ \cdot) + (f_ \cdot)(x_ )]^2)^{p_}, x\_Symbol] \rightarrow \text{With}\{\text{ff} = \text{FreeFactors}[\text{Tan}[e + f \cdot x], x]\}, \text{Simp}[b \cdot (\text{ff}/f) \text{ Subst}[\text{Int}[(b + b \cdot \text{ff}^2 \cdot x^2)^{p-1}], x], x, \text{Tan}[e + f \cdot x]/\text{ff}], x] \text{ /; FreeQ}\{b, e, f, p\}, x \ \&\& \text{ !IntegerQ}[p]$

## Maple [C] (warning: unable to verify)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.12 (sec) , antiderivative size = 27, normalized size of antiderivative = 0.63

method	result
default	$-\frac{\sqrt{4} (3 \cos(x)^5 - 10 \cos(x)^3 + 15 \cos(x) + 8) \text{csgn}(\csc(x))}{30}$
risch	$-\frac{i e^{6ix}}{160(e^{2ix}-1)\sqrt{-\frac{e^{2ix}}{(e^{2ix}-1)^2}}} - \frac{5ie^{2ix}}{16\sqrt{-\frac{e^{2ix}}{(e^{2ix}-1)^2}}(e^{2ix}-1)} - \frac{5i}{16(e^{2ix}-1)\sqrt{-\frac{e^{2ix}}{(e^{2ix}-1)^2}}} + \frac{5ie^{-2ix}}{96(e^{2ix}-1)\sqrt{-\frac{e^{2ix}}{(e^{2ix}-1)^2}}} + \dots$

input  $\text{int}(1/(\csc(x)^2)^{(5/2)}, x, \text{method}=\_RETURNVERBOSE)$

output  $-1/30 \cdot 4^{(1/2)} \cdot (3 \cdot \cos(x)^5 - 10 \cdot \cos(x)^3 + 15 \cdot \cos(x) + 8) \cdot \text{csgn}(\csc(x))$

**Fricas [A] (verification not implemented)**

Time = 0.07 (sec) , antiderivative size = 17, normalized size of antiderivative = 0.40

$$\int \frac{1}{\csc^2(x)^{5/2}} dx = -\frac{1}{5} \cos(x)^5 + \frac{2}{3} \cos(x)^3 - \cos(x)$$

input `integrate(1/(csc(x)^2)^(5/2),x, algorithm="fricas")`output `-1/5*cos(x)^5 + 2/3*cos(x)^3 - cos(x)`**Sympy [A] (verification not implemented)**

Time = 1.27 (sec) , antiderivative size = 46, normalized size of antiderivative = 1.07

$$\int \frac{1}{\csc^2(x)^{5/2}} dx = -\frac{8 \cot^5(x)}{15 (\csc^2(x))^{\frac{5}{2}}} - \frac{4 \cot^3(x)}{3 (\csc^2(x))^{\frac{5}{2}}} - \frac{\cot(x)}{(\csc^2(x))^{\frac{5}{2}}}$$

input `integrate(1/(csc(x)**2)**(5/2),x)`output `-8*cot(x)**5/(15*(csc(x)**2)**(5/2)) - 4*cot(x)**3/(3*(csc(x)**2)**(5/2)) - cot(x)/(csc(x)**2)**(5/2)`**Maxima [A] (verification not implemented)**

Time = 0.14 (sec) , antiderivative size = 17, normalized size of antiderivative = 0.40

$$\int \frac{1}{\csc^2(x)^{5/2}} dx = -\frac{1}{80} \cos(5x) + \frac{5}{48} \cos(3x) - \frac{5}{8} \cos(x)$$

input `integrate(1/(csc(x)^2)^(5/2),x, algorithm="maxima")`output `-1/80*cos(5*x) + 5/48*cos(3*x) - 5/8*cos(x)`

**Giac [A] (verification not implemented)**

Time = 0.14 (sec) , antiderivative size = 61, normalized size of antiderivative = 1.42

$$\int \frac{1}{\csc^2(x)^{5/2}} dx = \frac{16 \left( \frac{5(\cos(x)-1)\operatorname{sgn}(\sin(x))}{\cos(x)+1} - \frac{10(\cos(x)-1)^2\operatorname{sgn}(\sin(x))}{(\cos(x)+1)^2} - \operatorname{sgn}(\sin(x)) \right)}{15 \left( \frac{\cos(x)-1}{\cos(x)+1} - 1 \right)^5} + \frac{16}{15} \operatorname{sgn}(\sin(x))$$

input `integrate(1/(csc(x)^2)^(5/2),x, algorithm="giac")`output `-16/15*(5*(cos(x) - 1)*sgn(sin(x))/(cos(x) + 1) - 10*(cos(x) - 1)^2*sgn(sin(x))/(cos(x) + 1)^2 - sgn(sin(x)))/((cos(x) - 1)/(cos(x) + 1) - 1)^5 + 16/15*sgn(sin(x))`**Mupad [F(-1)]**

Timed out.

$$\int \frac{1}{\csc^2(x)^{5/2}} dx = \int \frac{1}{\left(\frac{1}{\sin(x)^2}\right)^{5/2}} dx$$

input `int(1/(1/sin(x)^2)^(5/2),x)`output `int(1/(1/sin(x)^2)^(5/2), x)`

**Reduce [B] (verification not implemented)**

Time = 0.17 (sec) , antiderivative size = 22, normalized size of antiderivative = 0.51

$$\int \frac{1}{\csc^2(x)^{5/2}} dx = -\frac{\cos(x) \sin(x)^4}{5} - \frac{4 \cos(x) \sin(x)^2}{15} - \frac{8 \cos(x)}{15} + \frac{8}{15}$$

input `int(1/(csc(x)^2)^(5/2),x)`

output `( - 3*cos(x)*sin(x)**4 - 4*cos(x)*sin(x)**2 - 8*cos(x) + 8)/15`

### 3.46 $\int \frac{1}{\csc^2(x)^{7/2}} dx$

Optimal result . . . . .	301
Mathematica [A] (verified) . . . . .	301
Rubi [A] (verified) . . . . .	302
Maple [C] (warning: unable to verify) . . . . .	303
Fricas [A] (verification not implemented) . . . . .	304
Sympy [A] (verification not implemented) . . . . .	304
Maxima [A] (verification not implemented) . . . . .	305
Giac [A] (verification not implemented) . . . . .	305
Mupad [F(-1)] . . . . .	306
Reduce [B] (verification not implemented) . . . . .	306

#### Optimal result

Integrand size = 8, antiderivative size = 57

$$\int \frac{1}{\csc^2(x)^{7/2}} dx = -\frac{\cot(x)}{7 \csc^2(x)^{7/2}} - \frac{6 \cot(x)}{35 \csc^2(x)^{5/2}} - \frac{8 \cot(x)}{35 \csc^2(x)^{3/2}} - \frac{16 \cot(x)}{35 \sqrt{\csc^2(x)}}$$

output `-1/7*cot(x)/(csc(x)^2)^(7/2)-6/35*cot(x)/(csc(x)^2)^(5/2)-8/35*cot(x)/(csc(x)^2)^(3/2)-16/35*cot(x)/(csc(x)^2)^(1/2)`

#### Mathematica [A] (verified)

Time = 0.05 (sec) , antiderivative size = 37, normalized size of antiderivative = 0.65

$$\int \frac{1}{\csc^2(x)^{7/2}} dx = \frac{(-1225 \cos(x) + 245 \cos(3x) - 49 \cos(5x) + 5 \cos(7x)) \csc(x)}{2240 \sqrt{\csc^2(x)}}$$

input `Integrate[(Csc[x]^2)^(-7/2),x]`

output `((-1225*Cos[x] + 245*Cos[3*x] - 49*Cos[5*x] + 5*Cos[7*x])*Csc[x])/(2240*Sqrt[Csc[x]^2])`

**Rubi [A] (verified)**

Time = 0.21 (sec) , antiderivative size = 75, normalized size of antiderivative = 1.32, number of steps used = 7, number of rules used = 6,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.750$ , Rules used = {3042, 4610, 209, 209, 209, 208}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{1}{\csc^2(x)^{7/2}} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{1}{\left(\sec\left(x + \frac{\pi}{2}\right)\right)^{7/2}} dx \\
 & \quad \downarrow \text{4610} \\
 & - \int \frac{1}{(\cot^2(x) + 1)^{9/2}} d \cot(x) \\
 & \quad \downarrow \text{209} \\
 & -\frac{6}{7} \int \frac{1}{(\cot^2(x) + 1)^{7/2}} d \cot(x) - \frac{\cot(x)}{7(\cot^2(x) + 1)^{7/2}} \\
 & \quad \downarrow \text{209} \\
 & -\frac{6}{7} \left( \frac{4}{5} \int \frac{1}{(\cot^2(x) + 1)^{5/2}} d \cot(x) + \frac{\cot(x)}{5(\cot^2(x) + 1)^{5/2}} \right) - \frac{\cot(x)}{7(\cot^2(x) + 1)^{7/2}} \\
 & \quad \downarrow \text{209} \\
 & -\frac{6}{7} \left( \frac{4}{5} \left( \frac{2}{3} \int \frac{1}{(\cot^2(x) + 1)^{3/2}} d \cot(x) + \frac{\cot(x)}{3(\cot^2(x) + 1)^{3/2}} \right) + \frac{\cot(x)}{5(\cot^2(x) + 1)^{5/2}} \right) - \\
 & \quad \frac{\cot(x)}{7(\cot^2(x) + 1)^{7/2}} \\
 & \quad \downarrow \text{208} \\
 & -\frac{\cot(x)}{7(\cot^2(x) + 1)^{7/2}} - \frac{6}{7} \left( \frac{\cot(x)}{5(\cot^2(x) + 1)^{5/2}} + \frac{4}{5} \left( \frac{2 \cot(x)}{3\sqrt{\cot^2(x) + 1}} + \frac{\cot(x)}{3(\cot^2(x) + 1)^{3/2}} \right) \right)
 \end{aligned}$$

input `Int[(Csc[x]^2)^(-7/2), x]`

output 
$$-1/7*\text{Cot}[x]/(1 + \text{Cot}[x]^2)^{7/2} - (6*(\text{Cot}[x]/(5*(1 + \text{Cot}[x]^2)^{5/2})) + (4*(\text{Cot}[x]/(3*(1 + \text{Cot}[x]^2)^{3/2})) + (2*\text{Cot}[x])/(3*\text{Sqrt}[1 + \text{Cot}[x]^2]))) / 5) / 7$$

**Defintions of rubi rules used**

rule 208 `Int[((a_) + (b_.)*(x_)^2)^(-3/2), x_Symbol] := Simp[x/(a*Sqrt[a + b*x^2]), x] /; FreeQ[{a, b}, x]`

rule 209 `Int[((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] := Simp[(-x)*((a + b*x^2)^(p + 1)/(2*a*(p + 1))), x] + Simp[(2*p + 3)/(2*a*(p + 1)) Int[(a + b*x^2)^(p + 1), x], x] /; FreeQ[{a, b}, x] && ILtQ[p + 3/2, 0]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 4610 `Int[((b_.)*sec[(e_.) + (f_.)*(x_)]^2)^(p_), x_Symbol] := With[{ff = FreeFactors[Tan[e + f*x], x]}, Simp[b*(ff/f) Subst[Int[(b + b*ff^2*x^2)^(p - 1), x], x, Tan[e + f*x]/ff], x] /; FreeQ[{b, e, f, p}, x] && !IntegerQ[p]`

**Maple [C] (warning: unable to verify)**

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.13 (sec) , antiderivative size = 33, normalized size of antiderivative = 0.58

method	result
default	$\frac{\sqrt{4} (5 \cos(x)^7 - 21 \cos(x)^5 + 35 \cos(x)^3 - 35 \cos(x) - 16) \operatorname{csgn}(\csc(x))}{70}$
risch	$\frac{ie^{8ix}}{896(e^{2ix}-1)\sqrt{-\frac{e^{2ix}}{(e^{2ix}-1)^2}}} - \frac{35ie^{2ix}}{128\sqrt{-\frac{e^{2ix}}{(e^{2ix}-1)^2}}(e^{2ix}-1)} - \frac{35i}{128(e^{2ix}-1)\sqrt{-\frac{e^{2ix}}{(e^{2ix}-1)^2}}} + \frac{7ie^{-2ix}}{128(e^{2ix}-1)\sqrt{-\frac{e^{2ix}}{(e^{2ix}-1)^2}}} - \dots$



input `int(1/(csc(x)^2)^(7/2),x,method=_RETURNVERBOSE)`

output `1/70*4^(1/2)*(5*cos(x)^7-21*cos(x)^5+35*cos(x)^3-35*cos(x)-16)*csgn(csc(x))`

### Fricas [A] (verification not implemented)

Time = 0.07 (sec) , antiderivative size = 21, normalized size of antiderivative = 0.37

$$\int \frac{1}{\csc^2(x)^{7/2}} dx = \frac{1}{7} \cos(x)^7 - \frac{3}{5} \cos(x)^5 + \cos(x)^3 - \cos(x)$$

input `integrate(1/(csc(x)^2)^(7/2),x, algorithm="fricas")`

output `1/7*cos(x)^7 - 3/5*cos(x)^5 + cos(x)^3 - cos(x)`

### Sympy [A] (verification not implemented)

Time = 9.54 (sec) , antiderivative size = 61, normalized size of antiderivative = 1.07

$$\int \frac{1}{\csc^2(x)^{7/2}} dx = -\frac{16 \cot^7(x)}{35 (\csc^2(x))^{7/2}} - \frac{8 \cot^5(x)}{5 (\csc^2(x))^{7/2}} - \frac{2 \cot^3(x)}{(\csc^2(x))^{7/2}} - \frac{\cot(x)}{(\csc^2(x))^{7/2}}$$

input `integrate(1/(csc(x)**2)**(7/2),x)`

output `-16*cot(x)**7/(35*(csc(x)**2)**(7/2)) - 8*cot(x)**5/(5*(csc(x)**2)**(7/2)) - 2*cot(x)**3/(csc(x)**2)**(7/2) - cot(x)/(csc(x)**2)**(7/2)`

**Maxima [A] (verification not implemented)**

Time = 0.15 (sec) , antiderivative size = 23, normalized size of antiderivative = 0.40

$$\int \frac{1}{\csc^2(x)^{7/2}} dx = \frac{1}{448} \cos(7x) - \frac{7}{320} \cos(5x) + \frac{7}{64} \cos(3x) - \frac{35}{64} \cos(x)$$

input `integrate(1/(csc(x)^2)^(7/2),x, algorithm="maxima")`output `1/448*cos(7*x) - 7/320*cos(5*x) + 7/64*cos(3*x) - 35/64*cos(x)`**Giac [A] (verification not implemented)**

Time = 0.15 (sec) , antiderivative size = 78, normalized size of antiderivative = 1.37

$$\int \frac{1}{\csc^2(x)^{7/2}} dx = \frac{32 \left( \frac{7(\cos(x)-1)\operatorname{sgn}(\sin(x))}{\cos(x)+1} - \frac{21(\cos(x)-1)^2\operatorname{sgn}(\sin(x))}{(\cos(x)+1)^2} + \frac{35(\cos(x)-1)^3\operatorname{sgn}(\sin(x))}{(\cos(x)+1)^3} - \operatorname{sgn}(\sin(x)) \right)}{35 \left( \frac{\cos(x)-1}{\cos(x)+1} - 1 \right)^7} + \frac{32}{35} \operatorname{sgn}(\sin(x))$$

input `integrate(1/(csc(x)^2)^(7/2),x, algorithm="giac")`output `-32/35*(7*(cos(x) - 1)*sgn(sin(x))/(cos(x) + 1) - 21*(cos(x) - 1)^2*sgn(sin(x))/(cos(x) + 1)^2 + 35*(cos(x) - 1)^3*sgn(sin(x))/(cos(x) + 1)^3 - sgn(sin(x)))/((cos(x) - 1)/(cos(x) + 1) - 1)^7 + 32/35*sgn(sin(x))`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{1}{\csc^2(x)^{7/2}} dx = \int \frac{1}{\left(\frac{1}{\sin(x)^2}\right)^{7/2}} dx$$

input `int(1/(1/sin(x)^2)^(7/2),x)`output `int(1/(1/sin(x)^2)^(7/2), x)`**Reduce [B] (verification not implemented)**

Time = 0.17 (sec) , antiderivative size = 30, normalized size of antiderivative = 0.53

$$\int \frac{1}{\csc^2(x)^{7/2}} dx = -\frac{\cos(x) \sin(x)^6}{7} - \frac{6 \cos(x) \sin(x)^4}{35} - \frac{8 \cos(x) \sin(x)^2}{35} - \frac{16 \cos(x)}{35} + \frac{16}{35}$$

input `int(1/(csc(x)^2)^(7/2),x)`output `( - 5*cos(x)*sin(x)**6 - 6*cos(x)*sin(x)**4 - 8*cos(x)*sin(x)**2 - 16*cos(x) + 16)/35`

### 3.47 $\int (a \csc^2(x))^{7/2} dx$

Optimal result . . . . .	307
Mathematica [A] (verified) . . . . .	307
Rubi [A] (verified) . . . . .	308
Maple [A] (verified) . . . . .	310
Fricas [A] (verification not implemented) . . . . .	310
Sympy [F(-1)] . . . . .	311
Maxima [B] (verification not implemented) . . . . .	311
Giac [B] (verification not implemented) . . . . .	312
Mupad [F(-1)] . . . . .	313
Reduce [B] (verification not implemented) . . . . .	313

#### Optimal result

Integrand size = 10, antiderivative size = 84

$$\int (a \csc^2(x))^{7/2} dx = -\frac{5}{16}a^{7/2}\operatorname{arctanh}\left(\frac{\sqrt{a} \cot(x)}{\sqrt{a \csc^2(x)}}\right) - \frac{5}{16}a^3 \cot(x)\sqrt{a \csc^2(x)} - \frac{5}{24}a^2 \cot(x) (a \csc^2(x))^{3/2} - \frac{1}{6}a \cot(x) (a \csc^2(x))^{5/2}$$

```
output -5/16*a^(7/2)*arctanh(a^(1/2)*cot(x)/(a*csc(x)^2)^(1/2))-5/16*a^3*cot(x)*(a*csc(x)^2)^(1/2)-5/24*a^2*cot(x)*(a*csc(x)^2)^(3/2)-1/6*a*cot(x)*(a*csc(x)^2)^(5/2)
```

#### Mathematica [A] (verified)

Time = 0.26 (sec) , antiderivative size = 61, normalized size of antiderivative = 0.73

$$\int (a \csc^2(x))^{7/2} dx = \frac{a^3 \csc^5(x)\sqrt{a \csc^2(x)}(-396 \cos(x) + 170 \cos(3x) - 30 \cos(5x) + 480(-\log(\cos(\frac{x}{2}))))}{1536}$$

```
input Integrate[(a*Csc[x]^2)^(7/2),x]
```

output

```
(a^3*Csc[x]^5*Sqrt[a*Csc[x]^2]*(-396*Cos[x] + 170*Cos[3*x] - 30*Cos[5*x] +
480*(-Log[Cos[x/2]] + Log[Sin[x/2]])*Sin[x]^6)/1536
```

**Rubi [A] (verified)**

Time = 0.23 (sec) , antiderivative size = 100, normalized size of antiderivative = 1.19, number of steps used = 8, number of rules used = 7,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.700$ , Rules used = {3042, 4610, 211, 211, 211, 224, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int (a \csc^2(x))^{7/2} dx$$

$$\downarrow \text{3042}$$

$$\int \left( a \sec \left( x + \frac{\pi}{2} \right)^2 \right)^{7/2} dx$$

$$\downarrow \text{4610}$$

$$-a \int (a \cot^2(x) + a)^{5/2} d \cot(x)$$

$$\downarrow \text{211}$$

$$-a \left( \frac{5}{6} a \int (a \cot^2(x) + a)^{3/2} d \cot(x) + \frac{1}{6} \cot(x) (a \cot^2(x) + a)^{5/2} \right)$$

$$\downarrow \text{211}$$

$$-a \left( \frac{5}{6} a \left( \frac{3}{4} a \int \sqrt{a \cot^2(x) + a} d \cot(x) + \frac{1}{4} \cot(x) (a \cot^2(x) + a)^{3/2} \right) + \frac{1}{6} \cot(x) (a \cot^2(x) + a)^{5/2} \right)$$

$$\downarrow \text{211}$$

$$-a \left( \frac{5}{6} a \left( \frac{3}{4} a \left( \frac{1}{2} a \int \frac{1}{\sqrt{a \cot^2(x) + a}} d \cot(x) + \frac{1}{2} \cot(x) \sqrt{a \cot^2(x) + a} \right) + \frac{1}{4} \cot(x) (a \cot^2(x) + a)^{3/2} \right) + \frac{1}{6} \cot(x) (a \cot^2(x) + a)^{5/2} \right)$$

$$\downarrow \text{224}$$

$$-a \left( \frac{5}{6} a \left( \frac{3}{4} a \left( \frac{1}{2} a \int \frac{1}{1 - \frac{a \cot^2(x)}{a \cot^2(x) + a}} d \frac{\cot(x)}{\sqrt{a \cot^2(x) + a}} + \frac{1}{2} \cot(x) \sqrt{a \cot^2(x) + a} \right) + \frac{1}{4} \cot(x) (a \cot^2(x) + a)^{3/2} \right) \right)$$

↓ 219

$$-a \left( \frac{5}{6} a \left( \frac{3}{4} a \left( \frac{1}{2} \sqrt{a} \operatorname{arctanh} \left( \frac{\sqrt{a} \cot(x)}{\sqrt{a \cot^2(x) + a}} \right) + \frac{1}{2} \cot(x) \sqrt{a \cot^2(x) + a} \right) + \frac{1}{4} \cot(x) (a \cot^2(x) + a)^{3/2} \right) + \frac{1}{6} \right)$$

input `Int[(a*Csc[x]^2)^(7/2),x]`

output `-(a*((Cot[x]*(a + a*Cot[x]^2)^(5/2))/6 + (5*a*((Cot[x]*(a + a*Cot[x]^2)^(3/2))/4 + (3*a*((Sqrt[a]*ArcTanh[(Sqrt[a]*Cot[x])/Sqrt[a + a*Cot[x]^2]])/2 + (Cot[x]*Sqrt[a + a*Cot[x]^2])/2))/4))/6)`

### Defintions of rubi rules used

rule 211 `Int[((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] := Simp[x*((a + b*x^2)^p/(2*p + 1)), x] + Simp[2*a*(p/(2*p + 1)) Int[(a + b*x^2)^(p - 1), x], x] /; FreeQ[{a, b}, x] && GtQ[p, 0] && (IntegerQ[4*p] || IntegerQ[6*p])`

rule 219 `Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (GtQ[a, 0] || LtQ[b, 0])`

rule 224 `Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] := Subst[Int[1/(1 - b*x^2), x], x, x/Sqrt[a + b*x^2]] /; FreeQ[{a, b}, x] && !GtQ[a, 0]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 4610

```
Int[((b_.)*sec[(e_.) + (f_.)*(x_)]^2)^(p_), x_Symbol] := With[{ff = FreeFac
tors[Tan[e + f*x], x]}, Simp[b*(ff/f) Subst[Int[(b + b*ff^2*x^2)^(p - 1),
x], x, Tan[e + f*x]/ff], x]] /; FreeQ[{b, e, f, p}, x] && !IntegerQ[p]
```

**Maple [A] (verified)**

Time = 0.23 (sec) , antiderivative size = 52, normalized size of antiderivative = 0.62

method	result
default	$\frac{\sqrt{4} \left( 15 \ln(-\cot(x) + \csc(x)) \sin(x)^6 - 15 \cos(x)^5 + 40 \cos(x)^3 - 33 \cos(x) \right) \sqrt{a \csc(x)^2} a^3 \csc(x)^5}{96}$
risch	$-\frac{ia^3 \sqrt{-\frac{ae^{2ix}}{(e^{2ix}-1)^2}} (15e^{10ix} - 85e^{8ix} + 198e^{6ix} + 198e^{4ix} - 85e^{2ix} + 15)}{24(e^{2ix}-1)^5} + \frac{5a^3 \sqrt{-\frac{ae^{2ix}}{(e^{2ix}-1)^2}} \ln(e^{ix}-1) \sin(x)}{8} - \frac{5a^3 \sqrt{-\frac{ae^{2ix}}{(e^{2ix}-1)^2}}}{(e^{2ix}-1)^5}$

input

```
int((a*csc(x)^2)^(7/2),x,method=_RETURNVERBOSE)
```

output

```
1/96*4^(1/2)*(15*ln(-cot(x)+csc(x))*sin(x)^6-15*cos(x)^5+40*cos(x)^3-33*co
s(x))*(a*csc(x)^2)^(1/2)*a^3*csc(x)^5
```

**Fricas [A] (verification not implemented)**

Time = 0.08 (sec) , antiderivative size = 106, normalized size of antiderivative = 1.26

$$\int (a \csc^2(x))^{7/2} dx = \frac{(30 a^3 \cos(x)^5 - 80 a^3 \cos(x)^3 + 66 a^3 \cos(x) + 15 (a^3 \cos(x)^6 - 3 a^3 \cos(x)^4 + 3 a^3 \cos(x)^2 - a^3) \log(-\cos(x) - 1) / (\cos(x) + 1)) \sqrt{-a / (\cos(x)^2 - 1)}}{96 (\cos(x)^4 - 2 \cos(x)^2 + 1) \sin(x)}$$

input

```
integrate((a*csc(x)^2)^(7/2),x, algorithm="fricas")
```

output

```
-1/96*(30*a^3*cos(x)^5 - 80*a^3*cos(x)^3 + 66*a^3*cos(x) + 15*(a^3*cos(x)^
6 - 3*a^3*cos(x)^4 + 3*a^3*cos(x)^2 - a^3)*log(-(cos(x) - 1)/(cos(x) + 1)
)*sqrt(-a/(cos(x)^2 - 1))/((cos(x)^4 - 2*cos(x)^2 + 1)*sin(x))
```

**Sympy [F(-1)]**

Timed out.

$$\int (a \csc^2(x))^{7/2} dx = \text{Timed out}$$

input `integrate((a*csc(x)**2)**(7/2),x)`output `Timed out`**Maxima [B] (verification not implemented)**Leaf count of result is larger than twice the leaf count of optimal. 2183 vs.  $2(64) = 128$ .

Time = 2.20 (sec) , antiderivative size = 2183, normalized size of antiderivative = 25.99

$$\int (a \csc^2(x))^{7/2} dx = \text{Too large to display}$$

input `integrate((a*csc(x)^2)^(7/2),x, algorithm="maxima")`



output

```

1/48*(1020*a^3*cos(3*x)*sin(2*x) - 180*a^3*cos(x)*sin(2*x) + 180*a^3*cos(2
*x)*sin(x) - 30*a^3*sin(x) + 15*(a^3*cos(12*x)^2 + 36*a^3*cos(10*x)^2 + 22
5*a^3*cos(8*x)^2 + 400*a^3*cos(6*x)^2 + 225*a^3*cos(4*x)^2 + 36*a^3*cos(2*
x)^2 + a^3*sin(12*x)^2 + 36*a^3*sin(10*x)^2 + 225*a^3*sin(8*x)^2 + 400*a^3
*sin(6*x)^2 + 225*a^3*sin(4*x)^2 - 180*a^3*sin(4*x)*sin(2*x) + 36*a^3*sin(
2*x)^2 - 12*a^3*cos(2*x) + a^3 - 2*(6*a^3*cos(10*x) - 15*a^3*cos(8*x) + 20
*a^3*cos(6*x) - 15*a^3*cos(4*x) + 6*a^3*cos(2*x) - a^3)*cos(12*x) - 12*(15
*a^3*cos(8*x) - 20*a^3*cos(6*x) + 15*a^3*cos(4*x) - 6*a^3*cos(2*x) + a^3)*
cos(10*x) - 30*(20*a^3*cos(6*x) - 15*a^3*cos(4*x) + 6*a^3*cos(2*x) - a^3)*
cos(8*x) - 40*(15*a^3*cos(4*x) - 6*a^3*cos(2*x) + a^3)*cos(6*x) - 30*(6*a^
3*cos(2*x) - a^3)*cos(4*x) - 2*(6*a^3*sin(10*x) - 15*a^3*sin(8*x) + 20*a^3
*sin(6*x) - 15*a^3*sin(4*x) + 6*a^3*sin(2*x))*sin(12*x) - 12*(15*a^3*sin(8
*x) - 20*a^3*sin(6*x) + 15*a^3*sin(4*x) - 6*a^3*sin(2*x))*sin(10*x) - 30*(
20*a^3*sin(6*x) - 15*a^3*sin(4*x) + 6*a^3*sin(2*x))*sin(8*x) - 120*(5*a^3*
sin(4*x) - 2*a^3*sin(2*x))*sin(6*x))*arctan2(sin(x), cos(x) + 1) - 15*(a^3
*cos(12*x)^2 + 36*a^3*cos(10*x)^2 + 225*a^3*cos(8*x)^2 + 400*a^3*cos(6*x)^
2 + 225*a^3*cos(4*x)^2 + 36*a^3*cos(2*x)^2 + a^3*sin(12*x)^2 + 36*a^3*sin(
10*x)^2 + 225*a^3*sin(8*x)^2 + 400*a^3*sin(6*x)^2 + 225*a^3*sin(4*x)^2 - 1
80*a^3*sin(4*x)*sin(2*x) + 36*a^3*sin(2*x)^2 - 12*a^3*cos(2*x) + a^3 - 2*(
6*a^3*cos(10*x) - 15*a^3*cos(8*x) + 20*a^3*cos(6*x) - 15*a^3*cos(4*x) +...

```

**Giac [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 164 vs.  $2(64) = 128$ .

Time = 0.16 (sec) , antiderivative size = 164, normalized size of antiderivative = 1.95

$$\int (a \csc^2(x))^{7/2} dx = \frac{1}{384} \left( 60 a^3 \log \left( -\frac{\cos(x) - 1}{\cos(x) + 1} \right) \operatorname{sgn}(\sin(x)) - \frac{45 a^3 (\cos(x) - 1) \operatorname{sgn}(\sin(x))}{\cos(x) + 1} + \frac{9 a^3}{\cos(x) + 1} \right)$$

input

```
integrate((a*csc(x)^2)^(7/2),x, algorithm="giac")
```

output

```
1/384*(60*a^3*log(-(cos(x) - 1)/(cos(x) + 1))*sgn(sin(x)) - 45*a^3*(cos(x)
- 1)*sgn(sin(x))/(cos(x) + 1) + 9*a^3*(cos(x) - 1)^2*sgn(sin(x))/(cos(x)
+ 1)^2 - a^3*(cos(x) - 1)^3*sgn(sin(x))/(cos(x) + 1)^3 + (a^3*sgn(sin(x))
- 9*a^3*(cos(x) - 1)*sgn(sin(x))/(cos(x) + 1) + 45*a^3*(cos(x) - 1)^2*sgn(
sin(x))/(cos(x) + 1)^2 - 110*a^3*(cos(x) - 1)^3*sgn(sin(x))/(cos(x) + 1)^3
)*(cos(x) + 1)^3/(cos(x) - 1)^3)*sqrt(a)
```

**Mupad [F(-1)]**

Timed out.

$$\int (a \csc^2(x))^{7/2} dx = \int \left( \frac{a}{\sin(x)^2} \right)^{7/2} dx$$

input

```
int((a/sin(x)^2)^(7/2),x)
```

output

```
int((a/sin(x)^2)^(7/2), x)
```

**Reduce [B] (verification not implemented)**

Time = 0.17 (sec) , antiderivative size = 43, normalized size of antiderivative = 0.51

$$\int (a \csc^2(x))^{7/2} dx = \frac{\sqrt{a} a^3 (-15 \cos(x) \sin(x)^4 - 10 \cos(x) \sin(x)^2 - 8 \cos(x) + 15 \log(\tan(\frac{x}{2})) \sin(x))}{48 \sin(x)^6}$$

input

```
int((a*csc(x)^2)^(7/2),x)
```

output

```
(sqrt(a)*a**3*( - 15*cos(x)*sin(x)**4 - 10*cos(x)*sin(x)**2 - 8*cos(x) + 1
5*log(tan(x/2))*sin(x)**6))/(48*sin(x)**6)
```

### 3.48 $\int (a \csc^2(x))^{5/2} dx$

Optimal result	314
Mathematica [A] (verified)	314
Rubi [A] (verified)	315
Maple [A] (verified)	317
Fricas [A] (verification not implemented)	317
Sympy [F]	318
Maxima [B] (verification not implemented)	318
Giac [B] (verification not implemented)	319
Mupad [F(-1)]	320
Reduce [B] (verification not implemented)	320

#### Optimal result

Integrand size = 10, antiderivative size = 65

$$\int (a \csc^2(x))^{5/2} dx = -\frac{3}{8}a^{5/2} \operatorname{arctanh}\left(\frac{\sqrt{a} \cot(x)}{\sqrt{a \csc^2(x)}}\right) - \frac{3}{8}a^2 \cot(x) \sqrt{a \csc^2(x)} - \frac{1}{4}a \cot(x) (a \csc^2(x))^{3/2}$$

output

$$-3/8*a^{(5/2)}*\operatorname{arctanh}(a^{(1/2)}*\cot(x)/(a*\csc(x)^2)^{(1/2)})-3/8*a^2*\cot(x)*(a*\csc(x)^2)^{(1/2)}-1/4*a*\cot(x)*(a*\csc(x)^2)^{(3/2)}$$

#### Mathematica [A] (verified)

Time = 0.18 (sec) , antiderivative size = 51, normalized size of antiderivative = 0.78

$$\int (a \csc^2(x))^{5/2} dx = \frac{1}{64} (a \csc^2(x))^{5/2} \sin(x) \left( -22 \cos(x) + 6 \left( \cos(3x) + 4 \left( -\log \left( \cos \left( \frac{x}{2} \right) \right) + \log \left( \sin \left( \frac{x}{2} \right) \right) \right) \right)$$

input

$$\operatorname{Integrate}[(a*\operatorname{Csc}[x]^2)^{(5/2)}, x]$$

output

$$((a*\operatorname{Csc}[x]^2)^{(5/2)}*\operatorname{Sin}[x]*(-22*\operatorname{Cos}[x] + 6*(\operatorname{Cos}[3*x] + 4*(-\operatorname{Log}[\operatorname{Cos}[x/2]] + \operatorname{Log}[\operatorname{Sin}[x/2]]))*\operatorname{Sin}[x]^4))/64$$

**Rubi [A] (verified)**

Time = 0.22 (sec) , antiderivative size = 76, normalized size of antiderivative = 1.17, number of steps used = 7, number of rules used = 6,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.600$ , Rules used = {3042, 4610, 211, 211, 224, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int (a \csc^2(x))^{5/2} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \left( a \sec \left( x + \frac{\pi}{2} \right)^2 \right)^{5/2} dx \\
 & \quad \downarrow \text{4610} \\
 & -a \int (a \cot^2(x) + a)^{3/2} d \cot(x) \\
 & \quad \downarrow \text{211} \\
 & -a \left( \frac{3}{4} a \int \sqrt{a \cot^2(x) + a} d \cot(x) + \frac{1}{4} \cot(x) (a \cot^2(x) + a)^{3/2} \right) \\
 & \quad \downarrow \text{211} \\
 & -a \left( \frac{3}{4} a \left( \frac{1}{2} a \int \frac{1}{\sqrt{a \cot^2(x) + a}} d \cot(x) + \frac{1}{2} \cot(x) \sqrt{a \cot^2(x) + a} \right) + \frac{1}{4} \cot(x) (a \cot^2(x) + a)^{3/2} \right) \\
 & \quad \downarrow \text{224} \\
 & -a \left( \frac{3}{4} a \left( \frac{1}{2} a \int \frac{1}{1 - \frac{a \cot^2(x)}{a \cot^2(x) + a}} d \frac{\cot(x)}{\sqrt{a \cot^2(x) + a}} + \frac{1}{2} \cot(x) \sqrt{a \cot^2(x) + a} \right) + \frac{1}{4} \cot(x) (a \cot^2(x) + a)^{3/2} \right) \\
 & \quad \downarrow \text{219} \\
 & -a \left( \frac{3}{4} a \left( \frac{1}{2} \sqrt{a} \operatorname{arctanh} \left( \frac{\sqrt{a} \cot(x)}{\sqrt{a \cot^2(x) + a}} \right) + \frac{1}{2} \cot(x) \sqrt{a \cot^2(x) + a} \right) + \frac{1}{4} \cot(x) (a \cot^2(x) + a)^{3/2} \right)
 \end{aligned}$$

input `Int[(a*Csc[x]^2)^(5/2),x]`

output `-(a*((Cot[x]*(a + a*Cot[x]^2)^(3/2))/4 + (3*a*((Sqrt[a]*ArcTanh[(Sqrt[a]*Cot[x])/Sqrt[a + a*Cot[x]^2]])/2 + (Cot[x]*Sqrt[a + a*Cot[x]^2])/2))/4))`

