

# Computer Algebra Independent Integration Tests

Summer 2024

5-Inverse-trig-functions/5.6-Inverse-cosecant/288-5.6

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# CHAPTER 1

## INTRODUCTION

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This report gives the result of running the computer algebra independent integration test. The download section in on the main webpage contains links to download the problems in plain text format used for all CAS systems. The number of integrals in this report is [ 49 ]. This is test number [ 288 ].

## 1.1 Listing of CAS systems tested

The following are the CAS systems tested:

1. Mathematica 14 (January 9, 2024) on windows 10 pro.
2. Rubi 4.17.3 (Sept 25, 2023) on Mathematica 14 on windows 10m pro.
3. Maple 2024 (March 1, 2024) on windows 10 pro.
4. Maxima 5.47 (June 1, 2023) using Lisp SBCL 2.4.0 on Linux Manjaro 23.1.2 KDE via sagemath 10.3.
5. FriCAS 1.3.10 built with sbcl 2.3.11 (January 10, 2024) on Linux Manjaro 23.1.2 KDE via sagemath 10.3.
6. Giac/Xcas 1.9.0-99 on Linux via sagemath 10.3.
7. Sympy 1.12 using Python 3.11.6 (Nov 14 2023, 09:36:21) [GCC 13.2.1 20230801] on Linux Manjaro 23.1.2 KDE.
8. Mupad using Matlab 2021a with Symbolic Math Toolbox Version 8.7 on windows 10.
9. Reduce CSL rev 6687 (January 9, 2024) on Linux Manjaro 23.1.2 KDE.

Maxima and Fricas and Giac are called using Sagemath. This was done using Sagemath `integrate` command by changing the name of the algorithm to use the different CAS systems.

Sympy was run directly in Python not via sagemath.

Reduce was called directly.

## 1.2 Results

Important note: A number of problems in this test suite have no antiderivative in closed form. This means the antiderivative of these integrals can not be expressed in terms of elementary, special functions or Hypergeometric2F1 functions. RootSum and RootOf are not allowed. If a CAS returns the above integral unevaluated within the time limit, then the result is counted as passed and assigned an A grade.

However, if CAS times out, then it is assigned an F grade even if the integral is not integrable, as this implies CAS could not determine that the integral is not integrable in the time limit.

If a CAS returns an antiderivative to such an integral, it is assigned an A grade automatically and this special result is listed in the introduction section of each individual test report to make it easy to identify as this can be important result to investigate.

The results given in in the table below reflects the above.

System	% solved	% Failed
Rubi	100.00 ( 49 )	0.00 ( 0 )
Mathematica	100.00 ( 49 )	0.00 ( 0 )
Maple	73.47 ( 36 )	26.53 ( 13 )
Fricas	55.10 ( 27 )	44.90 ( 22 )
Giac	51.02 ( 25 )	48.98 ( 24 )
Maxima	30.61 ( 15 )	69.39 ( 34 )
Mupad	24.49 ( 12 )	75.51 ( 37 )
Sympy	24.49 ( 12 )	75.51 ( 37 )
Reduce	4.08 ( 2 )	95.92 ( 47 )

Table 1.1: Percentage solved for each CAS

The table below gives additional break down of the grading of quality of the antiderivatives generated by each CAS. The grading is given using the letters A,B,C and F with A being the best quality. The grading is accomplished by comparing the antiderivative generated with the optimal antiderivatives included in the test suite. The following table describes the meaning of these grades.

grade	description
A	Integral was solved and antiderivative is optimal in quality and leaf size.
B	Integral was solved and antiderivative is optimal in quality but leaf size is larger than twice the optimal antiderivatives leaf size.
C	Integral was solved and antiderivative is non-optimal in quality. This can be due to one or more of the following reasons <ol style="list-style-type: none"> <li>1. antiderivative contains a hypergeometric function and the optimal antiderivative does not.</li> <li>2. antiderivative contains a special function and the optimal antiderivative does not.</li> <li>3. antiderivative contains the imaginary unit and the optimal antiderivative does not.</li> </ol>
F	Integral was not solved. Either the integral was returned unevaluated within the time limit, or it timed out, or CAS hanged or crashed or an exception was raised.

Table 1.2: Description of grading applied to integration result

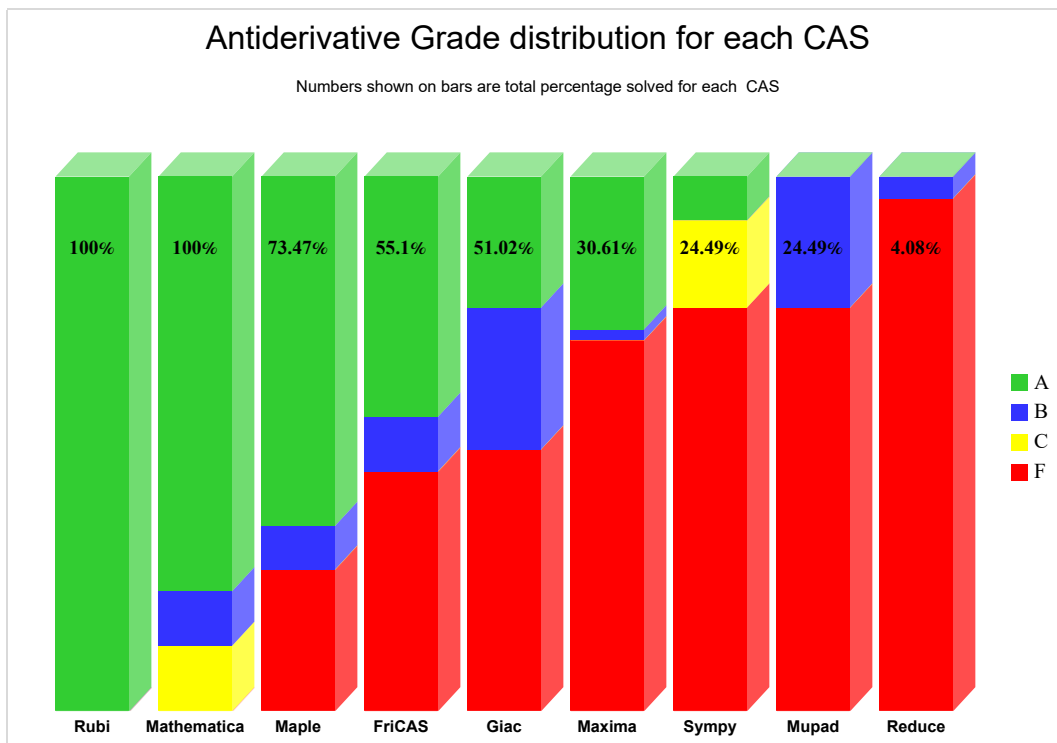
Grading is implemented for all CAS systems. Based on the above, the following table summarizes the grading for this test suite.

System	% A grade	% B grade	% C grade	% F grade
Rubi	100.000	0.000	0.000	0.000
Mathematica	77.551	10.204	12.245	0.000
Maple	65.306	8.163	0.000	26.531
Fricas	44.898	10.204	0.000	44.898
Maxima	28.571	2.041	0.000	69.388
Giac	24.490	26.531	0.000	48.980
Sympy	8.163	0.000	16.327	75.510
Mupad	0.000	24.490	0.000	75.510
Reduce	0.000	4.082	0.000	95.918

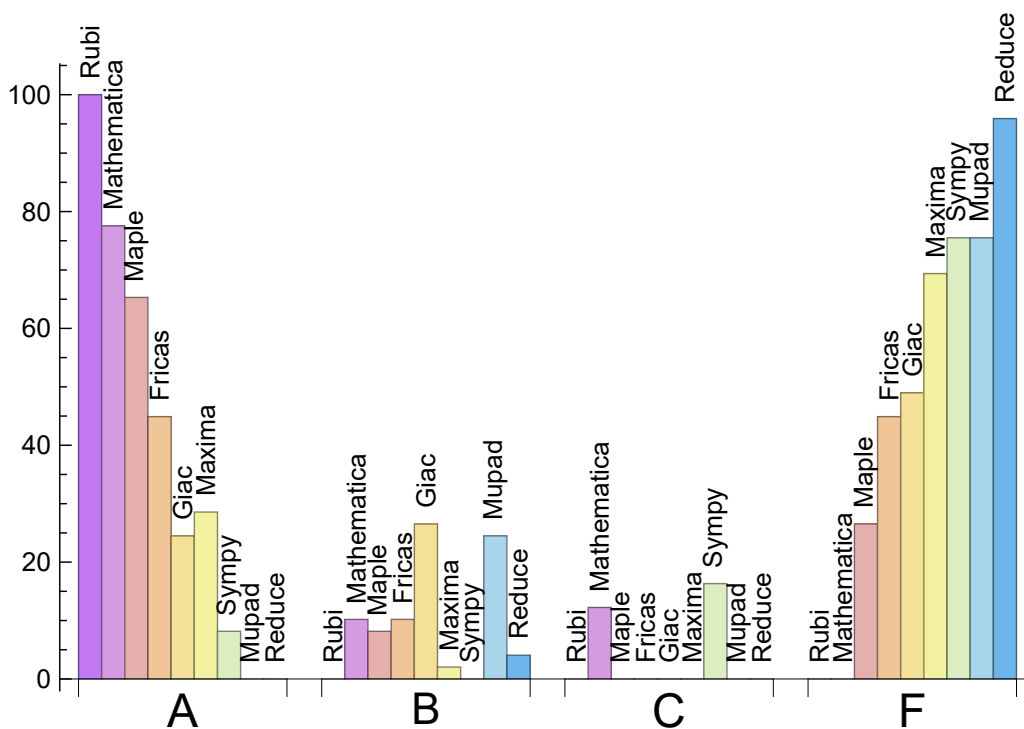
Table 1.3: Antiderivative Grade distribution of each CAS



The following is a Bar chart illustration of the data in the above table.



The figure below compares the grades of the CAS systems.



The following table shows the distribution of the different types of failures for each CAS. There are 3 types failures. The first is when CAS returns the input within the time limit, which means it could not solve it. This is the typical failure and given as **F**.

The second failure is due to time out. CAS could not solve the integral within the 3 minutes time limit which is assigned. This is assigned **F(-1)**.

The third is due to an exception generated, indicated as **F(-2)**. This most likely indicates an interface problem between sagemath and the CAS (applicable only to FriCAS, Maxima and Giac) or it could be an indication of an internal error in the CAS itself. This type of error requires more investigation to determine the cause.

System	Number failed	Percentage normal failure	Percentage time-out failure	Percentage exception failure
Rubi	0	0.00	0.00	0.00
Mathematica	0	0.00	0.00	0.00
Maple	13	100.00	0.00	0.00
Fricas	22	90.91	0.00	9.09
Giac	24	91.67	0.00	8.33
Maxima	34	100.00	0.00	0.00
Mupad	37	0.00	100.00	0.00
Sympy	37	91.89	8.11	0.00
Reduce	47	100.00	0.00	0.00

Table 1.4: Failure statistics for each CAS

### 1.3 Time and leaf size Performance

The table below summarizes the performance of each CAS system in terms of time used and leaf size of results.

Mean size is the average leaf size produced by the CAS (before any normalization). The Normalized mean is relative to the mean size of the optimal anti-derivative given in the input files.

For example, if CAS has **Normalized mean** of 3, then the mean size of its leaf size is 3 times as large as the mean size of the optimal leaf size.

Median size is value of leaf size where half the values are larger than this and half are smaller (before any normalization). i.e. The Middle value.

Similarly the **Normalized median** is relative to the median leaf size of the optimal.

For example, if a CAS has Normalized median of 1.2, then its median is 1.2 as large as the median leaf size of the optimal.

System	Mean time (sec)
Maxima	0.08
Fricas	0.13
Giac	0.15
Reduce	0.23
Mathematica	0.44
Rubi	0.52
Maple	0.62
Mupad	0.99
Sympy	12.43

Table 1.5: Time performance for each CAS

System	Mean size	Normalized mean	Median size	Normalized median
Reduce	33.00	0.88	33.00	0.88
Mupad	41.50	0.91	40.00	0.92
Maxima	50.80	1.20	52.00	1.14
Sympy	67.67	1.56	56.00	1.67
Fricas	121.11	1.34	51.00	0.83
Rubi	131.06	1.05	77.00	1.02
Giac	155.48	1.80	81.00	1.76
Mathematica	169.61	1.50	99.00	1.11
Maple	218.53	1.61	138.50	1.60

Table 1.6: Leaf size performance for each CAS

# 1.4 Performance based on number of rules Rubi used

This section shows how each CAS performed based on the number of rules Rubi needed to solve the same integral. One diagram is given for each CAS.

On the  $y$  axis is the percentage solved which Rubi itself needed the number of rules given the  $x$  axis. These plots show that as more rules are needed then most CAS system percentage of solving decreases which indicates the integral is becoming more complicated to solve.

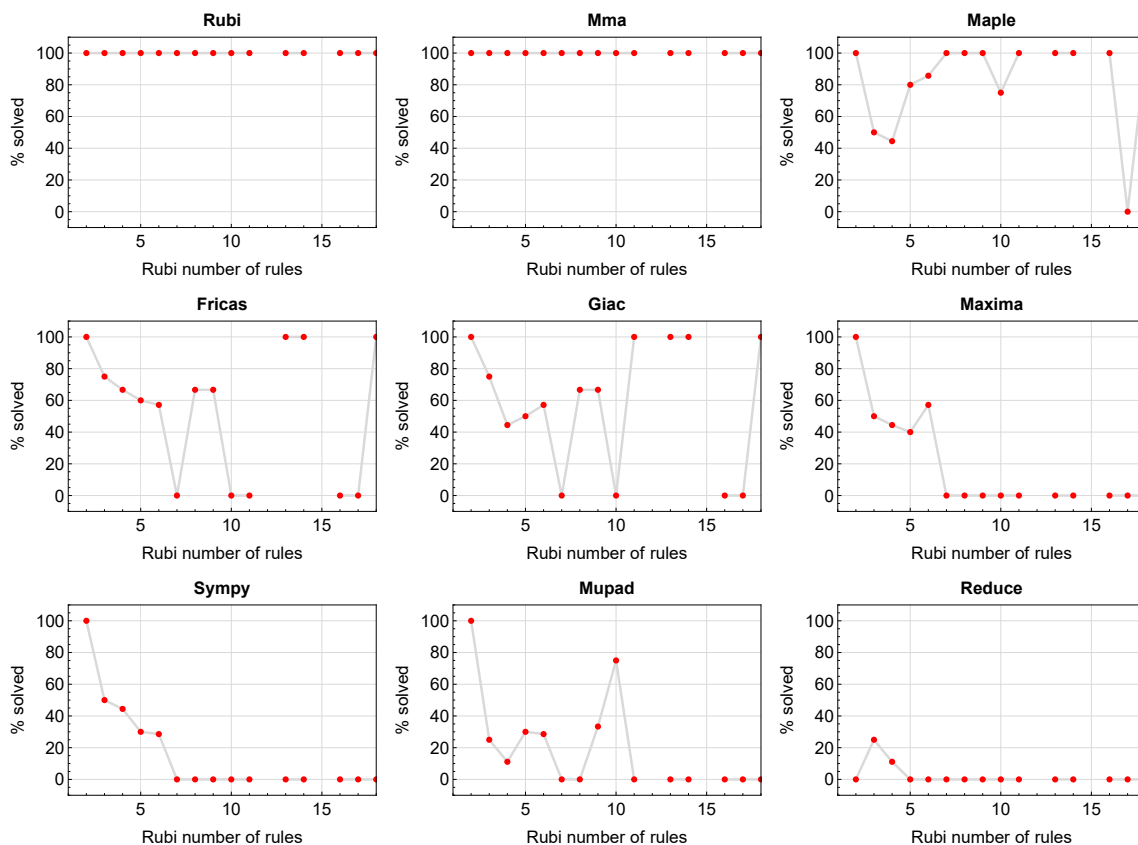


Figure 1.1: Solving statistics per number of Rubi rules used

## 1.5 Performance based on number of steps Rubi used

This section shows how each CAS performed based on the number of steps Rubi needed to solve the same integral. Note that the number of steps Rubi needed can be much higher than the number of rules, as the same rule could be used more than once.

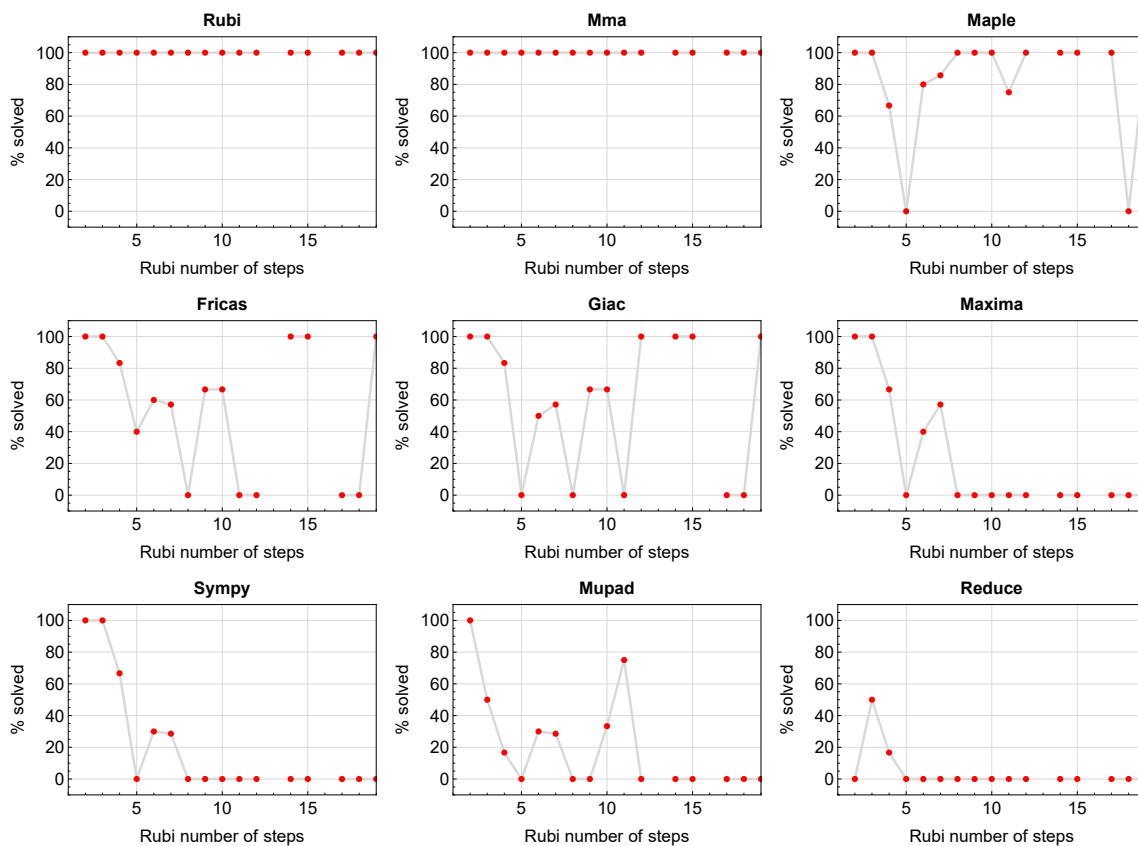


Figure 1.2: Solving statistics per number of Rubi steps used

The above diagram show that the percentage of solved intergals decreases for most CAS systems as the number of steps increases. As expected, for integrals that required less steps by Rubi, CAS systems had more success which indicates the integral was not as hard to solve. As Rubi needed more steps to solve the integral, the solved percentage decreased for most CAS systems which indicates the integral is becoming harder to solve.

## 1.6 Solved integrals histogram based on leaf size of result

The following shows the distribution of solved integrals for each CAS system based on leaf size of the antiderivatives produced by each CAS. It shows that most integrals solved produced leaf size less than about 100 to 150. The bin size used is 40.

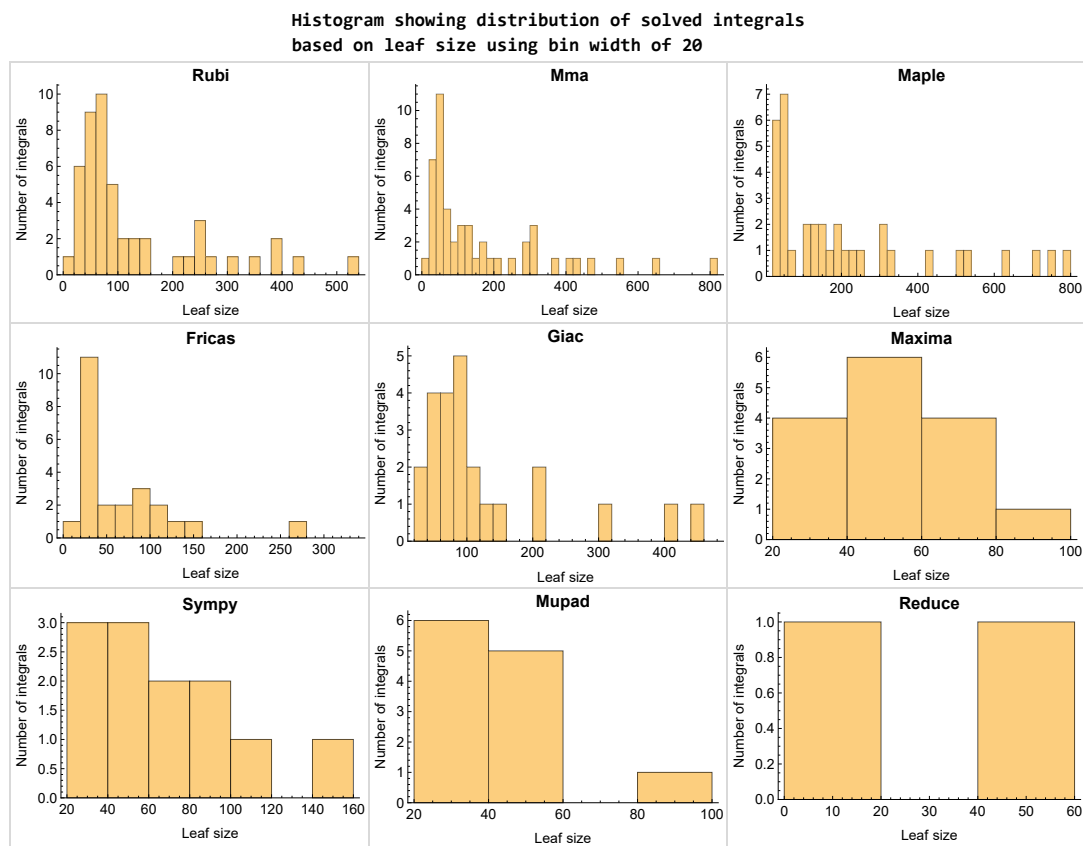


Figure 1.3: Solved integrals based on leaf size distribution

## 1.7 Solved integrals histogram based on CPU time used

The following shows the distribution of solved integrals for each CAS system based on CPU time used in seconds. The bin size used is 0.1 second.

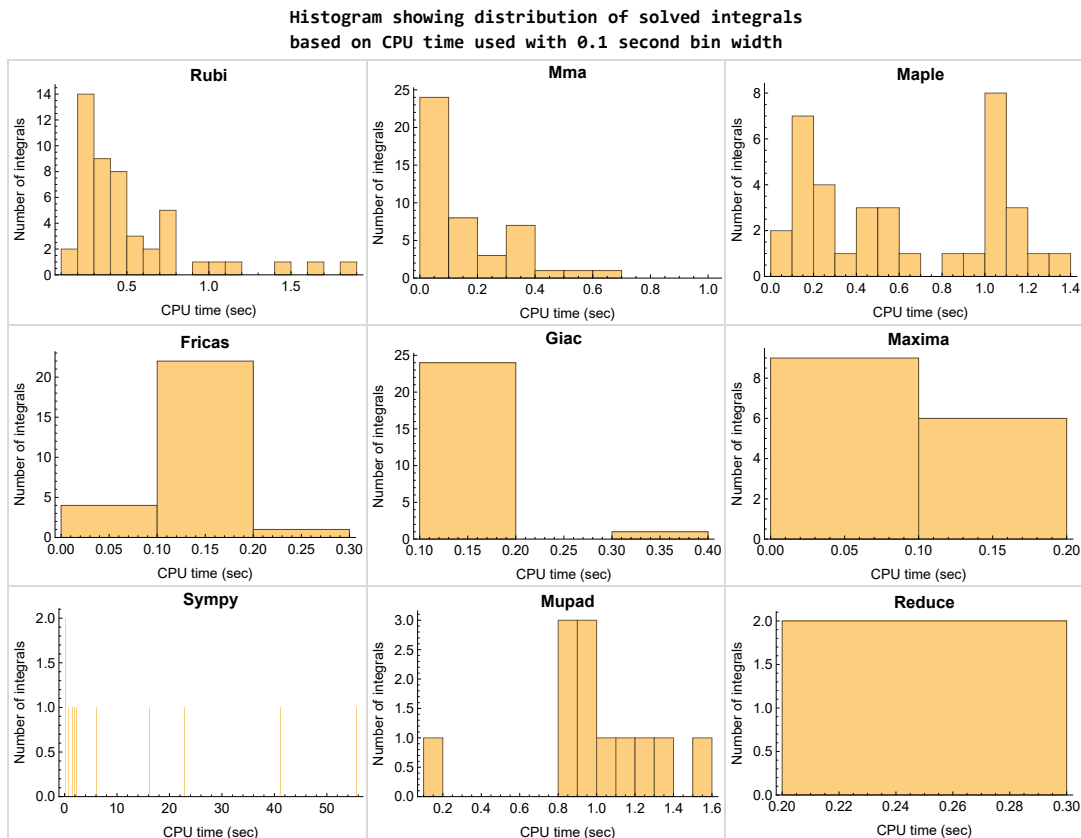


Figure 1.4: Solved integrals histogram based on CPU time used



## 1.8 Leaf size vs. CPU time used

The following shows the relation between the CPU time used to solve an integral and the leaf size of the antiderivative.

The result for Fracas, Maxima and Giac is shifted more to the right than the other CAS system due to the use of sagemath to call them, which causes an initial slight delay in the timing to start the integration due to overhead of starting a new process each time. This should also be taken into account when looking at the timing of these three CAS systems. Direct calls not using sagemath would result in faster timings, but current implementation uses sagemath as this makes testing much easier to do.

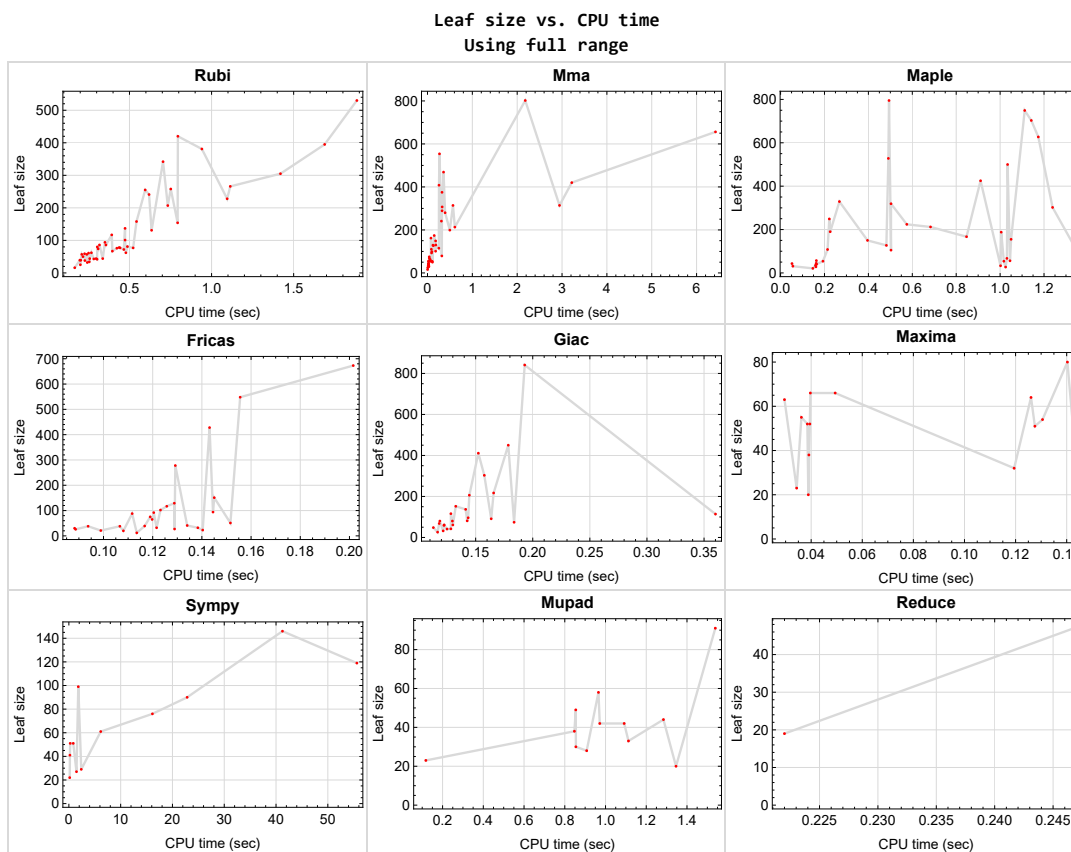


Figure 1.5: Leaf size vs. CPU time. Full range

## 1.9 list of integrals with no known antiderivative

{}

## 1.10 List of integrals solved by CAS but has no known antiderivative

Rubi {}

Mathematica {}

Maple {}

Maxima {}

Fricas {}

Sympy {}

Giac {}

Reduce {}

Mupad {}

## 1.11 list of integrals solved by CAS but failed verification

The following are integrals solved by CAS but the verification phase failed to verify the anti-derivative produced is correct. This does not necessarily mean that the anti-derivative is wrong as additional methods of verification might be needed, or more time is needed (3 minutes time limit was used). These integrals are listed here to make it possible to do further investigation to determine why the result could not be verified.

Rubi {1, 6, 16, 21, 30, 35, 38, 39, 40, 49}

Mathematica {27, 28, 32, 33}

Maple {}

**Maxima** Verification phase not currently implemented.

**Fricas** Verification phase not currently implemented.

**Sympy** Verification phase not currently implemented.

**Giac** Verification phase not currently implemented.

**Reduce** Verification phase not currently implemented.

**Mupad** Verification phase not currently implemented.

## 1.12 Timing

The command `AbsoluteTiming[]` was used in Mathematica to obtain the elapsed time for each integrate call. In Maple, the command `Usage` was used as in the following example

```
cpu_time := Usage(assign ('result_of_int',int(expr,x)),output='realtime')
```

For all other CAS systems, the elapsed time to complete each integral was found by taking the difference between the time after the call completed from the time before the call was made. This was done using Python's `time.time()` call.

All elapsed times shown are in seconds. A time limit of 3 CPU minutes was used for each integral. If the integrate command did not complete within this time limit, the integral was aborted and considered to have failed and assigned an F grade. The time used by failed integrals due to time out was not counted in the final statistics.

## 1.13 Verification

A verification phase was applied on the result of integration for **Rubi** and **Mathematica**.

Future version of this report will implement verification for the other CAS systems. For the integrals whose result was not run through a verification phase, it is assumed that the antiderivative was correct.

Verification phase also had 3 minutes time out. An integral whose result was not verified could still be correct, but further investigation is needed on those integrals. These integrals were marked in the summary table below and also in each integral separate section so they are easy to identify and locate.

## 1.14 Important notes about some of the results

### Important note about Maxima results

Since tests were run in a batch mode, and using an automated script, then any integral where Maxima needed an interactive response from the user to answer a question during the evaluation of the integral will fail.

The exception raised is `ValueError`. Therefore Maxima results is lower than what would result if Maxima was run directly and each question was answered correctly.

The percentage of such failures were not counted for each test file, but for an example, for the `Timofeev` test file, there were about 14 such integrals out of total 705, or about 2 percent. This percentage can be higher or lower depending on the specific input test file.

Such integrals can be identified by looking at the output of the integration in each section for Maxima. The exception message will indicate the cause of error.

Maxima `integrate` was run using SageMath with the following settings set by default

```
'besselexpand : true'  
'display2d : false'  
'domain : complex'  
'keepfloat : true'  
'load(to_poly_solve)'  
'load(simplify_sum)'  
'load(abs_integrate)' 'load(diag)'
```

SageMath automatic loading of Maxima `abs_integrate` was found to cause some problems. So the following code was added to disable this effect.

```
from sage.interfaces.maxima_lib import maxima_lib  
maxima_lib.set('extra_definite_integration_methods', '[]')  
maxima_lib.set('extra_integration_methods', '[]')
```

See <https://ask.sagemath.org/question/43088/integrate-results-that-are-different-from-using-maxima/> for reference.

## Important note about FriCAS result

There were few integrals which failed due to SageMath interface and not because FriCAS system could not do the integration.

These will fail With error `Exception raised: NotImplementedError`.

The number of such cases seems to be very small. About 1 or 2 percent of all integrals. These can be identified by looking at the exception message given in the result.

## Important note about finding leaf size of antiderivative

For Mathematica, Rubi, and Maple, the builtin system function `LeafSize` was used to find the leaf size of each antiderivative.

The other CAS systems (SageMath and Sympy) do not have special builtin function for this purpose at this time. Therefore the leaf size for Fricas and Sympy antiderivative was determined using the following function, thanks to user `slelievre` at [https://ask.sagemath.org/question/57123/could-we-have-a-leaf\\_count-function-in-base-sagemath/](https://ask.sagemath.org/question/57123/could-we-have-a-leaf_count-function-in-base-sagemath/)

```
def tree_size(expr):
    r"""
    Return the tree size of this expression.
    """
    if expr not in SR:
        # deal with lists, tuples, vectors
        return 1 + sum(tree_size(a) for a in expr)
    expr = SR(expr)
    x, aa = expr.operator(), expr.operands()
    if x is None:
        return 1
    else:
        return 1 + sum(tree_size(a) for a in aa)
```

For Sympy, which was called directly from Python, the following code was used to obtain the leafsize of its result

```
try:  
    # 1.7 is a fudge factor since it is low side from actual leaf count  
    leafCount = round(1.7*count_ops(anti))  
  
except Exception as ee:  
    leafCount = 1
```

### Important note about Mupad results

Matlab's symbolic toolbox does not have a leaf count function to measure the size of the antiderivative. Maple was used to determine the leaf size of Mupad output by post processing Mupad result.

Currently no grading of the antiderivative for Mupad is implemented. If it can integrate the problem, it was assigned a B grade automatically as a placeholder. In the future, when grading function is implemented for Mupad, the tests will be rerun again.

The following is an example of using Matlab's symbolic toolbox (Mupad) to solve an integral

```
integrand = evalin(symengine, 'cos(x)*sin(x)')  
the_variable = evalin(symengine, 'x')  
anti = int(integrand, the_variable)
```

Which gives  $\sin(x)^2/2$

# 1.15 Current tree layout of integration tests

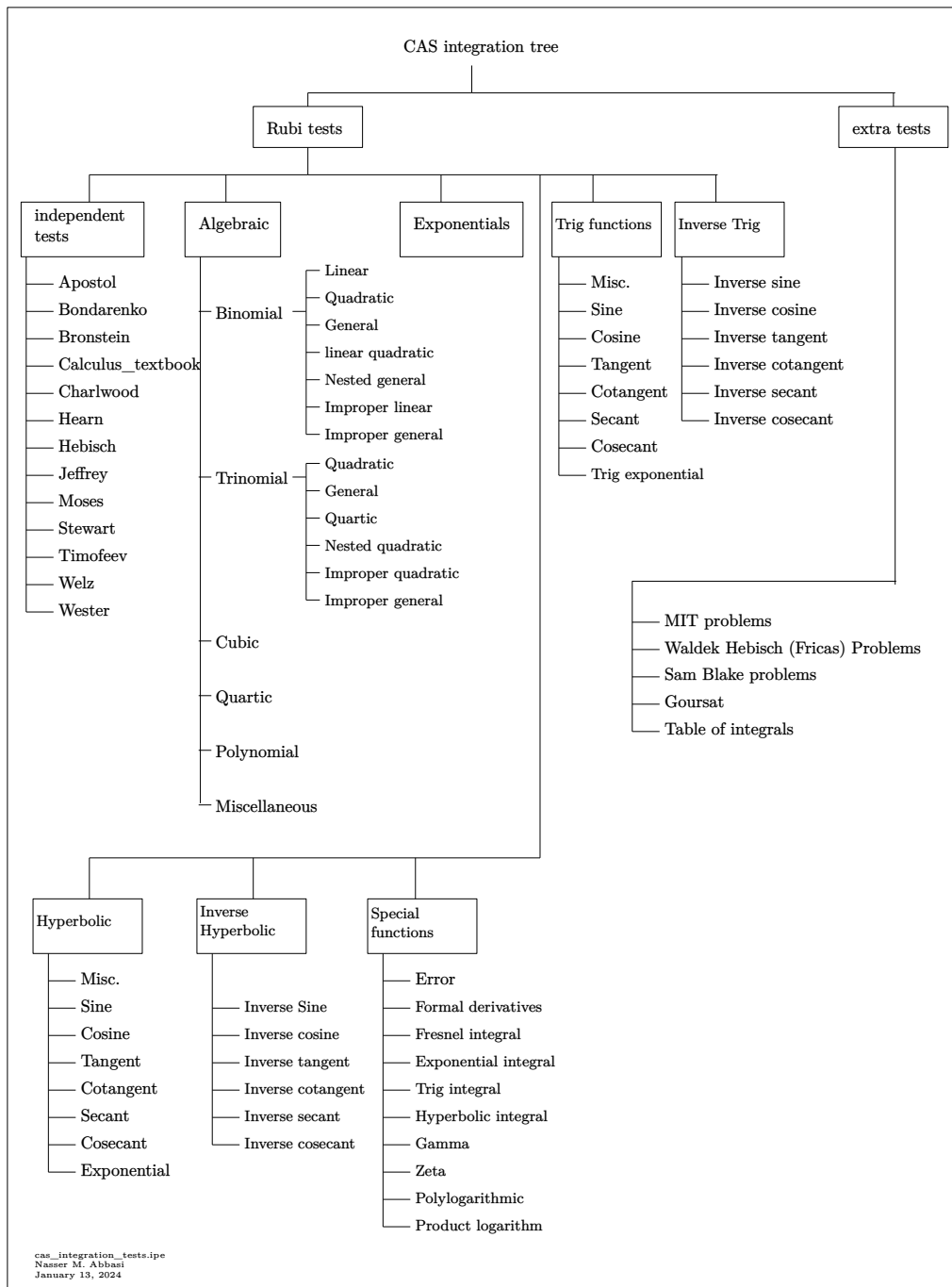
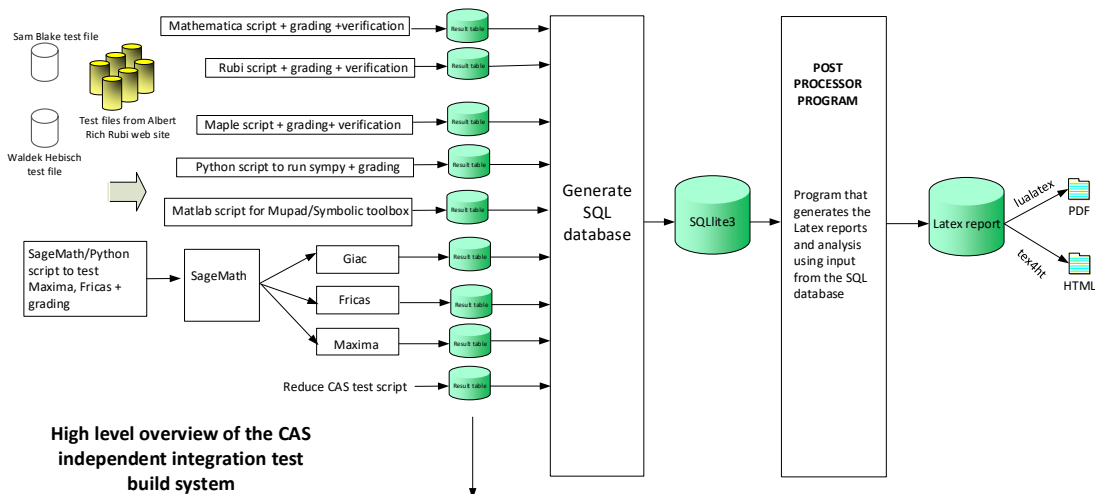


Figure 1.6: CAS integration tests tree

# 1.16 Design of the test system

The following diagram gives a high level view of the current test build system.



**High level overview of the CAS independent integration test build system**

One record (line) per one integral result. The line is CSV comma separated. This is description of each record

1. integer, the problem number.
2. integer. 0 for failed, 1 for passed, -1 for timeout, -2 for CAS specific exception. (this is not the grade field)
3. integer. Leaf size of result.
4. integer. Leaf size of the optimal antiderivative.
5. number. CPU time used to solve this integral. 0 if failed.
6. string. The integral in Latex format
7. string. The input used in CAS own syntax.
8. string. The result (antiderivative) produced by CAS in Latex format
9. string. The optimal antiderivative in Latex format.
10. integer. 0 or 1. Indicates if problem has known antiderivative or not
11. String. The result (antiderivative) in CAS own syntax.
12. String. The grade of the antiderivative. Can be "A", "B", "C", or "E"
13. String. Small string description of why the grade was given.
14. integer. 1 if result was verified or 0 if not verified. (For mma, rubi and maple only)

*The following fields are present only in Rubi Table file*

15. integer. Number of steps used.
16. integer. Number of rules used.
17. integer. Integrand leaf size.
18. real number. Ratio. Field 16 over field 17
19. String of form "{n,n,...}" which is list of the rules used by Rubi
20. String. The optimal antiderivative in Mathematica syntax

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Design note



# CHAPTER 2

## DETAILED SUMMARY TABLES OF RESULTS

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## 2.1 List of integrals sorted by grade for each CAS

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### Rubi

**A grade** { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49 }

**B grade** { }

**C grade** { }

**F normal fail** { }

**F(-1) timedout fail** { }

**F(-2) exception fail** { }

### Mma

**A grade** { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 17, 18, 19, 20, 22, 27, 28, 29, 30, 31, 33, 34, 35, 36, 37, 41, 42, 43, 44, 45, 46, 47, 48, 49 }

**B grade** { 13, 32, 38, 39, 40 }

**C grade** { 16, 21, 23, 24, 25, 26 }

**F normal fail** { }

**F(-1) timedout fail** { }

**F(-2) exception fail** { }

## Maple

**A grade** { 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 27, 28, 29, 30, 32, 33, 34, 35, 38, 40, 49 }

**B grade** { 22, 24, 25, 26 }

**C grade** { }

**F normal fail** { 1, 31, 36, 37, 39, 41, 42, 43, 44, 45, 46, 47, 48 }

**F(-1) timedout fail** { }

**F(-2) exception fail** { }

## Fricas

**A grade** { 2, 3, 4, 5, 7, 8, 9, 10, 11, 14, 15, 17, 18, 19, 20, 24, 25, 26, 45, 46, 47, 48 }

**B grade** { 13, 21, 23, 38, 39 }

**C grade** { }

**F normal fail** { 1, 6, 12, 22, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 41, 42, 43, 44, 49 }

**F(-1) timedout fail** { }

**F(-2) exception fail** { 16, 40 }

## Maxima

**A grade** { 2, 3, 4, 5, 7, 9, 10, 11, 13, 14, 15, 21, 38, 39 }

**B grade** { 8 }

**C grade** { }

**F normal fail** { 1, 6, 12, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49 }

**F(-1) timedout fail** { }

**F(-2) exception fail** { }

## Giac

**A grade** { 7, 8, 9, 10, 11, 14, 15, 20, 23, 39, 45, 49 }

**B grade** { 2, 3, 4, 5, 13, 17, 18, 19, 21, 24, 25, 26, 38 }

**C grade** { }

**F normal fail** { 1, 12, 16, 22, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 40, 41, 42, 43, 44, 46, 47, 48 }

**F(-1) timedout fail** { }

**F(-2) exception fail** { 6, 27 }

## Mupad

**A grade** { }

**B grade** { 1, 5, 6, 7, 10, 11, 12, 13, 21, 38, 39, 40 }

**C grade** { }

**F normal fail** { }

**F(-1) timedout fail** { 2, 3, 4, 8, 9, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 41, 42, 43, 44, 45, 46, 47, 48, 49 }

**F(-2) exception fail** { }

## Sympy

**A grade** { 9, 10, 11, 15 }

**B grade** { }

**C grade** { 2, 3, 4, 5, 7, 8, 13, 14 }

**F normal fail** { 1, 6, 12, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49 }

**F(-1) timedout fail** { 22, 38, 39 }

**F(-2) exception fail** { }

## Reduce

**A grade** { }

**B grade** { 10, 11 }

**C grade** { }

**F normal fail** { 1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49 }

**F(-1) timedout fail** { }

**F(-2) exception fail** { }

## 2.2 Detailed conclusion table per each integral for all CAS systems

Detailed conclusion table per each integral is given by the table below. The elapsed time is in seconds. For failed result it is given as **F(-1)** if the failure was due to timeout. It is given as **F(-2)** if the failure was due to an exception being raised, which could indicate a bug in the system. If the failure was due to integral not being evaluated within the time limit, then it is given as **F**.

In this table, the column **N.S.** means **normalized size** and is defined as  $\frac{\text{antiderivative leaf size}}{\text{optimal antiderivative leaf size}}$ . To make the table fit the page, the name **Mathematica** was abbreviated to **MMA**.

Problem 1	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	B
verified	N/A	<b>No</b>	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	62	72	56	0	0	0	0	0	12	58
N.S.	1	1.16	0.90	0.00	0.00	0.00	0.00	0.00	0.19	0.94
time (sec)	N/A	0.466	0.034	0.000	0.000	0.000	0.000	0.000	0.218	0.965

Problem 2	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	C	B	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	58	59	40	40	66	32	119	152	9	0
N.S.	1	1.02	0.69	0.69	1.14	0.55	2.05	2.62	0.16	0.00
time (sec)	N/A	0.227	0.019	0.164	0.040	0.122	55.606	0.133	0.236	0.000

Problem 3	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	C	B	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	47	50	35	35	52	27	90	116	9	0
N.S.	1	1.06	0.74	0.74	1.11	0.57	1.91	2.47	0.19	0.00
time (sec)	N/A	0.217	0.016	0.159	0.039	0.129	22.838	0.128	0.245	0.000

Problem 4	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	C	B	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	36	39	27	28	38	20	61	80	7	0
N.S.	1	1.08	0.75	0.78	1.06	0.56	1.69	2.22	0.19	0.00
time (sec)	N/A	0.198	0.016	0.161	0.039	0.108	6.149	0.130	0.243	0.000

Problem 5	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	C	B	<b>F</b>	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	16	16	16	21	20	12	29	41	5	20
N.S.	1	1.00	1.00	1.31	1.25	0.75	1.81	2.56	0.31	1.25
time (sec)	N/A	0.167	0.002	0.147	0.039	0.114	2.393	0.125	0.240	1.346

Problem 6	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-2)</b>	<b>F</b>	B
verified	N/A	<b>No</b>	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	56	62	54	105	0	0	0	0	9	42
N.S.	1	1.11	0.96	1.88	0.00	0.00	0.00	0.00	0.16	0.75
time (sec)	N/A	0.478	0.027	0.503	0.000	0.000	0.000	0.000	0.233	1.092

Problem 7	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	C	A	<b>F</b>	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	38	39	32	44	51	21	76	32	9	28
N.S.	1	1.03	0.84	1.16	1.34	0.55	2.00	0.84	0.24	0.74
time (sec)	N/A	0.204	0.020	0.162	0.128	0.099	16.131	0.122	0.234	0.907

Problem 8	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	B	A	C	A	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	54	57	55	57	80	30	146	59	9	0
N.S.	1	1.06	1.02	1.06	1.48	0.56	2.70	1.09	0.17	0.00
time (sec)	N/A	0.209	0.033	0.163	0.140	0.088	41.243	0.122	0.243	0.000

Problem 9	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	A	A	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	56	62	42	56	54	39	51	68	12	0
N.S.	1	1.11	0.75	1.00	0.96	0.70	0.91	1.21	0.21	0.00
time (sec)	N/A	0.268	0.025	1.045	0.131	0.117	0.256	0.118	0.224	0.000

Problem 10	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	A	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	47	57	44	67	46	38	41	48	47	38
N.S.	1	1.21	0.94	1.43	0.98	0.81	0.87	1.02	1.00	0.81
time (sec)	N/A	0.240	0.018	1.031	0.143	0.107	0.202	0.113	0.247	0.847

Problem 11	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	A	A	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	25	25	25	27	23	26	22	26	19	23
N.S.	1	1.00	1.00	1.08	0.92	1.04	0.88	1.04	0.76	0.92
time (sec)	N/A	0.201	0.007	1.025	0.034	0.089	0.162	0.117	0.222	0.118



Problem 12	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	F	F	F	F	F	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	59	67	54	125	0	0	0	0	12	49
N.S.	1	1.14	0.92	2.12	0.00	0.00	0.00	0.00	0.20	0.83
time (sec)	N/A	0.396	0.028	1.337	0.000	0.000	0.000	0.000	0.224	0.854

Problem 13	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	B	A	A	B	C	B	F	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	32	32	93	31	52	65	27	61	12	30
N.S.	1	1.00	2.91	0.97	1.62	2.03	0.84	1.91	0.38	0.94
time (sec)	N/A	0.243	0.091	0.056	0.039	0.120	1.471	0.130	0.253	0.854

Problem 14	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	C	A	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	38	38	35	33	32	38	51	61	12	0
N.S.	1	1.00	0.92	0.87	0.84	1.00	1.34	1.61	0.32	0.00
time (sec)	N/A	0.228	0.019	1.002	0.119	0.094	0.847	0.122	0.223	0.000

Problem 15	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	A	A	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	60	61	69	54	64	92	99	80	12	0
N.S.	1	1.02	1.15	0.90	1.07	1.53	1.65	1.33	0.20	0.00
time (sec)	N/A	0.251	0.040	1.017	0.126	0.120	1.819	0.119	0.241	0.000

Problem 16	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	C	A	<b>F</b>	<b>F(-2)</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	<b>No</b>	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	69	81	63	155	0	0	0	0	12	0
N.S.	1	1.17	0.91	2.25	0.00	0.00	0.00	0.00	0.17	0.00
time (sec)	N/A	0.488	0.059	1.050	0.000	0.000	0.000	0.000	0.227	0.000

Problem 17	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	<b>F</b>	A	<b>F</b>	B	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	197	207	174	329	0	151	0	411	12	0
N.S.	1	1.05	0.88	1.67	0.00	0.77	0.00	2.09	0.06	0.00
time (sec)	N/A	0.733	0.150	0.268	0.000	0.145	0.000	0.152	0.231	0.000

Problem 18	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	<b>F</b>	A	<b>F</b>	B	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	155	158	149	249	0	129	0	303	12	0
N.S.	1	1.02	0.96	1.61	0.00	0.83	0.00	1.95	0.08	0.00
time (sec)	N/A	0.543	0.181	0.222	0.000	0.129	0.000	0.158	0.253	0.000

Problem 19	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	<b>F</b>	A	<b>F</b>	B	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	116	117	129	190	0	117	0	206	12	0
N.S.	1	1.01	1.11	1.64	0.00	1.01	0.00	1.78	0.10	0.00
time (sec)	N/A	0.392	0.121	0.226	0.000	0.126	0.000	0.144	0.231	0.000

Problem 20	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	<b>F</b>	A	<b>F</b>	A	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	79	78	110	108	0	102	0	137	10	0
N.S.	1	0.99	1.39	1.37	0.00	1.29	0.00	1.73	0.13	0.00
time (sec)	N/A	0.437	0.086	0.214	0.000	0.123	0.000	0.141	0.238	0.000

Problem 21	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	C	A	A	B	<b>F</b>	B	<b>F</b>	B
verified	N/A	<b>No</b>	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	36	34	469	43	55	75	0	81	8	33
N.S.	1	0.94	13.03	1.19	1.53	2.08	0.00	2.25	0.22	0.92
time (sec)	N/A	0.255	0.358	0.052	0.036	0.119	0.000	0.143	0.229	1.113

Problem 22	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	B	<b>F</b>	<b>F</b>	<b>F(-1)</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	210	266	375	627	0	0	0	0	12	0
N.S.	1	1.27	1.79	2.99	0.00	0.00	0.00	0.00	0.06	0.00
time (sec)	N/A	1.114	0.324	1.174	0.000	0.000	0.000	0.000	0.226	0.000

Problem 23	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	C	A	<b>F</b>	B	<b>F</b>	A	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	69	76	115	127	0	278	0	96	12	0
N.S.	1	1.10	1.67	1.84	0.00	4.03	0.00	1.39	0.17	0.00
time (sec)	N/A	0.423	0.253	0.482	0.000	0.129	0.000	0.143	0.225	0.000

Problem 24	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	C	B	<b>F</b>	A	<b>F</b>	B	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	123	154	199	319	0	428	0	217	12	0
N.S.	1	1.25	1.62	2.59	0.00	3.48	0.00	1.76	0.10	0.00
time (sec)	N/A	0.793	0.498	0.503	0.000	0.143	0.000	0.166	0.234	0.000

Problem 25	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	C	B	<b>F</b>	A	<b>F</b>	B	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	180	228	241	528	0	548	0	450	12	0
N.S.	1	1.27	1.34	2.93	0.00	3.04	0.00	2.50	0.07	0.00
time (sec)	N/A	1.095	0.310	0.491	0.000	0.156	0.000	0.179	0.247	0.000

Problem 26	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	C	B	<b>F</b>	A	<b>F</b>	B	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	239	305	307	795	0	673	0	841	12	0
N.S.	1	1.28	1.28	3.33	0.00	2.82	0.00	3.52	0.05	0.00
time (sec)	N/A	1.420	0.327	0.494	0.000	0.202	0.000	0.193	0.281	0.000

Problem 27	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-2)</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	<b>No</b>	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	366	342	420	703	0	0	0	0	14	0
N.S.	1	0.93	1.15	1.92	0.00	0.00	0.00	0.00	0.04	0.00
time (sec)	N/A	0.704	3.217	1.142	0.000	0.000	0.000	0.000	0.228	0.000

Problem 28	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	<b>No</b>	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	272	255	314	500	0	0	0	0	14	0
N.S.	1	0.94	1.15	1.84	0.00	0.00	0.00	0.00	0.05	0.00
time (sec)	N/A	0.596	2.944	1.034	0.000	0.000	0.000	0.000	0.231	0.000

Problem 29	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	145	137	213	212	0	0	0	0	12	0
N.S.	1	0.94	1.47	1.46	0.00	0.00	0.00	0.00	0.08	0.00
time (sec)	N/A	0.473	0.610	0.683	0.000	0.000	0.000	0.000	0.243	0.000

Problem 30	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	<b>No</b>	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	86	77	99	150	0	0	0	0	10	0
N.S.	1	0.90	1.15	1.74	0.00	0.00	0.00	0.00	0.12	0.00
time (sec)	N/A	0.443	0.097	0.396	0.000	0.000	0.000	0.000	0.218	0.000

Problem 31	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	324	395	408	0	0	0	0	0	14	0
N.S.	1	1.22	1.26	0.00	0.00	0.00	0.00	0.00	0.04	0.00
time (sec)	N/A	1.689	0.257	0.000	0.000	0.000	0.000	0.000	0.220	0.000

Problem 32	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	B	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	<b>No</b>	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	254	258	802	302	0	0	0	0	14	0
N.S.	1	1.02	3.16	1.19	0.00	0.00	0.00	0.00	0.06	0.00
time (sec)	N/A	0.752	2.180	1.239	0.000	0.000	0.000	0.000	0.231	0.000

Problem 33	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	<b>No</b>	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	464	420	656	749	0	0	0	0	14	0
N.S.	1	0.91	1.41	1.61	0.00	0.00	0.00	0.00	0.03	0.00
time (sec)	N/A	0.795	6.421	1.112	0.000	0.000	0.000	0.000	0.213	0.000

Problem 34	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	264	241	314	425	0	0	0	0	12	0
N.S.	1	0.91	1.19	1.61	0.00	0.00	0.00	0.00	0.05	0.00
time (sec)	N/A	0.619	0.568	0.911	0.000	0.000	0.000	0.000	0.208	0.000

Problem 35	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	<b>No</b>	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	140	131	162	224	0	0	0	0	10	0
N.S.	1	0.94	1.16	1.60	0.00	0.00	0.00	0.00	0.07	0.00
time (sec)	N/A	0.635	0.077	0.575	0.000	0.000	0.000	0.000	0.221	0.000

Problem 36	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	448	530	554	0	0	0	0	0	14	0
N.S.	1	1.18	1.24	0.00	0.00	0.00	0.00	0.00	0.03	0.00
time (sec)	N/A	1.885	0.268	0.000	0.000	0.000	0.000	0.000	0.234	0.000

Problem 37	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	378	381	289	0	0	0	0	0	14	0
N.S.	1	1.01	0.76	0.00	0.00	0.00	0.00	0.00	0.04	0.00
time (sec)	N/A	0.941	0.325	0.000	0.000	0.000	0.000	0.000	0.229	0.000

Problem 38	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	B	A	A	B	<b>F(-1)</b>	B	<b>F</b>	B
verified	N/A	<b>No</b>	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	48	44	127	54	63	88	0	91	14	44
N.S.	1	0.92	2.65	1.12	1.31	1.83	0.00	1.90	0.29	0.92
time (sec)	N/A	0.297	0.132	0.193	0.030	0.112	0.000	0.164	0.210	1.284

Problem 39	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	B	<b>F</b>	A	B	<b>F(-1)</b>	A	<b>F</b>	B
verified	N/A	<b>No</b>	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	48	44	130	0	66	94	0	74	17	42
N.S.	1	0.92	2.71	0.00	1.38	1.96	0.00	1.54	0.35	0.88
time (sec)	N/A	0.337	0.185	0.000	0.049	0.145	0.000	0.184	0.225	0.972

Problem 40	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	B	A	<b>F</b>	<b>F(-2)</b>	<b>F</b>	<b>F</b>	<b>F</b>	B
verified	N/A	<b>No</b>	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	85	101	280	188	0	0	0	0	12	91
N.S.	1	1.19	3.29	2.21	0.00	0.00	0.00	0.00	0.14	1.07
time (sec)	N/A	0.473	0.393	1.005	0.000	0.000	0.000	0.000	0.227	1.539

Problem 41	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	95	94	79	0	0	0	0	0	12	0
N.S.	1	0.99	0.83	0.00	0.00	0.00	0.00	0.00	0.13	0.00
time (sec)	N/A	0.351	0.320	0.000	0.000	0.000	0.000	0.000	0.243	0.000

Problem 42	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	87	86	101	0	0	0	0	0	10	0
N.S.	1	0.99	1.16	0.00	0.00	0.00	0.00	0.00	0.11	0.00
time (sec)	N/A	0.357	0.181	0.000	0.000	0.000	0.000	0.000	0.234	0.000

Problem 43	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	87	86	54	0	0	0	0	0	8	0
N.S.	1	0.99	0.62	0.00	0.00	0.00	0.00	0.00	0.09	0.00
time (sec)	N/A	0.317	0.072	0.000	0.000	0.000	0.000	0.000	0.224	0.000



Problem 44	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	43	43	75	0	0	0	0	0	12	0
N.S.	1	1.00	1.74	0.00	0.00	0.00	0.00	0.00	0.28	0.00
time (sec)	N/A	0.283	0.039	0.000	0.000	0.000	0.000	0.000	0.219	0.000

Problem 45	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	A	<b>F</b>	A	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	39	44	33	0	0	23	0	42	12	0
N.S.	1	1.13	0.85	0.00	0.00	0.59	0.00	1.08	0.31	0.00
time (sec)	N/A	0.258	0.028	0.000	0.000	0.140	0.000	0.128	0.239	0.000

Problem 46	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	41	42	30	0	0	32	0	0	12	0
N.S.	1	1.02	0.73	0.00	0.00	0.78	0.00	0.00	0.29	0.00
time (sec)	N/A	0.303	0.031	0.000	0.000	0.138	0.000	0.000	0.284	0.000

Problem 47	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	<b>F</b>	<b>F</b>	A	<b>F</b>	<b>F</b>	<b>F</b>	<b>F(-1)</b>
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	84	80	54	0	0	41	0	0	12	0
N.S.	1	0.95	0.64	0.00	0.00	0.49	0.00	0.00	0.14	0.00
time (sec)	N/A	0.304	0.105	0.000	0.000	0.134	0.000	0.000	0.260	0.000

Problem 48	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	F	F	A	F	F	F	F(-1)
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	81	74	50	0	0	51	0	0	12	0
N.S.	1	0.91	0.62	0.00	0.00	0.63	0.00	0.00	0.15	0.00
time (sec)	N/A	0.309	0.111	0.000	0.000	0.152	0.000	0.000	0.277	0.000

Problem 49	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	F	F	F	A	F	F(-1)
verified	N/A	No	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	69	77	59	167	0	0	0	114	21	0
N.S.	1	1.12	0.86	2.42	0.00	0.00	0.00	1.65	0.30	0.00
time (sec)	N/A	0.522	0.059	0.847	0.000	0.000	0.000	0.360	0.309	0.000

## 2.3 Detailed conclusion table specific for Rubi results

The following table is specific to Rubi only. It gives additional statistics for each integral. the column **steps** is the number of steps used by Rubi to obtain the antiderivative. The **rules** column is the number of unique rules used. The **integrand size** column is the leaf size of the integrand. Finally the ratio  $\frac{\text{number of rules}}{\text{integrand size}}$  is also given. The larger this ratio is, the harder the integral is to solve. In this test file, problem number [26] had the largest ratio of [1.80000000000000004]

Table 2.1: Rubi specific breakdown of results for each integral

#	grade	number of steps used	number of unique rules	normalized antiderivative leaf size	integrand leaf size	$\frac{\text{number of rules}}{\text{integrand leaf size}}$
1	A	11	10	1.16	10	1.000
2	A	4	4	1.02	10	0.400
3	A	4	4	1.06	10	0.400
4	A	4	4	1.08	8	0.500
5	A	2	2	1.00	6	0.333
6	A	11	10	1.11	10	1.000
7	A	6	5	1.03	10	0.500
8	A	7	6	1.06	10	0.600
9	A	6	5	1.11	10	0.500
10	A	4	4	1.21	8	0.500
11	A	3	3	1.00	6	0.500
12	A	10	9	1.14	10	0.900
13	A	6	5	1.00	10	0.500
14	A	3	3	1.00	10	0.300
15	A	7	6	1.02	10	0.600
16	A	11	10	1.17	10	1.000
17	A	10	9	1.05	10	0.900
18	A	9	8	1.02	10	0.800
19	A	6	5	1.01	10	0.500
20	A	10	9	0.99	8	1.125
21	A	6	5	0.94	6	0.833

Continued on next page

Table 2.1 – continued from previous page

#	grade	number of steps used	number of unique rules	normalized antiderivative leaf size	integrand leaf size	$\frac{\text{number of rules}}{\text{integrand leaf size}}$
22	A	17	16	1.27	10	1.600
23	A	9	8	1.10	10	0.800
24	A	14	13	1.25	10	1.300
25	A	15	14	1.27	10	1.400
26	A	19	18	1.28	10	1.800
27	A	7	6	0.93	12	0.500
28	A	6	5	0.94	12	0.417
29	A	7	6	0.94	10	0.600
30	A	8	7	0.90	8	0.875
31	A	18	17	1.22	12	1.417
32	A	6	5	1.02	12	0.417
33	A	6	5	0.91	12	0.417
34	A	7	6	0.91	10	0.600
35	A	9	8	0.94	8	1.000
36	A	18	17	1.18	12	1.417
37	A	6	5	1.01	12	0.417
38	A	7	6	0.92	12	0.500
39	A	7	6	0.92	14	0.429
40	A	11	10	1.19	10	1.000
41	A	5	4	0.99	10	0.400
42	A	5	4	0.99	8	0.500
43	A	4	3	0.99	6	0.500
44	A	5	4	1.00	10	0.400
45	A	4	3	1.13	10	0.300
46	A	6	5	1.02	10	0.500
47	A	5	4	0.95	10	0.400
48	A	5	4	0.91	10	0.400
49	A	12	11	1.12	19	0.579

# CHAPTER 3

## LISTING OF INTEGRALS

3.1	$\int \frac{\csc^{-1}(ax^5)}{x} dx$ . . . . .	46
3.2	$\int x^3 \csc^{-1}(\sqrt{x}) dx$ . . . . .	53
3.3	$\int x^2 \csc^{-1}(\sqrt{x}) dx$ . . . . .	59
3.4	$\int x \csc^{-1}(\sqrt{x}) dx$ . . . . .	65
3.5	$\int \csc^{-1}(\sqrt{x}) dx$ . . . . .	71
3.6	$\int \frac{\csc^{-1}(\sqrt{x})}{x} dx$ . . . . .	76
3.7	$\int \frac{\csc^{-1}(\sqrt{x})}{x^2} dx$ . . . . .	83
3.8	$\int \frac{\csc^{-1}(\sqrt{x})}{x^3} dx$ . . . . .	89
3.9	$\int x^2 \csc^{-1}\left(\frac{a}{x}\right) dx$ . . . . .	96
3.10	$\int x \csc^{-1}\left(\frac{a}{x}\right) dx$ . . . . .	102
3.11	$\int \csc^{-1}\left(\frac{a}{x}\right) dx$ . . . . .	108
3.12	$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x} dx$ . . . . .	113
3.13	$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^2} dx$ . . . . .	119
3.14	$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^3} dx$ . . . . .	125
3.15	$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^4} dx$ . . . . .	130
3.16	$\int \frac{\csc^{-1}(ax^n)}{x} dx$ . . . . .	136
3.17	$\int x^4 \csc^{-1}(a + bx) dx$ . . . . .	143
3.18	$\int x^3 \csc^{-1}(a + bx) dx$ . . . . .	151
3.19	$\int x^2 \csc^{-1}(a + bx) dx$ . . . . .	159
3.20	$\int x \csc^{-1}(a + bx) dx$ . . . . .	166
3.21	$\int \csc^{-1}(a + bx) dx$ . . . . .	173
3.22	$\int \frac{\csc^{-1}(a+bx)}{x} dx$ . . . . .	179
3.23	$\int \frac{\csc^{-1}(a+bx)}{x^2} dx$ . . . . .	189
3.24	$\int \frac{\csc^{-1}(a+bx)}{x^3} dx$ . . . . .	197
3.25	$\int \frac{\csc^{-1}(a+bx)}{x^4} dx$ . . . . .	206

3.26	$\int \frac{\csc^{-1}(a+bx)}{x^5} dx$	217
3.27	$\int x^3 \csc^{-1}(a+bx)^2 dx$	230
3.28	$\int x^2 \csc^{-1}(a+bx)^2 dx$	238
3.29	$\int x \csc^{-1}(a+bx)^2 dx$	245
3.30	$\int \csc^{-1}(a+bx)^2 dx$	251
3.31	$\int \frac{\csc^{-1}(a+bx)^2}{x} dx$	257
3.32	$\int \frac{\csc^{-1}(a+bx)^2}{x^2} dx$	268
3.33	$\int x^2 \csc^{-1}(a+bx)^3 dx$	275
3.34	$\int x \csc^{-1}(a+bx)^3 dx$	284
3.35	$\int \csc^{-1}(a+bx)^3 dx$	291
3.36	$\int \frac{\csc^{-1}(a+bx)^3}{x} dx$	299
3.37	$\int \frac{\csc^{-1}(a+bx)^3}{x^2} dx$	312
3.38	$\int x^3 \csc^{-1}(a+bx^4) dx$	319
3.39	$\int x^{-1+n} \csc^{-1}(a+bx^n) dx$	326
3.40	$\int \csc^{-1}(ce^{a+bx}) dx$	332
3.41	$\int e^{\csc^{-1}(ax)} x^2 dx$	340
3.42	$\int e^{\csc^{-1}(ax)} x dx$	345
3.43	$\int e^{\csc^{-1}(ax)} dx$	350
3.44	$\int \frac{e^{\csc^{-1}(ax)}}{x} dx$	355
3.45	$\int \frac{e^{\csc^{-1}(ax)}}{x^2} dx$	360
3.46	$\int \frac{e^{\csc^{-1}(ax)}}{x^3} dx$	365
3.47	$\int \frac{e^{\csc^{-1}(ax)}}{x^4} dx$	370
3.48	$\int \frac{e^{\csc^{-1}(ax)}}{x^5} dx$	375
3.49	$\int \frac{\csc^{-1}(a+bx)}{\frac{a}{b}+dx} dx$	380

### 3.1 $\int \frac{\csc^{-1}(ax^5)}{x} dx$

Optimal result	46
Mathematica [A] (verified)	46
Rubi [A] (warning: unable to verify)	47
Maple [F]	50
Fricas [F]	50
Sympy [F]	50
Maxima [F]	51
Giac [F]	51
Mupad [B] (verification not implemented)	51
Reduce [F]	52

#### Optimal result

Integrand size = 10, antiderivative size = 62

$$\int \frac{\csc^{-1}(ax^5)}{x} dx = \frac{1}{10}i \csc^{-1}(ax^5)^2 - \frac{1}{5} \csc^{-1}(ax^5) \log\left(1 - e^{2i \csc^{-1}(ax^5)}\right) + \frac{1}{10}i \text{PolyLog}\left(2, e^{2i \csc^{-1}(ax^5)}\right)$$

output

```
1/10*I*arccsc(a*x^5)^2-1/5*arccsc(a*x^5)*ln(1-(I/a/x^5+(1-1/a^2/x^10)^(1/2))
)^2)+1/10*I*polylog(2,(I/a/x^5+(1-1/a^2/x^10)^(1/2))^2)
```

#### Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 56, normalized size of antiderivative = 0.90

$$\int \frac{\csc^{-1}(ax^5)}{x} dx = \frac{1}{10}i \left( \csc^{-1}(ax^5) \left( \csc^{-1}(ax^5) + 2i \log\left(1 - e^{2i \csc^{-1}(ax^5)}\right)\right) + \text{PolyLog}\left(2, e^{2i \csc^{-1}(ax^5)}\right) \right)$$

input

```
Integrate[ArcCsc[a*x^5]/x,x]
```

output

```
(I/10)*(ArcCsc[a*x^5]*(ArcCsc[a*x^5] + (2*I)*Log[1 - E^((2*I)*ArcCsc[a*x^5]
)]) + PolyLog[2, E^((2*I)*ArcCsc[a*x^5])])
```

**Rubi [A] (warning: unable to verify)**

Time = 0.47 (sec) , antiderivative size = 72, normalized size of antiderivative = 1.16, number of steps used = 11, number of rules used = 10,  $\frac{\text{number of rules}}{\text{integrand size}} = 1.000$ , Rules used = {7282, 5742, 5136, 3042, 25, 4200, 25, 2620, 2715, 2838}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{\csc^{-1}(ax^5)}{x} dx \\
 & \quad \downarrow \text{7282} \\
 & \frac{1}{5} \int \frac{\csc^{-1}(ax^5)}{x^5} dx^5 \\
 & \quad \downarrow \text{5742} \\
 & -\frac{1}{5} \int \frac{\arcsin\left(\frac{1}{ax^5}\right)}{x^5} d\frac{1}{x^5} \\
 & \quad \downarrow \text{5136} \\
 & -\frac{1}{5} \int a \sqrt{1 - \frac{1}{a^2 x^{10}}} x^5 \arcsin\left(\frac{1}{ax^5}\right) d\arcsin\left(\frac{1}{ax^5}\right) \\
 & \quad \downarrow \text{3042} \\
 & -\frac{1}{5} \int -\arcsin\left(\frac{1}{ax^5}\right) \tan\left(\arcsin\left(\frac{1}{ax^5}\right) + \frac{\pi}{2}\right) d\arcsin\left(\frac{1}{ax^5}\right) \\
 & \quad \downarrow \text{25} \\
 & \frac{1}{5} \int \arcsin\left(\frac{1}{ax^5}\right) \tan\left(\arcsin\left(\frac{1}{ax^5}\right) + \frac{\pi}{2}\right) d\arcsin\left(\frac{1}{ax^5}\right) \\
 & \quad \downarrow \text{4200} \\
 & \frac{1}{5} \left( \frac{ix^{10}}{2} - 2i \int -\frac{e^{2i \arcsin\left(\frac{1}{ax^5}\right)} \arcsin\left(\frac{1}{ax^5}\right)}{1 - e^{2i \arcsin\left(\frac{1}{ax^5}\right)}} d\arcsin\left(\frac{1}{ax^5}\right) \right)
 \end{aligned}$$



$$\downarrow 25$$

$$\frac{1}{5} \left( 2i \int \frac{e^{2i \arcsin\left(\frac{1}{ax^5}\right)} \arcsin\left(\frac{1}{ax^5}\right)}{1 - e^{2i \arcsin\left(\frac{1}{ax^5}\right)}} d \arcsin\left(\frac{1}{ax^5}\right) + \frac{ix^{10}}{2} \right)$$

$$\downarrow 2620$$

$$\frac{1}{5} \left( 2i \left( \frac{1}{2} i \arcsin\left(\frac{1}{ax^5}\right) \log\left(1 - e^{2i \arcsin\left(\frac{1}{ax^5}\right)}\right) - \frac{1}{2} i \int \log\left(1 - e^{2i \arcsin\left(\frac{1}{ax^5}\right)}\right) d \arcsin\left(\frac{1}{ax^5}\right) \right) + \frac{ix^{10}}{2} \right)$$

$$\downarrow 2715$$

$$\frac{1}{5} \left( 2i \left( \frac{1}{2} i \arcsin\left(\frac{1}{ax^5}\right) \log\left(1 - e^{2i \arcsin\left(\frac{1}{ax^5}\right)}\right) - \frac{1}{4} \int e^{2i \arcsin\left(\frac{1}{ax^5}\right)} \log\left(1 - e^{2i \arcsin\left(\frac{1}{ax^5}\right)}\right) de^{2i \arcsin\left(\frac{1}{ax^5}\right)} \right) + \frac{ix^{10}}{2} \right)$$

$$\downarrow 2838$$

$$\frac{1}{5} \left( 2i \left( \frac{1}{4} \text{PolyLog}\left(2, e^{2i \arcsin\left(\frac{1}{ax^5}\right)}\right) + \frac{1}{2} i \arcsin\left(\frac{1}{ax^5}\right) \log\left(1 - e^{2i \arcsin\left(\frac{1}{ax^5}\right)}\right) \right) + \frac{ix^{10}}{2} \right)$$

input `Int[ArcCsc[a*x^5]/x, x]`

output `((I/2)*x^10 + (2*I)*((I/2)*ArcSin[1/(a*x^5)]*Log[1 - E^((2*I)*ArcSin[1/(a*x^5)])] + PolyLog[2, E^((2*I)*ArcSin[1/(a*x^5)])]/4))/5`

### Defintions of rubi rules used

rule 25 `Int[-(Fx_), x_Symbol] :> Simp[Identity[-1] Int[Fx, x], x]`

rule 2620 `Int[(((F_)^((g_)*((e_) + (f_)*(x_))))^(n_)*((c_) + (d_)*(x_))^(m_))/((a_) + (b_)*((F_)^((g_)*((e_) + (f_)*(x_))))^(n_)), x_Symbol] :> Simp[(((c + d*x)^m/(b*f*g*n*Log[F]))*Log[1 + b*((F^(g*(e + f*x)))^n/a)], x] - Simp[d*(m/(b*f*g*n*Log[F])) Int[(c + d*x)^(m - 1)*Log[1 + b*((F^(g*(e + f*x)))^n/a)], x], x] /; FreeQ[{F, a, b, c, d, e, f, g, n}, x] && IGtQ[m, 0]`

rule 2715 `Int[Log[(a_) + (b_)*((F_)^((e_)*((c_) + (d_)*(x_))))^(n_)], x_Symbol]
:> Simp[1/(d*e*n*Log[F]) Subst[Int[Log[a + b*x]/x, x], x, (F^(e*(c + d*x)
))^n], x] /; FreeQ[{F, a, b, c, d, e, n}, x] && GtQ[a, 0]`

rule 2838 `Int[Log[(c_)*((d_) + (e_)*(x_)^(n_))]/(x_), x_Symbol] :> Simp[-PolyLog[2
, (-c)*e*x^n]/n, x] /; FreeQ[{c, d, e, n}, x] && EqQ[c*d, 1]`

rule 3042 `Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]`

rule 4200 `Int[((c_) + (d_)*(x_)^(m_))*tan[(e_) + Pi*(k_) + (f_)*(x_)], x_Symbol
] :> Simp[I*((c + d*x)^(m + 1)/(d*(m + 1))), x] - Simp[2*I Int[(c + d*x)^
m*E^(2*I*k*Pi)*(E^(2*I*(e + f*x)))/(1 + E^(2*I*k*Pi)*E^(2*I*(e + f*x))), x]
, x] /; FreeQ[{c, d, e, f}, x] && IntegerQ[4*k] && IGtQ[m, 0]`

rule 5136 `Int[((a_) + ArcSin[(c_)*(x_)])*(b_)^(n_)/(x_), x_Symbol] :> Subst[Int[(
a + b*x)^n*Cot[x], x], x, ArcSin[c*x]] /; FreeQ[{a, b, c}, x] && IGtQ[n, 0]`

rule 5742 `Int[((a_) + ArcCsc[(c_)*(x_)])*(b_)/(x_), x_Symbol] :> -Subst[Int[(a + b
*ArcSin[x/c])/x, x], x, 1/x] /; FreeQ[{a, b, c}, x]`

rule 7282 `Int[(u_)/(x_), x_Symbol] :> With[{lst = PowerVariableExpn[u, 0, x]}, Simp[1
/lst[[2]] Subst[Int[NormalizeIntegrand[Simplify[lst[[1]]/x], x], x], x, (
lst[[3]]*x)^lst[[2]]], x] /; !FalseQ[lst] && NeQ[lst[[2]], 0] /; NonsumQ[
u] && !RationalFunctionQ[u, x]`

**Maple [F]**

$$\int \frac{\operatorname{arccsc}(ax^5)}{x} dx$$

input `int(arccsc(a*x^5)/x,x)`

output `int(arccsc(a*x^5)/x,x)`

**Fricas [F]**

$$\int \frac{\csc^{-1}(ax^5)}{x} dx = \int \frac{\operatorname{arccsc}(ax^5)}{x} dx$$

input `integrate(arccsc(a*x^5)/x,x, algorithm="fricas")`

output `integral(arccsc(a*x^5)/x, x)`

**Sympy [F]**

$$\int \frac{\csc^{-1}(ax^5)}{x} dx = \int \frac{\operatorname{acsc}(ax^5)}{x} dx$$

input `integrate(acsc(a*x**5)/x,x)`

output `Integral(acsc(a*x**5)/x, x)`

**Maxima [F]**

$$\int \frac{\csc^{-1}(ax^5)}{x} dx = \int \frac{\operatorname{arccsc}(ax^5)}{x} dx$$

input `integrate(arccsc(a*x^5)/x,x, algorithm="maxima")`

output `5*a^2*integrate(sqrt(a*x^5 + 1)*sqrt(a*x^5 - 1)*log(x)/(a^4*x^11 - a^2*x), x) - 5*I*a^2*integrate(log(x)/(a^4*x^11 - a^2*x), x) + (arctan2(1, sqrt(a*x^5 + 1)*sqrt(a*x^5 - 1)) + I*log(a))*log(x) - 1/2*I*log(a^2*x^10)*log(x) + 1/2*I*log(a*x^5 + 1)*log(x) + 1/2*I*log(-a*x^5 + 1)*log(x) + 5/2*I*log(x)^2 + 1/10*I*dilog(a*x^5) + 1/10*I*dilog(-a*x^5)`

**Giac [F]**

$$\int \frac{\csc^{-1}(ax^5)}{x} dx = \int \frac{\operatorname{arccsc}(ax^5)}{x} dx$$

input `integrate(arccsc(a*x^5)/x,x, algorithm="giac")`

output `integrate(arccsc(a*x^5)/x, x)`

**Mupad [B] (verification not implemented)**

Time = 0.97 (sec) , antiderivative size = 58, normalized size of antiderivative = 0.94

$$\int \frac{\csc^{-1}(ax^5)}{x} dx = -\frac{\ln\left(1 - e^{\operatorname{asin}\left(\frac{1}{ax^5}\right)2i}\right) \operatorname{asin}\left(\frac{1}{ax^5}\right)}{5} + \frac{\operatorname{polylog}\left(2, e^{\operatorname{asin}\left(\frac{1}{ax^5}\right)2i}\right) \operatorname{li}}{10} + \frac{\operatorname{asin}\left(\frac{1}{ax^5}\right)^2 \operatorname{li}}{10}$$

input `int(asin(1/(a*x^5))/x,x)`

output `(polylog(2, exp(asin(1/(a*x^5))*2i))*1i)/10 - (log(1 - exp(asin(1/(a*x^5))*2i))*asin(1/(a*x^5)))/5 + (asin(1/(a*x^5))^2*1i)/10`

### Reduce [F]

$$\int \frac{\csc^{-1}(ax^5)}{x} dx = \int \frac{\operatorname{acsc}(ax^5)}{x} dx$$

input `int(acsc(a*x^5)/x,x)`

output `int(acsc(a*x**5)/x,x)`

## 3.2 $\int x^3 \csc^{-1}(\sqrt{x}) dx$

Optimal result	53
Mathematica [A] (verified)	53
Rubi [A] (verified)	54
Maple [A] (verified)	55
Fricas [A] (verification not implemented)	56
Sympy [C] (verification not implemented)	56
Maxima [A] (verification not implemented)	57
Giac [B] (verification not implemented)	57
Mupad [F(-1)]	58
Reduce [F]	58

### Optimal result

Integrand size = 10, antiderivative size = 58

$$\int x^3 \csc^{-1}(\sqrt{x}) dx = \frac{\sqrt{-1+x}}{4} + \frac{1}{4}(-1+x)^{3/2} + \frac{3}{20}(-1+x)^{5/2} + \frac{1}{28}(-1+x)^{7/2} + \frac{1}{4}x^4 \csc^{-1}(\sqrt{x})$$

output

```
1/4*(-1+x)^(1/2)+1/4*(-1+x)^(3/2)+3/20*(-1+x)^(5/2)+1/28*(-1+x)^(7/2)+1/4*x^4*arccsc(x^(1/2))
```

### Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 40, normalized size of antiderivative = 0.69

$$\int x^3 \csc^{-1}(\sqrt{x}) dx = \frac{1}{140}\sqrt{-1+x}(16+8x+6x^2+5x^3) + \frac{1}{4}x^4 \csc^{-1}(\sqrt{x})$$

input

```
Integrate[x^3*ArcCsc[Sqrt[x]],x]
```

output

```
(Sqrt[-1 + x]*(16 + 8*x + 6*x^2 + 5*x^3))/140 + (x^4*ArcCsc[Sqrt[x]])/4
```

**Rubi [A] (verified)**

Time = 0.23 (sec) , antiderivative size = 59, normalized size of antiderivative = 1.02, number of steps used = 4, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.400$ , Rules used = {5794, 27, 53, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int x^3 \csc^{-1}(\sqrt{x}) \, dx \\
 & \quad \downarrow \text{5794} \\
 & \frac{1}{4} \int \frac{x^3}{2\sqrt{x-1}} dx + \frac{1}{4} x^4 \csc^{-1}(\sqrt{x}) \\
 & \quad \downarrow \text{27} \\
 & \frac{1}{8} \int \frac{x^3}{\sqrt{x-1}} dx + \frac{1}{4} x^4 \csc^{-1}(\sqrt{x}) \\
 & \quad \downarrow \text{53} \\
 & \frac{1}{8} \int \left( (x-1)^{5/2} + 3(x-1)^{3/2} + 3\sqrt{x-1} + \frac{1}{\sqrt{x-1}} \right) dx + \frac{1}{4} x^4 \csc^{-1}(\sqrt{x}) \\
 & \quad \downarrow \text{2009} \\
 & \frac{1}{4} x^4 \csc^{-1}(\sqrt{x}) + \frac{1}{8} \left( \frac{2}{7} (x-1)^{7/2} + \frac{6}{5} (x-1)^{5/2} + 2(x-1)^{3/2} + 2\sqrt{x-1} \right)
 \end{aligned}$$

input `Int [x^3*ArcCsc [Sqrt [x]] , x]`

output `(2*Sqrt [-1 + x] + 2*(-1 + x)^(3/2) + (6*(-1 + x)^(5/2))/5 + (2*(-1 + x)^(7/2))/7)/8 + (x^4*ArcCsc [Sqrt [x]])/4`

## Definitions of rubi rules used

rule 27 `Int[(a_)*(Fx_), x_Symbol] := Simp[a Int[Fx, x], x] /; FreeQ[a, x] && !MatchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]`

rule 53 `Int[((a_.) + (b_.)*(x_))^(m_.)*((c_.) + (d_.)*(x_))^(n_.), x_Symbol] := Int[ExpandIntegrand[(a + b*x)^m*(c + d*x)^n, x], x] /; FreeQ[{a, b, c, d, n}, x] && IGtQ[m, 0] && (!IntegerQ[n] || (EqQ[c, 0] && LeQ[7*m + 4*n + 4, 0]) || LtQ[9*m + 5*(n + 1), 0] || GtQ[m + n + 2, 0])`

rule 2009 `Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]`

rule 5794 `Int[((a_.) + ArcCsc[u_]*(b_.))*((c_.) + (d_.)*(x_))^(m_.), x_Symbol] := Simp[(c + d*x)^(m + 1)*((a + b*ArcCsc[u])/(d*(m + 1))), x] + Simp[b*(u/(d*(m + 1)*Sqrt[u^2])) Int[SimplifyIntegrand[(c + d*x)^(m + 1)*(D[u, x]/(u*Sqrt[u^2 - 1])), x], x], x] /; FreeQ[{a, b, c, d, m}, x] && NeQ[m, -1] && InverseFunctionFreeQ[u, x] && !FunctionOfQ[(c + d*x)^(m + 1), u, x] && !FunctionOfExponentialQ[u, x]`

## Maple [A] (verified)

Time = 0.16 (sec) , antiderivative size = 40, normalized size of antiderivative = 0.69

method	result	size
parts	$\frac{x^4 \operatorname{arccsc}(\sqrt{x})}{4} + \frac{\sqrt{\frac{-1+x}{x}} \sqrt{x} (5x^3+6x^2+8x+16)}{140}$	40
derivativedivides	$\frac{x^4 \operatorname{arccsc}(\sqrt{x})}{4} + \frac{(-1+x)(5x^3+6x^2+8x+16)}{140\sqrt{\frac{-1+x}{x}} \sqrt{x}}$	43
default	$\frac{x^4 \operatorname{arccsc}(\sqrt{x})}{4} + \frac{(-1+x)(5x^3+6x^2+8x+16)}{140\sqrt{\frac{-1+x}{x}} \sqrt{x}}$	43

input `int(x^3*arccsc(x^(1/2)),x,method=_RETURNVERBOSE)`



output `1/4*x^4*arccsc(x^(1/2))+1/140*((-1+x)/x)^(1/2)*x^(1/2)*(5*x^3+6*x^2+8*x+16)`

### Fricas [A] (verification not implemented)

Time = 0.12 (sec) , antiderivative size = 32, normalized size of antiderivative = 0.55

$$\int x^3 \csc^{-1}(\sqrt{x}) dx = \frac{1}{4} x^4 \operatorname{arccsc}(\sqrt{x}) + \frac{1}{140} (5x^3 + 6x^2 + 8x + 16) \sqrt{x-1}$$

input `integrate(x^3*arccsc(x^(1/2)),x, algorithm="fricas")`

output `1/4*x^4*arccsc(sqrt(x)) + 1/140*(5*x^3 + 6*x^2 + 8*x + 16)*sqrt(x - 1)`

### Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 55.61 (sec) , antiderivative size = 119, normalized size of antiderivative = 2.05

$$\int x^3 \csc^{-1}(\sqrt{x}) dx = \frac{x^4 \operatorname{acsc}(\sqrt{x})}{4} + \frac{\begin{cases} \frac{2x^3\sqrt{x-1}}{7} + \frac{12x^2\sqrt{x-1}}{35} + \frac{16x\sqrt{x-1}}{35} + \frac{32\sqrt{x-1}}{35} & \text{for } |x| > 1 \\ \frac{2ix^3\sqrt{1-x}}{7} + \frac{12ix^2\sqrt{1-x}}{35} + \frac{16ix\sqrt{1-x}}{35} + \frac{32i\sqrt{1-x}}{35} & \text{otherwise} \end{cases}}{8}$$

input `integrate(x**3*acsc(x**(1/2)),x)`

output `x**4*acsc(sqrt(x))/4 + Piecewise((2*x**3*sqrt(x - 1)/7 + 12*x**2*sqrt(x - 1)/35 + 16*x*sqrt(x - 1)/35 + 32*sqrt(x - 1)/35, Abs(x) > 1), (2*I*x**3*sqrt(1 - x)/7 + 12*I*x**2*sqrt(1 - x)/35 + 16*I*x*sqrt(1 - x)/35 + 32*I*sqrt(1 - x)/35, True))/8`

**Maxima [A] (verification not implemented)**

Time = 0.04 (sec) , antiderivative size = 66, normalized size of antiderivative = 1.14

$$\int x^3 \csc^{-1}(\sqrt{x}) dx = \frac{1}{28} x^{\frac{7}{2}} \left(-\frac{1}{x} + 1\right)^{\frac{7}{2}} + \frac{3}{20} x^{\frac{5}{2}} \left(-\frac{1}{x} + 1\right)^{\frac{5}{2}} + \frac{1}{4} x^4 \operatorname{arccsc}(\sqrt{x}) + \frac{1}{4} x^{\frac{3}{2}} \left(-\frac{1}{x} + 1\right)^{\frac{3}{2}} + \frac{1}{4} \sqrt{x} \sqrt{-\frac{1}{x} + 1}$$

input `integrate(x^3*arccsc(x^(1/2)),x, algorithm="maxima")`

output `1/28*x^(7/2)*(-1/x + 1)^(7/2) + 3/20*x^(5/2)*(-1/x + 1)^(5/2) + 1/4*x^4*arccsc(sqrt(x)) + 1/4*x^(3/2)*(-1/x + 1)^(3/2) + 1/4*sqrt(x)*sqrt(-1/x + 1)`

**Giac [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 152 vs. 2(38) = 76.

Time = 0.13 (sec) , antiderivative size = 152, normalized size of antiderivative = 2.62

$$\int x^3 \csc^{-1}(\sqrt{x}) dx = \frac{1}{3584} x^{\frac{7}{2}} \left(\sqrt{-\frac{1}{x} + 1} - 1\right)^7 + \frac{7}{2560} x^{\frac{5}{2}} \left(\sqrt{-\frac{1}{x} + 1} - 1\right)^5 + \frac{1}{4} x^4 \arcsin\left(\frac{1}{\sqrt{x}}\right) + \frac{7}{512} x^{\frac{3}{2}} \left(\sqrt{-\frac{1}{x} + 1} - 1\right)^3 + \frac{35}{512} \sqrt{x} \left(\sqrt{-\frac{1}{x} + 1} - 1\right) - \frac{1225 x^3 \left(\sqrt{-\frac{1}{x} + 1} - 1\right)^6 + 245 x^2 \left(\sqrt{-\frac{1}{x} + 1} - 1\right)^4 + 49 x \left(\sqrt{-\frac{1}{x} + 1} - 1\right)^2 + 5}{17920 x^{\frac{7}{2}} \left(\sqrt{-\frac{1}{x} + 1} - 1\right)^7}$$

input `integrate(x^3*arccsc(x^(1/2)),x, algorithm="giac")`

output

```
1/3584*x^(7/2)*(sqrt(-1/x + 1) - 1)^7 + 7/2560*x^(5/2)*(sqrt(-1/x + 1) - 1)^5 + 1/4*x^4*arcsin(1/sqrt(x)) + 7/512*x^(3/2)*(sqrt(-1/x + 1) - 1)^3 + 35/512*sqrt(x)*(sqrt(-1/x + 1) - 1) - 1/17920*(1225*x^3*(sqrt(-1/x + 1) - 1)^6 + 245*x^2*(sqrt(-1/x + 1) - 1)^4 + 49*x*(sqrt(-1/x + 1) - 1)^2 + 5)/(x^(7/2)*(sqrt(-1/x + 1) - 1)^7)
```

**Mupad [F(-1)]**

Timed out.

$$\int x^3 \csc^{-1}(\sqrt{x}) dx = \int x^3 \operatorname{asin}\left(\frac{1}{\sqrt{x}}\right) dx$$

input

```
int(x^3*asin(1/x^(1/2)),x)
```

output

```
int(x^3*asin(1/x^(1/2)), x)
```

**Reduce [F]**

$$\int x^3 \csc^{-1}(\sqrt{x}) dx = \int \operatorname{acsc}(\sqrt{x}) x^3 dx$$

input

```
int(x^3*acsc(x^(1/2)),x)
```

output

```
int(acsc(sqrt(x))*x**3,x)
```

### 3.3 $\int x^2 \csc^{-1}(\sqrt{x}) dx$

Optimal result	59
Mathematica [A] (verified)	59
Rubi [A] (verified)	60
Maple [A] (verified)	61
Fricas [A] (verification not implemented)	62
Sympy [C] (verification not implemented)	62
Maxima [A] (verification not implemented)	62
Giac [B] (verification not implemented)	63
Mupad [F(-1)]	64
Reduce [F]	64

#### Optimal result

Integrand size = 10, antiderivative size = 47

$$\int x^2 \csc^{-1}(\sqrt{x}) dx = \frac{\sqrt{-1+x}}{3} + \frac{2}{9}(-1+x)^{3/2} + \frac{1}{15}(-1+x)^{5/2} + \frac{1}{3}x^3 \csc^{-1}(\sqrt{x})$$

output

```
1/3*(-1+x)^(1/2)+2/9*(-1+x)^(3/2)+1/15*(-1+x)^(5/2)+1/3*x^3*arccsc(x^(1/2))
```

#### Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 35, normalized size of antiderivative = 0.74

$$\int x^2 \csc^{-1}(\sqrt{x}) dx = \frac{1}{45}\sqrt{-1+x}(8+4x+3x^2) + \frac{1}{3}x^3 \csc^{-1}(\sqrt{x})$$

input

```
Integrate[x^2*ArcCsc[Sqrt[x]],x]
```

output

```
(Sqrt[-1+x]*(8+4*x+3*x^2))/45+(x^3*ArcCsc[Sqrt[x]])/3
```

**Rubi [A] (verified)**

Time = 0.22 (sec) , antiderivative size = 50, normalized size of antiderivative = 1.06, number of steps used = 4, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.400$ , Rules used = {5794, 27, 53, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int x^2 \csc^{-1}(\sqrt{x}) \, dx \\
 & \quad \downarrow \text{5794} \\
 & \frac{1}{3} \int \frac{x^2}{2\sqrt{x-1}} dx + \frac{1}{3} x^3 \csc^{-1}(\sqrt{x}) \\
 & \quad \downarrow \text{27} \\
 & \frac{1}{6} \int \frac{x^2}{\sqrt{x-1}} dx + \frac{1}{3} x^3 \csc^{-1}(\sqrt{x}) \\
 & \quad \downarrow \text{53} \\
 & \frac{1}{6} \int \left( (x-1)^{3/2} + 2\sqrt{x-1} + \frac{1}{\sqrt{x-1}} \right) dx + \frac{1}{3} x^3 \csc^{-1}(\sqrt{x}) \\
 & \quad \downarrow \text{2009} \\
 & \frac{1}{3} x^3 \csc^{-1}(\sqrt{x}) + \frac{1}{6} \left( \frac{2}{5} (x-1)^{5/2} + \frac{4}{3} (x-1)^{3/2} + 2\sqrt{x-1} \right)
 \end{aligned}$$

input `Int [x^2*ArcCsc [Sqrt [x]] , x]`

output `(2*Sqrt [-1 + x] + (4*(-1 + x)^(3/2))/3 + (2*(-1 + x)^(5/2))/5)/6 + (x^3*ArcCsc [Sqrt [x]])/3`

## Definitions of rubi rules used

rule 27 `Int[(a_)*(F_x_), x_Symbol] := Simp[a Int[F_x, x], x] /; FreeQ[a, x] && !MatchQ[F_x, (b_)*(G_x_) /; FreeQ[b, x]]`

rule 53 `Int[((a_.) + (b_.)*(x_))^(m_.)*((c_.) + (d_.)*(x_))^(n_.), x_Symbol] := Int[ExpandIntegrand[(a + b*x)^m*(c + d*x)^n, x], x] /; FreeQ[{a, b, c, d, n}, x] && IGtQ[m, 0] && (!IntegerQ[n] || (EqQ[c, 0] && LeQ[7*m + 4*n + 4, 0]) || LtQ[9*m + 5*(n + 1), 0] || GtQ[m + n + 2, 0])`

rule 2009 `Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]`

rule 5794 `Int[((a_.) + ArcCsc[u_]*(b_.))*((c_.) + (d_.)*(x_))^(m_.), x_Symbol] := Simp[(c + d*x)^(m + 1)*((a + b*ArcCsc[u])/(d*(m + 1))), x] + Simp[b*(u/(d*(m + 1)*Sqrt[u^2])) Int[SimplifyIntegrand[(c + d*x)^(m + 1)*(D[u, x]/(u*Sqrt[u^2 - 1])), x], x], x] /; FreeQ[{a, b, c, d, m}, x] && NeQ[m, -1] && InverseFunctionFreeQ[u, x] && !FunctionOfQ[(c + d*x)^(m + 1), u, x] && !FunctionOfExponentialQ[u, x]`

## Maple [A] (verified)

Time = 0.16 (sec) , antiderivative size = 35, normalized size of antiderivative = 0.74

method	result	size
parts	$\frac{x^3 \operatorname{arccsc}(\sqrt{x})}{3} + \frac{\sqrt{\frac{-1+x}{x}} \sqrt{x} (3x^2+4x+8)}{45}$	35
derivativedivides	$\frac{x^3 \operatorname{arccsc}(\sqrt{x})}{3} + \frac{(-1+x)(3x^2+4x+8)}{45\sqrt{\frac{-1+x}{x}} \sqrt{x}}$	38
default	$\frac{x^3 \operatorname{arccsc}(\sqrt{x})}{3} + \frac{(-1+x)(3x^2+4x+8)}{45\sqrt{\frac{-1+x}{x}} \sqrt{x}}$	38

input `int(x^2*arccsc(x^(1/2)),x,method=_RETURNVERBOSE)`

output `1/3*x^3*arccsc(x^(1/2))+1/45*((-1+x)/x)^(1/2)*x^(1/2)*(3*x^2+4*x+8)`

**Fricas [A] (verification not implemented)**

Time = 0.13 (sec) , antiderivative size = 27, normalized size of antiderivative = 0.57

$$\int x^2 \csc^{-1}(\sqrt{x}) dx = \frac{1}{3} x^3 \operatorname{arccsc}(\sqrt{x}) + \frac{1}{45} (3x^2 + 4x + 8)\sqrt{x-1}$$

input `integrate(x^2*arccsc(x^(1/2)),x, algorithm="fricas")`

output `1/3*x^3*arccsc(sqrt(x)) + 1/45*(3*x^2 + 4*x + 8)*sqrt(x - 1)`

**Sympy [C] (verification not implemented)**

Result contains complex when optimal does not.

Time = 22.84 (sec) , antiderivative size = 90, normalized size of antiderivative = 1.91

$$\int x^2 \csc^{-1}(\sqrt{x}) dx = \frac{x^3 \operatorname{acsc}(\sqrt{x})}{3} + \frac{\begin{cases} \frac{2x^2\sqrt{x-1}}{5} + \frac{8x\sqrt{x-1}}{15} + \frac{16\sqrt{x-1}}{15} & \text{for } |x| > 1 \\ \frac{2ix^2\sqrt{1-x}}{5} + \frac{8ix\sqrt{1-x}}{15} + \frac{16i\sqrt{1-x}}{15} & \text{otherwise} \end{cases}}{6}$$

input `integrate(x**2*acsc(x**(1/2)),x)`

output `x**3*acsc(sqrt(x))/3 + Piecewise((2*x**2*sqrt(x - 1)/5 + 8*x*sqrt(x - 1)/15 + 16*sqrt(x - 1)/15, Abs(x) > 1), (2*I*x**2*sqrt(1 - x)/5 + 8*I*x*sqrt(1 - x)/15 + 16*I*sqrt(1 - x)/15, True))/6`

**Maxima [A] (verification not implemented)**

Time = 0.04 (sec) , antiderivative size = 52, normalized size of antiderivative = 1.11

$$\int x^2 \csc^{-1}(\sqrt{x}) dx = \frac{1}{15} x^{\frac{5}{2}} \left(-\frac{1}{x} + 1\right)^{\frac{5}{2}} + \frac{1}{3} x^3 \operatorname{arccsc}(\sqrt{x}) + \frac{2}{9} x^{\frac{3}{2}} \left(-\frac{1}{x} + 1\right)^{\frac{3}{2}} + \frac{1}{3} \sqrt{x} \sqrt{-\frac{1}{x} + 1}$$

input `integrate(x^2*arccsc(x^(1/2)),x, algorithm="maxima")`

output `1/15*x^(5/2)*(-1/x + 1)^(5/2) + 1/3*x^3*arccsc(sqrt(x)) + 2/9*x^(3/2)*(-1/x + 1)^(3/2) + 1/3*sqrt(x)*sqrt(-1/x + 1)`

### Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 116 vs.  $2(31) = 62$ .

Time = 0.13 (sec) , antiderivative size = 116, normalized size of antiderivative = 2.47

$$\int x^2 \csc^{-1}(\sqrt{x}) dx = \frac{1}{480} x^{\frac{5}{2}} \left( \sqrt{-\frac{1}{x} + 1} - 1 \right)^5 + \frac{5}{288} x^{\frac{3}{2}} \left( \sqrt{-\frac{1}{x} + 1} - 1 \right)^3 + \frac{1}{3} x^3 \arcsin\left(\frac{1}{\sqrt{x}}\right) + \frac{5}{48} \sqrt{x} \left( \sqrt{-\frac{1}{x} + 1} - 1 \right) - \frac{150 x^2 \left( \sqrt{-\frac{1}{x} + 1} - 1 \right)^4 + 25 x \left( \sqrt{-\frac{1}{x} + 1} - 1 \right)^2 + 3}{1440 x^{\frac{5}{2}} \left( \sqrt{-\frac{1}{x} + 1} - 1 \right)^5}$$

input `integrate(x^2*arccsc(x^(1/2)),x, algorithm="giac")`

output `1/480*x^(5/2)*(sqrt(-1/x + 1) - 1)^5 + 5/288*x^(3/2)*(sqrt(-1/x + 1) - 1)^3 + 1/3*x^3*arcsin(1/sqrt(x)) + 5/48*sqrt(x)*(sqrt(-1/x + 1) - 1) - 1/1440*(150*x^2*(sqrt(-1/x + 1) - 1)^4 + 25*x*(sqrt(-1/x + 1) - 1)^2 + 3)/(x^(5/2)*(sqrt(-1/x + 1) - 1)^5)`



**Mupad [F(-1)]**

Timed out.

$$\int x^2 \csc^{-1}(\sqrt{x}) dx = \int x^2 \operatorname{asin}\left(\frac{1}{\sqrt{x}}\right) dx$$

input `int(x^2*asin(1/x^(1/2)),x)`output `int(x^2*asin(1/x^(1/2)), x)`**Reduce [F]**

$$\int x^2 \csc^{-1}(\sqrt{x}) dx = \int \operatorname{acsc}(\sqrt{x}) x^2 dx$$

input `int(x^2*acsc(x^(1/2)),x)`output `int(acsc(sqrt(x))*x**2,x)`

### 3.4 $\int x \csc^{-1}(\sqrt{x}) dx$

Optimal result	65
Mathematica [A] (verified)	65
Rubi [A] (verified)	66
Maple [A] (verified)	67
Fricas [A] (verification not implemented)	68
Sympy [C] (verification not implemented)	68
Maxima [A] (verification not implemented)	68
Giac [B] (verification not implemented)	69
Mupad [F(-1)]	69
Reduce [F]	70

#### Optimal result

Integrand size = 8, antiderivative size = 36

$$\int x \csc^{-1}(\sqrt{x}) dx = \frac{\sqrt{-1+x}}{2} + \frac{1}{6}(-1+x)^{3/2} + \frac{1}{2}x^2 \csc^{-1}(\sqrt{x})$$

output

```
1/2*(-1+x)^(1/2)+1/6*(-1+x)^(3/2)+1/2*x^2*arccsc(x^(1/2))
```

#### Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 27, normalized size of antiderivative = 0.75

$$\int x \csc^{-1}(\sqrt{x}) dx = \frac{1}{6}(\sqrt{-1+x}(2+x) + 3x^2 \csc^{-1}(\sqrt{x}))$$

input

```
Integrate[x*ArcCsc[Sqrt[x]],x]
```

output

```
(Sqrt[-1 + x]*(2 + x) + 3*x^2*ArcCsc[Sqrt[x]])/6
```

**Rubi [A] (verified)**

Time = 0.20 (sec) , antiderivative size = 39, normalized size of antiderivative = 1.08, number of steps used = 4, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {5794, 27, 53, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int x \csc^{-1}(\sqrt{x}) \, dx \\
 & \quad \downarrow \text{5794} \\
 & \frac{1}{2} \int \frac{x}{2\sqrt{x-1}} \, dx + \frac{1}{2} x^2 \csc^{-1}(\sqrt{x}) \\
 & \quad \downarrow \text{27} \\
 & \frac{1}{4} \int \frac{x}{\sqrt{x-1}} \, dx + \frac{1}{2} x^2 \csc^{-1}(\sqrt{x}) \\
 & \quad \downarrow \text{53} \\
 & \frac{1}{4} \int \left( \sqrt{x-1} + \frac{1}{\sqrt{x-1}} \right) \, dx + \frac{1}{2} x^2 \csc^{-1}(\sqrt{x}) \\
 & \quad \downarrow \text{2009} \\
 & \frac{1}{2} x^2 \csc^{-1}(\sqrt{x}) + \frac{1}{4} \left( \frac{2}{3} (x-1)^{3/2} + 2\sqrt{x-1} \right)
 \end{aligned}$$

input `Int[x*ArcCsc[Sqrt[x]],x]`

output `(2*Sqrt[-1 + x] + (2*(-1 + x)^(3/2))/3)/4 + (x^2*ArcCsc[Sqrt[x]])/2`

## Definitions of rubi rules used

rule 27 `Int[(a_)*(F_x_), x_Symbol] := Simp[a Int[F_x, x], x] /; FreeQ[a, x] && !MatchQ[F_x, (b_)*(G_x_) /; FreeQ[b, x]]`

rule 53 `Int[((a_.) + (b_.)*(x_))^(m_.)*((c_.) + (d_.)*(x_))^(n_.), x_Symbol] := Int[ExpandIntegrand[(a + b*x)^m*(c + d*x)^n, x], x] /; FreeQ[{a, b, c, d, n}, x] && IGtQ[m, 0] && (!IntegerQ[n] || (EqQ[c, 0] && LeQ[7*m + 4*n + 4, 0]) || LtQ[9*m + 5*(n + 1), 0] || GtQ[m + n + 2, 0])`

rule 2009 `Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]`

rule 5794 `Int[((a_.) + ArcCsc[u_]*(b_.))*((c_.) + (d_.)*(x_))^(m_.), x_Symbol] := Simp[(c + d*x)^(m + 1)*((a + b*ArcCsc[u])/(d*(m + 1))), x] + Simp[b*(u/(d*(m + 1)*Sqrt[u^2])) Int[SimplifyIntegrand[(c + d*x)^(m + 1)*(D[u, x]/(u*Sqrt[u^2 - 1])), x], x] /; FreeQ[{a, b, c, d, m}, x] && NeQ[m, -1] && InverseFunctionFreeQ[u, x] && !FunctionOfQ[(c + d*x)^(m + 1), u, x] && !FunctionOfExponentialQ[u, x]`

## Maple [A] (verified)

Time = 0.16 (sec) , antiderivative size = 28, normalized size of antiderivative = 0.78

method	result	size
parts	$\frac{x^2 \operatorname{arccsc}(\sqrt{x})}{2} + \frac{\sqrt{\frac{-1+x}{x}} \sqrt{x} (x+2)}{6}$	28
derivativedivides	$\frac{x^2 \operatorname{arccsc}(\sqrt{x})}{2} + \frac{(-1+x)(x+2)}{6\sqrt{\frac{-1+x}{x}} \sqrt{x}}$	31
default	$\frac{x^2 \operatorname{arccsc}(\sqrt{x})}{2} + \frac{(-1+x)(x+2)}{6\sqrt{\frac{-1+x}{x}} \sqrt{x}}$	31

input `int(x*arccsc(x^(1/2)),x,method=_RETURNVERBOSE)`

output `1/2*x^2*arccsc(x^(1/2))+1/6*((-1+x)/x)^(1/2)*x^(1/2)*(x+2)`

**Fricas [A] (verification not implemented)**

Time = 0.11 (sec) , antiderivative size = 20, normalized size of antiderivative = 0.56

$$\int x \csc^{-1}(\sqrt{x}) dx = \frac{1}{2} x^2 \operatorname{arccsc}(\sqrt{x}) + \frac{1}{6} (x+2)\sqrt{x-1}$$

input `integrate(x*arccsc(x^(1/2)),x, algorithm="fricas")`

output `1/2*x^2*arccsc(sqrt(x)) + 1/6*(x + 2)*sqrt(x - 1)`

**Sympy [C] (verification not implemented)**

Result contains complex when optimal does not.

Time = 6.15 (sec) , antiderivative size = 61, normalized size of antiderivative = 1.69

$$\int x \csc^{-1}(\sqrt{x}) dx = \frac{x^2 \operatorname{acsc}(\sqrt{x})}{2} + \frac{\begin{cases} \frac{2x\sqrt{x-1}}{3} + \frac{4\sqrt{x-1}}{3} & \text{for } |x| > 1 \\ \frac{2ix\sqrt{1-x}}{3} + \frac{4i\sqrt{1-x}}{3} & \text{otherwise} \end{cases}}{4}$$

input `integrate(x*acsc(x**(1/2)),x)`

output `x**2*acsc(sqrt(x))/2 + Piecewise((2*x*sqrt(x - 1)/3 + 4*sqrt(x - 1)/3, Abs(x) > 1), (2*I*x*sqrt(1 - x)/3 + 4*I*sqrt(1 - x)/3, True))/4`

**Maxima [A] (verification not implemented)**

Time = 0.04 (sec) , antiderivative size = 38, normalized size of antiderivative = 1.06

$$\int x \csc^{-1}(\sqrt{x}) dx = \frac{1}{6} x^{\frac{3}{2}} \left( -\frac{1}{x} + 1 \right)^{\frac{3}{2}} + \frac{1}{2} x^2 \operatorname{arccsc}(\sqrt{x}) + \frac{1}{2} \sqrt{x} \sqrt{-\frac{1}{x} + 1}$$

input `integrate(x*arccsc(x^(1/2)),x, algorithm="maxima")`

output  $1/6*x^{(3/2)}*(-1/x + 1)^{(3/2)} + 1/2*x^2*\text{arccsc}(\text{sqrt}(x)) + 1/2*\text{sqrt}(x)*\text{sqrt}(-1/x + 1)$

### Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 80 vs.  $2(24) = 48$ .

Time = 0.13 (sec) , antiderivative size = 80, normalized size of antiderivative = 2.22

$$\int x \csc^{-1}(\sqrt{x}) dx = \frac{1}{48} x^{\frac{3}{2}} \left( \sqrt{-\frac{1}{x} + 1} - 1 \right)^3 + \frac{1}{2} x^2 \arcsin\left(\frac{1}{\sqrt{x}}\right) + \frac{3}{16} \sqrt{x} \left( \sqrt{-\frac{1}{x} + 1} - 1 \right) - \frac{9x \left( \sqrt{-\frac{1}{x} + 1} - 1 \right)^2 + 1}{48 x^{\frac{3}{2}} \left( \sqrt{-\frac{1}{x} + 1} - 1 \right)^3}$$

input `integrate(x*arccsc(x^(1/2)),x, algorithm="giac")`

output  $1/48*x^{(3/2)}*(\text{sqrt}(-1/x + 1) - 1)^3 + 1/2*x^2*\arcsin(1/\text{sqrt}(x)) + 3/16*\text{sqrt}(x)*(\text{sqrt}(-1/x + 1) - 1) - 1/48*(9*x*(\text{sqrt}(-1/x + 1) - 1)^2 + 1)/(x^{(3/2)}*(\text{sqrt}(-1/x + 1) - 1)^3)$

### Mupad [F(-1)]

Timed out.

$$\int x \csc^{-1}(\sqrt{x}) dx = \int x \text{asin}\left(\frac{1}{\sqrt{x}}\right) dx$$

input `int(x*asin(1/x^(1/2)),x)`

output `int(x*asin(1/x^(1/2)), x)`

**Reduce [F]**

$$\int x \csc^{-1}(\sqrt{x}) dx = \int \operatorname{acsc}(\sqrt{x}) x dx$$

input `int(x*acsc(x^(1/2)),x)`

output `int(acsc(sqrt(x))*x,x)`

### 3.5 $\int \csc^{-1}(\sqrt{x}) dx$

Optimal result	71
Mathematica [A] (verified)	71
Rubi [A] (verified)	72
Maple [A] (verified)	73
Fricas [A] (verification not implemented)	73
Sympy [C] (verification not implemented)	73
Maxima [A] (verification not implemented)	74
Giac [B] (verification not implemented)	74
Mupad [B] (verification not implemented)	75
Reduce [F]	75

#### Optimal result

Integrand size = 6, antiderivative size = 16

$$\int \csc^{-1}(\sqrt{x}) dx = \sqrt{-1+x} + x \csc^{-1}(\sqrt{x})$$

output `(-1+x)^(1/2)+x*arccsc(x^(1/2))`

#### Mathematica [A] (verified)

Time = 0.00 (sec) , antiderivative size = 16, normalized size of antiderivative = 1.00

$$\int \csc^{-1}(\sqrt{x}) dx = \sqrt{-1+x} + x \csc^{-1}(\sqrt{x})$$

input `Integrate[ArcCsc[Sqrt[x]],x]`

output `Sqrt[-1 + x] + x*ArcCsc[Sqrt[x]]`



**Rubi [A] (verified)**

Time = 0.17 (sec) , antiderivative size = 16, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.333$ , Rules used = {5792, 17}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \csc^{-1}(\sqrt{x}) dx$$

$$\downarrow 5792$$

$$\int \frac{1}{2\sqrt{x-1}} dx + x \csc^{-1}(\sqrt{x})$$

$$\downarrow 17$$

$$\sqrt{x-1} + x \csc^{-1}(\sqrt{x})$$

input `Int[ArcCsc[Sqrt[x]], x]`

output `Sqrt[-1 + x] + x*ArcCsc[Sqrt[x]]`

**Defintions of rubi rules used**

rule 17 `Int[(c_.)*((a_.) + (b_.)*(x_)^(m_.), x_Symbol] :> Simp[c*((a + b*x)^(m + 1))/(b*(m + 1))], x] /; FreeQ[{a, b, c, m}, x] && NeQ[m, -1]`

rule 5792 `Int[ArcCsc[u_], x_Symbol] :> Simp[x*ArcCsc[u], x] + Simp[u/Sqrt[u^2] Int[SimplifyIntegrand[x*(D[u, x]/(u*Sqrt[u^2 - 1]))], x], x], x] /; InverseFunctionFreeQ[u, x] && !FunctionOfExponentialQ[u, x]`

**Maple [A] (verified)**

Time = 0.15 (sec) , antiderivative size = 21, normalized size of antiderivative = 1.31

method	result	size
parts	$x \operatorname{arccsc}(\sqrt{x}) + \sqrt{x} \sqrt{\frac{-1+x}{x}}$	21
derivativedivides	$x \operatorname{arccsc}(\sqrt{x}) + \frac{-1+x}{\sqrt{\frac{-1+x}{x}} \sqrt{x}}$	24
default	$x \operatorname{arccsc}(\sqrt{x}) + \frac{-1+x}{\sqrt{\frac{-1+x}{x}} \sqrt{x}}$	24

input `int(arccsc(x^(1/2)),x,method=_RETURNVERBOSE)`

output `x*arccsc(x^(1/2))+x^(1/2)*((-1+x)/x)^(1/2)`

**Fricas [A] (verification not implemented)**

Time = 0.11 (sec) , antiderivative size = 12, normalized size of antiderivative = 0.75

$$\int \csc^{-1}(\sqrt{x}) dx = x \operatorname{arccsc}(\sqrt{x}) + \sqrt{x-1}$$

input `integrate(arccsc(x^(1/2)),x, algorithm="fricas")`

output `x*arccsc(sqrt(x)) + sqrt(x - 1)`

**Sympy [C] (verification not implemented)**

Result contains complex when optimal does not.