### Defintions of rubi rules used

rule 211 `Int[((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] := Simp[x*((a + b*x^2)^p/(2*p + 1)), x] + Simp[2*a*(p/(2*p + 1)) Int[(a + b*x^2)^(p - 1), x], x] /; FreeQ[{a, b}, x] && GtQ[p, 0] && (IntegerQ[4*p] || IntegerQ[6*p])`

rule 219 `Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (GtQ[a, 0] || LtQ[b, 0])`

rule 224 `Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] := Subst[Int[1/(1 - b*x^2), x], x, x/Sqrt[a + b*x^2]] /; FreeQ[{a, b}, x] && !GtQ[a, 0]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 4610 `Int[((b_.)*sec[(e_.) + (f_.)*(x_)]^2)^(p_), x_Symbol] := With[{ff = FreeFactors[Tan[e + f*x], x]}, Simp[b*(ff/f) Subst[Int[(b + b*ff^2*x^2)^(p - 1), x], x, Tan[e + f*x]/ff], x] /; FreeQ[{b, e, f, p}, x] && !IntegerQ[p]`

**Maple [A] (verified)**

Time = 0.14 (sec) , antiderivative size = 40, normalized size of antiderivative = 0.62

method	result
default	$\left( \frac{3 \ln(-\cot(x) + \csc(x)) \sin(x)}{8} + \frac{3 \cot(x)^3}{8} - \frac{5 \csc(x)^2 \cot(x)}{8} \right) \sqrt{a \csc(x)^2 a^2}$
risch	$-\frac{ia^2 \sqrt{-\frac{ae^{2ix}}{(e^{2ix}-1)^2}} (3e^{6ix} - 11e^{4ix} - 11e^{2ix} + 3)}{4(e^{2ix}-1)^3} - \frac{3a^2 \sqrt{-\frac{ae^{2ix}}{(e^{2ix}-1)^2}} \ln(e^{ix}+1) \sin(x)}{4} + \frac{3a^2 \sqrt{-\frac{ae^{2ix}}{(e^{2ix}-1)^2}} \ln(e^{ix}-1) \sin(x)}{4}$

input `int((a*csc(x)^2)^(5/2),x,method=_RETURNVERBOSE)`output `(3/8*ln(-cot(x)+csc(x))*sin(x)+3/8*cot(x)^3-5/8*csc(x)^2*cot(x))*(a*csc(x)^2)^(1/2)*a^2`**Fricas [A] (verification not implemented)**

Time = 0.08 (sec) , antiderivative size = 80, normalized size of antiderivative = 1.23

$$\int (a \csc^2(x))^{5/2} dx = \frac{\left( 6a^2 \cos(x)^3 - 10a^2 \cos(x) + 3(a^2 \cos(x)^4 - 2a^2 \cos(x)^2 + a^2) \log\left(-\frac{\cos(x)-1}{\cos(x)+1}\right) \right) \sqrt{-\frac{a}{\cos(x)^2-1}}}{16(\cos(x)^2-1)\sin(x)}$$

input `integrate((a*csc(x)^2)^(5/2),x, algorithm="fricas")`output `-1/16*(6*a^2*cos(x)^3 - 10*a^2*cos(x) + 3*(a^2*cos(x)^4 - 2*a^2*cos(x)^2 + a^2)*log(-(cos(x) - 1)/(cos(x) + 1)))*sqrt(-a/(cos(x)^2 - 1))/((cos(x)^2 - 1)*sin(x))`

**Sympy [F]**

$$\int (a \csc^2(x))^{5/2} dx = \int (a \csc^2(x))^{\frac{5}{2}} dx$$

input `integrate((a*csc(x)**2)**(5/2),x)`

output `Integral((a*csc(x)**2)**(5/2), x)`

**Maxima [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 1113 vs. 2(49) = 98.

Time = 0.30 (sec) , antiderivative size = 1113, normalized size of antiderivative = 17.12

$$\int (a \csc^2(x))^{5/2} dx = \text{Too large to display}$$

input `integrate((a*csc(x)^2)^(5/2),x, algorithm="maxima")`

output

```

1/8*(88*a^2*cos(3*x)*sin(2*x) - 24*a^2*cos(x)*sin(2*x) + 24*a^2*cos(2*x)*s
in(x) - 6*a^2*sin(x) + 3*(a^2*cos(8*x)^2 + 16*a^2*cos(6*x)^2 + 36*a^2*cos(
4*x)^2 + 16*a^2*cos(2*x)^2 + a^2*sin(8*x)^2 + 16*a^2*sin(6*x)^2 + 36*a^2*s
in(4*x)^2 - 48*a^2*sin(4*x)*sin(2*x) + 16*a^2*sin(2*x)^2 - 8*a^2*cos(2*x)
+ a^2 - 2*(4*a^2*cos(6*x) - 6*a^2*cos(4*x) + 4*a^2*cos(2*x) - a^2)*cos(8*x
) - 8*(6*a^2*cos(4*x) - 4*a^2*cos(2*x) + a^2)*cos(6*x) - 12*(4*a^2*cos(2*x
) - a^2)*cos(4*x) - 4*(2*a^2*sin(6*x) - 3*a^2*sin(4*x) + 2*a^2*sin(2*x))*s
in(8*x) - 16*(3*a^2*sin(4*x) - 2*a^2*sin(2*x))*sin(6*x))*arctan2(sin(x), c
os(x) + 1) - 3*(a^2*cos(8*x)^2 + 16*a^2*cos(6*x)^2 + 36*a^2*cos(4*x)^2 + 1
6*a^2*cos(2*x)^2 + a^2*sin(8*x)^2 + 16*a^2*sin(6*x)^2 + 36*a^2*sin(4*x)^2
- 48*a^2*sin(4*x)*sin(2*x) + 16*a^2*sin(2*x)^2 - 8*a^2*cos(2*x) + a^2 - 2*
(4*a^2*cos(6*x) - 6*a^2*cos(4*x) + 4*a^2*cos(2*x) - a^2)*cos(8*x) - 8*(6*a
^2*cos(4*x) - 4*a^2*cos(2*x) + a^2)*cos(6*x) - 12*(4*a^2*cos(2*x) - a^2)*c
os(4*x) - 4*(2*a^2*sin(6*x) - 3*a^2*sin(4*x) + 2*a^2*sin(2*x))*sin(8*x) -
16*(3*a^2*sin(4*x) - 2*a^2*sin(2*x))*sin(6*x))*arctan2(sin(x), cos(x) - 1)
- 2*(3*a^2*sin(7*x) - 11*a^2*sin(5*x) - 11*a^2*sin(3*x) + 3*a^2*sin(x))*c
os(8*x) - 12*(2*a^2*sin(6*x) - 3*a^2*sin(4*x) + 2*a^2*sin(2*x))*cos(7*x) -
8*(11*a^2*sin(5*x) + 11*a^2*sin(3*x) - 3*a^2*sin(x))*cos(6*x) - 44*(3*a^2
*sin(4*x) - 2*a^2*sin(2*x))*cos(5*x) + 12*(11*a^2*sin(3*x) - 3*a^2*sin(x))
*cos(4*x) + 2*(3*a^2*cos(7*x) - 11*a^2*cos(5*x) - 11*a^2*cos(3*x) + 3*a...

```

### Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 106 vs.  $2(49) = 98$ .

Time = 0.15 (sec) , antiderivative size = 106, normalized size of antiderivative = 1.63

$$\int (a \csc^2(x))^{5/2} dx = \frac{1}{64} \left( 12 \log \left( -\frac{\cos(x) - 1}{\cos(x) + 1} \right) \operatorname{sgn}(\sin(x)) + \frac{\left( \frac{8(\cos(x)-1)\operatorname{sgn}(\sin(x))}{\cos(x)+1} - \frac{18(\cos(x)-1)^2\operatorname{sgn}(\sin(x))}{(\cos(x)+1)^2} \right)}{(\cos(x) - 1)}$$

input

```
integrate((a*csc(x)^2)^(5/2),x, algorithm="giac")
```

output

```

1/64*(12*log(-(cos(x) - 1)/(cos(x) + 1))*sgn(sin(x)) + (8*(cos(x) - 1)*sgn
(sin(x))/(cos(x) + 1) - 18*(cos(x) - 1)^2*sgn(sin(x))/(cos(x) + 1)^2 - sgn
(sin(x)))*(cos(x) + 1)^2/(cos(x) - 1)^2 - 8*(cos(x) - 1)*sgn(sin(x))/(cos(
x) + 1) + (cos(x) - 1)^2*sgn(sin(x))/(cos(x) + 1)^2)*a^(5/2)

```



**Mupad [F(-1)]**

Timed out.

$$\int (a \csc^2(x))^{5/2} dx = \int \left( \frac{a}{\sin(x)^2} \right)^{5/2} dx$$

input `int((a/sin(x)^2)^(5/2),x)`output `int((a/sin(x)^2)^(5/2), x)`**Reduce [B] (verification not implemented)**

Time = 0.18 (sec) , antiderivative size = 35, normalized size of antiderivative = 0.54

$$\int (a \csc^2(x))^{5/2} dx = \frac{\sqrt{a} a^2 (-3 \cos(x) \sin(x)^2 - 2 \cos(x) + 3 \log(\tan(\frac{x}{2})) \sin(x)^4)}{8 \sin(x)^4}$$

input `int((a*csc(x)^2)^(5/2),x)`output `(sqrt(a)*a**2*(- 3*cos(x)*sin(x)**2 - 2*cos(x) + 3*log(tan(x/2))*sin(x)**4))/(8*sin(x)**4)`

### 3.49 $\int (a \csc^2(x))^{3/2} dx$

Optimal result	321
Mathematica [A] (verified)	321
Rubi [A] (verified)	322
Maple [A] (verified)	323
Fricas [A] (verification not implemented)	324
Sympy [F]	324
Maxima [B] (verification not implemented)	324
Giac [B] (verification not implemented)	325
Mupad [F(-1)]	325
Reduce [B] (verification not implemented)	326

#### Optimal result

Integrand size = 10, antiderivative size = 46

$$\int (a \csc^2(x))^{3/2} dx = -\frac{1}{2}a^{3/2}\operatorname{arctanh}\left(\frac{\sqrt{a}\cot(x)}{\sqrt{a \csc^2(x)}}\right) - \frac{1}{2}a \cot(x)\sqrt{a \csc^2(x)}$$

output

```
-1/2*a^(3/2)*arctanh(a^(1/2)*cot(x)/(a*csc(x)^2)^(1/2))-1/2*a*cot(x)*(a*csc(x)^2)^(1/2)
```

#### Mathematica [A] (verified)

Time = 0.07 (sec) , antiderivative size = 39, normalized size of antiderivative = 0.85

$$\int (a \csc^2(x))^{3/2} dx = -\frac{1}{2}a\sqrt{a \csc^2(x)}\left(\cot(x)\csc(x) + \log\left(\cos\left(\frac{x}{2}\right)\right) - \log\left(\sin\left(\frac{x}{2}\right)\right)\right)\sin(x)$$

input

```
Integrate[(a*Csc[x]^2)^(3/2),x]
```

output

```
-1/2*(a*Sqrt[a*Csc[x]^2]*(Cot[x]*Csc[x] + Log[Cos[x/2]] - Log[Sin[x/2]])*Sin[x])
```

**Rubi [A] (verified)**

Time = 0.20 (sec) , antiderivative size = 52, normalized size of antiderivative = 1.13, number of steps used = 6, number of rules used = 5,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {3042, 4610, 211, 224, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int (a \csc^2(x))^{3/2} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \left( a \sec \left( x + \frac{\pi}{2} \right)^2 \right)^{3/2} dx \\
 & \quad \downarrow \text{4610} \\
 & -a \int \sqrt{a \cot^2(x) + a} d \cot(x) \\
 & \quad \downarrow \text{211} \\
 & -a \left( \frac{1}{2} a \int \frac{1}{\sqrt{a \cot^2(x) + a}} d \cot(x) + \frac{1}{2} \cot(x) \sqrt{a \cot^2(x) + a} \right) \\
 & \quad \downarrow \text{224} \\
 & -a \left( \frac{1}{2} a \int \frac{1}{1 - \frac{a \cot^2(x)}{a \cot^2(x) + a}} d \frac{\cot(x)}{\sqrt{a \cot^2(x) + a}} + \frac{1}{2} \cot(x) \sqrt{a \cot^2(x) + a} \right) \\
 & \quad \downarrow \text{219} \\
 & -a \left( \frac{1}{2} \sqrt{a} \operatorname{arctanh} \left( \frac{\sqrt{a} \cot(x)}{\sqrt{a \cot^2(x) + a}} \right) + \frac{1}{2} \cot(x) \sqrt{a \cot^2(x) + a} \right)
 \end{aligned}$$

input `Int[(a*Csc[x]^2)^(3/2),x]`

output `-(a*((Sqrt[a]*ArcTanh[(Sqrt[a]*Cot[x])/Sqrt[a + a*Cot[x]^2]])/2 + (Cot[x]*Sqrt[a + a*Cot[x]^2])/2))`

## Definitions of rubi rules used

rule 211  $\text{Int}[(a_ + (b_ \cdot)(x_ )^2)^{p_ }, x\_Symbol] \rightarrow \text{Simp}[x*((a + b*x^2)^p/(2*p + 1)), x] + \text{Simp}[2*a*(p/(2*p + 1)) \text{Int}[(a + b*x^2)^{p - 1}, x], x] /; \text{FreeQ}[\{a, b\}, x] \ \&\& \ \text{GtQ}[p, 0] \ \&\& \ (\text{IntegerQ}[4*p] \ || \ \text{IntegerQ}[6*p])$

rule 219  $\text{Int}[(a_ + (b_ \cdot)(x_ )^2)^{-1}, x\_Symbol] \rightarrow \text{Simp}[(1/(\text{Rt}[a, 2]*\text{Rt}[-b, 2]))*\text{ArcTanh}[\text{Rt}[-b, 2]*(x/\text{Rt}[a, 2])], x] /; \text{FreeQ}[\{a, b\}, x] \ \&\& \ \text{NegQ}[a/b] \ \&\& \ (\text{GtQ}[a, 0] \ || \ \text{LtQ}[b, 0])$

rule 224  $\text{Int}[1/\text{Sqrt}[(a_ + (b_ \cdot)(x_ )^2), x\_Symbol] \rightarrow \text{Subst}[\text{Int}[1/(1 - b*x^2), x], x, x/\text{Sqrt}[a + b*x^2]] /; \text{FreeQ}[\{a, b\}, x] \ \&\& \ !\text{GtQ}[a, 0]$

rule 3042  $\text{Int}[u_ , x\_Symbol] \rightarrow \text{Int}[\text{DeactivateTrig}[u, x], x] /; \text{FunctionOfTrigOfLinearQ}[u, x]$

rule 4610  $\text{Int}[(b_ \cdot)\text{sec}[(e_ \cdot) + (f_ \cdot)(x_ )]^2)^{p_ }, x\_Symbol] \rightarrow \text{With}[\{\text{ff} = \text{FreeFactors}[\text{Tan}[e + f*x], x]\}, \text{Simp}[b*(\text{ff}/f) \text{Subst}[\text{Int}[(b + b*\text{ff}^2*x^2)^{p - 1}, x], x, \text{Tan}[e + f*x]/\text{ff}], x] /; \text{FreeQ}[\{b, e, f, p\}, x] \ \&\& \ !\text{IntegerQ}[p]$

## Maple [A] (verified)

Time = 0.10 (sec) , antiderivative size = 28, normalized size of antiderivative = 0.61

method	result
default	$\left(\frac{\ln(-\cot(x)+\csc(x))\sin(x)}{2} - \frac{\cot(x)}{2}\right) a\sqrt{a\csc(x)^2}$
risch	$-\frac{ia\sqrt{-\frac{ae^{2ix}}{(e^{2ix}-1)^2}}(e^{2ix}+1)}{e^{2ix}-1} + a\sqrt{-\frac{ae^{2ix}}{(e^{2ix}-1)^2}} \ln(e^{ix}-1)\sin(x) - a\sqrt{-\frac{ae^{2ix}}{(e^{2ix}-1)^2}} \ln(e^{ix}+1)\sin(x)$

input  $\text{int}((a*\csc(x)^2)^{(3/2}), x, \text{method}=\_RETURNVERBOSE)$

output  $(1/2*\ln(-\cot(x)+\csc(x))*\sin(x)-1/2*\cot(x))*a*(a*\csc(x)^2)^{(1/2)}$

**Fricas [A] (verification not implemented)**

Time = 0.08 (sec) , antiderivative size = 49, normalized size of antiderivative = 1.07

$$\int (a \csc^2(x))^{3/2} dx = -\frac{\left(2a \cos(x) + (a \cos(x)^2 - a) \log\left(-\frac{\cos(x)-1}{\cos(x)+1}\right)\right) \sqrt{-\frac{a}{\cos(x)^2-1}}}{4 \sin(x)}$$

input `integrate((a*csc(x)^2)^(3/2),x, algorithm="fricas")`

output `-1/4*(2*a*cos(x) + (a*cos(x)^2 - a)*log(-(cos(x) - 1)/(cos(x) + 1)))*sqrt(-a/(cos(x)^2 - 1))/sin(x)`

**Sympy [F]**

$$\int (a \csc^2(x))^{3/2} dx = \int (a \csc^2(x))^{\frac{3}{2}} dx$$

input `integrate((a*csc(x)**2)**(3/2),x)`

output `Integral((a*csc(x)**2)**(3/2), x)`

**Maxima [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 318 vs. 2(34) = 68.

Time = 0.18 (sec) , antiderivative size = 318, normalized size of antiderivative = 6.91

$$\int (a \csc^2(x))^{3/2} dx = \frac{(4a \cos(3x) \sin(2x) + 4a \cos(x) \sin(2x) - 4a \cos(2x) \sin(x) - (a \cos(4x)^2 + 4a \cos(2x)^2 + a \sin(4x))) \sqrt{-\frac{a}{\cos(x)^2-1}}}{4 \sin(x)}$$

input `integrate((a*csc(x)^2)^(3/2),x, algorithm="maxima")`

output

```
-1/2*(4*a*cos(3*x)*sin(2*x) + 4*a*cos(x)*sin(2*x) - 4*a*cos(2*x)*sin(x) -
(a*cos(4*x)^2 + 4*a*cos(2*x)^2 + a*sin(4*x)^2 - 4*a*sin(4*x)*sin(2*x) + 4*
a*sin(2*x)^2 - 2*(2*a*cos(2*x) - a)*cos(4*x) - 4*a*cos(2*x) + a)*arctan2(s
in(x), cos(x) + 1) + (a*cos(4*x)^2 + 4*a*cos(2*x)^2 + a*sin(4*x)^2 - 4*a*s
in(4*x)*sin(2*x) + 4*a*sin(2*x)^2 - 2*(2*a*cos(2*x) - a)*cos(4*x) - 4*a*co
s(2*x) + a)*arctan2(sin(x), cos(x) - 1) + 2*(a*sin(3*x) + a*sin(x))*cos(4*
x) - 2*(a*cos(3*x) + a*cos(x))*sin(4*x) - 2*(2*a*cos(2*x) - a)*sin(3*x) +
2*a*sin(x)*sqrt(-a)/(2*(2*cos(2*x) - 1)*cos(4*x) - cos(4*x)^2 - 4*cos(2*x
)^2 - sin(4*x)^2 + 4*sin(4*x)*sin(2*x) - 4*sin(2*x)^2 + 4*cos(2*x) - 1)
```

**Giac [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 72 vs.  $2(34) = 68$ .

Time = 0.12 (sec) , antiderivative size = 72, normalized size of antiderivative = 1.57

$$\int (a \csc^2(x))^{3/2} dx = \frac{1}{8} \left( 2 \log \left( -\frac{\cos(x) - 1}{\cos(x) + 1} \right) \operatorname{sgn}(\sin(x)) - \frac{\left( \frac{2(\cos(x) - 1)\operatorname{sgn}(\sin(x))}{\cos(x) + 1} - \operatorname{sgn}(\sin(x)) \right) (\cos(x) - 1)}{\cos(x) - 1} \right)$$

input

```
integrate((a*csc(x)^2)^(3/2),x, algorithm="giac")
```

output

```
1/8*(2*log(-(cos(x) - 1)/(cos(x) + 1))*sgn(sin(x)) - (2*(cos(x) - 1)*sgn(s
in(x))/(cos(x) + 1) - sgn(sin(x))*(cos(x) + 1)/(cos(x) - 1) - (cos(x) - 1
)*sgn(sin(x))/(cos(x) + 1))*a^(3/2)
```

**Mupad [F(-1)]**

Timed out.

$$\int (a \csc^2(x))^{3/2} dx = \int \left( \frac{a}{\sin(x)^2} \right)^{3/2} dx$$

input

```
int((a/sin(x)^2)^(3/2),x)
```

output

```
int((a/sin(x)^2)^(3/2), x)
```

**Reduce [B] (verification not implemented)**

Time = 0.18 (sec) , antiderivative size = 24, normalized size of antiderivative = 0.52

$$\int (a \csc^2(x))^{3/2} dx = \frac{\sqrt{a} a (-\cos(x) + \log(\tan(\frac{x}{2}))) \sin(x)^2}{2 \sin(x)^2}$$

input `int((a*csc(x)^2)^(3/2),x)`

output `(sqrt(a)*a*(-cos(x)+log(tan(x/2))*sin(x)**2))/(2*sin(x)**2)`

### 3.50 $\int \sqrt{a \csc^2(x)} dx$

Optimal result	327
Mathematica [A] (verified)	327
Rubi [A] (verified)	328
Maple [A] (verified)	329
Fricas [A] (verification not implemented)	330
Sympy [F]	330
Maxima [A] (verification not implemented)	330
Giac [A] (verification not implemented)	331
Mupad [F(-1)]	331
Reduce [B] (verification not implemented)	331

#### Optimal result

Integrand size = 10, antiderivative size = 26

$$\int \sqrt{a \csc^2(x)} dx = -\sqrt{a} \operatorname{arctanh} \left( \frac{\sqrt{a} \cot(x)}{\sqrt{a \csc^2(x)}} \right)$$

output

```
-a^(1/2)*arctanh(a^(1/2)*cot(x)/(a*csc(x)^2)^(1/2))
```

#### Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 17, normalized size of antiderivative = 0.65

$$\int \sqrt{a \csc^2(x)} dx = -\operatorname{arctanh}(\cos(x)) \sqrt{a \csc^2(x)} \sin(x)$$

input

```
Integrate[Sqrt[a*Csc[x]^2],x]
```

output

```
-(ArcTanh[Cos[x]]*Sqrt[a*Csc[x]^2]*Sin[x])
```



**Rubi [A] (verified)**

Time = 0.20 (sec) , antiderivative size = 28, normalized size of antiderivative = 1.08, number of steps used = 5, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.400$ , Rules used = {3042, 4610, 224, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \sqrt{a \csc^2(x)} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \sqrt{a \sec\left(x + \frac{\pi}{2}\right)^2} dx \\
 & \quad \downarrow \text{4610} \\
 & -a \int \frac{1}{\sqrt{a \cot^2(x) + a}} d \cot(x) \\
 & \quad \downarrow \text{224} \\
 & -a \int \frac{1}{1 - \frac{a \cot^2(x)}{a \cot^2(x) + a}} d \frac{\cot(x)}{\sqrt{a \cot^2(x) + a}} \\
 & \quad \downarrow \text{219} \\
 & -\sqrt{a} \operatorname{arctanh}\left(\frac{\sqrt{a} \cot(x)}{\sqrt{a \cot^2(x) + a}}\right)
 \end{aligned}$$

input `Int [Sqrt [a*Csc [x]^2] , x]`

output `-(Sqrt [a]*ArcTanh [(Sqrt [a]*Cot [x])/Sqrt [a + a*Cot [x]^2]])`

## Definitions of rubi rules used

rule 219 `Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (GtQ[a, 0] || LtQ[b, 0])`

rule 224 `Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] := Subst[Int[1/(1 - b*x^2), x], x, x/Sqrt[a + b*x^2]] /; FreeQ[{a, b}, x] && !GtQ[a, 0]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 4610 `Int[((b_.)*sec[(e_.) + (f_.)*(x_)]^2)^(p_), x_Symbol] := With[{ff = FreeFactors[Tan[e + f*x], x]}, Simp[b*(ff/f) Subst[Int[(b + b*ff^2*x^2)^(p - 1), x], x, Tan[e + f*x]/ff], x] /; FreeQ[{b, e, f, p}, x] && !IntegerQ[p]`

## Maple [A] (verified)

Time = 0.06 (sec) , antiderivative size = 20, normalized size of antiderivative = 0.77

method	result	size
default	$\ln(-\cot(x) + \csc(x)) \sin(x) \sqrt{a \csc(x)^2}$	20
risch	$2\sqrt{-\frac{ae^{2ix}}{(e^{2ix}-1)^2}} \ln(e^{ix}-1) \sin(x) - 2\sqrt{-\frac{ae^{2ix}}{(e^{2ix}-1)^2}} \ln(e^{ix}+1) \sin(x)$	64

input `int((a*csc(x)^2)^(1/2), x, method=_RETURNVERBOSE)`

output `ln(-cot(x)+csc(x))*sin(x)*(a*csc(x)^2)^(1/2)`

**Fricas [A] (verification not implemented)**

Time = 0.08 (sec) , antiderivative size = 64, normalized size of antiderivative = 2.46

$$\int \sqrt{a \csc^2(x)} dx$$

$$= \left[ \frac{1}{2} \sqrt{-\frac{a}{\cos(x)^2 - 1}} \log\left(-\frac{\cos(x) - 1}{\cos(x) + 1}\right) \sin(x), \sqrt{-a} \arctan\left(\frac{\sqrt{-a} \sqrt{-\frac{a}{\cos(x)^2 - 1}} \cos(x) \sin(x)}{a}\right) \right]$$

input `integrate((a*csc(x)^2)^(1/2),x, algorithm="fricas")`output `[1/2*sqrt(-a/(cos(x)^2 - 1))*log(-(cos(x) - 1)/(cos(x) + 1))*sin(x), sqrt(-a)*arctan(sqrt(-a)*sqrt(-a/(cos(x)^2 - 1))*cos(x)*sin(x)/a)]`**Sympy [F]**

$$\int \sqrt{a \csc^2(x)} dx = \int \sqrt{a \csc^2(x)} dx$$

input `integrate((a*csc(x)**2)**(1/2),x)`output `Integral(sqrt(a*csc(x)**2), x)`**Maxima [A] (verification not implemented)**

Time = 0.15 (sec) , antiderivative size = 24, normalized size of antiderivative = 0.92

$$\int \sqrt{a \csc^2(x)} dx = -\sqrt{-a}(\arctan(\sin(x), \cos(x) + 1) - \arctan(\sin(x), \cos(x) - 1))$$

input `integrate((a*csc(x)^2)^(1/2),x, algorithm="maxima")`output `-sqrt(-a)*(arctan2(sin(x), cos(x) + 1) - arctan2(sin(x), cos(x) - 1))`

**Giac [A] (verification not implemented)**

Time = 0.17 (sec) , antiderivative size = 13, normalized size of antiderivative = 0.50

$$\int \sqrt{a \csc^2(x)} dx = \sqrt{a} \log \left( \left| \tan \left( \frac{1}{2} x \right) \right| \right) \operatorname{sgn}(\sin(x))$$

input `integrate((a*csc(x)^2)^(1/2),x, algorithm="giac")`

output `sqrt(a)*log(abs(tan(1/2*x)))*sgn(sin(x))`

**Mupad [F(-1)]**

Timed out.

$$\int \sqrt{a \csc^2(x)} dx = \int \sqrt{\frac{a}{\sin(x)^2}} dx$$

input `int((a/sin(x)^2)^(1/2),x)`

output `int((a/sin(x)^2)^(1/2), x)`

**Reduce [B] (verification not implemented)**

Time = 0.16 (sec) , antiderivative size = 8, normalized size of antiderivative = 0.31

$$\int \sqrt{a \csc^2(x)} dx = \sqrt{a} \log \left( \tan \left( \frac{x}{2} \right) \right)$$

input `int((a*csc(x)^2)^(1/2),x)`

output `sqrt(a)*log(tan(x/2))`

### 3.51 $\int \frac{1}{\sqrt{a \csc^2(x)}} dx$

Optimal result	332
Mathematica [A] (verified)	332
Rubi [A] (verified)	333
Maple [A] (verified)	334
Fricas [A] (verification not implemented)	334
Sympy [A] (verification not implemented)	335
Maxima [A] (verification not implemented)	335
Giac [B] (verification not implemented)	335
Mupad [B] (verification not implemented)	336
Reduce [B] (verification not implemented)	336

#### Optimal result

Integrand size = 10, antiderivative size = 14

$$\int \frac{1}{\sqrt{a \csc^2(x)}} dx = -\frac{\cot(x)}{\sqrt{a \csc^2(x)}}$$

output

```
-cot(x)/(a*csc(x)^2)^(1/2)
```

#### Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 14, normalized size of antiderivative = 1.00

$$\int \frac{1}{\sqrt{a \csc^2(x)}} dx = -\frac{\cot(x)}{\sqrt{a \csc^2(x)}}$$

input

```
Integrate[1/Sqrt[a*Csc[x]^2],x]
```

output

```
-(Cot[x]/Sqrt[a*Csc[x]^2])
```

**Rubi [A] (verified)**

Time = 0.19 (sec) , antiderivative size = 16, normalized size of antiderivative = 1.14, number of steps used = 4, number of rules used = 3,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.300$ , Rules used = {3042, 4610, 208}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{array}{c}
 \int \frac{1}{\sqrt{a \csc^2(x)}} dx \\
 \downarrow \text{3042} \\
 \int \frac{1}{\sqrt{a \sec(x + \frac{\pi}{2})^2}} dx \\
 \downarrow \text{4610} \\
 -a \int \frac{1}{(a \cot^2(x) + a)^{3/2}} d \cot(x) \\
 \downarrow \text{208} \\
 -\frac{\cot(x)}{\sqrt{a \cot^2(x) + a}}
 \end{array}$$

input `Int[1/Sqrt[a*Csc[x]^2],x]`

output `-(Cot[x]/Sqrt[a + a*Cot[x]^2])`

**Defintions of rubi rules used**

rule 208 `Int[((a_) + (b_.)*(x_)^2)^(-3/2), x_Symbol] := Simp[x/(a*Sqrt[a + b*x^2]), x] /; FreeQ[{a, b}, x]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 4610

```
Int[((b_.)*sec[(e_.) + (f_.)*(x_)]^2)^(p_), x_Symbol] := With[{ff = FreeFac
tors[Tan[e + f*x], x]}, Simp[b*(ff/f) Subst[Int[(b + b*ff^2*x^2)^(p - 1),
x], x, Tan[e + f*x]/ff], x]] /; FreeQ[{b, e, f, p}, x] && !IntegerQ[p]
```

**Maple [A] (verified)**

Time = 0.06 (sec) , antiderivative size = 18, normalized size of antiderivative = 1.29

method	result	size
default	$\frac{\sin(x)}{\sqrt{a \csc(x)^2 (\cos(x)-1)}}$	18
risch	$-\frac{ie^{2ix}}{2\sqrt{-\frac{ae^{2ix}}{(e^{2ix}-1)^2}}(e^{2ix}-1)} - \frac{i}{2(e^{2ix}-1)\sqrt{-\frac{ae^{2ix}}{(e^{2ix}-1)^2}}}$	69

input

```
int(1/(a*csc(x)^2)^(1/2),x,method=_RETURNVERBOSE)
```

output

```
sin(x)/(a*csc(x)^2)^(1/2)/(cos(x)-1)
```

**Fricas [A] (verification not implemented)**

Time = 0.07 (sec) , antiderivative size = 22, normalized size of antiderivative = 1.57

$$\int \frac{1}{\sqrt{a \csc^2(x)}} dx = -\frac{\sqrt{-\frac{a}{\cos(x)^2-1}} \cos(x) \sin(x)}{a}$$

input

```
integrate(1/(a*csc(x)^2)^(1/2),x, algorithm="fricas")
```

output

```
-sqrt(-a/(cos(x)^2 - 1))*cos(x)*sin(x)/a
```

**Sympy [A] (verification not implemented)**

Time = 0.29 (sec) , antiderivative size = 14, normalized size of antiderivative = 1.00

$$\int \frac{1}{\sqrt{a \csc^2(x)}} dx = -\frac{\cot(x)}{\sqrt{a \csc^2(x)}}$$

input `integrate(1/(a*csc(x)**2)**(1/2),x)`

output `-cot(x)/sqrt(a*csc(x)**2)`

**Maxima [A] (verification not implemented)**

Time = 0.11 (sec) , antiderivative size = 13, normalized size of antiderivative = 0.93

$$\int \frac{1}{\sqrt{a \csc^2(x)}} dx = -\frac{1}{\sqrt{\tan(x)^2 + 1}\sqrt{a}}$$

input `integrate(1/(a*csc(x)^2)^(1/2),x, algorithm="maxima")`

output `-1/(sqrt(tan(x)^2 + 1)*sqrt(a))`

**Giac [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 34 vs. 2(12) = 24.

Time = 0.16 (sec) , antiderivative size = 34, normalized size of antiderivative = 2.43

$$\int \frac{1}{\sqrt{a \csc^2(x)}} dx = \frac{2 \operatorname{sgn}(\sin(x))}{\sqrt{a}} + \frac{2}{\sqrt{a} \left( \frac{\cos(x)-1}{\cos(x)+1} - 1 \right) \operatorname{sgn}(\sin(x))}$$

input `integrate(1/(a*csc(x)^2)^(1/2),x, algorithm="giac")`

output `2*sgn(sin(x))/sqrt(a) + 2/(sqrt(a)*((cos(x) - 1)/(cos(x) + 1) - 1)*sgn(sin(x)))`



**Mupad [B] (verification not implemented)**

Time = 14.90 (sec) , antiderivative size = 15, normalized size of antiderivative = 1.07

$$\int \frac{1}{\sqrt{a} \csc^2(x)} dx = -\frac{\sin(2x)}{2\sqrt{a} \sqrt{\sin(x)^2}}$$

input `int(1/(a/sin(x)^2)^(1/2),x)`output `-sin(2*x)/(2*a^(1/2)*(sin(x)^2)^(1/2))`**Reduce [B] (verification not implemented)**

Time = 0.16 (sec) , antiderivative size = 12, normalized size of antiderivative = 0.86

$$\int \frac{1}{\sqrt{a} \csc^2(x)} dx = \frac{\sqrt{a}(-\cos(x) + 1)}{a}$$

input `int(1/(a*csc(x)^2)^(1/2),x)`output `(sqrt(a)*(-cos(x) + 1))/a`

### 3.52 $\int \frac{1}{(a \csc^2(x))^{3/2}} dx$

Optimal result	337
Mathematica [A] (verified)	337
Rubi [A] (verified)	338
Maple [A] (verified)	339
Fricas [A] (verification not implemented)	340
Sympy [A] (verification not implemented)	340
Maxima [F]	340
Giac [A] (verification not implemented)	341
Mupad [F(-1)]	341
Reduce [B] (verification not implemented)	341

#### Optimal result

Integrand size = 10, antiderivative size = 36

$$\int \frac{1}{(a \csc^2(x))^{3/2}} dx = -\frac{\cot(x)}{3(a \csc^2(x))^{3/2}} - \frac{2 \cot(x)}{3a\sqrt{a \csc^2(x)}}$$

output `-1/3*cot(x)/(a*csc(x)^2)^(3/2)-2/3*cot(x)/a/(a*csc(x)^2)^(1/2)`

#### Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 27, normalized size of antiderivative = 0.75

$$\int \frac{1}{(a \csc^2(x))^{3/2}} dx = \frac{(-9 \cos(x) + \cos(3x)) \csc^3(x)}{12 (a \csc^2(x))^{3/2}}$$

input `Integrate[(a*Csc[x]^2)^(-3/2),x]`

output `((-9*Cos[x] + Cos[3*x])*Csc[x]^3)/(12*(a*Csc[x]^2)^(3/2))`

**Rubi [A] (verified)**

Time = 0.20 (sec) , antiderivative size = 46, normalized size of antiderivative = 1.28, number of steps used = 5, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.400$ , Rules used = {3042, 4610, 209, 208}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{1}{(a \csc^2(x))^{3/2}} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{1}{\left(a \sec\left(x + \frac{\pi}{2}\right)\right)^{3/2}} dx \\
 & \quad \downarrow \text{4610} \\
 & -a \int \frac{1}{(a \cot^2(x) + a)^{5/2}} d \cot(x) \\
 & \quad \downarrow \text{209} \\
 & -a \left( \frac{2 \int \frac{1}{(a \cot^2(x) + a)^{3/2}} d \cot(x)}{3a} + \frac{\cot(x)}{3a (a \cot^2(x) + a)^{3/2}} \right) \\
 & \quad \downarrow \text{208} \\
 & -a \left( \frac{2 \cot(x)}{3a^2 \sqrt{a \cot^2(x) + a}} + \frac{\cot(x)}{3a (a \cot^2(x) + a)^{3/2}} \right)
 \end{aligned}$$

input `Int[(a*Csc[x]^2)^(-3/2),x]`

output `-(a*(Cot[x]/(3*a*(a + a*Cot[x]^2)^(3/2)) + (2*Cot[x])/(3*a^2*Sqrt[a + a*Cot[x]^2])))`

## Definitions of rubi rules used

rule 208 `Int[((a_) + (b_.)*(x_)^2)^(-3/2), x_Symbol] := Simp[x/(a*Sqrt[a + b*x^2]), x] /; FreeQ[{a, b}, x]`

rule 209 `Int[((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] := Simp[(-x)*((a + b*x^2)^(p + 1)/(2*a*(p + 1))), x] + Simp[(2*p + 3)/(2*a*(p + 1)) Int[(a + b*x^2)^(p + 1), x], x] /; FreeQ[{a, b}, x] && ILtQ[p + 3/2, 0]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 4610 `Int[((b_.)*sec[(e_.) + (f_.)*(x_)]^2)^(p_), x_Symbol] := With[{ff = FreeFactors[Tan[e + f*x], x]}, Simp[b*(ff/f) Subst[Int[(b + b*ff^2*x^2)^(p - 1), x], x, Tan[e + f*x]/ff], x] /; FreeQ[{b, e, f, p}, x] && !IntegerQ[p]`

## Maple [A] (verified)

Time = 0.07 (sec) , antiderivative size = 33, normalized size of antiderivative = 0.92

method	result	size
default	$\frac{\sqrt{4} (\cos(x)^2 \cot(x) - 3 \cot(x) - 2 \csc(x))}{6a \sqrt{a \csc(x)^2}}$	33
risch	$\frac{ie^{4ix}}{24a(e^{2ix}-1)\sqrt{-\frac{ae^{2ix}}{(e^{2ix}-1)^2}}} - \frac{3ie^{2ix}}{8a(e^{2ix}-1)\sqrt{-\frac{ae^{2ix}}{(e^{2ix}-1)^2}}} - \frac{3i}{8\sqrt{-\frac{ae^{2ix}}{(e^{2ix}-1)^2}}(e^{2ix}-1)a} + \frac{ie^{-2ix}}{24a(e^{2ix}-1)\sqrt{-\frac{ae^{2ix}}{(e^{2ix}-1)^2}}}$	155

input `int(1/(a*csc(x)^2)^(3/2),x,method=_RETURNVERBOSE)`

output `1/6*4^(1/2)/a/(a*csc(x)^2)^(1/2)*(cos(x)^2*cot(x)-3*cot(x)-2*csc(x))`

**Fricas [A] (verification not implemented)**

Time = 0.07 (sec) , antiderivative size = 29, normalized size of antiderivative = 0.81

$$\int \frac{1}{(a \csc^2(x))^{3/2}} dx = \frac{(\cos(x)^3 - 3 \cos(x)) \sqrt{-\frac{a}{\cos(x)^2 - 1}} \sin(x)}{3 a^2}$$

input `integrate(1/(a*csc(x)^2)^(3/2),x, algorithm="fricas")`output `1/3*(cos(x)^3 - 3*cos(x))*sqrt(-a/(cos(x)^2 - 1))*sin(x)/a^2`**Sympy [A] (verification not implemented)**

Time = 0.43 (sec) , antiderivative size = 32, normalized size of antiderivative = 0.89

$$\int \frac{1}{(a \csc^2(x))^{3/2}} dx = -\frac{2 \cot^3(x)}{3 (a \csc^2(x))^{\frac{3}{2}}} - \frac{\cot(x)}{(a \csc^2(x))^{\frac{3}{2}}}$$

input `integrate(1/(a*csc(x)**2)**(3/2),x)`output `-2*cot(x)**3/(3*(a*csc(x)**2)**(3/2)) - cot(x)/(a*csc(x)**2)**(3/2)`**Maxima [F]**

$$\int \frac{1}{(a \csc^2(x))^{3/2}} dx = \int \frac{1}{(a \csc(x)^2)^{\frac{3}{2}}} dx$$

input `integrate(1/(a*csc(x)^2)^(3/2),x, algorithm="maxima")`output `integrate((a*csc(x)^2)^(-3/2), x)`

**Giac [A] (verification not implemented)**

Time = 0.17 (sec) , antiderivative size = 52, normalized size of antiderivative = 1.44

$$\int \frac{1}{(a \csc^2(x))^{3/2}} dx = \frac{4 \left( \frac{\operatorname{sgn}(\sin(x))}{\sqrt{a}} - \frac{\frac{3(\cos(x)-1)-1}{\cos(x)+1}}{\sqrt{a} \left( \frac{\cos(x)-1}{\cos(x)+1} - 1 \right)^3 \operatorname{sgn}(\sin(x))} \right)}{3a}$$

input `integrate(1/(a*csc(x)^2)^(3/2),x, algorithm="giac")`output `4/3*(sgn(sin(x))/sqrt(a) - (3*(cos(x) - 1)/(cos(x) + 1) - 1)/(sqrt(a)*((cos(x) - 1)/(cos(x) + 1) - 1)^3*sgn(sin(x))))/a`**Mupad [F(-1)]**

Timed out.

$$\int \frac{1}{(a \csc^2(x))^{3/2}} dx = \int \frac{1}{\left( \frac{a}{\sin(x)^2} \right)^{3/2}} dx$$

input `int(1/(a/sin(x)^2)^(3/2),x)`output `int(1/(a/sin(x)^2)^(3/2), x)`**Reduce [B] (verification not implemented)**

Time = 0.17 (sec) , antiderivative size = 21, normalized size of antiderivative = 0.58

$$\int \frac{1}{(a \csc^2(x))^{3/2}} dx = \frac{\sqrt{a} (-\cos(x) \sin(x))^2 - 2 \cos(x) + 2}{3a^2}$$

input `int(1/(a*csc(x)^2)^(3/2),x)`output `(sqrt(a)*(-cos(x)*sin(x)**2 - 2*cos(x) + 2))/(3*a**2)`

### 3.53 $\int \frac{1}{(a \csc^2(x))^{5/2}} dx$

Optimal result	342
Mathematica [A] (verified)	342
Rubi [A] (verified)	343
Maple [A] (verified)	344
Fricas [A] (verification not implemented)	345
Sympy [A] (verification not implemented)	345
Maxima [F]	346
Giac [A] (verification not implemented)	346
Mupad [F(-1)]	346
Reduce [B] (verification not implemented)	347

#### Optimal result

Integrand size = 10, antiderivative size = 55

$$\int \frac{1}{(a \csc^2(x))^{5/2}} dx = -\frac{\cot(x)}{5 (a \csc^2(x))^{5/2}} - \frac{4 \cot(x)}{15a (a \csc^2(x))^{3/2}} - \frac{8 \cot(x)}{15a^2 \sqrt{a \csc^2(x)}}$$

output

$-1/5*\cot(x)/(a*\csc(x)^2)^{(5/2)}-4/15*\cot(x)/a/(a*\csc(x)^2)^{(3/2)}-8/15*\cot(x)/a^2/(a*\csc(x)^2)^{(1/2)}$

#### Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 36, normalized size of antiderivative = 0.65

$$\int \frac{1}{(a \csc^2(x))^{5/2}} dx = -\frac{(150 \cos(x) - 25 \cos(3x) + 3 \cos(5x)) \sqrt{a \csc^2(x)} \sin(x)}{240a^3}$$

input

`Integrate[(a*Csc[x]^2)^(-5/2),x]`

output

$-1/240*((150*\cos[x] - 25*\cos[3*x] + 3*\cos[5*x])*sqrt[a*\csc[x]^2]*\sin[x])/a^3$

**Rubi [A] (verified)**

Time = 0.21 (sec) , antiderivative size = 75, normalized size of antiderivative = 1.36, number of steps used = 6, number of rules used = 5,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {3042, 4610, 209, 209, 208}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{1}{(a \csc^2(x))^{5/2}} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{1}{\left(a \sec\left(x + \frac{\pi}{2}\right)\right)^{5/2}} dx \\
 & \quad \downarrow \text{4610} \\
 & -a \int \frac{1}{(a \cot^2(x) + a)^{7/2}} d \cot(x) \\
 & \quad \downarrow \text{209} \\
 & -a \left( \frac{4 \int \frac{1}{(a \cot^2(x) + a)^{5/2}} d \cot(x)}{5a} + \frac{\cot(x)}{5a (a \cot^2(x) + a)^{5/2}} \right) \\
 & \quad \downarrow \text{209} \\
 & -a \left( \frac{4 \left( \frac{2 \int \frac{1}{(a \cot^2(x) + a)^{3/2}} d \cot(x)}{3a} + \frac{\cot(x)}{3a (a \cot^2(x) + a)^{3/2}} \right)}{5a} + \frac{\cot(x)}{5a (a \cot^2(x) + a)^{5/2}} \right) \\
 & \quad \downarrow \text{208} \\
 & -a \left( \frac{4 \left( \frac{2 \cot(x)}{3a^2 \sqrt{a \cot^2(x) + a}} + \frac{\cot(x)}{3a (a \cot^2(x) + a)^{3/2}} \right)}{5a} + \frac{\cot(x)}{5a (a \cot^2(x) + a)^{5/2}} \right)
 \end{aligned}$$

input

```
Int[(a*Csc[x]^2)^(-5/2),x]
```



output 
$$-\frac{(a \cot(x) / (5a(a + a \cot(x)^2)^{5/2}) + (4 \cot(x) / (3a(a + a \cot(x)^2)^{3/2}) + (2 \cot(x) / (3a^2 \sqrt{a + a \cot(x)^2}))) / (5a))}{(5a)}$$

### Defintions of rubi rules used

rule 208 
$$\text{Int}[(a_ + (b_ \cdot (x_ )^2)^{-3/2}), x\_Symbol] \rightarrow \text{Simp}[x/(a \sqrt{a + b x^2}), x] \text{ /; FreeQ}[\{a, b\}, x]$$

rule 209 
$$\text{Int}[(a_ + (b_ \cdot (x_ )^2)^{p_}), x\_Symbol] \rightarrow \text{Simp}[(-x) \cdot (a + b x^2)^{p+1} / (2a(p+1)), x] + \text{Simp}[(2p+3)/(2a(p+1)) \text{ Int}[(a + b x^2)^{p+1}], x], x] \text{ /; FreeQ}[\{a, b\}, x] \ \&\& \ \text{ILtQ}[p + 3/2, 0]$$

rule 3042 
$$\text{Int}[u_, x\_Symbol] \rightarrow \text{Int}[\text{DeactivateTrig}[u, x], x] \text{ /; FunctionOfTrigOfLinearQ}[u, x]$$

rule 4610 
$$\text{Int}[(b_ \cdot \sec[e_ + (f_ \cdot (x_ )^2)^{p_}), x\_Symbol] \rightarrow \text{With}[\{ff = \text{FreeFactors}[\text{Tan}[e + f x], x]\}, \text{Simp}[b \cdot (ff/f) \text{ Subst}[\text{Int}[(b + b ff^2 x^2)^{p-1}], x], x, \text{Tan}[e + f x]/ff], x] \text{ /; FreeQ}[\{b, e, f, p\}, x] \ \&\& \ \text{!IntegerQ}[p]$$

### Maple [A] (verified)

Time = 0.10 (sec) , antiderivative size = 37, normalized size of antiderivative = 0.67

method	result
default	$-\frac{\sqrt{4} (3 \cos(x)^5 - 10 \cos(x)^3 + 15 \cos(x) + 8) \csc(x)}{30a^2 \sqrt{a \csc(x)^2}}$
risch	$-\frac{ie^{6ix}}{160a^2(e^{2ix}-1)\sqrt{-\frac{ae^{2ix}}{(e^{2ix}-1)^2}}} - \frac{5ie^{2ix}}{16a^2(e^{2ix}-1)\sqrt{-\frac{ae^{2ix}}{(e^{2ix}-1)^2}}} - \frac{5i}{16\sqrt{-\frac{ae^{2ix}}{(e^{2ix}-1)^2}}(e^{2ix}-1)a^2} + \frac{5ie^{-2ix}}{96a^2(e^{2ix}-1)\sqrt{-\frac{ae^{2ix}}{(e^{2ix}-1)^2}}}$

input 
$$\text{int}(1/(a \csc(x)^2)^{5/2}, x, \text{method}=\_RETURNVERBOSE)$$

output

```
-1/30*4^(1/2)*(3*cos(x)^5-10*cos(x)^3+15*cos(x)+8)/a^2/(a*csc(x)^2)^(1/2)*
csc(x)
```

**Fricas [A] (verification not implemented)**

Time = 0.07 (sec) , antiderivative size = 37, normalized size of antiderivative = 0.67

$$\int \frac{1}{(a \csc^2(x))^{5/2}} dx = -\frac{(3 \cos(x)^5 - 10 \cos(x)^3 + 15 \cos(x)) \sqrt{-\frac{a}{\cos(x)^2 - 1}} \sin(x)}{15 a^3}$$

input

```
integrate(1/(a*csc(x)^2)^(5/2),x, algorithm="fricas")
```

output

```
-1/15*(3*cos(x)^5 - 10*cos(x)^3 + 15*cos(x))*sqrt(-a/(cos(x)^2 - 1))*sin(x)
)/a^3
```

**Sympy [A] (verification not implemented)**

Time = 1.47 (sec) , antiderivative size = 51, normalized size of antiderivative = 0.93

$$\int \frac{1}{(a \csc^2(x))^{5/2}} dx = -\frac{8 \cot^5(x)}{15 (a \csc^2(x))^{\frac{5}{2}}} - \frac{4 \cot^3(x)}{3 (a \csc^2(x))^{\frac{5}{2}}} - \frac{\cot(x)}{(a \csc^2(x))^{\frac{5}{2}}}$$

input

```
integrate(1/(a*csc(x)**2)**(5/2),x)
```

output

```
-8*cot(x)**5/(15*(a*csc(x)**2)**(5/2)) - 4*cot(x)**3/(3*(a*csc(x)**2)**(5/
2)) - cot(x)/(a*csc(x)**2)**(5/2)
```

**Maxima [F]**

$$\int \frac{1}{(a \csc^2(x))^{5/2}} dx = \int \frac{1}{(a \csc(x)^2)^{5/2}} dx$$

input `integrate(1/(a*csc(x)^2)^(5/2),x, algorithm="maxima")`

output `integrate((a*csc(x)^2)^(-5/2), x)`

**Giac [A] (verification not implemented)**

Time = 0.16 (sec) , antiderivative size = 66, normalized size of antiderivative = 1.20

$$\int \frac{1}{(a \csc^2(x))^{5/2}} dx = \frac{16 \left( \frac{\operatorname{sgn}(\sin(x))}{\sqrt{a}} - \frac{\frac{5(\cos(x)-1)}{\cos(x)+1} - \frac{10(\cos(x)-1)^2-1}{(\cos(x)+1)^2} - 1}{\sqrt{a} \left( \frac{\cos(x)-1}{\cos(x)+1} - 1 \right)^5 \operatorname{sgn}(\sin(x))} \right)}{15 a^2}$$

input `integrate(1/(a*csc(x)^2)^(5/2),x, algorithm="giac")`

output `16/15*(sgn(sin(x))/sqrt(a) - (5*(cos(x) - 1)/(cos(x) + 1) - 10*(cos(x) - 1)^2/(cos(x) + 1)^2 - 1)/(sqrt(a)*((cos(x) - 1)/(cos(x) + 1) - 1)^5*sgn(sin(x))))/a^2`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{1}{(a \csc^2(x))^{5/2}} dx = \int \frac{1}{\left(\frac{a}{\sin(x)^2}\right)^{5/2}} dx$$

input `int(1/(a/sin(x)^2)^(5/2),x)`

output `int(1/(a/sin(x)^2)^(5/2), x)`

**Reduce [B] (verification not implemented)**

Time = 0.17 (sec) , antiderivative size = 29, normalized size of antiderivative = 0.53

$$\int \frac{1}{(a \csc^2(x))^{5/2}} dx = \frac{\sqrt{a} (-3 \cos(x) \sin(x)^4 - 4 \cos(x) \sin(x)^2 - 8 \cos(x) + 8)}{15a^3}$$

input `int(1/(a*csc(x)^2)^(5/2),x)`

output `(sqrt(a)*(- 3*cos(x)*sin(x)**4 - 4*cos(x)*sin(x)**2 - 8*cos(x) + 8))/(15*a**3)`

**3.54**      $\int \frac{1}{(a \csc^2(x))^{7/2}} dx$

Optimal result	348
Mathematica [A] (verified)	348
Rubi [A] (verified)	349
Maple [A] (verified)	351
Fricas [A] (verification not implemented)	351
Sympy [A] (verification not implemented)	352
Maxima [F]	352
Giac [A] (verification not implemented)	353
Mupad [F(-1)]	353
Reduce [B] (verification not implemented)	353

**Optimal result**

Integrand size = 10, antiderivative size = 74

$$\int \frac{1}{(a \csc^2(x))^{7/2}} dx = -\frac{\cot(x)}{7(a \csc^2(x))^{7/2}} - \frac{6 \cot(x)}{35a(a \csc^2(x))^{5/2}} - \frac{8 \cot(x)}{35a^2(a \csc^2(x))^{3/2}} - \frac{16 \cot(x)}{35a^3 \sqrt{a \csc^2(x)}}$$

output -1/7\*cot(x)/(a\*csc(x)^2)^(7/2)-6/35\*cot(x)/a/(a\*csc(x)^2)^(5/2)-8/35\*cot(x)/a^2/(a\*csc(x)^2)^(3/2)-16/35\*cot(x)/a^3/(a\*csc(x)^2)^(1/2)

**Mathematica [A] (verified)**

Time = 0.06 (sec) , antiderivative size = 42, normalized size of antiderivative = 0.57

$$\int \frac{1}{(a \csc^2(x))^{7/2}} dx = \frac{(-1225 \cos(x) + 245 \cos(3x) - 49 \cos(5x) + 5 \cos(7x)) \sqrt{a \csc^2(x)} \sin(x)}{2240a^4}$$

input Integrate[(a\*Csc[x]^2)^(-7/2),x]

output

```
((-1225*Cos[x] + 245*Cos[3*x] - 49*Cos[5*x] + 5*Cos[7*x])*Sqrt[a*Csc[x]^2]
*Sin[x])/(2240*a^4)
```

**Rubi [A] (verified)**

Time = 0.23 (sec) , antiderivative size = 104, normalized size of antiderivative = 1.41, number of steps used = 7, number of rules used = 6,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.600$ , Rules used = {3042, 4610, 209, 209, 209, 208}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{1}{(a \csc^2(x))^{7/2}} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{1}{\left(a \sec\left(x + \frac{\pi}{2}\right)\right)^{7/2}} dx \\
 & \quad \downarrow \text{4610} \\
 & -a \int \frac{1}{(a \cot^2(x) + a)^{9/2}} d \cot(x) \\
 & \quad \downarrow \text{209} \\
 & -a \left( \frac{6 \int \frac{1}{(a \cot^2(x) + a)^{7/2}} d \cot(x)}{7a} + \frac{\cot(x)}{7a (a \cot^2(x) + a)^{7/2}} \right) \\
 & \quad \downarrow \text{209} \\
 & -a \left( \frac{6 \left( \frac{4 \int \frac{1}{(a \cot^2(x) + a)^{5/2}} d \cot(x)}{5a} + \frac{\cot(x)}{5a (a \cot^2(x) + a)^{5/2}} \right)}{7a} + \frac{\cot(x)}{7a (a \cot^2(x) + a)^{7/2}} \right) \\
 & \quad \downarrow \text{209}
 \end{aligned}$$

$$-a \left( \frac{6 \left( \frac{4 \left( \frac{2 \int \frac{1}{(a \cot^2(x)+a)^{3/2}} d \cot(x)}{3a} + \frac{\cot(x)}{3a(a \cot^2(x)+a)^{3/2}} \right)}{5a} + \frac{\cot(x)}{5a(a \cot^2(x)+a)^{5/2}} \right)}{7a} + \frac{\cot(x)}{7a(a \cot^2(x)+a)^{7/2}} \right)$$

↓ 208

$$-a \left( \frac{6 \left( \frac{4 \left( \frac{2 \cot(x)}{3a^2 \sqrt{a \cot^2(x)+a}} + \frac{\cot(x)}{3a(a \cot^2(x)+a)^{3/2}} \right)}{5a} + \frac{\cot(x)}{5a(a \cot^2(x)+a)^{5/2}} \right)}{7a} + \frac{\cot(x)}{7a(a \cot^2(x)+a)^{7/2}} \right)$$

input `Int[(a*Csc[x]^2)^(-7/2), x]`

output `-(a*(Cot[x]/(7*a*(a + a*Cot[x]^2)^(7/2)) + (6*(Cot[x]/(5*a*(a + a*Cot[x]^2)^(5/2)) + (4*(Cot[x]/(3*a*(a + a*Cot[x]^2)^(3/2)) + (2*Cot[x]/(3*a^2*Sqrt[a + a*Cot[x]^2])))/(5*a)))/(7*a))`

**Defintions of rubi rules used**

rule 208 `Int[((a_) + (b_.)*(x_)^2)^(-3/2), x_Symbol] := Simp[x/(a*Sqrt[a + b*x^2]), x] /; FreeQ[{a, b}, x]`

rule 209 `Int[((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] := Simp[(-x)*((a + b*x^2)^(p + 1)/(2*a*(p + 1))), x] + Simp[(2*p + 3)/(2*a*(p + 1)) Int[(a + b*x^2)^(p + 1), x], x] /; FreeQ[{a, b}, x] && ILtQ[p + 3/2, 0]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 4610 `Int[((b_.)*sec[(e_.) + (f_.)*(x_)]^2)^(p_), x_Symbol] := With[{ff = FreeFactors[Tan[e + f*x], x]}, Simp[b*(ff/f) Subst[Int[(b + b*ff^2*x^2)^(p - 1), x], x, Tan[e + f*x]/ff], x] /; FreeQ[{b, e, f, p}, x] && !IntegerQ[p]`

### Maple [A] (verified)

Time = 0.10 (sec) , antiderivative size = 43, normalized size of antiderivative = 0.58

method	result
default	$\frac{\sqrt{4} (5 \cos(x)^7 - 21 \cos(x)^5 + 35 \cos(x)^3 - 35 \cos(x) - 16) \csc(x)}{70a^3 \sqrt{a \csc(x)^2}}$
risch	$\frac{ie^{8ix}}{896a^3(e^{2ix}-1)\sqrt{-\frac{ae^{2ix}}{(e^{2ix}-1)^2}}} - \frac{35ie^{2ix}}{128a^3(e^{2ix}-1)\sqrt{-\frac{ae^{2ix}}{(e^{2ix}-1)^2}}} - \frac{35i}{128\sqrt{-\frac{ae^{2ix}}{(e^{2ix}-1)^2}}(e^{2ix}-1)a^3} + \frac{7ie^{-2ix}}{128a^3(e^{2ix}-1)\sqrt{-\frac{ae^{2ix}}{(e^{2ix}-1)^2}}}$

input `int(1/(a*csc(x)^2)^(7/2),x,method=_RETURNVERBOSE)`

output `1/70*4^(1/2)*(5*cos(x)^7-21*cos(x)^5+35*cos(x)^3-35*cos(x)-16)/a^3/(a*csc(x)^2)^(1/2)*csc(x)`

### Fricas [A] (verification not implemented)

Time = 0.08 (sec) , antiderivative size = 43, normalized size of antiderivative = 0.58

$$\int \frac{1}{(a \csc^2(x))^{7/2}} dx = \frac{(5 \cos(x)^7 - 21 \cos(x)^5 + 35 \cos(x)^3 - 35 \cos(x)) \sqrt{-\frac{a}{\cos(x)^2 - 1}} \sin(x)}{35 a^4}$$

input `integrate(1/(a*csc(x)^2)^(7/2),x, algorithm="fricas")`



output  $\frac{1}{35}(5\cos(x)^7 - 21\cos(x)^5 + 35\cos(x)^3 - 35\cos(x))\sqrt{-a/(\cos(x)^2 - 1)}\sin(x)/a^4$

### Sympy [A] (verification not implemented)

Time = 12.34 (sec) , antiderivative size = 68, normalized size of antiderivative = 0.92

$$\int \frac{1}{(a \csc^2(x))^{7/2}} dx = -\frac{16 \cot^7(x)}{35 (a \csc^2(x))^{7/2}} - \frac{8 \cot^5(x)}{5 (a \csc^2(x))^{7/2}} - \frac{2 \cot^3(x)}{(a \csc^2(x))^{7/2}} - \frac{\cot(x)}{(a \csc^2(x))^{7/2}}$$

input `integrate(1/(a*csc(x)**2)**(7/2),x)`

output  $-16*\cot(x)**7/(35*(a*csc(x)**2)**(7/2)) - 8*\cot(x)**5/(5*(a*csc(x)**2)**(7/2)) - 2*\cot(x)**3/(a*csc(x)**2)**(7/2) - \cot(x)/(a*csc(x)**2)**(7/2)$

### Maxima [F]

$$\int \frac{1}{(a \csc^2(x))^{7/2}} dx = \int \frac{1}{(a \csc(x)^2)^{7/2}} dx$$

input `integrate(1/(a*csc(x)^2)^(7/2),x, algorithm="maxima")`

output `integrate((a*csc(x)^2)^(-7/2), x)`

**Giac [A] (verification not implemented)**

Time = 0.13 (sec) , antiderivative size = 76, normalized size of antiderivative = 1.03

$$\int \frac{1}{(a \csc^2(x))^{7/2}} dx = \frac{32 \operatorname{sgn}(\sin(x))}{35 a^{7/2}} - \frac{32 \left( \frac{7(\cos(x)-1)}{\cos(x)+1} - \frac{21(\cos(x)-1)^2}{(\cos(x)+1)^2} + \frac{35(\cos(x)-1)^3}{(\cos(x)+1)^3} - 1 \right)}{35 a^{7/2} \left( \frac{\cos(x)-1}{\cos(x)+1} - 1 \right)^7 \operatorname{sgn}(\sin(x))}$$

input `integrate(1/(a*csc(x)^2)^(7/2),x, algorithm="giac")`output `32/35*sgn(sin(x))/a^(7/2) - 32/35*(7*(cos(x) - 1)/(cos(x) + 1) - 21*(cos(x) - 1)^2/(cos(x) + 1)^2 + 35*(cos(x) - 1)^3/(cos(x) + 1)^3 - 1)/(a^(7/2))*((cos(x) - 1)/(cos(x) + 1) - 1)^7*sgn(sin(x))`**Mupad [F(-1)]**

Timed out.

$$\int \frac{1}{(a \csc^2(x))^{7/2}} dx = \int \frac{1}{\left( \frac{a}{\sin(x)^2} \right)^{7/2}} dx$$

input `int(1/(a/sin(x)^2)^(7/2),x)`output `int(1/(a/sin(x)^2)^(7/2), x)`**Reduce [B] (verification not implemented)**

Time = 0.17 (sec) , antiderivative size = 37, normalized size of antiderivative = 0.50

$$\int \frac{1}{(a \csc^2(x))^{7/2}} dx = \frac{\sqrt{a} (-5 \cos(x) \sin(x)^6 - 6 \cos(x) \sin(x)^4 - 8 \cos(x) \sin(x)^2 - 16 \cos(x) + 16)}{35 a^4}$$

input `int(1/(a*csc(x)^2)^(7/2),x)`

output  $(\sqrt{a} * (-5 * \cos(x) * \sin(x)**6 - 6 * \cos(x) * \sin(x)**4 - 8 * \cos(x) * \sin(x)**2 - 16 * \cos(x) + 16)) / (35 * a**4)$

### 3.55 $\int (a \csc^3(x))^{5/2} dx$

Optimal result	355
Mathematica [A] (verified)	355
Rubi [A] (verified)	356
Maple [C] (verified)	359
Fricas [C] (verification not implemented)	359
Sympy [F]	360
Maxima [F]	360
Giac [F]	361
Mupad [F(-1)]	361
Reduce [F]	361

#### Optimal result

Integrand size = 10, antiderivative size = 123

$$\int (a \csc^3(x))^{5/2} dx = -\frac{154}{585}a^2 \cot(x) \sqrt{a \csc^3(x)} - \frac{22}{117}a^2 \cot(x) \csc^2(x) \sqrt{a \csc^3(x)} - \frac{2}{13}a^2 \cot(x) \csc^4(x) \sqrt{a \csc^3(x)} - \frac{154}{195}a^2 \cos(x) \sqrt{a \csc^3(x)} \sin(x) + \frac{154}{195}a^2 \sqrt{a \csc^3(x)} E\left(\frac{\pi}{4} - \frac{x}{2} \middle| 2\right) \sin^{\frac{3}{2}}(x)$$

output

```
-154/585*a^2*cot(x)*(a*csc(x)^3)^(1/2)-22/117*a^2*cot(x)*csc(x)^2*(a*csc(x)^3)^(1/2)-2/13*a^2*cot(x)*csc(x)^4*(a*csc(x)^3)^(1/2)-154/195*a^2*cos(x)*(a*csc(x)^3)^(1/2)*sin(x)+154/195*a^2*(a*csc(x)^3)^(1/2)*EllipticE(cos(1/4*Pi+1/2*x),2^(1/2))*sin(x)^(3/2)
```

#### Mathematica [A] (verified)

Time = 0.15 (sec) , antiderivative size = 58, normalized size of antiderivative = 0.47

$$\int (a \csc^3(x))^{5/2} dx = \frac{(a \csc^3(x))^{5/2} \left( 29568 E\left(\frac{1}{4}(\pi - 2x) \middle| 2\right) \sin^{\frac{15}{2}}(x) - 9414 \sin(2x) + 5346 \sin(4x) - 169 \right)}{37440}$$

input

```
Integrate[(a*Csc[x]^3)^(5/2),x]
```

output

```
((a*Csc[x]^3)^(5/2)*(29568*EllipticE[(Pi - 2*x)/4, 2]*Sin[x]^(15/2) - 9414
*Sin[2*x] + 5346*Sin[4*x] - 1694*Sin[6*x] + 231*Sin[8*x]))/37440
```

**Rubi [A] (verified)**

Time = 0.66 (sec) , antiderivative size = 121, normalized size of antiderivative = 0.98, number of steps used = 14, number of rules used = 14,  $\frac{\text{number of rules}}{\text{integrand size}} = 1.400$ , Rules used = {3042, 4611, 3042, 4255, 3042, 4255, 3042, 4255, 3042, 4258, 3042, 3119}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int (a \csc^3(x))^{5/2} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \left(-a \sec\left(x + \frac{\pi}{2}\right)\right)^{5/2} dx \\
 & \quad \downarrow \text{4611} \\
 & \frac{a^2 \sqrt{a \csc^3(x)} \int (-\csc(x))^{15/2} dx}{(-\csc(x))^{3/2}} \\
 & \quad \downarrow \text{3042} \\
 & \frac{a^2 \sqrt{a \csc^3(x)} \int (-\csc(x))^{15/2} dx}{(-\csc(x))^{3/2}} \\
 & \quad \downarrow \text{4255} \\
 & \frac{a^2 \sqrt{a \csc^3(x)} \left(\frac{11}{13} \int (-\csc(x))^{11/2} dx + \frac{2}{13} \cos(x) (-\csc(x))^{13/2}\right)}{(-\csc(x))^{3/2}} \\
 & \quad \downarrow \text{3042} \\
 & \frac{a^2 \sqrt{a \csc^3(x)} \left(\frac{11}{13} \int (-\csc(x))^{11/2} dx + \frac{2}{13} \cos(x) (-\csc(x))^{13/2}\right)}{(-\csc(x))^{3/2}} \\
 & \quad \downarrow \text{4255} \\
 & \frac{a^2 \sqrt{a \csc^3(x)} \left(\frac{11}{13} \left(\frac{7}{9} \int (-\csc(x))^{7/2} dx + \frac{2}{9} \cos(x) (-\csc(x))^{9/2}\right) + \frac{2}{13} \cos(x) (-\csc(x))^{13/2}\right)}{(-\csc(x))^{3/2}}
 \end{aligned}$$

$$\begin{aligned} & \downarrow 3042 \\ & \frac{a^2 \sqrt{a \csc^3(x)} \left( \frac{11}{13} \left( \frac{7}{9} \int (-\csc(x))^{7/2} dx + \frac{2}{9} \cos(x) (-\csc(x))^{9/2} \right) + \frac{2}{13} \cos(x) (-\csc(x))^{13/2} \right)}{(-\csc(x))^{3/2}} \\ & \downarrow 4255 \\ & \frac{a^2 \sqrt{a \csc^3(x)} \left( \frac{11}{13} \left( \frac{7}{9} \left( \frac{3}{5} \int (-\csc(x))^{3/2} dx + \frac{2}{5} \cos(x) (-\csc(x))^{5/2} \right) + \frac{2}{9} \cos(x) (-\csc(x))^{9/2} \right) + \frac{2}{13} \cos(x) (-\csc(x))^{13/2} \right)}{(-\csc(x))^{3/2}} \\ & \downarrow 3042 \\ & \frac{a^2 \sqrt{a \csc^3(x)} \left( \frac{11}{13} \left( \frac{7}{9} \left( \frac{3}{5} \int (-\csc(x))^{3/2} dx + \frac{2}{5} \cos(x) (-\csc(x))^{5/2} \right) + \frac{2}{9} \cos(x) (-\csc(x))^{9/2} \right) + \frac{2}{13} \cos(x) (-\csc(x))^{13/2} \right)}{(-\csc(x))^{3/2}} \\ & \downarrow 4255 \\ & \frac{a^2 \sqrt{a \csc^3(x)} \left( \frac{11}{13} \left( \frac{7}{9} \left( \frac{3}{5} \left( 2 \cos(x) \sqrt{-\csc(x)} - \int \frac{1}{\sqrt{-\csc(x)}} dx \right) + \frac{2}{5} \cos(x) (-\csc(x))^{5/2} \right) + \frac{2}{9} \cos(x) (-\csc(x))^{9/2} \right) \right)}{(-\csc(x))^{3/2}} \\ & \downarrow 3042 \\ & \frac{a^2 \sqrt{a \csc^3(x)} \left( \frac{11}{13} \left( \frac{7}{9} \left( \frac{3}{5} \left( 2 \cos(x) \sqrt{-\csc(x)} - \int \frac{1}{\sqrt{-\csc(x)}} dx \right) + \frac{2}{5} \cos(x) (-\csc(x))^{5/2} \right) + \frac{2}{9} \cos(x) (-\csc(x))^{9/2} \right) \right)}{(-\csc(x))^{3/2}} \\ & \downarrow 4258 \\ & \frac{a^2 \sqrt{a \csc^3(x)} \left( \frac{11}{13} \left( \frac{7}{9} \left( \frac{3}{5} \left( 2 \cos(x) \sqrt{-\csc(x)} - \frac{\int \sqrt{\sin(x)} dx}{\sqrt{\sin(x)} \sqrt{-\csc(x)}} \right) + \frac{2}{5} \cos(x) (-\csc(x))^{5/2} \right) + \frac{2}{9} \cos(x) (-\csc(x))^{9/2} \right) \right)}{(-\csc(x))^{3/2}} \\ & \downarrow 3042 \\ & \frac{a^2 \sqrt{a \csc^3(x)} \left( \frac{11}{13} \left( \frac{7}{9} \left( \frac{3}{5} \left( 2 \cos(x) \sqrt{-\csc(x)} - \frac{\int \sqrt{\sin(x)} dx}{\sqrt{\sin(x)} \sqrt{-\csc(x)}} \right) + \frac{2}{5} \cos(x) (-\csc(x))^{5/2} \right) + \frac{2}{9} \cos(x) (-\csc(x))^{9/2} \right) \right)}{(-\csc(x))^{3/2}} \\ & \downarrow 3119 \\ & \frac{a^2 \sqrt{a \csc^3(x)} \left( \frac{2}{13} \cos(x) (-\csc(x))^{13/2} + \frac{11}{13} \left( \frac{2}{9} \cos(x) (-\csc(x))^{9/2} + \frac{7}{9} \left( \frac{2}{5} \cos(x) (-\csc(x))^{5/2} + \frac{3}{5} \left( 2 \cos(x) \sqrt{-\csc(x)} - \frac{\int \sqrt{\sin(x)} dx}{\sqrt{\sin(x)} \sqrt{-\csc(x)}} \right) \right) \right) \right)}{(-\csc(x))^{3/2}} \end{aligned}$$

input `Int[(a*Csc[x]^3)^(5/2),x]`

output `(a^2*Sqrt[a*Csc[x]^3]*((2*Cos[x]*(-Csc[x])^(13/2))/13 + (11*((2*Cos[x]*(-Csc[x])^(9/2))/9 + (7*((2*Cos[x]*(-Csc[x])^(5/2))/5 + (3*(2*Cos[x]*Sqrt[-Csc[x]] + (2*EllipticE[Pi/4 - x/2, 2])/(Sqrt[-Csc[x]]*Sqrt[Sin[x]])))/5))/9)/13))/(-Csc[x])^(3/2)`

### Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3119 `Int[Sqrt[sin[(c_.) + (d_.)*(x_)]], x_Symbol] := Simp[(2/d)*EllipticE[(1/2)*(c - Pi/2 + d*x), 2], x] /; FreeQ[{c, d}, x]`

rule 4255 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] := Simp[(-b)*Cos[c + d*x]*((b*Csc[c + d*x])^(n - 1)/(d*(n - 1))), x] + Simp[b^2*((n - 2)/(n - 1)) Int[(b*Csc[c + d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[2*n]`

rule 4258 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] := Simp[(b*Csc[c + d*x])^n*Sin[c + d*x]^n Int[1/Sin[c + d*x]^n, x], x] /; FreeQ[{b, c, d}, x] && EqQ[n^2, 1/4]`

rule 4611 `Int[((b_.)*((c_.)*sec[(e_.) + (f_.)*(x_)])^(n_))^(p_), x_Symbol] := Simp[b^IntPart[p]*((b*(c*Sec[e + f*x])^n)^FracPart[p]/(c*Sec[e + f*x])^(n*FracPart[p])) Int[(c*Sec[e + f*x])^(n*p), x], x] /; FreeQ[{b, c, e, f, n, p}, x] && !IntegerQ[p]`

**Maple [C] (verified)**

Result contains complex when optimal does not.