Time = 2.39 (sec) , antiderivative size = 29, normalized size of antiderivative = 1.81

$$\int \csc^{-1}(\sqrt{x}) dx = x \operatorname{acsc}(\sqrt{x}) + \frac{\begin{cases} 2\sqrt{x-1} & \text{for } |x| > 1 \\ 2i\sqrt{1-x} & \text{otherwise} \end{cases}}{2}$$

input `integrate(acsc(x**(1/2)),x)`

output `x*acsc(sqrt(x)) + Piecewise((2*sqrt(x - 1), Abs(x) > 1), (2*I*sqrt(1 - x), True))/2`

### Maxima [A] (verification not implemented)

Time = 0.04 (sec) , antiderivative size = 20, normalized size of antiderivative = 1.25

$$\int \csc^{-1}(\sqrt{x}) dx = x \operatorname{arccsc}(\sqrt{x}) + \sqrt{x} \sqrt{-\frac{1}{x} + 1}$$

input `integrate(arccsc(x^(1/2)),x, algorithm="maxima")`

output `x*arccsc(sqrt(x)) + sqrt(x)*sqrt(-1/x + 1)`

### Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 41 vs.  $2(12) = 24$ .

Time = 0.12 (sec) , antiderivative size = 41, normalized size of antiderivative = 2.56

$$\int \csc^{-1}(\sqrt{x}) dx = x \arcsin\left(\frac{1}{\sqrt{x}}\right) + \frac{1}{2} \sqrt{x} \left( \sqrt{-\frac{1}{x} + 1} - 1 \right) - \frac{1}{2 \sqrt{x} \left( \sqrt{-\frac{1}{x} + 1} - 1 \right)}$$

input `integrate(arccsc(x^(1/2)),x, algorithm="giac")`

output `x*arcsin(1/sqrt(x)) + 1/2*sqrt(x)*(sqrt(-1/x + 1) - 1) - 1/2/(sqrt(x)*(sqrt(-1/x + 1) - 1))`

**Mupad [B] (verification not implemented)**

Time = 1.35 (sec) , antiderivative size = 20, normalized size of antiderivative = 1.25

$$\int \csc^{-1}(\sqrt{x}) dx = x \operatorname{asin}\left(\frac{1}{\sqrt{x}}\right) + \sqrt{x} \sqrt{1 - \frac{1}{x}}$$

input `int(asin(1/x^(1/2)),x)`

output `x*asin(1/x^(1/2)) + x^(1/2)*(1 - 1/x)^(1/2)`

**Reduce [F]**

$$\int \csc^{-1}(\sqrt{x}) dx = \int \operatorname{acsc}(\sqrt{x}) dx$$

input `int(acsc(x^(1/2)),x)`

output `int(acsc(sqrt(x)),x)`

### 3.6 $\int \frac{\csc^{-1}(\sqrt{x})}{x} dx$

Optimal result . . . . .	76
Mathematica [A] (verified) . . . . .	76
Rubi [A] (warning: unable to verify) . . . . .	77
Maple [A] (verified) . . . . .	80
Fricas [F] . . . . .	80
Sympy [F] . . . . .	80
Maxima [F] . . . . .	81
Giac [F(-2)] . . . . .	81
Mupad [B] (verification not implemented) . . . . .	81
Reduce [F] . . . . .	82

#### Optimal result

Integrand size = 10, antiderivative size = 56

$$\int \frac{\csc^{-1}(\sqrt{x})}{x} dx = i \csc^{-1}(\sqrt{x})^2 - 2 \csc^{-1}(\sqrt{x}) \log(1 - e^{2i \csc^{-1}(\sqrt{x})}) + i \text{PolyLog}(2, e^{2i \csc^{-1}(\sqrt{x})})$$

output

```
I*arccsc(x^(1/2))^2-2*arccsc(x^(1/2))*ln(1-(I/x^(1/2)+(1-1/x)^(1/2))^2)+I*
polylog(2,(I/x^(1/2)+(1-1/x)^(1/2))^2)
```

#### Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 54, normalized size of antiderivative = 0.96

$$\int \frac{\csc^{-1}(\sqrt{x})}{x} dx = i \left( \csc^{-1}(\sqrt{x}) \left( \csc^{-1}(\sqrt{x}) + 2i \log(1 - e^{2i \csc^{-1}(\sqrt{x})}) \right) + \text{PolyLog}(2, e^{2i \csc^{-1}(\sqrt{x})}) \right)$$

input

```
Integrate[ArcCsc[Sqrt[x]]/x,x]
```

output

```
I*(ArcCsc[Sqrt[x]]*(ArcCsc[Sqrt[x]] + (2*I)*Log[1 - E^((2*I)*ArcCsc[Sqrt[x]])])) + PolyLog[2, E^((2*I)*ArcCsc[Sqrt[x]])]
```

**Rubi [A] (warning: unable to verify)**

Time = 0.48 (sec) , antiderivative size = 62, normalized size of antiderivative = 1.11, number of steps used = 11, number of rules used = 10,  $\frac{\text{number of rules}}{\text{integrand size}} = 1.000$ , Rules used = {7267, 5742, 5136, 3042, 25, 4200, 25, 2620, 2715, 2838}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{\csc^{-1}(\sqrt{x})}{x} dx \\
 & \quad \downarrow \text{7267} \\
 & 2 \int \frac{\csc^{-1}(\sqrt{x})}{\sqrt{x}} d\sqrt{x} \\
 & \quad \downarrow \text{5742} \\
 & -2 \int \frac{\arcsin\left(\frac{1}{\sqrt{x}}\right)}{\sqrt{x}} d\frac{1}{\sqrt{x}} \\
 & \quad \downarrow \text{5136} \\
 & -2 \int \sqrt{1 - \frac{1}{x}} \sqrt{x} \arcsin\left(\frac{1}{\sqrt{x}}\right) d\arcsin\left(\frac{1}{\sqrt{x}}\right) \\
 & \quad \downarrow \text{3042} \\
 & -2 \int -\arcsin\left(\frac{1}{\sqrt{x}}\right) \tan\left(\arcsin\left(\frac{1}{\sqrt{x}}\right) + \frac{\pi}{2}\right) d\arcsin\left(\frac{1}{\sqrt{x}}\right) \\
 & \quad \downarrow \text{25} \\
 & 2 \int \arcsin\left(\frac{1}{\sqrt{x}}\right) \tan\left(\arcsin\left(\frac{1}{\sqrt{x}}\right) + \frac{\pi}{2}\right) d\arcsin\left(\frac{1}{\sqrt{x}}\right) \\
 & \quad \downarrow \text{4200} \\
 & -2 \left( 2i \int -\frac{e^{2i \arcsin\left(\frac{1}{\sqrt{x}}\right)} \arcsin\left(\frac{1}{\sqrt{x}}\right)}{1 - e^{2i \arcsin\left(\frac{1}{\sqrt{x}}\right)}} d\arcsin\left(\frac{1}{\sqrt{x}}\right) - \frac{ix}{2} \right)
 \end{aligned}$$

$$\begin{aligned}
& \downarrow 25 \\
& -2 \left( -2i \int \frac{e^{2i \arcsin\left(\frac{1}{\sqrt{x}}\right)} \arcsin\left(\frac{1}{\sqrt{x}}\right)}{1 - e^{2i \arcsin\left(\frac{1}{\sqrt{x}}\right)}} d \arcsin\left(\frac{1}{\sqrt{x}}\right) - \frac{ix}{2} \right) \\
& \downarrow 2620 \\
& -2 \left( -2i \left( \frac{1}{2} i \arcsin\left(\frac{1}{\sqrt{x}}\right) \log\left(1 - e^{2i \arcsin\left(\frac{1}{\sqrt{x}}\right)}\right) - \frac{1}{2} i \int \log\left(1 - e^{2i \arcsin\left(\frac{1}{\sqrt{x}}\right)}\right) d \arcsin\left(\frac{1}{\sqrt{x}}\right) \right) - \frac{ix}{2} \right) \\
& \downarrow 2715 \\
& -2 \left( -2i \left( \frac{1}{2} i \arcsin\left(\frac{1}{\sqrt{x}}\right) \log\left(1 - e^{2i \arcsin\left(\frac{1}{\sqrt{x}}\right)}\right) - \frac{1}{4} \int e^{2i \arcsin\left(\frac{1}{\sqrt{x}}\right)} \log\left(1 - e^{2i \arcsin\left(\frac{1}{\sqrt{x}}\right)}\right) de^{2i \arcsin\left(\frac{1}{\sqrt{x}}\right)} \right) \right) \\
& \downarrow 2838 \\
& -2 \left( -2i \left( \frac{1}{4} \text{PolyLog}\left(2, e^{2i \arcsin\left(\frac{1}{\sqrt{x}}\right)}\right) + \frac{1}{2} i \arcsin\left(\frac{1}{\sqrt{x}}\right) \log\left(1 - e^{2i \arcsin\left(\frac{1}{\sqrt{x}}\right)}\right) \right) \right) - \frac{ix}{2}
\end{aligned}$$

input `Int[ArcCsc[Sqrt[x]]/x,x]`

output `-2*((-1/2*I)*x - (2*I)*((I/2)*ArcSin[1/Sqrt[x]]*Log[1 - E^((2*I)*ArcSin[1/Sqrt[x]])]) + PolyLog[2, E^((2*I)*ArcSin[1/Sqrt[x]])]/4)`

### Defintions of rubi rules used

rule 25 `Int[-(Fx_), x_Symbol] := Simp[Identity[-1] Int[Fx, x], x]`

rule 2620 `Int[(((F_)^((g_)*((e_) + (f_)*(x_)))^(n_))*((c_) + (d_)*(x_))^(m_))/((a_) + (b_)*((F_)^((g_)*((e_) + (f_)*(x_)))^(n_))), x_Symbol] := Simp[((c + d*x)^m/(b*f*g*n*Log[F]))*Log[1 + b*((F^(g*(e + f*x)))^n/a)], x] - Simp[d*(m/(b*f*g*n*Log[F])) Int[(c + d*x)^(m - 1)*Log[1 + b*((F^(g*(e + f*x)))^n/a)], x], x] /; FreeQ[{F, a, b, c, d, e, f, g, n}, x] && IGtQ[m, 0]`

rule 2715 `Int[Log[(a_) + (b_)*((F_)^((e_)*((c_) + (d_)*(x_))))^(n_)], x_Symbol]
:> Simp[1/(d*e*n*Log[F]) Subst[Int[Log[a + b*x]/x, x], x, (F^(e*(c + d*x)
))^n], x] /; FreeQ[{F, a, b, c, d, e, n}, x] && GtQ[a, 0]`

rule 2838 `Int[Log[(c_)*((d_) + (e_)*(x_)^(n_))]/(x_), x_Symbol] :> Simp[-PolyLog[2
, (-c)*e*x^n]/n, x] /; FreeQ[{c, d, e, n}, x] && EqQ[c*d, 1]`

rule 3042 `Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]`

rule 4200 `Int[((c_) + (d_)*(x_)^(m_))*tan[(e_) + Pi*(k_) + (f_)*(x_)], x_Symbol
] :> Simp[I*((c + d*x)^(m + 1)/(d*(m + 1))), x] - Simp[2*I Int[(c + d*x)^
m*E^(2*I*k*Pi)*(E^(2*I*(e + f*x)))/(1 + E^(2*I*k*Pi)*E^(2*I*(e + f*x))), x]
, x] /; FreeQ[{c, d, e, f}, x] && IntegerQ[4*k] && IGtQ[m, 0]`

rule 5136 `Int[((a_) + ArcSin[(c_)*(x_)]*(b_))^(n_)/(x_), x_Symbol] :> Subst[Int[(
a + b*x)^n*Cot[x], x], x, ArcSin[c*x]] /; FreeQ[{a, b, c}, x] && IGtQ[n, 0]`

rule 5742 `Int[((a_) + ArcCsc[(c_)*(x_)]*(b_))/(x_), x_Symbol] :> -Subst[Int[(a + b
*ArcSin[x/c])/x, x], x, 1/x] /; FreeQ[{a, b, c}, x]`

rule 7267 `Int[u_, x_Symbol] :> With[{lst = SubstForFractionalPowerOfLinear[u, x]}, Si
mp[lst[[2]]*lst[[4]] Subst[Int[lst[[1]], x], x, lst[[3]]^(1/lst[[2])]], x
] /; !FalseQ[lst] && SubstForFractionalPowerQ[u, lst[[3]], x]`



**Maple [A] (verified)**

Time = 0.50 (sec) , antiderivative size = 105, normalized size of antiderivative = 1.88

method	result
derivativedivides	$i \operatorname{arccsc}(\sqrt{x})^2 - 2 \operatorname{arccsc}(\sqrt{x}) \ln\left(1 - \frac{i}{\sqrt{x}} - \sqrt{1 - \frac{1}{x}}\right) + 2i \operatorname{polylog}\left(2, \frac{i}{\sqrt{x}} + \sqrt{1 - \frac{1}{x}}\right)$
default	$i \operatorname{arccsc}(\sqrt{x})^2 - 2 \operatorname{arccsc}(\sqrt{x}) \ln\left(1 - \frac{i}{\sqrt{x}} - \sqrt{1 - \frac{1}{x}}\right) + 2i \operatorname{polylog}\left(2, \frac{i}{\sqrt{x}} + \sqrt{1 - \frac{1}{x}}\right)$

input `int(arccsc(x^(1/2))/x,x,method=_RETURNVERBOSE)`

output `I*arccsc(x^(1/2))^2-2*arccsc(x^(1/2))*ln(1-I/x^(1/2)-(1-1/x)^(1/2))+2*I*polylog(2,I/x^(1/2)+(1-1/x)^(1/2))-2*arccsc(x^(1/2))*ln(1+I/x^(1/2)+(1-1/x)^(1/2))+2*I*polylog(2,-I/x^(1/2)-(1-1/x)^(1/2))`

**Fricas [F]**

$$\int \frac{\csc^{-1}(\sqrt{x})}{x} dx = \int \frac{\operatorname{arccsc}(\sqrt{x})}{x} dx$$

input `integrate(arccsc(x^(1/2))/x,x, algorithm="fricas")`

output `integral(arccsc(sqrt(x))/x, x)`

**Sympy [F]**

$$\int \frac{\csc^{-1}(\sqrt{x})}{x} dx = \int \frac{\operatorname{acsc}(\sqrt{x})}{x} dx$$

input `integrate(acsc(x**(1/2))/x,x)`

output `Integral(acsc(sqrt(x))/x, x)`

**Maxima [F]**

$$\int \frac{\csc^{-1}(\sqrt{x})}{x} dx = \int \frac{\operatorname{arccsc}(\sqrt{x})}{x} dx$$

input `integrate(arccsc(x^(1/2))/x,x, algorithm="maxima")`

output `integrate(arccsc(sqrt(x))/x, x)`

**Giac [F(-2)]**

Exception generated.

$$\int \frac{\csc^{-1}(\sqrt{x})}{x} dx = \text{Exception raised: NotImplementedError}$$

input `integrate(arccsc(x^(1/2))/x,x, algorithm="giac")`

output `Exception raised: NotImplementedError >> unable to parse Giac output: Invalid series expansion: non tractable function asin at +infinity`

**Mupad [B] (verification not implemented)**

Time = 1.09 (sec) , antiderivative size = 42, normalized size of antiderivative = 0.75

$$\int \frac{\csc^{-1}(\sqrt{x})}{x} dx = \operatorname{polylog}\left(2, e^{\operatorname{asin}\left(\frac{1}{\sqrt{x}}\right) 2i}\right) \operatorname{li} + \operatorname{asin}\left(\frac{1}{\sqrt{x}}\right)^2 \operatorname{li} - 2 \ln\left(1 - e^{\operatorname{asin}\left(\frac{1}{\sqrt{x}}\right) 2i}\right) \operatorname{asin}\left(\frac{1}{\sqrt{x}}\right)$$

input `int(asin(1/x^(1/2))/x,x)`

output `polylog(2, exp(asin(1/x^(1/2))*2i))*li + asin(1/x^(1/2))^2*li - 2*log(1 - exp(asin(1/x^(1/2))*2i))*asin(1/x^(1/2))`

**Reduce [F]**

$$\int \frac{\csc^{-1}(\sqrt{x})}{x} dx = \int \frac{\operatorname{acsc}(\sqrt{x})}{x} dx$$

input `int(acsc(x^(1/2))/x,x)`

output `int(acsc(sqrt(x))/x,x)`

### 3.7 $\int \frac{\csc^{-1}(\sqrt{x})}{x^2} dx$

Optimal result . . . . .	83
Mathematica [A] (verified) . . . . .	83
Rubi [A] (verified) . . . . .	84
Maple [A] (verified) . . . . .	86
Fricas [A] (verification not implemented) . . . . .	86
Sympy [C] (verification not implemented) . . . . .	87
Maxima [A] (verification not implemented) . . . . .	87
Giac [A] (verification not implemented) . . . . .	88
Mupad [B] (verification not implemented) . . . . .	88
Reduce [F] . . . . .	88

#### Optimal result

Integrand size = 10, antiderivative size = 38

$$\int \frac{\csc^{-1}(\sqrt{x})}{x^2} dx = -\frac{\sqrt{-1+x}}{2x} - \frac{\csc^{-1}(\sqrt{x})}{x} - \frac{1}{2} \arctan(\sqrt{-1+x})$$

output `-1/2*(-1+x)^(1/2)/x-arcscsc(x^(1/2))/x-1/2*arctan((-1+x)^(1/2))`

#### Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 32, normalized size of antiderivative = 0.84

$$\int \frac{\csc^{-1}(\sqrt{x})}{x^2} dx = -\frac{\sqrt{-1+x} + 2 \csc^{-1}(\sqrt{x}) - x \arcsin\left(\frac{1}{\sqrt{x}}\right)}{2x}$$

input `Integrate[ArcCsc[Sqrt[x]]/x^2,x]`

output `-1/2*(Sqrt[-1 + x] + 2*ArcCsc[Sqrt[x]] - x*ArcSin[1/Sqrt[x]])/x`

**Rubi [A] (verified)**

Time = 0.20 (sec) , antiderivative size = 39, normalized size of antiderivative = 1.03, number of steps used = 6, number of rules used = 5,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {5794, 27, 52, 73, 216}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{\csc^{-1}(\sqrt{x})}{x^2} dx \\
 & \quad \downarrow \text{5794} \\
 & - \int \frac{1}{2\sqrt{x-1}x^2} dx - \frac{\csc^{-1}(\sqrt{x})}{x} \\
 & \quad \downarrow \text{27} \\
 & -\frac{1}{2} \int \frac{1}{\sqrt{x-1}x^2} dx - \frac{\csc^{-1}(\sqrt{x})}{x} \\
 & \quad \downarrow \text{52} \\
 & \frac{1}{2} \left( -\frac{1}{2} \int \frac{1}{\sqrt{x-1}x} dx - \frac{\sqrt{x-1}}{x} \right) - \frac{\csc^{-1}(\sqrt{x})}{x} \\
 & \quad \downarrow \text{73} \\
 & \frac{1}{2} \left( - \int \frac{1}{x} d\sqrt{x-1} - \frac{\sqrt{x-1}}{x} \right) - \frac{\csc^{-1}(\sqrt{x})}{x} \\
 & \quad \downarrow \text{216} \\
 & \frac{1}{2} \left( -\arctan(\sqrt{x-1}) - \frac{\sqrt{x-1}}{x} \right) - \frac{\csc^{-1}(\sqrt{x})}{x}
 \end{aligned}$$

input `Int[ArcCsc[Sqrt[x]]/x^2,x]`

output `-(ArcCsc[Sqrt[x]]/x) + (-(Sqrt[-1 + x]/x) - ArcTan[Sqrt[-1 + x]])/2`

## Definitions of rubi rules used

rule 27 `Int[(a_)*(F_x_), x_Symbol] := Simp[a Int[F_x, x], x] /; FreeQ[a, x] && !MatchQ[F_x, (b_)*(G_x_)] /; FreeQ[b, x]`

rule 52 `Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] := Simp[(a + b*x)^(m + 1)*((c + d*x)^(n + 1)/((b*c - a*d)*(m + 1))), x] - Simp[d*((m + n + 2)/((b*c - a*d)*(m + 1))) Int[(a + b*x)^(m + 1)*(c + d*x)^n, x], x] /; FreeQ[{a, b, c, d, n}, x] && ILtQ[m, -1] && FractionQ[n] && LtQ[n, 0]`

rule 73 `Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] := With[{p = Denominator[m]}, Simp[p/b Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) + d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Denominator[m]] && IntLinearQ[a, b, c, d, m, n, x]`

rule 216 `Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(1/(Rt[a, 2]*Rt[b, 2]))*ArcTan[Rt[b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] && (GtQ[a, 0] || GtQ[b, 0])`

rule 5794 `Int[((a_.) + ArcCsc[u_]*(b_.))*((c_.) + (d_.)*(x_))^(m_.), x_Symbol] := Simp[(c + d*x)^(m + 1)*((a + b*ArcCsc[u])/(d*(m + 1))), x] + Simp[b*(u/(d*(m + 1)*Sqrt[u^2])) Int[SimplifyIntegrand[(c + d*x)^(m + 1)*(D[u, x]/(u*Sqrt[u^2 - 1]))], x], x] /; FreeQ[{a, b, c, d, m}, x] && NeQ[m, -1] && InverseFunctionFreeQ[u, x] && !FunctionOfQ[(c + d*x)^(m + 1), u, x] && !FunctionOfExponentialQ[u, x]`

**Maple [A] (verified)**

Time = 0.16 (sec) , antiderivative size = 44, normalized size of antiderivative = 1.16

method	result	size
parts	$-\frac{\operatorname{arccsc}(\sqrt{x})}{x} - \frac{\sqrt{\frac{-1+x}{x}} (\arctan(\sqrt{-1+x})x + \sqrt{-1+x})}{2\sqrt{x}\sqrt{-1+x}}$	44
derivativedivides	$-\frac{\operatorname{arccsc}(\sqrt{x})}{x} + \frac{\sqrt{-1+x} (\arctan(\frac{1}{\sqrt{-1+x}})x - \sqrt{-1+x})}{2\sqrt{\frac{-1+x}{x}} x^{\frac{3}{2}}}$	46
default	$-\frac{\operatorname{arccsc}(\sqrt{x})}{x} + \frac{\sqrt{-1+x} (\arctan(\frac{1}{\sqrt{-1+x}})x - \sqrt{-1+x})}{2\sqrt{\frac{-1+x}{x}} x^{\frac{3}{2}}}$	46

input `int(arccsc(x^(1/2))/x^2,x,method=_RETURNVERBOSE)`output  $-\operatorname{arccsc}(x^{1/2})/x - 1/2 * ((-1+x)/x)^{1/2} / x^{1/2} * (\arctan((-1+x)^{1/2}) * x + (-1+x)^{1/2}) / (-1+x)^{1/2}$ **Fricas [A] (verification not implemented)**

Time = 0.10 (sec) , antiderivative size = 21, normalized size of antiderivative = 0.55

$$\int \frac{\csc^{-1}(\sqrt{x})}{x^2} dx = \frac{(x-2) \operatorname{arccsc}(\sqrt{x}) - \sqrt{x-1}}{2x}$$

input `integrate(arccsc(x^(1/2))/x^2,x, algorithm="fricas")`output  $1/2 * ((x-2) * \operatorname{arccsc}(\operatorname{sqrt}(x)) - \operatorname{sqrt}(x-1)) / x$

**Sympy [C] (verification not implemented)**

Result contains complex when optimal does not.

Time = 16.13 (sec) , antiderivative size = 76, normalized size of antiderivative = 2.00

$$\int \frac{\csc^{-1}(\sqrt{x})}{x^2} dx = -\frac{\begin{cases} i \operatorname{acosh}\left(\frac{1}{\sqrt{x}}\right) - \frac{i}{\sqrt{x}\sqrt{-1+\frac{1}{x}}} + \frac{i}{x^{\frac{3}{2}}\sqrt{-1+\frac{1}{x}}} & \text{for } \frac{1}{|x|} > 1 \\ -\operatorname{asin}\left(\frac{1}{\sqrt{x}}\right) + \frac{\sqrt{1-\frac{1}{x}}}{\sqrt{x}} & \text{otherwise} \end{cases}}{2} - \frac{\operatorname{acsc}(\sqrt{x})}{x}$$

input `integrate(acsc(x**(1/2))/x**2,x)`

output `-Piecewise((I*acosh(1/sqrt(x)) - I/(sqrt(x)*sqrt(-1 + 1/x)) + I/(x**(3/2)*sqrt(-1 + 1/x)), 1/Abs(x) > 1), (-asin(1/sqrt(x)) + sqrt(1 - 1/x)/sqrt(x), True))/2 - acsc(sqrt(x))/x`

**Maxima [A] (verification not implemented)**

Time = 0.13 (sec) , antiderivative size = 51, normalized size of antiderivative = 1.34

$$\int \frac{\csc^{-1}(\sqrt{x})}{x^2} dx = \frac{\sqrt{x}\sqrt{-\frac{1}{x}+1}}{2\left(x\left(\frac{1}{x}-1\right)-1\right)} - \frac{\operatorname{arccsc}(\sqrt{x})}{x} - \frac{1}{2} \arctan\left(\sqrt{x}\sqrt{-\frac{1}{x}+1}\right)$$

input `integrate(arccsc(x^(1/2))/x^2,x, algorithm="maxima")`

output `1/2*sqrt(x)*sqrt(-1/x + 1)/(x*(1/x - 1) - 1) - arccsc(sqrt(x))/x - 1/2*arc tan(sqrt(x)*sqrt(-1/x + 1))`



**Giac [A] (verification not implemented)**

Time = 0.12 (sec) , antiderivative size = 32, normalized size of antiderivative = 0.84

$$\int \frac{\csc^{-1}(\sqrt{x})}{x^2} dx = -\left(\frac{1}{x} - 1\right) \arcsin\left(\frac{1}{\sqrt{x}}\right) - \frac{\sqrt{-\frac{1}{x} + 1}}{2\sqrt{x}} - \frac{1}{2} \arcsin\left(\frac{1}{\sqrt{x}}\right)$$

input `integrate(arccsc(x^(1/2))/x^2,x, algorithm="giac")`

output `-(1/x - 1)*arcsin(1/sqrt(x)) - 1/2*sqrt(-1/x + 1)/sqrt(x) - 1/2*arcsin(1/sqrt(x))`

**Mupad [B] (verification not implemented)**

Time = 0.91 (sec) , antiderivative size = 28, normalized size of antiderivative = 0.74

$$\int \frac{\csc^{-1}(\sqrt{x})}{x^2} dx = -\frac{\sqrt{1 - \frac{1}{x}}}{2\sqrt{x}} - \frac{\operatorname{asin}\left(\frac{1}{\sqrt{x}}\right) \left(\frac{2}{x} - 1\right)}{2}$$

input `int(asin(1/x^(1/2))/x^2,x)`

output `-(1 - 1/x)^(1/2)/(2*x^(1/2)) - (asin(1/x^(1/2))*(2/x - 1))/2`

**Reduce [F]**

$$\int \frac{\csc^{-1}(\sqrt{x})}{x^2} dx = \int \frac{\operatorname{acsc}(\sqrt{x})}{x^2} dx$$

input `int(acsc(x^(1/2))/x^2,x)`

output `int(acsc(sqrt(x))/x**2,x)`

### 3.8 $\int \frac{\csc^{-1}(\sqrt{x})}{x^3} dx$

Optimal result . . . . .	89
Mathematica [A] (verified) . . . . .	89
Rubi [A] (verified) . . . . .	90
Maple [A] (verified) . . . . .	92
Fricas [A] (verification not implemented) . . . . .	92
Sympy [C] (verification not implemented) . . . . .	93
Maxima [B] (verification not implemented) . . . . .	93
Giac [A] (verification not implemented) . . . . .	94
Mupad [F(-1)] . . . . .	94
Reduce [F] . . . . .	95

#### Optimal result

Integrand size = 10, antiderivative size = 54

$$\int \frac{\csc^{-1}(\sqrt{x})}{x^3} dx = -\frac{\sqrt{-1+x}}{8x^2} - \frac{3\sqrt{-1+x}}{16x} - \frac{\csc^{-1}(\sqrt{x})}{2x^2} - \frac{3}{16} \arctan(\sqrt{-1+x})$$

output `-1/8*(-1+x)^(1/2)/x^2-3/16*(-1+x)^(1/2)/x-1/2*arccsc(x^(1/2))/x^2-3/16*arctan((-1+x)^(1/2))`

#### Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 55, normalized size of antiderivative = 1.02

$$\int \frac{\csc^{-1}(\sqrt{x})}{x^3} dx = \left(-\frac{1}{8x^{3/2}} - \frac{3}{16\sqrt{x}}\right) \sqrt{\frac{-1+x}{x}} - \frac{\csc^{-1}(\sqrt{x})}{2x^2} + \frac{3}{16} \arcsin\left(\frac{1}{\sqrt{x}}\right)$$

input `Integrate[ArcCsc[Sqrt[x]]/x^3,x]`

output `(-1/8*1/x^(3/2) - 3/(16*Sqrt[x]))*Sqrt[(-1 + x)/x] - ArcCsc[Sqrt[x]]/(2*x^2) + (3*ArcSin[1/Sqrt[x]])/16`

**Rubi [A] (verified)**

Time = 0.21 (sec) , antiderivative size = 57, normalized size of antiderivative = 1.06, number of steps used = 7, number of rules used = 6,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.600$ , Rules used = {5794, 27, 52, 52, 73, 216}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{\csc^{-1}(\sqrt{x})}{x^3} dx \\
 & \quad \downarrow \text{5794} \\
 & -\frac{1}{2} \int \frac{1}{2\sqrt{x-1}x^3} dx - \frac{\csc^{-1}(\sqrt{x})}{2x^2} \\
 & \quad \downarrow \text{27} \\
 & -\frac{1}{4} \int \frac{1}{\sqrt{x-1}x^3} dx - \frac{\csc^{-1}(\sqrt{x})}{2x^2} \\
 & \quad \downarrow \text{52} \\
 & \frac{1}{4} \left( -\frac{3}{4} \int \frac{1}{\sqrt{x-1}x^2} dx - \frac{\sqrt{x-1}}{2x^2} \right) - \frac{\csc^{-1}(\sqrt{x})}{2x^2} \\
 & \quad \downarrow \text{52} \\
 & \frac{1}{4} \left( -\frac{3}{4} \left( \frac{1}{2} \int \frac{1}{\sqrt{x-1}x} dx + \frac{\sqrt{x-1}}{x} \right) - \frac{\sqrt{x-1}}{2x^2} \right) - \frac{\csc^{-1}(\sqrt{x})}{2x^2} \\
 & \quad \downarrow \text{73} \\
 & \frac{1}{4} \left( -\frac{3}{4} \left( \int \frac{1}{x} d\sqrt{x-1} + \frac{\sqrt{x-1}}{x} \right) - \frac{\sqrt{x-1}}{2x^2} \right) - \frac{\csc^{-1}(\sqrt{x})}{2x^2} \\
 & \quad \downarrow \text{216} \\
 & \frac{1}{4} \left( -\frac{3}{4} \left( \arctan(\sqrt{x-1}) + \frac{\sqrt{x-1}}{x} \right) - \frac{\sqrt{x-1}}{2x^2} \right) - \frac{\csc^{-1}(\sqrt{x})}{2x^2}
 \end{aligned}$$

input `Int[ArcCsc[Sqrt[x]]/x^3,x]`

output

```
-1/2*ArcCsc[Sqrt[x]]/x^2 + (-1/2*Sqrt[-1 + x]/x^2 - (3*(Sqrt[-1 + x]/x + ArcTan[Sqrt[-1 + x]]))/4)/4
```

### Defintions of rubi rules used

rule 27

```
Int[(a_)*(F_x_), x_Symbol] := Simp[a Int[F_x, x], x] /; FreeQ[a, x] && !MatchQ[F_x, (b_)*(G_x_)] /; FreeQ[b, x]
```

rule 52

```
Int[((a_.) + (b_.)*(x_)^(m_))*((c_.) + (d_.)*(x_)^(n_)), x_Symbol] := Simp[(a + b*x)^(m + 1)*((c + d*x)^(n + 1))/((b*c - a*d)*(m + 1)), x] - Simp[d*((m + n + 2)/((b*c - a*d)*(m + 1)) Int[(a + b*x)^(m + 1)*(c + d*x)^n, x], x] /; FreeQ[{a, b, c, d, n}, x] && ILtQ[m, -1] && FractionQ[n] && LtQ[n, 0]
```

rule 73

```
Int[((a_.) + (b_.)*(x_)^(m_))*((c_.) + (d_.)*(x_)^(n_)), x_Symbol] := With[{p = Denominator[m]}, Simp[p/b Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) + d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x] /; FreeQ[{a, b, c, d}, x] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Denominator[m]] && IntegerQ[a, b, c, d, m, n, x]
```

rule 216

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(1/(Rt[a, 2]*Rt[b, 2]))*ArcTan[Rt[b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] && (GtQ[a, 0] || GtQ[b, 0])
```

rule 5794

```
Int[((a_.) + ArcCsc[u_]*(b_.))*((c_.) + (d_.)*(x_)^(m_.)), x_Symbol] := Simp[(c + d*x)^(m + 1)*((a + b*ArcCsc[u])/(d*(m + 1))), x] + Simp[b*(u/(d*(m + 1)*Sqrt[u^2])) Int[SimplifyIntegrand[(c + d*x)^(m + 1)*(D[u, x]/(u*Sqrt[u^2 - 1]))], x], x] /; FreeQ[{a, b, c, d, m}, x] && NeQ[m, -1] && InverseFunctionFreeQ[u, x] && !FunctionOfQ[(c + d*x)^(m + 1), u, x] && !FunctionOfExponentialQ[u, x]
```

**Maple [A] (verified)**

Time = 0.16 (sec) , antiderivative size = 57, normalized size of antiderivative = 1.06

method	result	size
derivativedivides	$-\frac{\operatorname{arccsc}(\sqrt{x})}{2x^2} + \frac{\sqrt{-1+x} \left( 3 \arctan\left(\frac{1}{\sqrt{-1+x}}\right) x^2 - 3\sqrt{-1+x} x - 2\sqrt{-1+x} \right)}{16 \sqrt{\frac{-1+x}{x}} x^{\frac{5}{2}}}$	57
default	$-\frac{\operatorname{arccsc}(\sqrt{x})}{2x^2} + \frac{\sqrt{-1+x} \left( 3 \arctan\left(\frac{1}{\sqrt{-1+x}}\right) x^2 - 3\sqrt{-1+x} x - 2\sqrt{-1+x} \right)}{16 \sqrt{\frac{-1+x}{x}} x^{\frac{5}{2}}}$	57
parts	$-\frac{\operatorname{arccsc}(\sqrt{x})}{2x^2} - \frac{\sqrt{\frac{-1+x}{x}} \left( 3 \arctan(\sqrt{-1+x}) x^2 + 3\sqrt{-1+x} x + 2\sqrt{-1+x} \right)}{16x^{\frac{3}{2}} \sqrt{-1+x}}$	57

input `int(arccsc(x^(1/2))/x^3,x,method=_RETURNVERBOSE)`output `-1/2*arccsc(x^(1/2))/x^2+1/16*(-1+x)^(1/2)*(3*arctan(1/(-1+x)^(1/2))*x^2-3*(-1+x)^(1/2)*x-2*(-1+x)^(1/2))/((-1+x)/x)^(1/2)/x^(5/2)`**Fricas [A] (verification not implemented)**

Time = 0.09 (sec) , antiderivative size = 30, normalized size of antiderivative = 0.56

$$\int \frac{\csc^{-1}(\sqrt{x})}{x^3} dx = \frac{(3x^2 - 8) \operatorname{arccsc}(\sqrt{x}) - (3x + 2)\sqrt{x-1}}{16x^2}$$

input `integrate(arccsc(x^(1/2))/x^3,x, algorithm="fricas")`output `1/16*((3*x^2 - 8)*arccsc(sqrt(x)) - (3*x + 2)*sqrt(x - 1))/x^2`

**Sympy [C] (verification not implemented)**

Result contains complex when optimal does not.

Time = 41.24 (sec) , antiderivative size = 146, normalized size of antiderivative = 2.70

$$\int \frac{\csc^{-1}(\sqrt{x})}{x^3} dx = - \frac{\begin{cases} \frac{3i \operatorname{acosh}\left(\frac{1}{\sqrt{x}}\right)}{4} - \frac{3i}{4\sqrt{x}\sqrt{-1+\frac{1}{x}}} + \frac{i}{4x^{\frac{3}{2}}\sqrt{-1+\frac{1}{x}}} + \frac{i}{2x^{\frac{5}{2}}\sqrt{-1+\frac{1}{x}}} & \text{for } \frac{1}{|x|} > 1 \\ -\frac{3 \operatorname{asin}\left(\frac{1}{\sqrt{x}}\right)}{4} + \frac{3}{4\sqrt{x}\sqrt{1-\frac{1}{x}}} - \frac{1}{4x^{\frac{3}{2}}\sqrt{1-\frac{1}{x}}} - \frac{1}{2x^{\frac{5}{2}}\sqrt{1-\frac{1}{x}}} & \text{otherwise} \end{cases}}{4} - \frac{\operatorname{acsc}(\sqrt{x})}{2x^2}$$

input `integrate(acsc(x**(1/2))/x**3,x)`

output `-Piecewise((3*I*acosh(1/sqrt(x))/4 - 3*I/(4*sqrt(x)*sqrt(-1 + 1/x)) + I/(4*x**(3/2)*sqrt(-1 + 1/x)) + I/(2*x**(5/2)*sqrt(-1 + 1/x)), 1/Abs(x) > 1), (-3*asin(1/sqrt(x))/4 + 3/(4*sqrt(x)*sqrt(1 - 1/x)) - 1/(4*x**(3/2)*sqrt(1 - 1/x)) - 1/(2*x**(5/2)*sqrt(1 - 1/x)), True))/4 - acsc(sqrt(x))/(2*x**2)`

**Maxima [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 80 vs. 2(38) = 76.

Time = 0.14 (sec) , antiderivative size = 80, normalized size of antiderivative = 1.48

$$\int \frac{\csc^{-1}(\sqrt{x})}{x^3} dx = - \frac{3x^{\frac{3}{2}}\left(-\frac{1}{x} + 1\right)^{\frac{3}{2}} + 5\sqrt{x}\sqrt{-\frac{1}{x} + 1}}{16\left(x^2\left(\frac{1}{x} - 1\right)^2 - 2x\left(\frac{1}{x} - 1\right) + 1\right)} - \frac{\operatorname{arccsc}(\sqrt{x})}{2x^2} - \frac{3}{16} \arctan\left(\sqrt{x}\sqrt{-\frac{1}{x} + 1}\right)$$

input `integrate(arccsc(x^(1/2))/x^3,x, algorithm="maxima")`

output

```
-1/16*(3*x^(3/2)*(-1/x + 1)^(3/2) + 5*sqrt(x)*sqrt(-1/x + 1))/(x^2*(1/x - 1)^2 - 2*x*(1/x - 1) + 1) - 1/2*arccsc(sqrt(x))/x^2 - 3/16*arctan(sqrt(x)*sqrt(-1/x + 1))
```

**Giac [A] (verification not implemented)**

Time = 0.12 (sec) , antiderivative size = 59, normalized size of antiderivative = 1.09

$$\int \frac{\csc^{-1}(\sqrt{x})}{x^3} dx = -\frac{1}{2} \left(\frac{1}{x} - 1\right)^2 \arcsin\left(\frac{1}{\sqrt{x}}\right) - \left(\frac{1}{x} - 1\right) \arcsin\left(\frac{1}{\sqrt{x}}\right) + \frac{\left(-\frac{1}{x} + 1\right)^{\frac{3}{2}}}{8\sqrt{x}} - \frac{5\sqrt{-\frac{1}{x} + 1}}{16\sqrt{x}} - \frac{5}{16} \arcsin\left(\frac{1}{\sqrt{x}}\right)$$

input

```
integrate(arccsc(x^(1/2))/x^3,x, algorithm="giac")
```

output

```
-1/2*(1/x - 1)^2*arcsin(1/sqrt(x)) - (1/x - 1)*arcsin(1/sqrt(x)) + 1/8*(-1/x + 1)^(3/2)/sqrt(x) - 5/16*sqrt(-1/x + 1)/sqrt(x) - 5/16*arcsin(1/sqrt(x))
```

**Mupad [F(-1)]**

Timed out.

$$\int \frac{\csc^{-1}(\sqrt{x})}{x^3} dx = \int \frac{\operatorname{asin}\left(\frac{1}{\sqrt{x}}\right)}{x^3} dx$$

input

```
int(asin(1/x^(1/2))/x^3,x)
```

output

```
int(asin(1/x^(1/2))/x^3, x)
```

**Reduce [F]**

$$\int \frac{\csc^{-1}(\sqrt{x})}{x^3} dx = \int \frac{\operatorname{acsc}(\sqrt{x})}{x^3} dx$$

input `int(acsc(x^(1/2))/x^3,x)`

output `int(acsc(sqrt(x))/x**3,x)`



### 3.9 $\int x^2 \csc^{-1} \left( \frac{a}{x} \right) dx$

Optimal result	96
Mathematica [A] (verified)	96
Rubi [A] (verified)	97
Maple [A] (verified)	98
Fricas [A] (verification not implemented)	99
Sympy [A] (verification not implemented)	99
Maxima [A] (verification not implemented)	100
Giac [A] (verification not implemented)	100
Mupad [F(-1)]	100
Reduce [F]	101

#### Optimal result

Integrand size = 10, antiderivative size = 56

$$\int x^2 \csc^{-1} \left( \frac{a}{x} \right) dx = \frac{1}{3} a^3 \sqrt{1 - \frac{x^2}{a^2}} - \frac{1}{9} a^3 \left( 1 - \frac{x^2}{a^2} \right)^{3/2} + \frac{1}{3} x^3 \arcsin \left( \frac{x}{a} \right)$$

output  $1/3*a^3*(1-x^2/a^2)^{(1/2)}-1/9*a^3*(1-x^2/a^2)^{(3/2)}+1/3*x^3*\arcsin(x/a)$

#### Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 42, normalized size of antiderivative = 0.75

$$\int x^2 \csc^{-1} \left( \frac{a}{x} \right) dx = \frac{1}{9} a (2a^2 + x^2) \sqrt{1 - \frac{x^2}{a^2}} + \frac{1}{3} x^3 \csc^{-1} \left( \frac{a}{x} \right)$$

input `Integrate[x^2*ArcCsc[a/x],x]`

output  $(a*(2*a^2 + x^2)*\text{Sqrt}[1 - x^2/a^2])/9 + (x^3*\text{ArcCsc}[a/x])/3$

**Rubi [A] (verified)**

Time = 0.27 (sec) , antiderivative size = 62, normalized size of antiderivative = 1.11, number of steps used = 6, number of rules used = 5,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {5788, 5138, 243, 53, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int x^2 \csc^{-1}\left(\frac{a}{x}\right) dx \\
 & \quad \downarrow \text{5788} \\
 & \int x^2 \arcsin\left(\frac{x}{a}\right) dx \\
 & \quad \downarrow \text{5138} \\
 & \frac{1}{3}x^3 \arcsin\left(\frac{x}{a}\right) - \frac{\int \frac{x^3}{\sqrt{1-\frac{x^2}{a^2}}} dx}{3a} \\
 & \quad \downarrow \text{243} \\
 & \frac{1}{3}x^3 \arcsin\left(\frac{x}{a}\right) - \frac{\int \frac{x^2}{\sqrt{1-\frac{x^2}{a^2}}} dx^2}{6a} \\
 & \quad \downarrow \text{53} \\
 & \frac{1}{3}x^3 \arcsin\left(\frac{x}{a}\right) - \frac{\int \left(\frac{a^2}{\sqrt{1-\frac{x^2}{a^2}}} - a^2 \sqrt{1-\frac{x^2}{a^2}}\right) dx^2}{6a} \\
 & \quad \downarrow \text{2009} \\
 & \frac{1}{3}x^3 \arcsin\left(\frac{x}{a}\right) - \frac{\frac{2}{3}a^4 \left(1-\frac{x^2}{a^2}\right)^{3/2} - 2a^4 \sqrt{1-\frac{x^2}{a^2}}}{6a}
 \end{aligned}$$

input `Int [x^2*ArcCsc [a/x] , x]`

output `-1/6*(-2*a^4*sqrt[1 - x^2/a^2] + (2*a^4*(1 - x^2/a^2)^(3/2))/3)/a + (x^3*ArcSin[x/a])/3`

## Definitions of rubi rules used

- rule 53  $\text{Int}[(a_.) + (b_.)(x_)^{(m_.)}*((c_.) + (d_.)(x_)^{(n_.)}], x\_Symbol] \rightarrow \text{Int}[\text{ExpandIntegrand}[(a + b*x)^m*(c + d*x)^n, x], x] /; \text{FreeQ}[\{a, b, c, d, n\}, x] \ \&\& \ \text{IGtQ}[m, 0] \ \&\& \ (\ !\text{IntegerQ}[n] \ || \ (\text{EqQ}[c, 0] \ \&\& \ \text{LeQ}[7*m + 4*n + 4, 0]) \ || \ \text{LtQ}[9*m + 5*(n + 1), 0] \ || \ \text{GtQ}[m + n + 2, 0])$
- rule 243  $\text{Int}[(x_)^{(m_.)}*((a_) + (b_.)(x_)^2)^{(p_.)}], x\_Symbol] \rightarrow \text{Simp}[1/2 \ \text{Subst}[\text{Int}[x^{(m-1)/2}*(a + b*x)^p, x], x, x^2], x] /; \text{FreeQ}[\{a, b, m, p\}, x] \ \&\& \ \text{IntegerQ}[(m-1)/2]$
- rule 2009  $\text{Int}[u_, x\_Symbol] \rightarrow \text{Simp}[\text{IntSum}[u, x], x] /; \text{SumQ}[u]$
- rule 5138  $\text{Int}[(a_.) + \text{ArcSin}[c_.)(x_)]*(b_.)^{(n_.)}*((d_.)(x_)^{(m_.)}], x\_Symbol] \rightarrow \text{Simp}[(d*x)^{m+1}*((a + b*\text{ArcSin}[c*x])^n/(d*(m+1))), x] - \text{Simp}[b*c*(n/(d*(m+1))) \ \text{Int}[(d*x)^{m+1}*((a + b*\text{ArcSin}[c*x])^{n-1}/\text{Sqrt}[1 - c^2*x^2]), x], x] /; \text{FreeQ}[\{a, b, c, d, m\}, x] \ \&\& \ \text{IGtQ}[n, 0] \ \&\& \ \text{NeQ}[m, -1]$
- rule 5788  $\text{Int}[\text{ArcCsc}[(c_.)/((a_.) + (b_.)(x_)^{(n_.)})]^{(m_.)}*(u_.), x\_Symbol] \rightarrow \text{Int}[u*\text{ArcSin}[a/c + b*(x^n/c)]^m, x] /; \text{FreeQ}[\{a, b, c, n, m\}, x]$

## Maple [A] (verified)

Time = 1.04 (sec) , antiderivative size = 56, normalized size of antiderivative = 1.00

method	result	size
parts	$\frac{x^3 \operatorname{arccsc}\left(\frac{a}{x}\right)}{3} - \frac{x^2 a^2 \sqrt{1 - \frac{x^2}{a^2}}}{3} - \frac{2a^4 \sqrt{1 - \frac{x^2}{a^2}}}{3a}$	56
derivativedivides	$-a^3 \left( -\frac{x^3 \operatorname{arccsc}\left(\frac{a}{x}\right)}{3a^3} - \frac{\left(\frac{a^2}{x^2} - 1\right) \left(\frac{2a^2}{x^2} + 1\right) x^4}{9 \sqrt{\frac{\left(\frac{a^2}{x^2} - 1\right) x^2}{a^2}} a^4} \right)$	66
default	$-a^3 \left( -\frac{x^3 \operatorname{arccsc}\left(\frac{a}{x}\right)}{3a^3} - \frac{\left(\frac{a^2}{x^2} - 1\right) \left(\frac{2a^2}{x^2} + 1\right) x^4}{9 \sqrt{\frac{\left(\frac{a^2}{x^2} - 1\right) x^2}{a^2}} a^4} \right)$	66

input `int(x^2*arccsc(a/x),x,method=_RETURNVERBOSE)`

output `1/3*x^3*arccsc(a/x)-1/3/a*(-1/3*x^2*a^2*(1-x^2/a^2)^(1/2)-2/3*a^4*(1-x^2/a^2)^(1/2))`

### Fricas [A] (verification not implemented)

Time = 0.12 (sec) , antiderivative size = 39, normalized size of antiderivative = 0.70

$$\int x^2 \csc^{-1} \left( \frac{a}{x} \right) dx = \frac{1}{3} x^3 \operatorname{arccsc} \left( \frac{a}{x} \right) + \frac{1}{9} (2 a^2 x + x^3) \sqrt{\frac{a^2 - x^2}{x^2}}$$

input `integrate(x^2*arccsc(a/x),x, algorithm="fricas")`

output `1/3*x^3*arccsc(a/x) + 1/9*(2*a^2*x + x^3)*sqrt((a^2 - x^2)/x^2)`

### Sympy [A] (verification not implemented)

Time = 0.26 (sec) , antiderivative size = 51, normalized size of antiderivative = 0.91

$$\int x^2 \csc^{-1} \left( \frac{a}{x} \right) dx = \begin{cases} \frac{2a^3 \sqrt{1 - \frac{x^2}{a^2}}}{9} + \frac{ax^2 \sqrt{1 - \frac{x^2}{a^2}}}{9} + \frac{x^3 \operatorname{acsc} \left( \frac{a}{x} \right)}{3} & \text{for } a \neq 0 \\ \infty x^3 & \text{otherwise} \end{cases}$$

input `integrate(x**2*acsc(a/x),x)`

output `Piecewise(((2*a**3*sqrt(1 - x**2/a**2)/9 + a*x**2*sqrt(1 - x**2/a**2)/9 + x**3*acsc(a/x)/3, Ne(a, 0)), (zoo*x**3, True))`

**Maxima [A] (verification not implemented)**

Time = 0.13 (sec) , antiderivative size = 54, normalized size of antiderivative = 0.96

$$\int x^2 \csc^{-1}\left(\frac{a}{x}\right) dx = \frac{1}{3} x^3 \operatorname{arccsc}\left(\frac{a}{x}\right) + \frac{2a^4 \sqrt{-\frac{x^2}{a^2} + 1} + a^2 x^2 \sqrt{-\frac{x^2}{a^2} + 1}}{9a}$$

input `integrate(x^2*arccsc(a/x),x, algorithm="maxima")`output `1/3*x^3*arccsc(a/x) + 1/9*(2*a^4*sqrt(-x^2/a^2 + 1) + a^2*x^2*sqrt(-x^2/a^2 + 1))/a`**Giac [A] (verification not implemented)**

Time = 0.12 (sec) , antiderivative size = 68, normalized size of antiderivative = 1.21

$$\int x^2 \csc^{-1}\left(\frac{a}{x}\right) dx = \frac{1}{3} a^2 x \left(\frac{x^2}{a^2} - 1\right) \arcsin\left(\frac{x}{a}\right) - \frac{1}{9} a^3 \left(-\frac{x^2}{a^2} + 1\right)^{\frac{3}{2}} + \frac{1}{3} a^2 x \arcsin\left(\frac{x}{a}\right) + \frac{1}{3} a^3 \sqrt{-\frac{x^2}{a^2} + 1}$$

input `integrate(x^2*arccsc(a/x),x, algorithm="giac")`output `1/3*a^2*x*(x^2/a^2 - 1)*arcsin(x/a) - 1/9*a^3*(-x^2/a^2 + 1)^(3/2) + 1/3*a^2*x*arcsin(x/a) + 1/3*a^3*sqrt(-x^2/a^2 + 1)`**Mupad [F(-1)]**

Timed out.

$$\int x^2 \csc^{-1}\left(\frac{a}{x}\right) dx = \begin{cases} \frac{x^3 \operatorname{asin}\left(\frac{x}{a}\right)}{3} + \frac{\sqrt{a^2 - x^2} (2a^2 + x^2)}{9} & \text{if } 0 < a \\ \int x^2 \operatorname{asin}\left(\frac{x}{a}\right) dx & \text{if } -0 < a \end{cases}$$

input `int(x^2*asin(x/a),x)`

output `piecewise(0 < a, (x^3*asin(x/a))/3 + ((a^2 - x^2)^(1/2)*(2*a^2 + x^2))/9, ~0 < a, int(x^2*asin(x/a), x))`

### Reduce [F]

$$\int x^2 \csc^{-1} \left( \frac{a}{x} \right) dx = \int \operatorname{acsc} \left( \frac{a}{x} \right) x^2 dx$$

input `int(x^2*acsc(a/x), x)`

output `int(acsc(a/x)*x**2, x)`

### 3.10 $\int x \csc^{-1} \left( \frac{a}{x} \right) dx$

Optimal result	102
Mathematica [A] (verified)	102
Rubi [A] (verified)	103
Maple [A] (verified)	104
Fricas [A] (verification not implemented)	105
Sympy [A] (verification not implemented)	106
Maxima [A] (verification not implemented)	106
Giac [A] (verification not implemented)	107
Mupad [B] (verification not implemented)	107
Reduce [B] (verification not implemented)	107

#### Optimal result

Integrand size = 8, antiderivative size = 47

$$\int x \csc^{-1} \left( \frac{a}{x} \right) dx = \frac{1}{4}ax\sqrt{1 - \frac{x^2}{a^2}} - \frac{1}{4}a^2 \arcsin \left( \frac{x}{a} \right) + \frac{1}{2}x^2 \arcsin \left( \frac{x}{a} \right)$$

output `1/4*a*x*(1-x^2/a^2)^(1/2)-1/4*a^2*arcsin(x/a)+1/2*x^2*arcsin(x/a)`

#### Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 44, normalized size of antiderivative = 0.94

$$\int x \csc^{-1} \left( \frac{a}{x} \right) dx = \frac{1}{4} \left( ax\sqrt{1 - \frac{x^2}{a^2}} + 2x^2 \csc^{-1} \left( \frac{a}{x} \right) - a^2 \arcsin \left( \frac{x}{a} \right) \right)$$

input `Integrate[x*ArcCsc[a/x],x]`

output `(a*x*Sqrt[1 - x^2/a^2] + 2*x^2*ArcCsc[a/x] - a^2*ArcSin[x/a])/4`

**Rubi [A] (verified)**

Time = 0.24 (sec) , antiderivative size = 57, normalized size of antiderivative = 1.21, number of steps used = 4, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {5788, 5138, 262, 223}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int x \csc^{-1}\left(\frac{a}{x}\right) dx \\
 & \quad \downarrow \text{5788} \\
 & \int x \arcsin\left(\frac{x}{a}\right) dx \\
 & \quad \downarrow \text{5138} \\
 & \frac{1}{2}x^2 \arcsin\left(\frac{x}{a}\right) - \frac{\int \frac{x^2}{\sqrt{1-\frac{x^2}{a^2}}} dx}{2a} \\
 & \quad \downarrow \text{262} \\
 & \frac{1}{2}x^2 \arcsin\left(\frac{x}{a}\right) - \frac{\frac{1}{2}a^2 \int \frac{1}{\sqrt{1-\frac{x^2}{a^2}}} dx - \frac{1}{2}a^2 x \sqrt{1-\frac{x^2}{a^2}}}{2a} \\
 & \quad \downarrow \text{223} \\
 & \frac{1}{2}x^2 \arcsin\left(\frac{x}{a}\right) - \frac{\frac{1}{2}a^3 \arcsin\left(\frac{x}{a}\right) - \frac{1}{2}a^2 x \sqrt{1-\frac{x^2}{a^2}}}{2a}
 \end{aligned}$$

input

```
Int[x*ArcCsc[a/x], x]
```

output

```
(x^2*ArcSin[x/a])/2 - (-1/2*(a^2*x*sqrt[1 - x^2/a^2]) + (a^3*ArcSin[x/a])/2)/(2*a)
```



**Defintions of rubi rules used**

rule 223 `Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] := Simp[ArcSin[Rt[-b, 2]*(x/Sqrt[a])]/Rt[-b, 2], x] /; FreeQ[{a, b}, x] && GtQ[a, 0] && NegQ[b]`

rule 262 `Int[((c_.)*(x_))^(m_)*((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] := Simp[c*(c*x)^(m - 1)*((a + b*x^2)^(p + 1)/(b*(m + 2*p + 1))), x] - Simp[a*c^2*((m - 1)/(b*(m + 2*p + 1))) Int[(c*x)^(m - 2)*(a + b*x^2)^p, x], x] /; FreeQ[{a, b, c, p}, x] && GtQ[m, 2 - 1] && NeQ[m + 2*p + 1, 0] && IntBinomialQ[a, b, c, 2, m, p, x]`

rule 5138 `Int[((a_.) + ArcSin[(c_.)*(x_)]*(b_.))^(n_.)*((d_.)*(x_)^(m_.), x_Symbol] := Simp[(d*x)^(m + 1)*((a + b*ArcSin[c*x])^n/(d*(m + 1))), x] - Simp[b*c*(n/(d*(m + 1))) Int[(d*x)^(m + 1)*((a + b*ArcSin[c*x])^(n - 1)/Sqrt[1 - c^2*x^2]), x], x] /; FreeQ[{a, b, c, d, m}, x] && IGtQ[n, 0] && NeQ[m, -1]`

rule 5788 `Int[ArcCsc[(c_.)/((a_.) + (b_.)*(x_)^(n_.))]^(m_.)*(u_.), x_Symbol] := Int[u*ArcSin[a/c + b*(x^n/c)]^m, x] /; FreeQ[{a, b, c, n, m}, x]`

**Maple [A] (verified)**

Time = 1.03 (sec) , antiderivative size = 67, normalized size of antiderivative = 1.43

method	result	size
parts	$\frac{x^2 \operatorname{arccsc}\left(\frac{a}{x}\right)}{2} - \frac{-\frac{x a^2 \sqrt{1-\frac{x^2}{a^2}}}{2} + \frac{a^2 \arctan\left(\frac{\sqrt{\frac{1}{a^2}} x}{\sqrt{1-\frac{x^2}{a^2}}}\right)}{2\sqrt{\frac{1}{a^2}}}}{2a}$	67
derivativedivides	$-a^2 \left( -\frac{x^2 \operatorname{arccsc}\left(\frac{a}{x}\right)}{2a^2} + \frac{\sqrt{\frac{a^2}{x^2}-1} \left( \frac{\arctan\left(\frac{1}{\sqrt{\frac{a^2}{x^2}-1}}\right) a^2}{x^2} - \sqrt{\frac{a^2}{x^2}-1} x^3 \right)}{4\sqrt{\frac{\left(\frac{a^2}{x^2}-1\right) x^2}{a^2}} a^3} \right)$	91
default	$-a^2 \left( -\frac{x^2 \operatorname{arccsc}\left(\frac{a}{x}\right)}{2a^2} + \frac{\sqrt{\frac{a^2}{x^2}-1} \left( \frac{\arctan\left(\frac{1}{\sqrt{\frac{a^2}{x^2}-1}}\right) a^2}{x^2} - \sqrt{\frac{a^2}{x^2}-1} x^3 \right)}{4\sqrt{\frac{\left(\frac{a^2}{x^2}-1\right) x^2}{a^2}} a^3} \right)$	91