Time = 0.92 (sec) , antiderivative size = 185, normalized size of antiderivative = 1.50

method	result
default	$-\frac{\sqrt{8} \sqrt{a \csc(x)^3} a^2 \left( (-462 \cos(x) - 462) \sin(x) \sqrt{1+i \cot(x) - i \csc(x)} \sqrt{1-i \cot(x) + i \csc(x)} \sqrt{i(-\cot(x) + \csc(x))} \operatorname{EllipticE}\left(\sqrt{1+i \cot(x) - i \csc(x)}\right) \right)}{\dots}$

input `int((a*csc(x)^3)^(5/2),x,method=_RETURNVERBOSE)`

output 
$$-1/1170*8^{(1/2)}*(a*\csc(x)^3)^{(1/2)}*a^2*((-462*\cos(x)-462)*\sin(x)*(1+I*\cot(x)-I*\csc(x))^{(1/2)}*(1-I*\cot(x)+I*\csc(x))^{(1/2)}*(I*(-\cot(x)+\csc(x)))^{(1/2)}*\operatorname{EllipticE}((1+I*\cot(x)-I*\csc(x))^{(1/2)},1/2*2^{(1/2)})+(231*\cos(x)+231)*\sin(x)*(1+I*\cot(x)-I*\csc(x))^{(1/2)}*(1-I*\cot(x)+I*\csc(x))^{(1/2)}*(I*(-\cot(x)+\csc(x))))^{(1/2)}*\operatorname{EllipticF}((1+I*\cot(x)-I*\csc(x))^{(1/2)},1/2*2^{(1/2)})+2^{(1/2)}*(231*\sin(x)+77*\cot(x)+55*\csc(x)^2*\cot(x)+45*\csc(x)^4*\cot(x)))$$

**Fricas [C] (verification not implemented)**

Result contains complex when optimal does not.

Time = 0.08 (sec) , antiderivative size = 161, normalized size of antiderivative = 1.31

$$\int (a \csc^3(x))^{5/2} dx = \frac{231 (a^2 \cos(x)^4 - 2 a^2 \cos(x)^2 + a^2) \sqrt{2i a \sin(x)} \operatorname{weierstrassZeta}(4, 0, \operatorname{weierstrassPInverse}(4, 0, \cos(x))) - \dots}{\dots}$$

input `integrate((a*csc(x)^3)^(5/2),x, algorithm="fricas")`



output

```
-1/585*(231*(a^2*cos(x)^4 - 2*a^2*cos(x)^2 + a^2)*sqrt(2*I*a)*sin(x)*weierstrassZeta(4, 0, weierstrassPInverse(4, 0, cos(x) + I*sin(x))) + 231*(a^2*cos(x)^4 - 2*a^2*cos(x)^2 + a^2)*sqrt(-2*I*a)*sin(x)*weierstrassZeta(4, 0, weierstrassPInverse(4, 0, cos(x) - I*sin(x))) - 2*(231*a^2*cos(x)^7 - 770*a^2*cos(x)^5 + 902*a^2*cos(x)^3 - 408*a^2*cos(x))*sqrt(-a/((cos(x)^2 - 1)*sin(x))))/((cos(x)^4 - 2*cos(x)^2 + 1)*sin(x))
```

**Sympy [F]**

$$\int (a \csc^3(x))^{5/2} dx = \int (a \csc^3(x))^{5/2} dx$$

input

```
integrate((a*csc(x)**3)**(5/2),x)
```

output

```
Integral((a*csc(x)**3)**(5/2), x)
```

**Maxima [F]**

$$\int (a \csc^3(x))^{5/2} dx = \int (a \csc^3(x))^{5/2} dx$$

input

```
integrate((a*csc(x)^3)^(5/2),x, algorithm="maxima")
```

output

```
integrate((a*csc(x)^3)^(5/2), x)
```

**Giac [F]**

$$\int (a \csc^3(x))^{5/2} dx = \int (a \csc(x)^3)^{5/2} dx$$

input `integrate((a*csc(x)^3)^(5/2),x, algorithm="giac")`

output `integrate((a*csc(x)^3)^(5/2), x)`

**Mupad [F(-1)]**

Timed out.

$$\int (a \csc^3(x))^{5/2} dx = \int \left( \frac{a}{\sin(x)^3} \right)^{5/2} dx$$

input `int((a/sin(x)^3)^(5/2),x)`

output `int((a/sin(x)^3)^(5/2), x)`

**Reduce [F]**

$$\int (a \csc^3(x))^{5/2} dx = \sqrt{a} \left( \int \sqrt{\csc(x)} \csc(x)^7 dx \right) a^2$$

input `int((a*csc(x)^3)^(5/2),x)`

output `sqrt(a)*int(sqrt(csc(x))*csc(x)**7,x)*a**2`

### 3.56 $\int (a \csc^3(x))^{3/2} dx$

Optimal result	362
Mathematica [A] (verified)	362
Rubi [A] (verified)	363
Maple [C] (verified)	365
Fricas [C] (verification not implemented)	366
Sympy [F]	366
Maxima [F]	367
Giac [F]	367
Mupad [F(-1)]	367
Reduce [F]	368

#### Optimal result

Integrand size = 10, antiderivative size = 71

$$\int (a \csc^3(x))^{3/2} dx = -\frac{10}{21}a \cos(x) \sqrt{a \csc^3(x)} - \frac{2}{7}a \cot(x) \csc(x) \sqrt{a \csc^3(x)} - \frac{10}{21}a \sqrt{a \csc^3(x)} \operatorname{EllipticF}\left(\frac{\pi}{4} - \frac{x}{2}, 2\right) \sin^{\frac{3}{2}}(x)$$

output

```
-10/21*a*cos(x)*(a*csc(x)^3)^(1/2)-2/7*a*cot(x)*csc(x)*(a*csc(x)^3)^(1/2)+
10/21*a*(a*csc(x)^3)^(1/2)*InverseJacobiAM(-1/4*Pi+1/2*x,2^(1/2))*sin(x)^(
3/2)
```

#### Mathematica [A] (verified)

Time = 0.10 (sec) , antiderivative size = 46, normalized size of antiderivative = 0.65

$$\int (a \csc^3(x))^{3/2} dx = -\frac{1}{84}(a \csc^3(x))^{3/2} \left( 40 \operatorname{EllipticF}\left(\frac{1}{4}(\pi - 2x), 2\right) \sin^{\frac{9}{2}}(x) + 22 \sin(2x) - 5 \sin(4x) \right)$$

input

```
Integrate[(a*Csc[x]^3)^(3/2),x]
```

output

```
-1/84*((a*Csc[x]^3)^(3/2)*(40*EllipticF[(Pi - 2*x)/4, 2]*Sin[x]^(9/2) + 22
*Sin[2*x] - 5*Sin[4*x]))
```

**Rubi [A] (verified)**

Time = 0.46 (sec) , antiderivative size = 86, normalized size of antiderivative = 1.21, number of steps used = 10, number of rules used = 10,  $\frac{\text{number of rules}}{\text{integrand size}} = 1.000$ , Rules used = {3042, 4611, 3042, 4255, 3042, 4255, 3042, 4258, 3042, 3120}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int (a \csc^3(x))^{3/2} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \left(-a \sec\left(x + \frac{\pi}{2}\right)\right)^{3/2} dx \\
 & \quad \downarrow \text{4611} \\
 & -\frac{a\sqrt{a \csc^3(x)} \int (-\csc(x))^{9/2} dx}{(-\csc(x))^{3/2}} \\
 & \quad \downarrow \text{3042} \\
 & -\frac{a\sqrt{a \csc^3(x)} \int (-\csc(x))^{9/2} dx}{(-\csc(x))^{3/2}} \\
 & \quad \downarrow \text{4255} \\
 & -\frac{a\sqrt{a \csc^3(x)}\left(\frac{5}{7} \int (-\csc(x))^{5/2} dx + \frac{2}{7} \cos(x)(-\csc(x))^{7/2}\right)}{(-\csc(x))^{3/2}} \\
 & \quad \downarrow \text{3042} \\
 & -\frac{a\sqrt{a \csc^3(x)}\left(\frac{5}{7} \int (-\csc(x))^{5/2} dx + \frac{2}{7} \cos(x)(-\csc(x))^{7/2}\right)}{(-\csc(x))^{3/2}} \\
 & \quad \downarrow \text{4255} \\
 & -\frac{a\sqrt{a \csc^3(x)}\left(\frac{5}{7}\left(\frac{1}{3} \int \sqrt{-\csc(x)} dx + \frac{2}{3} \cos(x)(-\csc(x))^{3/2}\right) + \frac{2}{7} \cos(x)(-\csc(x))^{7/2}\right)}{(-\csc(x))^{3/2}}
 \end{aligned}$$

$$\begin{aligned}
& \downarrow 3042 \\
& \frac{a\sqrt{a\csc^3(x)}\left(\frac{5}{7}\left(\frac{1}{3}\int\sqrt{-\csc(x)}dx+\frac{2}{3}\cos(x)(-\csc(x))^{3/2}\right)+\frac{2}{7}\cos(x)(-\csc(x))^{7/2}\right)}{(-\csc(x))^{3/2}} \\
& \downarrow 4258 \\
& \frac{a\sqrt{a\csc^3(x)}\left(\frac{5}{7}\left(\frac{1}{3}\sqrt{\sin(x)}\sqrt{-\csc(x)}\int\frac{1}{\sqrt{\sin(x)}}dx+\frac{2}{3}\cos(x)(-\csc(x))^{3/2}\right)+\frac{2}{7}\cos(x)(-\csc(x))^{7/2}\right)}{(-\csc(x))^{3/2}} \\
& \downarrow 3042 \\
& \frac{a\sqrt{a\csc^3(x)}\left(\frac{5}{7}\left(\frac{1}{3}\sqrt{\sin(x)}\sqrt{-\csc(x)}\int\frac{1}{\sqrt{\sin(x)}}dx+\frac{2}{3}\cos(x)(-\csc(x))^{3/2}\right)+\frac{2}{7}\cos(x)(-\csc(x))^{7/2}\right)}{(-\csc(x))^{3/2}} \\
& \downarrow 3120 \\
& \frac{a\sqrt{a\csc^3(x)}\left(\frac{2}{7}\cos(x)(-\csc(x))^{7/2}+\frac{5}{7}\left(\frac{2}{3}\cos(x)(-\csc(x))^{3/2}-\frac{2}{3}\sqrt{\sin(x)}\sqrt{-\csc(x)}\operatorname{EllipticF}\left(\frac{\pi}{4}-\frac{x}{2},2\right)\right)\right)}{(-\csc(x))^{3/2}}
\end{aligned}$$

input `Int[(a*Csc[x]^3)^(3/2),x]`

output `-((a*Sqrt[a*Csc[x]^3]*((2*Cos[x]*(-Csc[x])^(7/2))/7+(5*((2*Cos[x]*(-Csc[x])^(3/2))/3-(2*Sqrt[-Csc[x]]*EllipticF[Pi/4-x/2,2]*Sqrt[Sin[x]])/3))/7))/(-Csc[x])^(3/2)`

### Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3120 `Int[1/Sqrt[sin[(c_.)+(d_.)*(x_)]], x_Symbol] := Simp[(2/d)*EllipticF[(1/2)*(c-Pi/2+d*x), 2], x] /; FreeQ[{c, d}, x]`

rule 4255 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] := Simp[(-b)*Cos[c + d*x]*((b*Csc[c + d*x])^(n - 1)/(d*(n - 1))), x] + Simp[b^2*((n - 2)/(n - 1)) Int[(b*Csc[c + d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[2*n]`

rule 4258 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] := Simp[(b*Csc[c + d*x])^n*Sin[c + d*x]^n Int[1/Sin[c + d*x]^n, x], x] /; FreeQ[{b, c, d}, x] && EqQ[n^2, 1/4]`

rule 4611 `Int[((b_.)*((c_.)*sec[(e_.) + (f_.)*(x_)])^(n_))^(p_), x_Symbol] := Simp[b^IntPart[p]*((b*(c*Sec[e + f*x])^n)^FracPart[p]/(c*Sec[e + f*x])^(n*FracPart[p])) Int[(c*Sec[e + f*x])^(n*p), x], x] /; FreeQ[{b, c, e, f, n, p}, x] && !IntegerQ[p]`

## Maple [C] (verified)

Result contains complex when optimal does not.

Time = 0.37 (sec) , antiderivative size = 103, normalized size of antiderivative = 1.45

method	result
default	$-\frac{\sqrt{8} a \sqrt{a \csc(x)^3} \left( i \sin(x) (-5 \cos(x) - 5) \sqrt{1 + i \cot(x) - i \csc(x)} \sqrt{1 - i \cot(x) + i \csc(x)} \sqrt{i(-\cot(x) + \csc(x))} \operatorname{EllipticF}\left(\sqrt{1 + i \cot(x) - i \csc(x)}, \frac{1}{2}\right) \right)}{42}$

input `int((a*csc(x)^3)^(3/2),x,method=_RETURNVERBOSE)`

output `-1/42*8^(1/2)*a*(a*csc(x)^3)^(1/2)*(I*sin(x)*(-5*cos(x)-5)*(1+I*cot(x)-I*csc(x))^(1/2)*(1-I*cot(x)+I*csc(x))^(1/2)*(I*(-cot(x)+csc(x)))^(1/2)*EllipticF((1+I*cot(x)-I*csc(x))^(1/2),1/2*2^(1/2))+(-5*cos(x)^2+8)*2^(1/2)*csc(x))*cot(x)`

**Fricas [C] (verification not implemented)**

Result contains complex when optimal does not.

Time = 0.09 (sec) , antiderivative size = 99, normalized size of antiderivative = 1.39

$$\int (a \csc^3(x))^{3/2} dx =$$

$$\frac{5 (i a \cos(x)^2 - i a) \sqrt{2i a} \operatorname{weierstrassPInverse}(4, 0, \cos(x) + i \sin(x)) + 5 (-i a \cos(x)^2 + i a) \sqrt{-2i a} \operatorname{weierstrassPInverse}(4, 0, \cos(x) - i \sin(x)) + 2*(5*a*\cos(x)^3 - 8*a*\cos(x))*\sqrt{-a/((\cos(x)^2 - 1)*\sin(x))}}{21 (\cos(x)^2 - 1)}$$

input `integrate((a*csc(x)^3)^(3/2),x, algorithm="fricas")`

output `-1/21*(5*(I*a*cos(x)^2 - I*a)*sqrt(2*I*a)*weierstrassPInverse(4, 0, cos(x) + I*sin(x)) + 5*(-I*a*cos(x)^2 + I*a)*sqrt(-2*I*a)*weierstrassPInverse(4, 0, cos(x) - I*sin(x)) + 2*(5*a*cos(x)^3 - 8*a*cos(x))*sqrt(-a/((cos(x)^2 - 1)*sin(x))))/(cos(x)^2 - 1)`

**Sympy [F]**

$$\int (a \csc^3(x))^{3/2} dx = \int (a \csc^3(x))^{\frac{3}{2}} dx$$

input `integrate((a*csc(x)**3)**(3/2),x)`

output `Integral((a*csc(x)**3)**(3/2), x)`

**Maxima [F]**

$$\int (a \csc^3(x))^{3/2} dx = \int (a \csc(x)^3)^{\frac{3}{2}} dx$$

input `integrate((a*csc(x)^3)^(3/2),x, algorithm="maxima")`

output `integrate((a*csc(x)^3)^(3/2), x)`

**Giac [F]**

$$\int (a \csc^3(x))^{3/2} dx = \int (a \csc(x)^3)^{\frac{3}{2}} dx$$

input `integrate((a*csc(x)^3)^(3/2),x, algorithm="giac")`

output `integrate((a*csc(x)^3)^(3/2), x)`

**Mupad [F(-1)]**

Timed out.

$$\int (a \csc^3(x))^{3/2} dx = \int \left( \frac{a}{\sin(x)^3} \right)^{3/2} dx$$

input `int((a/sin(x)^3)^(3/2),x)`

output `int((a/sin(x)^3)^(3/2), x)`



**Reduce [F]**

$$\int (a \csc^3(x))^{3/2} dx = \sqrt{a} \left( \int \sqrt{\csc(x)} \csc(x)^4 dx \right) a$$

input `int((a*csc(x)^3)^(3/2),x)`

output `sqrt(a)*int(sqrt(csc(x))*csc(x)**4,x)*a`

### 3.57 $\int \sqrt{a \csc^3(x)} dx$

Optimal result	369
Mathematica [A] (verified)	369
Rubi [A] (verified)	370
Maple [C] (verified)	372
Fricas [C] (verification not implemented)	372
Sympy [F]	373
Maxima [F]	373
Giac [F]	373
Mupad [F(-1)]	374
Reduce [F]	374

#### Optimal result

Integrand size = 10, antiderivative size = 48

$$\int \sqrt{a \csc^3(x)} dx = -2 \cos(x) \sqrt{a \csc^3(x)} \sin(x) + 2 \sqrt{a \csc^3(x)} E\left(\frac{\pi}{4} - \frac{x}{2} \middle| 2\right) \sin^{\frac{3}{2}}(x)$$

output

$-2*\cos(x)*(a*\csc(x)^3)^{(1/2)}*\sin(x)+2*(a*\csc(x)^3)^{(1/2)}*EllipticE(\cos(1/4*\pi+1/2*x),2^{(1/2)})*\sin(x)^{(3/2)}$

#### Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 46, normalized size of antiderivative = 0.96

$$\int \sqrt{a \csc^3(x)} dx = -2 \cos(x) \sqrt{a \csc^3(x)} \sin(x) + 2 \sqrt{a \csc^3(x)} E\left(\frac{1}{4}(\pi - 2x) \middle| 2\right) \sin^{\frac{3}{2}}(x)$$

input

`Integrate[Sqrt[a*Csc[x]^3],x]`

output

$-2*\cos[x]*\sqrt{a*\csc[x]^3}*\sin[x] + 2*\sqrt{a*\csc[x]^3}*EllipticE[(\pi - 2*x)/4, 2]*\sin[x]^{(3/2)}$

**Rubi [A] (verified)**

Time = 0.37 (sec) , antiderivative size = 61, normalized size of antiderivative = 1.27, number of steps used = 8, number of rules used = 8,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.800$ , Rules used = {3042, 4611, 3042, 4255, 3042, 4258, 3042, 3119}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \sqrt{a \csc^3(x)} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \sqrt{-a \sec\left(x + \frac{\pi}{2}\right)^3} dx \\
 & \quad \downarrow \text{4611} \\
 & \frac{\sqrt{a \csc^3(x)} \int (-\csc(x))^{3/2} dx}{(-\csc(x))^{3/2}} \\
 & \quad \downarrow \text{3042} \\
 & \frac{\sqrt{a \csc^3(x)} \int (-\csc(x))^{3/2} dx}{(-\csc(x))^{3/2}} \\
 & \quad \downarrow \text{4255} \\
 & \frac{\sqrt{a \csc^3(x)} \left( 2 \cos(x) \sqrt{-\csc(x)} - \int \frac{1}{\sqrt{-\csc(x)}} dx \right)}{(-\csc(x))^{3/2}} \\
 & \quad \downarrow \text{3042} \\
 & \frac{\sqrt{a \csc^3(x)} \left( 2 \cos(x) \sqrt{-\csc(x)} - \int \frac{1}{\sqrt{-\csc(x)}} dx \right)}{(-\csc(x))^{3/2}} \\
 & \quad \downarrow \text{4258} \\
 & \frac{\sqrt{a \csc^3(x)} \left( 2 \cos(x) \sqrt{-\csc(x)} - \frac{\int \sqrt{\sin(x)} dx}{\sqrt{\sin(x)} \sqrt{-\csc(x)}} \right)}{(-\csc(x))^{3/2}} \\
 & \quad \downarrow \text{3042} \\
 & \frac{\sqrt{a \csc^3(x)} \left( 2 \cos(x) \sqrt{-\csc(x)} - \frac{\int \sqrt{\sin(x)} dx}{\sqrt{\sin(x)} \sqrt{-\csc(x)}} \right)}{(-\csc(x))^{3/2}}
 \end{aligned}$$

$$\downarrow \text{3119}$$

$$\frac{\sqrt{a \csc^3(x)} \left( 2 \cos(x) \sqrt{-\csc(x)} + \frac{2E\left(\frac{\pi}{4} - \frac{x}{2} \mid 2\right)}{\sqrt{\sin(x)} \sqrt{-\csc(x)}} \right)}{(-\csc(x))^{3/2}}$$

input `Int[Sqrt[a*Csc[x]^3], x]`

output `(Sqrt[a*Csc[x]^3]*(2*Cos[x]*Sqrt[-Csc[x]] + (2*EllipticE[Pi/4 - x/2, 2]))/(Sqrt[-Csc[x]*Sqrt[Sin[x]])))/(-Csc[x])^(3/2)`

### Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] :=> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3119 `Int[Sqrt[sin[(c_.) + (d_.)*(x_)]], x_Symbol] :=> Simp[(2/d)*EllipticE[(1/2)*(c - Pi/2 + d*x), 2], x] /; FreeQ[{c, d}, x]`

rule 4255 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^n], x_Symbol] :=> Simp[(-b)*Cos[c + d*x]*((b*Csc[c + d*x])^(n - 1)/(d*(n - 1))), x] + Simp[b^2*((n - 2)/(n - 1)) Int[(b*Csc[c + d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[2*n]`

rule 4258 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^n], x_Symbol] :=> Simp[(b*Csc[c + d*x])^n*Sin[c + d*x]^n Int[1/Sin[c + d*x]^n, x], x] /; FreeQ[{b, c, d}, x] && EqQ[n^2, 1/4]`

rule 4611 `Int[((b_.)*((c_.)*sec[(e_.) + (f_.)*(x_)])^n))^p], x_Symbol] :=> Simp[b^IntPart[p]*((b*(c*Sec[e + f*x])^n)^FracPart[p]/(c*Sec[e + f*x])^(n*FracPart[p])) Int[(c*Sec[e + f*x])^(n*p), x], x] /; FreeQ[{b, c, e, f, n, p}, x] && !IntegerQ[p]`

**Maple [C] (verified)**

Result contains complex when optimal does not.

Time = 0.14 (sec) , antiderivative size = 155, normalized size of antiderivative = 3.23

method	result
default	$\frac{\sqrt{8} \sin(x) \left( \sqrt{1+i(\cot(x)-\csc(x))} \sqrt{1-i(\cot(x)-\csc(x))} \sqrt{-i(\cot(x)-\csc(x))} (-\cos(x)-1) \operatorname{EllipticF}\left(\sqrt{1+i \cot(x)-i \csc(x)}, \frac{\sqrt{2}}{2}\right) \right)}{\dots}$

input `int((a*csc(x)^3)^(1/2),x,method=_RETURNVERBOSE)`

output  $\frac{1}{2} \cdot 8^{1/2} \cdot \sin(x) \cdot \left( (1+I \cdot (\cot(x)-\csc(x)))^{1/2} \cdot (1-I \cdot (\cot(x)-\csc(x)))^{1/2} \right) \cdot (-I \cdot (\cot(x)-\csc(x)))^{1/2} \cdot (-\cos(x)-1) \cdot \operatorname{EllipticF}\left( (1+I \cdot \cot(x)-I \cdot \csc(x))^{1/2}, 1/2 \cdot 2^{1/2} \right) + 2 \cdot (1+I \cdot (\cot(x)-\csc(x)))^{1/2} \cdot (1-I \cdot (\cot(x)-\csc(x)))^{1/2} \right) \cdot (-I \cdot (\cot(x)-\csc(x)))^{1/2} \cdot (\cos(x)+1) \cdot \operatorname{EllipticE}\left( (1+I \cdot \cot(x)-I \cdot \csc(x))^{1/2}, 1/2 \cdot 2^{1/2} \right) - 2^{1/2} \cdot (a \cdot \csc(x)^3)^{1/2}$

**Fricas [C] (verification not implemented)**

Result contains complex when optimal does not.

Time = 0.08 (sec) , antiderivative size = 64, normalized size of antiderivative = 1.33

$$\int \sqrt{a \csc^3(x)} dx = -2 \sqrt{-\frac{a}{(\cos(x)^2 - 1) \sin(x)}} \cos(x) \sin(x) - \sqrt{2i a} \operatorname{weierstrassZeta}(4, 0, \operatorname{weierstrassPInverse}(4, 0, \cos(x) + i \sin(x))) - \sqrt{-2i a} \operatorname{weierstrassZeta}(4, 0, \operatorname{weierstrassPInverse}(4, 0, \cos(x) - i \sin(x)))$$

input `integrate((a*csc(x)^3)^(1/2),x, algorithm="fricas")`

output  $-2 \cdot \sqrt{-a / ((\cos(x)^2 - 1) \cdot \sin(x))} \cdot \cos(x) \cdot \sin(x) - \sqrt{2 \cdot I \cdot a} \cdot \operatorname{weierstrassZeta}(4, 0, \operatorname{weierstrassPInverse}(4, 0, \cos(x) + I \cdot \sin(x))) - \sqrt{-2 \cdot I \cdot a} \cdot \operatorname{weierstrassZeta}(4, 0, \operatorname{weierstrassPInverse}(4, 0, \cos(x) - I \cdot \sin(x)))$

**Sympy [F]**

$$\int \sqrt{a \csc^3(x)} dx = \int \sqrt{a \csc(x)^3} dx$$

input `integrate((a*csc(x)**3)**(1/2),x)`

output `Integral(sqrt(a*csc(x)**3), x)`

**Maxima [F]**

$$\int \sqrt{a \csc^3(x)} dx = \int \sqrt{a \csc(x)^3} dx$$

input `integrate((a*csc(x)^3)^(1/2),x, algorithm="maxima")`

output `integrate(sqrt(a*csc(x)^3), x)`

**Giac [F]**

$$\int \sqrt{a \csc^3(x)} dx = \int \sqrt{a \csc(x)^3} dx$$

input `integrate((a*csc(x)^3)^(1/2),x, algorithm="giac")`

output `integrate(sqrt(a*csc(x)^3), x)`

**Mupad [F(-1)]**

Timed out.

$$\int \sqrt{a \csc^3(x)} dx = \int \sqrt{\frac{a}{\sin(x)^3}} dx$$

input `int((a/sin(x)^3)^(1/2),x)`output `int((a/sin(x)^3)^(1/2), x)`**Reduce [F]**

$$\int \sqrt{a \csc^3(x)} dx = \sqrt{a} \left( \int \sqrt{\csc(x)} \csc(x) dx \right)$$

input `int((a*csc(x)^3)^(1/2),x)`output `sqrt(a)*int(sqrt(csc(x))*csc(x),x)`

### 3.58 $\int \frac{1}{\sqrt{a \csc^3(x)}} dx$

Optimal result	375
Mathematica [A] (verified)	375
Rubi [A] (verified)	376
Maple [C] (verified)	378
Fricas [C] (verification not implemented)	378
Sympy [F]	379
Maxima [F]	379
Giac [F]	379
Mupad [F(-1)]	380
Reduce [F]	380

#### Optimal result

Integrand size = 10, antiderivative size = 50

$$\int \frac{1}{\sqrt{a \csc^3(x)}} dx = -\frac{2 \cot(x)}{3\sqrt{a \csc^3(x)}} - \frac{2 \operatorname{EllipticF}\left(\frac{\pi}{4} - \frac{x}{2}, 2\right)}{3\sqrt{a \csc^3(x)} \sin^{\frac{3}{2}}(x)}$$

output `-2/3*cot(x)/(a*csc(x)^3)^(1/2)+2/3*InverseJacobiAM(-1/4*Pi+1/2*x,2^(1/2))/(a*csc(x)^3)^(1/2)/sin(x)^(3/2)`

#### Mathematica [A] (verified)

Time = 0.05 (sec) , antiderivative size = 38, normalized size of antiderivative = 0.76

$$\int \frac{1}{\sqrt{a \csc^3(x)}} dx = \frac{-2 \cot(x) - \frac{2 \operatorname{EllipticF}\left(\frac{1}{4}(\pi - 2x), 2\right)}{\sin^{\frac{3}{2}}(x)}}{3\sqrt{a \csc^3(x)}}$$

input `Integrate[1/Sqrt[a*Csc[x]^3],x]`

output `(-2*Cot[x] - (2*EllipticF[(Pi - 2*x)/4, 2])/Sin[x]^(3/2))/(3*Sqrt[a*Csc[x]^3])`



**Rubi [A] (verified)**

Time = 0.37 (sec) , antiderivative size = 65, normalized size of antiderivative = 1.30, number of steps used = 8, number of rules used = 8,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.800$ , Rules used = {3042, 4611, 3042, 4256, 3042, 4258, 3042, 3120}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{1}{\sqrt{a \csc^3(x)}} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{1}{\sqrt{-a \sec(x + \frac{\pi}{2})^3}} dx \\
 & \quad \downarrow \text{4611} \\
 & \frac{(-\csc(x))^{3/2} \int \frac{1}{(-\csc(x))^{3/2}} dx}{\sqrt{a \csc^3(x)}} \\
 & \quad \downarrow \text{3042} \\
 & \frac{(-\csc(x))^{3/2} \int \frac{1}{(-\csc(x))^{3/2}} dx}{\sqrt{a \csc^3(x)}} \\
 & \quad \downarrow \text{4256} \\
 & \frac{(-\csc(x))^{3/2} \left( \frac{1}{3} \int \sqrt{-\csc(x)} dx + \frac{2 \cos(x)}{3\sqrt{-\csc(x)}} \right)}{\sqrt{a \csc^3(x)}} \\
 & \quad \downarrow \text{3042} \\
 & \frac{(-\csc(x))^{3/2} \left( \frac{1}{3} \int \sqrt{-\csc(x)} dx + \frac{2 \cos(x)}{3\sqrt{-\csc(x)}} \right)}{\sqrt{a \csc^3(x)}} \\
 & \quad \downarrow \text{4258} \\
 & \frac{(-\csc(x))^{3/2} \left( \frac{1}{3} \sqrt{\sin(x)} \sqrt{-\csc(x)} \int \frac{1}{\sqrt{\sin(x)}} dx + \frac{2 \cos(x)}{3\sqrt{-\csc(x)}} \right)}{\sqrt{a \csc^3(x)}} \\
 & \quad \downarrow \text{3042}
 \end{aligned}$$

$$\frac{(-\csc(x))^{3/2} \left( \frac{1}{3} \sqrt{\sin(x)} \sqrt{-\csc(x)} \int \frac{1}{\sqrt{\sin(x)}} dx + \frac{2 \cos(x)}{3 \sqrt{-\csc(x)}} \right)}{\sqrt{a \csc^3(x)}}$$

↓ 3120

$$\frac{(-\csc(x))^{3/2} \left( \frac{2 \cos(x)}{3 \sqrt{-\csc(x)}} - \frac{2}{3} \sqrt{\sin(x)} \sqrt{-\csc(x)} \operatorname{EllipticF} \left( \frac{\pi}{4} - \frac{x}{2}, 2 \right) \right)}{\sqrt{a \csc^3(x)}}$$

input `Int[1/Sqrt[a*Csc[x]^3],x]`

output `((-Csc[x])^(3/2)*((2*Cos[x])/(3*Sqrt[-Csc[x]]) - (2*Sqrt[-Csc[x]]*EllipticF[Pi/4 - x/2, 2]*Sqrt[Sin[x]]/3))/Sqrt[a*Csc[x]^3]`

### Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] :=> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3120 `Int[1/Sqrt[sin[(c_.) + (d_.)*(x_)]], x_Symbol] :=> Simp[(2/d)*EllipticF[(1/2)*(c - Pi/2 + d*x), 2], x] /; FreeQ[{c, d}, x]`

rule 4256 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^n, x_Symbol] :=> Simp[Cos[c + d*x]*((b*Csc[c + d*x])^(n + 1)/(b*d^n)), x] + Simp[(n + 1)/(b^2*n) Int[(b*Csc[c + d*x])^(n + 2), x], x] /; FreeQ[{b, c, d}, x] && LtQ[n, -1] && IntegerQ[2*n]`

rule 4258 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^n, x_Symbol] :=> Simp[(b*Csc[c + d*x])^n*Sin[c + d*x]^n Int[1/Sin[c + d*x]^n, x], x] /; FreeQ[{b, c, d}, x] && EqQ[n^2, 1/4]`

rule 4611

```
Int[((b_.)*((c_.)*sec[(e_.) + (f_.)*(x_)])^(n_))^(p_), x_Symbol] := Simp[b^
IntPart[p]*((b*(c*Sec[e + f*x])^n)^FracPart[p]/(c*Sec[e + f*x])^(n*FracPart
[p])) Int[(c*Sec[e + f*x])^(n*p), x], x] /; FreeQ[{b, c, e, f, n, p}, x]
&& !IntegerQ[p]
```

**Maple [C] (verified)**

Result contains complex when optimal does not.

Time = 0.17 (sec) , antiderivative size = 93, normalized size of antiderivative = 1.86

method	result
default	$\frac{\sqrt{8} \left( i(\cos(x)+1)\sqrt{1+i\cot(x)-i\csc(x)} \sqrt{1-i\cot(x)+i\csc(x)} \sqrt{i(-\cot(x)+\csc(x))} \operatorname{EllipticF}\left(\sqrt{1+i\cot(x)-i\csc(x)}, \frac{\sqrt{2}}{2}\right) \csc(x)^2 \right)}{6\sqrt{a\csc(x)^3}}$

input

```
int(1/(a*csc(x)^3)^(1/2),x,method=_RETURNVERBOSE)
```

output

```
1/6*8^(1/2)/(a*csc(x)^3)^(1/2)*(I*(cos(x)+1)*(1+I*cot(x)-I*csc(x))^(1/2)*
(1-I*cot(x)+I*csc(x))^(1/2)*(I*(-cot(x)+csc(x)))^(1/2)*EllipticF((1+I*cot(x)
)-I*csc(x))^(1/2),1/2*2^(1/2))*csc(x)^2-cot(x)*2^(1/2))
```

**Fricas [C] (verification not implemented)**

Result contains complex when optimal does not.

Time = 0.08 (sec) , antiderivative size = 68, normalized size of antiderivative = 1.36

$$\int \frac{1}{\sqrt{a \csc^3(x)}} dx$$

$$= \frac{2(\cos(x)^3 - \cos(x)) \sqrt{-\frac{a}{(\cos(x)^2 - 1)\sin(x)}} - i\sqrt{2i} \operatorname{aweierstrassPInverse}(4, 0, \cos(x) + i\sin(x)) + i\sqrt{-2a}}{3a}$$

input

```
integrate(1/(a*csc(x)^3)^(1/2),x, algorithm="fricas")
```

output `1/3*(2*(cos(x)^3 - cos(x))*sqrt(-a/((cos(x)^2 - 1)*sin(x))) - I*sqrt(2*I*a)*weierstrassPInverse(4, 0, cos(x) + I*sin(x)) + I*sqrt(-2*I*a)*weierstrassPInverse(4, 0, cos(x) - I*sin(x)))/a`

### Sympy [F]

$$\int \frac{1}{\sqrt{a \csc^3(x)}} dx = \int \frac{1}{\sqrt{a \csc^3(x)}} dx$$

input `integrate(1/(a*csc(x)**3)**(1/2),x)`

output `Integral(1/sqrt(a*csc(x)**3), x)`

### Maxima [F]

$$\int \frac{1}{\sqrt{a \csc^3(x)}} dx = \int \frac{1}{\sqrt{a \csc^3(x)}} dx$$

input `integrate(1/(a*csc(x)^3)^(1/2),x, algorithm="maxima")`

output `integrate(1/sqrt(a*csc(x)^3), x)`

### Giac [F]

$$\int \frac{1}{\sqrt{a \csc^3(x)}} dx = \int \frac{1}{\sqrt{a \csc^3(x)}} dx$$

input `integrate(1/(a*csc(x)^3)^(1/2),x, algorithm="giac")`

output `integrate(1/sqrt(a*csc(x)^3), x)`

### Mupad [F(-1)]

Timed out.

$$\int \frac{1}{\sqrt{a \csc^3(x)}} dx = \int \frac{1}{\sqrt{\frac{a}{\sin(x)^3}}} dx$$

input `int(1/(a/sin(x)^3)^(1/2),x)`

output `int(1/(a/sin(x)^3)^(1/2), x)`

### Reduce [F]

$$\int \frac{1}{\sqrt{a \csc^3(x)}} dx = \frac{\sqrt{a} \left( \int \frac{\sqrt{\csc(x)}}{\csc(x)^2} dx \right)}{a}$$

input `int(1/(a*csc(x)^3)^(1/2),x)`

output `(sqrt(a)*int(sqrt(csc(x))/csc(x)**2,x))/a`

**3.59**  $\int \frac{1}{(a \csc^3(x))^{3/2}} dx$

Optimal result	381
Mathematica [A] (verified)	381
Rubi [A] (verified)	382
Maple [C] (verified)	384
Fricas [C] (verification not implemented)	385
Sympy [F]	385
Maxima [F]	386
Giac [F]	386
Mupad [F(-1)]	386
Reduce [F]	387

**Optimal result**

Integrand size = 10, antiderivative size = 79

$$\int \frac{1}{(a \csc^3(x))^{3/2}} dx = -\frac{14 \cos(x)}{45a \sqrt{a \csc^3(x)}} - \frac{14E\left(\frac{\pi}{4} - \frac{x}{2} \mid 2\right)}{15a \sqrt{a \csc^3(x)} \sin^{\frac{3}{2}}(x)} - \frac{2 \cos(x) \sin^2(x)}{9a \sqrt{a \csc^3(x)}}$$

output `-14/45*cos(x)/a/(a*csc(x)^3)^(1/2)-14/15*EllipticE(cos(1/4*Pi+1/2*x),2^(1/2))/a/(a*csc(x)^3)^(1/2)/sin(x)^(3/2)-2/9*cos(x)*sin(x)^2/a/(a*csc(x)^3)^(1/2)`

**Mathematica [A] (verified)**

Time = 0.08 (sec) , antiderivative size = 52, normalized size of antiderivative = 0.66

$$\int \frac{1}{(a \csc^3(x))^{3/2}} dx = \frac{-84E\left(\frac{1}{4}(\pi - 2x) \mid 2\right) + (-33 \cos(x) + 5 \cos(3x)) \sin^{\frac{3}{2}}(x)}{90 (a \csc^3(x))^{3/2} \sin^{\frac{9}{2}}(x)}$$

input `Integrate[(a*Csc[x]^3)^(-3/2),x]`

output `(-84*EllipticE[(Pi - 2*x)/4, 2] + (-33*Cos[x] + 5*Cos[3*x])*Sin[x]^(3/2))/(90*(a*Csc[x]^3)^(3/2)*Sin[x]^(9/2))`

**Rubi [A] (verified)**

Time = 0.44 (sec) , antiderivative size = 88, normalized size of antiderivative = 1.11, number of steps used = 10, number of rules used = 10,  $\frac{\text{number of rules}}{\text{integrand size}} = 1.000$ , Rules used = {3042, 4611, 3042, 4256, 3042, 4256, 3042, 4258, 3042, 3119}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{1}{(a \csc^3(x))^{3/2}} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{1}{\left(-a \sec\left(x + \frac{\pi}{2}\right)\right)^{3/2}} dx \\
 & \quad \downarrow \text{4611} \\
 & -\frac{(-\csc(x))^{3/2} \int \frac{1}{(-\csc(x))^{9/2}} dx}{a \sqrt{a \csc^3(x)}} \\
 & \quad \downarrow \text{3042} \\
 & -\frac{(-\csc(x))^{3/2} \int \frac{1}{(-\csc(x))^{9/2}} dx}{a \sqrt{a \csc^3(x)}} \\
 & \quad \downarrow \text{4256} \\
 & -\frac{(-\csc(x))^{3/2} \left( \frac{7}{9} \int \frac{1}{(-\csc(x))^{5/2}} dx + \frac{2 \cos(x)}{9(-\csc(x))^{7/2}} \right)}{a \sqrt{a \csc^3(x)}} \\
 & \quad \downarrow \text{3042} \\
 & -\frac{(-\csc(x))^{3/2} \left( \frac{7}{9} \int \frac{1}{(-\csc(x))^{5/2}} dx + \frac{2 \cos(x)}{9(-\csc(x))^{7/2}} \right)}{a \sqrt{a \csc^3(x)}} \\
 & \quad \downarrow \text{4256} \\
 & -\frac{(-\csc(x))^{3/2} \left( \frac{7}{9} \left( \frac{3}{5} \int \frac{1}{\sqrt{-\csc(x)}} dx + \frac{2 \cos(x)}{5(-\csc(x))^{3/2}} \right) + \frac{2 \cos(x)}{9(-\csc(x))^{7/2}} \right)}{a \sqrt{a \csc^3(x)}} \\
 & \quad \downarrow \text{3042}
 \end{aligned}$$

$$\begin{aligned}
& \frac{(-\csc(x))^{3/2} \left( \frac{7}{9} \left( \frac{3}{5} \int \frac{1}{\sqrt{-\csc(x)}} dx + \frac{2 \cos(x)}{5(-\csc(x))^{3/2}} \right) + \frac{2 \cos(x)}{9(-\csc(x))^{7/2}} \right)}{a \sqrt{a \csc^3(x)}} \\
& \quad \downarrow 4258 \\
& \frac{(-\csc(x))^{3/2} \left( \frac{7}{9} \left( \frac{3 \int \sqrt{\sin(x)} dx}{5 \sqrt{\sin(x)} \sqrt{-\csc(x)}} + \frac{2 \cos(x)}{5(-\csc(x))^{3/2}} \right) + \frac{2 \cos(x)}{9(-\csc(x))^{7/2}} \right)}{a \sqrt{a \csc^3(x)}} \\
& \quad \downarrow 3042 \\
& \frac{(-\csc(x))^{3/2} \left( \frac{7}{9} \left( \frac{3 \int \sqrt{\sin(x)} dx}{5 \sqrt{\sin(x)} \sqrt{-\csc(x)}} + \frac{2 \cos(x)}{5(-\csc(x))^{3/2}} \right) + \frac{2 \cos(x)}{9(-\csc(x))^{7/2}} \right)}{a \sqrt{a \csc^3(x)}} \\
& \quad \downarrow 3119 \\
& \frac{(-\csc(x))^{3/2} \left( \frac{2 \cos(x)}{9(-\csc(x))^{7/2}} + \frac{7}{9} \left( \frac{2 \cos(x)}{5(-\csc(x))^{3/2}} - \frac{6E\left(\frac{\pi}{4} - \frac{\pi}{2} \mid 2\right)}{5 \sqrt{\sin(x)} \sqrt{-\csc(x)}} \right) \right)}{a \sqrt{a \csc^3(x)}}
\end{aligned}$$

input `Int[(a*Csc[x]^3)^(-3/2),x]`

output `-((( -Csc[x])^(3/2)*((2*Cos[x])/(9*(-Csc[x])^(7/2)) + (7*((2*Cos[x])/(5*(-Csc[x])^(3/2)) - (6*EllipticE[Pi/4 - x/2, 2])/(5*sqrt[-Csc[x])*sqrt[Sin[x]])))/9))/(a*sqrt[a*Csc[x]^3]))`

### Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3119 `Int[Sqrt[sin[(c_.) + (d.)*(x_)]], x_Symbol] := Simp[(2/d)*EllipticE[(1/2)*(c - Pi/2 + d*x), 2], x] /; FreeQ[{c, d}, x]`

rule 4256 `Int[(csc[(c_.) + (d.)*(x_)]*(b_.))^n, x_Symbol] := Simp[Cos[c + d*x]*((b*Csc[c + d*x])^(n + 1)/(b*d^n)), x] + Simp[(n + 1)/(b^2*n) Int[(b*Csc[c + d*x])^(n + 2), x], x] /; FreeQ[{b, c, d}, x] && LtQ[n, -1] && IntegerQ[2*n]`



rule 4258

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] := Simp[(b*Csc[c + d*x]
)^(n)*Sin[c + d*x]^n Int[1/Sin[c + d*x]^n, x], x] /; FreeQ[{b, c, d}, x] &&
EqQ[n^2, 1/4]
```

rule 4611

```
Int[((b_.)*((c_.)*sec[(e_.) + (f_.)*(x_)])^(n_))^(p_), x_Symbol] := Simp[b^
IntPart[p]*((b*(c*Sec[e + f*x])^(n))^FracPart[p]/(c*Sec[e + f*x])^(n*FracPart
[p])) Int[(c*Sec[e + f*x])^(n*p), x], x] /; FreeQ[{b, c, e, f, n, p}, x]
&& !IntegerQ[p]
```

### Maple [C] (verified)

Result contains complex when optimal does not.

Time = 0.46 (sec) , antiderivative size = 178, normalized size of antiderivative = 2.25

method	result
default	$-\frac{\sqrt{8} \left( (42 \cos(x) + 42) \sqrt{1 + i \cot(x) - i \csc(x)} \sqrt{1 - i \cot(x) + i \csc(x)} \sqrt{i(-\cot(x) + \csc(x))} \operatorname{EllipticE}\left(\sqrt{1 + i \cot(x) - i \csc(x)}, \frac{\sqrt{2}}{2}\right) + \dots \right)}{\dots}$

input

```
int(1/(a*csc(x)^3)^(3/2), x, method=_RETURNVERBOSE)
```

output

```
-1/90*8^(1/2)*((42*cos(x)+42)*(1+I*cot(x)-I*csc(x))^(1/2)*(1-I*cot(x)+I*cs
c(x))^(1/2)*(I*(-cot(x)+csc(x)))^(1/2)*EllipticE((1+I*cot(x)-I*csc(x))^(1/
2), 1/2*2^(1/2)))+(-21*cos(x)-21)*(1+I*cot(x)-I*csc(x))^(1/2)*(1-I*cot(x)+I*
csc(x))^(1/2)*(I*(-cot(x)+csc(x)))^(1/2)*EllipticF((1+I*cot(x)-I*csc(x))^(
1/2), 1/2*2^(1/2)))+(5*cos(x)^5-17*cos(x)^3+33*cos(x)-21)*2^(1/2))/a/(a*csc(
x)^3)^(1/2)*csc(x)^2
```

**Fricas [C] (verification not implemented)**

Result contains complex when optimal does not.

Time = 0.09 (sec) , antiderivative size = 84, normalized size of antiderivative = 1.06

$$\int \frac{1}{(a \csc^3(x))^{3/2}} dx =$$

$$2 (5 \cos(x)^5 - 17 \cos(x)^3 + 12 \cos(x)) \sqrt{-\frac{a}{(\cos(x)^2 - 1) \sin(x)}} \sin(x) - 21 \sqrt{2i} a \text{weierstrassZeta}(4, 0, \text{weier}$$


---

input `integrate(1/(a*csc(x)^3)^(3/2),x, algorithm="fricas")`

output `-1/45*(2*(5*cos(x)^5 - 17*cos(x)^3 + 12*cos(x))*sqrt(-a/((cos(x)^2 - 1)*sin(x)))*sin(x) - 21*sqrt(2*I*a)*weierstrassZeta(4, 0, weierstrassPInverse(4, 0, cos(x) + I*sin(x))) - 21*sqrt(-2*I*a)*weierstrassZeta(4, 0, weierstrassPInverse(4, 0, cos(x) - I*sin(x))))/a^2`

**Sympy [F]**

$$\int \frac{1}{(a \csc^3(x))^{3/2}} dx = \int \frac{1}{(a \csc^3(x))^{\frac{3}{2}}} dx$$

input `integrate(1/(a*csc(x)**3)**(3/2),x)`

output `Integral((a*csc(x)**3)**(-3/2), x)`

**Maxima [F]**

$$\int \frac{1}{(a \csc^3(x))^{3/2}} dx = \int \frac{1}{(a \csc(x)^3)^{3/2}} dx$$

input `integrate(1/(a*csc(x)^3)^(3/2),x, algorithm="maxima")`

output `integrate((a*csc(x)^3)^(-3/2), x)`

**Giac [F]**

$$\int \frac{1}{(a \csc^3(x))^{3/2}} dx = \int \frac{1}{(a \csc(x)^3)^{3/2}} dx$$

input `integrate(1/(a*csc(x)^3)^(3/2),x, algorithm="giac")`

output `integrate((a*csc(x)^3)^(-3/2), x)`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{1}{(a \csc^3(x))^{3/2}} dx = \int \frac{1}{\left(\frac{a}{\sin(x)^3}\right)^{3/2}} dx$$

input `int(1/(a/sin(x)^3)^(3/2),x)`

output `int(1/(a/sin(x)^3)^(3/2), x)`

Reduce [F]

$$\int \frac{1}{(a \csc^3(x))^{3/2}} dx = \frac{\sqrt{a} \left( \int \frac{\sqrt{\csc(x)}}{\csc(x)^5} dx \right)}{a^2}$$

input `int(1/(a*csc(x)^3)^(3/2),x)`

output `(sqrt(a)*int(sqrt(csc(x))/csc(x)**5,x))/a**2`

**3.60**  $\int \frac{1}{(a \csc^3(x))^{5/2}} dx$

Optimal result	388
Mathematica [A] (verified)	388
Rubi [A] (verified)	389
Maple [C] (verified)	392
Fricas [C] (verification not implemented)	392
Sympy [F]	393
Maxima [F]	393
Giac [F]	394
Mupad [F(-1)]	394
Reduce [F]	394

**Optimal result**

Integrand size = 10, antiderivative size = 123

$$\int \frac{1}{(a \csc^3(x))^{5/2}} dx = -\frac{26 \cot(x)}{77a^2 \sqrt{a \csc^3(x)}} - \frac{26 \operatorname{EllipticF}\left(\frac{\pi}{4} - \frac{x}{2}, 2\right)}{77a^2 \sqrt{a \csc^3(x)} \sin^{\frac{3}{2}}(x)}$$

$$-\frac{78 \cos(x) \sin(x)}{385a^2 \sqrt{a \csc^3(x)}} - \frac{26 \cos(x) \sin^3(x)}{165a^2 \sqrt{a \csc^3(x)}} - \frac{2 \cos(x) \sin^5(x)}{15a^2 \sqrt{a \csc^3(x)}}$$

output

```
-26/77*cot(x)/a^2/(a*csc(x)^3)^(1/2)+26/77*InverseJacobiAM(-1/4*Pi+1/2*x,2
^(1/2))/a^2/(a*csc(x)^3)^(1/2)/sin(x)^(3/2)-78/385*cos(x)*sin(x)/a^2/(a*cs
c(x)^3)^(1/2)-26/165*cos(x)*sin(x)^3/a^2/(a*csc(x)^3)^(1/2)-2/15*cos(x)*si
n(x)^5/a^2/(a*csc(x)^3)^(1/2)
```

**Mathematica [A] (verified)**

Time = 0.12 (sec) , antiderivative size = 63, normalized size of antiderivative = 0.51

$$\int \frac{1}{(a \csc^3(x))^{5/2}} dx = \frac{\sqrt{a \csc^3(x)} \sin(x) \left( 24960 \operatorname{EllipticF}\left(\frac{1}{4}(\pi - 2x), 2\right) \sqrt{\sin(x)} + 19122 \sin(2x) - 4406 \sin(4x) + 826 \sin(6x) \right)}{73920a^3}$$

input `Integrate[(a*Csc[x]^3)^(-5/2),x]`

output `-1/73920*(Sqrt[a*Csc[x]^3]*Sin[x]*(24960*EllipticF[(Pi - 2*x)/4, 2]*Sqrt[Sin[x]] + 19122*Sin[2*x] - 4406*Sin[4*x] + 826*Sin[6*x] - 77*Sin[8*x]))/a^3`

### Rubi [A] (verified)

Time = 0.63 (sec) , antiderivative size = 125, normalized size of antiderivative = 1.02, number of steps used = 14, number of rules used = 14,  $\frac{\text{number of rules}}{\text{integrand size}} = 1.400$ , Rules used = {3042, 4611, 3042, 4256, 3042, 4256, 3042, 4256, 3042, 4256, 3042, 4258, 3042, 3120}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{1}{(a \csc^3(x))^{5/2}} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{1}{(-a \sec(x + \frac{\pi}{2}))^{5/2}} dx \\
 & \quad \downarrow \text{4611} \\
 & \frac{(-\csc(x))^{3/2} \int \frac{1}{(-\csc(x))^{15/2}} dx}{a^2 \sqrt{a \csc^3(x)}} \\
 & \quad \downarrow \text{3042} \\
 & \frac{(-\csc(x))^{3/2} \int \frac{1}{(-\csc(x))^{15/2}} dx}{a^2 \sqrt{a \csc^3(x)}} \\
 & \quad \downarrow \text{4256} \\
 & \frac{(-\csc(x))^{3/2} \left( \frac{13}{15} \int \frac{1}{(-\csc(x))^{11/2}} dx + \frac{2 \cos(x)}{15(-\csc(x))^{13/2}} \right)}{a^2 \sqrt{a \csc^3(x)}} \\
 & \quad \downarrow \text{3042}
 \end{aligned}$$

$$\frac{(-\csc(x))^{3/2} \left( \frac{13}{15} \int \frac{1}{(-\csc(x))^{11/2}} dx + \frac{2 \cos(x)}{15(-\csc(x))^{13/2}} \right)}{a^2 \sqrt{a \csc^3(x)}}$$

↓ 4256

$$\frac{(-\csc(x))^{3/2} \left( \frac{13}{15} \left( \frac{9}{11} \int \frac{1}{(-\csc(x))^{7/2}} dx + \frac{2 \cos(x)}{11(-\csc(x))^{9/2}} \right) + \frac{2 \cos(x)}{15(-\csc(x))^{13/2}} \right)}{a^2 \sqrt{a \csc^3(x)}}$$

↓ 3042

$$\frac{(-\csc(x))^{3/2} \left( \frac{13}{15} \left( \frac{9}{11} \int \frac{1}{(-\csc(x))^{7/2}} dx + \frac{2 \cos(x)}{11(-\csc(x))^{9/2}} \right) + \frac{2 \cos(x)}{15(-\csc(x))^{13/2}} \right)}{a^2 \sqrt{a \csc^3(x)}}$$

↓ 4256

$$\frac{(-\csc(x))^{3/2} \left( \frac{13}{15} \left( \frac{9}{11} \left( \frac{5}{7} \int \frac{1}{(-\csc(x))^{3/2}} dx + \frac{2 \cos(x)}{7(-\csc(x))^{5/2}} \right) + \frac{2 \cos(x)}{11(-\csc(x))^{9/2}} \right) + \frac{2 \cos(x)}{15(-\csc(x))^{13/2}} \right)}{a^2 \sqrt{a \csc^3(x)}}$$

↓ 3042

$$\frac{(-\csc(x))^{3/2} \left( \frac{13}{15} \left( \frac{9}{11} \left( \frac{5}{7} \int \frac{1}{(-\csc(x))^{3/2}} dx + \frac{2 \cos(x)}{7(-\csc(x))^{5/2}} \right) + \frac{2 \cos(x)}{11(-\csc(x))^{9/2}} \right) + \frac{2 \cos(x)}{15(-\csc(x))^{13/2}} \right)}{a^2 \sqrt{a \csc^3(x)}}$$

↓ 4256

$$\frac{(-\csc(x))^{3/2} \left( \frac{13}{15} \left( \frac{9}{11} \left( \frac{5}{7} \left( \frac{1}{3} \int \sqrt{-\csc(x)} dx + \frac{2 \cos(x)}{3\sqrt{-\csc(x)}} \right) + \frac{2 \cos(x)}{7(-\csc(x))^{5/2}} \right) + \frac{2 \cos(x)}{11(-\csc(x))^{9/2}} \right) + \frac{2 \cos(x)}{15(-\csc(x))^{13/2}} \right)}{a^2 \sqrt{a \csc^3(x)}}$$

↓ 3042

$$\frac{(-\csc(x))^{3/2} \left( \frac{13}{15} \left( \frac{9}{11} \left( \frac{5}{7} \left( \frac{1}{3} \int \sqrt{-\csc(x)} dx + \frac{2 \cos(x)}{3\sqrt{-\csc(x)}} \right) + \frac{2 \cos(x)}{7(-\csc(x))^{5/2}} \right) + \frac{2 \cos(x)}{11(-\csc(x))^{9/2}} \right) + \frac{2 \cos(x)}{15(-\csc(x))^{13/2}} \right)}{a^2 \sqrt{a \csc^3(x)}}$$

↓ 4258

$$\frac{(-\csc(x))^{3/2} \left( \frac{13}{15} \left( \frac{9}{11} \left( \frac{5}{7} \left( \frac{1}{3} \sqrt{\sin(x)} \sqrt{-\csc(x)} \int \frac{1}{\sqrt{\sin(x)}} dx + \frac{2 \cos(x)}{3\sqrt{-\csc(x)}} \right) + \frac{2 \cos(x)}{7(-\csc(x))^{5/2}} \right) + \frac{2 \cos(x)}{11(-\csc(x))^{9/2}} \right) + \frac{2 \cos(x)}{15(-\csc(x))^{13/2}} \right)}{a^2 \sqrt{a \csc^3(x)}}$$

↓ 3042

$$\frac{(-\csc(x))^{3/2} \left( \frac{13}{15} \left( \frac{9}{11} \left( \frac{5}{7} \left( \frac{1}{3} \sqrt{\sin(x)} \sqrt{-\csc(x)} \int \frac{1}{\sqrt{\sin(x)}} dx + \frac{2 \cos(x)}{3 \sqrt{-\csc(x)}} \right) + \frac{2 \cos(x)}{7 (-\csc(x))^{5/2}} \right) + \frac{2 \cos(x)}{11 (-\csc(x))^{9/2}} \right) + \frac{2 \cos(x)}{15 (-\csc(x))^{13/2}} \right)}{a^2 \sqrt{a \csc^3(x)}}$$

↓ 3120

$$\frac{(-\csc(x))^{3/2} \left( \frac{2 \cos(x)}{15 (-\csc(x))^{13/2}} + \frac{13}{15} \left( \frac{2 \cos(x)}{11 (-\csc(x))^{9/2}} + \frac{9}{11} \left( \frac{2 \cos(x)}{7 (-\csc(x))^{5/2}} + \frac{5}{7} \left( \frac{2 \cos(x)}{3 \sqrt{-\csc(x)}} - \frac{2}{3} \sqrt{\sin(x)} \sqrt{-\csc(x)} \operatorname{EllipticF}\left[\frac{\pi}{4} - \frac{x}{2}, 2\right] \sqrt{\sin(x)} \right) \right) \right)}{a^2 \sqrt{a \csc^3(x)}}$$

input

```
Int[(a*Csc[x]^3)^(-5/2),x]
```

output

```
((-Csc[x])^(3/2)*((2*Cos[x])/(15*(-Csc[x])^(13/2)) + (13*((2*Cos[x])/(11*(-Csc[x])^(9/2)) + (9*((2*Cos[x])/(7*(-Csc[x])^(5/2)) + (5*((2*Cos[x])/(3*Sqrt[-Csc[x])) - (2*Sqrt[-Csc[x])*EllipticF[Pi/4 - x/2, 2]*Sqrt[Sin[x]])/3)/7))/11))/15))/(a^2*Sqrt[a*Csc[x]^3])
```

### Defintions of rubi rules used

rule 3042

```
Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]
```

rule 3120

```
Int[1/Sqrt[sin[(c_.) + (d_.)*(x_)]], x_Symbol] := Simp[(2/d)*EllipticF[(1/2)*(c - Pi/2 + d*x), 2], x] /; FreeQ[{c, d}, x]
```

rule 4256

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^n, x_Symbol] := Simp[Cos[c + d*x]*((b*Csc[c + d*x])^(n + 1)/(b*d^n), x] + Simp[(n + 1)/(b^2*n) Int[(b*Csc[c + d*x])^(n + 2), x], x] /; FreeQ[{b, c, d}, x] && LtQ[n, -1] && IntegerQ[2*n]
```

rule 4258

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^n, x_Symbol] := Simp[(b*Csc[c + d*x])^n*Sin[c + d*x]^n Int[1/Sin[c + d*x]^n, x], x] /; FreeQ[{b, c, d}, x] && EqQ[n^2, 1/4]
```



rule 4611

```
Int[((b_.)*((c_.)*sec[(e_.) + (f_.)*(x_)])^(n_))^(p_), x_Symbol] := Simp[b^
IntPart[p]*((b*(c*Sec[e + f*x])^n)^FracPart[p]/(c*Sec[e + f*x])^(n*FracPart
[p])) Int[(c*Sec[e + f*x])^(n*p), x], x] /; FreeQ[{b, c, e, f, n, p}, x]
&& !IntegerQ[p]
```

**Maple [C] (verified)**

Result contains complex when optimal does not.

Time = 1.02 (sec) , antiderivative size = 117, normalized size of antiderivative = 0.95

method	result
default	$\frac{\sqrt{8} \left( \sqrt{2} \left( 77 \cos(x)^6 - 322 \cos(x)^4 + 530 \cos(x)^2 - 480 \right) \cot(x) + i(195 \cos(x) + 195) \sqrt{1+i \cot(x) - i \csc(x)} \sqrt{1-i \cot(x) + i \csc(x)} \sqrt{i(-\dots)} \right)}{2310a^2 \sqrt{a \csc(x)^3}}$

input

```
int(1/(a*csc(x)^3)^(5/2),x,method=_RETURNVERBOSE)
```

output

```
1/2310*8^(1/2)/a^2/(a*csc(x)^3)^(1/2)*(2^(1/2)*(77*cos(x)^6-322*cos(x)^4+5
30*cos(x)^2-480)*cot(x)+I*(195*cos(x)+195)*(1+I*cot(x)-I*csc(x))^(1/2)*(1-
I*cot(x)+I*csc(x))^(1/2)*(I*(-cot(x)+csc(x)))^(1/2)*EllipticF((1+I*cot(x)-
I*csc(x))^(1/2),1/2*2^(1/2))*csc(x)^2)
```

**Fricas [C] (verification not implemented)**

Result contains complex when optimal does not.

Time = 0.10 (sec) , antiderivative size = 88, normalized size of antiderivative = 0.72

$$\int \frac{1}{(a \csc^3(x))^{5/2}} dx = \frac{2(77 \cos(x)^9 - 399 \cos(x)^7 + 852 \cos(x)^5 - 1010 \cos(x)^3 + 480 \cos(x)) \sqrt{-\frac{a}{(\cos(x)^2-1) \sin(x)}} + 195i \sqrt{\dots}}{\dots}$$

input

```
integrate(1/(a*csc(x)^3)^(5/2),x, algorithm="fricas")
```

output

```
-1/1155*(2*(77*cos(x)^9 - 399*cos(x)^7 + 852*cos(x)^5 - 1010*cos(x)^3 + 480*cos(x))*sqrt(-a/((cos(x)^2 - 1)*sin(x))) + 195*I*sqrt(2*I*a)*weierstrassPInverse(4, 0, cos(x) + I*sin(x)) - 195*I*sqrt(-2*I*a)*weierstrassPInverse(4, 0, cos(x) - I*sin(x)))/a^3
```

**Sympy [F]**

$$\int \frac{1}{(a \csc^3(x))^{5/2}} dx = \int \frac{1}{(a \csc^3(x))^{5/2}} dx$$

input

```
integrate(1/(a*csc(x)**3)**(5/2),x)
```

output

```
Integral((a*csc(x)**3)**(-5/2), x)
```

**Maxima [F]**

$$\int \frac{1}{(a \csc^3(x))^{5/2}} dx = \int \frac{1}{(a \csc(x)^3)^{5/2}} dx$$

input

```
integrate(1/(a*csc(x)^3)^(5/2),x, algorithm="maxima")
```

output

```
integrate((a*csc(x)^3)^(-5/2), x)
```

**Giac [F]**

$$\int \frac{1}{(a \csc^3(x))^{5/2}} dx = \int \frac{1}{(a \csc(x)^3)^{5/2}} dx$$

input `integrate(1/(a*csc(x)^3)^(5/2),x, algorithm="giac")`

output `integrate((a*csc(x)^3)^(-5/2), x)`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{1}{(a \csc^3(x))^{5/2}} dx = \int \frac{1}{\left(\frac{a}{\sin(x)^3}\right)^{5/2}} dx$$

input `int(1/(a/sin(x)^3)^(5/2),x)`

output `int(1/(a/sin(x)^3)^(5/2), x)`

**Reduce [F]**

$$\int \frac{1}{(a \csc^3(x))^{5/2}} dx = \frac{\sqrt{a} \left( \int \frac{\sqrt{\csc(x)}}{\csc(x)^8} dx \right)}{a^3}$$

input `int(1/(a*csc(x)^3)^(5/2),x)`

output `(sqrt(a)*int(sqrt(csc(x))/csc(x)**8,x))/a**3`

### 3.61 $\int (a \csc^4(x))^{7/2} dx$

Optimal result . . . . .	395
Mathematica [A] (verified) . . . . .	396
Rubi [A] (verified) . . . . .	396
Maple [A] (verified) . . . . .	398
Fricas [A] (verification not implemented) . . . . .	398
Sympy [F(-1)] . . . . .	399
Maxima [A] (verification not implemented) . . . . .	399
Giac [A] (verification not implemented) . . . . .	399
Mupad [B] (verification not implemented) . . . . .	400
Reduce [B] (verification not implemented) . . . . .	401

#### Optimal result

Integrand size = 10, antiderivative size = 164

$$\begin{aligned} \int (a \csc^4(x))^{7/2} dx &= -2a^3 \cos^2(x) \cot(x) \sqrt{a \csc^4(x)} \\ &\quad - 3a^3 \cos^2(x) \cot^3(x) \sqrt{a \csc^4(x)} - \frac{20}{7} a^3 \cos^2(x) \cot^5(x) \sqrt{a \csc^4(x)} \\ &\quad - \frac{5}{3} a^3 \cos^2(x) \cot^7(x) \sqrt{a \csc^4(x)} - \frac{6}{11} a^3 \cos^2(x) \cot^9(x) \sqrt{a \csc^4(x)} \\ &\quad - \frac{1}{13} a^3 \cos^2(x) \cot^{11}(x) \sqrt{a \csc^4(x)} - a^3 \cos(x) \sqrt{a \csc^4(x)} \sin(x) \end{aligned}$$

output

```
-2*a^3*cos(x)^2*cot(x)*(a*csc(x)^4)^(1/2)-3*a^3*cos(x)^2*cot(x)^3*(a*csc(x)^4)^(1/2)-20/7*a^3*cos(x)^2*cot(x)^5*(a*csc(x)^4)^(1/2)-5/3*a^3*cos(x)^2*cot(x)^7*(a*csc(x)^4)^(1/2)-6/11*a^3*cos(x)^2*cot(x)^9*(a*csc(x)^4)^(1/2)-1/13*a^3*cos(x)^2*cot(x)^11*(a*csc(x)^4)^(1/2)-a^3*cos(x)*(a*csc(x)^4)^(1/2)*sin(x)
```

**Mathematica [A] (verified)**

Time = 0.07 (sec) , antiderivative size = 59, normalized size of antiderivative = 0.36

$$\int (a \csc^4(x))^{7/2} dx = \frac{a^3 \cos(x) \sqrt{a \csc^4(x)} (1024 + 512 \csc^2(x) + 384 \csc^4(x) + 320 \csc^6(x) + 280 \csc^8(x) + 252 \csc^{10}(x) + 231 \csc^{12}(x))}{3003}$$

input

```
Integrate[(a*Csc[x]^4)^(7/2),x]
```

output

```
-1/3003*(a^3*Cos[x]*Sqrt[a*Csc[x]^4]*(1024 + 512*Csc[x]^2 + 384*Csc[x]^4 + 320*Csc[x]^6 + 280*Csc[x]^8 + 252*Csc[x]^10 + 231*Csc[x]^12)*Sin[x])
```

**Rubi [A] (verified)**

Time = 0.27 (sec) , antiderivative size = 66, normalized size of antiderivative = 0.40, number of steps used = 6, number of rules used = 5,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {3042, 4611, 3042, 4254, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int (a \csc^4(x))^{7/2} dx \\ & \quad \downarrow \text{3042} \\ & \int \left( a \sec \left( x + \frac{\pi}{2} \right)^4 \right)^{7/2} dx \\ & \quad \downarrow \text{4611} \\ & a^3 \sin^2(x) \sqrt{a \csc^4(x)} \int \csc^{14}(x) dx \\ & \quad \downarrow \text{3042} \\ & a^3 \sin^2(x) \sqrt{a \csc^4(x)} \int \csc(x)^{14} dx \\ & \quad \downarrow \text{4254} \end{aligned}$$

$$-a^3 \sin^2(x) \sqrt{a \csc^4(x)} \int (\cot^{12}(x) + 6 \cot^{10}(x) + 15 \cot^8(x) + 20 \cot^6(x) + 15 \cot^4(x) + 6 \cot^2(x) + 1) d \cot(x)$$

↓ 2009

$$-a^3 \sin^2(x) \left( \frac{\cot^{13}(x)}{13} + \frac{6 \cot^{11}(x)}{11} + \frac{5 \cot^9(x)}{3} + \frac{20 \cot^7(x)}{7} + 3 \cot^5(x) + 2 \cot^3(x) + \cot(x) \right) \sqrt{a \csc^4(x)}$$

input `Int[(a*Csc[x]^4)^(7/2),x]`

output `-(a^3*(Cot[x] + 2*Cot[x]^3 + 3*Cot[x]^5 + (20*Cot[x]^7)/7 + (5*Cot[x]^9)/3 + (6*Cot[x]^11)/11 + Cot[x]^13/13)*Sqrt[a*Csc[x]^4]*Sin[x]^2)`

### Defintions of rubi rules used

rule 2009 `Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 4254 `Int[csc[(c_.) + (d_.)*(x_)]^(n_), x_Symbol] := Simp[-d^(-1) Subst[Int[Exp andIntegrand[(1 + x^2)^(n/2 - 1), x], x], x, Cot[c + d*x]], x] /; FreeQ[{c, d}, x] && IGtQ[n/2, 0]`

rule 4611 `Int[((b_.)*((c_.)*sec[(e_.) + (f_.)*(x_)]^(n_)))^(p_), x_Symbol] := Simp[b^IntPart[p]*((b*(c*Sec[e + f*x])^n)^FracPart[p]/(c*Sec[e + f*x])^(n*FracPart[p])) Int[(c*Sec[e + f*x])^(n*p), x], x] /; FreeQ[{b, c, e, f, n, p}, x] && !IntegerQ[p]`

**Maple [A] (verified)**

Time = 0.59 (sec) , antiderivative size = 61, normalized size of antiderivative = 0.37

method	result
default	$-\frac{\sqrt{16} \left( 1024 \cos(x)^{12} - 6656 \cos(x)^{10} + 18304 \cos(x)^8 - 27456 \cos(x)^6 + 24024 \cos(x)^4 - 12012 \cos(x)^2 + 3003 \right) \sqrt{a \csc(x)^4 a^3 \cot(x)}}{12012}$
risch	$\frac{2048ia^3 \sqrt{\frac{ae^{4ix}}{(e^{2ix}-1)^4}} (1716 e^{10ix} - 1287 e^{8ix} + 715 e^{6ix} - 286 e^{4ix} - 13 + 79 \cos(2x) + 77i \sin(2x))}{3003(e^{2ix}-1)^{11}}$

input `int((a*csc(x)^4)^(7/2),x,method=_RETURNVERBOSE)`

output 
$$-1/12012*16^{(1/2)}*(1024*\cos(x)^{12}-6656*\cos(x)^{10}+18304*\cos(x)^8-27456*\cos(x)^6+24024*\cos(x)^4-12012*\cos(x)^2+3003)*(a*csc(x)^4)^{(1/2)}*a^3*\cot(x)*csc(x)^{10}$$

**Fricas [A] (verification not implemented)**

Time = 0.08 (sec) , antiderivative size = 118, normalized size of antiderivative = 0.72

$$\int (a \csc^4(x))^{7/2} dx = \frac{(1024 a^3 \cos(x)^{13} - 6656 a^3 \cos(x)^{11} + 18304 a^3 \cos(x)^9 - 27456 a^3 \cos(x)^7 + 24024 a^3 \cos(x)^5 - 12012 a^3 \cos(x)^3 + 3003 a^3 \cos(x)) \sqrt{a/(\cos(x)^4 - 2\cos(x)^2 + 1)}}{3003 (\cos(x)^{10} - 5 \cos(x)^8 + 10 \cos(x)^6 - 10 \cos(x)^4 + 5 \cos(x)^2 - 1) \sin(x)}$$

input `integrate((a*csc(x)^4)^(7/2),x, algorithm="fricas")`

output 
$$1/3003*(1024*a^3*\cos(x)^{13} - 6656*a^3*\cos(x)^{11} + 18304*a^3*\cos(x)^9 - 27456*a^3*\cos(x)^7 + 24024*a^3*\cos(x)^5 - 12012*a^3*\cos(x)^3 + 3003*a^3*\cos(x))\sqrt{a/(\cos(x)^4 - 2*\cos(x)^2 + 1)}/((\cos(x)^{10} - 5*\cos(x)^8 + 10*\cos(x)^6 - 10*\cos(x)^4 + 5*\cos(x)^2 - 1)*\sin(x))$$

**Sympy [F(-1)]**

Timed out.

$$\int (a \csc^4(x))^{7/2} dx = \text{Timed out}$$

input `integrate((a*csc(x)**4)**(7/2),x)`output `Timed out`**Maxima [A] (verification not implemented)**

Time = 0.12 (sec) , antiderivative size = 66, normalized size of antiderivative = 0.40

$$\int (a \csc^4(x))^{7/2} dx = \frac{3003 a^{7/2} \tan(x)^{12} + 6006 a^{7/2} \tan(x)^{10} + 9009 a^{7/2} \tan(x)^8 + 8580 a^{7/2} \tan(x)^6 + 5005 a^{7/2} \tan(x)^4 + 1638 a^{7/2} \tan(x)^2 + 231 a^{7/2}}{3003 \tan(x)^{13}}$$

input `integrate((a*csc(x)^4)^(7/2),x, algorithm="maxima")`output `-1/3003*(3003*a^(7/2)*tan(x)^12 + 6006*a^(7/2)*tan(x)^10 + 9009*a^(7/2)*tan(x)^8 + 8580*a^(7/2)*tan(x)^6 + 5005*a^(7/2)*tan(x)^4 + 1638*a^(7/2)*tan(x)^2 + 231*a^(7/2))/tan(x)^13`**Giac [A] (verification not implemented)**

Time = 0.18 (sec) , antiderivative size = 69, normalized size of antiderivative = 0.42

$$\int (a \csc^4(x))^{7/2} dx = \frac{(3003 a^3 \tan(x)^{12} + 6006 a^3 \tan(x)^{10} + 9009 a^3 \tan(x)^8 + 8580 a^3 \tan(x)^6 + 5005 a^3 \tan(x)^4 + 1638 a^3 \tan(x)^2 + 231 a^3)}{3003 \tan(x)^{13}}$$

input `integrate((a*csc(x)^4)^(7/2),x, algorithm="giac")`



output