```
input int(x*arccsc(a/x),x,method=_RETURNVERBOSE)
```

```
output 1/2*x^2*arccsc(a/x)-1/2/a*(-1/2*x*a^2*(1-x^2/a^2)^(1/2)+1/2*a^2/(1/a^2)^(1/2)*arctan((1/a^2)^(1/2)*x/(1-x^2/a^2)^(1/2)))
```

**Fricas [A] (verification not implemented)**

Time = 0.11 (sec) , antiderivative size = 38, normalized size of antiderivative = 0.81

$$\int x \operatorname{csc}^{-1}\left(\frac{a}{x}\right) dx = \frac{1}{4} x^2 \sqrt{\frac{a^2 - x^2}{x^2}} - \frac{1}{4} (a^2 - 2x^2) \operatorname{arccsc}\left(\frac{a}{x}\right)$$

```
input integrate(x*arccsc(a/x),x, algorithm="fricas")
```

output `1/4*x^2*sqrt((a^2 - x^2)/x^2) - 1/4*(a^2 - 2*x^2)*arccsc(a/x)`

### Sympy [A] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 41, normalized size of antiderivative = 0.87

$$\int x \csc^{-1} \left( \frac{a}{x} \right) dx = \begin{cases} -\frac{a^2 \operatorname{arccsc} \left( \frac{a}{x} \right)}{4} + \frac{ax \sqrt{1 - \frac{x^2}{a^2}}}{4} + \frac{x^2 \operatorname{arccsc} \left( \frac{a}{x} \right)}{2} & \text{for } a \neq 0 \\ \tilde{\infty} x^2 & \text{otherwise} \end{cases}$$

input `integrate(x*acsc(a/x),x)`

output `Piecewise((-a**2*acsc(a/x)/4 + a*x*sqrt(1 - x**2/a**2)/4 + x**2*acsc(a/x)/2, Ne(a, 0)), (zoo*x**2, True))`

### Maxima [A] (verification not implemented)

Time = 0.14 (sec) , antiderivative size = 46, normalized size of antiderivative = 0.98

$$\int x \csc^{-1} \left( \frac{a}{x} \right) dx = \frac{1}{2} x^2 \operatorname{arccsc} \left( \frac{a}{x} \right) - \frac{a^3 \arcsin \left( \frac{x}{a} \right) - a^2 x \sqrt{-\frac{x^2}{a^2} + 1}}{4 a}$$

input `integrate(x*arccsc(a/x),x, algorithm="maxima")`

output `1/2*x^2*arccsc(a/x) - 1/4*(a^3*arcsin(x/a) - a^2*x*sqrt(-x^2/a^2 + 1))/a`

**Giac [A] (verification not implemented)**

Time = 0.11 (sec) , antiderivative size = 48, normalized size of antiderivative = 1.02

$$\int x \csc^{-1}\left(\frac{a}{x}\right) dx = \frac{1}{2} a^2 \left(\frac{x^2}{a^2} - 1\right) \arcsin\left(\frac{x}{a}\right) + \frac{1}{4} a^2 \arcsin\left(\frac{x}{a}\right) + \frac{1}{4} a x \sqrt{-\frac{x^2}{a^2} + 1}$$

input `integrate(x*arccsc(a/x),x, algorithm="giac")`

output `1/2*a^2*(x^2/a^2 - 1)*arcsin(x/a) + 1/4*a^2*arcsin(x/a) + 1/4*a*x*sqrt(-x^2/a^2 + 1)`

**Mupad [B] (verification not implemented)**

Time = 0.85 (sec) , antiderivative size = 38, normalized size of antiderivative = 0.81

$$\int x \csc^{-1}\left(\frac{a}{x}\right) dx = \frac{a^2 \arcsin\left(\frac{x}{a}\right) \left(\frac{2x^2}{a^2} - 1\right)}{4} + \frac{a x \sqrt{1 - \frac{x^2}{a^2}}}{4}$$

input `int(x*asin(x/a),x)`

output `(a^2*asin(x/a)*((2*x^2)/a^2 - 1))/4 + (a*x*(1 - x^2/a^2)^(1/2))/4`

**Reduce [B] (verification not implemented)**

Time = 0.25 (sec) , antiderivative size = 47, normalized size of antiderivative = 1.00

$$\int x \csc^{-1}\left(\frac{a}{x}\right) dx = -\frac{\operatorname{acsc}\left(\frac{a}{x}\right) a^2}{2} + \frac{\operatorname{acsc}\left(\frac{a}{x}\right) x^2}{2} + \frac{\operatorname{asin}\left(\frac{x}{a}\right) a^2}{4} + \frac{\sqrt{a^2 - x^2} x}{4}$$

input `int(x*acsc(a/x),x)`

output `( - 2*acsc(a/x)*a**2 + 2*acsc(a/x)*x**2 + asin(x/a)*a**2 + sqrt(a**2 - x**2)*x)/4`

### 3.11 $\int \csc^{-1} \left( \frac{a}{x} \right) dx$

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Reduce [B] (verification not implemented)	112

#### Optimal result

Integrand size = 6, antiderivative size = 25

$$\int \csc^{-1} \left( \frac{a}{x} \right) dx = a \sqrt{1 - \frac{x^2}{a^2}} + x \arcsin \left( \frac{x}{a} \right)$$

output `a*(1-x^2/a^2)^(1/2)+x*arcsin(x/a)`

#### Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 25, normalized size of antiderivative = 1.00

$$\int \csc^{-1} \left( \frac{a}{x} \right) dx = a \sqrt{1 - \frac{x^2}{a^2}} + x \csc^{-1} \left( \frac{a}{x} \right)$$

input `Integrate[ArcCsc[a/x], x]`

output `a*Sqrt[1 - x^2/a^2] + x*ArcCsc[a/x]`

**Rubi [A] (verified)**

Time = 0.20 (sec) , antiderivative size = 25, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {5788, 5130, 241}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \csc^{-1}\left(\frac{a}{x}\right) dx \\ & \quad \downarrow \text{5788} \\ & \int \arcsin\left(\frac{x}{a}\right) dx \\ & \quad \downarrow \text{5130} \\ & x \arcsin\left(\frac{x}{a}\right) - \frac{\int \frac{x}{\sqrt{1-\frac{x^2}{a^2}}} dx}{a} \\ & \quad \downarrow \text{241} \\ & a\sqrt{1-\frac{x^2}{a^2}} + x \arcsin\left(\frac{x}{a}\right) \end{aligned}$$

input `Int[ArcCsc[a/x], x]`

output `a*Sqrt[1 - x^2/a^2] + x*ArcSin[x/a]`

**Defintions of rubi rules used**

rule 241 `Int[(x_)*((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] := Simp[(a + b*x^2)^(p + 1)/(2*b*(p + 1)), x] /; FreeQ[{a, b, p}, x] && NeQ[p, -1]`

rule 5130 `Int[((a_.) + ArcSin[(c_.)*(x_)])*(b_.))^(n_.), x_Symbol] := Simp[x*(a + b*ArcSin[c*x])^n, x] - Simp[b*c*n Int[x*((a + b*ArcSin[c*x])^(n - 1)/Sqrt[1 - c^2*x^2]), x], x] /; FreeQ[{a, b, c}, x] && GtQ[n, 0]`

rule 5788

```
Int[ArcCsc[(c_.)/((a_.) + (b_.)*(x_)^(n_.))]^(m_.)*(u_.), x_Symbol] := Int[
u*ArcSin[a/c + b*(x^n/c)]^m, x] /; FreeQ[{a, b, c, n, m}, x]
```

**Maple [A] (verified)**

Time = 1.02 (sec) , antiderivative size = 27, normalized size of antiderivative = 1.08

method	result	size
parts	$x \operatorname{arccsc}\left(\frac{a}{x}\right) + a\sqrt{\frac{a^2-x^2}{a^2}}$	27
derivativedivides	$-a\left(-\frac{x \operatorname{arccsc}\left(\frac{a}{x}\right)}{a} - \frac{x^2\left(\frac{a^2}{x^2}-1\right)}{\sqrt{\frac{\left(\frac{a^2}{x^2}-1\right)x^2}{a^2}} a^2}\right)$	52
default	$-a\left(-\frac{x \operatorname{arccsc}\left(\frac{a}{x}\right)}{a} - \frac{x^2\left(\frac{a^2}{x^2}-1\right)}{\sqrt{\frac{\left(\frac{a^2}{x^2}-1\right)x^2}{a^2}} a^2}\right)$	52

```
input int(arccsc(a/x),x,method=_RETURNVERBOSE)
```

```
output x*arccsc(a/x)+a*((a^2-x^2)/a^2)^(1/2)
```

**Fricas [A] (verification not implemented)**

Time = 0.09 (sec) , antiderivative size = 26, normalized size of antiderivative = 1.04

$$\int \csc^{-1}\left(\frac{a}{x}\right) dx = x \operatorname{arccsc}\left(\frac{a}{x}\right) + x\sqrt{\frac{a^2-x^2}{x^2}}$$

```
input integrate(arccsc(a/x),x, algorithm="fricas")
```

```
output x*arccsc(a/x) + x*sqrt((a^2 - x^2)/x^2)
```

**Sympy [A] (verification not implemented)**

Time = 0.16 (sec) , antiderivative size = 22, normalized size of antiderivative = 0.88

$$\int \csc^{-1}\left(\frac{a}{x}\right) dx = \begin{cases} a\sqrt{1 - \frac{x^2}{a^2}} + x \operatorname{arccsc}\left(\frac{a}{x}\right) & \text{for } a \neq 0 \\ \tilde{\infty}x & \text{otherwise} \end{cases}$$

input `integrate(acsc(a/x), x)`output `Piecewise((a*sqrt(1 - x**2/a**2) + x*acsc(a/x), Ne(a, 0)), (zoo*x, True))`**Maxima [A] (verification not implemented)**

Time = 0.03 (sec) , antiderivative size = 23, normalized size of antiderivative = 0.92

$$\int \csc^{-1}\left(\frac{a}{x}\right) dx = x \operatorname{arccsc}\left(\frac{a}{x}\right) + a\sqrt{-\frac{x^2}{a^2} + 1}$$

input `integrate(arccsc(a/x), x, algorithm="maxima")`output `x*arccsc(a/x) + a*sqrt(-x^2/a^2 + 1)`**Giac [A] (verification not implemented)**

Time = 0.12 (sec) , antiderivative size = 26, normalized size of antiderivative = 1.04

$$\int \csc^{-1}\left(\frac{a}{x}\right) dx = a\left(\frac{x \arcsin\left(\frac{x}{a}\right)}{a} + \sqrt{-\frac{x^2}{a^2} + 1}\right)$$

input `integrate(arccsc(a/x), x, algorithm="giac")`output `a*(x*arcsin(x/a)/a + sqrt(-x^2/a^2 + 1))`



**Mupad [B] (verification not implemented)**

Time = 0.12 (sec) , antiderivative size = 23, normalized size of antiderivative = 0.92

$$\int \csc^{-1}\left(\frac{a}{x}\right) dx = a \sqrt{1 - \frac{x^2}{a^2}} + x \operatorname{asin}\left(\frac{x}{a}\right)$$

input `int(asin(x/a),x)`

output `a*(1 - x^2/a^2)^(1/2) + x*asin(x/a)`

**Reduce [B] (verification not implemented)**

Time = 0.22 (sec) , antiderivative size = 19, normalized size of antiderivative = 0.76

$$\int \csc^{-1}\left(\frac{a}{x}\right) dx = a \operatorname{csc}\left(\frac{a}{x}\right) x + \sqrt{a^2 - x^2}$$

input `int(acsc(a/x),x)`

output `acsc(a/x)*x + sqrt(a**2 - x**2)`

### 3.12 $\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x} dx$

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Mupad [B] (verification not implemented) . . . . .	118
Reduce [F] . . . . .	118

#### Optimal result

Integrand size = 10, antiderivative size = 59

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x} dx = -\frac{1}{2}i \arcsin\left(\frac{x}{a}\right)^2 + \arcsin\left(\frac{x}{a}\right) \log\left(1 - e^{2i \arcsin\left(\frac{x}{a}\right)}\right) - \frac{1}{2}i \operatorname{PolyLog}\left(2, e^{2i \arcsin\left(\frac{x}{a}\right)}\right)$$

output `-1/2*I*arcsin(x/a)^2+arcsin(x/a)*ln(1-(I*x/a+(1-x^2/a^2)^(1/2))^2)-1/2*I*Polylog(2,(I*x/a+(1-x^2/a^2)^(1/2))^2)`

#### Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 54, normalized size of antiderivative = 0.92

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x} dx = \csc^{-1}\left(\frac{a}{x}\right) \log\left(1 - e^{2i \csc^{-1}\left(\frac{a}{x}\right)}\right) - \frac{1}{2}i \left( \csc^{-1}\left(\frac{a}{x}\right)^2 + \operatorname{PolyLog}\left(2, e^{2i \csc^{-1}\left(\frac{a}{x}\right)}\right) \right)$$

input `Integrate[ArcCsc[a/x]/x,x]`

output

$$\text{ArcCsc}[a/x] \cdot \text{Log}[1 - E^{((2*I)*\text{ArcCsc}[a/x])}] - (I/2) \cdot (\text{ArcCsc}[a/x]^2 + \text{PolyLog}[2, E^{((2*I)*\text{ArcCsc}[a/x])}])$$
**Rubi [A] (verified)**

Time = 0.40 (sec) , antiderivative size = 67, normalized size of antiderivative = 1.14, number of steps used = 10, number of rules used = 9,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.900$ , Rules used = {5788, 5136, 3042, 25, 4200, 25, 2620, 2715, 2838}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x} dx \\ & \quad \downarrow 5788 \\ & \int \frac{\arcsin\left(\frac{x}{a}\right)}{x} dx \\ & \quad \downarrow 5136 \\ & \int \frac{a\sqrt{1-\frac{x^2}{a^2}} \arcsin\left(\frac{x}{a}\right)}{x} d\arcsin\left(\frac{x}{a}\right) \\ & \quad \downarrow 3042 \\ & \int -\arcsin\left(\frac{x}{a}\right) \tan\left(\arcsin\left(\frac{x}{a}\right) + \frac{\pi}{2}\right) d\arcsin\left(\frac{x}{a}\right) \\ & \quad \downarrow 25 \\ & -\int \arcsin\left(\frac{x}{a}\right) \tan\left(\arcsin\left(\frac{x}{a}\right) + \frac{\pi}{2}\right) d\arcsin\left(\frac{x}{a}\right) \\ & \quad \downarrow 4200 \\ & 2i \int -\frac{e^{2i \arcsin\left(\frac{x}{a}\right)} \arcsin\left(\frac{x}{a}\right)}{1 - e^{2i \arcsin\left(\frac{x}{a}\right)}} d\arcsin\left(\frac{x}{a}\right) - \frac{1}{2}i \arcsin\left(\frac{x}{a}\right)^2 \\ & \quad \downarrow 25 \\ & -2i \int \frac{e^{2i \arcsin\left(\frac{x}{a}\right)} \arcsin\left(\frac{x}{a}\right)}{1 - e^{2i \arcsin\left(\frac{x}{a}\right)}} d\arcsin\left(\frac{x}{a}\right) - \frac{1}{2}i \arcsin\left(\frac{x}{a}\right)^2 \end{aligned}$$

$$\begin{aligned} & \downarrow 2620 \\ & -2i \left( \frac{1}{2} i \arcsin \left( \frac{x}{a} \right) \log \left( 1 - e^{2i \arcsin \left( \frac{x}{a} \right)} \right) - \frac{1}{2} i \int \log \left( 1 - e^{2i \arcsin \left( \frac{x}{a} \right)} \right) d \arcsin \left( \frac{x}{a} \right) \right) - \\ & \qquad \qquad \qquad \frac{1}{2} i \arcsin \left( \frac{x}{a} \right)^2 \end{aligned}$$

$$\begin{aligned} & \downarrow 2715 \\ & -2i \left( \frac{1}{2} i \arcsin \left( \frac{x}{a} \right) \log \left( 1 - e^{2i \arcsin \left( \frac{x}{a} \right)} \right) - \frac{1}{4} \int e^{-2i \arcsin \left( \frac{x}{a} \right)} \log \left( 1 - e^{2i \arcsin \left( \frac{x}{a} \right)} \right) d e^{2i \arcsin \left( \frac{x}{a} \right)} \right) - \\ & \qquad \qquad \qquad \frac{1}{2} i \arcsin \left( \frac{x}{a} \right)^2 \end{aligned}$$

$$\begin{aligned} & \downarrow 2838 \\ & -2i \left( \frac{1}{4} \text{PolyLog} \left( 2, e^{2i \arcsin \left( \frac{x}{a} \right)} \right) + \frac{1}{2} i \arcsin \left( \frac{x}{a} \right) \log \left( 1 - e^{2i \arcsin \left( \frac{x}{a} \right)} \right) \right) - \frac{1}{2} i \arcsin \left( \frac{x}{a} \right)^2 \end{aligned}$$

input `Int[ArcCsc[a/x]/x,x]`

output `(-1/2*I)*ArcSin[x/a]^2 - (2*I)*((I/2)*ArcSin[x/a]*Log[1 - E^((2*I)*ArcSin[x/a])]) + PolyLog[2, E^((2*I)*ArcSin[x/a])/4]`

### Defintions of rubi rules used

rule 25 `Int[-(Fx_), x_Symbol] := Simp[Identity[-1] Int[Fx, x], x]`

rule 2620 `Int[(((F_)^((g_)*((e_) + (f_)*(x_))))^(n_)*((c_) + (d_)*(x_))^(m_))/((a_) + (b_)*((F_)^((g_)*((e_) + (f_)*(x_))))^(n_)), x_Symbol] := Simp[((c + d*x)^m/(b*f*g*n*Log[F]))*Log[1 + b*((F^(g*(e + f*x)))^n/a)], x] - Simp[d*(m/(b*f*g*n*Log[F])) Int[(c + d*x)^(m - 1)*Log[1 + b*((F^(g*(e + f*x)))^n/a)], x], x] /; FreeQ[{F, a, b, c, d, e, f, g, n}, x] && IGtQ[m, 0]`

rule 2715 `Int[Log[(a_) + (b_)*((F_)^((e_)*((c_) + (d_)*(x_))))^(n_)], x_Symbol] := Simp[1/(d*e*n*Log[F]) Subst[Int[Log[a + b*x]/x, x], x, (F^(e*(c + d*x)))^n], x] /; FreeQ[{F, a, b, c, d, e, n}, x] && GtQ[a, 0]`

rule 2838  $\text{Int}[\text{Log}[(c\_)*(d\_)+(e\_)*(x\_)^{(n\_)}]/(x\_), x\_Symbol] \rightarrow \text{Simp}[-\text{PolyLog}[2, (-c)*e*x^n/n, x] /; \text{FreeQ}[\{c, d, e, n\}, x] \ \&\& \ \text{EqQ}[c*d, 1]$

rule 3042  $\text{Int}[u_, x\_Symbol] \rightarrow \text{Int}[\text{DeactivateTrig}[u, x], x] /; \text{FunctionOfTrigOfLinearQ}[u, x]$

rule 4200  $\text{Int}[(c\_)+(d\_)*(x\_)^{(m\_)*\tan[(e\_)+\text{Pi}*(k\_)+(f\_)*(x\_)]}, x\_Symbol] \rightarrow \text{Simp}[I*(c+d*x)^{(m+1)}/(d*(m+1)), x] - \text{Simp}[2*I \text{Int}[(c+d*x)^m * E^{(2*I*k*Pi)} * (E^{(2*I*(e+f*x))}/(1+E^{(2*I*k*Pi)}*E^{(2*I*(e+f*x))}))], x] /; \text{FreeQ}[\{c, d, e, f\}, x] \ \&\& \ \text{IntegerQ}[4*k] \ \&\& \ \text{IGtQ}[m, 0]$

rule 5136  $\text{Int}[(a\_)+\text{ArcSin}[(c\_)*(x\_)]*(b\_)]^{(n\_)} / (x_), x\_Symbol] \rightarrow \text{Subst}[\text{Int}[(a+b*x)^n * \text{Cot}[x], x], x, \text{ArcSin}[c*x]] /; \text{FreeQ}[\{a, b, c\}, x] \ \&\& \ \text{IGtQ}[n, 0]$

rule 5788  $\text{Int}[\text{ArcCsc}[(c\_)/((a\_)+(b\_)*(x\_)^{(n\_)}))]^{(m\_)}*(u_), x\_Symbol] \rightarrow \text{Int}[u*\text{ArcSin}[a/c+b*(x^n/c)]^m, x] /; \text{FreeQ}[\{a, b, c, n, m\}, x]$

## Maple [A] (verified)

Time = 1.34 (sec) , antiderivative size = 125, normalized size of antiderivative = 2.12

method	result
derivativedivides	$-\frac{i \operatorname{arccsc}\left(\frac{a}{x}\right)^2}{2} + \operatorname{arccsc}\left(\frac{a}{x}\right) \ln\left(1 + \frac{ix}{a} + \sqrt{1 - \frac{x^2}{a^2}}\right) - i \operatorname{polylog}\left(2, -\frac{ix}{a} - \sqrt{1 - \frac{x^2}{a^2}}\right) +$
default	$-\frac{i \operatorname{arccsc}\left(\frac{a}{x}\right)^2}{2} + \operatorname{arccsc}\left(\frac{a}{x}\right) \ln\left(1 + \frac{ix}{a} + \sqrt{1 - \frac{x^2}{a^2}}\right) - i \operatorname{polylog}\left(2, -\frac{ix}{a} - \sqrt{1 - \frac{x^2}{a^2}}\right) +$

input  $\text{int}(\operatorname{arccsc}(a/x)/x, x, \text{method}=\_RETURNVERBOSE)$

output  $-1/2*I*\operatorname{arccsc}(a/x)^2 + \operatorname{arccsc}(a/x)*\ln(1+I*x/a+(1-x^2/a^2)^{(1/2)}) - I*\operatorname{polylog}(2, -I/a*x-(1-x^2/a^2)^{(1/2)}) + \operatorname{arccsc}(a/x)*\ln(1-I/a*x-(1-x^2/a^2)^{(1/2)}) - I*\operatorname{polylog}(2, I*x/a+(1-x^2/a^2)^{(1/2)})$

**Fricas [F]**

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x} dx = \int \frac{\operatorname{arccsc}\left(\frac{a}{x}\right)}{x} dx$$

input `integrate(arccsc(a/x)/x,x, algorithm="fricas")`

output `integral(arccsc(a/x)/x, x)`

**Sympy [F]**

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x} dx = \int \frac{\operatorname{acsc}\left(\frac{a}{x}\right)}{x} dx$$

input `integrate(acsc(a/x)/x,x)`

output `Integral(acsc(a/x)/x, x)`

**Maxima [F]**

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x} dx = \int \frac{\operatorname{arccsc}\left(\frac{a}{x}\right)}{x} dx$$

input `integrate(arccsc(a/x)/x,x, algorithm="maxima")`

output `integrate(arccsc(a/x)/x, x)`

**Giac [F]**

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x} dx = \int \frac{\operatorname{arccsc}\left(\frac{a}{x}\right)}{x} dx$$

input `integrate(arccsc(a/x)/x,x, algorithm="giac")`

output `integrate(arccsc(a/x)/x, x)`

**Mupad [B] (verification not implemented)**

Time = 0.85 (sec) , antiderivative size = 49, normalized size of antiderivative = 0.83

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x} dx = -\frac{\operatorname{polylog}\left(2, e^{\operatorname{asin}\left(\frac{x}{a}\right) 2i}\right) 1i}{2} + \ln\left(1 - e^{\operatorname{asin}\left(\frac{x}{a}\right) 2i}\right) \operatorname{asin}\left(\frac{x}{a}\right) - \frac{\operatorname{asin}\left(\frac{x}{a}\right)^2 1i}{2}$$

input `int(asin(x/a)/x,x)`

output `log(1 - exp(asin(x/a)*2i))*asin(x/a) - (polylog(2, exp(asin(x/a)*2i))*1i)/2 - (asin(x/a)^2*1i)/2`

**Reduce [F]**

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x} dx = \int \frac{\operatorname{acsc}\left(\frac{a}{x}\right)}{x} dx$$

input `int(acsc(a/x)/x,x)`

output `int(acsc(a/x)/x,x)`

### 3.13 $\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^2} dx$

Optimal result . . . . .	119
Mathematica [B] (verified) . . . . .	119
Rubi [A] (verified) . . . . .	120
Maple [A] (verified) . . . . .	122
Fricas [B] (verification not implemented) . . . . .	122
Sympy [C] (verification not implemented) . . . . .	123
Maxima [A] (verification not implemented) . . . . .	123
Giac [B] (verification not implemented) . . . . .	124
Mupad [B] (verification not implemented) . . . . .	124
Reduce [F] . . . . .	124

#### Optimal result

Integrand size = 10, antiderivative size = 32

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^2} dx = -\frac{\arcsin\left(\frac{x}{a}\right)}{x} - \frac{\operatorname{arctanh}\left(\sqrt{1 - \frac{x^2}{a^2}}\right)}{a}$$

output `-arcsin(x/a)/x-arctanh((1-x^2/a^2)^(1/2))/a`

#### Mathematica [B] (verified)

Leaf count is larger than twice the leaf count of optimal. 93 vs. 2(32) = 64.

Time = 0.09 (sec) , antiderivative size = 93, normalized size of antiderivative = 2.91

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^2} dx = -\frac{\csc^{-1}\left(\frac{a}{x}\right)}{x} - \frac{\sqrt{-1 + \frac{a^2}{x^2}} x \left( -\log\left(1 - \frac{a}{\sqrt{-1 + \frac{a^2}{x^2}} x}\right) + \log\left(1 + \frac{a}{\sqrt{-1 + \frac{a^2}{x^2}} x}\right) \right)}{2a^2 \sqrt{1 - \frac{x^2}{a^2}}}$$

input `Integrate[ArcCsc[a/x]/x^2,x]`



output

$$-(\text{ArcCsc}[a/x]/x) - (\text{Sqrt}[-1 + a^2/x^2]*x*(-\text{Log}[1 - a/(\text{Sqrt}[-1 + a^2/x^2]*x)] + \text{Log}[1 + a/(\text{Sqrt}[-1 + a^2/x^2]*x)]))/(2*a^2*\text{Sqrt}[1 - x^2/a^2])$$

**Rubi [A] (verified)**

Time = 0.24 (sec) , antiderivative size = 32, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 5,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {5788, 5138, 243, 73, 221}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^2} dx \\ & \quad \downarrow \text{5788} \\ & \int \frac{\arcsin\left(\frac{x}{a}\right)}{x^2} dx \\ & \quad \downarrow \text{5138} \\ & \frac{\int \frac{1}{x\sqrt{1-\frac{x^2}{a^2}}} dx}{a} - \frac{\arcsin\left(\frac{x}{a}\right)}{x} \\ & \quad \downarrow \text{243} \\ & \frac{\int \frac{1}{x^2\sqrt{1-\frac{x^2}{a^2}}} dx^2}{2a} - \frac{\arcsin\left(\frac{x}{a}\right)}{x} \\ & \quad \downarrow \text{73} \\ & -a \int \frac{1}{a^2 - a^2x^4} d\sqrt{1 - \frac{x^2}{a^2}} - \frac{\arcsin\left(\frac{x}{a}\right)}{x} \\ & \quad \downarrow \text{221} \\ & -\frac{\operatorname{arctanh}\left(\sqrt{1 - \frac{x^2}{a^2}}\right)}{a} - \frac{\arcsin\left(\frac{x}{a}\right)}{x} \end{aligned}$$

input

$$\text{Int}[\text{ArcCsc}[a/x]/x^2, x]$$

output  $-(\text{ArcSin}[x/a]/x) - \text{ArcTanh}[\text{Sqrt}[1 - x^2/a^2]]/a$

### Defintions of rubi rules used

rule 73  $\text{Int}[(a_.) + (b_.)*(x_.)^{(m_.)}*((c_.) + (d_.)*(x_.)^{(n_.)}), x\_Symbol] \rightarrow \text{With}[\{p = \text{Denominator}[m]\}, \text{Simp}[p/b \text{ Subst}[\text{Int}[x^{(p*(m+1) - 1)}*(c - a*(d/b) + d*(x^p/b))^{(n)}, x], x, (a + b*x)^{(1/p)}, x]] /; \text{FreeQ}[\{a, b, c, d\}, x] \&\& \text{LtQ}[-1, m, 0] \&\& \text{LeQ}[-1, n, 0] \&\& \text{LeQ}[\text{Denominator}[n], \text{Denominator}[m]] \&\& \text{IntLinearQ}[a, b, c, d, m, n, x]$

rule 221  $\text{Int}[(a_.) + (b_.)*(x_.)^2)^{-1}, x\_Symbol] \rightarrow \text{Simp}[(\text{Rt}[-a/b, 2]/a)*\text{ArcTanh}[x/\text{Rt}[-a/b, 2]], x] /; \text{FreeQ}[\{a, b\}, x] \&\& \text{NegQ}[a/b]$

rule 243  $\text{Int}[(x_.)^{(m_.)}*((a_.) + (b_.)*(x_.)^2)^{(p_.)}), x\_Symbol] \rightarrow \text{Simp}[1/2 \text{ Subst}[\text{Int}[x^{((m-1)/2)*(a + b*x)^p}, x], x, x^2], x] /; \text{FreeQ}[\{a, b, m, p\}, x] \&\& \text{IntegerQ}[(m-1)/2]$

rule 5138  $\text{Int}[(a_.) + \text{ArcSin}[c_.*(x_.)]*(b_.)^{(n_.)}*((d_.)*(x_.)^{(m_.)}), x\_Symbol] \rightarrow \text{Simp}[(d*x)^{(m+1)}*((a + b*\text{ArcSin}[c*x])^n/(d*(m+1))), x] - \text{Simp}[b*c*(n/(d*(m+1))) \text{ Int}[(d*x)^{(m+1)}*((a + b*\text{ArcSin}[c*x])^{(n-1)}/\text{Sqrt}[1 - c^2*x^2]), x], x] /; \text{FreeQ}[\{a, b, c, d, m\}, x] \&\& \text{IGtQ}[n, 0] \&\& \text{NeQ}[m, -1]$

rule 5788  $\text{Int}[\text{ArcCsc}[c_./((a_.) + (b_.)*(x_.)^{(n_.)})]^{(m_.)}*(u_.), x\_Symbol] \rightarrow \text{Int}[u*\text{ArcSin}[a/c + b*(x^n/c)]^m, x] /; \text{FreeQ}[\{a, b, c, n, m\}, x]$

**Maple [A] (verified)**

Time = 0.06 (sec) , antiderivative size = 31, normalized size of antiderivative = 0.97

method	result	size
parts	$-\frac{\operatorname{arccsc}\left(\frac{a}{x}\right)}{x} - \frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\right)}{a}$	31
derivativedivides	$-\frac{\frac{\operatorname{arccsc}\left(\frac{a}{x}\right)a}{x} + \ln\left(\frac{a}{x} + \frac{a\sqrt{1-\frac{x^2}{a^2}}}{x}\right)}{a}$	42
default	$-\frac{\frac{\operatorname{arccsc}\left(\frac{a}{x}\right)a}{x} + \ln\left(\frac{a}{x} + \frac{a\sqrt{1-\frac{x^2}{a^2}}}{x}\right)}{a}$	42

input `int(arccsc(a/x)/x^2,x,method=_RETURNVERBOSE)`

output `-arccsc(a/x)/x-1/a*arctanh(1/(1-x^2/a^2)^(1/2))`

**Fricas [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 65 vs. 2(30) = 60.

Time = 0.12 (sec) , antiderivative size = 65, normalized size of antiderivative = 2.03

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^2} dx = -\frac{2a \operatorname{arccsc}\left(\frac{a}{x}\right) + x \log\left(x\sqrt{\frac{a^2-x^2}{x^2}} + a\right) - x \log\left(x\sqrt{\frac{a^2-x^2}{x^2}} - a\right)}{2ax}$$

input `integrate(arccsc(a/x)/x^2,x, algorithm="fricas")`

output `-1/2*(2*a*arccsc(a/x) + x*log(x*sqrt((a^2 - x^2)/x^2) + a) - x*log(x*sqrt((a^2 - x^2)/x^2) - a))/(a*x)`

**Sympy [C] (verification not implemented)**

Result contains complex when optimal does not.

Time = 1.47 (sec) , antiderivative size = 27, normalized size of antiderivative = 0.84

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^2} dx = -\frac{\operatorname{acsc}\left(\frac{a}{x}\right)}{x} + \frac{\begin{cases} -\operatorname{acosh}\left(\frac{a}{x}\right) & \text{for } \left|\frac{a^2}{x^2}\right| > 1 \\ i \operatorname{asin}\left(\frac{a}{x}\right) & \text{otherwise} \end{cases}}{a}$$

input `integrate(acsc(a/x)/x**2,x)`

output `-acsc(a/x)/x + Piecewise((-acosh(a/x), Abs(a**2/x**2) > 1), (I*asin(a/x), True))/a`

**Maxima [A] (verification not implemented)**

Time = 0.04 (sec) , antiderivative size = 52, normalized size of antiderivative = 1.62

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^2} dx = -\frac{\frac{2a \operatorname{arccsc}\left(\frac{a}{x}\right)}{x} + \log\left(\sqrt{-\frac{x^2}{a^2} + 1} + 1\right) - \log\left(-\sqrt{-\frac{x^2}{a^2} + 1} + 1\right)}{2a}$$

input `integrate(arccsc(a/x)/x^2,x, algorithm="maxima")`

output `-1/2*(2*a*arccsc(a/x)/x + log(sqrt(-x^2/a^2 + 1) + 1) - log(-sqrt(-x^2/a^2 + 1) + 1))/a`

**Giac [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 61 vs.  $2(30) = 60$ .

Time = 0.13 (sec) , antiderivative size = 61, normalized size of antiderivative = 1.91

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^2} dx = -\frac{a\left(\frac{\log(|a+\sqrt{a^2-x^2}|)}{a} - \frac{\log(|-a+\sqrt{a^2-x^2}|)}{a}\right)}{2|a|} - \frac{\arcsin\left(\frac{x}{a}\right)}{x}$$

input `integrate(arccsc(a/x)/x^2,x, algorithm="giac")`

output `-1/2*a*(log(abs(a + sqrt(a^2 - x^2)))/a - log(abs(-a + sqrt(a^2 - x^2)))/a)/abs(a) - arcsin(x/a)/x`

**Mupad [B] (verification not implemented)**

Time = 0.85 (sec) , antiderivative size = 30, normalized size of antiderivative = 0.94

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^2} dx = -\frac{\operatorname{asin}\left(\frac{x}{a}\right)}{x} - \frac{\operatorname{atanh}\left(\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\right)}{a}$$

input `int(asin(x/a)/x^2,x)`

output `- asin(x/a)/x - atanh(1/(1 - x^2/a^2)^(1/2))/a`

**Reduce [F]**

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^2} dx = \int \frac{\operatorname{acsc}\left(\frac{a}{x}\right)}{x^2} dx$$

input `int(acsc(a/x)/x^2,x)`

output `int(acsc(a/x)/x**2,x)`

### 3.14 $\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^3} dx$

Optimal result . . . . .	125
Mathematica [A] (verified) . . . . .	125
Rubi [A] (verified) . . . . .	126
Maple [A] (verified) . . . . .	127
Fricas [A] (verification not implemented) . . . . .	127
Sympy [C] (verification not implemented) . . . . .	128
Maxima [A] (verification not implemented) . . . . .	128
Giac [A] (verification not implemented) . . . . .	129
Mupad [F(-1)] . . . . .	129
Reduce [F] . . . . .	129

#### Optimal result

Integrand size = 10, antiderivative size = 38

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^3} dx = -\frac{\sqrt{1-\frac{x^2}{a^2}}}{2ax} - \frac{\arcsin\left(\frac{x}{a}\right)}{2x^2}$$

output

$$-1/2*(1-x^2/a^2)^(1/2)/a/x-1/2*\arcsin(x/a)/x^2$$

#### Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 35, normalized size of antiderivative = 0.92

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^3} dx = -\frac{x\sqrt{1-\frac{x^2}{a^2}} + a \csc^{-1}\left(\frac{a}{x}\right)}{2ax^2}$$

input

```
Integrate[ArcCsc[a/x]/x^3,x]
```

output

$$-1/2*(x*\sqrt{1-x^2/a^2} + a*\text{ArcCsc}[a/x])/(a*x^2)$$

**Rubi [A] (verified)**

Time = 0.23 (sec) , antiderivative size = 38, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.300$ , Rules used = {5788, 5138, 242}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^3} dx \\ & \quad \downarrow 5788 \\ & \int \frac{\arcsin\left(\frac{x}{a}\right)}{x^3} dx \\ & \quad \downarrow 5138 \\ & \int \frac{1}{x^2 \sqrt{1 - \frac{x^2}{a^2}}} dx - \frac{\arcsin\left(\frac{x}{a}\right)}{2x^2} \\ & \quad \downarrow 242 \\ & -\frac{\sqrt{1 - \frac{x^2}{a^2}}}{2ax} - \frac{\arcsin\left(\frac{x}{a}\right)}{2x^2} \end{aligned}$$

input `Int[ArcCsc[a/x]/x^3,x]`

output `-1/2*Sqrt[1 - x^2/a^2]/(a*x) - ArcSin[x/a]/(2*x^2)`

**Defintions of rubi rules used**

rule 242 `Int[((c_.)*(x_))^(m_.)*((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(c*x)^(m + 1)*((a + b*x^2)^(p + 1)/(a*c*(m + 1))), x] /; FreeQ[{a, b, c, m, p}, x] && EqQ[m + 2*p + 3, 0] && NeQ[m, -1]`

rule 5138  $\text{Int}[(a_.) + \text{ArcSin}[c_.](x_.)](b_.)^{(n_.)}((d_.)(x_.))^{(m_.)}, x\_Symbol] \rightarrow \text{Simp}[(d*x)^{(m+1)}((a + b*\text{ArcSin}[c*x])^n/(d*(m+1))), x] - \text{Simp}[b*c*(n/(d*(m+1))) \text{Int}[(d*x)^{(m+1)}((a + b*\text{ArcSin}[c*x])^{(n-1)}/\text{Sqrt}[1 - c^2*x^2]), x], x] /; \text{FreeQ}\{a, b, c, d, m\}, x \ \&\& \ \text{IGtQ}[n, 0] \ \&\& \ \text{NeQ}[m, -1]$

rule 5788  $\text{Int}[\text{ArcCsc}[(c_.)/((a_.) + (b_.)(x_.)^{(n_.)})]^{(m_.)}(u_.), x\_Symbol] \rightarrow \text{Int}[u*\text{ArcSin}[a/c + b*(x^n/c)]^m, x] /; \text{FreeQ}\{a, b, c, n, m\}, x]$

### Maple [A] (verified)

Time = 1.00 (sec) , antiderivative size = 33, normalized size of antiderivative = 0.87

method	result	size
parts	$-\frac{\text{arccsc}\left(\frac{a}{x}\right)}{2x^2} - \frac{\sqrt{1-\frac{x^2}{a^2}}}{2ax}$	33
derivativedivides	$-\frac{\frac{a^2 \text{arccsc}\left(\frac{a}{x}\right)}{2x^2} + \frac{x\left(\frac{a^2}{x^2}-1\right)}{2\sqrt{\frac{\left(\frac{a^2}{x^2}-1\right)x^2}{a^2}}}{a}}{a^2}$	54
default	$-\frac{\frac{a^2 \text{arccsc}\left(\frac{a}{x}\right)}{2x^2} + \frac{x\left(\frac{a^2}{x^2}-1\right)}{2\sqrt{\frac{\left(\frac{a^2}{x^2}-1\right)x^2}{a^2}}}{a}}{a^2}$	54

input `int(arccsc(a/x)/x^3,x,method=_RETURNVERBOSE)`

output  $-1/2*\text{arccsc}(a/x)/x^2-1/2*(1-x^2/a^2)^{(1/2)}/a/x$

### Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 38, normalized size of antiderivative = 1.00

$$\int \frac{\text{csc}^{-1}\left(\frac{a}{x}\right)}{x^3} dx = -\frac{a^2 \text{arccsc}\left(\frac{a}{x}\right) + x^2 \sqrt{\frac{a^2-x^2}{x^2}}}{2a^2x^2}$$

input `integrate(arccsc(a/x)/x^3,x, algorithm="fricas")`



output `-1/2*(a^2*arccsc(a/x) + x^2*sqrt((a^2 - x^2)/x^2))/(a^2*x^2)`

### Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 0.85 (sec) , antiderivative size = 51, normalized size of antiderivative = 1.34

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^3} dx = -\frac{\operatorname{acsc}\left(\frac{a}{x}\right)}{2x^2} + \frac{\begin{cases} -\frac{\sqrt{\frac{a^2}{x^2}-1}}{a} & \text{for } \left|\frac{a^2}{x^2}\right| > 1 \\ -\frac{i\sqrt{-\frac{a^2}{x^2}+1}}{a} & \text{otherwise} \end{cases}}{2a}$$

input `integrate(acsc(a/x)/x**3,x)`

output `-acsc(a/x)/(2*x**2) + Piecewise((-sqrt(a**2/x**2 - 1)/a, Abs(a**2/x**2) > 1), (-I*sqrt(-a**2/x**2 + 1)/a, True))/(2*a)`

### Maxima [A] (verification not implemented)

Time = 0.12 (sec) , antiderivative size = 32, normalized size of antiderivative = 0.84

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^3} dx = -\frac{\operatorname{arccsc}\left(\frac{a}{x}\right)}{2x^2} - \frac{\sqrt{-\frac{x^2}{a^2}+1}}{2ax}$$

input `integrate(arccsc(a/x)/x^3,x, algorithm="maxima")`

output `-1/2*arccsc(a/x)/x^2 - 1/2*sqrt(-x^2/a^2 + 1)/(a*x)`

**Giac [A] (verification not implemented)**

Time = 0.12 (sec) , antiderivative size = 61, normalized size of antiderivative = 1.61

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^3} dx = -\frac{a\left(\frac{a+\sqrt{a^2-x^2}}{a^2x} - \frac{x}{(a+\sqrt{a^2-x^2})a^2}\right)}{4|a|} - \frac{\arcsin\left(\frac{x}{a}\right)}{2x^2}$$

input `integrate(arccsc(a/x)/x^3,x, algorithm="giac")`

output `-1/4*a*((a + sqrt(a^2 - x^2))/(a^2*x) - x/((a + sqrt(a^2 - x^2))*a^2))/abs(a) - 1/2*arcsin(x/a)/x^2`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^3} dx = \int \frac{\operatorname{asin}\left(\frac{x}{a}\right)}{x^3} dx$$

input `int(asin(x/a)/x^3,x)`

output `int(asin(x/a)/x^3, x)`

**Reduce [F]**

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^3} dx = \int \frac{\operatorname{acsc}\left(\frac{a}{x}\right)}{x^3} dx$$

input `int(acsc(a/x)/x^3,x)`

output `int(acsc(a/x)/x**3,x)`

### 3.15 $\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^4} dx$

Optimal result	130
Mathematica [A] (verified)	130
Rubi [A] (verified)	131
Maple [A] (verified)	133
Fricas [A] (verification not implemented)	133
Sympy [A] (verification not implemented)	134
Maxima [A] (verification not implemented)	134
Giac [A] (verification not implemented)	135
Mupad [F(-1)]	135
Reduce [F]	135

#### Optimal result

Integrand size = 10, antiderivative size = 60

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^4} dx = -\frac{\sqrt{1-\frac{x^2}{a^2}}}{6ax^2} - \frac{\arcsin\left(\frac{x}{a}\right)}{3x^3} - \frac{\operatorname{arctanh}\left(\sqrt{1-\frac{x^2}{a^2}}\right)}{6a^3}$$

output  $-1/6*(1-x^2/a^2)^{(1/2)}/a/x^2-1/3*\arcsin(x/a)/x^3-1/6*\operatorname{arctanh}((1-x^2/a^2)^{(1/2)})/a^3$

#### Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 69, normalized size of antiderivative = 1.15

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^4} dx = -\frac{a^2x\sqrt{1-\frac{x^2}{a^2}} + 2a^3\csc^{-1}\left(\frac{a}{x}\right) - x^3\log(x) + x^3\log\left(1 + \sqrt{1-\frac{x^2}{a^2}}\right)}{6a^3x^3}$$

input `Integrate[ArcCsc[a/x]/x^4,x]`

output  $-1/6*(a^2*x*\operatorname{Sqrt}[1-x^2/a^2] + 2*a^3*\operatorname{ArcCsc}[a/x] - x^3*\operatorname{Log}[x] + x^3*\operatorname{Log}[1 + \operatorname{Sqrt}[1-x^2/a^2]])/(a^3*x^3)$

**Rubi [A] (verified)**

Time = 0.25 (sec) , antiderivative size = 61, normalized size of antiderivative = 1.02, number of steps used = 7, number of rules used = 6,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.600$ , Rules used = {5788, 5138, 243, 52, 73, 221}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^4} dx \\
 & \quad \downarrow \text{5788} \\
 & \int \frac{\arcsin\left(\frac{x}{a}\right)}{x^4} dx \\
 & \quad \downarrow \text{5138} \\
 & \frac{\int \frac{1}{x^3 \sqrt{1-\frac{x^2}{a^2}}} dx}{3a} - \frac{\arcsin\left(\frac{x}{a}\right)}{3x^3} \\
 & \quad \downarrow \text{243} \\
 & \frac{\int \frac{1}{x^4 \sqrt{1-\frac{x^2}{a^2}}} dx^2}{6a} - \frac{\arcsin\left(\frac{x}{a}\right)}{3x^3} \\
 & \quad \downarrow \text{52} \\
 & \frac{\int \frac{1}{x^2 \sqrt{1-\frac{x^2}{a^2}}} dx^2}{2a^2} - \frac{\sqrt{1-\frac{x^2}{a^2}}}{x^2} - \frac{\arcsin\left(\frac{x}{a}\right)}{3x^3} \\
 & \quad \downarrow \text{73} \\
 & - \int \frac{1}{a^2 - a^2 x^4} d\sqrt{1-\frac{x^2}{a^2}} - \frac{\sqrt{1-\frac{x^2}{a^2}}}{x^2} - \frac{\arcsin\left(\frac{x}{a}\right)}{3x^3} \\
 & \quad \downarrow \text{221} \\
 & - \frac{\operatorname{arctanh}\left(\sqrt{1-\frac{x^2}{a^2}}\right)}{a^2} - \frac{\sqrt{1-\frac{x^2}{a^2}}}{x^2} - \frac{\arcsin\left(\frac{x}{a}\right)}{3x^3}
 \end{aligned}$$

input

Int [ArcCsc [a/x] / x^4, x]

output 
$$-1/3*\text{ArcSin}[x/a]/x^3 + (-\text{Sqrt}[1 - x^2/a^2]/x^2) - \text{ArcTanh}[\text{Sqrt}[1 - x^2/a^2]]/a^2)/(6*a)$$

### Defintions of rubi rules used

rule 52 
$$\text{Int}[(a_.) + (b_.)*(x_.)^{(m_.)}*((c_.) + (d_.)*(x_.)^{(n_.)}, x\_Symbol] \rightarrow \text{Simp}[(a + b*x)^{(m + 1)}*((c + d*x)^{(n + 1)} / ((b*c - a*d)*(m + 1))), x] - \text{Simp}[d*((m + n + 2) / ((b*c - a*d)*(m + 1))) \text{Int}[(a + b*x)^{(m + 1)}*(c + d*x)^n, x], x] /; \text{FreeQ}\{a, b, c, d, n\}, x\} \&\& \text{ILtQ}[m, -1] \&\& \text{FractionQ}[n] \&\& \text{LtQ}[n, 0]$$

rule 73 
$$\text{Int}[(a_.) + (b_.)*(x_.)^{(m_.)}*((c_.) + (d_.)*(x_.)^{(n_.)}, x\_Symbol] \rightarrow \text{With}[\{p = \text{Denominator}[m]\}, \text{Simp}[p/b \text{Subst}[\text{Int}[x^{(p*(m + 1) - 1)}*(c - a*(d/b) + d*(x^p/b))^n, x], x, (a + b*x)^{(1/p)}, x]] /; \text{FreeQ}\{a, b, c, d\}, x\} \&\& \text{LtQ}[-1, m, 0] \&\& \text{LeQ}[-1, n, 0] \&\& \text{LeQ}[\text{Denominator}[n], \text{Denominator}[m]] \&\& \text{IntLinearQ}[a, b, c, d, m, n, x]$$

rule 221 
$$\text{Int}[(a_.) + (b_.)*(x_.)^2)^{-1}, x\_Symbol] \rightarrow \text{Simp}[(\text{Rt}[-a/b, 2]/a)*\text{ArcTanh}[x/\text{Rt}[-a/b, 2]], x] /; \text{FreeQ}\{a, b\}, x\} \&\& \text{NegQ}[a/b]$$

rule 243 
$$\text{Int}[(x_.)^{(m_.)}*((a_.) + (b_.)*(x_.)^2)^{(p_.)}, x\_Symbol] \rightarrow \text{Simp}[1/2 \text{Subst}[\text{Int}[x^{(m - 1)/2}*(a + b*x)^p, x], x, x^2], x] /; \text{FreeQ}\{a, b, m, p\}, x\} \&\& \text{IntegerQ}[(m - 1)/2]$$

rule 5138 
$$\text{Int}[(a_.) + \text{ArcSin}[c_.)*(x_.)]*(b_.)^{(n_.)}*((d_.)*(x_.)^{(m_.)}, x\_Symbol] \rightarrow \text{Simp}[(d*x)^{(m + 1)}*((a + b*\text{ArcSin}[c*x])^n / (d*(m + 1))), x] - \text{Simp}[b*c*(n / (d*(m + 1))) \text{Int}[(d*x)^{(m + 1)}*((a + b*\text{ArcSin}[c*x])^{(n - 1)} / \text{Sqrt}[1 - c^2*x^2]), x], x] /; \text{FreeQ}\{a, b, c, d, m\}, x\} \&\& \text{IGtQ}[n, 0] \&\& \text{NeQ}[m, -1]$$

rule 5788 
$$\text{Int}[\text{ArcCsc}[c_.] / ((a_.) + (b_.)*(x_.)^{(n_.)})]^{(m_.)}*(u_.), x\_Symbol] \rightarrow \text{Int}[u*\text{ArcSin}[a/c + b*(x^n/c)]^m, x] /; \text{FreeQ}\{a, b, c, n, m\}, x\}$$

**Maple [A] (verified)**

Time = 1.02 (sec) , antiderivative size = 54, normalized size of antiderivative = 0.90

method	result	size
parts	$-\frac{\operatorname{arccsc}\left(\frac{a}{x}\right)}{3x^3} + \frac{-\frac{\sqrt{1-x^2}}{2x^2} - \frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\right)}{2a^2}}{3a}$	54
derivativeldivides	$\frac{a^3 \operatorname{arccsc}\left(\frac{a}{x}\right) + \frac{\sqrt{\frac{a^2}{x^2}-1} \left( a \frac{\sqrt{\frac{a^2}{x^2}-1}}{x} + \ln\left(\frac{a}{x} + \sqrt{\frac{a^2}{x^2}-1}\right) \right) x}{6 \sqrt{\left(\frac{a^2}{x^2}-1\right) x^2} a}}{a^3}$	91
default	$\frac{a^3 \operatorname{arccsc}\left(\frac{a}{x}\right) + \frac{\sqrt{\frac{a^2}{x^2}-1} \left( a \frac{\sqrt{\frac{a^2}{x^2}-1}}{x} + \ln\left(\frac{a}{x} + \sqrt{\frac{a^2}{x^2}-1}\right) \right) x}{6 \sqrt{\left(\frac{a^2}{x^2}-1\right) x^2} a}}{a^3}$	91

input `int(arccsc(a/x)/x^4,x,method=_RETURNVERBOSE)`output `-1/3*arccsc(a/x)/x^3+1/3/a*(-1/2/x^2*(1-x^2/a^2)^(1/2)-1/2/a^2*arctanh(1/(1-x^2/a^2)^(1/2)))`**Fricas [A] (verification not implemented)**

Time = 0.12 (sec) , antiderivative size = 92, normalized size of antiderivative = 1.53

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^4} dx$$

$$= \frac{4a^3 \operatorname{arccsc}\left(\frac{a}{x}\right) + x^3 \log\left(x \sqrt{\frac{a^2-x^2}{x^2}} + a\right) - x^3 \log\left(x \sqrt{\frac{a^2-x^2}{x^2}} - a\right) + 2ax^2 \sqrt{\frac{a^2-x^2}{x^2}}}{12a^3x^3}$$

input `integrate(arccsc(a/x)/x^4,x, algorithm="fricas")`output `-1/12*(4*a^3*arccsc(a/x) + x^3*log(x*sqrt((a^2 - x^2)/x^2) + a) - x^3*log(x*sqrt((a^2 - x^2)/x^2) - a) + 2*a*x^2*sqrt((a^2 - x^2)/x^2))/(a^3*x^3)`

**Sympy [A] (verification not implemented)**

Time = 1.82 (sec) , antiderivative size = 99, normalized size of antiderivative = 1.65

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^4} dx = -\frac{\operatorname{acsc}\left(\frac{a}{x}\right)}{3x^3} + \frac{\begin{cases} -\frac{\sqrt{\frac{a^2}{x^2}-1}}{2ax} - \frac{\operatorname{acosh}\left(\frac{a}{x}\right)}{2a^2} & \text{for } \left|\frac{a^2}{x^2}\right| > 1 \\ \frac{ia}{2x^3\sqrt{-\frac{a^2}{x^2}+1}} - \frac{i}{2ax\sqrt{-\frac{a^2}{x^2}+1}} + \frac{i\operatorname{asin}\left(\frac{a}{x}\right)}{2a^2} & \text{otherwise} \end{cases}}{3a}$$

input `integrate(acsc(a/x)/x**4,x)`output `-acsc(a/x)/(3*x**3) + Piecewise((-sqrt(a**2/x**2 - 1)/(2*a*x) - acosh(a/x)/(2*a**2), Abs(a**2/x**2) > 1), (I*a/(2*x**3*sqrt(-a**2/x**2 + 1)) - I/(2*a*x*sqrt(-a**2/x**2 + 1)) + I*asin(a/x)/(2*a**2), True))/(3*a)`**Maxima [A] (verification not implemented)**

Time = 0.13 (sec) , antiderivative size = 64, normalized size of antiderivative = 1.07

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^4} dx = -\frac{\log\left(\frac{2\sqrt{-\frac{x^2}{a^2}+1}}{|x|} + \frac{2}{|x|}\right)}{6a} + \frac{\sqrt{-\frac{x^2}{a^2}+1}}{3x^3} - \frac{\operatorname{arccsc}\left(\frac{a}{x}\right)}{3x^3}$$

input `integrate(arccsc(a/x)/x^4,x, algorithm="maxima")`output `-1/6*(log(2*sqrt(-x^2/a^2 + 1)/abs(x) + 2/abs(x))/a^2 + sqrt(-x^2/a^2 + 1)/x^2)/a - 1/3*arccsc(a/x)/x^3`

**Giac [A] (verification not implemented)**

Time = 0.12 (sec) , antiderivative size = 80, normalized size of antiderivative = 1.33

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^4} dx = -\frac{a\left(\frac{\log\left(\left|a+\sqrt{a^2-x^2}\right|\right)}{a^3} - \frac{\log\left(\left|-a+\sqrt{a^2-x^2}\right|\right)}{a^3} + \frac{2\sqrt{a^2-x^2}}{a^2x^2}\right)}{12|a|} - \frac{\arcsin\left(\frac{x}{a}\right)}{3x^3}$$

input `integrate(arccsc(a/x)/x^4,x, algorithm="giac")`output `-1/12*a*(log(abs(a + sqrt(a^2 - x^2)))/a^3 - log(abs(-a + sqrt(a^2 - x^2)))/a^3 + 2*sqrt(a^2 - x^2)/(a^2*x^2))/abs(a) - 1/3*arcsin(x/a)/x^3`**Mupad [F(-1)]**

Timed out.

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^4} dx = \int \frac{\operatorname{asin}\left(\frac{x}{a}\right)}{x^4} dx$$

input `int(asin(x/a)/x^4,x)`output `int(asin(x/a)/x^4, x)`**Reduce [F]**

$$\int \frac{\csc^{-1}\left(\frac{a}{x}\right)}{x^4} dx = \int \frac{\operatorname{acsc}\left(\frac{a}{x}\right)}{x^4} dx$$

input `int(acsc(a/x)/x^4,x)`output `int(acsc(a/x)/x**4,x)`



### 3.16 $\int \frac{\csc^{-1}(ax^n)}{x} dx$

Optimal result	136
Mathematica [C] (verified)	136
Rubi [A] (warning: unable to verify)	137
Maple [A] (verified)	140
Fricas [F(-2)]	140
Sympy [F]	141
Maxima [F]	141
Giac [F]	141
Mupad [F(-1)]	142
Reduce [F]	142

#### Optimal result

Integrand size = 10, antiderivative size = 69

$$\int \frac{\csc^{-1}(ax^n)}{x} dx = \frac{i \csc^{-1}(ax^n)^2}{2n} - \frac{\csc^{-1}(ax^n) \log\left(1 - e^{2i \csc^{-1}(ax^n)}\right)}{n} + \frac{i \operatorname{PolyLog}\left(2, e^{2i \csc^{-1}(ax^n)}\right)}{2n}$$

output

$1/2*I*\operatorname{arccsc}(a*x^n)^2/n - \operatorname{arccsc}(a*x^n)*\ln(1 - (I/a/(x^n) + (1 - 1/a^2/(x^n)^2)^{(1/2)})^2)/n + 1/2*I*\operatorname{polylog}(2, (I/a/(x^n) + (1 - 1/a^2/(x^n)^2)^{(1/2)})^2)/n$

#### Mathematica [C] (verified)

Result contains higher order function than in optimal. Order 5 vs. order 4 in optimal.

Time = 0.06 (sec) , antiderivative size = 63, normalized size of antiderivative = 0.91

$$\int \frac{\csc^{-1}(ax^n)}{x} dx = -\frac{x^{-n} {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{x^{-2n}}{a^2}\right)}{an} + \left(\csc^{-1}(ax^n) - \arcsin\left(\frac{x^{-n}}{a}\right)\right) \log(x)$$

input

`Integrate[ArcCsc[a*x^n]/x, x]`

output