```
-1/3003*(3003*a^3*tan(x)^12 + 6006*a^3*tan(x)^10 + 9009*a^3*tan(x)^8 + 8580*a^3*tan(x)^6 + 5005*a^3*tan(x)^4 + 1638*a^3*tan(x)^2 + 231*a^3)*sqrt(a)/tan(x)^13
```

### Mupad [B] (verification not implemented)

Time = 19.96 (sec) , antiderivative size = 603, normalized size of antiderivative = 3.68

$$\int (a \csc^4(x))^{7/2} dx = \text{Too large to display}$$

input

```
int((a/sin(x)^4)^(7/2),x)
```

output

```
(a^3*(a/((exp(-x*1i)*1i)/2 - (exp(x*1i)*1i)/2)^4)^(1/2)*(6*exp(x*4i) - 4*exp(x*2i) - 4*exp(x*6i) + exp(x*8i) + 1)*2048i)/(7*(exp(x*2i) - 1)^7*(exp(x*2i) - 2*exp(x*4i) + exp(x*6i))) + (a^3*(a/((exp(-x*1i)*1i)/2 - (exp(x*1i)*1i)/2)^4)^(1/2)*(6*exp(x*4i) - 4*exp(x*2i) - 4*exp(x*6i) + exp(x*8i) + 1)*1536i)/((exp(x*2i) - 1)^8*(exp(x*2i) - 2*exp(x*4i) + exp(x*6i))) + (a^3*(a/((exp(-x*1i)*1i)/2 - (exp(x*1i)*1i)/2)^4)^(1/2)*(6*exp(x*4i) - 4*exp(x*2i) - 4*exp(x*6i) + exp(x*8i) + 1)*10240i)/(3*(exp(x*2i) - 1)^9*(exp(x*2i) - 2*exp(x*4i) + exp(x*6i))) + (a^3*(a/((exp(-x*1i)*1i)/2 - (exp(x*1i)*1i)/2)^4)^(1/2)*(6*exp(x*4i) - 4*exp(x*2i) - 4*exp(x*6i) + exp(x*8i) + 1)*4096i)/((exp(x*2i) - 1)^10*(exp(x*2i) - 2*exp(x*4i) + exp(x*6i))) + (a^3*(a/((exp(-x*1i)*1i)/2 - (exp(x*1i)*1i)/2)^4)^(1/2)*(6*exp(x*4i) - 4*exp(x*2i) - 4*exp(x*6i) + exp(x*8i) + 1)*30720i)/(11*(exp(x*2i) - 1)^11*(exp(x*2i) - 2*exp(x*4i) + exp(x*6i))) + (a^3*(a/((exp(-x*1i)*1i)/2 - (exp(x*1i)*1i)/2)^4)^(1/2)*(6*exp(x*4i) - 4*exp(x*2i) - 4*exp(x*6i) + exp(x*8i) + 1)*1024i)/((exp(x*2i) - 1)^12*(exp(x*2i) - 2*exp(x*4i) + exp(x*6i))) + (a^3*(a/((exp(-x*1i)*1i)/2 - (exp(x*1i)*1i)/2)^4)^(1/2)*(6*exp(x*4i) - 4*exp(x*2i) - 4*exp(x*6i) + exp(x*8i) + 1)*2048i)/(13*(exp(x*2i) - 1)^13*(exp(x*2i) - 2*exp(x*4i) + exp(x*6i)))
```

**Reduce [B] (verification not implemented)**

Time = 0.17 (sec) , antiderivative size = 51, normalized size of antiderivative = 0.31

$$\int (a \csc^4(x))^{7/2} dx = \frac{\sqrt{a} \cos(x) a^3 (-1024 \sin(x)^{12} - 512 \sin(x)^{10} - 384 \sin(x)^8 - 320 \sin(x)^6 - 280 \sin(x)^4 - 252 \sin(x)^2 - 231)}{3003 \sin(x)^{13}}$$

input `int((a*csc(x)^4)^(7/2),x)`

output `(sqrt(a)*cos(x)*a**3*( - 1024*sin(x)**12 - 512*sin(x)**10 - 384*sin(x)**8 - 320*sin(x)**6 - 280*sin(x)**4 - 252*sin(x)**2 - 231))/(3003*sin(x)**13)`

### 3.62 $\int (a \csc^4(x))^{5/2} dx$

Optimal result	402
Mathematica [A] (verified)	402
Rubi [A] (verified)	403
Maple [A] (verified)	404
Fricas [A] (verification not implemented)	405
Sympy [F]	405
Maxima [A] (verification not implemented)	406
Giac [A] (verification not implemented)	406
Mupad [B] (verification not implemented)	407
Reduce [B] (verification not implemented)	407

#### Optimal result

Integrand size = 10, antiderivative size = 118

$$\begin{aligned} \int (a \csc^4(x))^{5/2} dx &= -\frac{4}{3}a^2 \cos^2(x) \cot(x) \sqrt{a \csc^4(x)} \\ &\quad - \frac{6}{5}a^2 \cos^2(x) \cot^3(x) \sqrt{a \csc^4(x)} - \frac{4}{7}a^2 \cos^2(x) \cot^5(x) \sqrt{a \csc^4(x)} \\ &\quad - \frac{1}{9}a^2 \cos^2(x) \cot^7(x) \sqrt{a \csc^4(x)} - a^2 \cos(x) \sqrt{a \csc^4(x)} \sin(x) \end{aligned}$$

output

```
-4/3*a^2*cos(x)^2*cot(x)*(a*csc(x)^4)^(1/2)-6/5*a^2*cos(x)^2*cot(x)^3*(a*csc(x)^4)^(1/2)-4/7*a^2*cos(x)^2*cot(x)^5*(a*csc(x)^4)^(1/2)-1/9*a^2*cos(x)^2*cot(x)^7*(a*csc(x)^4)^(1/2)-a^2*cos(x)*(a*csc(x)^4)^(1/2)*sin(x)
```

#### Mathematica [A] (verified)

Time = 0.05 (sec) , antiderivative size = 47, normalized size of antiderivative = 0.40

$$\begin{aligned} \int (a \csc^4(x))^{5/2} dx &= -\frac{1}{315}a^2 \cos(x) \sqrt{a \csc^4(x)} (128 \\ &\quad + 64 \csc^2(x) + 48 \csc^4(x) + 40 \csc^6(x) + 35 \csc^8(x)) \sin(x) \end{aligned}$$

input

```
Integrate[(a*Csc[x]^4)^(5/2),x]
```

output

```
-1/315*(a^2*Cos[x]*Sqrt[a*Csc[x]^4]*(128 + 64*Csc[x]^2 + 48*Csc[x]^4 + 40*
Csc[x]^6 + 35*Csc[x]^8)*Sin[x])
```

**Rubi [A] (verified)**

Time = 0.26 (sec) , antiderivative size = 54, normalized size of antiderivative = 0.46, number of steps used = 6, number of rules used = 5,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {3042, 4611, 3042, 4254, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int (a \csc^4(x))^{5/2} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \left( a \sec \left( x + \frac{\pi}{2} \right)^4 \right)^{5/2} dx \\
 & \quad \downarrow \text{4611} \\
 & a^2 \sin^2(x) \sqrt{a \csc^4(x)} \int \csc^{10}(x) dx \\
 & \quad \downarrow \text{3042} \\
 & a^2 \sin^2(x) \sqrt{a \csc^4(x)} \int \csc(x)^{10} dx \\
 & \quad \downarrow \text{4254} \\
 & -a^2 \sin^2(x) \sqrt{a \csc^4(x)} \int (\cot^8(x) + 4 \cot^6(x) + 6 \cot^4(x) + 4 \cot^2(x) + 1) d \cot(x) \\
 & \quad \downarrow \text{2009} \\
 & -a^2 \sin^2(x) \left( \frac{\cot^9(x)}{9} + \frac{4 \cot^7(x)}{7} + \frac{6 \cot^5(x)}{5} + \frac{4 \cot^3(x)}{3} + \cot(x) \right) \sqrt{a \csc^4(x)}
 \end{aligned}$$

input

```
Int [(a*Csc[x]^4)^(5/2), x]
```

output  $-(a^2(\cot[x] + (4\cot[x]^3)/3 + (6\cot[x]^5)/5 + (4\cot[x]^7)/7 + \cot[x]^9/9)*\sqrt{a\csc[x]^4}\sin[x]^2)$

### Defintions of rubi rules used

rule 2009  $\text{Int}[u_, x\_Symbol] \rightarrow \text{Simp}[\text{IntSum}[u, x], x] \text{ ; SumQ}[u]$

rule 3042  $\text{Int}[u_, x\_Symbol] \rightarrow \text{Int}[\text{DeactivateTrig}[u, x], x] \text{ ; FunctionOfTrigOfLinearQ}[u, x]$

rule 4254  $\text{Int}[\csc[(c_.) + (d_.)(x_)]^{(n_)}, x\_Symbol] \rightarrow \text{Simp}[-d^{(-1)} \text{ Subst}[\text{Int}[\text{ExpandIntegrand}[(1 + x^2)^{(n/2 - 1)}, x], x], x, \cot[c + d*x]], x] \text{ ; FreeQ}[\{c, d\}, x] \ \&\& \ \text{IGtQ}[n/2, 0]$

rule 4611  $\text{Int}[(b_.)((c_.)*\sec[(e_.) + (f_.)(x_)]^{(n_)})^{(p_)}, x\_Symbol] \rightarrow \text{Simp}[b^{\text{IntPart}[p]} * ((b*(c*\sec[e + f*x])^n)^{\text{FracPart}[p]} / (c*\sec[e + f*x])^{(n*\text{FracPart}[p])}) \text{ Int}[(c*\sec[e + f*x])^{(n*p)}, x], x] \text{ ; FreeQ}[\{b, c, e, f, n, p\}, x] \ \&\& \ \text{!IntegerQ}[p]$

### Maple [A] (verified)

Time = 0.27 (sec) , antiderivative size = 49, normalized size of antiderivative = 0.42

method	result	size
default	$-\frac{\sqrt{16} \left( 128 \cos(x)^8 - 576 \cos(x)^6 + 1008 \cos(x)^4 - 840 \cos(x)^2 + 315 \right) \sqrt{a \csc(x)^4} a^2 \cot(x) \csc(x)^6}{1260}$	49
risch	$\frac{256ia^2 \sqrt{\frac{ae^{4ix}}{(e^{2ix}-1)^4}} (126e^{6ix} - 84e^{4ix} - 9 + 37\cos(2x) + 35i\sin(2x))}{315(e^{2ix}-1)^7}$	63

input  $\text{int}((a*\csc(x)^4)^{(5/2)}, x, \text{method}=\_RETURNVERBOSE)$

output 
$$-1/1260*16^{(1/2)}*(128*\cos(x)^8-576*\cos(x)^6+1008*\cos(x)^4-840*\cos(x)^2+315)*(\text{a*csc}(x)^4)^{(1/2)}*\text{a}^2*\cot(x)*\text{csc}(x)^6$$

### Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 88, normalized size of antiderivative = 0.75

$$\int (a \csc^4(x))^{5/2} dx = \frac{(128 a^2 \cos(x)^9 - 576 a^2 \cos(x)^7 + 1008 a^2 \cos(x)^5 - 840 a^2 \cos(x)^3 + 315 a^2 \cos(x)) \sqrt{a/(\cos(x)^4 - 2 \cos(x)^2 + 1)}}{315 (\cos(x)^6 - 3 \cos(x)^4 + 3 \cos(x)^2 - 1) \sin(x)}$$

input `integrate((a*csc(x)^4)^(5/2),x, algorithm="fricas")`

output 
$$1/315*(128*a^2*\cos(x)^9 - 576*a^2*\cos(x)^7 + 1008*a^2*\cos(x)^5 - 840*a^2*\cos(x)^3 + 315*a^2*\cos(x))*\text{sqrt}(a/(\cos(x)^4 - 2*\cos(x)^2 + 1))/((\cos(x)^6 - 3*\cos(x)^4 + 3*\cos(x)^2 - 1)*\sin(x))$$

### Sympy [F]

$$\int (a \csc^4(x))^{5/2} dx = \int (a \csc^4(x))^{\frac{5}{2}} dx$$

input `integrate((a*csc(x)**4)**(5/2),x)`

output `Integral((a*csc(x)**4)**(5/2), x)`

**Maxima [A] (verification not implemented)**

Time = 0.11 (sec) , antiderivative size = 48, normalized size of antiderivative = 0.41

$$\int (a \csc^4(x))^{5/2} dx = \frac{315 a^{5/2} \tan(x)^8 + 420 a^{5/2} \tan(x)^6 + 378 a^{5/2} \tan(x)^4 + 180 a^{5/2} \tan(x)^2 + 35 a^{5/2}}{315 \tan(x)^9}$$

input `integrate((a*csc(x)^4)^(5/2),x, algorithm="maxima")`

output `-1/315*(315*a^(5/2)*tan(x)^8 + 420*a^(5/2)*tan(x)^6 + 378*a^(5/2)*tan(x)^4 + 180*a^(5/2)*tan(x)^2 + 35*a^(5/2))/tan(x)^9`

**Giac [A] (verification not implemented)**

Time = 0.15 (sec) , antiderivative size = 35, normalized size of antiderivative = 0.30

$$\int (a \csc^4(x))^{5/2} dx = \frac{(315 \tan(x)^8 + 420 \tan(x)^6 + 378 \tan(x)^4 + 180 \tan(x)^2 + 35) a^{5/2}}{315 \tan(x)^9}$$

input `integrate((a*csc(x)^4)^(5/2),x, algorithm="giac")`

output `-1/315*(315*tan(x)^8 + 420*tan(x)^6 + 378*tan(x)^4 + 180*tan(x)^2 + 35)*a^(5/2)/tan(x)^9`

**Mupad [B] (verification not implemented)**

Time = 17.67 (sec) , antiderivative size = 121, normalized size of antiderivative = 1.03

$$\int (a \csc^4(x))^{5/2} dx = \frac{128 a^{5/2} (e^{x 46i} 1i - e^{x 48i} 9i + e^{x 50i} 36i - e^{x 52i} 84i + e^{x 54i} 126i)}{315 \left( \frac{e^{-x 2i}}{2} + \frac{e^{x 2i}}{2} - 1 \right) (e^{x 48i} - 7 e^{x 50i} + 21 e^{x 52i} - 35 e^{x 54i} + 35 e^{x 56i} - 21 e^{x 58i} + 7 e^{x 60i} - e^{x 62i})}$$

input `int((a/sin(x)^4)^(5/2),x)`output `(128*a^(5/2)*(exp(x*46i)*1i - exp(x*48i)*9i + exp(x*50i)*36i - exp(x*52i)*84i + exp(x*54i)*126i))/(315*(exp(-x*2i)/2 + exp(x*2i)/2 - 1)*(exp(x*48i) - 7*exp(x*50i) + 21*exp(x*52i) - 35*exp(x*54i) + 35*exp(x*56i) - 21*exp(x*58i) + 7*exp(x*60i) - exp(x*62i)))`**Reduce [B] (verification not implemented)**

Time = 0.17 (sec) , antiderivative size = 39, normalized size of antiderivative = 0.33

$$\int (a \csc^4(x))^{5/2} dx = \frac{\sqrt{a} \cos(x) a^2 (-128 \sin(x)^8 - 64 \sin(x)^6 - 48 \sin(x)^4 - 40 \sin(x)^2 - 35)}{315 \sin(x)^9}$$

input `int((a*csc(x)^4)^(5/2),x)`output `(sqrt(a)*cos(x)*a**2*(- 128*sin(x)**8 - 64*sin(x)**6 - 48*sin(x)**4 - 40*sin(x)**2 - 35))/(315*sin(x)**9)`



### 3.63 $\int (a \csc^4(x))^{3/2} dx$

Optimal result . . . . .	408
Mathematica [A] (verified) . . . . .	408
Rubi [A] (verified) . . . . .	409
Maple [A] (verified) . . . . .	410
Fricas [A] (verification not implemented) . . . . .	411
Sympy [F] . . . . .	411
Maxima [A] (verification not implemented) . . . . .	411
Giac [A] (verification not implemented) . . . . .	412
Mupad [B] (verification not implemented) . . . . .	412
Reduce [B] (verification not implemented) . . . . .	412

#### Optimal result

Integrand size = 10, antiderivative size = 62

$$\int (a \csc^4(x))^{3/2} dx = -\frac{2}{3}a \cos^2(x) \cot(x) \sqrt{a \csc^4(x)} - \frac{1}{5}a \cos^2(x) \cot^3(x) \sqrt{a \csc^4(x)} - a \cos(x) \sqrt{a \csc^4(x)} \sin(x)$$

output

```
-2/3*a*cos(x)^2*cot(x)*(a*csc(x)^4)^(1/2)-1/5*a*cos(x)^2*cot(x)^3*(a*csc(x)^4)^(1/2)-a*cos(x)*(a*csc(x)^4)^(1/2)*sin(x)
```

#### Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 33, normalized size of antiderivative = 0.53

$$\int (a \csc^4(x))^{3/2} dx = -\frac{1}{15}a \cos(x) \sqrt{a \csc^4(x)} (8 + 4 \csc^2(x) + 3 \csc^4(x)) \sin(x)$$

input

```
Integrate[(a*Csc[x]^4)^(3/2),x]
```

output

```
-1/15*(a*Cos[x]*Sqrt[a*Csc[x]^4]*(8 + 4*Csc[x]^2 + 3*Csc[x]^4)*Sin[x])
```

**Rubi [A] (verified)**

Time = 0.25 (sec) , antiderivative size = 36, normalized size of antiderivative = 0.58, number of steps used = 6, number of rules used = 5,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {3042, 4611, 3042, 4254, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int (a \csc^4(x))^{3/2} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \left( a \sec \left( x + \frac{\pi}{2} \right)^4 \right)^{3/2} dx \\
 & \quad \downarrow \text{4611} \\
 & a \sin^2(x) \sqrt{a \csc^4(x)} \int \csc^6(x) dx \\
 & \quad \downarrow \text{3042} \\
 & a \sin^2(x) \sqrt{a \csc^4(x)} \int \csc(x)^6 dx \\
 & \quad \downarrow \text{4254} \\
 & -a \sin^2(x) \sqrt{a \csc^4(x)} \int (\cot^4(x) + 2 \cot^2(x) + 1) d \cot(x) \\
 & \quad \downarrow \text{2009} \\
 & -a \sin^2(x) \left( \frac{\cot^5(x)}{5} + \frac{2 \cot^3(x)}{3} + \cot(x) \right) \sqrt{a \csc^4(x)}
 \end{aligned}$$

input `Int[(a*Csc[x]^4)^(3/2),x]`

output `-(a*(Cot[x] + (2*Cot[x]^3)/3 + Cot[x]^5/5)*Sqrt[a*Csc[x]^4]*Sin[x]^2)`

## Definitions of rubi rules used

rule 2009 `Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 4254 `Int[csc[(c_.) + (d_.)*(x_)]^(n_), x_Symbol] := Simp[-d^(-1) Subst[Int[Exp andIntegrand[(1 + x^2)^(n/2 - 1), x], x], x, Cot[c + d*x]], x] /; FreeQ[{c, d}, x] && IGtQ[n/2, 0]`

rule 4611 `Int[((b_.)*((c_.)*sec[(e_.) + (f_.)*(x_)])^(n_))^(p_), x_Symbol] := Simp[b^IntPart[p]*((b*(c*Sec[e + f*x])^n)^FracPart[p]/(c*Sec[e + f*x])^(n*FracPart[p])) Int[(c*Sec[e + f*x])^(n*p), x], x] /; FreeQ[{b, c, e, f, n, p}, x] && !IntegerQ[p]`

## Maple [A] (verified)

Time = 0.11 (sec) , antiderivative size = 35, normalized size of antiderivative = 0.56

method	result	size
default	$-\frac{\sqrt{16} (8 \cos(x)^4 - 20 \cos(x)^2 + 15) \sqrt{a \csc(x)^4} a \csc(x)^2 \cot(x)}{60}$	35
risch	$\frac{16ia \sqrt{\frac{a e^{4ix}}{(e^{2ix} - 1)^4}} (-5 + 11 \cos(2x) + 9i \sin(2x))}{15(e^{2ix} - 1)^3}$	47

input `int((a*csc(x)^4)^(3/2),x,method=_RETURNVERBOSE)`

output `-1/60*16^(1/2)*(8*cos(x)^4-20*cos(x)^2+15)*(a*csc(x)^4)^(1/2)*a*csc(x)^2*cot(x)`

**Fricas [A] (verification not implemented)**

Time = 0.07 (sec) , antiderivative size = 52, normalized size of antiderivative = 0.84

$$\int (a \csc^4(x))^{3/2} dx = \frac{(8 a \cos(x)^5 - 20 a \cos(x)^3 + 15 a \cos(x)) \sqrt{\frac{a}{\cos(x)^4 - 2 \cos(x)^2 + 1}}}{15 (\cos(x)^2 - 1) \sin(x)}$$

input `integrate((a*csc(x)^4)^(3/2),x, algorithm="fricas")`

output `1/15*(8*a*cos(x)^5 - 20*a*cos(x)^3 + 15*a*cos(x))*sqrt(a/(cos(x)^4 - 2*cos(x)^2 + 1))/((cos(x)^2 - 1)*sin(x))`

**Sympy [F]**

$$\int (a \csc^4(x))^{3/2} dx = \int (a \csc^4(x))^{\frac{3}{2}} dx$$

input `integrate((a*csc(x)**4)**(3/2),x)`

output `Integral((a*csc(x)**4)**(3/2), x)`

**Maxima [A] (verification not implemented)**

Time = 0.12 (sec) , antiderivative size = 30, normalized size of antiderivative = 0.48

$$\int (a \csc^4(x))^{3/2} dx = -\frac{15 a^{\frac{3}{2}} \tan(x)^4 + 10 a^{\frac{3}{2}} \tan(x)^2 + 3 a^{\frac{3}{2}}}{15 \tan(x)^5}$$

input `integrate((a*csc(x)^4)^(3/2),x, algorithm="maxima")`

output `-1/15*(15*a^(3/2)*tan(x)^4 + 10*a^(3/2)*tan(x)^2 + 3*a^(3/2))/tan(x)^5`

**Giac [A] (verification not implemented)**

Time = 0.14 (sec) , antiderivative size = 23, normalized size of antiderivative = 0.37

$$\int (a \csc^4(x))^{3/2} dx = -\frac{(15 \tan(x)^4 + 10 \tan(x)^2 + 3)a^{3/2}}{15 \tan(x)^5}$$

input `integrate((a*csc(x)^4)^(3/2),x, algorithm="giac")`output `-1/15*(15*tan(x)^4 + 10*tan(x)^2 + 3)*a^(3/2)/tan(x)^5`**Mupad [B] (verification not implemented)**

Time = 15.97 (sec) , antiderivative size = 44, normalized size of antiderivative = 0.71

$$\int (a \csc^4(x))^{3/2} dx = \frac{a^{3/2} 8i}{15} - \frac{4a^{3/2} (2 \sin(2x)^3 - 9 \sin(2x) + 3 \sin(4x) + 2i)}{15 (\cos(2x) - 1)^3}$$

input `int((a/sin(x)^4)^(3/2),x)`output `((a^(3/2)*8i)/15 - (4*a^(3/2)*(3*sin(4*x) - 9*sin(2*x) + 2*sin(2*x)^3 + 2i))/15)/(cos(2*x) - 1)^3`**Reduce [B] (verification not implemented)**

Time = 0.17 (sec) , antiderivative size = 25, normalized size of antiderivative = 0.40

$$\int (a \csc^4(x))^{3/2} dx = \frac{\sqrt{a} \cos(x) a (-8 \sin(x)^4 - 4 \sin(x)^2 - 3)}{15 \sin(x)^5}$$

input `int((a*csc(x)^4)^(3/2),x)`output `(sqrt(a)*cos(x)*a*(- 8*sin(x)**4 - 4*sin(x)**2 - 3))/(15*sin(x)**5)`

### 3.64 $\int \sqrt{a \csc^4(x)} dx$

Optimal result	413
Mathematica [A] (verified)	413
Rubi [A] (verified)	414
Maple [A] (verified)	415
Fricas [A] (verification not implemented)	416
Sympy [F]	416
Maxima [A] (verification not implemented)	416
Giac [A] (verification not implemented)	417
Mupad [B] (verification not implemented)	417
Reduce [B] (verification not implemented)	417

#### Optimal result

Integrand size = 10, antiderivative size = 16

$$\int \sqrt{a \csc^4(x)} dx = -\cos(x) \sqrt{a \csc^4(x)} \sin(x)$$

output `-cos(x)*(a*csc(x)^4)^(1/2)*sin(x)`

#### Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 16, normalized size of antiderivative = 1.00

$$\int \sqrt{a \csc^4(x)} dx = -\cos(x) \sqrt{a \csc^4(x)} \sin(x)$$

input `Integrate[Sqrt[a*Csc[x]^4],x]`

output `-(Cos[x]*Sqrt[a*Csc[x]^4]*Sin[x])`

**Rubi [A] (verified)**

Time = 0.23 (sec) , antiderivative size = 16, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 5,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {3042, 4611, 3042, 4254, 24}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \sqrt{a \csc^4(x)} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \sqrt{a \sec\left(x + \frac{\pi}{2}\right)^4} dx \\
 & \quad \downarrow \text{4611} \\
 & \sin^2(x) \sqrt{a \csc^4(x)} \int \csc^2(x) dx \\
 & \quad \downarrow \text{3042} \\
 & \sin^2(x) \sqrt{a \csc^4(x)} \int \csc(x)^2 dx \\
 & \quad \downarrow \text{4254} \\
 & \sin^2(x) \left(-\sqrt{a \csc^4(x)}\right) \int 1 d \cot(x) \\
 & \quad \downarrow \text{24} \\
 & \sin(x)(-\cos(x)) \sqrt{a \csc^4(x)}
 \end{aligned}$$

input `Int[Sqrt[a*Csc[x]^4],x]`

output `-(Cos[x]*Sqrt[a*Csc[x]^4]*Sin[x])`

## Definitions of rubi rules used

rule 24 `Int[a_, x_Symbol] := Simp[a*x, x] /; FreeQ[a, x]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 4254 `Int[csc[(c_.) + (d_.)*(x_)]^(n_), x_Symbol] := Simp[-d^(-1) Subst[Int[Exp andIntegrand[(1 + x^2)^(n/2 - 1), x], x], x, Cot[c + d*x]], x] /; FreeQ[{c, d}, x] && IGtQ[n/2, 0]`

rule 4611 `Int[((b_.)*((c_.)*sec[(e_.) + (f_.)*(x_)])^(n_))^(p_), x_Symbol] := Simp[b^IntPart[p]*((b*(c*Sec[e + f*x])^n)^FracPart[p]/(c*Sec[e + f*x])^(n*FracPart[p])) Int[(c*Sec[e + f*x])^(n*p), x], x] /; FreeQ[{b, c, e, f, n, p}, x] && !IntegerQ[p]`

## Maple [A] (verified)

Time = 0.07 (sec) , antiderivative size = 18, normalized size of antiderivative = 1.12

method	result	size
default	$-\frac{\sqrt{16} \cos(x) \sqrt{a \csc(x)^4} \sin(x)}{4}$	18
risch	$2i \sqrt{\frac{a e^{4ix}}{(e^{2ix}-1)^4}} (1 - e^{-2ix})$	31

input `int((a*csc(x)^4)^(1/2),x,method=_RETURNVERBOSE)`

output `-1/4*16^(1/2)*cos(x)*(a*csc(x)^4)^(1/2)*sin(x)`



**Fricas [A] (verification not implemented)**

Time = 0.07 (sec) , antiderivative size = 24, normalized size of antiderivative = 1.50

$$\int \sqrt{a \csc^4(x)} dx = -\sqrt{\frac{a}{\cos(x)^4 - 2 \cos(x)^2 + 1}} \cos(x) \sin(x)$$

input `integrate((a*csc(x)^4)^(1/2),x, algorithm="fricas")`output `-sqrt(a/(cos(x)^4 - 2*cos(x)^2 + 1))*cos(x)*sin(x)`**Sympy [F]**

$$\int \sqrt{a \csc^4(x)} dx = \int \sqrt{a \csc^4(x)} dx$$

input `integrate((a*csc(x)**4)**(1/2),x)`output `Integral(sqrt(a*csc(x)**4), x)`**Maxima [A] (verification not implemented)**

Time = 0.11 (sec) , antiderivative size = 9, normalized size of antiderivative = 0.56

$$\int \sqrt{a \csc^4(x)} dx = -\frac{\sqrt{a}}{\tan(x)}$$

input `integrate((a*csc(x)^4)^(1/2),x, algorithm="maxima")`output `-sqrt(a)/tan(x)`

**Giac [A] (verification not implemented)**

Time = 0.13 (sec) , antiderivative size = 9, normalized size of antiderivative = 0.56

$$\int \sqrt{a \csc^4(x)} dx = -\frac{\sqrt{a}}{\tan(x)}$$

input `integrate((a*csc(x)^4)^(1/2),x, algorithm="giac")`

output `-sqrt(a)/tan(x)`

**Mupad [B] (verification not implemented)**

Time = 15.12 (sec) , antiderivative size = 7, normalized size of antiderivative = 0.44

$$\int \sqrt{a \csc^4(x)} dx = -\sqrt{a} \cot(x)$$

input `int((a/sin(x)^4)^(1/2),x)`

output `-a^(1/2)*cot(x)`

**Reduce [B] (verification not implemented)**

Time = 0.17 (sec) , antiderivative size = 10, normalized size of antiderivative = 0.62

$$\int \sqrt{a \csc^4(x)} dx = -\frac{\sqrt{a} \cos(x)}{\sin(x)}$$

input `int((a*csc(x)^4)^(1/2),x)`

output `( - sqrt(a)*cos(x))/sin(x)`

### 3.65 $\int \frac{1}{\sqrt{a \csc^4(x)}} dx$

Optimal result	418
Mathematica [A] (verified)	418
Rubi [A] (verified)	419
Maple [A] (verified)	420
Fricas [A] (verification not implemented)	421
Sympy [F]	421
Maxima [A] (verification not implemented)	421
Giac [F(-2)]	422
Mupad [F(-1)]	422
Reduce [B] (verification not implemented)	422

#### Optimal result

Integrand size = 10, antiderivative size = 36

$$\int \frac{1}{\sqrt{a \csc^4(x)}} dx = -\frac{\cot(x)}{2\sqrt{a \csc^4(x)}} + \frac{x \csc^2(x)}{2\sqrt{a \csc^4(x)}}$$

output `-1/2*cot(x)/(a*csc(x)^4)^(1/2)+1/2*x*csc(x)^2/(a*csc(x)^4)^(1/2)`

#### Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 25, normalized size of antiderivative = 0.69

$$\int \frac{1}{\sqrt{a \csc^4(x)}} dx = \frac{-\cot(x) + x \csc^2(x)}{2\sqrt{a \csc^4(x)}}$$

input `Integrate[1/Sqrt[a*Csc[x]^4],x]`

output `(-Cot[x] + x*Csc[x]^2)/(2*Sqrt[a*Csc[x]^4])`

**Rubi [A] (verified)**

Time = 0.22 (sec) , antiderivative size = 29, normalized size of antiderivative = 0.81, number of steps used = 5, number of rules used = 5,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {3042, 4611, 3042, 3115, 24}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{1}{\sqrt{a \csc^4(x)}} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{1}{\sqrt{a \sec(x + \frac{\pi}{2})^4}} dx \\
 & \quad \downarrow \text{4611} \\
 & \frac{\csc^2(x) \int \sin^2(x) dx}{\sqrt{a \csc^4(x)}} \\
 & \quad \downarrow \text{3042} \\
 & \frac{\csc^2(x) \int \sin(x)^2 dx}{\sqrt{a \csc^4(x)}} \\
 & \quad \downarrow \text{3115} \\
 & \frac{\csc^2(x) \left( \frac{\int 1 dx}{2} - \frac{1}{2} \sin(x) \cos(x) \right)}{\sqrt{a \csc^4(x)}} \\
 & \quad \downarrow \text{24} \\
 & \frac{\csc^2(x) \left( \frac{x}{2} - \frac{1}{2} \sin(x) \cos(x) \right)}{\sqrt{a \csc^4(x)}}
 \end{aligned}$$

input `Int [1/Sqrt [a*Csc [x]^4] , x]`

output `(Csc [x]^2*(x/2 - (Cos [x]*Sin [x])/2))/Sqrt [a*Csc [x]^4]`

## Definitions of rubi rules used

rule 24 `Int[a_, x_Symbol] := Simp[a*x, x] /; FreeQ[a, x]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3115 `Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] := Simp[(-b)*Cos[c + d*x]*((b*Sin[c + d*x])^(n - 1)/(d*n)), x] + Simp[b^2*((n - 1)/n) Int[(b*Sin[c + d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[2*n]`

rule 4611 `Int[((b_.)*((c_.)*sec[(e_.) + (f_.)*(x_)])^(n_))^(p_), x_Symbol] := Simp[b^IntPart[p]*((b*(c*Sec[e + f*x])^n)^FracPart[p]/(c*Sec[e + f*x])^(n*FracPart[p])) Int[(c*Sec[e + f*x])^(n*p), x], x] /; FreeQ[{b, c, e, f, n, p}, x] && !IntegerQ[p]`

## Maple [A] (verified)

Time = 0.08 (sec) , antiderivative size = 24, normalized size of antiderivative = 0.67

method	result	size
default	$-\frac{\sqrt{16}(\cot(x) - x \csc(x)^2)}{8\sqrt{a \csc(x)^4}}$	24
risch	$-\frac{e^{2ix} x}{2\sqrt{\frac{a e^{4ix}}{(e^{2ix}-1)^4}} (e^{2ix}-1)^2} - \frac{i e^{4ix}}{8\sqrt{\frac{a e^{4ix}}{(e^{2ix}-1)^4}} (e^{2ix}-1)^2} + \frac{i}{8\sqrt{\frac{a e^{4ix}}{(e^{2ix}-1)^4}} (e^{2ix}-1)^2}$	102

input `int(1/(a*csc(x)^4)^(1/2), x, method=_RETURNVERBOSE)`

output `-1/8*16^(1/2)/(a*csc(x)^4)^(1/2)*(cot(x)-x*csc(x)^2)`

**Fricas [A] (verification not implemented)**

Time = 0.08 (sec) , antiderivative size = 46, normalized size of antiderivative = 1.28

$$\int \frac{1}{\sqrt{a \csc^4(x)}} dx = -\frac{(x \cos(x)^2 - (\cos(x)^3 - \cos(x)) \sin(x) - x) \sqrt{\frac{a}{\cos(x)^4 - 2 \cos(x)^2 + 1}}}{2a}$$

input `integrate(1/(a*csc(x)^4)^(1/2),x, algorithm="fricas")`

output `-1/2*(x*cos(x)^2 - (cos(x)^3 - cos(x))*sin(x) - x)*sqrt(a/(cos(x)^4 - 2*cos(x)^2 + 1))/a`

**Sympy [F]**

$$\int \frac{1}{\sqrt{a \csc^4(x)}} dx = \int \frac{1}{\sqrt{a \csc^4(x)}} dx$$

input `integrate(1/(a*csc(x)**4)**(1/2),x)`

output `Integral(1/sqrt(a*csc(x)**4), x)`

**Maxima [A] (verification not implemented)**

Time = 0.11 (sec) , antiderivative size = 25, normalized size of antiderivative = 0.69

$$\int \frac{1}{\sqrt{a \csc^4(x)}} dx = \frac{x}{2\sqrt{a}} - \frac{\tan(x)}{2(\sqrt{a} \tan(x)^2 + \sqrt{a})}$$

input `integrate(1/(a*csc(x)^4)^(1/2),x, algorithm="maxima")`

output `1/2*x/sqrt(a) - 1/2*tan(x)/(sqrt(a)*tan(x)^2 + sqrt(a))`

**Giac [F(-2)]**

Exception generated.

$$\int \frac{1}{\sqrt{a \csc^4(x)}} dx = \text{Exception raised: TypeError}$$

input `integrate(1/(a*csc(x)^4)^(1/2),x, algorithm="giac")`

output `Exception raised: TypeError >> an error occurred running a Giac command:INPUT:sage2:=int(sage0,sageVARx):;OUTPUT:sym2poly/r2sym(const gen & e,const index_m & i,const vecteur & l) Error: Bad Argument Value`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{1}{\sqrt{a \csc^4(x)}} dx = \int \frac{1}{\sqrt{\frac{a}{\sin(x)^4}}} dx$$

input `int(1/(a/sin(x)^4)^(1/2),x)`

output `int(1/(a/sin(x)^4)^(1/2), x)`

**Reduce [B] (verification not implemented)**

Time = 0.18 (sec) , antiderivative size = 15, normalized size of antiderivative = 0.42

$$\int \frac{1}{\sqrt{a \csc^4(x)}} dx = \frac{\sqrt{a}(-\cos(x)\sin(x) + x)}{2a}$$

input `int(1/(a*csc(x)^4)^(1/2),x)`

output `(sqrt(a)*(-cos(x)*sin(x) + x))/(2*a)`

### 3.66 $\int \frac{1}{(a \csc^4(x))^{3/2}} dx$

Optimal result	423
Mathematica [A] (verified)	423
Rubi [A] (verified)	424
Maple [A] (verified)	426
Fricas [A] (verification not implemented)	426
Sympy [F]	427
Maxima [A] (verification not implemented)	427
Giac [F(-2)]	427
Mupad [F(-1)]	428
Reduce [B] (verification not implemented)	428

#### Optimal result

Integrand size = 10, antiderivative size = 86

$$\int \frac{1}{(a \csc^4(x))^{3/2}} dx = -\frac{5 \cot(x)}{16a\sqrt{a \csc^4(x)}} + \frac{5x \csc^2(x)}{16a\sqrt{a \csc^4(x)}} - \frac{5 \cos(x) \sin(x)}{24a\sqrt{a \csc^4(x)}} - \frac{\cos(x) \sin^3(x)}{6a\sqrt{a \csc^4(x)}}$$

output

$$-5/16*\cot(x)/a/(a*\csc(x)^4)^{(1/2)}+5/16*x*\csc(x)^2/a/(a*\csc(x)^4)^{(1/2)}-5/24*\cos(x)*\sin(x)/a/(a*\csc(x)^4)^{(1/2)}-1/6*\cos(x)*\sin(x)^3/a/(a*\csc(x)^4)^{(1/2)}$$

#### Mathematica [A] (verified)

Time = 0.05 (sec) , antiderivative size = 38, normalized size of antiderivative = 0.44

$$\int \frac{1}{(a \csc^4(x))^{3/2}} dx = -\frac{\csc^6(x)(-60x + 45 \sin(2x) - 9 \sin(4x) + \sin(6x))}{192 (a \csc^4(x))^{3/2}}$$

input

`Integrate[(a*Csc[x]^4)^(-3/2),x]`



output

$$-1/192*(\text{Csc}[x]^6*(-60*x + 45*\text{Sin}[2*x] - 9*\text{Sin}[4*x] + \text{Sin}[6*x]))/(a*\text{Csc}[x]^4)^{(3/2)}$$

**Rubi [A] (verified)**

Time = 0.34 (sec) , antiderivative size = 62, normalized size of antiderivative = 0.72, number of steps used = 9, number of rules used = 9,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.900$ , Rules used = {3042, 4611, 3042, 3115, 3042, 3115, 3042, 3115, 24}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \frac{1}{(a \csc^4(x))^{3/2}} dx \\ & \quad \downarrow \text{3042} \\ & \int \frac{1}{\left(a \sec\left(x + \frac{\pi}{2}\right)^4\right)^{3/2}} dx \\ & \quad \downarrow \text{4611} \\ & \frac{\csc^2(x) \int \sin^6(x) dx}{a \sqrt{a \csc^4(x)}} \\ & \quad \downarrow \text{3042} \\ & \frac{\csc^2(x) \int \sin(x)^6 dx}{a \sqrt{a \csc^4(x)}} \\ & \quad \downarrow \text{3115} \\ & \frac{\csc^2(x) \left(\frac{5}{6} \int \sin^4(x) dx - \frac{1}{6} \sin^5(x) \cos(x)\right)}{a \sqrt{a \csc^4(x)}} \\ & \quad \downarrow \text{3042} \\ & \frac{\csc^2(x) \left(\frac{5}{6} \int \sin(x)^4 dx - \frac{1}{6} \sin^5(x) \cos(x)\right)}{a \sqrt{a \csc^4(x)}} \\ & \quad \downarrow \text{3115} \\ & \frac{\csc^2(x) \left(\frac{5}{6} \left(\frac{3}{4} \int \sin^2(x) dx - \frac{1}{4} \sin^3(x) \cos(x)\right) - \frac{1}{6} \sin^5(x) \cos(x)\right)}{a \sqrt{a \csc^4(x)}} \end{aligned}$$

$$\begin{array}{c}
 \downarrow 3042 \\
 \frac{\csc^2(x) \left( \frac{5}{6} \left( \frac{3}{4} \int \sin(x)^2 dx - \frac{1}{4} \sin^3(x) \cos(x) \right) - \frac{1}{6} \sin^5(x) \cos(x) \right)}{a \sqrt{a \csc^4(x)}} \\
 \downarrow 3115 \\
 \frac{\csc^2(x) \left( \frac{5}{6} \left( \frac{3}{4} \left( \frac{\int 1 dx}{2} - \frac{1}{2} \sin(x) \cos(x) \right) - \frac{1}{4} \sin^3(x) \cos(x) \right) - \frac{1}{6} \sin^5(x) \cos(x) \right)}{a \sqrt{a \csc^4(x)}} \\
 \downarrow 24 \\
 \frac{\csc^2(x) \left( \frac{5}{6} \left( \frac{3}{4} \left( \frac{x}{2} - \frac{1}{2} \sin(x) \cos(x) \right) - \frac{1}{4} \sin^3(x) \cos(x) \right) - \frac{1}{6} \sin^5(x) \cos(x) \right)}{a \sqrt{a \csc^4(x)}}
 \end{array}$$

input `Int[(a*Csc[x]^4)^(-3/2),x]`

output `(Csc[x]^2*(-1/6*(Cos[x]*Sin[x]^5) + (5*(-1/4*(Cos[x]*Sin[x]^3) + (3*(x/2 - (Cos[x]*Sin[x])/2))/4))/6))/(a*Sqrt[a*Csc[x]^4])`

### Defintions of rubi rules used

rule 24 `Int[a_, x_Symbol] := Simp[a*x, x] /; FreeQ[a, x]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3115 `Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] := Simp[(-b)*Cos[c + d*x]*((b*SIN[c + d*x])^(n - 1)/(d*n), x] + Simp[b^2*((n - 1)/n) Int[(b*SIN[c + d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[2*n]`

rule 4611 `Int[((b_.)*((c_.)*sec[(e_.) + (f_.)*(x_)])^(n_))^(p_), x_Symbol] := Simp[b^IntPart[p]*((b*(c*Sec[e + f*x])^n)^FracPart[p]/(c*Sec[e + f*x])^(n*FracPart[p])) Int[(c*Sec[e + f*x])^(n*p), x], x] /; FreeQ[{b, c, e, f, n, p}, x] && !IntegerQ[p]`

**Maple [A] (verified)**

Time = 0.17 (sec) , antiderivative size = 42, normalized size of antiderivative = 0.49

method	result
default	$-\frac{\sqrt{16} \left( (8 \cos(x)^4 - 26 \cos(x)^2 + 33) \cot(x) - 15x \csc(x)^2 \right)}{192a \sqrt{a \csc(x)^4}}$
risch	$-\frac{5 e^{2ix} x}{16a(e^{2ix}-1)^2 \sqrt{\frac{a e^{4ix}}{(e^{2ix}-1)^4}}} - \frac{i e^{8ix}}{384a(e^{2ix}-1)^2 \sqrt{\frac{a e^{4ix}}{(e^{2ix}-1)^4}}} + \frac{3i e^{6ix}}{128a(e^{2ix}-1)^2 \sqrt{\frac{a e^{4ix}}{(e^{2ix}-1)^4}}} - \frac{15i e^{4ix}}{128a(e^{2ix}-1)^2 \sqrt{\frac{a e^{4ix}}{(e^{2ix}-1)^4}}} + \dots$

input `int(1/(a*csc(x)^4)^(3/2),x,method=_RETURNVERBOSE)`output `-1/192*16^(1/2)/a/(a*csc(x)^4)^(1/2)*((8*cos(x)^4-26*cos(x)^2+33)*cot(x)-15*x*csc(x)^2)`**Fricas [A] (verification not implemented)**

Time = 0.08 (sec) , antiderivative size = 61, normalized size of antiderivative = 0.71

$$\int \frac{1}{(a \csc^4(x))^{3/2}} dx = \frac{(15x \cos(x)^2 - (8 \cos(x)^7 - 34 \cos(x)^5 + 59 \cos(x)^3 - 33 \cos(x)) \sin(x) - 15x) \sqrt{\frac{a}{\cos(x)^4 - 2 \cos(x)^2 + 1}}}{48 a^2}$$

input `integrate(1/(a*csc(x)^4)^(3/2),x, algorithm="fricas")`output `-1/48*(15*x*cos(x)^2 - (8*cos(x)^7 - 34*cos(x)^5 + 59*cos(x)^3 - 33*cos(x))*sin(x) - 15*x)*sqrt(a/(cos(x)^4 - 2*cos(x)^2 + 1))/a^2`

**Sympy [F]**

$$\int \frac{1}{(a \csc^4(x))^{3/2}} dx = \int \frac{1}{(a \csc^4(x))^{\frac{3}{2}}} dx$$

input `integrate(1/(a*csc(x)**4)**(3/2),x)`

output `Integral((a*csc(x)**4)**(-3/2), x)`

**Maxima [A] (verification not implemented)**

Time = 0.11 (sec) , antiderivative size = 58, normalized size of antiderivative = 0.67

$$\int \frac{1}{(a \csc^4(x))^{3/2}} dx = -\frac{33 \tan(x)^5 + 40 \tan(x)^3 + 15 \tan(x)}{48 \left( a^{\frac{3}{2}} \tan(x)^6 + 3 a^{\frac{3}{2}} \tan(x)^4 + 3 a^{\frac{3}{2}} \tan(x)^2 + a^{\frac{3}{2}} \right)} + \frac{5x}{16 a^{\frac{3}{2}}}$$

input `integrate(1/(a*csc(x)^4)^(3/2),x, algorithm="maxima")`

output `-1/48*(33*tan(x)^5 + 40*tan(x)^3 + 15*tan(x))/(a^(3/2)*tan(x)^6 + 3*a^(3/2)*tan(x)^4 + 3*a^(3/2)*tan(x)^2 + a^(3/2)) + 5/16*x/a^(3/2)`

**Giac [F(-2)]**

Exception generated.

$$\int \frac{1}{(a \csc^4(x))^{3/2}} dx = \text{Exception raised: TypeError}$$

input `integrate(1/(a*csc(x)^4)^(3/2),x, algorithm="giac")`

output `Exception raised: TypeError >> an error occurred running a Giac command:IN PUT:sage2:=int(sage0,sageVARx):;OUTPUT:sym2poly/r2sym(const gen & e,const index_m & i,const vecteur & l) Error: Bad Argument Value`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{1}{(a \csc^4(x))^{3/2}} dx = \int \frac{1}{\left(\frac{a}{\sin(x)^4}\right)^{3/2}} dx$$

input `int(1/(a/sin(x)^4)^(3/2),x)`output `int(1/(a/sin(x)^4)^(3/2), x)`**Reduce [B] (verification not implemented)**

Time = 0.18 (sec) , antiderivative size = 33, normalized size of antiderivative = 0.38

$$\int \frac{1}{(a \csc^4(x))^{3/2}} dx = \frac{\sqrt{a} (-8 \cos(x) \sin(x)^5 - 10 \cos(x) \sin(x)^3 - 15 \cos(x) \sin(x) + 15x)}{48a^2}$$

input `int(1/(a*csc(x)^4)^(3/2),x)`output `(sqrt(a)*(- 8*cos(x)*sin(x)**5 - 10*cos(x)*sin(x)**3 - 15*cos(x)*sin(x) + 15*x))/(48*a**2)`

**3.67**  $\int \frac{1}{(a \csc^4(x))^{5/2}} dx$

Optimal result	429
Mathematica [A] (verified)	429
Rubi [A] (verified)	430
Maple [A] (verified)	432
Fricas [A] (verification not implemented)	433
Sympy [F]	433
Maxima [A] (verification not implemented)	433
Giac [F(-2)]	434
Mupad [F(-1)]	434
Reduce [B] (verification not implemented)	435

**Optimal result**

Integrand size = 10, antiderivative size = 132

$$\int \frac{1}{(a \csc^4(x))^{5/2}} dx = -\frac{63 \cot(x)}{256a^2 \sqrt{a \csc^4(x)}} + \frac{63x \csc^2(x)}{256a^2 \sqrt{a \csc^4(x)}} - \frac{21 \cos(x) \sin(x)}{128a^2 \sqrt{a \csc^4(x)}} - \frac{21 \cos(x) \sin^3(x)}{160a^2 \sqrt{a \csc^4(x)}} - \frac{9 \cos(x) \sin^5(x)}{80a^2 \sqrt{a \csc^4(x)}} - \frac{\cos(x) \sin^7(x)}{10a^2 \sqrt{a \csc^4(x)}}$$

output `-63/256*cot(x)/a^2/(a*csc(x)^4)^(1/2)+63/256*x*csc(x)^2/a^2/(a*csc(x)^4)^(1/2)-21/128*cos(x)*sin(x)/a^2/(a*csc(x)^4)^(1/2)-21/160*cos(x)*sin(x)^3/a^2/(a*csc(x)^4)^(1/2)-9/80*cos(x)*sin(x)^5/a^2/(a*csc(x)^4)^(1/2)-1/10*cos(x)*sin(x)^7/a^2/(a*csc(x)^4)^(1/2)`

**Mathematica [A] (verified)**

Time = 0.10 (sec) , antiderivative size = 55, normalized size of antiderivative = 0.42

$$\int \frac{1}{(a \csc^4(x))^{5/2}} dx = \frac{\sqrt{a \csc^4(x)} \sin^2(x) (2520x - 2100 \sin(2x) + 600 \sin(4x) - 150 \sin(6x) + 25 \sin(8x))}{10240a^3}$$

input `Integrate[(a*Csc[x]^4)^(-5/2), x]`

output

```
(Sqrt[a*Csc[x]^4]*Sin[x]^2*(2520*x - 2100*Sin[2*x] + 600*Sin[4*x] - 150*Sin[6*x] + 25*Sin[8*x] - 2*Sin[10*x]))/(10240*a^3)
```

**Rubi [A] (verified)**

Time = 0.46 (sec) , antiderivative size = 92, normalized size of antiderivative = 0.70, number of steps used = 13, number of rules used = 13,  $\frac{\text{number of rules}}{\text{integrand size}} = 1.300$ , Rules used = {3042, 4611, 3042, 3115, 3042, 3115, 3042, 3115, 3042, 3115, 3042, 3115, 24}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{1}{(a \csc^4(x))^{5/2}} dx \\
 & \quad \downarrow \text{3042} \\
 & \int \frac{1}{\left(a \sec\left(x + \frac{\pi}{2}\right)^4\right)^{5/2}} dx \\
 & \quad \downarrow \text{4611} \\
 & \frac{\csc^2(x) \int \sin^{10}(x) dx}{a^2 \sqrt{a \csc^4(x)}} \\
 & \quad \downarrow \text{3042} \\
 & \frac{\csc^2(x) \int \sin(x)^{10} dx}{a^2 \sqrt{a \csc^4(x)}} \\
 & \quad \downarrow \text{3115} \\
 & \frac{\csc^2(x) \left(\frac{9}{10} \int \sin^8(x) dx - \frac{1}{10} \sin^9(x) \cos(x)\right)}{a^2 \sqrt{a \csc^4(x)}} \\
 & \quad \downarrow \text{3042} \\
 & \frac{\csc^2(x) \left(\frac{9}{10} \int \sin(x)^8 dx - \frac{1}{10} \sin^9(x) \cos(x)\right)}{a^2 \sqrt{a \csc^4(x)}} \\
 & \quad \downarrow \text{3115} \\
 & \frac{\csc^2(x) \left(\frac{9}{10} \left(\frac{7}{8} \int \sin^6(x) dx - \frac{1}{8} \sin^7(x) \cos(x)\right) - \frac{1}{10} \sin^9(x) \cos(x)\right)}{a^2 \sqrt{a \csc^4(x)}}
 \end{aligned}$$

$$\begin{aligned}
& \downarrow 3042 \\
& \frac{\csc^2(x) \left( \frac{9}{10} \left( \frac{7}{8} \int \sin(x)^6 dx - \frac{1}{8} \sin^7(x) \cos(x) \right) - \frac{1}{10} \sin^9(x) \cos(x) \right)}{a^2 \sqrt{a \csc^4(x)}} \\
& \downarrow 3115 \\
& \frac{\csc^2(x) \left( \frac{9}{10} \left( \frac{7}{8} \left( \frac{5}{6} \int \sin^4(x) dx - \frac{1}{6} \sin^5(x) \cos(x) \right) - \frac{1}{8} \sin^7(x) \cos(x) \right) - \frac{1}{10} \sin^9(x) \cos(x) \right)}{a^2 \sqrt{a \csc^4(x)}} \\
& \downarrow 3042 \\
& \frac{\csc^2(x) \left( \frac{9}{10} \left( \frac{7}{8} \left( \frac{5}{6} \int \sin(x)^4 dx - \frac{1}{6} \sin^5(x) \cos(x) \right) - \frac{1}{8} \sin^7(x) \cos(x) \right) - \frac{1}{10} \sin^9(x) \cos(x) \right)}{a^2 \sqrt{a \csc^4(x)}} \\
& \downarrow 3115 \\
& \frac{\csc^2(x) \left( \frac{9}{10} \left( \frac{7}{8} \left( \frac{5}{6} \left( \frac{3}{4} \int \sin^2(x) dx - \frac{1}{4} \sin^3(x) \cos(x) \right) - \frac{1}{6} \sin^5(x) \cos(x) \right) - \frac{1}{8} \sin^7(x) \cos(x) \right) - \frac{1}{10} \sin^9(x) \cos(x) \right)}{a^2 \sqrt{a \csc^4(x)}} \\
& \downarrow 3042 \\
& \frac{\csc^2(x) \left( \frac{9}{10} \left( \frac{7}{8} \left( \frac{5}{6} \left( \frac{3}{4} \int \sin(x)^2 dx - \frac{1}{4} \sin^3(x) \cos(x) \right) - \frac{1}{6} \sin^5(x) \cos(x) \right) - \frac{1}{8} \sin^7(x) \cos(x) \right) - \frac{1}{10} \sin^9(x) \cos(x) \right)}{a^2 \sqrt{a \csc^4(x)}} \\
& \downarrow 3115 \\
& \frac{\csc^2(x) \left( \frac{9}{10} \left( \frac{7}{8} \left( \frac{5}{6} \left( \frac{3}{4} \left( \frac{\int 1 dx}{2} - \frac{1}{2} \sin(x) \cos(x) \right) - \frac{1}{4} \sin^3(x) \cos(x) \right) - \frac{1}{6} \sin^5(x) \cos(x) \right) - \frac{1}{8} \sin^7(x) \cos(x) \right) - \frac{1}{10} \sin^9(x) \cos(x) \right)}{a^2 \sqrt{a \csc^4(x)}} \\
& \downarrow 24 \\
& \frac{\csc^2(x) \left( \frac{9}{10} \left( \frac{7}{8} \left( \frac{5}{6} \left( \frac{3}{4} \left( \frac{x}{2} - \frac{1}{2} \sin(x) \cos(x) \right) - \frac{1}{4} \sin^3(x) \cos(x) \right) - \frac{1}{6} \sin^5(x) \cos(x) \right) - \frac{1}{8} \sin^7(x) \cos(x) \right) - \frac{1}{10} \sin^9(x) \cos(x) \right)}{a^2 \sqrt{a \csc^4(x)}}
\end{aligned}$$

input `Int[(a*Csc[x]^4)^(-5/2),x]`

output `(Csc[x]^2*(-1/10*(Cos[x]*Sin[x]^9) + (9*(-1/8*(Cos[x]*Sin[x]^7) + (7*(-1/6*(Cos[x]*Sin[x]^5) + (5*(-1/4*(Cos[x]*Sin[x]^3) + (3*(x/2 - (Cos[x]*Sin[x])/2))/4))/6))/8))/10))/(a^2*Sqrt[a*Csc[x]^4])`



## Definitions of rubi rules used

rule 24 `Int[a_, x_Symbol] := Simp[a*x, x] /; FreeQ[a, x]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3115 `Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] := Simp[(-b)*Cos[c + d*x]*((b*Sin[c + d*x])^(n - 1)/(d*n)), x] + Simp[b^2*((n - 1)/n) Int[(b*Sin[c + d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[2*n]`

rule 4611 `Int[((b_.)*((c_.)*sec[(e_.) + (f_.)*(x_)])^(n_))^(p_), x_Symbol] := Simp[b^IntPart[p]*((b*(c*Sec[e + f*x])^n)^FracPart[p]/(c*Sec[e + f*x])^(n*FracPart[p])) Int[(c*Sec[e + f*x])^(n*p), x], x] /; FreeQ[{b, c, e, f, n, p}, x] && !IntegerQ[p]`

## Maple [A] (verified)

Time = 0.42 (sec) , antiderivative size = 54, normalized size of antiderivative = 0.41

method	result
default	$-\frac{\sqrt{16} \left( (128 \cos(x)^8 - 656 \cos(x)^6 + 1368 \cos(x)^4 - 1490 \cos(x)^2 + 965) \cot(x) - 315x \csc(x)^2 \right)}{5120a^2 \sqrt{a \csc(x)^4}}$
risch	$-\frac{63 e^{2ix} x}{256a^2 (e^{2ix} - 1)^2 \sqrt{\frac{a e^{4ix}}{(e^{2ix} - 1)^4}}} - \frac{ie^{12ix}}{10240a^2 (e^{2ix} - 1)^2 \sqrt{\frac{a e^{4ix}}{(e^{2ix} - 1)^4}}} + \frac{5ie^{10ix}}{4096a^2 (e^{2ix} - 1)^2 \sqrt{\frac{a e^{4ix}}{(e^{2ix} - 1)^4}}} - \frac{105ie^{4ix}}{1024a^2 (e^{2ix} - 1)^2 \sqrt{\frac{a e^{4ix}}{(e^{2ix} - 1)^4}}}$

input `int(1/(a*csc(x)^4)^(5/2), x, method=_RETURNVERBOSE)`

output `-1/5120*16^(1/2)/a^2/(a*csc(x)^4)^(1/2)*((128*cos(x)^8-656*cos(x)^6+1368*cos(x)^4-1490*cos(x)^2+965)*cot(x)-315*x*csc(x)^2)`

**Fricas [A] (verification not implemented)**

Time = 0.08 (sec) , antiderivative size = 73, normalized size of antiderivative = 0.55

$$\int \frac{1}{(a \csc^4(x))^{5/2}} dx = \frac{(315 x \cos(x)^2 - (128 \cos(x)^{11} - 784 \cos(x)^9 + 2024 \cos(x)^7 - 2858 \cos(x)^5 + 2455 \cos(x)^3 - 965 \cos(x)) \sin(x) - 315 x) \sqrt{a / (\cos(x)^4 - 2 \cos(x)^2 + 1)}}{1280 a^3}$$

input `integrate(1/(a*csc(x)^4)^(5/2),x, algorithm="fricas")`output `-1/1280*(315*x*cos(x)^2 - (128*cos(x)^11 - 784*cos(x)^9 + 2024*cos(x)^7 - 2858*cos(x)^5 + 2455*cos(x)^3 - 965*cos(x))*sin(x) - 315*x)*sqrt(a/(cos(x)^4 - 2*cos(x)^2 + 1))/a^3`**Sympy [F]**

$$\int \frac{1}{(a \csc^4(x))^{5/2}} dx = \int \frac{1}{(a \csc^4(x))^{\frac{5}{2}}} dx$$

input `integrate(1/(a*csc(x)**4)**(5/2),x)`output `Integral((a*csc(x)**4)**(-5/2), x)`**Maxima [A] (verification not implemented)**

Time = 0.12 (sec) , antiderivative size = 88, normalized size of antiderivative = 0.67

$$\int \frac{1}{(a \csc^4(x))^{5/2}} dx = \frac{965 \tan(x)^9 + 2370 \tan(x)^7 + 2688 \tan(x)^5 + 1470 \tan(x)^3 + 315 \tan(x)}{1280 \left( a^{\frac{5}{2}} \tan(x)^{10} + 5 a^{\frac{5}{2}} \tan(x)^8 + 10 a^{\frac{5}{2}} \tan(x)^6 + 10 a^{\frac{5}{2}} \tan(x)^4 + 5 a^{\frac{5}{2}} \tan(x)^2 + a^{\frac{5}{2}} \right)} + \frac{63 x}{256 a^{\frac{5}{2}}}$$

input `integrate(1/(a*csc(x)^4)^(5/2),x, algorithm="maxima")`

output `-1/1280*(965*tan(x)^9 + 2370*tan(x)^7 + 2688*tan(x)^5 + 1470*tan(x)^3 + 315*tan(x))/(a^(5/2)*tan(x)^10 + 5*a^(5/2)*tan(x)^8 + 10*a^(5/2)*tan(x)^6 + 10*a^(5/2)*tan(x)^4 + 5*a^(5/2)*tan(x)^2 + a^(5/2)) + 63/256*x/a^(5/2)`

### Giac [F(-2)]

Exception generated.

$$\int \frac{1}{(a \csc^4(x))^{5/2}} dx = \text{Exception raised: TypeError}$$

input `integrate(1/(a*csc(x)^4)^(5/2),x, algorithm="giac")`

output Exception raised: TypeError >> an error occurred running a Giac command:IN PUT:sage2:=int(sage0,sageVARx):;OUTPUT:sym2poly/r2sym(const gen & e,const index\_m & i,const vecteur & l) Error: Bad Argument Value

### Mupad [F(-1)]

Timed out.

$$\int \frac{1}{(a \csc^4(x))^{5/2}} dx = \int \frac{1}{\left(\frac{a}{\sin(x)^4}\right)^{5/2}} dx$$

input `int(1/(a/sin(x)^4)^(5/2),x)`

output `int(1/(a/sin(x)^4)^(5/2), x)`

**Reduce [B] (verification not implemented)**

Time = 0.17 (sec) , antiderivative size = 49, normalized size of antiderivative = 0.37

$$\int \frac{1}{(a \csc^4(x))^{5/2}} dx = \frac{\sqrt{a} (-128 \cos(x) \sin(x)^9 - 144 \cos(x) \sin(x)^7 - 168 \cos(x) \sin(x)^5 - 210 \cos(x) \sin(x)^3 - 315 \cos(x) \sin(x) + 315x)}{1280a^3}$$

input

```
int(1/(a*csc(x)^4)^(5/2),x)
```

output

```
(sqrt(a)*(- 128*cos(x)*sin(x)**9 - 144*cos(x)*sin(x)**7 - 168*cos(x)*sin(x)**5 - 210*cos(x)*sin(x)**3 - 315*cos(x)*sin(x) + 315*x))/(1280*a**3)
```

### 3.68 $\int ((b \csc(c + dx))^p)^n dx$

Optimal result	436
Mathematica [A] (verified)	436
Rubi [A] (verified)	437
Maple [F]	438
Fricas [F]	439
Sympy [F]	439
Maxima [F]	439
Giac [F]	440
Mupad [F(-1)]	440
Reduce [F]	440

#### Optimal result

Integrand size = 12, antiderivative size = 80

$$\int ((b \csc(c + dx))^p)^n dx = \frac{\cos(c + dx) ((b \csc(c + dx))^p)^n \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{1}{2}(1 - np), \frac{1}{2}(3 - np), \sin^2(c + dx)\right) \sin(c + dx)}{d(1 - np)\sqrt{\cos^2(c + dx)}}$$

output `cos(d*x+c)*((b*csc(d*x+c))^p)^n*hypergeom([1/2, -1/2*n*p+1/2], [-1/2*n*p+3/2], sin(d*x+c)^2)*sin(d*x+c)/d/(-n*p+1)/(cos(d*x+c)^2)^(1/2)`

#### Mathematica [A] (verified)

Time = 0.12 (sec) , antiderivative size = 71, normalized size of antiderivative = 0.89

$$\int ((b \csc(c + dx))^p)^n dx = \frac{\cos(c + dx) ((b \csc(c + dx))^p)^n \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{1}{2}(1 + np), \frac{3}{2}, \cos^2(c + dx)\right) \sin(c + dx) \sin^2(c + dx)}{d}$$

input `Integrate[((b*Csc[c + d*x])^p)^n,x]`

output

```

-((Cos[c + d*x]*((b*Csc[c + d*x])^p)^n*Hypergeometric2F1[1/2, (1 + n*p)/2,
3/2, Cos[c + d*x]^2]*Sin[c + d*x]*(Sin[c + d*x]^2)^((-1 + n*p)/2))/d

```

**Rubi [A] (verified)**

Time = 0.38 (sec) , antiderivative size = 80, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 6,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {3042, 4611, 3042, 4259, 3042, 3122}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
& \int ((b \csc(c + dx))^p)^n dx \\
& \quad \downarrow \text{3042} \\
& \int \left( (-b \sec\left(c + dx + \frac{\pi}{2}\right))^p \right)^n dx \\
& \quad \downarrow \text{4611} \\
& (b \csc(c + dx))^{-np} ((b \csc(c + dx))^p)^n \int (b \csc(c + dx))^{np} dx \\
& \quad \downarrow \text{3042} \\
& (b \csc(c + dx))^{-np} ((b \csc(c + dx))^p)^n \int (b \csc(c + dx))^{np} dx \\
& \quad \downarrow \text{4259} \\
& \left(\frac{\sin(c + dx)}{b}\right)^{np} ((b \csc(c + dx))^p)^n \int \left(\frac{\sin(c + dx)}{b}\right)^{-np} dx \\
& \quad \downarrow \text{3042} \\
& \left(\frac{\sin(c + dx)}{b}\right)^{np} ((b \csc(c + dx))^p)^n \int \left(\frac{\sin(c + dx)}{b}\right)^{-np} dx \\
& \quad \downarrow \text{3122}
\end{aligned}$$

$$\frac{\sin(c + dx) \cos(c + dx) ((b \csc(c + dx))^p)^n \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{1}{2}(1 - np), \frac{1}{2}(3 - np), \sin^2(c + dx)\right)}{d(1 - np) \sqrt{\cos^2(c + dx)}}$$

input `Int[((b*Csc[c + d*x])^p)^n,x]`

output `(Cos[c + d*x]*((b*Csc[c + d*x])^p)^n*Hypergeometric2F1[1/2, (1 - n*p)/2, (3 - n*p)/2, Sin[c + d*x]^2]*Sin[c + d*x])/(d*(1 - n*p)*Sqrt[Cos[c + d*x]^2])`

### Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3122 `Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] := Simp[Cos[c + d*x]*((b*Sin[c + d*x])^(n + 1)/(b*d*(n + 1)*Sqrt[Cos[c + d*x]^2]))*Hypergeometric2F1[1/2, (n + 1)/2, (n + 3)/2, Sin[c + d*x]^2], x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[2*n]`

rule 4259 `Int[(csc[(c_.) + (d_.)*(x_)])*(b_.))^(n_), x_Symbol] := Simp[(b*Csc[c + d*x])^(n - 1)*((Sin[c + d*x]/b)^(n - 1) Int[1/(Sin[c + d*x]/b)^n, x]), x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[n]`

rule 4611 `Int[((b_.)*((c_.)*sec[(e_.) + (f_.)*(x_)])^(n_))^(p_), x_Symbol] := Simp[b^IntPart[p]*((b*(c*Sec[e + f*x])^n)^FracPart[p]/(c*Sec[e + f*x])^(n*FracPart[p])) Int[(c*Sec[e + f*x])^(n*p), x], x] /; FreeQ[{b, c, e, f, n, p}, x] && !IntegerQ[p]`

### Maple [F]

$$\int ((b \csc(dx + c))^p)^n dx$$

input `int(((b*csc(d*x+c))^p)^n,x)`

output `int((b*csc(d*x+c))^p)^n,x`

### Fricas [F]

$$\int ((b \csc(c + dx))^p)^n dx = \int ((b \csc(dx + c))^p)^n dx$$

input `integrate((b*csc(d*x+c))^p)^n,x, algorithm="fricas")`

output `integral((b*csc(d*x + c))^p)^n, x`

### Sympy [F]

$$\int ((b \csc(c + dx))^p)^n dx = \int ((b \csc(c + dx))^p)^n dx$$

input `integrate((b*csc(d*x+c)**p)**n,x)`

output `Integral((b*csc(c + d*x)**p)**n, x)`

### Maxima [F]

$$\int ((b \csc(c + dx))^p)^n dx = \int ((b \csc(dx + c))^p)^n dx$$

input `integrate((b*csc(d*x+c))^p)^n,x, algorithm="maxima")`

output `integrate((b*csc(d*x + c))^p)^n, x`



**Giac [F]**

$$\int ((b \csc(c + dx))^p)^n dx = \int ((b \csc(dx + c))^p)^n dx$$

input `integrate(((b*csc(d*x+c))^p)^n,x, algorithm="giac")`

output `integrate(((b*csc(d*x + c))^p)^n, x)`

**Mupad [F(-1)]**

Timed out.

$$\int ((b \csc(c + dx))^p)^n dx = \int \left( \left( \frac{b}{\sin(c + dx)} \right)^p \right)^n dx$$

input `int(((b/sin(c + d*x))^p)^n,x)`

output `int(((b/sin(c + d*x))^p)^n, x)`

**Reduce [F]**

$$\int ((b \csc(c + dx))^p)^n dx = b^{np} \left( \int \csc(dx + c)^{np} dx \right)$$

input `int(((b*csc(d*x+c))^p)^n,x)`

output `b**(n*p)*int(csc(c + d*x)**(n*p),x)`

### 3.69 $\int (a(b \csc(c + dx))^p)^n dx$

Optimal result	441
Mathematica [A] (verified)	441
Rubi [A] (verified)	442
Maple [F]	443
Fricas [F]	444
Sympy [F]	444
Maxima [F]	444
Giac [F]	445
Mupad [F(-1)]	445
Reduce [F]	445

#### Optimal result

Integrand size = 14, antiderivative size = 82

$$\int (a(b \csc(c + dx))^p)^n dx = \frac{\cos(c + dx) (a(b \csc(c + dx))^p)^n \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{1}{2}(1 - np), \frac{1}{2}(3 - np), \sin^2(c + dx)\right) \sin(c + dx)}{d(1 - np)\sqrt{\cos^2(c + dx)}}$$

output

```
cos(d*x+c)*(a*(b*csc(d*x+c))^p)^n*hypergeom([1/2, -1/2*n*p+1/2], [-1/2*n*p+3/2], sin(d*x+c)^2)*sin(d*x+c)/d/(-n*p+1)/(cos(d*x+c)^2)^(1/2)
```

#### Mathematica [A] (verified)

Time = 0.07 (sec) , antiderivative size = 73, normalized size of antiderivative = 0.89

$$\int (a(b \csc(c + dx))^p)^n dx = \frac{\cos(c + dx) (a(b \csc(c + dx))^p)^n \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{1}{2}(1 + np), \frac{3}{2}, \cos^2(c + dx)\right) \sin(c + dx) \sin^2(c + dx)}{d}$$

input