```
-(HypergeometricPFQ[{1/2, 1/2, 1/2}, {3/2, 3/2}, 1/(a^2*x^(2*n))]/(a*n*x^n
)) + (ArcCsc[a*x^n] - ArcSin[1/(a*x^n)])*Log[x]
```

**Rubi [A] (warning: unable to verify)**

Time = 0.49 (sec) , antiderivative size = 81, normalized size of antiderivative = 1.17, number of steps used = 11, number of rules used = 10,  $\frac{\text{number of rules}}{\text{integrand size}} = 1.000$ , Rules used = {7282, 5742, 5136, 3042, 25, 4200, 25, 2620, 2715, 2838}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{\csc^{-1}(ax^n)}{x} dx \\
 & \quad \downarrow \text{7282} \\
 & \frac{\int x^{-n} \csc^{-1}(ax^n) dx^n}{n} \\
 & \quad \downarrow \text{5742} \\
 & -\frac{\int x^{-n} \arcsin\left(\frac{x^{-n}}{a}\right) dx^{-n}}{n} \\
 & \quad \downarrow \text{5136} \\
 & -\frac{\int ax^n \sqrt{1 - \frac{x^{-2n}}{a^2}} \arcsin\left(\frac{x^{-n}}{a}\right) d \arcsin\left(\frac{x^{-n}}{a}\right)}{n} \\
 & \quad \downarrow \text{3042} \\
 & -\frac{\int -\arcsin\left(\frac{x^{-n}}{a}\right) \tan\left(\arcsin\left(\frac{x^{-n}}{a}\right) + \frac{\pi}{2}\right) d \arcsin\left(\frac{x^{-n}}{a}\right)}{n} \\
 & \quad \downarrow \text{25} \\
 & \frac{\int \arcsin\left(\frac{x^{-n}}{a}\right) \tan\left(\arcsin\left(\frac{x^{-n}}{a}\right) + \frac{\pi}{2}\right) d \arcsin\left(\frac{x^{-n}}{a}\right)}{n} \\
 & \quad \downarrow \text{4200}
 \end{aligned}$$

$$\begin{array}{c}
 \frac{2i \int \frac{e^{2i \arcsin\left(\frac{x^{-n}}{a}\right)} \arcsin\left(\frac{x^{-n}}{a}\right)}{1 - e^{2i \arcsin\left(\frac{x^{-n}}{a}\right)}} d \arcsin\left(\frac{x^{-n}}{a}\right) - \frac{1}{2} i x^{2n}}{n} \\
 \downarrow 25 \\
 \frac{-2i \int \frac{e^{2i \arcsin\left(\frac{x^{-n}}{a}\right)} \arcsin\left(\frac{x^{-n}}{a}\right)}{1 - e^{2i \arcsin\left(\frac{x^{-n}}{a}\right)}} d \arcsin\left(\frac{x^{-n}}{a}\right) - \frac{1}{2} i x^{2n}}{n} \\
 \downarrow 2620 \\
 \frac{-2i \left( \frac{1}{2} i \arcsin\left(\frac{x^{-n}}{a}\right) \log\left(1 - e^{2i \arcsin\left(\frac{x^{-n}}{a}\right)}\right) - \frac{1}{2} i \int \log\left(1 - e^{2i \arcsin\left(\frac{x^{-n}}{a}\right)}\right) d \arcsin\left(\frac{x^{-n}}{a}\right) \right) - \frac{1}{2} i x^{2n}}{n} \\
 \downarrow 2715 \\
 \frac{-2i \left( \frac{1}{2} i \arcsin\left(\frac{x^{-n}}{a}\right) \log\left(1 - e^{2i \arcsin\left(\frac{x^{-n}}{a}\right)}\right) - \frac{1}{4} \int e^{2i \arcsin\left(\frac{x^{-n}}{a}\right)} \log\left(1 - e^{2i \arcsin\left(\frac{x^{-n}}{a}\right)}\right) d e^{2i \arcsin\left(\frac{x^{-n}}{a}\right)} \right) - \frac{1}{2} i x^{2n}}{n} \\
 \downarrow 2838 \\
 \frac{-2i \left( \frac{1}{4} \text{PolyLog}\left(2, e^{2i \arcsin\left(\frac{x^{-n}}{a}\right)}\right) + \frac{1}{2} i \arcsin\left(\frac{x^{-n}}{a}\right) \log\left(1 - e^{2i \arcsin\left(\frac{x^{-n}}{a}\right)}\right) \right) - \frac{1}{2} i x^{2n}}{n}
 \end{array}$$

input `Int[ArcCsc[a*x^n]/x, x]`

output `-((( -1/2*I)*x^(2*n) - (2*I)*((I/2)*ArcSin[1/(a*x^n)]*Log[1 - E^((2*I)*ArcSin[1/(a*x^n)])]) + PolyLog[2, E^((2*I)*ArcSin[1/(a*x^n)])])/4)/n`

## Definitions of rubi rules used

- rule 25  $\text{Int}[-(\text{Fx}_), \text{x\_Symbol}] \rightarrow \text{Simp}[\text{Identity}[-1] \quad \text{Int}[\text{Fx}, \text{x}], \text{x}]$
- rule 2620  $\text{Int}[(((\text{F}_)^{((\text{g}_)*((\text{e}_) + (\text{f}_)*(\text{x}_))))^{(\text{n}_)}*((\text{c}_) + (\text{d}_)*(\text{x}_))^{(\text{m}_)})/((\text{a}_) + (\text{b}_)*((\text{F}_)^{((\text{g}_)*((\text{e}_) + (\text{f}_)*(\text{x}_))))^{(\text{n}_)})), \text{x\_Symbol}] \rightarrow \text{Simp}[\text{Simp}[(\text{c} + \text{d*x})^{\text{m}}/(\text{b*f*g*n*Log[F]})*\text{Log}[1 + \text{b}*((\text{F}^{\text{g*(e + f*x)})}^{\text{n/a}})], \text{x}] - \text{Simp}[\text{d*(m/(b*f*g*n*Log[F]))} \quad \text{Int}[(\text{c} + \text{d*x})^{\text{m} - 1}*\text{Log}[1 + \text{b}*((\text{F}^{\text{g*(e + f*x)})}^{\text{n/a}})], \text{x}], \text{x}] /; \text{FreeQ}\{\text{F}, \text{a}, \text{b}, \text{c}, \text{d}, \text{e}, \text{f}, \text{g}, \text{n}\}, \text{x}\} \&\& \text{IGtQ}[\text{m}, 0]$
- rule 2715  $\text{Int}[\text{Log}[(\text{a}_) + (\text{b}_)*((\text{F}_)^{((\text{e}_)*((\text{c}_) + (\text{d}_)*(\text{x}_))))^{(\text{n}_)}], \text{x\_Symbol}] \rightarrow \text{Simp}[1/(\text{d*e*n*Log[F]}) \quad \text{Subst}[\text{Int}[\text{Log}[\text{a} + \text{b*x}]/\text{x}, \text{x}], \text{x}, (\text{F}^{\text{e*(c + d*x)})}^{\text{n}}], \text{x}] /; \text{FreeQ}\{\text{F}, \text{a}, \text{b}, \text{c}, \text{d}, \text{e}, \text{n}\}, \text{x}\} \&\& \text{GtQ}[\text{a}, 0]$
- rule 2838  $\text{Int}[\text{Log}[(\text{c}_)*((\text{d}_) + (\text{e}_)*(\text{x}_)^{(\text{n}_)})]/(\text{x}_), \text{x\_Symbol}] \rightarrow \text{Simp}[-\text{PolyLog}[2, (\text{-c})*\text{e*x}^{\text{n}}/\text{n}, \text{x}] /; \text{FreeQ}\{\text{c}, \text{d}, \text{e}, \text{n}\}, \text{x}\} \&\& \text{EqQ}[\text{c*d}, 1]$
- rule 3042  $\text{Int}[\text{u}_, \text{x\_Symbol}] \rightarrow \text{Int}[\text{DeactivateTrig}[\text{u}, \text{x}], \text{x}] /; \text{FunctionOfTrigOfLinearQ}[\text{u}, \text{x}]$
- rule 4200  $\text{Int}[((\text{c}_) + (\text{d}_)*(\text{x}_))^{(\text{m}_)}*\text{tan}[(\text{e}_) + \text{Pi}*(\text{k}_) + (\text{f}_)*(\text{x}_)], \text{x\_Symbol}] \rightarrow \text{Simp}[\text{I}*(\text{c} + \text{d*x})^{\text{m} + 1}/(\text{d*(m + 1)}), \text{x}] - \text{Simp}[2*\text{I} \quad \text{Int}[(\text{c} + \text{d*x})^{\text{m}}*\text{E}^{(2*\text{I}*k*\text{Pi})}*(\text{E}^{(2*\text{I}*(\text{e} + \text{f*x}))})/(1 + \text{E}^{(2*\text{I}*k*\text{Pi})}*\text{E}^{(2*\text{I}*(\text{e} + \text{f*x}))})], \text{x}], \text{x}] /; \text{FreeQ}\{\text{c}, \text{d}, \text{e}, \text{f}\}, \text{x}\} \&\& \text{IntegerQ}[4*k] \&\& \text{IGtQ}[\text{m}, 0]$
- rule 5136  $\text{Int}[((\text{a}_) + \text{ArcSin}[(\text{c}_)*(\text{x}_)]*(\text{b}_))^{(\text{n}_)}/(\text{x}_), \text{x\_Symbol}] \rightarrow \text{Subst}[\text{Int}[(\text{a} + \text{b*x})^{\text{n}}*\text{Cot}[\text{x}], \text{x}], \text{x}, \text{ArcSin}[\text{c*x}]] /; \text{FreeQ}\{\text{a}, \text{b}, \text{c}\}, \text{x}\} \&\& \text{IGtQ}[\text{n}, 0]$
- rule 5742  $\text{Int}[((\text{a}_) + \text{ArcCsc}[(\text{c}_)*(\text{x}_)]*(\text{b}_))^{(\text{n}_)}/(\text{x}_), \text{x\_Symbol}] \rightarrow -\text{Subst}[\text{Int}[(\text{a} + \text{b}*ArcSin[\text{x}/\text{c}])/x, \text{x}], \text{x}, 1/\text{x}] /; \text{FreeQ}\{\text{a}, \text{b}, \text{c}\}, \text{x}\}$

rule 7282

```
Int[(u_)/(x_), x_Symbol] := With[{lst = PowerVariableExpn[u, 0, x]}, Simp[1
/lst[[2]] Subst[Int[NormalizeIntegrand[Simplify[lst[[1]]/x], x], x], x, (
lst[[3]]*x)^lst[[2]]], x] /; !FalseQ[lst] && NeQ[lst[[2]], 0] /; NonsumQ[
u] && !RationalFunctionQ[u, x]
```

**Maple [A] (verified)**

Time = 1.05 (sec) , antiderivative size = 155, normalized size of antiderivative = 2.25

method	result
derivativedivides	$\frac{i \operatorname{arccsc}(a x^n)^2}{2} - \operatorname{arccsc}(a x^n) \ln \left( 1 + \frac{i x^{-n}}{a} + \sqrt{1 - \frac{x^{-2n}}{a^2}} \right) + i \operatorname{polylog} \left( 2, -\frac{i x^{-n}}{a} - \sqrt{1 - \frac{x^{-2n}}{a^2}} \right) - \operatorname{arccsc}(a x^n) \ln \left( 1 - \frac{i x^{-n}}{a} - \sqrt{1 - \frac{x^{-2n}}{a^2}} \right)$
default	$\frac{i \operatorname{arccsc}(a x^n)^2}{2} - \operatorname{arccsc}(a x^n) \ln \left( 1 + \frac{i x^{-n}}{a} + \sqrt{1 - \frac{x^{-2n}}{a^2}} \right) + i \operatorname{polylog} \left( 2, -\frac{i x^{-n}}{a} - \sqrt{1 - \frac{x^{-2n}}{a^2}} \right) - \operatorname{arccsc}(a x^n) \ln \left( 1 - \frac{i x^{-n}}{a} - \sqrt{1 - \frac{x^{-2n}}{a^2}} \right)$

input

```
int(arccsc(a*x^n)/x,x,method=_RETURNVERBOSE)
```

output

```
1/n*(1/2*I*arccsc(a*x^n)^2-arccsc(a*x^n)*ln(1+I/a/(x^n)+(1-1/a^2/(x^n)^2)^(1/2))+I*polylog(2,-I/a/(x^n)-(1-1/a^2/(x^n)^2)^(1/2))-arccsc(a*x^n)*ln(1-I/a/(x^n)-(1-1/a^2/(x^n)^2)^(1/2))+I*polylog(2,I/a/(x^n)+(1-1/a^2/(x^n)^2)^(1/2)))
```

**Fricas [F(-2)]**

Exception generated.

$$\int \frac{\operatorname{csc}^{-1}(a x^n)}{x} dx = \text{Exception raised: TypeError}$$

input

```
integrate(arccsc(a*x^n)/x,x, algorithm="fricas")
```

output

```
Exception raised: TypeError >> Error detected within library code: integrate: implementation incomplete (constant residues)
```

**Sympy [F]**

$$\int \frac{\csc^{-1}(ax^n)}{x} dx = \int \frac{\operatorname{acsc}(ax^n)}{x} dx$$

input `integrate(acsc(a*x**n)/x,x)`

output `Integral(acsc(a*x**n)/x, x)`

**Maxima [F]**

$$\int \frac{\csc^{-1}(ax^n)}{x} dx = \int \frac{\operatorname{arccsc}(ax^n)}{x} dx$$

input `integrate(arccsc(a*x^n)/x,x, algorithm="maxima")`

output `a^2*n*integrate(sqrt(a*x^n + 1)*sqrt(a*x^n - 1)*log(x)/(a^4*x*x^(2*n) - a^2*x), x) + arctan2(1, sqrt(a*x^n + 1)*sqrt(a*x^n - 1))*log(x)`

**Giac [F]**

$$\int \frac{\csc^{-1}(ax^n)}{x} dx = \int \frac{\operatorname{arccsc}(ax^n)}{x} dx$$

input `integrate(arccsc(a*x^n)/x,x, algorithm="giac")`

output `integrate(arccsc(a*x^n)/x, x)`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{\csc^{-1}(ax^n)}{x} dx = \int \frac{\operatorname{asin}\left(\frac{1}{ax^n}\right)}{x} dx$$

input `int(asin(1/(a*x^n))/x,x)`output `int(asin(1/(a*x^n))/x, x)`**Reduce [F]**

$$\int \frac{\csc^{-1}(ax^n)}{x} dx = \int \frac{\operatorname{acsc}(x^na)}{x} dx$$

input `int(acsc(a*x^n)/x,x)`output `int(acsc(x**n*a)/x,x)`

### 3.17 $\int x^4 \csc^{-1}(a + bx) dx$

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#### Optimal result

Integrand size = 10, antiderivative size = 197

$$\int x^4 \csc^{-1}(a + bx) dx = -\frac{a(20 + 53a^2)(a + bx)\sqrt{1 - \frac{1}{(a+bx)^2}}}{30b^5} - \frac{11ax^2(a + bx)\sqrt{1 - \frac{1}{(a+bx)^2}}}{60b^3} + \frac{x^3(a + bx)\sqrt{1 - \frac{1}{(a+bx)^2}}}{20b^2} + \frac{(9 + 58a^2)(a + bx)^2\sqrt{1 - \frac{1}{(a+bx)^2}}}{120b^5} + \frac{a^5 \csc^{-1}(a + bx)}{5b^5} + \frac{1}{5}x^5 \csc^{-1}(a + bx) + \frac{(3 + 40a^2 + 40a^4) \operatorname{arctanh}\left(\sqrt{1 - \frac{1}{(a+bx)^2}}\right)}{40b^5}$$

output

```
-1/30*a*(53*a^2+20)*(b*x+a)*(1-1/(b*x+a)^2)^(1/2)/b^5-11/60*a*x^2*(b*x+a)*
(1-1/(b*x+a)^2)^(1/2)/b^3+1/20*x^3*(b*x+a)*(1-1/(b*x+a)^2)^(1/2)/b^2+1/120
*(58*a^2+9)*(b*x+a)^2*(1-1/(b*x+a)^2)^(1/2)/b^5+1/5*a^5*arccsc(b*x+a)/b^5+
1/5*x^5*arccsc(b*x+a)+1/40*(40*a^4+40*a^2+3)*arctanh((1-1/(b*x+a)^2)^(1/2)
)/b^5
```



**Mathematica [A] (verified)**

Time = 0.15 (sec) , antiderivative size = 174, normalized size of antiderivative = 0.88

$$\int x^4 \csc^{-1}(a + bx) dx$$

$$= \frac{-\sqrt{\frac{-1+a^2+2abx+b^2x^2}{(a+bx)^2}}(a^2(71+154a^2)+2a(31+48a^2)bx-9(1+4a^2)b^2x^2+16ab^3x^3-6b^4x^4)+24b^5x^5}{120b^5}$$

input

```
Integrate[x^4*ArcCsc[a + b*x],x]
```

output

```
(-(Sqrt[(-1 + a^2 + 2*a*b*x + b^2*x^2)/(a + b*x)^2]*(a^2*(71 + 154*a^2) +
2*a*(31 + 48*a^2)*b*x - 9*(1 + 4*a^2)*b^2*x^2 + 16*a*b^3*x^3 - 6*b^4*x^4))
+ 24*b^5*x^5*ArcCsc[a + b*x] + 24*a^5*ArcSin[(a + b*x)^(-1)] + 3*(3 + 40*
a^2 + 40*a^4)*Log[(a + b*x)*(1 + Sqrt[(-1 + a^2 + 2*a*b*x + b^2*x^2)/(a +
b*x)^2])])/(120*b^5)
```

**Rubi [A] (verified)**

Time = 0.73 (sec) , antiderivative size = 207, normalized size of antiderivative = 1.05, number of steps used = 10, number of rules used = 9,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.900$ , Rules used = {5782, 4927, 3042, 4269, 3042, 4544, 3042, 4536, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x^4 \csc^{-1}(a + bx) dx$$

$$\downarrow 5782$$

$$\frac{\int b^4 x^4 (a + bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx) d \csc^{-1}(a + bx)}{b^5}$$

$$\downarrow 4927$$

$$\frac{-\frac{1}{5} \int -b^5 x^5 d \csc^{-1}(a + bx) - \frac{1}{5} b^5 x^5 \csc^{-1}(a + bx)}{b^5}$$

$$\downarrow 3042$$

$$\frac{-\frac{1}{5} \int (a - \csc(\csc^{-1}(a + bx)))^5 d \csc^{-1}(a + bx) - \frac{1}{5} b^5 x^5 \csc^{-1}(a + bx)}{b^5}$$

↓ 4269

$$\frac{\frac{1}{5} \left( -\frac{1}{4} \int b^2 x^2 (4a^3 + 11(a + bx)^2 a - 3(4a^2 + 1)(a + bx)) d \csc^{-1}(a + bx) - \frac{1}{4} b^3 x^3 (a + bx) \sqrt{1 - \frac{1}{(a+bx)^2}} \right) - \frac{1}{5} b^5}{b^5}$$

↓ 3042

$$\frac{\frac{1}{5} \left( -\frac{1}{4} \int (a - \csc(\csc^{-1}(a + bx)))^2 (4a^3 + 11 \csc(\csc^{-1}(a + bx))^2 a - 3(4a^2 + 1) \csc(\csc^{-1}(a + bx))) d \csc^{-1}(a + bx) \right)}{b^5}$$

↓ 4544

$$\frac{\frac{1}{5} \left( \frac{1}{4} \left( \frac{11}{3} a b^2 x^2 (a + bx) \sqrt{1 - \frac{1}{(a+bx)^2}} - \frac{1}{3} \int -bx (12a^4 - (48a^2 + 31)(a + bx)a + (58a^2 + 9)(a + bx)^2) d \csc^{-1}(a + bx) \right) \right)}{b^5}$$

↓ 3042

$$\frac{\frac{1}{5} \left( \frac{1}{4} \left( \frac{11}{3} a b^2 x^2 (a + bx) \sqrt{1 - \frac{1}{(a+bx)^2}} - \frac{1}{3} \int (a - \csc(\csc^{-1}(a + bx))) (12a^4 - (48a^2 + 31) \csc(\csc^{-1}(a + bx))) d \csc^{-1}(a + bx) \right) \right)}{b^5}$$

↓ 4536

$$\frac{\frac{1}{5} \left( \frac{1}{4} \left( \frac{1}{3} \left( -\frac{1}{2} \int (24a^5 + 4(53a^2 + 20)(a + bx)^2 a - 3(40a^4 + 40a^2 + 3)(a + bx)) d \csc^{-1}(a + bx) - \frac{1}{2} (58a^2 + 9) \right) \right) \right)}{b^5}$$

↓ 2009

$$\frac{\frac{1}{5} \left( \frac{1}{4} \left( \frac{1}{3} \left( -24a^5 \csc^{-1}(a + bx) + 4(53a^2 + 20) a(a + bx) \sqrt{1 - \frac{1}{(a+bx)^2}} - 3(40a^4 + 40a^2 + 3) \operatorname{arctanh} \left( \sqrt{1 - \frac{1}{(a+bx)^2}} \right) \right) \right) \right)}{b^5}$$

input

Int[x^4\*ArcCsc[a + b\*x],x]

output

```

-((-1/5*(b^5*x^5*ArcCsc[a + b*x]) + (-1/4*(b^3*x^3*(a + b*x)*Sqrt[1 - (a +
b*x)^(-2)])) + ((11*a*b^2*x^2*(a + b*x)*Sqrt[1 - (a + b*x)^(-2)])/3 + (-1/
2*((9 + 58*a^2)*(a + b*x)^2*Sqrt[1 - (a + b*x)^(-2)]) + (4*a*(20 + 53*a^2)
*(a + b*x)*Sqrt[1 - (a + b*x)^(-2)] - 24*a^5*ArcCsc[a + b*x] - 3*(3 + 40*a
^2 + 40*a^4)*ArcTanh[Sqrt[1 - (a + b*x)^(-2)]])/2)/3)/4)/5)/b^5)

```

### Defintions of rubi rules used

rule 2009

```
Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 3042

```
Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4269

```
Int[(csc[(c_) + (d_)*(x_)]*(b_) + (a_))^(n_), x_Symbol] := Simp[(-b^2)*C
ot[c + d*x]*((a + b*Csc[c + d*x])^(n - 2)/(d*(n - 1))), x] + Simp[1/(n - 1)
Int[(a + b*Csc[c + d*x])^(n - 3)*Simp[a^3*(n - 1) + (b*(b^2*(n - 2) + 3*
a^2*(n - 1))*Csc[c + d*x] + (a*b^2*(3*n - 4))*Csc[c + d*x]^2, x], x], x] /
; FreeQ[{a, b, c, d}, x] && NeQ[a^2 - b^2, 0] && GtQ[n, 2] && IntegerQ[2*n]
```

rule 4536

```
Int[((A_) + csc[(e_) + (f_)*(x_)]*(B_) + csc[(e_) + (f_)*(x_)]^2*(C_
))*csc[(e_) + (f_)*(x_)]*(b_) + (a_)), x_Symbol] := Simp[(-b)*C*Csc[e +
f*x]*(Cot[e + f*x]/(2*f)), x] + Simp[1/2 Int[Simp[2*A*a + (2*B*a + b*(2*
A + C))*Csc[e + f*x] + 2*(a*C + B*b)*Csc[e + f*x]^2, x], x], x] /; FreeQ[{a
, b, e, f, A, B, C}, x]
```

rule 4544

```
Int[((A_) + csc[(e_) + (f_)*(x_)]*(B_) + csc[(e_) + (f_)*(x_)]^2*(C_
))*csc[(e_) + (f_)*(x_)]*(b_) + (a_))^(m_), x_Symbol] := Simp[(-C)*Cot
[e + f*x]*((a + b*Csc[e + f*x])^m/(f*(m + 1))), x] + Simp[1/(m + 1) Int[(
a + b*Csc[e + f*x])^(m - 1)*Simp[a*A*(m + 1) + ((A*b + a*B)*(m + 1) + b*C*m
)*Csc[e + f*x] + (b*B*(m + 1) + a*C*m)*Csc[e + f*x]^2, x], x], x] /; FreeQ[
{a, b, e, f, A, B, C}, x] && NeQ[a^2 - b^2, 0] && IGtQ[2*m, 0]
```

rule 4927

```
Int[Cot[(c_.) + (d_.)*(x_)]*Csc[(c_.) + (d_.)*(x_)]*(Csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_.))^(n_.)*((e_.) + (f_.)*(x_))^(m_.), x_Symbol] := Simp[(-e + f*x)^m*((a + b*Csc[c + d*x])^(n + 1)/(b*d*(n + 1))), x] + Simp[f*(m/(b*d*(n + 1))) Int[(e + f*x)^(m - 1)*(a + b*Csc[c + d*x])^(n + 1), x], x] /; FreeQ[{a, b, c, d, e, f, n}, x] && IGtQ[m, 0] && NeQ[n, -1]
```

rule 5782

```
Int[((a_.) + ArcCsc[(c_) + (d_.)*(x_)]*(b_.))^(p_.)*((e_.) + (f_.)*(x_))^(m_.), x_Symbol] := Simp[-(d^(m + 1))^( -1) Subst[Int[(a + b*x)^p*Csc[x]*Cot[x]*(d*e - c*f + f*Csc[x])^m, x], x, ArcCsc[c + d*x]], x] /; FreeQ[{a, b, c, d, e, f}, x] && IGtQ[p, 0] && IntegerQ[m]
```

## Maple [A] (verified)

Time = 0.27 (sec) , antiderivative size = 329, normalized size of antiderivative = 1.67

method	result
derivativedivides	$-\frac{\operatorname{arccsc}(bx+a)a^5}{5} + \operatorname{arccsc}(bx+a)a^4(bx+a) - 2 \operatorname{arccsc}(bx+a)a^3(bx+a)^2 + 2 \operatorname{arccsc}(bx+a)a^2(bx+a)^3 - \operatorname{arccsc}(bx+a)a(bx+a)^4$
default	$-\frac{\operatorname{arccsc}(bx+a)a^5}{5} + \operatorname{arccsc}(bx+a)a^4(bx+a) - 2 \operatorname{arccsc}(bx+a)a^3(bx+a)^2 + 2 \operatorname{arccsc}(bx+a)a^2(bx+a)^3 - \operatorname{arccsc}(bx+a)a(bx+a)^4$
parts	$\frac{x^5 \operatorname{arccsc}(bx+a)}{5} + \frac{\sqrt{b^2x^2+2abx+a^2-1} \left( 6x^3\sqrt{b^2x^2+2abx+a^2-1} b^3\sqrt{b^2+24a^5} \arctan\left(\frac{1}{\sqrt{b^2x^2+2abx+a^2-1}}\right) \sqrt{b^2-24a^5} \right)}{\sqrt{b^2x^2+2abx+a^2-1}}$

input

```
int(x^4*arccsc(b*x+a), x, method=_RETURNVERBOSE)
```

output

```
1/b^5*(-1/5*arccsc(b*x+a)*a^5+arccsc(b*x+a)*a^4*(b*x+a)-2*arccsc(b*x+a)*a^3*(b*x+a)^2+2*arccsc(b*x+a)*a^2*(b*x+a)^3-arccsc(b*x+a)*a*(b*x+a)^4+1/5*arccsc(b*x+a)*(b*x+a)^5+1/120*((b*x+a)^2-1)^(1/2)*(24*a^5*arctan(1/((b*x+a)^2-1)^(1/2))+120*a^4*ln(b*x+a+((b*x+a)^2-1)^(1/2))-240*a^3*((b*x+a)^2-1)^(1/2)+120*a^2*(b*x+a)*((b*x+a)^2-1)^(1/2)-40*a*(b*x+a)^2*((b*x+a)^2-1)^(1/2)+6*(b*x+a)^3*((b*x+a)^2-1)^(1/2)+120*a^2*ln(b*x+a+((b*x+a)^2-1)^(1/2))-80*a*((b*x+a)^2-1)^(1/2)+9*(b*x+a)*((b*x+a)^2-1)^(1/2)+9*ln(b*x+a+((b*x+a)^2-1)^(1/2)))/(((b*x+a)^2-1)/(b*x+a)^2)^(1/2)/(b*x+a))
```

**Fricas [A] (verification not implemented)**

Time = 0.15 (sec) , antiderivative size = 151, normalized size of antiderivative = 0.77

$$\int x^4 \csc^{-1}(a + bx) dx$$

$$= \frac{24b^5x^5 \operatorname{arccsc}(bx + a) - 48a^5 \operatorname{arctan}(-bx - a + \sqrt{b^2x^2 + 2abx + a^2 - 1}) - 3(40a^4 + 40a^2 + 3) \log(-bx - a + \sqrt{b^2x^2 + 2abx + a^2 - 1})}{b^5}$$

input

```
integrate(x^4*arccsc(b*x+a),x, algorithm="fricas")
```

output

```
1/120*(24*b^5*x^5*arccsc(b*x + a) - 48*a^5*arctan(-b*x - a + sqrt(b^2*x^2 + 2*a*b*x + a^2 - 1)) - 3*(40*a^4 + 40*a^2 + 3)*log(-b*x - a + sqrt(b^2*x^2 + 2*a*b*x + a^2 - 1)) + (6*b^3*x^3 - 22*a*b^2*x^2 - 154*a^3 + (58*a^2 + 9)*b*x - 71*a)*sqrt(b^2*x^2 + 2*a*b*x + a^2 - 1))/b^5
```

**Sympy [F]**

$$\int x^4 \csc^{-1}(a + bx) dx = \int x^4 \operatorname{acsc}(a + bx) dx$$

input

```
integrate(x**4*acsc(b*x+a),x)
```

output

```
Integral(x**4*acsc(a + b*x), x)
```

**Maxima [F]**

$$\int x^4 \csc^{-1}(a + bx) dx = \int x^4 \operatorname{arccsc}(bx + a) dx$$

input `integrate(x^4*arccsc(b*x+a),x, algorithm="maxima")`

output `1/5*x^5*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) + integrate(1/5*(b^2*x^6 + a*b*x^5)*e^(1/2*log(b*x + a + 1) + 1/2*log(b*x + a - 1))/(b^2*x^2 + 2*a*b*x + a^2 + (b^2*x^2 + 2*a*b*x + a^2 - 1)*e^(log(b*x + a + 1) + log(b*x + a - 1)) - 1), x)`

**Giac [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 411 vs.  $2(173) = 346$ .

Time = 0.15 (sec) , antiderivative size = 411, normalized size of antiderivative = 2.09

$$\int x^4 \csc^{-1}(a + bx) dx =$$

$$-\frac{1}{960} b \left( \frac{192 (bx + a)^5 \left( \frac{5a}{bx+a} - \frac{10a^2}{(bx+a)^2} + \frac{10a^3}{(bx+a)^3} - \frac{5a^4}{(bx+a)^4} - 1 \right) \arcsin \left( -\frac{1}{(bx+a) \left( \frac{a}{bx+a} - 1 \right) - a} \right) + \frac{3 (bx + a)}{b^6} \right)$$

input `integrate(x^4*arccsc(b*x+a),x, algorithm="giac")`

output

```
-1/960*b*(192*(b*x + a)^5*(5*a/(b*x + a) - 10*a^2/(b*x + a)^2 + 10*a^3/(b*x + a)^3 - 5*a^4/(b*x + a)^4 - 1)*arcsin(-1/((b*x + a)*(a/(b*x + a) - 1) - a))/b^6 + (3*(b*x + a)^4*(sqrt(-1/(b*x + a)^2 + 1) - 1)^4 + 40*(b*x + a)^3*a*(sqrt(-1/(b*x + a)^2 + 1) - 1)^3 + 240*(b*x + a)^2*a^2*(sqrt(-1/(b*x + a)^2 + 1) - 1)^2 + 960*(b*x + a)*a^3*(sqrt(-1/(b*x + a)^2 + 1) - 1) + 24*(b*x + a)^2*(sqrt(-1/(b*x + a)^2 + 1) - 1)^2 + 360*(b*x + a)*a*(sqrt(-1/(b*x + a)^2 + 1) - 1) + 24*(40*a^4 + 40*a^2 + 3)*log(1/2*abs(b*x + a)*abs(-2*sqrt(-1/(b*x + a)^2 + 1) + 2)) - (120*(8*a^3 + 3*a)*(b*x + a)^3*(sqrt(-1/(b*x + a)^2 + 1) - 1)^3 + 24*(10*a^2 + 1)*(b*x + a)^2*(sqrt(-1/(b*x + a)^2 + 1) - 1)^2 + 40*(b*x + a)*a*(sqrt(-1/(b*x + a)^2 + 1) - 1) + 3)/((b*x + a)^4*(sqrt(-1/(b*x + a)^2 + 1) - 1)^4))/b^6)
```

**Mupad [F(-1)]**

Timed out.

$$\int x^4 \csc^{-1}(a + bx) dx = \int x^4 \operatorname{asin}\left(\frac{1}{a + bx}\right) dx$$

input

```
int(x^4*asin(1/(a + b*x)),x)
```

output

```
int(x^4*asin(1/(a + b*x)), x)
```

**Reduce [F]**

$$\int x^4 \csc^{-1}(a + bx) dx = \int \operatorname{acsc}(bx + a) x^4 dx$$

input

```
int(x^4*acsc(b*x+a),x)
```

output

```
int(acsc(a + b*x)*x**4,x)
```

### 3.18 $\int x^3 \csc^{-1}(a + bx) dx$

Optimal result	151
Mathematica [A] (verified)	152
Rubi [A] (verified)	152
Maple [A] (verified)	155
Fricas [A] (verification not implemented)	155
Sympy [F]	156
Maxima [F]	156
Giac [B] (verification not implemented)	157
Mupad [F(-1)]	157
Reduce [F]	158

#### Optimal result

Integrand size = 10, antiderivative size = 155

$$\int x^3 \csc^{-1}(a + bx) dx = \frac{(2 + 17a^2)(a + bx)\sqrt{1 - \frac{1}{(a+bx)^2}}}{12b^4} + \frac{x^2(a + bx)\sqrt{1 - \frac{1}{(a+bx)^2}}}{12b^2}$$

$$- \frac{a(a + bx)^2\sqrt{1 - \frac{1}{(a+bx)^2}}}{3b^4} - \frac{a^4 \csc^{-1}(a + bx)}{4b^4}$$

$$+ \frac{1}{4}x^4 \csc^{-1}(a + bx) - \frac{a(1 + 2a^2) \operatorname{arctanh}\left(\sqrt{1 - \frac{1}{(a+bx)^2}}\right)}{2b^4}$$

output

```
1/12*(17*a^2+2)*(b*x+a)*(1-1/(b*x+a)^2)^(1/2)/b^4+1/12*x^2*(b*x+a)*(1-1/(b
*x+a)^2)^(1/2)/b^2-1/3*a*(b*x+a)^2*(1-1/(b*x+a)^2)^(1/2)/b^4-1/4*a^4*arccs
c(b*x+a)/b^4+1/4*x^4*arccsc(b*x+a)-1/2*a*(2*a^2+1)*arctanh((1-1/(b*x+a)^2)
^(1/2))/b^4
```



**Mathematica [A] (verified)**

Time = 0.18 (sec) , antiderivative size = 149, normalized size of antiderivative = 0.96

$$\int x^3 \csc^{-1}(a + bx) dx$$

$$= \frac{\sqrt{\frac{-1+a^2+2abx+b^2x^2}{(a+bx)^2}}(2a + 13a^3 + 2bx + 9a^2bx - 3ab^2x^2 + b^3x^3) + 3b^4x^4 \csc^{-1}(a + bx) - 3a^4 \arcsin\left(\frac{1}{a+bx}\right)}{12b^4}$$

input `Integrate[x^3*ArcCsc[a + b*x],x]`

output

```
(Sqrt[(-1 + a^2 + 2*a*b*x + b^2*x^2)/(a + b*x)^2]*(2*a + 13*a^3 + 2*b*x +
9*a^2*b*x - 3*a*b^2*x^2 + b^3*x^3) + 3*b^4*x^4*ArcCsc[a + b*x] - 3*a^4*Arc
Sin[(a + b*x)^(-1)] - 6*a*(1 + 2*a^2)*Log[(a + b*x)*(1 + Sqrt[(-1 + a^2 +
2*a*b*x + b^2*x^2)/(a + b*x)^2])])/(12*b^4)
```

**Rubi [A] (verified)**

Time = 0.54 (sec) , antiderivative size = 158, normalized size of antiderivative = 1.02, number of steps used = 9, number of rules used = 8,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.800$ , Rules used = {5782, 25, 4927, 3042, 4269, 3042, 4536, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x^3 \csc^{-1}(a + bx) dx$$

$$\downarrow 5782$$

$$-\frac{\int b^3 x^3 (a + bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx) d \csc^{-1}(a + bx)}{b^4}$$

$$\downarrow 25$$

$$\frac{\int -b^3 x^3 (a + bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx) d \csc^{-1}(a + bx)}{b^4}$$

$$\downarrow 4927$$

$$\frac{\frac{1}{4} \int b^4 x^4 d \csc^{-1}(a + bx) - \frac{1}{4} b^4 x^4 \csc^{-1}(a + bx)}{b^4}$$

↓ 3042

$$\frac{\frac{1}{4} \int (a - \csc(\csc^{-1}(a + bx)))^4 d \csc^{-1}(a + bx) - \frac{1}{4} b^4 x^4 \csc^{-1}(a + bx)}{b^4}$$

↓ 4269

$$\frac{\frac{1}{4} \left( \frac{1}{3} \int -bx(3a^3 + 8(a + bx)^2 a - (9a^2 + 2)(a + bx)) d \csc^{-1}(a + bx) - \frac{1}{3} b^2 x^2 (a + bx) \sqrt{1 - \frac{1}{(a+bx)^2}} \right) - \frac{1}{4} b^4 x^4 \csc^{-1}(a + bx)}{b^4}$$

↓ 3042

$$\frac{\frac{1}{4} \left( \frac{1}{3} \int (a - \csc(\csc^{-1}(a + bx))) \left( 3a^3 + 8 \csc(\csc^{-1}(a + bx))^2 a - (9a^2 + 2) \csc(\csc^{-1}(a + bx)) \right) d \csc^{-1}(a + bx) \right) - \frac{1}{4} b^4 x^4 \csc^{-1}(a + bx)}{b^4}$$

↓ 4536

$$\frac{\frac{1}{4} \left( \frac{1}{3} \left( \frac{1}{2} \int (6a^4 - 12(2a^2 + 1)(a + bx)a + 2(17a^2 + 2)(a + bx)^2) d \csc^{-1}(a + bx) + 4a \sqrt{1 - \frac{1}{(a+bx)^2}} (a + bx)^2 \right) \right) - \frac{1}{4} b^4 x^4 \csc^{-1}(a + bx)}{b^4}$$

↓ 2009

$$\frac{\frac{1}{4} \left( \frac{1}{3} \left( \frac{1}{2} \left( 6a^4 \csc^{-1}(a + bx) + 12(2a^2 + 1) a \operatorname{arctanh} \left( \sqrt{1 - \frac{1}{(a+bx)^2}} \right) - 2(17a^2 + 2)(a + bx) \sqrt{1 - \frac{1}{(a+bx)^2}} \right) + 4a \sqrt{1 - \frac{1}{(a+bx)^2}} (a + bx)^2 \right) \right) - \frac{1}{4} b^4 x^4 \csc^{-1}(a + bx)}{b^4}$$

input `Int[x^3*ArcCsc[a + b*x],x]`

output `-((-1/4*(b^4*x^4*ArcCsc[a + b*x]) + (-1/3*(b^2*x^2*(a + b*x)*Sqrt[1 - (a + b*x)^(-2)])) + (4*a*(a + b*x)^2*Sqrt[1 - (a + b*x)^(-2)] + (-2*(2 + 17*a^2)*(a + b*x)*Sqrt[1 - (a + b*x)^(-2)] + 6*a^4*ArcCsc[a + b*x] + 12*a*(1 + 2*a^2)*ArcTanh[Sqrt[1 - (a + b*x)^(-2)]])/2)/3)/4)/b^4`

## Definitions of rubi rules used

- rule 25  $\text{Int}[-(\text{Fx}_), \text{x\_Symbol}] \rightarrow \text{Simp}[\text{Identity}[-1] \quad \text{Int}[\text{Fx}, \text{x}], \text{x}]$
- rule 2009  $\text{Int}[\text{u}_, \text{x\_Symbol}] \rightarrow \text{Simp}[\text{IntSum}[\text{u}, \text{x}], \text{x}] \text{ /; SumQ}[\text{u}]$
- rule 3042  $\text{Int}[\text{u}_, \text{x\_Symbol}] \rightarrow \text{Int}[\text{DeactivateTrig}[\text{u}, \text{x}], \text{x}] \text{ /; FunctionOfTrigOfLinearQ}[\text{u}, \text{x}]$
- rule 4269  $\text{Int}[(\text{csc}[(\text{c}_.) + (\text{d}_.) * (\text{x}_)] * (\text{b}_.) + (\text{a}_.))^{\text{n}_.}, \text{x\_Symbol}] \rightarrow \text{Simp}[(-\text{b}^2) * \text{Cot}[\text{c} + \text{d} * \text{x}] * ((\text{a} + \text{b} * \text{Csc}[\text{c} + \text{d} * \text{x}])^{\text{n} - 2} / (\text{d} * (\text{n} - 1))), \text{x}] + \text{Simp}[1 / (\text{n} - 1) \text{ Int}[(\text{a} + \text{b} * \text{Csc}[\text{c} + \text{d} * \text{x}])^{\text{n} - 3} * \text{Simp}[\text{a}^3 * (\text{n} - 1) + (\text{b} * (\text{b}^2 * (\text{n} - 2) + 3 * \text{a}^2 * (\text{n} - 1))) * \text{Csc}[\text{c} + \text{d} * \text{x}] + (\text{a} * \text{b}^2 * (3 * \text{n} - 4)) * \text{Csc}[\text{c} + \text{d} * \text{x}]^2, \text{x}], \text{x}], \text{x}] \text{ /; FreeQ}[\{\text{a}, \text{b}, \text{c}, \text{d}\}, \text{x}] \&\& \text{NeQ}[\text{a}^2 - \text{b}^2, 0] \&\& \text{GtQ}[\text{n}, 2] \&\& \text{IntegerQ}[2 * \text{n}]$
- rule 4536  $\text{Int}[(\text{A}_.) + \text{csc}[(\text{e}_.) + (\text{f}_.) * (\text{x}_)] * (\text{B}_.) + \text{csc}[(\text{e}_.) + (\text{f}_.) * (\text{x}_)]^2 * (\text{C}_.) * (\text{csc}[(\text{e}_.) + (\text{f}_.) * (\text{x}_)] * (\text{b}_.) + (\text{a}_.)), \text{x\_Symbol}] \rightarrow \text{Simp}[(-\text{b}) * \text{C} * \text{Csc}[\text{e} + \text{f} * \text{x}] * (\text{Cot}[\text{e} + \text{f} * \text{x}] / (2 * \text{f})), \text{x}] + \text{Simp}[1 / 2 \quad \text{Int}[\text{Simp}[2 * \text{A} * \text{a} + (2 * \text{B} * \text{a} + \text{b} * (2 * \text{A} + \text{C})) * \text{Csc}[\text{e} + \text{f} * \text{x}] + 2 * (\text{a} * \text{C} + \text{B} * \text{b}) * \text{Csc}[\text{e} + \text{f} * \text{x}]^2, \text{x}], \text{x}], \text{x}] \text{ /; FreeQ}[\{\text{a}, \text{b}, \text{e}, \text{f}, \text{A}, \text{B}, \text{C}\}, \text{x}]$
- rule 4927  $\text{Int}[\text{Cot}[(\text{c}_.) + (\text{d}_.) * (\text{x}_)] * \text{Csc}[(\text{c}_.) + (\text{d}_.) * (\text{x}_)] * (\text{Csc}[(\text{c}_.) + (\text{d}_.) * (\text{x}_)] * (\text{b}_.) + (\text{a}_.))^{\text{n}_.} * ((\text{e}_.) + (\text{f}_.) * (\text{x}_.))^{\text{m}_.}, \text{x\_Symbol}] \rightarrow \text{Simp}[(-\text{e} + \text{f} * \text{x})^{\text{m}} * ((\text{a} + \text{b} * \text{Csc}[\text{c} + \text{d} * \text{x}])^{\text{n} + 1} / (\text{b} * \text{d} * (\text{n} + 1))), \text{x}] + \text{Simp}[\text{f} * (\text{m} / (\text{b} * \text{d} * (\text{n} + 1))) \quad \text{Int}[(\text{e} + \text{f} * \text{x})^{\text{m} - 1} * (\text{a} + \text{b} * \text{Csc}[\text{c} + \text{d} * \text{x}])^{\text{n} + 1}, \text{x}], \text{x}] \text{ /; FreeQ}[\{\text{a}, \text{b}, \text{c}, \text{d}, \text{e}, \text{f}, \text{n}\}, \text{x}] \&\& \text{IGtQ}[\text{m}, 0] \&\& \text{NeQ}[\text{n}, -1]$
- rule 5782  $\text{Int}[(\text{a}_.) + \text{ArcCsc}[(\text{c}_.) + (\text{d}_.) * (\text{x}_)] * (\text{b}_.)]^{\text{p}_.} * ((\text{e}_.) + (\text{f}_.) * (\text{x}_.))^{\text{m}_.}, \text{x\_Symbol}] \rightarrow \text{Simp}[(-\text{d}^{\text{m} + 1})^{-1} \quad \text{Subst}[\text{Int}[(\text{a} + \text{b} * \text{x})^{\text{p}} * \text{Csc}[\text{x}] * \text{Cot}[\text{x}] * (\text{d} * \text{e} - \text{c} * \text{f} + \text{f} * \text{Csc}[\text{x}])^{\text{m}}, \text{x}], \text{x}, \text{ArcCsc}[\text{c} + \text{d} * \text{x}]], \text{x}] \text{ /; FreeQ}[\{\text{a}, \text{b}, \text{c}, \text{d}, \text{e}, \text{f}\}, \text{x}] \&\& \text{IGtQ}[\text{p}, 0] \&\& \text{IntegerQ}[\text{m}]$

### Maple [A] (verified)

Time = 0.22 (sec) , antiderivative size = 249, normalized size of antiderivative = 1.61

method	result
derivativedivides	$\frac{\operatorname{arccsc}(bx+a)a^4}{4} - \operatorname{arccsc}(bx+a)a^3(bx+a) + \frac{3 \operatorname{arccsc}(bx+a)a^2(bx+a)^2}{2} - \operatorname{arccsc}(bx+a)a(bx+a)^3 + \frac{\operatorname{arccsc}(bx+a)(bx+a)^4}{4} - \dots$
default	$\frac{\operatorname{arccsc}(bx+a)a^4}{4} - \operatorname{arccsc}(bx+a)a^3(bx+a) + \frac{3 \operatorname{arccsc}(bx+a)a^2(bx+a)^2}{2} - \operatorname{arccsc}(bx+a)a(bx+a)^3 + \frac{\operatorname{arccsc}(bx+a)(bx+a)^4}{4} - \dots$
parts	$\frac{x^4 \operatorname{arccsc}(bx+a)}{4} - \frac{\sqrt{b^2x^2+2abx+a^2-1} \left( 3a^4 \arctan\left(\frac{1}{\sqrt{b^2x^2+2abx+a^2-1}}\right) \sqrt{b^2-x^2} \sqrt{b^2x^2+2abx+a^2-1} b^2 \sqrt{b^2+12} \right)}{4}$

```
input int(x^3*arccsc(b*x+a), x, method=_RETURNVERBOSE)
```

```
output 1/b^4*(1/4*arccsc(b*x+a)*a^4-arccsc(b*x+a)*a^3*(b*x+a)+3/2*arccsc(b*x+a)*a^2*(b*x+a)^2-arccsc(b*x+a)*a*(b*x+a)^3+1/4*arccsc(b*x+a)*(b*x+a)^4-1/12*((b*x+a)^2-1)^(1/2)*(3*a^4*arctan(1/((b*x+a)^2-1)^(1/2))+12*a^3*ln(b*x+a+((b*x+a)^2-1)^(1/2))-18*a^2*((b*x+a)^2-1)^(1/2)+6*a*(b*x+a)*((b*x+a)^2-1)^(1/2)-(b*x+a)^2*((b*x+a)^2-1)^(1/2)+6*a*ln(b*x+a+((b*x+a)^2-1)^(1/2))-2*((b*x+a)^2-1)^(1/2))/(((b*x+a)^2-1)/(b*x+a)^2)^(1/2)/(b*x+a))
```

### Fricas [A] (verification not implemented)

Time = 0.13 (sec) , antiderivative size = 129, normalized size of antiderivative = 0.83

$$\int x^3 \operatorname{csc}^{-1}(a + bx) dx = \frac{3 b^4 x^4 \operatorname{arccsc}(bx + a) + 6 a^4 \arctan(-bx - a + \sqrt{b^2 x^2 + 2 abx + a^2 - 1}) + 6 (2 a^3 + a) \log(-bx - a + \sqrt{b^2 x^2 + 2 abx + a^2 - 1})}{12 b^4}$$

```
input integrate(x^3*arccsc(b*x+a), x, algorithm="fricas")
```

output

```
1/12*(3*b^4*x^4*arccsc(b*x + a) + 6*a^4*arctan(-b*x - a + sqrt(b^2*x^2 + 2
*a*b*x + a^2 - 1)) + 6*(2*a^3 + a)*log(-b*x - a + sqrt(b^2*x^2 + 2*a*b*x +
a^2 - 1)) + sqrt(b^2*x^2 + 2*a*b*x + a^2 - 1)*(b^2*x^2 - 4*a*b*x + 13*a^2
+ 2))/b^4
```

**Sympy [F]**

$$\int x^3 \csc^{-1}(a + bx) dx = \int x^3 \operatorname{acsc}(a + bx) dx$$

input

```
integrate(x**3*acsc(b*x+a),x)
```

output

```
Integral(x**3*acsc(a + b*x), x)
```

**Maxima [F]**

$$\int x^3 \csc^{-1}(a + bx) dx = \int x^3 \operatorname{arccsc}(bx + a) dx$$

input

```
integrate(x^3*arccsc(b*x+a),x, algorithm="maxima")
```

output

```
1/4*x^4*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) + integrate(1/4*(b
^2*x^5 + a*b*x^4)*e^(1/2*log(b*x + a + 1) + 1/2*log(b*x + a - 1))/(b^2*x^2
+ 2*a*b*x + a^2 + (b^2*x^2 + 2*a*b*x + a^2 - 1)*e^(log(b*x + a + 1) + log
(b*x + a - 1)) - 1), x)
```

**Giac [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 303 vs.  $2(135) = 270$ .

Time = 0.16 (sec) , antiderivative size = 303, normalized size of antiderivative = 1.95

$$\int x^3 \csc^{-1}(a + bx) dx =$$

$$-\frac{1}{96}b \left( \frac{24(bx+a)^4 \left( \frac{4a}{bx+a} - \frac{6a^2}{(bx+a)^2} + \frac{4a^3}{(bx+a)^3} - 1 \right) \arcsin \left( -\frac{1}{(bx+a) \left( \frac{a}{bx+a} - 1 \right) - a} \right) - (bx+a)^3 \left( \sqrt{-\frac{1}{(bx+a)^2 + 1}} - 1 \right)^3 + 12(bx+a)^2 a \left( \sqrt{-\frac{1}{(bx+a)^2 + 1}} - 1 \right) + 9(bx+a) \left( \sqrt{-\frac{1}{(bx+a)^2 + 1}} - 1 \right) + 48(2a^3 + a) \log \left( \frac{1}{2} \text{abs}(bx+a) \text{abs}(-2\sqrt{-\frac{1}{(bx+a)^2 + 1}} + 2) \right) - (9(8a^2 + 1)(bx+a)^2 \left( \sqrt{-\frac{1}{(bx+a)^2 + 1}} - 1 \right)^2 + 12(bx+a) a \left( \sqrt{-\frac{1}{(bx+a)^2 + 1}} - 1 \right) + 1) / ((bx+a)^3 \left( \sqrt{-\frac{1}{(bx+a)^2 + 1}} - 1 \right)^3) \right) / b^5$$

input `integrate(x^3*arccsc(b*x+a),x, algorithm="giac")`

output `-1/96*b*(24*(b*x + a)^4*(4*a/(b*x + a) - 6*a^2/(b*x + a)^2 + 4*a^3/(b*x + a)^3 - 1)*arcsin(-1/((b*x + a)*(a/(b*x + a) - 1) - a))/b^5 - ((b*x + a)^3*(sqrt(-1/(b*x + a)^2 + 1) - 1)^3 + 12*(b*x + a)^2*a*(sqrt(-1/(b*x + a)^2 + 1) - 1) + 9*(b*x + a)*(sqrt(-1/(b*x + a)^2 + 1) - 1) + 48*(2*a^3 + a)*log(1/2*abs(b*x + a)*abs(-2*sqrt(-1/(b*x + a)^2 + 1) + 2)) - (9*(8*a^2 + 1)*(b*x + a)^2*(sqrt(-1/(b*x + a)^2 + 1) - 1)^2 + 12*(b*x + a)*a*(sqrt(-1/(b*x + a)^2 + 1) - 1) + 1)/((b*x + a)^3*(sqrt(-1/(b*x + a)^2 + 1) - 1)^3))/b^5`

**Mupad [F(-1)]**

Timed out.

$$\int x^3 \csc^{-1}(a + bx) dx = \int x^3 \operatorname{asin} \left( \frac{1}{a + bx} \right) dx$$

input `int(x^3*asin(1/(a + b*x)),x)`

output `int(x^3*asin(1/(a + b*x)), x)`

**Reduce [F]**

$$\int x^3 \csc^{-1}(a + bx) dx = \int \operatorname{acsc}(bx + a) x^3 dx$$

input `int(x^3*acsc(b*x+a),x)`

output `int(acsc(a + b*x)*x**3,x)`

### 3.19 $\int x^2 \csc^{-1}(a + bx) dx$

Optimal result	159
Mathematica [A] (verified)	159
Rubi [A] (verified)	160
Maple [A] (verified)	162
Fricas [A] (verification not implemented)	162
Sympy [F]	163
Maxima [F]	163
Giac [B] (verification not implemented)	164
Mupad [F(-1)]	164
Reduce [F]	165

#### Optimal result

Integrand size = 10, antiderivative size = 116

$$\int x^2 \csc^{-1}(a + bx) dx = -\frac{5a(a + bx)\sqrt{1 - \frac{1}{(a+bx)^2}}}{6b^3} + \frac{x(a + bx)\sqrt{1 - \frac{1}{(a+bx)^2}}}{6b^2} + \frac{a^3 \csc^{-1}(a + bx)}{3b^3} + \frac{1}{3}x^3 \csc^{-1}(a + bx) + \frac{(1 + 6a^2) \operatorname{arctanh}\left(\sqrt{1 - \frac{1}{(a+bx)^2}}\right)}{6b^3}$$

output

```
-5/6*a*(b*x+a)*(1-1/(b*x+a)^2)^(1/2)/b^3+1/6*x*(b*x+a)*(1-1/(b*x+a)^2)^(1/2)/b^2+1/3*a^3*arccsc(b*x+a)/b^3+1/3*x^3*arccsc(b*x+a)+1/6*(6*a^2+1)*arctanh((1-1/(b*x+a)^2)^(1/2))/b^3
```

#### Mathematica [A] (verified)

Time = 0.12 (sec) , antiderivative size = 129, normalized size of antiderivative = 1.11

$$\int x^2 \csc^{-1}(a + bx) dx = \frac{(-5a^2 - 4abx + b^2x^2) \sqrt{\frac{-1+a^2+2abx+b^2x^2}{(a+bx)^2}} + 2b^3x^3 \csc^{-1}(a + bx) + 2a^3 \arcsin\left(\frac{1}{a+bx}\right) + (1 + 6a^2) \log\left((a + bx) \sqrt{1 - \frac{1}{(a+bx)^2}}\right)}{6b^3}$$



input `Integrate[x^2*ArcCsc[a + b*x],x]`

output `((-5*a^2 - 4*a*b*x + b^2*x^2)*Sqrt[(-1 + a^2 + 2*a*b*x + b^2*x^2)/(a + b*x)^2] + 2*b^3*x^3*ArcCsc[a + b*x] + 2*a^3*ArcSin[(a + b*x)^(-1)] + (1 + 6*a^2)*Log[(a + b*x)*(1 + Sqrt[(-1 + a^2 + 2*a*b*x + b^2*x^2)/(a + b*x)^2])))/(6*b^3)`

### Rubi [A] (verified)

Time = 0.39 (sec) , antiderivative size = 117, normalized size of antiderivative = 1.01, number of steps used = 6, number of rules used = 5,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {5782, 4927, 3042, 4269, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int x^2 \csc^{-1}(a + bx) dx \\
 & \quad \downarrow \text{5782} \\
 & - \frac{\int b^2 x^2 (a + bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx) d \csc^{-1}(a + bx)}{b^3} \\
 & \quad \downarrow \text{4927} \\
 & - \frac{-\frac{1}{3} \int -b^3 x^3 d \csc^{-1}(a + bx) - \frac{1}{3} b^3 x^3 \csc^{-1}(a + bx)}{b^3} \\
 & \quad \downarrow \text{3042} \\
 & - \frac{-\frac{1}{3} \int (a - \csc(\csc^{-1}(a + bx)))^3 d \csc^{-1}(a + bx) - \frac{1}{3} b^3 x^3 \csc^{-1}(a + bx)}{b^3} \\
 & \quad \downarrow \text{4269} \\
 & - \frac{\frac{1}{3} \left( -\frac{1}{2} \int (2a^3 + 5(a + bx)^2 a - (6a^2 + 1)(a + bx)) d \csc^{-1}(a + bx) - \frac{1}{2} bx \sqrt{1 - \frac{1}{(a+bx)^2}} (a + bx) \right) - \frac{1}{3} b^3 x^3 \csc^{-1}}{b^3} \\
 & \quad \downarrow \text{2009}
 \end{aligned}$$

$$\frac{\frac{1}{3}\left(\frac{1}{2}\left(-2a^3 \csc^{-1}(a+bx) - (6a^2+1) \operatorname{arctanh}\left(\sqrt{1-\frac{1}{(a+bx)^2}}\right) + 5a(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}\right) - \frac{1}{2}bx(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}\right)}{b^3}$$

input `Int[x^2*ArcCsc[a + b*x], x]`

output `-((-1/3*(b^3*x^3*ArcCsc[a + b*x]) + (-1/2*(b*x*(a + b*x)*Sqrt[1 - (a + b*x)^(-2)])) + (5*a*(a + b*x)*Sqrt[1 - (a + b*x)^(-2)] - 2*a^3*ArcCsc[a + b*x] - (1 + 6*a^2)*ArcTanh[Sqrt[1 - (a + b*x)^(-2)]])/2)/3)/b^3`

### Defintions of rubi rules used

rule 2009 `Int[u_, x_Symbol] :=> Simp[IntSum[u, x], x] /; SumQ[u]`

rule 3042 `Int[u_, x_Symbol] :=> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 4269 `Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_.))^(n_), x_Symbol] :=> Simp[(-b^2)*Cot[c + d*x]*((a + b*Csc[c + d*x])^(n - 2)/(d*(n - 1))), x] + Simp[1/(n - 1) Int[(a + b*Csc[c + d*x])^(n - 3)*Simp[a^3*(n - 1) + (b*(b^2*(n - 2) + 3*a^2*(n - 1))*Csc[c + d*x] + (a*b^2*(3*n - 4))*Csc[c + d*x]^2, x], x], x] /; FreeQ[{a, b, c, d}, x] && NeQ[a^2 - b^2, 0] && GtQ[n, 2] && IntegerQ[2*n]`

rule 4927 `Int[Cot[(c_.) + (d_.)*(x_)]*Csc[(c_.) + (d_.)*(x_)]*(Csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_.))^(n_.)*((e_.) + (f_.)*(x_))^(m_.), x_Symbol] :=> Simp[(-(e + f*x)^m)*((a + b*Csc[c + d*x])^(n + 1)/(b*d*(n + 1))), x] + Simp[f*(m/(b*d*(n + 1))) Int[(e + f*x)^(m - 1)*(a + b*Csc[c + d*x])^(n + 1), x], x] /; FreeQ[{a, b, c, d, e, f, n}, x] && IGtQ[m, 0] && NeQ[n, -1]`

rule 5782 `Int[((a_.) + ArcCsc[(c_.) + (d_.)*(x_)]*(b_.))^(p_.)*((e_.) + (f_.)*(x_))^(m_.), x_Symbol] :=> Simp[-(d^(m + 1))^(-1) Subst[Int[(a + b*x)^p*Csc[x]*Cot[x]*(d*e - c*f + f*Csc[x])^m, x], x, ArcCsc[c + d*x]], x] /; FreeQ[{a, b, c, d, e, f}, x] && IGtQ[p, 0] && IntegerQ[m]`

### Maple [A] (verified)

Time = 0.23 (sec) , antiderivative size = 190, normalized size of antiderivative = 1.64

method	result
derivativedivides	$-\frac{\operatorname{arccsc}(bx+a)a^3}{3} + \operatorname{arccsc}(bx+a)a^2(bx+a) - \operatorname{arccsc}(bx+a)a(bx+a)^2 + \frac{\operatorname{arccsc}(bx+a)(bx+a)^3}{3} + \frac{\sqrt{(bx+a)^2-1} \left( 2a^3 \arctan\left(\frac{1}{\sqrt{(bx+a)^2-1}}\right) \right)}{b^3}$
default	$-\frac{\operatorname{arccsc}(bx+a)a^3}{3} + \operatorname{arccsc}(bx+a)a^2(bx+a) - \operatorname{arccsc}(bx+a)a(bx+a)^2 + \frac{\operatorname{arccsc}(bx+a)(bx+a)^3}{3} + \frac{\sqrt{(bx+a)^2-1} \left( 2a^3 \arctan\left(\frac{1}{\sqrt{(bx+a)^2-1}}\right) \right)}{b^3}$
parts	$\frac{x^3 \operatorname{arccsc}(bx+a)}{3} + \frac{\sqrt{b^2x^2+2abx+a^2-1} \left( 2a^3 \arctan\left(\frac{1}{\sqrt{b^2x^2+2abx+a^2-1}}\right) \sqrt{b^2} + 6 \ln\left(\frac{b^2x+\sqrt{b^2x^2+2abx+a^2-1}\sqrt{b^2}}{\sqrt{b^2}}\right) \right)}{6b^3 \sqrt{b^2x^2+2abx+a^2-1}}$

```
input int(x^2*arccsc(b*x+a), x, method=_RETURNVERBOSE)
```

```
output 1/b^3*(-1/3*arccsc(b*x+a)*a^3+arccsc(b*x+a)*a^2*(b*x+a)-arccsc(b*x+a)*a*(b*x+a)^2+1/3*arccsc(b*x+a)*(b*x+a)^3+1/6*((b*x+a)^2-1)^(1/2)*(2*a^3*arctan(1/((b*x+a)^2-1)^(1/2))+6*a^2*ln(b*x+a+((b*x+a)^2-1)^(1/2))-6*a*((b*x+a)^2-1)^(1/2)+(b*x+a)*((b*x+a)^2-1)^(1/2)+ln(b*x+a+((b*x+a)^2-1)^(1/2)))/(((b*x+a)^2-1)/(b*x+a)^2)^(1/2)/(b*x+a))
```

### Fricas [A] (verification not implemented)

Time = 0.13 (sec) , antiderivative size = 117, normalized size of antiderivative = 1.01

$$\int x^2 \operatorname{csc}^{-1}(a + bx) dx = \frac{2b^3x^3 \operatorname{arccsc}(bx+a) - 4a^3 \arctan(-bx-a+\sqrt{b^2x^2+2abx+a^2-1}) - (6a^2+1) \log(-bx-a+\sqrt{b^2x^2+2abx+a^2-1})}{6b^3}$$

```
input integrate(x^2*arccsc(b*x+a), x, algorithm="fricas")
```

output

```
1/6*(2*b^3*x^3*arccsc(b*x + a) - 4*a^3*arctan(-b*x - a + sqrt(b^2*x^2 + 2*
a*b*x + a^2 - 1)) - (6*a^2 + 1)*log(-b*x - a + sqrt(b^2*x^2 + 2*a*b*x + a^
2 - 1)) + sqrt(b^2*x^2 + 2*a*b*x + a^2 - 1)*(b*x - 5*a))/b^3
```

**Sympy [F]**

$$\int x^2 \csc^{-1}(a + bx) dx = \int x^2 \operatorname{acsc}(a + bx) dx$$

input

```
integrate(x**2*acsc(b*x+a),x)
```

output

```
Integral(x**2*acsc(a + b*x), x)
```

**Maxima [F]**

$$\int x^2 \csc^{-1}(a + bx) dx = \int x^2 \operatorname{arccsc}(bx + a) dx$$

input

```
integrate(x^2*arccsc(b*x+a),x, algorithm="maxima")
```

output

```
1/3*x^3*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) + integrate(1/3*(b
^2*x^4 + a*b*x^3)*e^(1/2*log(b*x + a + 1) + 1/2*log(b*x + a - 1))/(b^2*x^2
+ 2*a*b*x + a^2 + (b^2*x^2 + 2*a*b*x + a^2 - 1)*e^(log(b*x + a + 1) + log
(b*x + a - 1)) - 1), x)
```

**Giac [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 206 vs.  $2(100) = 200$ .

Time = 0.14 (sec) , antiderivative size = 206, normalized size of antiderivative = 1.78

$$\int x^2 \csc^{-1}(a + bx) dx =$$

$$-\frac{1}{24} b \left( \frac{8 (bx + a)^3 \left( \frac{3a}{bx+a} - \frac{3a^2}{(bx+a)^2} - 1 \right) \arcsin \left( -\frac{1}{(bx+a) \left( \frac{a}{bx+a} - 1 \right) - a} \right)}{b^4} + \frac{(bx + a)^2 \left( \sqrt{-\frac{1}{(bx+a)^2} + 1} - 1 \right)}{b^4} \right)$$

input `integrate(x^2*arccsc(b*x+a),x, algorithm="giac")`

output `-1/24*b*(8*(b*x + a)^3*(3*a/(b*x + a) - 3*a^2/(b*x + a)^2 - 1)*arcsin(-1/((b*x + a)*(a/(b*x + a) - 1) - a))/b^4 + ((b*x + a)^2*(sqrt(-1/(b*x + a)^2 + 1) - 1)^2 + 12*(b*x + a)*a*(sqrt(-1/(b*x + a)^2 + 1) - 1) + 4*(6*a^2 + 1)*log(1/2*abs(b*x + a)*abs(-2*sqrt(-1/(b*x + a)^2 + 1) + 2)) - (12*(b*x + a)*a*(sqrt(-1/(b*x + a)^2 + 1) - 1) + 1)/((b*x + a)^2*(sqrt(-1/(b*x + a)^2 + 1) - 1)^2))/b^4)`

**Mupad [F(-1)]**

Timed out.

$$\int x^2 \csc^{-1}(a + bx) dx = \int x^2 \operatorname{asin} \left( \frac{1}{a + bx} \right) dx$$

input `int(x^2*asin(1/(a + b*x)),x)`

output `int(x^2*asin(1/(a + b*x)), x)`

**Reduce [F]**

$$\int x^2 \csc^{-1}(a + bx) dx = \int \operatorname{acsc}(bx + a) x^2 dx$$

input `int(x^2*acsc(b*x+a),x)`

output `int(acsc(a + b*x)*x**2,x)`

### 3.20 $\int x \csc^{-1}(a + bx) dx$

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#### Optimal result

Integrand size = 8, antiderivative size = 79

$$\int x \csc^{-1}(a + bx) dx = \frac{(a + bx)\sqrt{1 - \frac{1}{(a+bx)^2}}}{2b^2} - \frac{a^2 \csc^{-1}(a + bx)}{2b^2} + \frac{1}{2}x^2 \csc^{-1}(a + bx) - \frac{a \operatorname{arctanh}\left(\sqrt{1 - \frac{1}{(a+bx)^2}}\right)}{b^2}$$

output

```
1/2*(b*x+a)*(1-1/(b*x+a)^2)^(1/2)/b^2-1/2*a^2*arccsc(b*x+a)/b^2+1/2*x^2*arccsc(b*x+a)-a*arctanh((1-1/(b*x+a)^2)^(1/2))/b^2
```

#### Mathematica [A] (verified)

Time = 0.09 (sec) , antiderivative size = 110, normalized size of antiderivative = 1.39

$$\int x \csc^{-1}(a + bx) dx = \frac{(a + bx)\sqrt{\frac{-1+a^2+2abx+b^2x^2}{(a+bx)^2}} + b^2x^2 \csc^{-1}(a + bx) - a^2 \arcsin\left(\frac{1}{a+bx}\right) - 2a \log\left((a + bx)\left(1 + \sqrt{\frac{-1+a^2+2abx+b^2x^2}{(a+bx)^2}}\right)\right)}{2b^2}$$

input

```
Integrate[x*ArcCsc[a + b*x],x]
```

output

```
((a + b*x)*Sqrt[(-1 + a^2 + 2*a*b*x + b^2*x^2)/(a + b*x)^2] + b^2*x^2*ArcCsc[a + b*x] - a^2*ArcSin[(a + b*x)^(-1)] - 2*a*Log[(a + b*x)*(1 + Sqrt[(-1 + a^2 + 2*a*b*x + b^2*x^2)/(a + b*x)^2])])/(2*b^2)
```

**Rubi [A] (verified)**

Time = 0.44 (sec) , antiderivative size = 78, normalized size of antiderivative = 0.99, number of steps used = 10, number of rules used = 9,  $\frac{\text{number of rules}}{\text{integrand size}} = 1.125$ , Rules used = {5782, 25, 4927, 3042, 4260, 3042, 4254, 24, 4257}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int x \csc^{-1}(a + bx) dx \\
 & \quad \downarrow \text{5782} \\
 & - \frac{\int bx(a + bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx) d \csc^{-1}(a + bx)}{b^2} \\
 & \quad \downarrow \text{25} \\
 & - \frac{\int -bx(a + bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx) d \csc^{-1}(a + bx)}{b^2} \\
 & \quad \downarrow \text{4927} \\
 & - \frac{\frac{1}{2} \int b^2 x^2 d \csc^{-1}(a + bx) - \frac{1}{2} b^2 x^2 \csc^{-1}(a + bx)}{b^2} \\
 & \quad \downarrow \text{3042} \\
 & - \frac{\frac{1}{2} \int (a - \csc(\csc^{-1}(a + bx)))^2 d \csc^{-1}(a + bx) - \frac{1}{2} b^2 x^2 \csc^{-1}(a + bx)}{b^2} \\
 & \quad \downarrow \text{4260} \\
 & - \frac{\frac{1}{2} (-2a \int (a + bx) d \csc^{-1}(a + bx) + \int (a + bx)^2 d \csc^{-1}(a + bx) + a^2 \csc^{-1}(a + bx)) - \frac{1}{2} b^2 x^2 \csc^{-1}(a + bx)}{b^2} \\
 & \quad \downarrow \text{3042}
 \end{aligned}$$



$$\frac{\frac{1}{2} \left( -2a \int \csc(\csc^{-1}(a+bx)) d \csc^{-1}(a+bx) + \int \csc(\csc^{-1}(a+bx))^2 d \csc^{-1}(a+bx) + a^2 \csc^{-1}(a+bx) \right) - \frac{1}{2}}{b^2}$$

↓ 4254

$$\frac{\frac{1}{2} \left( -\int 1 d \left( (a+bx) \sqrt{1 - \frac{1}{(a+bx)^2}} \right) - 2a \int \csc(\csc^{-1}(a+bx)) d \csc^{-1}(a+bx) + a^2 \csc^{-1}(a+bx) \right) - \frac{1}{2} b^2 x^2 \csc^{-1}(a+bx)}{b^2}$$

↓ 24

$$\frac{\frac{1}{2} \left( -2a \int \csc(\csc^{-1}(a+bx)) d \csc^{-1}(a+bx) + a^2 \csc^{-1}(a+bx) - (a+bx) \sqrt{1 - \frac{1}{(a+bx)^2}} \right) - \frac{1}{2} b^2 x^2 \csc^{-1}(a+bx)}{b^2}$$

↓ 4257

$$\frac{\frac{1}{2} \left( a^2 \csc^{-1}(a+bx) + 2a \operatorname{arctanh} \left( \sqrt{1 - \frac{1}{(a+bx)^2}} \right) - (a+bx) \sqrt{1 - \frac{1}{(a+bx)^2}} \right) - \frac{1}{2} b^2 x^2 \csc^{-1}(a+bx)}{b^2}$$

input

Int[x\*ArcCsc[a + b\*x], x]

output

$$-\left( \frac{-1/2 * (b^2 * x^2 * \operatorname{ArcCsc}[a + b*x]) + \left( -(a + b*x) * \operatorname{Sqrt}[1 - (a + b*x)^{-2}] \right)}{2} \right) + a^2 * \operatorname{ArcCsc}[a + b*x] + 2 * a * \operatorname{ArcTanh}[\operatorname{Sqrt}[1 - (a + b*x)^{-2}]] / b^2$$
**Defintions of rubi rules used**

rule 24

Int[a\_, x\_Symbol] := Simp[a\*x, x] /; FreeQ[a, x]

rule 25

Int[-(Fx\_), x\_Symbol] := Simp[Identity[-1] Int[Fx, x], x]

rule 3042

Int[u\_, x\_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]

rule 4254  $\text{Int}[\text{csc}[(c_.) + (d_.)(x_.)]^{(n_.)}, x\_Symbol] \rightarrow \text{Simp}[-d^{(-1)} \text{Subst}[\text{Int}[\text{Exp}[\text{andIntegrand}[(1 + x^2)^{(n/2 - 1)}, x], x], x, \text{Cot}[c + d*x]], x] /; \text{FreeQ}[\{c, d\}, x] \ \&\& \ \text{IGtQ}[n/2, 0]$

rule 4257  $\text{Int}[\text{csc}[(c_.) + (d_.)(x_.)], x\_Symbol] \rightarrow \text{Simp}[-\text{ArcTanh}[\text{Cos}[c + d*x]]/d, x] /; \text{FreeQ}[\{c, d\}, x]$

rule 4260  $\text{Int}[(\text{csc}[(c_.) + (d_.)(x_.)]*(b_.) + (a_.))^{2}, x\_Symbol] \rightarrow \text{Simp}[a^{2*x}, x] + (\text{Simp}[2*a*b \text{Int}[\text{Csc}[c + d*x], x], x] + \text{Simp}[b^{2} \text{Int}[\text{Csc}[c + d*x]^{2}, x], x]) /; \text{FreeQ}[\{a, b, c, d\}, x]$

rule 4927  $\text{Int}[\text{Cot}[(c_.) + (d_.)(x_.)]*\text{Csc}[(c_.) + (d_.)(x_.)]*(\text{Csc}[(c_.) + (d_.)(x_.)]*(b_.) + (a_.))^{(n_.)}*((e_.) + (f_.)(x_.))^{(m_.)}, x\_Symbol] \rightarrow \text{Simp}[(- (e + f*x)^m)*((a + b*\text{Csc}[c + d*x])^{(n + 1)}/(b*d*(n + 1))), x] + \text{Simp}[f*(m/(b*d*(n + 1))) \text{Int}[(e + f*x)^{(m - 1)}*(a + b*\text{Csc}[c + d*x])^{(n + 1)}, x], x] /; \text{FreeQ}[\{a, b, c, d, e, f, n\}, x] \ \&\& \ \text{IGtQ}[m, 0] \ \&\& \ \text{NeQ}[n, -1]$

rule 5782  $\text{Int}[(a_.) + \text{ArcCsc}[(c_.) + (d_.)(x_.)]*(b_.))^{(p_.)}*((e_.) + (f_.)(x_.))^{(m_.)}, x\_Symbol] \rightarrow \text{Simp}[-(d^{(m + 1)})^{(-1)} \text{Subst}[\text{Int}[(a + b*x)^p*\text{Csc}[x]*\text{Cot}[x]*(d*e - c*f + f*\text{Csc}[x])^m, x], x, \text{ArcCsc}[c + d*x]], x] /; \text{FreeQ}[\{a, b, c, d, e, f\}, x] \ \&\& \ \text{IGtQ}[p, 0] \ \&\& \ \text{IntegerQ}[m]$

### Maple [A] (verified)

Time = 0.21 (sec) , antiderivative size = 108, normalized size of antiderivative = 1.37

method	result
derivativedivides	$\frac{\operatorname{arccsc}(bx+a)(bx+a)^2 - \operatorname{arccsc}(bx+a)a(bx+a) - \frac{\sqrt{(bx+a)^2-1} \left( 2a \ln(bx+a + \sqrt{(bx+a)^2-1}) - \sqrt{(bx+a)^2-1} \right)}{2(bx+a)\sqrt{\frac{(bx+a)^2-1}{(bx+a)^2}}}}{b^2}$
default	$\frac{\operatorname{arccsc}(bx+a)(bx+a)^2 - \operatorname{arccsc}(bx+a)a(bx+a) - \frac{\sqrt{(bx+a)^2-1} \left( 2a \ln(bx+a + \sqrt{(bx+a)^2-1}) - \sqrt{(bx+a)^2-1} \right)}{2(bx+a)\sqrt{\frac{(bx+a)^2-1}{(bx+a)^2}}}}{b^2}$
parts	$\frac{x^2 \operatorname{arccsc}(bx+a)}{2} - \frac{\sqrt{b^2x^2+2abx+a^2-1} \left( a^2 \arctan\left(\frac{1}{\sqrt{b^2x^2+2abx+a^2-1}}\right) \sqrt{b^2+2a} \ln\left(\frac{b^2x+\sqrt{b^2x^2+2abx+a^2-1}\sqrt{b^2+2a}}{\sqrt{b^2}}\right) \right)}{2b^2\sqrt{\frac{b^2x^2+2abx+a^2-1}{(bx+a)^2}} (bx+a)\sqrt{b^2}}$

```
input int(x*arccsc(b*x+a), x, method=_RETURNVERBOSE)
```

```
output 1/b^2*(1/2*arccsc(b*x+a)*(b*x+a)^2-arccsc(b*x+a)*a*(b*x+a)-1/2*((b*x+a)^2-1)^(1/2)*(2*a*ln(b*x+a+((b*x+a)^2-1)^(1/2))-((b*x+a)^2-1)^(1/2))/(b*x+a)/((b*x+a)^2-1)/(b*x+a)^2)^(1/2))
```

### Fricas [A] (verification not implemented)

Time = 0.12 (sec) , antiderivative size = 102, normalized size of antiderivative = 1.29

$$\int x \operatorname{csc}^{-1}(a + bx) dx = \frac{b^2x^2 \operatorname{arccsc}(bx + a) + 2a^2 \arctan(-bx - a + \sqrt{b^2x^2 + 2abx + a^2 - 1}) + 2a \log(-bx - a + \sqrt{b^2x^2 + 2abx + a^2 - 1})}{2b^2}$$

```
input integrate(x*arccsc(b*x+a), x, algorithm="fricas")
```

```
output 1/2*(b^2*x^2*arccsc(b*x + a) + 2*a^2*arctan(-b*x - a + sqrt(b^2*x^2 + 2*a*b*x + a^2 - 1)) + 2*a*log(-b*x - a + sqrt(b^2*x^2 + 2*a*b*x + a^2 - 1)) + sqrt(b^2*x^2 + 2*a*b*x + a^2 - 1))/b^2
```

**Sympy [F]**

$$\int x \csc^{-1}(a + bx) dx = \int x \operatorname{acsc}(a + bx) dx$$

input `integrate(x*acsc(b*x+a),x)`

output `Integral(x*acsc(a + b*x), x)`

**Maxima [F]**

$$\int x \csc^{-1}(a + bx) dx = \int x \operatorname{arccsc}(bx + a) dx$$

input `integrate(x*arccsc(b*x+a),x, algorithm="maxima")`

output `1/2*x^2*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) + integrate(1/2*(b^2*x^3 + a*b*x^2)*e^(1/2*log(b*x + a + 1) + 1/2*log(b*x + a - 1))/(b^2*x^2 + 2*a*b*x + a^2 + (b^2*x^2 + 2*a*b*x + a^2 - 1)*e^(log(b*x + a + 1) + log(b*x + a - 1)) - 1), x)`

**Giac [A] (verification not implemented)**

Time = 0.14 (sec) , antiderivative size = 137, normalized size of antiderivative = 1.73

$$\int x \csc^{-1}(a + bx) dx =$$

$$-\frac{1}{4} b \left( \frac{2 (bx + a)^2 \left( \frac{2a}{bx+a} - 1 \right) \arcsin \left( -\frac{1}{(bx+a) \left( \frac{a}{bx+a} - 1 \right) - a} \right)}{b^3} - \frac{(bx + a) \left( \sqrt{-\frac{1}{(bx+a)^2} + 1} - 1 \right) + 4a \log \left( \frac{1}{2} \right)}{b^3} \right)$$

input `integrate(x*arccsc(b*x+a),x, algorithm="giac")`

output

```
-1/4*b*(2*(b*x + a)^2*(2*a/(b*x + a) - 1)*arcsin(-1/((b*x + a)*(a/(b*x + a) - 1) - a))/b^3 - ((b*x + a)*(sqrt(-1/(b*x + a)^2 + 1) - 1) + 4*a*log(1/2*abs(b*x + a)*abs(-2*sqrt(-1/(b*x + a)^2 + 1) + 2)) - 1/((b*x + a)*(sqrt(-1/(b*x + a)^2 + 1) - 1)))/b^3)
```

**Mupad [F(-1)]**

Timed out.

$$\int x \csc^{-1}(a + bx) dx = \int x \operatorname{asin}\left(\frac{1}{a + bx}\right) dx$$

input

```
int(x*asin(1/(a + b*x)),x)
```

output

```
int(x*asin(1/(a + b*x)), x)
```

**Reduce [F]**

$$\int x \csc^{-1}(a + bx) dx = \int \operatorname{acsc}(bx + a) x dx$$

input

```
int(x*acsc(b*x+a),x)
```

output

```
int(acsc(a + b*x)*x,x)
```

### 3.21 $\int \csc^{-1}(a + bx) dx$

Optimal result	173
Mathematica [C] (verified)	173
Rubi [A] (warning: unable to verify)	174
Maple [A] (verified)	176
Fricas [B] (verification not implemented)	176
Sympy [F]	177
Maxima [A] (verification not implemented)	177
Giac [B] (verification not implemented)	177
Mupad [B] (verification not implemented)	178
Reduce [F]	178

#### Optimal result

Integrand size = 6, antiderivative size = 36

$$\int \csc^{-1}(a + bx) dx = \frac{(a + bx) \csc^{-1}(a + bx)}{b} + \frac{\operatorname{arctanh}\left(\sqrt{1 - \frac{1}{(a+bx)^2}}\right)}{b}$$

output

```
(b*x+a)*arccsc(b*x+a)/b+arctanh((1-1/(b*x+a)^2)^(1/2))/b
```

#### Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 0.36 (sec) , antiderivative size = 469, normalized size of antiderivative = 13.03

$$\int \csc^{-1}(a + bx) dx = x \csc^{-1}(a + bx) + (a + bx) \sqrt{\frac{-1+a^2+2abx+b^2x^2}{(a+bx)^2}} \left( \sqrt[4]{-1}(-i + \sqrt{-1 + a^2}) \sqrt{2i - ia^2 + 2\sqrt{-1 + a^2}} \arctan\left(\frac{(-1)^{3/4} \sqrt{2i - ia^2 + 2\sqrt{-1 + a^2}}}{a\sqrt{-1+a^2} - a\sqrt{-1+a^2}}\right) \right)$$

input

```
Integrate[ArcCsc[a + b*x], x]
```

output

```
x*ArcCsc[a + b*x] - ((a + b*x)*Sqrt[(-1 + a^2 + 2*a*b*x + b^2*x^2)/(a + b*x)^2]*((-1)^(1/4)*(-I + Sqrt[-1 + a^2])*Sqrt[2*I - I*a^2 + 2*Sqrt[-1 + a^2]])*ArcTan[((-1)^(3/4)*Sqrt[2*I - I*a^2 + 2*Sqrt[-1 + a^2]]*b*x)/(a*Sqrt[-1 + a^2] - a*Sqrt[-1 + a^2 + 2*a*b*x + b^2*x^2])] + (-1)^(3/4)*(I + Sqrt[-1 + a^2])*Sqrt[-2*I + I*a^2 + 2*Sqrt[-1 + a^2]]*ArcTan[((-1)^(1/4)*Sqrt[-2*I + I*a^2 + 2*Sqrt[-1 + a^2]]*b*x)/(a*Sqrt[-1 + a^2] - a*Sqrt[-1 + a^2 + 2*a*b*x + b^2*x^2])] + a*(a*ArcTan[(Sqrt[-1 + a^2]*b^2*x^2)/(a^4 + a^3*b*x + b^2*x^2 - a^2*(1 + Sqrt[-1 + a^2])*Sqrt[-1 + a^2 + 2*a*b*x + b^2*x^2])]) - Log[Sqrt[-1 + a^2] - b*x - Sqrt[-1 + a^2 + 2*a*b*x + b^2*x^2]] + Log[b^2*(Sqrt[-1 + a^2] + b*x - Sqrt[-1 + a^2 + 2*a*b*x + b^2*x^2])])]/(a*b*Sqrt[-1 + a^2 + 2*a*b*x + b^2*x^2])
```

**Rubi [A] (warning: unable to verify)**

Time = 0.26 (sec) , antiderivative size = 34, normalized size of antiderivative = 0.94, number of steps used = 6, number of rules used = 5,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.833$ , Rules used = {5774, 895, 798, 73, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \csc^{-1}(a + bx) dx \\
 & \quad \downarrow 5774 \\
 & \int \frac{1}{(a + bx)\sqrt{1 - \frac{1}{(a+bx)^2}}} dx + \frac{(a + bx) \csc^{-1}(a + bx)}{b} \\
 & \quad \downarrow 895 \\
 & \frac{\int \frac{1}{(a+bx)\sqrt{1 - \frac{1}{(a+bx)^2}}} d(a + bx)}{b} + \frac{(a + bx) \csc^{-1}(a + bx)}{b} \\
 & \quad \downarrow 798 \\
 & \frac{(a + bx) \csc^{-1}(a + bx)}{b} - \frac{\int \frac{(a+bx)^2}{\sqrt{-a-bx+1}} d\frac{1}{(a+bx)^2}}{2b} \\
 & \quad \downarrow 73
 \end{aligned}$$

$$\frac{\int \frac{1}{1-\frac{1}{(a+bx)^4}} d\sqrt{-a-bx+1}}{b} + \frac{(a+bx) \csc^{-1}(a+bx)}{b}$$

↓ 219

$$\frac{\operatorname{arctanh}(\sqrt{-a-bx+1})}{b} + \frac{(a+bx) \csc^{-1}(a+bx)}{b}$$

input `Int[ArcCsc[a + b*x], x]`

output `((a + b*x)*ArcCsc[a + b*x])/b + ArcTanh[Sqrt[1 - a - b*x]]/b`

### Definitions of rubi rules used

rule 73 `Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] := With[  
 {p = Denominator[m]}, Simp[p/b Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +  
 d*(x^p/b))^(n), x], x, (a + b*x)^(1/p)], x] /; FreeQ[{a, b, c, d}, x] && Lt  
 Q[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Denominator[m]] && IntL  
 inearQ[a, b, c, d, m, n, x]`

rule 219 `Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*  
 ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt  
 Q[a, 0] || LtQ[b, 0])`

rule 798 `Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] := Simp[1/n Subst  
 [Int[x^(Simplify[(m + 1)/n] - 1)*(a + b*x)^p, x], x, x^n], x] /; FreeQ[{a,  
 b, m, n, p}, x] && IntegerQ[Simplify[(m + 1)/n]]`

rule 895 `Int[(u_)^(m_.)*((a_) + (b_.)*(v_)^(n_))^(p_.), x_Symbol] := Simp[u^m/(Coeff  
 icient[v, x, 1]*v^m) Subst[Int[x^m*(a + b*x^n)^p, x], x, v], x] /; FreeQ[  
 {a, b, m, n, p}, x] && LinearPairQ[u, v, x]`

rule 5774 `Int[ArcCsc[(c_) + (d_.)*(x_)], x_Symbol] := Simp[(c + d*x)*(ArcCsc[c + d*x]  
 /d), x] + Int[1/((c + d*x)*Sqrt[1 - 1/(c + d*x)^2]), x] /; FreeQ[{c, d}, x]`



**Maple [A] (verified)**

Time = 0.05 (sec) , antiderivative size = 43, normalized size of antiderivative = 1.19

method	result
derivativedivides	$\frac{\operatorname{arccsc}(bx+a)(bx+a)+\ln\left(bx+a+(bx+a)\sqrt{1-\frac{1}{(bx+a)^2}}\right)}{b}$
default	$\frac{\operatorname{arccsc}(bx+a)(bx+a)+\ln\left(bx+a+(bx+a)\sqrt{1-\frac{1}{(bx+a)^2}}\right)}{b}$
parts	$x \operatorname{arccsc}(bx+a) + \frac{\sqrt{b^2x^2+2abx+a^2-1} \left( a \arctan\left(\frac{1}{\sqrt{b^2x^2+2abx+a^2-1}}\right) \sqrt{b^2} + \ln\left(\frac{b^2x+\sqrt{b^2x^2+2abx+a^2-1}}{\sqrt{b^2}}\right) \right)}{b\sqrt{\frac{b^2x^2+2abx+a^2-1}{(bx+a)^2}}(bx+a)\sqrt{b^2}}$

input `int(arccsc(b*x+a),x,method=_RETURNVERBOSE)`output `1/b*(arccsc(b*x+a)*(b*x+a)+ln(b*x+a+(b*x+a)*(1-1/(b*x+a)^2)^(1/2)))`**Fricas [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 75 vs. 2(34) = 68.

Time = 0.12 (sec) , antiderivative size = 75, normalized size of antiderivative = 2.08

$$\int \operatorname{csc}^{-1}(a+bx) dx$$

$$= \frac{bx \operatorname{arccsc}(bx+a) - 2a \arctan(-bx-a+\sqrt{b^2x^2+2abx+a^2-1}) - \log(-bx-a+\sqrt{b^2x^2+2abx+a^2-1})}{b}$$

input `integrate(arccsc(b*x+a),x, algorithm="fricas")`output `(b*x*arccsc(b*x+a) - 2*a*arctan(-b*x - a + sqrt(b^2*x^2 + 2*a*b*x + a^2 - 1)) - log(-b*x - a + sqrt(b^2*x^2 + 2*a*b*x + a^2 - 1)))/b`

**Sympy [F]**

$$\int \csc^{-1}(a + bx) dx = \int \operatorname{acsc}(a + bx) dx$$

input `integrate(acsc(b*x+a), x)`

output `Integral(acsc(a + b*x), x)`

**Maxima [A] (verification not implemented)**

Time = 0.04 (sec) , antiderivative size = 55, normalized size of antiderivative = 1.53

$$\int \csc^{-1}(a + bx) dx$$

$$= \frac{2(bx + a) \operatorname{arccsc}(bx + a) + \log\left(\sqrt{-\frac{1}{(bx+a)^2} + 1} + 1\right) - \log\left(-\sqrt{-\frac{1}{(bx+a)^2} + 1} + 1\right)}{2b}$$

input `integrate(arccsc(b*x+a), x, algorithm="maxima")`

output `1/2*(2*(b*x + a)*arccsc(b*x + a) + log(sqrt(-1/(b*x + a)^2 + 1) + 1) - log(-sqrt(-1/(b*x + a)^2 + 1) + 1))/b`

**Giac [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 81 vs. 2(34) = 68.

Time = 0.14 (sec) , antiderivative size = 81, normalized size of antiderivative = 2.25

$$\int \csc^{-1}(a + bx) dx$$

$$= \frac{1}{2} b \left( \frac{2(bx + a) \arcsin\left(-\frac{1}{(bx+a)\left(\frac{a}{bx+a}-1\right)-a}\right)}{b^2} + \frac{\log\left(\sqrt{-\frac{1}{(bx+a)^2} + 1} + 1\right) - \log\left(-\sqrt{-\frac{1}{(bx+a)^2} + 1} + 1\right)}{b^2} \right)$$

input `integrate(arccsc(b*x+a),x, algorithm="giac")`

output `1/2*b*(2*(b*x + a)*arcsin(-1/((b*x + a)*(a/(b*x + a) - 1) - a))/b^2 + (log(sqrt(-1/(b*x + a)^2 + 1) + 1) - log(-sqrt(-1/(b*x + a)^2 + 1) + 1))/b^2)`

### Mupad [B] (verification not implemented)

Time = 1.11 (sec) , antiderivative size = 33, normalized size of antiderivative = 0.92

$$\int \csc^{-1}(a + bx) dx = \frac{\operatorname{atanh}\left(\frac{1}{\sqrt{1 - \frac{1}{(a+bx)^2}}}\right) + \operatorname{asin}\left(\frac{1}{a+bx}\right) (a + bx)}{b}$$

input `int(asin(1/(a + b*x)),x)`

output `(atanh(1/(1 - 1/(a + b*x)^2)^(1/2)) + asin(1/(a + b*x))*(a + b*x))/b`

### Reduce [F]

$$\int \csc^{-1}(a + bx) dx = \int \operatorname{acsc}(bx + a) dx$$

input `int(acsc(b*x+a),x)`

output `int(acsc(a + b*x),x)`

### 3.22 $\int \frac{\csc^{-1}(a+bx)}{x} dx$

Optimal result	179
Mathematica [A] (verified)	180
Rubi [A] (verified)	181
Maple [B] (verified)	186
Fricas [F]	187
Sympy [F(-1)]	187
Maxima [F]	187
Giac [F]	188
Mupad [F(-1)]	188
Reduce [F]	188

#### Optimal result

Integrand size = 10, antiderivative size = 210

$$\begin{aligned} \int \frac{\csc^{-1}(a+bx)}{x} dx &= \csc^{-1}(a+bx) \log\left(1 + \frac{iae^{i \csc^{-1}(a+bx)}}{1 - \sqrt{1-a^2}}\right) \\ &\quad + \csc^{-1}(a+bx) \log\left(1 + \frac{iae^{i \csc^{-1}(a+bx)}}{1 + \sqrt{1-a^2}}\right) \\ &\quad - \csc^{-1}(a+bx) \log\left(1 - e^{2i \csc^{-1}(a+bx)}\right) \\ &\quad - i \operatorname{PolyLog}\left(2, -\frac{iae^{i \csc^{-1}(a+bx)}}{1 - \sqrt{1-a^2}}\right) \\ &\quad - i \operatorname{PolyLog}\left(2, -\frac{iae^{i \csc^{-1}(a+bx)}}{1 + \sqrt{1-a^2}}\right) + \frac{1}{2}i \operatorname{PolyLog}\left(2, e^{2i \csc^{-1}(a+bx)}\right) \end{aligned}$$

output

```
arccsc(b*x+a)*ln(1+I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1-(-a^2+1)^(1/2)))
+arccsc(b*x+a)*ln(1+I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1+(-a^2+1)^(1/2)))
-arccsc(b*x+a)*ln(1-(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))^2)-I*polylog(2,
-I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1-(-a^2+1)^(1/2)))-I*polylog(2,-I*
a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1+(-a^2+1)^(1/2)))+1/2*I*polylog(2,(I
/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))^2)
```

**Mathematica [A] (verified)**

Time = 0.32 (sec) , antiderivative size = 375, normalized size of antiderivative = 1.79

$$\begin{aligned}
& \int \frac{\csc^{-1}(a+bx)}{x} dx \\
&= \frac{1}{8} \left( i(\pi - 2 \csc^{-1}(a+bx))^2 \right. \\
&\quad - 32i \arcsin \left( \frac{\sqrt{-1+a}}{\sqrt{2}} \right) \arctan \left( \frac{(1+a) \cot \left( \frac{1}{4}(\pi + 2 \csc^{-1}(a+bx)) \right)}{\sqrt{1-a^2}} \right) - 4 \left( \pi \right. \\
&\quad \left. - 2 \csc^{-1}(a+bx) + 4 \arcsin \left( \frac{\sqrt{-1+a}}{\sqrt{2}} \right) \right) \log \left( 1 + \frac{i(-1 + \sqrt{1-a^2}) e^{-i \csc^{-1}(a+bx)}}{a} \right) \\
&\quad - 4 \left( \pi - 2 \csc^{-1}(a+bx) - 4 \arcsin \left( \frac{\sqrt{-1+a}}{\sqrt{2}} \right) \right) \log \left( 1 \right. \\
&\quad \left. - \frac{i(1 + \sqrt{1-a^2}) e^{-i \csc^{-1}(a+bx)}}{a} \right) - 8 \csc^{-1}(a+bx) \log \left( 1 - e^{2i \csc^{-1}(a+bx)} \right) \\
&\quad + 4(\pi - 2 \csc^{-1}(a+bx)) \log \left( \frac{bx}{a+bx} \right) + 8 \csc^{-1}(a+bx) \log \left( \frac{bx}{a+bx} \right) \\
&\quad + 8i \left( \text{PolyLog} \left( 2, -\frac{i(-1 + \sqrt{1-a^2}) e^{-i \csc^{-1}(a+bx)}}{a} \right) \right. \\
&\quad \left. + \text{PolyLog} \left( 2, \frac{i(1 + \sqrt{1-a^2}) e^{-i \csc^{-1}(a+bx)}}{a} \right) \right) \\
&\quad \left. + 4i \left( \csc^{-1}(a+bx)^2 + \text{PolyLog} \left( 2, e^{2i \csc^{-1}(a+bx)} \right) \right) \right)
\end{aligned}$$

input

Integrate[ArcCsc[a + b\*x]/x, x]

output

```
(I*(Pi - 2*ArcCsc[a + b*x])^2 - (32*I)*ArcSin[Sqrt[(-1 + a)/a]/Sqrt[2]]*ArcTan[(((1 + a)*Cot[(Pi + 2*ArcCsc[a + b*x])/4])/Sqrt[1 - a^2]] - 4*(Pi - 2*ArcCsc[a + b*x] + 4*ArcSin[Sqrt[(-1 + a)/a]/Sqrt[2]])*Log[1 + (I*(-1 + Sqrt[1 - a^2]))/(a*E^(I*ArcCsc[a + b*x]))] - 4*(Pi - 2*ArcCsc[a + b*x] - 4*ArcSin[Sqrt[(-1 + a)/a]/Sqrt[2]])*Log[1 - (I*(1 + Sqrt[1 - a^2]))/(a*E^(I*ArcCsc[a + b*x]))] - 8*ArcCsc[a + b*x]*Log[1 - E^((2*I)*ArcCsc[a + b*x])] + 4*(Pi - 2*ArcCsc[a + b*x])*Log[(b*x)/(a + b*x)] + 8*ArcCsc[a + b*x]*Log[(b*x)/(a + b*x)] + (8*I)*(PolyLog[2, ((-I)*(-1 + Sqrt[1 - a^2]))/(a*E^(I*ArcCsc[a + b*x]))] + PolyLog[2, (I*(1 + Sqrt[1 - a^2]))/(a*E^(I*ArcCsc[a + b*x]))]) + (4*I)*(ArcCsc[a + b*x]^2 + PolyLog[2, E^((2*I)*ArcCsc[a + b*x])])]/8
```

**Rubi [A] (verified)**

Time = 1.11 (sec) , antiderivative size = 266, normalized size of antiderivative = 1.27, number of steps used = 17, number of rules used = 16,  $\frac{\text{number of rules}}{\text{integrand size}} = 1.600$ , Rules used = {5782, 25, 5063, 5040, 25, 3042, 25, 4200, 25, 2620, 2715, 2838, 5030, 2620, 2715, 2838}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc^{-1}(a + bx)}{x} dx$$

$$\downarrow 5782$$

$$- \int \frac{(a + bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx)}{bx} d \csc^{-1}(a + bx)$$

$$\downarrow 25$$

$$\int - \frac{(a + bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx)}{bx} d \csc^{-1}(a + bx)$$

$$\downarrow 5063$$

$$\int \frac{(a + bx) \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx)}{\frac{a}{a+bx} - 1} d \csc^{-1}(a + bx)$$

$$\downarrow 5040$$

$$\begin{aligned}
& a \int -\frac{\sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)}{1 - \frac{a}{a+bx}} d \csc^{-1}(a+bx) - \int (a+bx) \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a+bx) d \csc^{-1}(a+bx) \\
& \quad \downarrow 25 \\
& - \int (a+bx) \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a+bx) d \csc^{-1}(a+bx) - \\
& \quad a \int \frac{\sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)}{1 - \frac{a}{a+bx}} d \csc^{-1}(a+bx) \\
& \quad \downarrow 3042 \\
& -a \int \frac{\sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)}{1 - \frac{a}{a+bx}} d \csc^{-1}(a+bx) - \int -\csc^{-1}(a+bx) \tan \left( \csc^{-1}(a+bx) + \frac{\pi}{2} \right) d \csc^{-1}(a+bx) \\
& \quad \downarrow 25 \\
& \int \csc^{-1}(a+bx) \tan \left( \csc^{-1}(a+bx) + \frac{\pi}{2} \right) d \csc^{-1}(a+bx) - \\
& \quad a \int \frac{\sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)}{1 - \frac{a}{a+bx}} d \csc^{-1}(a+bx) \\
& \quad \downarrow 4200 \\
& -2i \int -\frac{e^{2i \csc^{-1}(a+bx)} \csc^{-1}(a+bx)}{1 - e^{2i \csc^{-1}(a+bx)}} d \csc^{-1}(a+bx) - \\
& \quad a \int \frac{\sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)}{1 - \frac{a}{a+bx}} d \csc^{-1}(a+bx) + \frac{1}{2} i \csc^{-1}(a+bx)^2 \\
& \quad \downarrow 25 \\
& 2i \int \frac{e^{2i \csc^{-1}(a+bx)} \csc^{-1}(a+bx)}{1 - e^{2i \csc^{-1}(a+bx)}} d \csc^{-1}(a+bx) - a \int \frac{\sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)}{1 - \frac{a}{a+bx}} d \csc^{-1}(a+bx) \\
& \quad \quad \quad + \frac{1}{2} i \csc^{-1}(a+bx)^2 \\
& \quad \downarrow 2620
\end{aligned}$$

$$\begin{aligned}
& -a \int \frac{\sqrt{1 - \frac{1}{(a+bx)^2}} \operatorname{csc}^{-1}(a+bx)}{1 - \frac{a}{a+bx}} d \operatorname{csc}^{-1}(a+bx) + \\
& 2i \left( \frac{1}{2} i \operatorname{csc}^{-1}(a+bx) \log \left( 1 - e^{2i \operatorname{csc}^{-1}(a+bx)} \right) - \frac{1}{2} i \int \log \left( 1 - e^{2i \operatorname{csc}^{-1}(a+bx)} \right) d \operatorname{csc}^{-1}(a+bx) \right) + \\
& \qquad \qquad \qquad \frac{1}{2} i \operatorname{csc}^{-1}(a+bx)^2
\end{aligned}$$

↓ 2715

$$\begin{aligned}
& -a \int \frac{\sqrt{1 - \frac{1}{(a+bx)^2}} \operatorname{csc}^{-1}(a+bx)}{1 - \frac{a}{a+bx}} d \operatorname{csc}^{-1}(a+bx) + \\
& 2i \left( \frac{1}{2} i \operatorname{csc}^{-1}(a+bx) \log \left( 1 - e^{2i \operatorname{csc}^{-1}(a+bx)} \right) - \frac{1}{4} \int e^{-2i \operatorname{csc}^{-1}(a+bx)} \log \left( 1 - e^{2i \operatorname{csc}^{-1}(a+bx)} \right) d e^{2i \operatorname{csc}^{-1}(a+bx)} \right) + \\
& \qquad \qquad \qquad \frac{1}{2} i \operatorname{csc}^{-1}(a+bx)^2
\end{aligned}$$

↓ 2838

$$\begin{aligned}
& -a \int \frac{\sqrt{1 - \frac{1}{(a+bx)^2}} \operatorname{csc}^{-1}(a+bx)}{1 - \frac{a}{a+bx}} d \operatorname{csc}^{-1}(a+bx) + \\
& 2i \left( \frac{1}{4} \operatorname{PolyLog} \left( 2, e^{2i \operatorname{csc}^{-1}(a+bx)} \right) + \frac{1}{2} i \operatorname{csc}^{-1}(a+bx) \log \left( 1 - e^{2i \operatorname{csc}^{-1}(a+bx)} \right) \right) + \frac{1}{2} i \operatorname{csc}^{-1}(a+ \\
& \qquad \qquad \qquad bx)^2
\end{aligned}$$

↓ 5030

$$\begin{aligned}
& -a \left( \int \frac{e^{i \operatorname{csc}^{-1}(a+bx)} \operatorname{csc}^{-1}(a+bx)}{i e^{i \operatorname{csc}^{-1}(a+bx)} a - \sqrt{1-a^2} + 1} d \operatorname{csc}^{-1}(a+bx) + \int \frac{e^{i \operatorname{csc}^{-1}(a+bx)} \operatorname{csc}^{-1}(a+bx)}{i e^{i \operatorname{csc}^{-1}(a+bx)} a + \sqrt{1-a^2} + 1} d \operatorname{csc}^{-1}(a+bx) + \frac{i \operatorname{csc}^{-1}(a+bx)}{a} \right) + \\
& 2i \left( \frac{1}{4} \operatorname{PolyLog} \left( 2, e^{2i \operatorname{csc}^{-1}(a+bx)} \right) + \frac{1}{2} i \operatorname{csc}^{-1}(a+bx) \log \left( 1 - e^{2i \operatorname{csc}^{-1}(a+bx)} \right) \right) + \frac{1}{2} i \operatorname{csc}^{-1}(a+ \\
& \qquad \qquad \qquad bx)^2
\end{aligned}$$

↓ 2620

$$\begin{aligned}
& -a \left( \frac{\int \log \left( \frac{i e^{i \operatorname{csc}^{-1}(a+bx)} a}{1 - \sqrt{1-a^2}} + 1 \right) d \operatorname{csc}^{-1}(a+bx)}{a} + \frac{\int \log \left( \frac{i e^{i \operatorname{csc}^{-1}(a+bx)} a}{\sqrt{1-a^2} + 1} + 1 \right) d \operatorname{csc}^{-1}(a+bx)}{a} - \frac{\operatorname{csc}^{-1}(a+bx) \log \left( 1 - e^{2i \operatorname{csc}^{-1}(a+bx)} \right)}{a} \right) + \\
& 2i \left( \frac{1}{4} \operatorname{PolyLog} \left( 2, e^{2i \operatorname{csc}^{-1}(a+bx)} \right) + \frac{1}{2} i \operatorname{csc}^{-1}(a+bx) \log \left( 1 - e^{2i \operatorname{csc}^{-1}(a+bx)} \right) \right) + \frac{1}{2} i \operatorname{csc}^{-1}(a+ \\
& \qquad \qquad \qquad bx)^2
\end{aligned}$$

↓ 2715



$$\begin{aligned}
& -a \left( \frac{i \int e^{-i \csc^{-1}(a+bx)} \log \left( \frac{ie^{i \csc^{-1}(a+bx)} a}{1-\sqrt{1-a^2}} + 1 \right) de^{i \csc^{-1}(a+bx)}}{a} - \frac{i \int e^{-i \csc^{-1}(a+bx)} \log \left( \frac{ie^{i \csc^{-1}(a+bx)} a}{\sqrt{1-a^2}+1} + 1 \right) de^{i \csc^{-1}(a+bx)}}{a} \right) \\
& 2i \left( \frac{1}{4} \text{PolyLog} \left( 2, e^{2i \csc^{-1}(a+bx)} \right) + \frac{1}{2} i \csc^{-1}(a+bx) \log \left( 1 - e^{2i \csc^{-1}(a+bx)} \right) \right) + \frac{1}{2} i \csc^{-1}(a+bx)^2 \\
& \quad \downarrow \text{2838} \\
& -a \left( \frac{i \text{PolyLog} \left( 2, -\frac{iae^{i \csc^{-1}(a+bx)}}{1-\sqrt{1-a^2}} \right)}{a} + \frac{i \text{PolyLog} \left( 2, -\frac{iae^{i \csc^{-1}(a+bx)}}{\sqrt{1-a^2}+1} \right)}{a} - \frac{\csc^{-1}(a+bx) \log \left( 1 + \frac{iae^{i \csc^{-1}(a+bx)}}{1-\sqrt{1-a^2}} \right)}{a} \right) \\
& 2i \left( \frac{1}{4} \text{PolyLog} \left( 2, e^{2i \csc^{-1}(a+bx)} \right) + \frac{1}{2} i \csc^{-1}(a+bx) \log \left( 1 - e^{2i \csc^{-1}(a+bx)} \right) \right) + \frac{1}{2} i \csc^{-1}(a+bx)^2
\end{aligned}$$

input `Int[ArcCsc[a + b*x]/x, x]`

output `(I/2)*ArcCsc[a + b*x]^2 - a*(((I/2)*ArcCsc[a + b*x]^2)/a - (ArcCsc[a + b*x]*Log[1 + (I*a*E^(I*ArcCsc[a + b*x]))/(1 - Sqrt[1 - a^2])])/a - (ArcCsc[a + b*x]*Log[1 + (I*a*E^(I*ArcCsc[a + b*x]))/(1 + Sqrt[1 - a^2])])/a + (I*PolyLog[2, ((-I)*a*E^(I*ArcCsc[a + b*x]))/(1 - Sqrt[1 - a^2])])/a + (I*PolyLog[2, ((-I)*a*E^(I*ArcCsc[a + b*x]))/(1 + Sqrt[1 - a^2])])/a) + (2*I)*((I/2)*ArcCsc[a + b*x]*Log[1 - E^((2*I)*ArcCsc[a + b*x])] + PolyLog[2, E^((2*I)*ArcCsc[a + b*x])])/4)`

### Defintions of rubi rules used

rule 25 `Int[-(Fx_), x_Symbol] := Simp[Identity[-1] Int[Fx, x], x]`

rule 2620 `Int[(((F_)^((g_)*((e_) + (f_)*(x_))))^(n_)*((c_) + (d_)*(x_))^(m_))/((a_) + (b_)*((F_)^((g_)*((e_) + (f_)*(x_))))^(n_)), x_Symbol] := Simp[(((c + d*x)^m/(b*f*g*n*Log[F]))*Log[1 + b*((F^(g*(e + f*x)))^n/a)], x] - Simp[d*(m/(b*f*g*n*Log[F])) Int[(c + d*x)^(m - 1)*Log[1 + b*((F^(g*(e + f*x)))^n/a)], x], x] /; FreeQ[{F, a, b, c, d, e, f, g, n}, x] && IGtQ[m, 0]`

rule 2715  $\text{Int}[\text{Log}[(a_) + (b_)*((F_)^((e_)*((c_) + (d_)*(x_)))^((n_))], x\_Symbol] \rightarrow \text{Simp}[1/(d*e*n*\text{Log}[F]) \text{ Subst}[\text{Int}[\text{Log}[a + b*x]/x, x], x, (F^{e*(c + d*x)})^n], x] /;$   $\text{FreeQ}\{F, a, b, c, d, e, n\}, x\} \&\& \text{GtQ}[a, 0]$

rule 2838  $\text{Int}[\text{Log}[(c_)*((d_) + (e_)*(x_)^(n_))]/(x_), x\_Symbol] \rightarrow \text{Simp}[-\text{PolyLog}[2, (-c)*e*x^n]/n, x] /;$   $\text{FreeQ}\{c, d, e, n\}, x\} \&\& \text{EqQ}[c*d, 1]$

rule 3042  $\text{Int}[u_, x\_Symbol] \rightarrow \text{Int}[\text{DeactivateTrig}[u, x], x] /;$   $\text{FunctionOfTrigOfLinearQ}[u, x]$

rule 4200  $\text{Int}[(c_) + (d_)*(x_)^(m_)*\tan[(e_) + \text{Pi}*(k_) + (f_)*(x_)], x\_Symbol] \rightarrow \text{Simp}[I*((c + d*x)^(m + 1)/(d*(m + 1))), x] - \text{Simp}[2*I \text{ Int}[(c + d*x)^m * E^{(2*I*k*Pi)}*(E^{(2*I*(e + f*x))}/(1 + E^{(2*I*k*Pi)}*E^{(2*I*(e + f*x))})], x] /;$   $\text{FreeQ}\{c, d, e, f\}, x\} \&\& \text{IntegerQ}[4*k] \&\& \text{IGtQ}[m, 0]$

rule 5030  $\text{Int}[(\text{Cos}[(c_) + (d_)*(x_)]*((e_) + (f_)*(x_)^(m_)))/((a_) + (b_)*\text{Sin}[(c_) + (d_)*(x_)]), x\_Symbol] \rightarrow \text{Simp}[(-I)*((e + f*x)^(m + 1)/(b*f*(m + 1))), x] + (\text{Int}[(e + f*x)^m*(E^{I*(c + d*x)})/(a - \text{Rt}[a^2 - b^2, 2] - I*b*E^{I*(c + d*x)}), x] + \text{Int}[(e + f*x)^m*(E^{I*(c + d*x)})/(a + \text{Rt}[a^2 - b^2, 2] - I*b*E^{I*(c + d*x)}), x]) /;$   $\text{FreeQ}\{a, b, c, d, e, f\}, x\} \&\& \text{IGtQ}[m, 0] \&\& \text{PosQ}[a^2 - b^2]$

rule 5040  $\text{Int}[(\text{Cot}[(c_) + (d_)*(x_)]^(n_)*((e_) + (f_)*(x_)^(m_)))/((a_) + (b_)*\text{Sin}[(c_) + (d_)*(x_)]), x\_Symbol] \rightarrow \text{Simp}[1/a \text{ Int}[(e + f*x)^m*\text{Cot}[c + d*x]^n, x], x] - \text{Simp}[b/a \text{ Int}[(e + f*x)^m*\text{Cos}[c + d*x]*(\text{Cot}[c + d*x]^(n - 1)/(a + b*\text{Sin}[c + d*x])), x], x] /;$   $\text{FreeQ}\{a, b, c, d, e, f\}, x\} \&\& \text{IGtQ}[m, 0] \&\& \text{IGtQ}[n, 0]$

rule 5063  $\text{Int}[(e_ + (f_)*(x_)^(m_)*F_)[(c_) + (d_)*(x_)]^(n_)*(G_)[(c_) + (d_)*(x_)]^(p_)/(Csc[(c_) + (d_)*(x_)]*(b_) + (a_)), x\_Symbol] \rightarrow \text{Int}[(e + f*x)^m*\text{Sin}[c + d*x]*F[c + d*x]^n*(G[c + d*x]^p/(b + a*\text{Sin}[c + d*x])), x] /;$   $\text{FreeQ}\{a, b, c, d, e, f\}, x\} \&\& \text{TrigQ}[F] \&\& \text{TrigQ}[G] \&\& \text{IntegersQ}[m, n, p]$

rule 5782