```
Integrate[(a*(b*Csc[c + d*x]))^p]^n,x]
```

output

```

-((Cos[c + d*x]*(a*(b*Csc[c + d*x])^p)^n*Hypergeometric2F1[1/2, (1 + n*p)/
2, 3/2, Cos[c + d*x]^2]*Sin[c + d*x]*(Sin[c + d*x]^2)^((-1 + n*p)/2))/d

```

**Rubi [A] (verified)**

Time = 0.36 (sec) , antiderivative size = 82, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 6,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.429$ , Rules used = {3042, 4611, 3042, 4259, 3042, 3122}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int (a(b \csc(c + dx))^p)^n dx \\
 & \quad \downarrow \text{3042} \\
 & \int \left( a \left( -b \sec \left( c + dx + \frac{\pi}{2} \right) \right)^p \right)^n dx \\
 & \quad \downarrow \text{4611} \\
 & (b \csc(c + dx))^{-np} (a(b \csc(c + dx))^p)^n \int (b \csc(c + dx))^{np} dx \\
 & \quad \downarrow \text{3042} \\
 & (b \csc(c + dx))^{-np} (a(b \csc(c + dx))^p)^n \int (b \csc(c + dx))^{np} dx \\
 & \quad \downarrow \text{4259} \\
 & \left( \frac{\sin(c + dx)}{b} \right)^{np} (a(b \csc(c + dx))^p)^n \int \left( \frac{\sin(c + dx)}{b} \right)^{-np} dx \\
 & \quad \downarrow \text{3042} \\
 & \left( \frac{\sin(c + dx)}{b} \right)^{np} (a(b \csc(c + dx))^p)^n \int \left( \frac{\sin(c + dx)}{b} \right)^{-np} dx \\
 & \quad \downarrow \text{3122}
 \end{aligned}$$

$$\frac{\sin(c + dx) \cos(c + dx) \operatorname{Hypergeometric2F1} \left( \frac{1}{2}, \frac{1}{2}(1 - np), \frac{1}{2}(3 - np), \sin^2(c + dx) \right) (a(b \csc(c + dx))^p)^n}{d(1 - np) \sqrt{\cos^2(c + dx)}}$$

input `Int[(a*(b*Csc[c + d*x])^p)^n,x]`

output `(Cos[c + d*x]*(a*(b*Csc[c + d*x])^p)^n*Hypergeometric2F1[1/2, (1 - n*p)/2, (3 - n*p)/2, Sin[c + d*x]^2]*Sin[c + d*x])/(d*(1 - n*p)*Sqrt[Cos[c + d*x]^2])`

### Defintions of rubi rules used

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3122 `Int[((b_.)*sin[(c_.) + (d_.)*(x_)])^(n_), x_Symbol] := Simp[Cos[c + d*x]*((b*Sin[c + d*x])^(n + 1)/(b*d*(n + 1)*Sqrt[Cos[c + d*x]^2]))*Hypergeometric2F1[1/2, (n + 1)/2, (n + 3)/2, Sin[c + d*x]^2], x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[2*n]`

rule 4259 `Int[(csc[(c_.) + (d_.)*(x_)])*(b_.)^(n_), x_Symbol] := Simp[(b*Csc[c + d*x])^(n - 1)*((Sin[c + d*x]/b)^(n - 1) Int[1/(Sin[c + d*x]/b)^n, x]), x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[n]`

rule 4611 `Int[((b_.)*((c_.)*sec[(e_.) + (f_.)*(x_)])^(n_))^(p_), x_Symbol] := Simp[b^IntPart[p]*((b*(c*Sec[e + f*x])^n)^FracPart[p]/(c*Sec[e + f*x])^(n*FracPart[p])) Int[(c*Sec[e + f*x])^(n*p), x], x] /; FreeQ[{b, c, e, f, n, p}, x] && !IntegerQ[p]`

### Maple [F]

$$\int (a(b \csc(dx + c))^p)^n dx$$

input `int((a*(b*csc(d*x+c))^p)^n,x)`

output `int((a*(b*csc(d*x+c))^p)^n,x)`

### Fricas [F]

$$\int (a(b \csc(c + dx))^p)^n dx = \int ((b \csc(dx + c))^p a)^n dx$$

input `integrate((a*(b*csc(d*x+c))^p)^n,x, algorithm="fricas")`

output `integral(((b*csc(d*x + c))^p*a)^n, x)`

### Sympy [F]

$$\int (a(b \csc(c + dx))^p)^n dx = \int (a(b \csc(c + dx))^p)^n dx$$

input `integrate((a*(b*csc(d*x+c))**p)**n,x)`

output `Integral((a*(b*csc(c + d*x))**p)**n, x)`

### Maxima [F]

$$\int (a(b \csc(c + dx))^p)^n dx = \int ((b \csc(dx + c))^p a)^n dx$$

input `integrate((a*(b*csc(d*x+c))^p)^n,x, algorithm="maxima")`

output `integrate(((b*csc(d*x + c))^p*a)^n, x)`

**Giac [F]**

$$\int (a(b \csc(c + dx))^p)^n dx = \int ((b \csc(dx + c))^p a)^n dx$$

input `integrate((a*(b*csc(d*x+c))^p)^n,x, algorithm="giac")`

output `integrate(((b*csc(d*x + c))^p*a)^n, x)`

**Mupad [F(-1)]**

Timed out.

$$\int (a(b \csc(c + dx))^p)^n dx = \int \left( a \left( \frac{b}{\sin(c + dx)} \right)^p \right)^n dx$$

input `int((a*(b/sin(c + d*x))^p)^n,x)`

output `int((a*(b/sin(c + d*x))^p)^n, x)`

**Reduce [F]**

$$\int (a(b \csc(c + dx))^p)^n dx = b^{np} a^n \left( \int \csc(dx + c)^{np} dx \right)$$

input `int((a*(b*csc(d*x+c))^p)^n,x)`

output `b**(n*p)*a**n*int(csc(c + d*x)**(n*p),x)`

### 3.70 $\int (a \csc(e + fx))^m (b \csc(e + fx))^n dx$

Optimal result	446
Mathematica [A] (verified)	446
Rubi [A] (verified)	447
Maple [F]	448
Fricas [F]	449
Sympy [F]	449
Maxima [F]	449
Giac [F]	450
Mupad [F(-1)]	450
Reduce [F]	450

#### Optimal result

Integrand size = 21, antiderivative size = 91

$$\int (a \csc(e + fx))^m (b \csc(e + fx))^n dx$$

$$= \frac{a \cos(e + fx) (a \csc(e + fx))^{-1+m} (b \csc(e + fx))^n \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{1}{2}(1 - m - n), \frac{1}{2}(3 - m - n), \frac{1}{\cos^2(e + fx)}\right)}{f(1 - m - n) \sqrt{\cos^2(e + fx)}}$$

output `a*cos(f*x+e)*(a*csc(f*x+e))(-1+m)*(b*csc(f*x+e))n*hypergeom([1/2, 1/2-1/2*m-1/2*n], [3/2-1/2*m-1/2*n], sin(f*x+e)2/f/(1-m-n)/(cos(f*x+e)2)(1/2))`

#### Mathematica [A] (verified)

Time = 0.36 (sec) , antiderivative size = 77, normalized size of antiderivative = 0.85

$$\int (a \csc(e + fx))^m (b \csc(e + fx))^n dx =$$

$$\frac{\cos(e + fx) (a \csc(e + fx))^m (b \csc(e + fx))^n \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{1}{2}(1 + m + n), \frac{3}{2}, \cos^2(e + fx)\right)}{f}$$

input `Integrate[(a*Csc[e + f*x])m*(b*Csc[e + f*x])n,x]`

output

```

-((Cos[e + f*x]*(a*Csc[e + f*x])^m*(b*Csc[e + f*x])^n*Hypergeometric2F1[1/
2, (1 + m + n)/2, 3/2, Cos[e + f*x]^2]*Sin[e + f*x]*(Sin[e + f*x]^2)^((-1
+ m + n)/2))/f)

```

**Rubi [A] (verified)**

Time = 0.36 (sec) , antiderivative size = 91, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.238$ , Rules used = {2034, 3042, 4259, 3042, 3122}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
& \int (a \csc(e + fx))^m (b \csc(e + fx))^n dx \\
& \quad \downarrow \text{2034} \\
& (a \csc(e + fx))^{-n} (b \csc(e + fx))^n \int (a \csc(e + fx))^{m+n} dx \\
& \quad \downarrow \text{3042} \\
& (a \csc(e + fx))^{-n} (b \csc(e + fx))^n \int (a \csc(e + fx))^{m+n} dx \\
& \quad \downarrow \text{4259} \\
& (a \csc(e + fx))^{m-1} (b \csc(e + fx))^n \left( \frac{\sin(e + fx)}{a} \right)^{m+n-1} \int \left( \frac{\sin(e + fx)}{a} \right)^{-m-n} dx \\
& \quad \downarrow \text{3042} \\
& (a \csc(e + fx))^{m-1} (b \csc(e + fx))^n \left( \frac{\sin(e + fx)}{a} \right)^{m+n-1} \int \left( \frac{\sin(e + fx)}{a} \right)^{-m-n} dx \\
& \quad \downarrow \text{3122} \\
& \frac{a \cos(e + fx) (a \csc(e + fx))^{m-1} (b \csc(e + fx))^n \operatorname{Hypergeometric2F1}\left(\frac{1}{2}, \frac{1}{2}(-m - n + 1), \frac{1}{2}(-m - n + 3), \sin^2(e + fx)\right)}{f(-m - n + 1) \sqrt{\cos^2(e + fx)}}
\end{aligned}$$

input

```

Int[(a*Csc[e + f*x])^m*(b*Csc[e + f*x])^n,x]

```



output

```
(a*cos[e + f*x]*(a*csc[e + f*x])^(-1 + m)*(b*csc[e + f*x])^n*Hypergeometric2F1[1/2, (1 - m - n)/2, (3 - m - n)/2, Sin[e + f*x]^2])/(f*(1 - m - n)*Sqrt[Cos[e + f*x]^2])
```

### Definitions of rubi rules used

rule 2034

```
Int[(Fv_)*((a_)*(v_))^(m_)*((b_)*(v_))^(n_), x_Symbol] := Simp[b^IntPart[n]*((b*v)^FracPart[n]/(a^IntPart[n]*(a*v)^FracPart[n])) Int[(a*v)^(m + n)*Fv, x], x] /; FreeQ[{a, b, m, n}, x] && !IntegerQ[m] && !IntegerQ[n] && !IntegerQ[m + n]
```

rule 3042

```
Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]
```

rule 3122

```
Int[((b_)*sin[(c_) + (d_)*(x_)])^(n_), x_Symbol] := Simp[Cos[c + d*x]*((b*Sin[c + d*x])^(n + 1)/(b*d*(n + 1)*Sqrt[Cos[c + d*x]^2]))*Hypergeometric2F1[1/2, (n + 1)/2, (n + 3)/2, Sin[c + d*x]^2], x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[2*n]
```

rule 4259

```
Int[(csc[(c_) + (d_)*(x_)])*(b_)^(n_), x_Symbol] := Simp[(b*Csc[c + d*x])^(n - 1)*((Sin[c + d*x]/b)^(n - 1) Int[1/(Sin[c + d*x]/b)^n, x]), x] /; FreeQ[{b, c, d, n}, x] && !IntegerQ[n]
```

### Maple **[F]**

$$\int (a \csc (fx + e))^m (b \csc (fx + e))^n dx$$

input

```
int((a*csc(f*x+e))^m*(b*csc(f*x+e))^n,x)
```

output

```
int((a*csc(f*x+e))^m*(b*csc(f*x+e))^n,x)
```

**Fricas [F]**

$$\int (a \csc(e + fx))^m (b \csc(e + fx))^n dx = \int (a \csc(fx + e))^m (b \csc(fx + e))^n dx$$

input `integrate((a*csc(f*x+e))^m*(b*csc(f*x+e))^n,x, algorithm="fricas")`

output `integral((a*csc(f*x + e))^m*(b*csc(f*x + e))^n, x)`

**Sympy [F]**

$$\int (a \csc(e + fx))^m (b \csc(e + fx))^n dx = \int (a \csc(e + fx))^m (b \csc(e + fx))^n dx$$

input `integrate((a*csc(f*x+e))**m*(b*csc(f*x+e))**n,x)`

output `Integral((a*csc(e + f*x))**m*(b*csc(e + f*x))**n, x)`

**Maxima [F]**

$$\int (a \csc(e + fx))^m (b \csc(e + fx))^n dx = \int (a \csc(fx + e))^m (b \csc(fx + e))^n dx$$

input `integrate((a*csc(f*x+e))^m*(b*csc(f*x+e))^n,x, algorithm="maxima")`

output `integrate((a*csc(f*x + e))^m*(b*csc(f*x + e))^n, x)`

**Giac [F]**

$$\int (a \csc(e + fx))^m (b \csc(e + fx))^n dx = \int (a \csc(fx + e))^m (b \csc(fx + e))^n dx$$

input `integrate((a*csc(f*x+e))^m*(b*csc(f*x+e))^n,x, algorithm="giac")`

output `integrate((a*csc(f*x + e))^m*(b*csc(f*x + e))^n, x)`

**Mupad [F(-1)]**

Timed out.

$$\int (a \csc(e + fx))^m (b \csc(e + fx))^n dx = \int \left( \frac{a}{\sin(e + fx)} \right)^m \left( \frac{b}{\sin(e + fx)} \right)^n dx$$

input `int((a/sin(e + f*x))^m*(b/sin(e + f*x))^n,x)`

output `int((a/sin(e + f*x))^m*(b/sin(e + f*x))^n, x)`

**Reduce [F]**

$$\int (a \csc(e + fx))^m (b \csc(e + fx))^n dx = b^n a^m \left( \int \csc(fx + e)^{m+n} dx \right)$$

input `int((a*csc(f*x+e))^m*(b*csc(f*x+e))^n,x)`

output `b**n*a**m*int(csc(e + f*x)**(m + n),x)`

# CHAPTER 4

## APPENDIX

4.1	Listing of Grading functions . . . . .	451
4.2	Links to plain text integration problems used in this report for each CAS .	469

### 4.1 Listing of Grading functions

The following are the current version of the grading functions used for grading the quality of the antiderivative with reference to the optimal antiderivative included in the test suite.

There is a version for Maple and for Mathematica/Rubi. There is a version for grading Sympy and version for use with Sagemath.

The following are links to the current source code.

The following are the listings of source code of the grading functions.

#### Mathematica and Rubi grading function

```
(* Original version thanks to Albert Rich emailed on 03/21/2017 *)
(* ::Package:: *)

(* Nasser: April 7, 2022. add second output which gives reason for the grade *)
(*                               Small rewrite of logic in main function to make it*)
(*                               match Maple's logic. No change in functionality otherwise*)

(* ::Subsection:: *)
(*GradeAntiderivative[result,optimal]*)
```

```

(* ::Text:: *)
(*If result and optimal are mathematical expressions, *)
(*      GradeAntiderivative[result,optimal] returns*)
(* "F" if the result fails to integrate an expression that*)
(*      is integrable*)
(* "C" if result involves higher level functions than necessary*)
(* "B" if result is more than twice the size of the optimal*)
(*      antiderivative*)
(* "A" if result can be considered optimal*)

GradeAntiderivative[result_,optimal_] := Module[{expnResult,expnOptimal,leafCountResult,leafCountOptimal},
  expnResult = ExpnType[result];
  expnOptimal = ExpnType[optimal];
  leafCountResult = LeafCount[result];
  leafCountOptimal = LeafCount[optimal];

  (*Print["expnResult=",expnResult," expnOptimal=",expnOptimal];*)
  If[expnResult<=expnOptimal,
    If[Not[FreeQ[result,Complex]], (*result contains complex*)
      If[Not[FreeQ[optimal,Complex]], (*optimal contains complex*)
        If[leafCountResult<=2*leafCountOptimal,
          finalresult={"A",""}
          ,(*ELSE*)
          finalresult={"B","Both result and optimal contain complex but leaf count
          ]
        ,(*ELSE*)
        finalresult={"C","Result contains complex when optimal does not."}
      ]
    ,(*ELSE*)(*result does not contains complex*)
    If[leafCountResult<=2*leafCountOptimal,
      finalresult={"A",""}
      ,(*ELSE*)
      finalresult={"B","Leaf count is larger than twice the leaf count of optimal.
      ]
    ]
  ,(*ELSE*)(*expnResult>expnOptimal*)
  If[FreeQ[result,Integrate] && FreeQ[result,Int],
    finalresult={"C","Result contains higher order function than in optimal. Order "
  ,

```

```

        finalresult={"F","Contains unresolved integral."}
    ]
];

finalresult
]

(* ::Text:: *)
(*The following summarizes the type number assigned an *)
(*expression based on the functions it involves*)
(*1 = rational function*)
(*2 = algebraic function*)
(*3 = elementary function*)
(*4 = special function*)
(*5 = hyperpergeometric function*)
(*6 = appell function*)
(*7 = rootsum function*)
(*8 = integrate function*)
(*9 = unknown function*)

ExpnType[expn_] :=
  If[AtomQ[expn],
    1,
  If[ListQ[expn],
    Max[Map[ExpnType, expn]],
  If[Head[expn]===Power,
    If[IntegerQ[expn[[2]]],
      ExpnType[expn[[1]]],
    If[Head[expn[[2]]]===Rational,
      If[IntegerQ[expn[[1]]] || Head[expn[[1]]]===Rational,
        1,
        Max[ExpnType[expn[[1]], 2]],
      Max[ExpnType[expn[[1]], ExpnType[expn[[2]], 3]],
    If[Head[expn]===Plus || Head[expn]===Times,
      Max[ExpnType[First[expn]], ExpnType[Rest[expn]]],
    If[ElementaryFunctionQ[Head[expn]],
      Max[3, ExpnType[expn[[1]]]],
    If[SpecialFunctionQ[Head[expn]],
      Apply[Max, Append[Map[ExpnType, Apply[List, expn]], 4]],
    If[HypergeometricFunctionQ[Head[expn]],

```

```

    Apply [Max, Append [Map [ExpnType, Apply [List, expn]], 5]],
    If [AppellFunctionQ [Head [expn]],
        Apply [Max, Append [Map [ExpnType, Apply [List, expn]], 6]],
        If [Head [expn] === RootSum,
            Apply [Max, Append [Map [ExpnType, Apply [List, expn]], 7]],
            If [Head [expn] === Integrate || Head [expn] === Int,
                Apply [Max, Append [Map [ExpnType, Apply [List, expn]], 8]],
                9]]]]]]]]]]]

```

```

ElementaryFunctionQ [func_] :=
  MemberQ [{
    Exp, Log,
    Sin, Cos, Tan, Cot, Sec, Csc,
    ArcSin, ArcCos, ArcTan, ArcCot, ArcSec, ArcCsc,
    Sinh, Cosh, Tanh, Coth, Sech, Csch,
    ArcSinh, ArcCosh, ArcTanh, ArcCoth, ArcSech, ArcCsch
  }, func]

```

```

SpecialFunctionQ [func_] :=
  MemberQ [{
    Erf, Erfc, Erfi,
    FresnelS, FresnelC,
    ExpIntegralE, ExpIntegralEi, LogIntegral,
    SinIntegral, CosIntegral, SinhIntegral, CoshIntegral,
    Gamma, LogGamma, PolyGamma,
    Zeta, PolyLog, ProductLog,
    EllipticF, EllipticE, EllipticPi
  }, func]

```

```

HypergeometricFunctionQ [func_] :=
  MemberQ [{Hypergeometric1F1, Hypergeometric2F1, HypergeometricPFQ}, func]

```

```

AppellFunctionQ [func_] :=
  MemberQ [{AppellF1}, func]

```

## Maple grading function

```

# File: GradeAntiderivative.mpl
# Original version thanks to Albert Rich emailed on 03/21/2017

#Nasser 03/22/2017 Use Maple leaf count instead since buildin
#Nasser 03/23/2017 missing 'ln' for ElementaryFunctionQ added
#Nasser 03/24/2017 corrected the check for complex result
#Nasser 10/27/2017 check for leafsize and do not call ExpnType()
#
#                   if leaf size is "too large". Set at 500,000
#Nasser 12/22/2019 Added debug flag, added 'dilog' to special functions
#
#                   see problem 156, file Apostol_Problems
#Nasser 4/07/2022  add second output which gives reason for the grade

GradeAntiderivative := proc(result,optimal)
local leaf_count_result,
      leaf_count_optimal,
      ExpnType_result,
      ExpnType_optimal,
      debug:=false;

      leaf_count_result:=leafcount(result);
      #do NOT call ExpnType() if leaf size is too large. Recursion problem
      if leaf_count_result > 500000 then
          return "B","result has leaf size over 500,000. Avoiding possible recursion issue";
      fi;

      leaf_count_optimal := leafcount(optimal);
      ExpnType_result   := ExpnType(result);
      ExpnType_optimal := ExpnType(optimal);

      if debug then
          print("ExpnType_result",ExpnType_result," ExpnType_optimal=",ExpnType_optimal);
      fi;

# If result and optimal are mathematical expressions,
# GradeAntiderivative[result,optimal] returns
# "F" if the result fails to integrate an expression that
#     is integrable
# "C" if result involves higher level functions than necessary
# "B" if result is more than twice the size of the optimal

```



```

#   antiderivative
#   "A" if result can be considered optimal

#This check below actually is not needed, since I only
#call this grading only for passed integrals. i.e. I check
#for "F" before calling this. But no harm of keeping it here.
#just in case.

if not type(result,freeof('int')) then
    return "F","Result contains unresolved integral";
fi;

if ExpnType_result<=ExpnType_optimal then
    if debug then
        print("ExpnType_result<=ExpnType_optimal");
    fi;
    if is_contains_complex(result) then
        if is_contains_complex(optimal) then
            if debug then
                print("both result and optimal complex");
            fi;
            if leaf_count_result<=2*leaf_count_optimal then
                return "A"," ";
            else
                return "B",cat("Both result and optimal contain complex but leaf count of
                                convert(leaf_count_result,string)," vs. $2 (" ,
                                convert(leaf_count_optimal,string)," ) = ",convert(2*leaf

        end if
    else #result contains complex but optimal is not
        if debug then
            print("result contains complex but optimal is not");
        fi;
        return "C","Result contains complex when optimal does not.";
    fi;
else # result do not contain complex
    # this assumes optimal do not as well. No check is needed here.
    if debug then
        print("result do not contain complex, this assumes optimal do not as well
    fi;

```

```
    if leaf_count_result<=2*leaf_count_optimal then
      if debug then
        print("leaf_count_result<=2*leaf_count_optimal");
      fi;
      return "A"," ";
    else
      if debug then
        print("leaf_count_result>2*leaf_count_optimal");
      fi;
      return "B",cat("Leaf count of result is larger than twice the leaf count of
                    convert(leaf_count_result,string)," $ vs. $2(",
                    convert(leaf_count_optimal,string),")=",convert(2*leaf_co
      fi;
    fi;
  else #ExpnType(result) > ExpnType(optimal)
    if debug then
      print("ExpnType(result) > ExpnType(optimal)");
    fi;
    return "C",cat("Result contains higher order function than in optimal. Order ",
                  convert(ExpnType_result,string)," vs. order ",
                  convert(ExpnType_optimal,string),".");
  fi;
end proc:

#
# is_contains_complex(result)
# takes expressions and returns true if it contains "I" else false
#
#Nasser 032417
is_contains_complex:= proc(expression)
  return (has(expression,I));
end proc:

# The following summarizes the type number assigned an expression
# based on the functions it involves
# 1 = rational function
# 2 = algebraic function
# 3 = elementary function
# 4 = special function
# 5 = hyperpergeometric function
```

```

# 6 = appell function
# 7 = rootsum function
# 8 = integrate function
# 9 = unknown function

ExpnType := proc(expn)
  if type(expn,'atomic') then
    1
  elif type(expn,'list') then
    apply(max,map(ExpnType,expn))
  elif type(expn,'sqrt') then
    if type(op(1,expn),'rational') then
      1
    else
      max(2,ExpnType(op(1,expn)))
    end if
  elif type(expn,'^^') then
    if type(op(2,expn),'integer') then
      ExpnType(op(1,expn))
    elif type(op(2,expn),'rational') then
      if type(op(1,expn),'rational') then
        1
      else
        max(2,ExpnType(op(1,expn)))
      end if
    else
      max(3,ExpnType(op(1,expn)),ExpnType(op(2,expn)))
    end if
  elif type(expn,'+'') or type(expn,'*') then
    max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
  elif ElementaryFunctionQ(op(0,expn)) then
    max(3,ExpnType(op(1,expn)))
  elif SpecialFunctionQ(op(0,expn)) then
    max(4,apply(max,map(ExpnType,[op(expn)])))
  elif HypergeometricFunctionQ(op(0,expn)) then
    max(5,apply(max,map(ExpnType,[op(expn)])))
  elif AppellFunctionQ(op(0,expn)) then
    max(6,apply(max,map(ExpnType,[op(expn)])))
  elif op(0,expn)='int' then
    max(8,apply(max,map(ExpnType,[op(expn)]))) else
  9
end proc

```

```

    end if
end proc:

ElementaryFunctionQ := proc(func)
  member(func, [
    exp, log, ln,
    sin, cos, tan, cot, sec, csc,
    arcsin, arccos, arctan, arccot, arcsec, arccsc,
    sinh, cosh, tanh, coth, sech, csch,
    arcsinh, arccosh, arctanh, arccoth, arcsech, arccsch])
end proc:

SpecialFunctionQ := proc(func)
  member(func, [
    erf, erfc, erfi,
    FresnelS, FresnelC,
    Ei, Ei, Li, Si, Ci, Shi, Chi,
    GAMMA, lnGAMMA, Psi, Zeta, polylog, dilog, LambertW,
    EllipticF, EllipticE, EllipticPi])
end proc:

HypergeometricFunctionQ := proc(func)
  member(func, [Hypergeometric1F1, hypergeom, HypergeometricPFQ])
end proc:

AppellFunctionQ := proc(func)
  member(func, [AppellF1])
end proc:

# u is a sum or product. rest(u) returns all but the
# first term or factor of u.
rest := proc(u) local v;
  if nops(u)=2 then
    op(2,u)
  else
    apply(op(0,u), op(2..nops(u),u))
  end if
end proc:

#leafcount(u) returns the number of nodes in u.

```

```
#Nasser 3/23/17 Replaced by build-in leafCount from package in Maple
leafcount := proc(u)
  MmaTranslator[Mma][LeafCount](u);
end proc:
```

## Sympy grading function

```
#Dec 24, 2019. Nasser M. Abbasi:
#      Port of original Maple grading function by
#      Albert Rich to use with Sympy/Python
#Dec 27, 2019 Nasser. Added `RootSum`. See problem 177, Timofeev file
#      added 'exp_polar'
from sympy import *

def leaf_count(expr):
  #sympy do not have leaf count function. This is approximation
  return round(1.7*count_ops(expr))

def is_sqrt(expr):
  if isinstance(expr,Pow):
    if expr.args[1] == Rational(1,2):
      return True
    else:
      return False
  else:
    return False

def is_elementary_function(func):
  return func in [exp,log,ln,sin,cos,tan,cot,sec,csc,
    asin,acos,atan,acot,asec,acsc,sinh,cosh,tanh,coth,sech,csch,
    asinh,acosh,atanh,acoth,asech,acsch
  ]

def is_special_function(func):
  return func in [ erf,erfc,erfi,
    fresnels,fresnelc,Ei,Ei,Li,Si,Ci,Shi,Chi,
    gamma,loggamma,digamma,zeta,polylog,LambertW,
    elliptic_f,elliptic_e,elliptic_pi,exp_polar
  ]
```

```

def is_hypergeometric_function(func):
    return func in [hyper]

def is_appell_function(func):
    return func in [appellf1]

def is_atom(expn):
    try:
        if expn.isAtom or isinstance(expn,int) or isinstance(expn,float):
            return True
        else:
            return False

    except AttributeError as error:
        return False

def expnType(expn):
    debug=False
    if debug:
        print("expn=",expn,"type(expn)=",type(expn))

    if is_atom(expn):
        return 1
    elif isinstance(expn,list):
        return max(map(expnType, expn)) #apply(max,map(ExpnType,expn))
    elif is_sqrt(expn):
        if isinstance(expn.args[0],Rational): #type(op(1,expn),'rational')
            return 1
        else:
            return max(2,expnType(expn.args[0])) #max(2,ExpnType(op(1,expn)))
    elif isinstance(expn,Pow): #type(expn,'^')
        if isinstance(expn.args[1],Integer): #type(op(2,expn),'integer')
            return expnType(expn.args[0]) #ExpnType(op(1,expn))
        elif isinstance(expn.args[1],Rational): #type(op(2,expn),'rational')
            if isinstance(expn.args[0],Rational): #type(op(1,expn),'rational')
                return 1
            else:
                return max(2,expnType(expn.args[0])) #max(2,ExpnType(op(1,expn)))
        else:
            return max(3,expnType(expn.args[0]),expnType(expn.args[1])) #max(3,ExpnType(op(1,expn)),ExpnType(op(2,expn)))
    elif isinstance(expn,Add) or isinstance(expn,Mul): #type(expn,'+') or type(expn,'*')

```

```

    m1 = expnType(expn.args[0])
    m2 = expnType(list(expn.args[1:]))
    return max(m1,m2) #max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
elif is_elementary_function(expn.func): #ElementaryFunctionQ(op(0,expn))
    return max(3,expnType(expn.args[0])) #max(3,ExpnType(op(1,expn)))
elif is_special_function(expn.func): #SpecialFunctionQ(op(0,expn))
    m1 = max(map(expnType, list(expn.args)))
    return max(4,m1) #max(4,apply(max,map(ExpnType,[op(expn)])))
elif is_hypergeometric_function(expn.func): #HypergeometricFunctionQ(op(0,expn))
    m1 = max(map(expnType, list(expn.args)))
    return max(5,m1) #max(5,apply(max,map(ExpnType,[op(expn)])))
elif is_appell_function(expn.func):
    m1 = max(map(expnType, list(expn.args)))
    return max(6,m1) #max(5,apply(max,map(ExpnType,[op(expn)])))
elif isinstance(expn,RootSum):
    m1 = max(map(expnType, list(expn.args))) #Apply[Max,Append[Map[ExpnType,Apply[List,expn]],7]],
    return max(7,m1)
elif str(expn).find("Integral") != -1:
    m1 = max(map(expnType, list(expn.args)))
    return max(8,m1) #max(5,apply(max,map(ExpnType,[op(expn)])))
else:
    return 9

#main function
def grade_antiderivative(result,optimal):

    #print("Enter grade_antiderivative for sagemath")
    #print("Enter grade_antiderivative, result=",result," optimal=",optimal)

    leaf_count_result = leaf_count(result)
    leaf_count_optimal = leaf_count(optimal)

    #print("leaf_count_result=",leaf_count_result)
    #print("leaf_count_optimal=",leaf_count_optimal)

    expnType_result = expnType(result)
    expnType_optimal = expnType(optimal)

    if str(result).find("Integral") != -1:
        grade = "F"
        grade_annotation = ""

```

```

else:
    if expnType_result <= expnType_optimal:
        if result.has(I):
            if optimal.has(I): #both result and optimal complex
                if leaf_count_result <= 2*leaf_count_optimal:
                    grade = "A"
                    grade_annotation = ""
                else:
                    grade = "B"
                    grade_annotation = "Both result and optimal contain complex but leaf count of result is lar
            else: #result contains complex but optimal is not
                grade = "C"
                grade_annotation = "Result contains complex when optimal does not."
        else: # result do not contain complex, this assumes optimal do not as well
            if leaf_count_result <= 2*leaf_count_optimal:
                grade = "A"
                grade_annotation = ""
            else:
                grade = "B"
                grade_annotation = "Leaf count of result is larger than twice the leaf count of optimal. "+str(
        else:
            grade = "C"
            grade_annotation = "Result contains higher order function than in optimal. Order "+str(ExpnType

#print("Before returning. grade=",grade, " grade_annotation=",grade_annotation)

return grade, grade_annotation

```

## SageMath grading function

```

#Dec 24, 2019. Nasser: Ported original Maple grading function by
#       Albert Rich to use with Sagemath. This is used to
#       grade Fricas, Giac and Maxima results.
#Dec 24, 2019. Nasser: Added 'exp_integral_e' and 'sng', 'sin_integral'
#       'arctan2', 'floor', 'abs', 'log_integral'
#June 4, 2022 Made default grade_annotation "none" instead of "" due
#       issue later when reading the file.
#July 14, 2022. Added ellipticF. This is until they fix sagemath, then remove it.

```



```

from sage.all import *
from sage.symbolic.operators import add_vararg, mul_vararg

debug=False;

def tree_size(expr):
    r"""
    Return the tree size of this expression.
    """
    #print("Enter tree_size, expr is ",expr)

    if expr not in SR:
        # deal with lists, tuples, vectors
        return 1 + sum(tree_size(a) for a in expr)
    expr = SR(expr)
    x, aa = expr.operator(), expr.operands()
    if x is None:
        return 1
    else:
        return 1 + sum(tree_size(a) for a in aa)

def is_sqrt(expr):
    if expr.operator() == operator.pow: #isinstance(expr,Pow):
        if expr.operands()[1]==1/2: #expr.args[1] == Rational(1,2):
            if debug: print ("expr is sqrt")
            return True
        else:
            return False
    else:
        return False

def is_elementary_function(func):
    #debug=False
    m = func.name() in ['exp','log','ln',
        'sin','cos','tan','cot','sec','csc',
        'arcsin','arccos','arctan','arccot','arcsec','arccsc',
        'sinh','cosh','tanh','coth','sech','csch',
        'arcsinh','arccosh','arctanh','arcoth','arcsech','arcsch','sgn',
        'arctan2','floor','abs'
    ]
    if debug:

```

```

    if m:
        print ("func ", func , " is elementary_function")
    else:
        print ("func ", func , " is NOT elementary_function")

    return m

def is_special_function(func):
    #debug=False
    if debug:
        print ("type(func)=", type(func))

    m= func.name() in ['erf','erfc','erfi','fresnel_sin','fresnel_cos','Ei',
        'Ei','Li','Si','sin_integral','Ci','cos_integral','Shi','sinh_integral',
        'Chi','cosh_integral','gamma','log_gamma','psi','zeta',
        'polylog','lambert_w','elliptic_f','elliptic_e','ellipticF',
        'elliptic_pi','exp_integral_e','log_integral',
        'weierstrassPInverse','weierstrass','weierstrassP','weierstrassZeta',
        'weierstrassPPrime','weierstrassSigma']

    if debug:
        print ("m=",m)
    if m:
        print ("func ", func , " is special_function")
    else:
        print ("func ", func , " is NOT special_function")

    return m

def is_hypergeometric_function(func):
    return func.name() in ['hypergeometric','hypergeometric_M','hypergeometric_U']

def is_appell_function(func):
    return func.name() in ['hypergeometric'] #[appellf1] can't find this in sagemath

def is_atom(expn):

    #debug=False

```

```

if debug:
    print ("Enter is_atom, expn=",expn)

if not hasattr(expn, 'parent'):
    return False

#thanks to answer at https://ask.sagemath.org/question/49179/what-is-sagemath-equivalent-to-atomic
try:
    if expn.parent() is SR:
        return expn.operator() is None
    if expn.parent() in (ZZ, QQ, AA, QQbar):
        return expn in expn.parent() # Should always return True
    if hasattr(expn.parent(),"base_ring") and hasattr(expn.parent(),"gens"):
        return expn in expn.parent().base_ring() or expn in expn.parent().gens()

    return False

except AttributeError as error:
    print ("Exception,AttributeError in is_atom")
    print ("caught exception" , type(error).__name__ )
    return False

def expnType(expn):

    if debug:
        print (">>>>>Enter expnType, expn=", expn)
        print (">>>>>is_atom(expn)=", is_atom(expn))

    if is_atom(expn):
        return 1
    elif type(expn)==list: #instance(expn,list):
        return max(map(expnType, expn)) #apply(max,map(ExpnType,expn))
    elif is_sqrt(expn):
        if type(expn.operands()[0])==Rational: #type(instance(expn.args[0],Rational):
            return 1
        else:
            return max(2,expnType(expn.operands()[0])) #max(2,expnType(expn.args[0]))
    elif expn.operator() == operator.pow: #instance(expn,Pow)
        if type(expn.operands()[1])==Integer: #instance(expn.args[1],Integer)

```

```

    return expnType(expn.operands()[0]) #expnType(expn.args[0])
elif type(expn.operands()[1])==Rational: #isinstance(expn.args[1],Rational)
    if type(expn.operands()[0])==Rational: #isinstance(expn.args[0],Rational)
        return 1
    else:
        return max(2,expnType(expn.operands()[0])) #max(2,expnType(expn.args[0]))
else:
    return max(3,expnType(expn.operands()[0]),expnType(expn.operands()[1])) #max(3,expnType(expn
elif expn.operator() == add_vararg or expn.operator() == mul_vararg: #isinstance(expn,Add) or isins
    m1 = expnType(expn.operands()[0]) #expnType(expn.args[0])
    m2 = expnType(expn.operands()[1:]) #expnType(list(expn.args[1:]))
    return max(m1,m2) #max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
elif is_elementary_function(expn.operator()): #is_elementary_function(expn.func)
    return max(3,expnType(expn.operands()[0]))
elif is_special_function(expn.operator()): #is_special_function(expn.func)
    m1 = max(map(expnType, expn.operands())) #max(map(expnType, list(expn.args)))
    return max(4,m1) #max(4,m1)
elif is_hypergeometric_function(expn.operator()): #is_hypergeometric_function(expn.func)
    m1 = max(map(expnType, expn.operands())) #max(map(expnType, list(expn.args)))
    return max(5,m1) #max(5,m1)
elif is_appell_function(expn.operator()):
    m1 = max(map(expnType, expn.operands())) #max(map(expnType, list(expn.args)))
    return max(6,m1) #max(6,m1)
elif str(expn).find("Integral") != -1: #this will never happen, since it
    #is checked before calling the grading function that is passed.
    #but kept it here.
    m1 = max(map(expnType, expn.operands())) #max(map(expnType, list(expn.args)))
    return max(8,m1) #max(5,apply(max,map(ExpnType,[op(expn)])))
else:
    return 9

#main function
def grade_antiderivative(result,optimal):

if debug:
    print ("Enter grade_antiderivative for sagemath")
    print ("Enter grade_antiderivative, result=",result)
    print ("Enter grade_antiderivative, optimal=",optimal)
    print ("type(anti)=", type(result))
    print ("type(optimal)=", type(optimal))

```

```

leaf_count_result = tree_size(result) #leaf_count(result)
leaf_count_optimal = tree_size(optimal) #leaf_count(optimal)

#if debug: print ("leaf_count_result=", leaf_count_result, "leaf_count_optimal=",leaf_count_optimal)

expnType_result = expnType(result)
expnType_optimal = expnType(optimal)

if debug: print ("expnType_result=", expnType_result, "expnType_optimal=",expnType_optimal)

if expnType_result <= expnType_optimal:
    if result.has(I):
        if optimal.has(I): #both result and optimal complex
            if leaf_count_result <= 2*leaf_count_optimal:
                grade = "A"
                grade_annotation = "none"
            else:
                grade = "B"
                grade_annotation = "Both result and optimal contain complex but leaf count of result is larger"
        else: #result contains complex but optimal is not
            grade = "C"
            grade_annotation = "Result contains complex when optimal does not."
    else: # result do not contain complex, this assumes optimal do not as well
        if leaf_count_result <= 2*leaf_count_optimal:
            grade = "A"
            grade_annotation = "none"
        else:
            grade = "B"
            grade_annotation = "Leaf count of result is larger than twice the leaf count of optimal. "+str(leaf_count_result - 2*leaf_count_optimal)
else:
    grade = "C"
    grade_annotation = "Result contains higher order function than in optimal. Order "+str(expnType_result - expnType_optimal)

print("Before returning. grade=",grade, " grade_annotation=",grade_annotation)

return grade, grade_annotation

```

## 4.2 Links to plain text integration problems used in this report for each CAS

1. Mathematica integration problems as .m file
2. Maple integration problems as .txt file
3. Sagemath integration problems as .sage file
4. Reduce integration problems as .txt file
5. Mupad integration problems as .txt file
6. Sympy integration problems as .py file