```
Int[((a_.) + ArcCsc[(c_) + (d_.)*(x_)]*(b_.))^(p_.)*((e_.) + (f_.)*(x_))^(m
_.), x_Symbol] := Simp[-(d^(m + 1))^(-1) Subst[Int[(a + b*x)^p*Csc[x]*Cot
[x]*(d*e - c*f + f*Csc[x])^m, x], x, ArcCsc[c + d*x]], x] /; FreeQ[{a, b, c
, d, e, f}, x] && IGtQ[p, 0] && IntegerQ[m]
```

### Maple [B] (verified)

Both result and optimal contain complex but leaf count of result is larger than twice the leaf count of optimal. 626 vs. 2(269) = 538.

Time = 1.17 (sec) , antiderivative size = 627, normalized size of antiderivative = 2.99

method	result
derivativedivides	$\frac{\operatorname{arccsc}(bx+a) \ln\left(\frac{-\left(\frac{i}{bx+a} + \sqrt{1 - \frac{1}{(bx+a)^2}}\right) a + i + \sqrt{a^2 - 1}}{i + \sqrt{a^2 - 1}}\right)}{a^2 - 1} - \frac{\operatorname{arccsc}(bx+a) \ln\left(\frac{-\left(\frac{i}{bx+a} + \sqrt{1 - \frac{1}{(bx+a)^2}}\right) a + i - \sqrt{a^2 - 1}}{i - \sqrt{a^2 - 1}}\right)}{a^2 - 1}$
default	$\frac{\operatorname{arccsc}(bx+a) \ln\left(\frac{-\left(\frac{i}{bx+a} + \sqrt{1 - \frac{1}{(bx+a)^2}}\right) a + i + \sqrt{a^2 - 1}}{i + \sqrt{a^2 - 1}}\right)}{a^2 - 1} - \frac{\operatorname{arccsc}(bx+a) \ln\left(\frac{-\left(\frac{i}{bx+a} + \sqrt{1 - \frac{1}{(bx+a)^2}}\right) a + i - \sqrt{a^2 - 1}}{i - \sqrt{a^2 - 1}}\right)}{a^2 - 1}$

input

```
int(arccsc(b*x+a)/x,x,method=_RETURNVERBOSE)
```

output

```
-arccsc(b*x+a)/(a^2-1)*ln((-I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))*a+I+(a^2-1)^(1/2))/(I+(a^2-1)^(1/2))-arccsc(b*x+a)/(a^2-1)*ln((-I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))*a+I-(a^2-1)^(1/2))/(I-(a^2-1)^(1/2))+I/(a^2-1)*dilog((-I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))*a+I+(a^2-1)^(1/2))/(I+(a^2-1)^(1/2))+I/(a^2-1)*dilog((-I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))*a+I-(a^2-1)^(1/2))/(I-(a^2-1)^(1/2))+arccsc(b*x+a)/(a^2-1)*ln((-I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))*a+I+(a^2-1)^(1/2))/(I+(a^2-1)^(1/2))*a^2+arccsc(b*x+a)/(a^2-1)*ln((-I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))*a+I-(a^2-1)^(1/2))/(I-(a^2-1)^(1/2))*a^2+I*dilog(1+I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))-I*dilog(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))-I/(a^2-1)*dilog((-I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))*a+I+(a^2-1)^(1/2))/(I+(a^2-1)^(1/2))*a^2-I/(a^2-1)*dilog((-I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))*a+I-(a^2-1)^(1/2))/(I-(a^2-1)^(1/2))*a^2-arccsc(b*x+a)*ln(1+I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))
```

**Fricas [F]**

$$\int \frac{\csc^{-1}(a + bx)}{x} dx = \int \frac{\operatorname{arccsc}(bx + a)}{x} dx$$

input `integrate(arccsc(b*x+a)/x,x, algorithm="fricas")`

output `integral(arccsc(b*x + a)/x, x)`

**Sympy [F(-1)]**

Timed out.

$$\int \frac{\csc^{-1}(a + bx)}{x} dx = \text{Timed out}$$

input `integrate(acsc(b*x+a)/x,x)`

output `Timed out`

**Maxima [F]**

$$\int \frac{\csc^{-1}(a + bx)}{x} dx = \int \frac{\operatorname{arccsc}(bx + a)}{x} dx$$

input `integrate(arccsc(b*x+a)/x,x, algorithm="maxima")`

output `integrate(arccsc(b*x + a)/x, x)`

**Giac [F]**

$$\int \frac{\csc^{-1}(a + bx)}{x} dx = \int \frac{\operatorname{arccsc}(bx + a)}{x} dx$$

input `integrate(arccsc(b*x+a)/x,x, algorithm="giac")`

output `integrate(arccsc(b*x + a)/x, x)`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{\csc^{-1}(a + bx)}{x} dx = \int \frac{\operatorname{asin}\left(\frac{1}{a+bx}\right)}{x} dx$$

input `int(asin(1/(a + b*x))/x,x)`

output `int(asin(1/(a + b*x))/x, x)`

**Reduce [F]**

$$\int \frac{\csc^{-1}(a + bx)}{x} dx = \int \frac{\operatorname{acsc}(bx + a)}{x} dx$$

input `int(acsc(b*x+a)/x,x)`

output `int(acsc(a + b*x)/x,x)`

### 3.23 $\int \frac{\csc^{-1}(a+bx)}{x^2} dx$

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#### Optimal result

Integrand size = 10, antiderivative size = 69

$$\int \frac{\csc^{-1}(a + bx)}{x^2} dx = -\frac{b \csc^{-1}(a + bx)}{a} - \frac{\csc^{-1}(a + bx)}{x} - \frac{2b \arctan\left(\frac{a - \tan\left(\frac{1}{2} \csc^{-1}(a + bx)\right)}{\sqrt{1 - a^2}}\right)}{a\sqrt{1 - a^2}}$$

output

```
-b*arccsc(b*x+a)/a-arccsc(b*x+a)/x-2*b*arctan((a-tan(1/2*arccsc(b*x+a)))/(-a^2+1)^(1/2))/a/(-a^2+1)^(1/2)
```

### Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 0.25 (sec) , antiderivative size = 115, normalized size of antiderivative = 1.67

$$\int \frac{\csc^{-1}(a + bx)}{x^2} dx = -\frac{\csc^{-1}(a + bx)}{x} + b \left( -\arcsin\left(\frac{1}{a+bx}\right) + \frac{i \log\left(2 \left( \frac{-ia(-1+a^2+abx) - a(a+bx)\sqrt{\frac{-1+a^2+2abx+b^2x^2}{(a+bx)^2}}}{\sqrt{1-a^2}} \right)}{bx} \right)}{\sqrt{1-a^2}} \right) + \frac{\phantom{b \left( -\arcsin\left(\frac{1}{a+bx}\right) + \frac{i \log\left(2 \left( \frac{-ia(-1+a^2+abx) - a(a+bx)\sqrt{\frac{-1+a^2+2abx+b^2x^2}{(a+bx)^2}}}{\sqrt{1-a^2}} \right)}{bx} \right)}{\sqrt{1-a^2}} \right)}}{a}$$

input `Integrate[ArcCsc[a + b*x]/x^2,x]`

output `-(ArcCsc[a + b*x]/x) + (b*(-ArcSin[(a + b*x)^(-1)] + (I*Log[(2*((( -I)*a*(-1 + a^2 + a*b*x))/Sqrt[1 - a^2] - a*(a + b*x)*Sqrt[(-1 + a^2 + 2*a*b*x + b^2*x^2)/(a + b*x)^2]))/(b*x)])/Sqrt[1 - a^2]))/a`

### Rubi [A] (verified)

Time = 0.42 (sec) , antiderivative size = 76, normalized size of antiderivative = 1.10, number of steps used = 9, number of rules used = 8,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.800$ , Rules used = {5782, 4927, 3042, 4270, 3042, 3139, 1083, 217}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc^{-1}(a + bx)}{x^2} dx \downarrow 5782$$

$$\begin{aligned}
& -b \int \frac{(a+bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \operatorname{csc}^{-1}(a+bx)}{b^2 x^2} d \operatorname{csc}^{-1}(a+bx) \\
& \quad \downarrow 4927 \\
& -b \left( \int -\frac{1}{bx} d \operatorname{csc}^{-1}(a+bx) + \frac{\operatorname{csc}^{-1}(a+bx)}{bx} \right) \\
& \quad \downarrow 3042 \\
& -b \left( \int \frac{1}{a - \operatorname{csc}(\operatorname{csc}^{-1}(a+bx))} d \operatorname{csc}^{-1}(a+bx) + \frac{\operatorname{csc}^{-1}(a+bx)}{bx} \right) \\
& \quad \downarrow 4270 \\
& -b \left( -\frac{\int \frac{1}{1 - \frac{a}{a+bx}} d \operatorname{csc}^{-1}(a+bx)}{a} + \frac{\operatorname{csc}^{-1}(a+bx)}{a} + \frac{\operatorname{csc}^{-1}(a+bx)}{bx} \right) \\
& \quad \downarrow 3042 \\
& -b \left( -\frac{\int \frac{1}{1 - a \sin(\operatorname{csc}^{-1}(a+bx))} d \operatorname{csc}^{-1}(a+bx)}{a} + \frac{\operatorname{csc}^{-1}(a+bx)}{a} + \frac{\operatorname{csc}^{-1}(a+bx)}{bx} \right) \\
& \quad \downarrow 3139 \\
& -b \left( -\frac{2 \int \frac{1}{\tan^2(\frac{1}{2} \operatorname{csc}^{-1}(a+bx)) - 2a \tan(\frac{1}{2} \operatorname{csc}^{-1}(a+bx)) + 1} d \tan(\frac{1}{2} \operatorname{csc}^{-1}(a+bx))}{a} + \frac{\operatorname{csc}^{-1}(a+bx)}{a} + \frac{\operatorname{csc}^{-1}(a+bx)}{bx} \right) \\
& \quad \downarrow 1083 \\
& -b \left( \frac{4 \int \frac{1}{-(2 \tan(\frac{1}{2} \operatorname{csc}^{-1}(a+bx)) - 2a)^2 - 4(1-a^2)} d(2 \tan(\frac{1}{2} \operatorname{csc}^{-1}(a+bx)) - 2a)}{a} + \frac{\operatorname{csc}^{-1}(a+bx)}{a} + \frac{\operatorname{csc}^{-1}(a+bx)}{bx} \right) \\
& \quad \downarrow 217 \\
& -b \left( -\frac{2 \arctan\left(\frac{2 \tan(\frac{1}{2} \operatorname{csc}^{-1}(a+bx)) - 2a}{2\sqrt{1-a^2}}\right)}{a\sqrt{1-a^2}} + \frac{\operatorname{csc}^{-1}(a+bx)}{a} + \frac{\operatorname{csc}^{-1}(a+bx)}{bx} \right)
\end{aligned}$$

input

Int[ArcCsc[a + b\*x]/x^2, x]



output

$$-(b*(\text{ArcCsc}[a + b*x]/a + \text{ArcCsc}[a + b*x]/(b*x) - (2*\text{ArcTan}[(-2*a + 2*\text{Tan}[\text{ArcCsc}[a + b*x]/2)]/(2*\text{Sqrt}[1 - a^2])))/(a*\text{Sqrt}[1 - a^2]))$$

### Defintions of rubi rules used

rule 217

$$\text{Int}[(a_ + (b_)*(x_)^2)^{-1}, x\_Symbol] \rightarrow \text{Simp}[(-\text{Rt}[-a, 2]*\text{Rt}[-b, 2])^{-1})*\text{ArcTan}[\text{Rt}[-b, 2]*(x/\text{Rt}[-a, 2])], x] /; \text{FreeQ}\{a, b, x\} \ \&\& \ \text{PosQ}\{a/b\} \ \&\& \ (\text{LtQ}\{a, 0\} \ || \ \text{LtQ}\{b, 0\})$$

rule 1083

$$\text{Int}[(a_ + (b_)*(x_) + (c_)*(x_)^2)^{-1}, x\_Symbol] \rightarrow \text{Simp}[-2 \ \text{Subst}[\text{Int}[1/\text{Simp}[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; \text{FreeQ}\{a, b, c, x\}$$

rule 3042

$$\text{Int}[u_, x\_Symbol] \rightarrow \text{Int}[\text{DeactivateTrig}[u, x], x] /; \text{FunctionOfTrigOfLinearQ}[u, x]$$

rule 3139

$$\text{Int}[(a_ + (b_)*\sin[(c_ + (d_)*(x_))] )^{-1}, x\_Symbol] \rightarrow \text{With}\{e = \text{FreeFactors}[\text{Tan}[(c + d*x)/2], x]\}, \text{Simp}[2*(e/d) \ \text{Subst}[\text{Int}[1/(a + 2*b*e*x + a*e^2*x^2), x], x, \text{Tan}[(c + d*x)/2]/e], x] /; \text{FreeQ}\{a, b, c, d, x\} \ \&\& \ \text{NeQ}[a^2 - b^2, 0]$$

rule 4270

$$\text{Int}[(\text{csc}[(c_ + (d_)*(x_)]*(b_ + (a_))^{-1}), x\_Symbol] \rightarrow \text{Simp}[x/a, x] - \text{Simp}[1/a \ \text{Int}[1/(1 + (a/b)*\sin[c + d*x]), x], x] /; \text{FreeQ}\{a, b, c, d, x\} \ \&\& \ \text{NeQ}[a^2 - b^2, 0]$$

rule 4927

$$\text{Int}[\text{Cot}[(c_ + (d_)*(x_)]*\text{Csc}[(c_ + (d_)*(x_)]*(\text{Csc}[(c_ + (d_)*(x_)]*(b_ + (a_))^{(n_)}*((e_ + (f_)*(x_))^{(m_)}), x\_Symbol] \rightarrow \text{Simp}[(-e + f*x)^m*(a + b*\text{Csc}[c + d*x])^{(n + 1)}/(b*d*(n + 1))), x] + \text{Simp}[f*(m/(b*d*(n + 1))) \ \text{Int}[(e + f*x)^{(m - 1)}*(a + b*\text{Csc}[c + d*x])^{(n + 1)}, x], x] /; \text{FreeQ}\{a, b, c, d, e, f, n\}, x\} \ \&\& \ \text{IGtQ}[m, 0] \ \&\& \ \text{NeQ}[n, -1]$$

rule 5782

```
Int[((a_.) + ArcCsc[(c_) + (d_.)*(x_)]*(b_.))^(p_.)*((e_.) + (f_.)*(x_))^(m_.), x_Symbol] :> Simp[-(d^(m + 1))^(-1) Subst[Int[(a + b*x)^p*Csc[x]*Cot[x]*(d*e - c*f + f*Csc[x])^m, x], x, ArcCsc[c + d*x]], x] /; FreeQ[{a, b, c, d, e, f}, x] && IGtQ[p, 0] && IntegerQ[m]
```

### Maple [A] (verified)

Time = 0.48 (sec) , antiderivative size = 127, normalized size of antiderivative = 1.84

method	result
derivativedivides	$b \left( -\frac{\operatorname{arccsc}(bx+a)}{bx} - \frac{\sqrt{(bx+a)^2-1} \left( \arctan\left(\frac{1}{\sqrt{(bx+a)^2-1}}\right) \sqrt{a^2-1} - \ln\left(\frac{2\sqrt{a^2-1}\sqrt{(bx+a)^2-1+2(bx+a)a-2}}{bx}\right) \right)}{\sqrt{\frac{(bx+a)^2-1}{(bx+a)^2}} (bx+a)a\sqrt{a^2-1}} \right)$
default	$b \left( -\frac{\operatorname{arccsc}(bx+a)}{bx} - \frac{\sqrt{(bx+a)^2-1} \left( \arctan\left(\frac{1}{\sqrt{(bx+a)^2-1}}\right) \sqrt{a^2-1} - \ln\left(\frac{2\sqrt{a^2-1}\sqrt{(bx+a)^2-1+2(bx+a)a-2}}{bx}\right) \right)}{\sqrt{\frac{(bx+a)^2-1}{(bx+a)^2}} (bx+a)a\sqrt{a^2-1}} \right)$
parts	$-\frac{\operatorname{arccsc}(bx+a)}{x} - \frac{b\sqrt{b^2x^2+2abx+a^2-1} \left( \arctan\left(\frac{1}{\sqrt{b^2x^2+2abx+a^2-1}}\right) \sqrt{a^2-1} - \ln\left(\frac{2a^2-2+2abx+2\sqrt{a^2-1}\sqrt{b^2x^2+2abx+a^2-1}}{x}\right) \right)}{\sqrt{\frac{b^2x^2+2abx+a^2-1}{(bx+a)^2}} (bx+a)a\sqrt{a^2-1}}$

```
input int(arccsc(b*x+a)/x^2,x,method=_RETURNVERBOSE)
```

```
output b*(-1/b/x*arccsc(b*x+a)-((b*x+a)^2-1)^(1/2)*(arctan(1/((b*x+a)^2-1)^(1/2))
*(a^2-1)^(1/2)-ln(2*((a^2-1)^(1/2)*((b*x+a)^2-1)^(1/2)+(b*x+a)*a-1)/b/x))/
(((b*x+a)^2-1)/(b*x+a)^2)^(1/2)/(b*x+a)/a/(a^2-1)^(1/2))
```

**Fricas [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 132 vs.  $2(63) = 126$ .

Time = 0.13 (sec) , antiderivative size = 278, normalized size of antiderivative = 4.03

$$\int \frac{\csc^{-1}(a + bx)}{x^2} dx$$

$$= \left[ \frac{2(a^2 - 1)bx \arctan(-bx - a + \sqrt{b^2x^2 + 2abx + a^2 - 1}) + \sqrt{a^2 - 1}bx \log\left(\frac{a^2bx + a^3 + \sqrt{b^2x^2 + 2abx + a^2 - 1}(a^2 - 1)}{(a^3 - a)x}\right)}{(a^3 - a)x} \right]$$

input `integrate(arccsc(b*x+a)/x^2,x, algorithm="fricas")`

output `[(2*(a^2 - 1)*b*x*arctan(-b*x - a + sqrt(b^2*x^2 + 2*a*b*x + a^2 - 1)) + sqrt(a^2 - 1)*b*x*log((a^2*b*x + a^3 + sqrt(b^2*x^2 + 2*a*b*x + a^2 - 1)*(a^2 + sqrt(a^2 - 1)*a - 1) + (a*b*x + a^2 - 1)*sqrt(a^2 - 1) - a)/x) - (a^3 - a)*arccsc(b*x + a))/((a^3 - a)*x), (2*(a^2 - 1)*b*x*arctan(-b*x - a + sqrt(b^2*x^2 + 2*a*b*x + a^2 - 1)) - 2*sqrt(-a^2 + 1)*b*x*arctan(-(sqrt(-a^2 + 1)*b*x - sqrt(b^2*x^2 + 2*a*b*x + a^2 - 1)*sqrt(-a^2 + 1))/(a^2 - 1)) - (a^3 - a)*arccsc(b*x + a))/((a^3 - a)*x)]`

**Sympy [F]**

$$\int \frac{\csc^{-1}(a + bx)}{x^2} dx = \int \frac{\operatorname{acsc}(a + bx)}{x^2} dx$$

input `integrate(acsc(b*x+a)/x**2,x)`

output `Integral(acsc(a + b*x)/x**2, x)`

**Maxima [F]**

$$\int \frac{\csc^{-1}(a + bx)}{x^2} dx = \int \frac{\operatorname{arccsc}(bx + a)}{x^2} dx$$

input `integrate(arccsc(b*x+a)/x^2,x, algorithm="maxima")`

output `-(x*integrate((b^2*x + a*b)*e^(1/2*log(b*x + a + 1) + 1/2*log(b*x + a - 1)) / (b^2*x^3 + 2*a*b*x^2 + (a^2 - 1)*x + (b^2*x^3 + 2*a*b*x^2 + (a^2 - 1)*x) * e^(log(b*x + a + 1) + log(b*x + a - 1))), x) + arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)))/x`

**Giac [A] (verification not implemented)**

Time = 0.14 (sec) , antiderivative size = 96, normalized size of antiderivative = 1.39

$$\int \frac{\csc^{-1}(a + bx)}{x^2} dx = -b \left( \frac{2 \arctan \left( \frac{(bx+a) \left( \sqrt{-\frac{1}{(bx+a)^2} + 1} - 1 \right) + a}{\sqrt{-a^2 + 1}} \right)}{\sqrt{-a^2 + 1} a} - \frac{\arcsin \left( -\frac{1}{(bx+a) \left( \frac{a}{bx+a} - 1 \right) - a} \right)}{a \left( \frac{a}{bx+a} - 1 \right)} \right)$$

input `integrate(arccsc(b*x+a)/x^2,x, algorithm="giac")`

output `-b*(2*arctan(((b*x + a)*(sqrt(-1/(b*x + a)^2 + 1) - 1) + a)/sqrt(-a^2 + 1)) / (sqrt(-a^2 + 1)*a) - arcsin(-1/((b*x + a)*(a/(b*x + a) - 1) - a)) / (a*(a/(b*x + a) - 1)))`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{\csc^{-1}(a + bx)}{x^2} dx = \int \frac{\operatorname{asin}\left(\frac{1}{a+bx}\right)}{x^2} dx$$

input `int(asin(1/(a + b*x))/x^2,x)`output `int(asin(1/(a + b*x))/x^2, x)`**Reduce [F]**

$$\int \frac{\csc^{-1}(a + bx)}{x^2} dx = \int \frac{\operatorname{acsc}(bx + a)}{x^2} dx$$

input `int(acsc(b*x+a)/x^2,x)`output `int(acsc(a + b*x)/x**2,x)`

### 3.24 $\int \frac{\csc^{-1}(a+bx)}{x^3} dx$

Optimal result . . . . .	197
Mathematica [C] (verified) . . . . .	197
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Reduce [F] . . . . .	205

#### Optimal result

Integrand size = 10, antiderivative size = 123

$$\int \frac{\csc^{-1}(a + bx)}{x^3} dx = -\frac{b(a + bx)\sqrt{1 - \frac{1}{(a+bx)^2}}}{2a(1 - a^2)x} + \frac{b^2 \csc^{-1}(a + bx)}{2a^2} - \frac{\csc^{-1}(a + bx)}{2x^2} + \frac{(1 - 2a^2)b^2 \arctan\left(\frac{a - \tan(\frac{1}{2} \csc^{-1}(a+bx))}{\sqrt{1-a^2}}\right)}{a^2(1 - a^2)^{3/2}}$$

output

```
-1/2*b*(b*x+a)*(1-1/(b*x+a)^2)^(1/2)/a/(-a^2+1)/x+1/2*b^2*arccsc(b*x+a)/a^2-1/2*arccsc(b*x+a)/x^2+(-2*a^2+1)*b^2*arctan((a-tan(1/2*arccsc(b*x+a)))/(-a^2+1)^(1/2))/a^2/(-a^2+1)^(3/2)
```

#### Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 0.50 (sec) , antiderivative size = 199, normalized size of antiderivative = 1.62

$$\int \frac{\csc^{-1}(a + bx)}{x^3} dx$$

$$= \frac{bx(a+bx)\sqrt{\frac{-1+a^2+2abx+b^2x^2}{(a+bx)^2}}}{a(-1+a^2)} - \csc^{-1}(a + bx) + \frac{b^2x^2 \arcsin\left(\frac{1}{a+bx}\right)}{a^2} + \frac{i(-1+2a^2)b^2x^2 \log\left(\frac{4(-1+a)a^2(1+a)\left(\frac{i(-1+a^2+abx)}{\sqrt{1-a^2}}\right) + (a+bx)}{(-1+2a^2)b^2x}\right)}{2x^2 a^2(1-a^2)^{3/2}}$$

input `Integrate[ArcCsc[a + b*x]/x^3,x]`

output `((b*x*(a + b*x)*Sqrt[(-1 + a^2 + 2*a*b*x + b^2*x^2)/(a + b*x)^2])/(a*(-1 + a^2)) - ArcCsc[a + b*x] + (b^2*x^2*ArcSin[(a + b*x)^(-1)])/a^2 + (I*(-1 + 2*a^2)*b^2*x^2*Log[(4*(-1 + a)*a^2*(1 + a)*((I*(-1 + a^2 + a*b*x))/Sqrt[1 - a^2] + (a + b*x)*Sqrt[(-1 + a^2 + 2*a*b*x + b^2*x^2)/(a + b*x)^2]])/((-1 + 2*a^2)*b^2*x)))/(a^2*(1 - a^2)^(3/2)))/(2*x^2)`

### Rubi [A] (verified)

Time = 0.79 (sec) , antiderivative size = 154, normalized size of antiderivative = 1.25, number of steps used = 14, number of rules used = 13,  $\frac{\text{number of rules}}{\text{integrand size}} = 1.300$ , Rules used = {5782, 25, 4927, 3042, 4272, 3042, 4407, 3042, 4318, 3042, 3139, 1083, 217}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\csc^{-1}(a + bx)}{x^3} dx$$

↓ 5782

$$-b^2 \int \frac{(a + bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx)}{b^3 x^3} d \csc^{-1}(a + bx)$$

↓ 25

$$b^2 \int -\frac{(a + bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx)}{b^3 x^3} d \csc^{-1}(a + bx)$$

$$\begin{aligned}
& \downarrow 4927 \\
& -b^2 \left( \frac{\csc^{-1}(a+bx)}{2b^2x^2} - \frac{1}{2} \int \frac{1}{b^2x^2} d \csc^{-1}(a+bx) \right) \\
& \downarrow 3042 \\
& -b^2 \left( \frac{\csc^{-1}(a+bx)}{2b^2x^2} - \frac{1}{2} \int \frac{1}{(a - \csc(\csc^{-1}(a+bx)))^2} d \csc^{-1}(a+bx) \right) \\
& \downarrow 4272 \\
& -b^2 \left( \frac{1}{2} \left( \frac{(a+bx) \sqrt{1 - \frac{1}{(a+bx)^2}}}{a(1-a^2)bx} - \frac{\int -\frac{-a^2 - (a+bx)a+1}{bx} d \csc^{-1}(a+bx)}{a(1-a^2)} \right) + \frac{\csc^{-1}(a+bx)}{2b^2x^2} \right) \\
& \downarrow 3042 \\
& -b^2 \left( \frac{1}{2} \left( \frac{(a+bx) \sqrt{1 - \frac{1}{(a+bx)^2}}}{a(1-a^2)bx} - \frac{\int \frac{-a^2 - \csc(\csc^{-1}(a+bx))a+1}{a - \csc(\csc^{-1}(a+bx))} d \csc^{-1}(a+bx)}{a(1-a^2)} \right) + \frac{\csc^{-1}(a+bx)}{2b^2x^2} \right) \\
& \downarrow 4407 \\
& -b^2 \left( \frac{1}{2} \left( \frac{(a+bx) \sqrt{1 - \frac{1}{(a+bx)^2}}}{a(1-a^2)bx} - \frac{\frac{(1-2a^2) \int -\frac{a+bx}{bx} d \csc^{-1}(a+bx)}{a} + \frac{(1-a^2) \csc^{-1}(a+bx)}{a}}{a(1-a^2)} \right) + \frac{\csc^{-1}(a+bx)}{2b^2x^2} \right) \\
& \downarrow 3042 \\
& -b^2 \left( \frac{1}{2} \left( \frac{(a+bx) \sqrt{1 - \frac{1}{(a+bx)^2}}}{a(1-a^2)bx} - \frac{\frac{(1-2a^2) \int \frac{\csc(\csc^{-1}(a+bx))}{a - \csc(\csc^{-1}(a+bx))} d \csc^{-1}(a+bx)}{a} + \frac{(1-a^2) \csc^{-1}(a+bx)}{a}}{a(1-a^2)} \right) + \frac{\csc^{-1}(a+bx)}{2b^2x^2} \right) \\
& \downarrow 4318 \\
& -b^2 \left( \frac{1}{2} \left( \frac{(a+bx) \sqrt{1 - \frac{1}{(a+bx)^2}}}{a(1-a^2)bx} - \frac{\frac{(1-a^2) \csc^{-1}(a+bx)}{a} - \frac{(1-2a^2) \int \frac{1}{1 - \frac{1}{a+bx}} d \csc^{-1}(a+bx)}{a}}{a(1-a^2)} \right) + \frac{\csc^{-1}(a+bx)}{2b^2x^2} \right) \\
& \downarrow 3042
\end{aligned}$$



$$-b^2 \left( \frac{1}{2} \left( \frac{(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{a(1-a^2)bx} - \frac{(1-a^2)\csc^{-1}(a+bx)}{a} - \frac{(1-2a^2)\int\frac{1}{1-a\sin(\csc^{-1}(a+bx))}d\csc^{-1}(a+bx)}{a(1-a^2)} \right) + \frac{\csc^{-1}(a+bx)}{2b^2x^2} \right)$$

↓ 3139

$$-b^2 \left( \frac{1}{2} \left( \frac{(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{a(1-a^2)bx} - \frac{(1-a^2)\csc^{-1}(a+bx)}{a} - \frac{2(1-2a^2)\int\frac{1}{\tan^2(\frac{1}{2}\csc^{-1}(a+bx))-2a\tan(\frac{1}{2}\csc^{-1}(a+bx))+1}d\tan(\frac{1}{2}\csc^{-1}(a+bx))}{a(1-a^2)} \right) + \frac{\csc^{-1}(a+bx)}{2b^2x^2} \right)$$

↓ 1083

$$-b^2 \left( \frac{1}{2} \left( \frac{(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{a(1-a^2)bx} - \frac{4(1-2a^2)\int\frac{1}{-(2\tan(\frac{1}{2}\csc^{-1}(a+bx))-2a)^2-4(1-a^2)}d(2\tan(\frac{1}{2}\csc^{-1}(a+bx))-2a)}{a(1-a^2)} + \frac{(1-a^2)\csc^{-1}(a+bx)}{a} \right) + \frac{\csc^{-1}(a+bx)}{2b^2x^2} \right)$$

↓ 217

$$-b^2 \left( \frac{1}{2} \left( \frac{(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{a(1-a^2)bx} - \frac{(1-a^2)\csc^{-1}(a+bx)}{a} - \frac{2(1-2a^2)\arctan\left(\frac{2\tan(\frac{1}{2}\csc^{-1}(a+bx))-2a}{2\sqrt{1-a^2}}\right)}{a\sqrt{1-a^2}} \right) + \frac{\csc^{-1}(a+bx)}{2b^2x^2} \right)$$

input

```
Int[ArcCsc[a + b*x]/x^3,x]
```

output

```
-(b^2*(ArcCsc[a + b*x]/(2*b^2*x^2) + (((a + b*x)*Sqrt[1 - (a + b*x)^(-2)])
/(a*(1 - a^2)*b*x) - (((1 - a^2)*ArcCsc[a + b*x])/a - (2*(1 - 2*a^2)*ArcTan[(-2*a + 2*Tan[ArcCsc[a + b*x]/2])/(2*Sqrt[1 - a^2])])/(a*Sqrt[1 - a^2]))
/(a*(1 - a^2)))/2)
```

## Definitions of rubi rules used

- rule 25  $\text{Int}[-(\text{Fx}_), \text{x\_Symbol}] \rightarrow \text{Simp}[\text{Identity}[-1] \quad \text{Int}[\text{Fx}, \text{x}], \text{x}]$
- rule 217  $\text{Int}[(\text{a}_) + (\text{b}_) \cdot (\text{x}_)^2)^{-1}, \text{x\_Symbol}] \rightarrow \text{Simp}[(-\text{Rt}[-\text{a}, 2] \cdot \text{Rt}[-\text{b}, 2])^{-1}) \cdot \text{ArcTan}[\text{Rt}[-\text{b}, 2] \cdot (\text{x}/\text{Rt}[-\text{a}, 2])], \text{x}] /; \text{FreeQ}[\{\text{a}, \text{b}\}, \text{x}] \ \&\& \ \text{PosQ}[\text{a}/\text{b}] \ \&\& \ (\text{LtQ}[\text{a}, 0] \ || \ \text{LtQ}[\text{b}, 0])$
- rule 1083  $\text{Int}[(\text{a}_) + (\text{b}_) \cdot (\text{x}_) + (\text{c}_) \cdot (\text{x}_)^2)^{-1}, \text{x\_Symbol}] \rightarrow \text{Simp}[-2 \quad \text{Subst}[\text{Int}[1/\text{Simp}[\text{b}^2 - 4 \cdot \text{a} \cdot \text{c} - \text{x}^2, \text{x}], \text{x}], \text{x}, \text{b} + 2 \cdot \text{c} \cdot \text{x}], \text{x}] /; \text{FreeQ}[\{\text{a}, \text{b}, \text{c}\}, \text{x}]$
- rule 3042  $\text{Int}[\text{u}_, \text{x\_Symbol}] \rightarrow \text{Int}[\text{DeactivateTrig}[\text{u}, \text{x}], \text{x}] /; \text{FunctionOfTrigOfLinearQ}[\text{u}, \text{x}]$
- rule 3139  $\text{Int}[(\text{a}_) + (\text{b}_) \cdot \sin[(\text{c}_) + (\text{d}_) \cdot (\text{x}_)]]^{-1}, \text{x\_Symbol}] \rightarrow \text{With}[\{\text{e} = \text{FreeFactors}[\text{Tan}[(\text{c} + \text{d} \cdot \text{x})/2], \text{x}]\}, \text{Simp}[2 \cdot (\text{e}/\text{d}) \quad \text{Subst}[\text{Int}[1/(\text{a} + 2 \cdot \text{b} \cdot \text{e} \cdot \text{x} + \text{a} \cdot \text{e}^2 \cdot \text{x}^2), \text{x}], \text{x}, \text{Tan}[(\text{c} + \text{d} \cdot \text{x})/2]/\text{e}], \text{x}]] /; \text{FreeQ}[\{\text{a}, \text{b}, \text{c}, \text{d}\}, \text{x}] \ \&\& \ \text{NeQ}[\text{a}^2 - \text{b}^2, 0]$
- rule 4272  $\text{Int}[(\text{csc}[(\text{c}_) + (\text{d}_) \cdot (\text{x}_)] \cdot (\text{b}_) + (\text{a}_))^{(\text{n}_)}, \text{x\_Symbol}] \rightarrow \text{Simp}[\text{b}^2 \cdot \text{Cot}[\text{c} + \text{d} \cdot \text{x}] \cdot ((\text{a} + \text{b} \cdot \text{Csc}[\text{c} + \text{d} \cdot \text{x}])^{(\text{n} + 1)} / (\text{a} \cdot \text{d} \cdot (\text{n} + 1) \cdot (\text{a}^2 - \text{b}^2))), \text{x}] + \text{Simp}[1/(\text{a} \cdot (\text{n} + 1) \cdot (\text{a}^2 - \text{b}^2)) \quad \text{Int}[(\text{a} + \text{b} \cdot \text{Csc}[\text{c} + \text{d} \cdot \text{x}])^{(\text{n} + 1)} \cdot \text{Simp}[(\text{a}^2 - \text{b}^2) \cdot (\text{n} + 1) - \text{a} \cdot \text{b} \cdot (\text{n} + 1) \cdot \text{Csc}[\text{c} + \text{d} \cdot \text{x}] + \text{b}^2 \cdot (\text{n} + 2) \cdot \text{Csc}[\text{c} + \text{d} \cdot \text{x}]^2, \text{x}], \text{x}], \text{x}] /; \text{FreeQ}[\{\text{a}, \text{b}, \text{c}, \text{d}\}, \text{x}] \ \&\& \ \text{NeQ}[\text{a}^2 - \text{b}^2, 0] \ \&\& \ \text{LtQ}[\text{n}, -1] \ \&\& \ \text{IntegerQ}[2 \cdot \text{n}]$
- rule 4318  $\text{Int}[\text{csc}[(\text{e}_) + (\text{f}_) \cdot (\text{x}_)] / (\text{csc}[(\text{e}_) + (\text{f}_) \cdot (\text{x}_)] \cdot (\text{b}_) + (\text{a}_)), \text{x\_Symbol}] \rightarrow \text{Simp}[1/\text{b} \quad \text{Int}[1/(1 + (\text{a}/\text{b}) \cdot \text{Sin}[\text{e} + \text{f} \cdot \text{x}]), \text{x}], \text{x}] /; \text{FreeQ}[\{\text{a}, \text{b}, \text{e}, \text{f}\}, \text{x}] \ \&\& \ \text{NeQ}[\text{a}^2 - \text{b}^2, 0]$

```
rule 4407 Int[(csc[(e_.) + (f_.)*(x_)]*(d_.) + (c_))/(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_)), x_Symbol] := Simp[c*(x/a), x] - Simp[(b*c - a*d)/a Int[Csc[e + f*
x]/(a + b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c
- a*d, 0]
```

```
rule 4927 Int[Cot[(c_.) + (d_.)*(x_)]*Csc[(c_.) + (d_.)*(x_)]*(Csc[(c_.) + (d_.)*(x_)
]*(b_.) + (a_))^(n_.)*((e_.) + (f_.)*(x_))^(m_.), x_Symbol] := Simp[(-e +
f*x)^m*((a + b*Csc[c + d*x])^(n + 1)/(b*d*(n + 1))), x] + Simp[f*(m/(b*d*(
n + 1))) Int[(e + f*x)^(m - 1)*(a + b*Csc[c + d*x])^(n + 1), x], x] /; Fr
eeQ[{a, b, c, d, e, f, n}, x] && IGtQ[m, 0] && NeQ[n, -1]
```

```
rule 5782 Int[((a_.) + ArcCsc[(c_.) + (d_.)*(x_)]*(b_.))^(p_.)*((e_.) + (f_.)*(x_))^(m
_.), x_Symbol] := Simp[-(d^(m + 1))^( -1) Subst[Int[(a + b*x)^p*Csc[x]*Cot
[x]*(d*e - c*f + f*Csc[x])^m, x], x, ArcCsc[c + d*x]], x] /; FreeQ[{a, b, c
, d, e, f}, x] && IGtQ[p, 0] && IntegerQ[m]
```

**Maple [B] (verified)**

Leaf count of result is larger than twice the leaf count of optimal. 318 vs. 2(109) = 218.

Time = 0.50 (sec) , antiderivative size = 319, normalized size of antiderivative = 2.59

method	result
parts	$-\frac{\operatorname{arccsc}(bx+a)}{2x^2} + \frac{b\sqrt{b^2x^2+2abx+a^2-1} \left( (a^2-1)^{\frac{3}{2}} \arctan\left(\frac{1}{\sqrt{b^2x^2+2abx+a^2-1}}\right) a^2bx - 2 \ln\left(\frac{2a^2-2+2abx+2\sqrt{a^2-1}}{x}\right) \right)}{2x^2}$
derivativedivides	$b^2 \left( -\frac{\operatorname{arccsc}(bx+a)}{2b^2x^2} - \frac{\sqrt{(bx+a)^2-1} \left( \arctan\left(\frac{1}{\sqrt{(bx+a)^2-1}}\right) (a^2-1)^{\frac{3}{2}} a^3 - \arctan\left(\frac{1}{\sqrt{(bx+a)^2-1}}\right) (a^2-1)^{\frac{3}{2}} a^2 (bx) \right)}{2b^2x^2} \right)$
default	$b^2 \left( -\frac{\operatorname{arccsc}(bx+a)}{2b^2x^2} - \frac{\sqrt{(bx+a)^2-1} \left( \arctan\left(\frac{1}{\sqrt{(bx+a)^2-1}}\right) (a^2-1)^{\frac{3}{2}} a^3 - \arctan\left(\frac{1}{\sqrt{(bx+a)^2-1}}\right) (a^2-1)^{\frac{3}{2}} a^2 (bx) \right)}{2b^2x^2} \right)$

```
input int(arccsc(b*x+a)/x^3,x,method=_RETURNVERBOSE)
```

output

```
-1/2*arccsc(b*x+a)/x^2+1/2*b*(b^2*x^2+2*a*b*x+a^2-1)^(1/2)*((a^2-1)^(3/2)*
arctan(1/(b^2*x^2+2*a*b*x+a^2-1)^(1/2))*a^2*b*x-2*ln(2*(a*b*x+(a^2-1)^(1/2)
)*(b^2*x^2+2*a*b*x+a^2-1)^(1/2)+a^2-1)/x)*a^4*b*x-b*arctan(1/(b^2*x^2+2*a*
b*x+a^2-1)^(1/2))*x*(a^2-1)^(3/2)+(a^2-1)^(3/2)*(b^2*x^2+2*a*b*x+a^2-1)^(1
/2)*a+3*ln(2*(a*b*x+(a^2-1)^(1/2)*(b^2*x^2+2*a*b*x+a^2-1)^(1/2)+a^2-1)/x)*
a^2*b*x-b*ln(2*(a*b*x+(a^2-1)^(1/2)*(b^2*x^2+2*a*b*x+a^2-1)^(1/2)+a^2-1)/x
)*x)/((b^2*x^2+2*a*b*x+a^2-1)/(b*x+a)^2)^(1/2)/(b*x+a)/a^2/(a^2-1)^(5/2)/x
```

**Fricas [A] (verification not implemented)**

Time = 0.14 (sec) , antiderivative size = 428, normalized size of antiderivative = 3.48

$$\int \frac{\csc^{-1}(a+bx)}{x^3} dx$$

$$= \left[ \frac{(2a^2-1)\sqrt{a^2-1}b^2x^2 \log\left(\frac{a^2bx+a^3+\sqrt{b^2x^2+2abx+a^2-1}(a^2-\sqrt{a^2-1}a-1)-(abx+a^2-1)\sqrt{a^2-1}-a}{x}\right) - 2(a^4-2a^2+1)\sqrt{a^2-1}}{\dots} \right]$$

input

```
integrate(arccsc(b*x+a)/x^3,x, algorithm="fricas")
```

output

```
[1/2*((2*a^2 - 1)*sqrt(a^2 - 1)*b^2*x^2*log((a^2*b*x + a^3 + sqrt(b^2*x^2
+ 2*a*b*x + a^2 - 1)*(a^2 - sqrt(a^2 - 1)*a - 1) - (a*b*x + a^2 - 1)*sqrt(
a^2 - 1) - a)/x) - 2*(a^4 - 2*a^2 + 1)*b^2*x^2*arctan(-b*x - a + sqrt(b^2*x
^2 + 2*a*b*x + a^2 - 1)) + (a^3 - a)*b^2*x^2 + sqrt(b^2*x^2 + 2*a*b*x + a
^2 - 1)*(a^3 - a)*b*x - (a^6 - 2*a^4 + a^2)*arccsc(b*x + a))/((a^6 - 2*a^4
+ a^2)*x^2), 1/2*(2*(2*a^2 - 1)*sqrt(-a^2 + 1)*b^2*x^2*arctan(-(sqrt(-a^2
+ 1)*b*x - sqrt(b^2*x^2 + 2*a*b*x + a^2 - 1)*sqrt(-a^2 + 1))/(a^2 - 1)) -
2*(a^4 - 2*a^2 + 1)*b^2*x^2*arctan(-b*x - a + sqrt(b^2*x^2 + 2*a*b*x + a^
2 - 1)) + (a^3 - a)*b^2*x^2 + sqrt(b^2*x^2 + 2*a*b*x + a^2 - 1)*(a^3 - a)*
b*x - (a^6 - 2*a^4 + a^2)*arccsc(b*x + a))/((a^6 - 2*a^4 + a^2)*x^2)]
```

**Sympy [F]**

$$\int \frac{\csc^{-1}(a + bx)}{x^3} dx = \int \frac{\operatorname{acsc}(a + bx)}{x^3} dx$$

input `integrate(acsc(b*x+a)/x**3,x)`

output `Integral(acsc(a + b*x)/x**3, x)`

**Maxima [F]**

$$\int \frac{\csc^{-1}(a + bx)}{x^3} dx = \int \frac{\operatorname{arccsc}(bx + a)}{x^3} dx$$

input `integrate(arccsc(b*x+a)/x^3,x, algorithm="maxima")`

output `-1/2*(2*x^2*integrate(1/2*(b^2*x + a*b)*e^(1/2*log(b*x + a + 1) + 1/2*log(b*x + a - 1))/(b^2*x^4 + 2*a*b*x^3 + (a^2 - 1)*x^2 + (b^2*x^4 + 2*a*b*x^3 + (a^2 - 1)*x^2)*e^(log(b*x + a + 1) + log(b*x + a - 1))), x) + arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1))/x^2`

**Giac [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 217 vs. 2(108) = 216.

Time = 0.17 (sec) , antiderivative size = 217, normalized size of antiderivative = 1.76

$$\int \frac{\csc^{-1}(a + bx)}{x^3} dx = \frac{1}{2} b \left( \frac{2(2a^2b - b) \arctan\left(\frac{(bx+a)\left(\sqrt{-\frac{1}{(bx+a)^2} + 1} - 1\right) + a}{\sqrt{-a^2 + 1}}\right)}{(a^4 - a^2)\sqrt{-a^2 + 1}} + \frac{2\left((bx+a)ab\left(\sqrt{-\frac{1}{(bx+a)^2} + 1} - 1\right) + 2(bx+a)a\right)}{\left((bx+a)^2\left(\sqrt{-\frac{1}{(bx+a)^2} + 1} - 1\right)^2 + 2(bx+a)a\right)} \right)$$

input `integrate(arccsc(b*x+a)/x^3,x, algorithm="giac")`

output `1/2*b*(2*(2*a^2*b - b)*arctan(((b*x + a)*(sqrt(-1/(b*x + a)^2 + 1) - 1) + a)/sqrt(-a^2 + 1)))/((a^4 - a^2)*sqrt(-a^2 + 1)) + 2*((b*x + a)*a*b*(sqrt(-1/(b*x + a)^2 + 1) - 1) + b)/(((b*x + a)^2*(sqrt(-1/(b*x + a)^2 + 1) - 1)^2 + 2*(b*x + a)*a*(sqrt(-1/(b*x + a)^2 + 1) - 1) + 1)*(a^3 - a)) - (2*a*b/(b*x + a) - b)*arcsin(-1/((b*x + a)*(a/(b*x + a) - 1) - a))/(a^2*(a/(b*x + a) - 1)^2))`

### Mupad [F(-1)]

Timed out.

$$\int \frac{\csc^{-1}(a + bx)}{x^3} dx = \int \frac{\operatorname{asin}\left(\frac{1}{a+bx}\right)}{x^3} dx$$

input `int(asin(1/(a + b*x))/x^3,x)`

output `int(asin(1/(a + b*x))/x^3, x)`

### Reduce [F]

$$\int \frac{\csc^{-1}(a + bx)}{x^3} dx = \int \frac{\operatorname{acsc}(bx + a)}{x^3} dx$$

input `int(acsc(b*x+a)/x^3,x)`

output `int(acsc(a + b*x)/x**3,x)`

### 3.25 $\int \frac{\csc^{-1}(a+bx)}{x^4} dx$

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#### Optimal result

Integrand size = 10, antiderivative size = 180

$$\int \frac{\csc^{-1}(a+bx)}{x^4} dx = -\frac{b(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{6a(1-a^2)x^2} + \frac{(2-5a^2)b^2(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{6a^2(1-a^2)^2x} - \frac{b^3 \csc^{-1}(a+bx)}{3a^3} - \frac{\csc^{-1}(a+bx)}{3x^3} - \frac{(2-5a^2+6a^4)b^3 \arctan\left(\frac{a-\tan(\frac{1}{2}\csc^{-1}(a+bx))}{\sqrt{1-a^2}}\right)}{3a^3(1-a^2)^{5/2}}$$

output

```
-1/6*b*(b*x+a)*(1-1/(b*x+a)^2)^(1/2)/a/(-a^2+1)/x^2+1/6*(-5*a^2+2)*b^2*(b*x+a)*(1-1/(b*x+a)^2)^(1/2)/a^2/(-a^2+1)^2/x-1/3*b^3*arccsc(b*x+a)/a^3-1/3*arccsc(b*x+a)/x^3-1/3*(6*a^4-5*a^2+2)*b^3*arctan((a-tan(1/2*arccsc(b*x+a)))/(-a^2+1)^(1/2))/a^3/(-a^2+1)^(5/2)
```

**Mathematica [C] (verified)**

Result contains complex when optimal does not.

Time = 0.31 (sec) , antiderivative size = 241, normalized size of antiderivative = 1.34

$$\int \frac{\csc^{-1}(a+bx)}{x^4} dx$$

$$= \frac{1}{6} \left( \frac{b \sqrt{\frac{-1+a^2+2abx+b^2x^2}{(a+bx)^2}} (a^4 + abx - 4a^3bx + 2b^2x^2 - a^2(1 + 5b^2x^2))}{a^2(-1+a^2)^2 x^2} - \frac{2 \csc^{-1}(a+bx)}{x^3} - \frac{2b^3 \arcsin\left(\frac{1}{a+bx}\right)}{a^3} + \frac{i(2-5a^2+6a^4)b^3 \log\left(\frac{12a^3(-1+a^2)^2 \left(-\frac{i(-1+a^2+abx)}{\sqrt{1-a^2}} - (a+bx) \sqrt{\frac{-1+a^2+2abx+b^2x^2}{(a+bx)^2}}\right)}{(2-5a^2+6a^4)b^3x}\right)}{a^3(1-a^2)^{5/2}} \right)$$

input `Integrate[ArcCsc[a + b*x]/x^4,x]`

output `((b*Sqrt[(-1 + a^2 + 2*a*b*x + b^2*x^2)/(a + b*x)^2]*(a^4 + a*b*x - 4*a^3*b*x + 2*b^2*x^2 - a^2*(1 + 5*b^2*x^2)))/(a^2*(-1 + a^2)^2*x^2) - (2*ArcCsc[a + b*x])/x^3 - (2*b^3*ArcSin[(a + b*x)^(-1)])/a^3 + (I*(2 - 5*a^2 + 6*a^4)*b^3*Log[(12*a^3*(-1 + a^2)^2*((-I)*(-1 + a^2 + a*b*x))/Sqrt[1 - a^2] - (a + b*x)*Sqrt[(-1 + a^2 + 2*a*b*x + b^2*x^2)/(a + b*x)^2]])/((2 - 5*a^2 + 6*a^4)*b^3*x))/(a^3*(1 - a^2)^(5/2))/6`



**Rubi [A] (verified)**

Time = 1.09 (sec) , antiderivative size = 228, normalized size of antiderivative = 1.27, number of steps used = 15, number of rules used = 14,  $\frac{\text{number of rules}}{\text{integrand size}} = 1.400$ , Rules used = {5782, 4927, 3042, 4272, 3042, 4548, 3042, 4407, 3042, 4318, 3042, 3139, 1083, 217}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{\csc^{-1}(a+bx)}{x^4} dx \\
 & \quad \downarrow \text{5782} \\
 & -b^3 \int \frac{(a+bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)}{b^4 x^4} d \csc^{-1}(a+bx) \\
 & \quad \downarrow \text{4927} \\
 & -b^3 \left( \frac{1}{3} \int -\frac{1}{b^3 x^3} d \csc^{-1}(a+bx) + \frac{\csc^{-1}(a+bx)}{3b^3 x^3} \right) \\
 & \quad \downarrow \text{3042} \\
 & -b^3 \left( \frac{1}{3} \int \frac{1}{(a - \csc(\csc^{-1}(a+bx)))^3} d \csc^{-1}(a+bx) + \frac{\csc^{-1}(a+bx)}{3b^3 x^3} \right) \\
 & \quad \downarrow \text{4272} \\
 & -b^3 \left( \frac{1}{3} \left( \int \frac{-(a+bx)^2 - 2a(a+bx) + 2(1-a^2)}{b^2 x^2} d \csc^{-1}(a+bx) + \frac{\sqrt{1 - \frac{1}{(a+bx)^2}}(a+bx)}{2a(1-a^2)b^2 x^2} \right) + \frac{\csc^{-1}(a+bx)}{3b^3 x^3} \right) \\
 & \quad \downarrow \text{3042} \\
 & -b^3 \left( \frac{1}{3} \left( \int \frac{-\csc(\csc^{-1}(a+bx))^2 - 2a \csc(\csc^{-1}(a+bx)) + 2(1-a^2)}{(a - \csc(\csc^{-1}(a+bx)))^2} d \csc^{-1}(a+bx) + \frac{\sqrt{1 - \frac{1}{(a+bx)^2}}(a+bx)}{2a(1-a^2)b^2 x^2} \right) + \frac{\csc^{-1}(a+bx)}{3b^3 x^3} \right) \\
 & \quad \downarrow \text{4548}
 \end{aligned}$$

$$-b^3 \left( \frac{1}{3} \left( \frac{\int \frac{2(1-a^2)^2 - a(1-4a^2)(a+bx)}{bx} d \csc^{-1}(a+bx)}{a(1-a^2)} - \frac{(2-5a^2)(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{a(1-a^2)bx} + \frac{\sqrt{1-\frac{1}{(a+bx)^2}}(a+bx)}{2a(1-a^2)b^2x^2} \right) + \frac{\csc^{-1}(a+bx)}{3b^3x^3} \right)$$

↓ 3042

$$-b^3 \left( \frac{1}{3} \left( \frac{\int \frac{2(1-a^2)^2 - a(1-4a^2) \csc(\csc^{-1}(a+bx))}{a - \csc(\csc^{-1}(a+bx))} d \csc^{-1}(a+bx)}{a(1-a^2)} - \frac{(2-5a^2)(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{a(1-a^2)bx} + \frac{\sqrt{1-\frac{1}{(a+bx)^2}}(a+bx)}{2a(1-a^2)b^2x^2} \right) + \frac{\csc^{-1}(a+bx)}{3b^3x^3} \right)$$

↓ 4407

$$-b^3 \left( \frac{1}{3} \left( \frac{\left( \frac{(6a^4-5a^2+2) \int \frac{-\frac{a+bx}{bx} d \csc^{-1}(a+bx)}{a} + \frac{2(1-a^2)^2 \csc^{-1}(a+bx)}{a} \right)}{a(1-a^2)} - \frac{(2-5a^2)(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{a(1-a^2)bx} + \frac{\sqrt{1-\frac{1}{(a+bx)^2}}(a+bx)}{2a(1-a^2)b^2x^2} \right) + \frac{\csc^{-1}(a+bx)}{3b^3x^3} \right)$$

↓ 3042

$$-b^3 \left( \frac{1}{3} \left( \frac{\left( \frac{(6a^4-5a^2+2) \int \frac{\csc(\csc^{-1}(a+bx))}{a - \csc(\csc^{-1}(a+bx))} d \csc^{-1}(a+bx)}{a} + \frac{2(1-a^2)^2 \csc^{-1}(a+bx)}{a} \right)}{a(1-a^2)} - \frac{(2-5a^2)(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{a(1-a^2)bx} + \frac{\sqrt{1-\frac{1}{(a+bx)^2}}(a+bx)}{2a(1-a^2)b^2x^2} \right) + \frac{\csc^{-1}(a+bx)}{3b^3x^3} \right)$$

↓ 4318

$$-b^3 \left( \frac{1}{3} \left( \frac{\left( \frac{2(1-a^2)^2 \csc^{-1}(a+bx)}{a} - \frac{(6a^4-5a^2+2) \int \frac{1}{1-\frac{1}{a+bx}} d \csc^{-1}(a+bx)}{a} \right)}{a(1-a^2)} - \frac{(2-5a^2)(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{a(1-a^2)bx} + \frac{\sqrt{1-\frac{1}{(a+bx)^2}}(a+bx)}{2a(1-a^2)b^2x^2} \right) + \frac{\csc^{-1}(a+bx)}{3b^3x^3} \right)$$

↓ 3042

$$-b^3 \left( \frac{1}{3} \left( \frac{\left( \frac{2(1-a^2)^2 \csc^{-1}(a+bx)}{a} - \frac{(6a^4-5a^2+2) \int \frac{1}{1-a \sin(\csc^{-1}(a+bx))} d \csc^{-1}(a+bx)}{a} \right)}{a(1-a^2)} - \frac{(2-5a^2)(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{a(1-a^2)bx} + \frac{\sqrt{1-\frac{1}{(a+bx)^2}}(a+bx)}{2a(1-a^2)b^2x^2} \right) + \frac{\csc^{-1}(a+bx)}{3b^3x^3} \right)$$

↓ 3139

$$-b^3 \left( \frac{1}{3} \left( \frac{\frac{2(1-a^2)^2 \csc^{-1}(a+bx)}{a} - \frac{2(6a^4-5a^2+2) \int \frac{1}{\tan^2(\frac{1}{2} \csc^{-1}(a+bx)) - 2a \tan(\frac{1}{2} \csc^{-1}(a+bx)) + 1} dx \tan(\frac{1}{2} \csc^{-1}(a+bx))}{a(1-a^2)}}{2a(1-a^2)} - \frac{(2-5a^2)(a+bx)\sqrt{1-a^2}}{a(1-a^2)bx} \right) \right)$$

↓ 1083

$$-b^3 \left( \frac{1}{3} \left( \frac{\frac{4(6a^4-5a^2+2) \int \frac{1}{-(2 \tan(\frac{1}{2} \csc^{-1}(a+bx)) - 2a)^2 - 4(1-a^2)} dx (2 \tan(\frac{1}{2} \csc^{-1}(a+bx)) - 2a)}{a(1-a^2)} + \frac{2(1-a^2)^2 \csc^{-1}(a+bx)}{a}}{2a(1-a^2)} - \frac{(2-5a^2)(a+bx)\sqrt{1-a^2}}{a(1-a^2)bx} \right) \right)$$

↓ 217

$$-b^3 \left( \frac{1}{3} \left( \frac{\sqrt{1 - \frac{1}{(a+bx)^2}}(a+bx)}{2a(1-a^2)b^2x^2} + \frac{\frac{2(1-a^2)^2 \csc^{-1}(a+bx)}{a} - \frac{2(6a^4-5a^2+2) \arctan\left(\frac{2 \tan(\frac{1}{2} \csc^{-1}(a+bx)) - 2a}{2\sqrt{1-a^2}}\right)}{a\sqrt{1-a^2}}}{a(1-a^2)}}{2a(1-a^2)} - \frac{(2-5a^2)(a+bx)\sqrt{1-a^2}}{a(1-a^2)bx} \right) \right)$$

input `Int[ArcCsc[a + b*x]/x^4,x]`

output `-(b^3*(ArcCsc[a + b*x]/(3*b^3*x^3) + (((a + b*x)*Sqrt[1 - (a + b*x)^(-2)])/(2*a*(1 - a^2)*b^2*x^2) + (-(((2 - 5*a^2)*(a + b*x)*Sqrt[1 - (a + b*x)^(-2)])/(a*(1 - a^2)*b*x) + ((2*(1 - a^2)^2*ArcCsc[a + b*x])/a - (2*(2 - 5*a^2 + 6*a^4)*ArcTan[(-2*a + 2*Tan[ArcCsc[a + b*x]/2)]/(2*Sqrt[1 - a^2]))/(a*Sqrt[1 - a^2]))/(a*(1 - a^2)))/(2*a*(1 - a^2)))/3))`

## Definitions of rubi rules used

- rule 217  $\text{Int}[(a_ + (b_ \cdot x)^2)^{-1}, x\_Symbol] \rightarrow \text{Simp}[(-\text{Rt}[-a, 2] \cdot \text{Rt}[-b, 2])^{-1} \cdot \text{ArcTan}[\text{Rt}[-b, 2] \cdot (x/\text{Rt}[-a, 2])], x] /;$   $\text{FreeQ}\{a, b, x\} \ \&\& \ \text{PosQ}[a/b] \ \&\& \ (\text{LtQ}[a, 0] \ || \ \text{LtQ}[b, 0])$
- rule 1083  $\text{Int}[(a_ + (b_ \cdot x) + (c_ \cdot x)^2)^{-1}, x\_Symbol] \rightarrow \text{Simp}[-2 \ \text{Subst}[\text{Int}[1/\text{Simp}[b^2 - 4 \cdot a \cdot c - x^2, x], x], x, b + 2 \cdot c \cdot x], x] /;$   $\text{FreeQ}\{a, b, c, x\}$
- rule 3042  $\text{Int}[u_, x\_Symbol] \rightarrow \text{Int}[\text{DeactivateTrig}[u, x], x] /;$   $\text{FunctionOfTrigOfLinearQ}[u, x]$
- rule 3139  $\text{Int}[(a_ + (b_ \cdot \sin[(c_ \cdot x) + (d_ \cdot x)])^{-1}, x\_Symbol] \rightarrow \text{With}\{e = \text{FreeFactors}[\text{Tan}[(c + d \cdot x)/2], x]\}, \text{Simp}[2 \cdot (e/d) \ \text{Subst}[\text{Int}[1/(a + 2 \cdot b \cdot e \cdot x + a \cdot e^2 \cdot x^2), x], x, \text{Tan}[(c + d \cdot x)/2]/e], x] /;$   $\text{FreeQ}\{a, b, c, d, x\} \ \&\& \ \text{NeQ}[a^2 - b^2, 0]$
- rule 4272  $\text{Int}[(\text{csc}[(c_ \cdot x) + (d_ \cdot x)] \cdot (b_ \cdot x) + (a_ \cdot x))^n, x\_Symbol] \rightarrow \text{Simp}[b^2 \cdot \text{Cot}[c + d \cdot x] \cdot ((a + b \cdot \text{Csc}[c + d \cdot x])^{n+1} / (a \cdot d \cdot (n+1) \cdot (a^2 - b^2))), x] + \text{Simp}[1 / (a \cdot (n+1) \cdot (a^2 - b^2)) \ \text{Int}[(a + b \cdot \text{Csc}[c + d \cdot x])^{n+1} \cdot \text{Simp}[(a^2 - b^2) \cdot (n+1) - a \cdot b \cdot (n+1) \cdot \text{Csc}[c + d \cdot x] + b^2 \cdot (n+2) \cdot \text{Csc}[c + d \cdot x]^2, x], x], x] /;$   $\text{FreeQ}\{a, b, c, d, x\} \ \&\& \ \text{NeQ}[a^2 - b^2, 0] \ \&\& \ \text{LtQ}[n, -1] \ \&\& \ \text{IntegerQ}[2 \cdot n]$
- rule 4318  $\text{Int}[\text{csc}[(e_ \cdot x) + (f_ \cdot x)] / (\text{csc}[(e_ \cdot x) + (f_ \cdot x)] \cdot (b_ \cdot x) + (a_ \cdot x)), x\_Symbol] \rightarrow \text{Simp}[1/b \ \text{Int}[1/(1 + (a/b) \cdot \text{Sin}[e + f \cdot x]), x], x] /;$   $\text{FreeQ}\{a, b, e, f, x\} \ \&\& \ \text{NeQ}[a^2 - b^2, 0]$
- rule 4407  $\text{Int}[(\text{csc}[(e_ \cdot x) + (f_ \cdot x)] \cdot (d_ \cdot x) + (c_ \cdot x)) / (\text{csc}[(e_ \cdot x) + (f_ \cdot x)] \cdot (b_ \cdot x) + (a_ \cdot x)), x\_Symbol] \rightarrow \text{Simp}[c \cdot (x/a), x] - \text{Simp}[(b \cdot c - a \cdot d) / a \ \text{Int}[\text{Csc}[e + f \cdot x] / (a + b \cdot \text{Csc}[e + f \cdot x]), x], x] /;$   $\text{FreeQ}\{a, b, c, d, e, f, x\} \ \&\& \ \text{NeQ}[b \cdot c - a \cdot d, 0]$

rule 4548

```
Int[((A_.) + csc[(e_.) + (f_.)*(x_.)]*(B_.) + csc[(e_.) + (f_.)*(x_.)]^2*(C_.
))*((csc[(e_.) + (f_.)*(x_.)]*(b_.) + (a_.))^(m_.), x_Symbol] := Simp[(A*b^2 -
a*b*B + a^2*C)*Cot[e + f*x]*((a + b*Csc[e + f*x])^(m + 1)/(a*f*(m + 1)*(a^2
- b^2))), x] + Simp[1/(a*(m + 1)*(a^2 - b^2)) Int[(a + b*Csc[e + f*x])^(
m + 1)*Simp[A*(a^2 - b^2)*(m + 1) - a*(A*b - a*B + b*C)*(m + 1)*Csc[e + f*x
] + (A*b^2 - a*b*B + a^2*C)*(m + 2)*Csc[e + f*x]^2, x], x], x] /; FreeQ[{a,
b, e, f, A, B, C}, x] && NeQ[a^2 - b^2, 0] && LtQ[m, -1]
```

rule 4927

```
Int[Cot[(c_.) + (d_.)*(x_.)]*Csc[(c_.) + (d_.)*(x_.)]*(Csc[(c_.) + (d_.)*(x_.)
]*(b_.) + (a_.))^(n_.)*((e_.) + (f_.)*(x_.))^(m_.), x_Symbol] := Simp[(-e +
f*x)^m*((a + b*Csc[c + d*x])^(n + 1)/(b*d*(n + 1))), x] + Simp[f*m/(b*d*(
n + 1)) Int[(e + f*x)^(m - 1)*(a + b*Csc[c + d*x])^(n + 1), x], x] /; Fr
eeQ[{a, b, c, d, e, f, n}, x] && IGtQ[m, 0] && NeQ[n, -1]
```

rule 5782

```
Int[((a_.) + ArcCsc[(c_.) + (d_.)*(x_.)]*(b_.))^(p_.)*((e_.) + (f_.)*(x_.))^(m
_.), x_Symbol] := Simp[-(d^(m + 1))^(m - 1) Subst[Int[(a + b*x)^p*Csc[x]*Cot
[x]*(d*e - c*f + f*Csc[x])^m, x], x, ArcCsc[c + d*x]], x] /; FreeQ[{a, b, c
, d, e, f}, x] && IGtQ[p, 0] && IntegerQ[m]
```

## Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 527 vs.  $2(160) = 320$ .

Time = 0.49 (sec) , antiderivative size = 528, normalized size of antiderivative = 2.93

method	result
parts	$-\frac{\operatorname{arccsc}(bx+a)}{3x^3} + \frac{b\sqrt{b^2x^2+2abx+a^2-1}}{3x^3} \left( -2(a^2-1)^{\frac{3}{2}} \arctan\left(\frac{1}{\sqrt{b^2x^2+2abx+a^2-1}}\right) a^4b^2x^2 + 6 \ln\left(\frac{2a^2-2+2abx+2\sqrt{b^2x^2+2abx+a^2-1}}{b^2x^2+2abx+a^2-1}\right) \right)$
derivativedivides	Expression too large to display
default	Expression too large to display

input

```
int(arccsc(b*x+a)/x^4,x,method=_RETURNVERBOSE)
```

output

```

-1/3*arccsc(b*x+a)/x^3+1/6*b*(b^2*x^2+2*a*b*x+a^2-1)^(1/2)*(-2*(a^2-1)^(3/2)*arctan(1/(b^2*x^2+2*a*b*x+a^2-1)^(1/2))*a^4*b^2*x^2+6*ln(2*(a*b*x+(a^2-1)^(1/2)*(b^2*x^2+2*a*b*x+a^2-1)^(1/2)+a^2-1)/x)*a^6*b^2*x^2+4*(a^2-1)^(3/2)*arctan(1/(b^2*x^2+2*a*b*x+a^2-1)^(1/2))*a^2*b^2*x^2-5*(b^2*x^2+2*a*b*x+a^2-1)^(1/2)*(a^2-1)^(3/2)*a^3*b*x-11*ln(2*(a*b*x+(a^2-1)^(1/2)*(b^2*x^2+2*a*b*x+a^2-1)^(1/2)+a^2-1)/x)*a^4*b^2*x^2+(b^2*x^2+2*a*b*x+a^2-1)^(1/2)*(a^2-1)^(3/2)*a^4-2*b^2*arctan(1/(b^2*x^2+2*a*b*x+a^2-1)^(1/2))*x^2*(a^2-1)^(3/2)+2*(b^2*x^2+2*a*b*x+a^2-1)^(1/2)*(a^2-1)^(3/2)*a*b*x+7*ln(2*(a*b*x+(a^2-1)^(1/2)*(b^2*x^2+2*a*b*x+a^2-1)^(1/2)+a^2-1)/x)*a^2*b^2*x^2-(b^2*x^2+2*a*b*x+a^2-1)^(1/2)*(a^2-1)^(3/2)*a^2-2*b^2*ln(2*(a*b*x+(a^2-1)^(1/2)*(b^2*x^2+2*a*b*x+a^2-1)^(1/2)+a^2-1)/x)*x^2)/((b^2*x^2+2*a*b*x+a^2-1)/(b*x+a)^2)^(1/2)/(b*x+a)/a^3/(a^2-1)^(7/2)/x^2

```

**Fricas [A] (verification not implemented)**

Time = 0.16 (sec) , antiderivative size = 548, normalized size of antiderivative = 3.04

$$\int \frac{\csc^{-1}(a+bx)}{x^4} dx$$

$$= \frac{(6a^4 - 5a^2 + 2)\sqrt{a^2 - 1}b^3x^3 \log\left(\frac{a^2bx + a^3 + \sqrt{b^2x^2 + 2abx + a^2 - 1}(a^2 + \sqrt{a^2 - 1}a - 1) + (abx + a^2 - 1)\sqrt{a^2 - 1} - a}{x}\right) + 4(a^6 - 3a^4 + 3a^2 - 1)\sqrt{a^2 - 1}b^3x^3 \arctan\left(-\frac{\sqrt{-a^2 + 1}bx - \sqrt{b^2x^2 + 2abx + a^2 - 1}\sqrt{-a^2 + 1}}{a^2 - 1}\right) - 4(a^6 - 3a^4 + 3a^2 - 1)\sqrt{a^2 - 1}b^3x^3}{2(6a^4 - 5a^2 + 2)\sqrt{-a^2 + 1}b^3x^3 \arctan\left(-\frac{\sqrt{-a^2 + 1}bx - \sqrt{b^2x^2 + 2abx + a^2 - 1}\sqrt{-a^2 + 1}}{a^2 - 1}\right) - 4(a^6 - 3a^4 + 3a^2 - 1)\sqrt{a^2 - 1}b^3x^3}$$

input

```
integrate(arccsc(b*x+a)/x^4,x, algorithm="fricas")
```

output

```
[1/6*((6*a^4 - 5*a^2 + 2)*sqrt(a^2 - 1)*b^3*x^3*log((a^2*b*x + a^3 + sqrt(
b^2*x^2 + 2*a*b*x + a^2 - 1)*(a^2 + sqrt(a^2 - 1)*a - 1) + (a*b*x + a^2 -
1)*sqrt(a^2 - 1) - a)/x) + 4*(a^6 - 3*a^4 + 3*a^2 - 1)*b^3*x^3*arctan(-b*x
- a + sqrt(b^2*x^2 + 2*a*b*x + a^2 - 1)) - (5*a^5 - 7*a^3 + 2*a)*b^3*x^3
- 2*(a^9 - 3*a^7 + 3*a^5 - a^3)*arccsc(b*x + a) - ((5*a^5 - 7*a^3 + 2*a)*b
^2*x^2 - (a^6 - 2*a^4 + a^2)*b*x)*sqrt(b^2*x^2 + 2*a*b*x + a^2 - 1))/((a^9
- 3*a^7 + 3*a^5 - a^3)*x^3), -1/6*(2*(6*a^4 - 5*a^2 + 2)*sqrt(-a^2 + 1)*b
^3*x^3*arctan(-(sqrt(-a^2 + 1)*b*x - sqrt(b^2*x^2 + 2*a*b*x + a^2 - 1)*sq
rt(-a^2 + 1))/(a^2 - 1)) - 4*(a^6 - 3*a^4 + 3*a^2 - 1)*b^3*x^3*arctan(-b*x
- a + sqrt(b^2*x^2 + 2*a*b*x + a^2 - 1)) + (5*a^5 - 7*a^3 + 2*a)*b^3*x^3
+ 2*(a^9 - 3*a^7 + 3*a^5 - a^3)*arccsc(b*x + a) + ((5*a^5 - 7*a^3 + 2*a)*b
^2*x^2 - (a^6 - 2*a^4 + a^2)*b*x)*sqrt(b^2*x^2 + 2*a*b*x + a^2 - 1))/((a^9
- 3*a^7 + 3*a^5 - a^3)*x^3)]
```

**Sympy [F]**

$$\int \frac{\csc^{-1}(a + bx)}{x^4} dx = \int \frac{\operatorname{acsc}(a + bx)}{x^4} dx$$

input

```
integrate(acsc(b*x+a)/x**4,x)
```

output

```
Integral(acsc(a + b*x)/x**4, x)
```

**Maxima [F]**

$$\int \frac{\csc^{-1}(a + bx)}{x^4} dx = \int \frac{\operatorname{arccsc}(bx + a)}{x^4} dx$$

input

```
integrate(arccsc(b*x+a)/x^4,x, algorithm="maxima")
```

output

```
-1/3*(3*x^3*integrate(1/3*(b^2*x + a*b)*e^(1/2*log(b*x + a + 1) + 1/2*log(
b*x + a - 1))/(b^2*x^5 + 2*a*b*x^4 + (a^2 - 1)*x^3 + (b^2*x^5 + 2*a*b*x^4
+ (a^2 - 1)*x^3)*e^(log(b*x + a + 1) + log(b*x + a - 1))), x) + arctan2(1,
sqrt(b*x + a + 1)*sqrt(b*x + a - 1))/x^3
```

**Giac [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 450 vs.  $2(156) = 312$ .

Time = 0.18 (sec) , antiderivative size = 450, normalized size of antiderivative = 2.50

$$\int \frac{\csc^{-1}(a + bx)}{x^4} dx = -\frac{1}{3} b \left( \frac{(6a^4b^2 - 5a^2b^2 + 2b^2) \arctan\left(\frac{(bx+a)\left(\sqrt{-\frac{1}{(bx+a)^2} + 1} - 1\right) + a}{\sqrt{-a^2 + 1}}\right)}{(a^7 - 2a^5 + a^3)\sqrt{-a^2 + 1}} + \frac{4(bx+a)^3 a^3 b^2 \left(\sqrt{-\frac{1}{(bx+a)^2} + 1} - 1\right)}{(a^7 - 2a^5 + a^3)\sqrt{-a^2 + 1}} \right)$$

input `integrate(arccsc(b*x+a)/x^4,x, algorithm="giac")`

output

```
-1/3*b*((6*a^4*b^2 - 5*a^2*b^2 + 2*b^2)*arctan(((b*x + a)*(sqrt(-1/(b*x + a)^2 + 1) - 1) + a)/sqrt(-a^2 + 1))/((a^7 - 2*a^5 + a^3)*sqrt(-a^2 + 1)) + (4*(b*x + a)^3*a^3*b^2*(sqrt(-1/(b*x + a)^2 + 1) - 1)^3 + 10*(b*x + a)^2*a^4*b^2*(sqrt(-1/(b*x + a)^2 + 1) - 1)^2 - (b*x + a)^3*a*b^2*(sqrt(-1/(b*x + a)^2 + 1) - 1)^3 + (b*x + a)^2*a^2*b^2*(sqrt(-1/(b*x + a)^2 + 1) - 1)^2 + 16*(b*x + a)*a^3*b^2*(sqrt(-1/(b*x + a)^2 + 1) - 1) - 2*(b*x + a)^2*b^2*(sqrt(-1/(b*x + a)^2 + 1) - 1)^2 - 7*(b*x + a)*a*b^2*(sqrt(-1/(b*x + a)^2 + 1) - 1) + 5*a^2*b^2 - 2*b^2)/((a^6 - 2*a^4 + a^2)*((b*x + a)^2*(sqrt(-1/(b*x + a)^2 + 1) - 1)^2 + 2*(b*x + a)*a*(sqrt(-1/(b*x + a)^2 + 1) - 1) + 1)^2) + (3*a*b^2/(b*x + a) - 3*a^2*b^2/(b*x + a)^2 - b^2)*arcsin(-1/((b*x + a)*(a/(b*x + a) - 1) - a))/(a^3*(a/(b*x + a) - 1)^3))
```

**Mupad [F(-1)]**

Timed out.

$$\int \frac{\csc^{-1}(a + bx)}{x^4} dx = \int \frac{\operatorname{asin}\left(\frac{1}{a+bx}\right)}{x^4} dx$$

input `int(asin(1/(a + b*x))/x^4,x)`

output

```
int(asin(1/(a + b*x))/x^4, x)
```



**Reduce [F]**

$$\int \frac{\csc^{-1}(a + bx)}{x^4} dx = \int \frac{\operatorname{acsc}(bx + a)}{x^4} dx$$

input `int(acsc(b*x+a)/x^4,x)`

output `int(acsc(a + b*x)/x**4,x)`

### 3.26 $\int \frac{\csc^{-1}(a+bx)}{x^5} dx$

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#### Optimal result

Integrand size = 10, antiderivative size = 239

$$\int \frac{\csc^{-1}(a+bx)}{x^5} dx = -\frac{b(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{12a(1-a^2)x^3} + \frac{(3-8a^2)b^2(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{24a^2(1-a^2)^2x^2}$$

$$- \frac{(6-17a^2+26a^4)b^3(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{24a^3(1-a^2)^3x}$$

$$+ \frac{b^4 \csc^{-1}(a+bx)}{4a^4} - \frac{\csc^{-1}(a+bx)}{4x^4}$$

$$+ \frac{(2-7a^2+8a^4-8a^6)b^4 \arctan\left(\frac{a-\tan(\frac{1}{2}\csc^{-1}(a+bx))}{\sqrt{1-a^2}}\right)}{4a^4(1-a^2)^{7/2}}$$

output

```
-1/12*b*(b*x+a)*(1-1/(b*x+a)^2)^(1/2)/a/(-a^2+1)/x^3+1/24*(-8*a^2+3)*b^2*(
b*x+a)*(1-1/(b*x+a)^2)^(1/2)/a^2/(-a^2+1)^2/x^2-1/24*(26*a^4-17*a^2+6)*b^3
*(b*x+a)*(1-1/(b*x+a)^2)^(1/2)/a^3/(-a^2+1)^3/x+1/4*b^4*arccsc(b*x+a)/a^4-
1/4*arccsc(b*x+a)/x^4+1/4*(-8*a^6+8*a^4-7*a^2+2)*b^4*arctan((a-tan(1/2*arc
csc(b*x+a)))/(-a^2+1)^(1/2))/a^4/(-a^2+1)^(7/2)
```

**Mathematica [C] (verified)**

Result contains complex when optimal does not.

Time = 0.33 (sec) , antiderivative size = 307, normalized size of antiderivative = 1.28

$$\int \frac{\csc^{-1}(a + bx)}{x^5} dx$$

$$= \frac{1}{8} \left( \frac{b \sqrt{\frac{-1+a^2+2abx+b^2x^2}{(a+bx)^2}} (2a^7 - 6a^6bx + 3ab^2x^2 + 6b^3x^3 + a^3(2 - 6b^2x^2) + 2a^5(-2 + 9b^2x^2) + a^4bx(7 + 26b^2x^2) - a^2(bx + 17b^3x^3))}{3a^3(-1 + a^2)^3 x^3} - \frac{2 \csc^{-1}(a + bx)}{x^4} + \frac{2b^4 \arcsin\left(\frac{1}{a+bx}\right)}{a^4} + \frac{i(-2 + 7a^2 - 8a^4 + 8a^6) b^4 \log\left(\frac{16a^4(-1+a^2)^3 \left(\frac{i(-1+a^2+abx)}{\sqrt{1-a^2}} + (a+bx) \sqrt{\frac{-1+a^2+2abx+b^2x^2}{(a+bx)^2}}\right)}{(-2+7a^2-8a^4+8a^6)b^4x}\right)}{a^4(1-a^2)^{7/2}} \right)$$

input `Integrate[ArcCsc[a + b*x]/x^5,x]`

output `((b*Sqrt[(-1 + a^2 + 2*a*b*x + b^2*x^2)/(a + b*x)^2]*(2*a^7 - 6*a^6*b*x + 3*a*b^2*x^2 + 6*b^3*x^3 + a^3*(2 - 6*b^2*x^2) + 2*a^5*(-2 + 9*b^2*x^2) + a^4*b*x*(7 + 26*b^2*x^2) - a^2*(b*x + 17*b^3*x^3)))/(3*a^3*(-1 + a^2)^3*x^3) - (2*ArcCsc[a + b*x])/x^4 + (2*b^4*ArcSin[(a + b*x)^(-1)])/a^4 + (I*(-2 + 7*a^2 - 8*a^4 + 8*a^6)*b^4*Log[(16*a^4*(-1 + a^2)^3*((I*(-1 + a^2 + a*b*x))/Sqrt[1 - a^2] + (a + b*x)*Sqrt[(-1 + a^2 + 2*a*b*x + b^2*x^2)/(a + b*x)^2])])/((-2 + 7*a^2 - 8*a^4 + 8*a^6)*b^4*x)]/(a^4*(1 - a^2)^(7/2)))/8`

**Rubi [A] (verified)**

Time = 1.42 (sec) , antiderivative size = 305, normalized size of antiderivative = 1.28, number of steps used = 19, number of rules used = 18,  $\frac{\text{number of rules}}{\text{integrand size}} = 1.800$ , Rules used = {5782, 25, 4927, 3042, 4272, 3042, 4548, 3042, 4548, 27, 3042, 4407, 3042, 4318, 3042, 3139, 1083, 217}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{\csc^{-1}(a+bx)}{x^5} dx \\
 & \quad \downarrow \text{5782} \\
 & -b^4 \int \frac{(a+bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)}{b^5 x^5} d \csc^{-1}(a+bx) \\
 & \quad \downarrow \text{25} \\
 & b^4 \int -\frac{(a+bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)}{b^5 x^5} d \csc^{-1}(a+bx) \\
 & \quad \downarrow \text{4927} \\
 & -b^4 \left( \frac{\csc^{-1}(a+bx)}{4b^4 x^4} - \frac{1}{4} \int \frac{1}{b^4 x^4} d \csc^{-1}(a+bx) \right) \\
 & \quad \downarrow \text{3042} \\
 & -b^4 \left( \frac{\csc^{-1}(a+bx)}{4b^4 x^4} - \frac{1}{4} \int \frac{1}{(a - \csc(\csc^{-1}(a+bx)))^4} d \csc^{-1}(a+bx) \right) \\
 & \quad \downarrow \text{4272} \\
 & -b^4 \left( \frac{1}{4} \left( \frac{(a+bx) \sqrt{1 - \frac{1}{(a+bx)^2}}}{3a(1-a^2)b^3 x^3} - \frac{\int -\frac{-2(a+bx)^2 - 3a(a+bx) + 3(1-a^2)}{b^3 x^3} d \csc^{-1}(a+bx)}{3a(1-a^2)} \right) + \frac{\csc^{-1}(a+bx)}{4b^4 x^4} \right) \\
 & \quad \downarrow \text{3042} \\
 & -b^4 \left( \frac{1}{4} \left( \frac{(a+bx) \sqrt{1 - \frac{1}{(a+bx)^2}}}{3a(1-a^2)b^3 x^3} - \frac{\int \frac{-2 \csc(\csc^{-1}(a+bx))^2 - 3a \csc(\csc^{-1}(a+bx)) + 3(1-a^2)}{(a - \csc(\csc^{-1}(a+bx)))^3} d \csc^{-1}(a+bx)}{3a(1-a^2)} \right) + \frac{\csc^{-1}(a+bx)}{4b^4 x^4} \right)
 \end{aligned}$$

$$\begin{aligned} & \downarrow 4548 \\ -b^4 & \left( \frac{1}{4} \left( \frac{(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{3a(1-a^2)b^3x^3} - \frac{\int \frac{6(1-a^2)^2 - (3-8a^2)(a+bx)^2 - 2a(1-6a^2)(a+bx)}{b^2x^2} d\csc^{-1}(a+bx)}{2a(1-a^2)} + \frac{(3-8a^2)\sqrt{1-\frac{1}{(a+bx)^2}}(a+bx)}{2a(1-a^2)b^2x^2} \right) \right) \end{aligned}$$

$$\begin{aligned} & \downarrow 3042 \\ -b^4 & \left( \frac{1}{4} \left( \frac{(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{3a(1-a^2)b^3x^3} - \frac{\int \frac{6(1-a^2)^2 - (3-8a^2)\csc(\csc^{-1}(a+bx))^2 - 2a(1-6a^2)\csc(\csc^{-1}(a+bx))}{(a-\csc(\csc^{-1}(a+bx)))^2} d\csc^{-1}(a+bx)}{2a(1-a^2)} + \frac{(3-8a^2)\sqrt{1-\frac{1}{(a+bx)^2}}(a+bx)}{2a(1-a^2)b^2x^2} \right) \right) \end{aligned}$$

$$\begin{aligned} & \downarrow 4548 \\ -b^4 & \left( \frac{1}{4} \left( \frac{(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{3a(1-a^2)b^3x^3} - \frac{\int -\frac{3(2(1-a^2)^3 - a(6a^4 - 2a^2 + 1)(a+bx))}{bx} d\csc^{-1}(a+bx)}{a(1-a^2)} - \frac{(26a^4 - 17a^2 + 6)(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{a(1-a^2)bx} + \frac{(3-8a^2)\sqrt{1-\frac{1}{(a+bx)^2}}(a+bx)}{2a(1-a^2)b^2x^2} \right) \right) \end{aligned}$$

$$\begin{aligned} & \downarrow 27 \\ -b^4 & \left( \frac{1}{4} \left( \frac{(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{3a(1-a^2)b^3x^3} - \frac{\int -\frac{2(1-a^2)^3 - a(6a^4 - 2a^2 + 1)(a+bx)}{bx} d\csc^{-1}(a+bx)}{a(1-a^2)} - \frac{(26a^4 - 17a^2 + 6)(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{a(1-a^2)bx} + \frac{(3-8a^2)\sqrt{1-\frac{1}{(a+bx)^2}}(a+bx)}{2a(1-a^2)b^2x^2} \right) \right) \end{aligned}$$

$$\begin{aligned} & \downarrow 3042 \\ -b^4 & \left( \frac{1}{4} \left( \frac{(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{3a(1-a^2)b^3x^3} - \frac{\int \frac{2(1-a^2)^3 - a(6a^4 - 2a^2 + 1)\csc(\csc^{-1}(a+bx))}{a-\csc(\csc^{-1}(a+bx))} d\csc^{-1}(a+bx)}{a(1-a^2)} - \frac{(26a^4 - 17a^2 + 6)(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{a(1-a^2)bx} + \frac{(3-8a^2)\sqrt{1-\frac{1}{(a+bx)^2}}(a+bx)}{2a(1-a^2)b^2x^2} \right) \right) \end{aligned}$$

$$\begin{array}{c}
 \downarrow 4407 \\
 -b^4 \left( \frac{1}{4} \left( \frac{(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{3a(1-a^2)b^3x^3} - \frac{3 \left( \frac{(-8a^6+8a^4-7a^2+2) \int -\frac{a+bx}{bx} d \csc^{-1}(a+bx)}{a} + \frac{2(1-a^2)^3 \csc^{-1}(a+bx)}{a} \right)}{a(1-a^2)} - \frac{(26a^4-17a^2+6)(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{a(1-a^2)bx} \right)}{2a(1-a^2)} \right) \\
 \hline
 3a(1-a^2)
 \end{array}$$

$$\begin{array}{c}
 \downarrow 3042 \\
 -b^4 \left( \frac{1}{4} \left( \frac{(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{3a(1-a^2)b^3x^3} - \frac{3 \left( \frac{(-8a^6+8a^4-7a^2+2) \int \frac{\csc(\csc^{-1}(a+bx))}{a-\csc(\csc^{-1}(a+bx))} d \csc^{-1}(a+bx)}{a} + \frac{2(1-a^2)^3 \csc^{-1}(a+bx)}{a} \right)}{a(1-a^2)} - \frac{(26a^4-17a^2+6)(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{a(1-a^2)bx} \right)}{2a(1-a^2)} \right) \\
 \hline
 3a(1-a^2)
 \end{array}$$

$$\begin{array}{c}
 \downarrow 4318 \\
 -b^4 \left( \frac{1}{4} \left( \frac{(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{3a(1-a^2)b^3x^3} - \frac{3 \left( \frac{2(1-a^2)^3 \csc^{-1}(a+bx)}{a} - \frac{(-8a^6+8a^4-7a^2+2) \int \frac{1}{1-\frac{a}{a+bx}} d \csc^{-1}(a+bx)}{a} \right)}{a(1-a^2)} - \frac{(26a^4-17a^2+6)(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{a(1-a^2)bx} \right)}{2a(1-a^2)} \right) \\
 \hline
 3a(1-a^2)
 \end{array}$$

$$\begin{array}{c}
 \downarrow 3042 \\
 -b^4 \left( \frac{1}{4} \left( \frac{(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{3a(1-a^2)b^3x^3} - \frac{3 \left( \frac{2(1-a^2)^3 \csc^{-1}(a+bx)}{a} - \frac{(-8a^6+8a^4-7a^2+2) \int \frac{1}{1-a \sin(\csc^{-1}(a+bx))} d \csc^{-1}(a+bx)}{a} \right)}{a(1-a^2)} - \frac{(26a^4-17a^2+6)(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{a(1-a^2)bx} \right)}{2a(1-a^2)} \right) \\
 \hline
 3a(1-a^2)
 \end{array}$$

$$\begin{array}{c}
 \downarrow \text{3139} \\
 -b^4 \left( \frac{1}{4} \left( \frac{(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{3a(1-a^2)b^3x^3} - \frac{\frac{2(1-a^2)^3 \csc^{-1}(a+bx)}{a} - \frac{2(-8a^6+8a^4-7a^2+2) \int \frac{1}{\tan^2(\frac{1}{2} \csc^{-1}(a+bx)) - 2a \tan(\frac{1}{2} \csc^{-1}(a+bx)) + 1} d \tan(\frac{1}{2} \csc^{-1}(a+bx))}{a(1-a^2)}}{2a(1-a^2)} \right) \right) \\
 \frac{\phantom{-b^4} \left( \frac{1}{4} \left( \frac{(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{3a(1-a^2)b^3x^3} - \frac{\frac{2(1-a^2)^3 \csc^{-1}(a+bx)}{a} - \frac{2(-8a^6+8a^4-7a^2+2) \int \frac{1}{\tan^2(\frac{1}{2} \csc^{-1}(a+bx)) - 2a \tan(\frac{1}{2} \csc^{-1}(a+bx)) + 1} d \tan(\frac{1}{2} \csc^{-1}(a+bx))}{a(1-a^2)}}{2a(1-a^2)} \right) \right)}{3a(1-a^2)}
 \end{array}$$

$$\begin{array}{c}
 \downarrow \text{1083} \\
 -b^4 \left( \frac{1}{4} \left( \frac{(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{3a(1-a^2)b^3x^3} - \frac{\frac{4(-8a^6+8a^4-7a^2+2) \int \frac{1}{(2 \tan(\frac{1}{2} \csc^{-1}(a+bx)) - 2a)^2 - 4(1-a^2)} d(2 \tan(\frac{1}{2} \csc^{-1}(a+bx)) - 2a)}{a(1-a^2)} + \frac{2(1-a^2)^3 \csc^{-1}(a+bx)}{a}}{2a(1-a^2)} \right) \right) \\
 \frac{\phantom{-b^4} \left( \frac{1}{4} \left( \frac{(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{3a(1-a^2)b^3x^3} - \frac{\frac{4(-8a^6+8a^4-7a^2+2) \int \frac{1}{(2 \tan(\frac{1}{2} \csc^{-1}(a+bx)) - 2a)^2 - 4(1-a^2)} d(2 \tan(\frac{1}{2} \csc^{-1}(a+bx)) - 2a)}{a(1-a^2)} + \frac{2(1-a^2)^3 \csc^{-1}(a+bx)}{a}}{2a(1-a^2)} \right) \right)}{3a(1-a^2)}
 \end{array}$$

$$\begin{array}{c}
 \downarrow \text{217} \\
 -b^4 \left( \frac{1}{4} \left( \frac{(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{3a(1-a^2)b^3x^3} - \frac{\frac{(3-8a^2)\sqrt{1-\frac{1}{(a+bx)^2}}(a+bx)}{2a(1-a^2)b^2x^2} + \frac{\frac{2(1-a^2)^3 \csc^{-1}(a+bx)}{a} - \frac{2(-8a^6+8a^4-7a^2+2) \arctan\left(\frac{2 \tan(\frac{1}{2} \csc^{-1}(a+bx))}{a\sqrt{1-a^2}}\right)}{a(1-a^2)}}{2a(1-a^2)}}{3a(1-a^2)} \right) \right) \\
 \frac{\phantom{-b^4} \left( \frac{1}{4} \left( \frac{(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}}{3a(1-a^2)b^3x^3} - \frac{\frac{(3-8a^2)\sqrt{1-\frac{1}{(a+bx)^2}}(a+bx)}{2a(1-a^2)b^2x^2} + \frac{\frac{2(1-a^2)^3 \csc^{-1}(a+bx)}{a} - \frac{2(-8a^6+8a^4-7a^2+2) \arctan\left(\frac{2 \tan(\frac{1}{2} \csc^{-1}(a+bx))}{a\sqrt{1-a^2}}\right)}{a(1-a^2)}}{2a(1-a^2)}}{3a(1-a^2)} \right) \right)}{3a(1-a^2)}
 \end{array}$$

input

```
Int[ArcCsc[a + b*x]/x^5,x]
```

output

$$\begin{aligned}
& -(b^4 * (\text{ArcCsc}[a + b*x] / (4*b^4*x^4) + (((a + b*x)*\text{Sqrt}[1 - (a + b*x)^{-2}])) \\
& / (3*a*(1 - a^2)*b^3*x^3) - (((3 - 8*a^2)*(a + b*x)*\text{Sqrt}[1 - (a + b*x)^{-2}]) \\
& ) / (2*a*(1 - a^2)*b^2*x^2) + (-(((6 - 17*a^2 + 26*a^4)*(a + b*x)*\text{Sqrt}[1 - \\
& (a + b*x)^{-2}])) / (a*(1 - a^2)*b*x)) + (3*((2*(1 - a^2)^3*\text{ArcCsc}[a + b*x]) / \\
& a - (2*(2 - 7*a^2 + 8*a^4 - 8*a^6)*\text{ArcTan}[(-2*a + 2*\text{Tan}[\text{ArcCsc}[a + b*x] / 2] \\
& ) / (2*\text{Sqrt}[1 - a^2])])) / (a*\text{Sqrt}[1 - a^2])) / (a*(1 - a^2)) / (2*a*(1 - a^2)) / \\
& (3*a*(1 - a^2)) / 4)
\end{aligned}$$

### Defintions of rubi rules used

rule 25 `Int[-(Fx_), x_Symbol] := Simp[Identity[-1] Int[Fx, x], x]`

rule 27 `Int[(a_)*(Fx_), x_Symbol] := Simp[a Int[Fx, x], x] /; FreeQ[a, x] && !MatchQ[Fx, (b_)*(Gx_)] /; FreeQ[b, x]`

rule 217 `Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(-Rt[-a, 2]*Rt[-b, 2])^(-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] && (LtQ[a, 0] || LtQ[b, 0])`

rule 1083 `Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] := Simp[-2 Subst[Int[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c}, x]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 3139 `Int[((a_) + (b_.)*sin[(c_.) + (d_.)*(x_)])^(-1), x_Symbol] := With[{e = FreeFactors[Tan[(c + d*x)/2], x]}, Simp[2*(e/d) Subst[Int[1/(a + 2*b*e*x + a*e^2*x^2), x], x, Tan[(c + d*x)/2]/e], x] /; FreeQ[{a, b, c, d}, x] && NeQ[a^2 - b^2, 0]`



rule 4272

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_.))^(n_), x_Symbol] := Simp[b^2*Cot[
c + d*x]*((a + b*Csc[c + d*x])^(n + 1)/(a*d*(n + 1)*(a^2 - b^2))), x] + Sim
p[1/(a*(n + 1)*(a^2 - b^2)) Int[(a + b*Csc[c + d*x])^(n + 1)*Simp[(a^2 -
b^2)*(n + 1) - a*b*(n + 1)*Csc[c + d*x] + b^2*(n + 2)*Csc[c + d*x]^2, x], x
], x] /; FreeQ[{a, b, c, d}, x] && NeQ[a^2 - b^2, 0] && LtQ[n, -1] && Integ
erQ[2*n]
```

rule 4318

```
Int[csc[(e_.) + (f_.)*(x_)]/(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_.)), x_Symbo
l] := Simp[1/b Int[1/(1 + (a/b)*Sin[e + f*x]), x], x] /; FreeQ[{a, b, e,
f}, x] && NeQ[a^2 - b^2, 0]
```

rule 4407

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.) + (c_.))/(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_.)), x_Symbol] := Simp[c*(x/a), x] - Simp[(b*c - a*d)/a Int[Csc[e + f*
x]/(a + b*Csc[e + f*x]), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b*c
- a*d, 0]
```

rule 4548

```
Int[((A_.) + csc[(e_.) + (f_.)*(x_)]*(B_.) + csc[(e_.) + (f_.)*(x_)]^2*(C_.
))*((csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_.))^(m_)), x_Symbol] := Simp[(A*b^2 -
a*b*B + a^2*C)*Cot[e + f*x]*((a + b*Csc[e + f*x])^(m + 1)/(a*f*(m + 1)*(a^2
- b^2))), x] + Simp[1/(a*(m + 1)*(a^2 - b^2)) Int[(a + b*Csc[e + f*x])^(
m + 1)*Simp[A*(a^2 - b^2)*(m + 1) - a*(A*b - a*B + b*C)*(m + 1)*Csc[e + f*x
] + (A*b^2 - a*b*B + a^2*C)*(m + 2)*Csc[e + f*x]^2, x], x], x] /; FreeQ[{a,
b, e, f, A, B, C}, x] && NeQ[a^2 - b^2, 0] && LtQ[m, -1]
```

rule 4927

```
Int[Cot[(c_.) + (d_.)*(x_)]*Csc[(c_.) + (d_.)*(x_)]*(Csc[(c_.) + (d_.)*(x_
)]*(b_.) + (a_.))^(n_.)*((e_.) + (f_.)*(x_))^(m_.), x_Symbol] := Simp[(-e +
f*x)^m*((a + b*Csc[c + d*x])^(n + 1)/(b*d*(n + 1))), x] + Simp[f*(m/(b*d*(
n + 1))) Int[(e + f*x)^(m - 1)*(a + b*Csc[c + d*x])^(n + 1), x], x] /; Fr
eeQ[{a, b, c, d, e, f, n}, x] && IGtQ[m, 0] && NeQ[n, -1]
```

rule 5782

```
Int[((a_.) + ArcCsc[(c_.) + (d_.)*(x_)]*(b_.))^(p_.)*((e_.) + (f_.)*(x_))^(m
_.), x_Symbol] := Simp[-(d^(m + 1))^(-1) Subst[Int[(a + b*x)^p*Csc[x]*Cot
[x]*(d*e - c*f + f*Csc[x])^m, x], x, ArcCsc[c + d*x]], x] /; FreeQ[{a, b, c
, d, e, f}, x] && IGtQ[p, 0] && IntegerQ[m]
```

**Maple [B] (verified)**

Leaf count of result is larger than twice the leaf count of optimal. 794 vs.  $2(215) = 430$ .

Time = 0.49 (sec) , antiderivative size = 795, normalized size of antiderivative = 3.33

method	result
parts	$-\frac{\operatorname{arccsc}(bx+a)}{4x^4} + \frac{b\sqrt{b^2x^2+2abx+a^2-1} \left(6(a^2-1)\right)^{\frac{3}{2}} \arctan\left(\frac{1}{\sqrt{b^2x^2+2abx+a^2-1}}\right) a^6 b^3 x^3 - 24 \ln\left(\frac{2a^2-2+2abx+2\sqrt{b^2x^2+2abx+a^2-1}}{b^2x^2+2abx+a^2-1}\right)}{4x^4}$
derivativedivides	Expression too large to display
default	Expression too large to display

input `int(arccsc(b*x+a)/x^5,x,method=_RETURNVERBOSE)`

output

$$\begin{aligned}
 & -1/4*\operatorname{arccsc}(b*x+a)/x^4 + 1/24*b*(b^2*x^2+2*a*b*x+a^2-1)^{(1/2)}*(6*(a^2-1)^{(3/2)} \\
 & * \operatorname{arctan}(1/(b^2*x^2+2*a*b*x+a^2-1)^{(1/2)})*a^6*b^3*x^3 - 24*\ln(2*(a*b*x+(a^2-1)^{(1/2)} \\
 & *(b^2*x^2+2*a*b*x+a^2-1)^{(1/2)}+a^2-1)/x)*a^8*b^3*x^3 - 18*(a^2-1)^{(3/2)} \\
 & * \operatorname{arctan}(1/(b^2*x^2+2*a*b*x+a^2-1)^{(1/2)})*a^4*b^3*x^3 + 26*(a^2-1)^{(3/2)}* \\
 & (b^2*x^2+2*a*b*x+a^2-1)^{(1/2)}*a^5*b^2*x^2 + 48*\ln(2*(a*b*x+(a^2-1)^{(1/2)}*(b^2 \\
 & *x^2+2*a*b*x+a^2-1)^{(1/2)}+a^2-1)/x)*a^6*b^3*x^3 + 18*(a^2-1)^{(3/2)}* \operatorname{arctan}(1 \\
 & / (b^2*x^2+2*a*b*x+a^2-1)^{(1/2)})*a^2*b^3*x^3 - 8*(a^2-1)^{(3/2)}*(b^2*x^2+2*a*b \\
 & *x+a^2-1)^{(1/2)}*a^6*b*x+2*(a^2-1)^{(3/2)}*(b^2*x^2+2*a*b*x+a^2-1)^{(1/2)}*a^7- \\
 & 17*(a^2-1)^{(3/2)}*(b^2*x^2+2*a*b*x+a^2-1)^{(1/2)}*a^3*b^2*x^2 - 45*\ln(2*(a*b*x+ \\
 & (a^2-1)^{(1/2)}*(b^2*x^2+2*a*b*x+a^2-1)^{(1/2)}+a^2-1)/x)*a^4*b^3*x^3 - 6*b^3* \operatorname{ar} \\
 & \operatorname{ctan}(1/(b^2*x^2+2*a*b*x+a^2-1)^{(1/2)})*x^3*(a^2-1)^{(3/2)} + 11*(a^2-1)^{(3/2)}*( \\
 & b^2*x^2+2*a*b*x+a^2-1)^{(1/2)}*a^4*b*x-4*(a^2-1)^{(3/2)}*(b^2*x^2+2*a*b*x+a^2- \\
 & 1)^{(1/2)}*a^5+6*(a^2-1)^{(3/2)}*(b^2*x^2+2*a*b*x+a^2-1)^{(1/2)}*a*b^2*x^2+27*\ln \\
 & (2*(a*b*x+(a^2-1)^{(1/2)}*(b^2*x^2+2*a*b*x+a^2-1)^{(1/2)}+a^2-1)/x)*a^2*b^3*x^ \\
 & 3-3*(a^2-1)^{(3/2)}*(b^2*x^2+2*a*b*x+a^2-1)^{(1/2)}*a^2*b*x+2*(a^2-1)^{(3/2)}*(b \\
 & ^2*x^2+2*a*b*x+a^2-1)^{(1/2)}*a^3-6*b^3*\ln(2*(a*b*x+(a^2-1)^{(1/2)}*(b^2*x^2+ \\
 & *a*b*x+a^2-1)^{(1/2)}+a^2-1)/x)*x^3)/((b^2*x^2+2*a*b*x+a^2-1)/(b*x+a)^2)^{(1/ \\
 & 2)/(b*x+a)/a^4/(a^2-1)^{(9/2)}/x^3
 \end{aligned}$$

**Fricas [A] (verification not implemented)**

Time = 0.20 (sec) , antiderivative size = 673, normalized size of antiderivative = 2.82

$$\int \frac{\csc^{-1}(a+bx)}{x^5} dx$$

$$= \left[ \frac{3(8a^6 - 8a^4 + 7a^2 - 2)\sqrt{a^2 - 1}b^4x^4 \log\left(\frac{a^2bx + a^3 + \sqrt{b^2x^2 + 2abx + a^2 - 1}(a^2 - \sqrt{a^2 - 1}a - 1) - (abx + a^2 - 1)\sqrt{a^2 - 1} - a}{x}\right)}{\dots} \right]$$

input `integrate(arccsc(b*x+a)/x^5,x, algorithm="fricas")`

output

```
[1/24*(3*(8*a^6 - 8*a^4 + 7*a^2 - 2)*sqrt(a^2 - 1)*b^4*x^4*log((a^2*b*x +
a^3 + sqrt(b^2*x^2 + 2*a*b*x + a^2 - 1)*(a^2 - sqrt(a^2 - 1)*a - 1) - (a*b
*x + a^2 - 1)*sqrt(a^2 - 1) - a)/x) - 12*(a^8 - 4*a^6 + 6*a^4 - 4*a^2 + 1)
*b^4*x^4*arctan(-b*x - a + sqrt(b^2*x^2 + 2*a*b*x + a^2 - 1)) + (26*a^7 -
43*a^5 + 23*a^3 - 6*a)*b^4*x^4 - 6*(a^12 - 4*a^10 + 6*a^8 - 4*a^6 + a^4)*a
rccsc(b*x + a) + ((26*a^7 - 43*a^5 + 23*a^3 - 6*a)*b^3*x^3 - (8*a^8 - 19*a
^6 + 14*a^4 - 3*a^2)*b^2*x^2 + 2*(a^9 - 3*a^7 + 3*a^5 - a^3)*b*x)*sqrt(b^2
*x^2 + 2*a*b*x + a^2 - 1))/((a^12 - 4*a^10 + 6*a^8 - 4*a^6 + a^4)*x^4), 1/
24*(6*(8*a^6 - 8*a^4 + 7*a^2 - 2)*sqrt(-a^2 + 1)*b^4*x^4*arctan(-(sqrt(-a
^2 + 1)*b*x - sqrt(b^2*x^2 + 2*a*b*x + a^2 - 1)*sqrt(-a^2 + 1))/(a^2 - 1))
- 12*(a^8 - 4*a^6 + 6*a^4 - 4*a^2 + 1)*b^4*x^4*arctan(-b*x - a + sqrt(b^2*
x^2 + 2*a*b*x + a^2 - 1)) + (26*a^7 - 43*a^5 + 23*a^3 - 6*a)*b^4*x^4 - 6*(
a^12 - 4*a^10 + 6*a^8 - 4*a^6 + a^4)*arccsc(b*x + a) + ((26*a^7 - 43*a^5 +
23*a^3 - 6*a)*b^3*x^3 - (8*a^8 - 19*a^6 + 14*a^4 - 3*a^2)*b^2*x^2 + 2*(a
^9 - 3*a^7 + 3*a^5 - a^3)*b*x)*sqrt(b^2*x^2 + 2*a*b*x + a^2 - 1))/((a^12 -
4*a^10 + 6*a^8 - 4*a^6 + a^4)*x^4)]
```

**Sympy [F]**

$$\int \frac{\csc^{-1}(a+bx)}{x^5} dx = \int \frac{\operatorname{acsc}(a+bx)}{x^5} dx$$

input `integrate(acsc(b*x+a)/x**5,x)`

output `Integral(acsc(a + b*x)/x**5, x)`

### Maxima [F]

$$\int \frac{\csc^{-1}(a + bx)}{x^5} dx = \int \frac{\operatorname{arccsc}(bx + a)}{x^5} dx$$

input `integrate(arccsc(b*x+a)/x^5,x, algorithm="maxima")`

output `-1/4*(4*x^4*integrate(1/4*(b^2*x + a*b)*e^(1/2*log(b*x + a + 1) + 1/2*log(b*x + a - 1))/(b^2*x^6 + 2*a*b*x^5 + (a^2 - 1)*x^4 + (b^2*x^6 + 2*a*b*x^5 + (a^2 - 1)*x^4)*e^(log(b*x + a + 1) + log(b*x + a - 1))), x) + arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1))/x^4`

### Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 841 vs. 2(209) = 418.

Time = 0.19 (sec) , antiderivative size = 841, normalized size of antiderivative = 3.52

$$\int \frac{\csc^{-1}(a + bx)}{x^5} dx = \text{Too large to display}$$

input `integrate(arccsc(b*x+a)/x^5,x, algorithm="giac")`

output

```

1/12*b*(3*(8*a^6*b^3 - 8*a^4*b^3 + 7*a^2*b^3 - 2*b^3)*arctan(((b*x + a)*(s
qrt(-1/(b*x + a)^2 + 1) - 1) + a)/sqrt(-a^2 + 1))/((a^10 - 3*a^8 + 3*a^6 -
a^4)*sqrt(-a^2 + 1)) + (18*(b*x + a)^5*a^5*b^3*(sqrt(-1/(b*x + a)^2 + 1)
- 1)^5 + 84*(b*x + a)^4*a^6*b^3*(sqrt(-1/(b*x + a)^2 + 1) - 1)^4 + 104*(b*
x + a)^3*a^7*b^3*(sqrt(-1/(b*x + a)^2 + 1) - 1)^3 - 6*(b*x + a)^5*a^3*b^3*
(sqrt(-1/(b*x + a)^2 + 1) - 1)^5 - 12*(b*x + a)^4*a^4*b^3*(sqrt(-1/(b*x +
a)^2 + 1) - 1)^4 + 88*(b*x + a)^3*a^5*b^3*(sqrt(-1/(b*x + a)^2 + 1) - 1)^3
+ 3*(b*x + a)^5*a*b^3*(sqrt(-1/(b*x + a)^2 + 1) - 1)^5 + 228*(b*x + a)^2*
a^6*b^3*(sqrt(-1/(b*x + a)^2 + 1) - 1)^2 - 3*(b*x + a)^4*a^2*b^3*(sqrt(-1/
(b*x + a)^2 + 1) - 1)^4 - 78*(b*x + a)^3*a^3*b^3*(sqrt(-1/(b*x + a)^2 + 1)
- 1)^3 - 114*(b*x + a)^2*a^4*b^3*(sqrt(-1/(b*x + a)^2 + 1) - 1)^2 + 6*(b*
x + a)^4*b^3*(sqrt(-1/(b*x + a)^2 + 1) - 1)^4 + 138*(b*x + a)*a^5*b^3*(sqr
t(-1/(b*x + a)^2 + 1) - 1) + 36*(b*x + a)^3*a*b^3*(sqrt(-1/(b*x + a)^2 + 1)
- 1)^3 + 24*(b*x + a)^2*a^2*b^3*(sqrt(-1/(b*x + a)^2 + 1) - 1)^2 - 96*(b
*x + a)*a^3*b^3*(sqrt(-1/(b*x + a)^2 + 1) - 1) + 26*a^4*b^3 + 12*(b*x + a)
^2*b^3*(sqrt(-1/(b*x + a)^2 + 1) - 1)^2 + 33*(b*x + a)*a*b^3*(sqrt(-1/(b*x
+ a)^2 + 1) - 1) - 17*a^2*b^3 + 6*b^3)/((a^9 - 3*a^7 + 3*a^5 - a^3)*((b*x
+ a)^2*(sqrt(-1/(b*x + a)^2 + 1) - 1)^2 + 2*(b*x + a)*a*(sqrt(-1/(b*x + a)
)^2 + 1) - 1) + 1)^3) - 3*(4*a*b^3/(b*x + a) - 6*a^2*b^3/(b*x + a)^2 + 4*a
^3*b^3/(b*x + a)^3 - b^3)*arcsin(-1/((b*x + a)*(a/(b*x + a) - 1) - a))/...

```

**Mupad [F(-1)]**

Timed out.

$$\int \frac{\csc^{-1}(a + bx)}{x^5} dx = \int \frac{\operatorname{asin}\left(\frac{1}{a+bx}\right)}{x^5} dx$$

input

```
int(asin(1/(a + b*x))/x^5,x)
```

output

```
int(asin(1/(a + b*x))/x^5, x)
```

**Reduce [F]**

$$\int \frac{\csc^{-1}(a + bx)}{x^5} dx = \int \frac{\operatorname{acsc}(bx + a)}{x^5} dx$$

input `int(acsc(b*x+a)/x^5,x)`

output `int(acsc(a + b*x)/x**5,x)`

**3.27**       $\int x^3 \csc^{-1}(a + bx)^2 dx$ 

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## Optimal result

Integrand size = 12, antiderivative size = 366

$$\begin{aligned}
 \int x^3 \csc^{-1}(a+bx)^2 dx = & -\frac{ax}{b^3} + \frac{(a+bx)^2}{12b^4} + \frac{(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)}{3b^4} \\
 & + \frac{3a^2(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)}{b^4} \\
 & - \frac{a(a+bx)^2\sqrt{1-\frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)}{b^4} \\
 & + \frac{(a+bx)^3\sqrt{1-\frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)}{6b^4} - \frac{a^4 \csc^{-1}(a+bx)^2}{4b^4} \\
 & + \frac{1}{4}x^4 \csc^{-1}(a+bx)^2 - \frac{2a \csc^{-1}(a+bx) \operatorname{arctanh}\left(e^{i \csc^{-1}(a+bx)}\right)}{b^4} \\
 & - \frac{4a^3 \csc^{-1}(a+bx) \operatorname{arctanh}\left(e^{i \csc^{-1}(a+bx)}\right)}{b^4} + \frac{\log(a+bx)}{3b^4} \\
 & + \frac{3a^2 \log(a+bx)}{b^4} + \frac{ia \operatorname{PolyLog}\left(2, -e^{i \csc^{-1}(a+bx)}\right)}{b^4} \\
 & + \frac{2ia^3 \operatorname{PolyLog}\left(2, -e^{i \csc^{-1}(a+bx)}\right)}{b^4} \\
 & - \frac{ia \operatorname{PolyLog}\left(2, e^{i \csc^{-1}(a+bx)}\right)}{b^4} \\
 & - \frac{2ia^3 \operatorname{PolyLog}\left(2, e^{i \csc^{-1}(a+bx)}\right)}{b^4}
 \end{aligned}$$

output

```

-a*x/b^3+1/12*(b*x+a)^2/b^4+1/3*(b*x+a)*(1-1/(b*x+a)^2)^(1/2)*arccsc(b*x+a
)/b^4+3*a^2*(b*x+a)*(1-1/(b*x+a)^2)^(1/2)*arccsc(b*x+a)/b^4-a*(b*x+a)^2*(1
-1/(b*x+a)^2)^(1/2)*arccsc(b*x+a)/b^4+1/6*(b*x+a)^3*(1-1/(b*x+a)^2)^(1/2)*
arccsc(b*x+a)/b^4-1/4*a^4*arccsc(b*x+a)^2/b^4+1/4*x^4*arccsc(b*x+a)^2-2*a*
arccsc(b*x+a)*arctanh(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/b^4-4*a^3*arccsc(b*
x+a)*arctanh(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/b^4+1/3*ln(b*x+a)/b^4+3*a^2*
ln(b*x+a)/b^4+I*a*polylog(2,-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))/b^4-2*I*a^3*
polylog(2,I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/b^4+2*I*a^3*polylog(2,-I/(b*x+a
)-(1-1/(b*x+a)^2)^(1/2))/b^4-I*a*polylog(2,I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2
))/b^4

```



**Mathematica [A] (warning: unable to verify)**

Time = 3.22 (sec) , antiderivative size = 420, normalized size of antiderivative = 1.15

$$\int x^3 \csc^{-1}(a + bx)^2 dx$$

$$= \frac{-16(6a - 2(1 + 9a^2) \csc^{-1}(a + bx) + 3(a + 2a^3) \csc^{-1}(a + bx)^2) \cot\left(\frac{1}{2} \csc^{-1}(a + bx)\right) + 2(2 - 24a \csc^{-1}(a + bx) + 3(3 + 36a^2) \csc^{-1}(a + bx)^2) \operatorname{Csc}\left[\frac{\csc^{-1}(a + bx)}{2}\right]^2 + 3 \csc^{-1}(a + bx)^2 \operatorname{Csc}\left[\frac{\csc^{-1}(a + bx)}{2}\right]^4 - (2 \csc^{-1}(a + bx) (-1 + 6a \csc^{-1}(a + bx)) \operatorname{Csc}\left[\frac{\csc^{-1}(a + bx)}{2}\right]^4) / (a + bx) - 64(1 + 9a^2) \operatorname{Log}[(a + bx)^{-1}] + 192(a + 2a^3) (\operatorname{ArcCsc}[a + bx] (\operatorname{Log}[1 - E^{(I \operatorname{ArcCsc}[a + bx])}] - \operatorname{Log}[1 + E^{(I \operatorname{ArcCsc}[a + bx])}])) + I (\operatorname{PolyLog}[2, -E^{(I \operatorname{ArcCsc}[a + bx])}] - \operatorname{PolyLog}[2, E^{(I \operatorname{ArcCsc}[a + bx])}])) + 2(2 + 24a \csc^{-1}(a + bx) + 3(3 + 36a^2) \csc^{-1}(a + bx)^2) \operatorname{Sec}\left[\frac{\csc^{-1}(a + bx)}{2}\right]^2 + 3 \csc^{-1}(a + bx)^2 \operatorname{Sec}\left[\frac{\csc^{-1}(a + bx)}{2}\right]^4 - 32(a + bx)^3 \csc^{-1}(a + bx) (1 + 6a \csc^{-1}(a + bx)) \operatorname{Sin}\left[\frac{\csc^{-1}(a + bx)}{2}\right]^4 - 16(6a + 2(1 + 9a^2) \csc^{-1}(a + bx) + 3(a + 2a^3) \csc^{-1}(a + bx)^2) \operatorname{Tan}\left[\frac{\csc^{-1}(a + bx)}{2}\right] / (192b^4)}$$

input

Integrate[x^3\*ArcCsc[a + b\*x]^2,x]

output

```
(-16*(6*a - 2*(1 + 9*a^2)*ArcCsc[a + b*x] + 3*(a + 2*a^3)*ArcCsc[a + b*x]^2)*Cot[ArcCsc[a + b*x]/2] + 2*(2 - 24*a*ArcCsc[a + b*x] + (3 + 36*a^2)*ArcCsc[a + b*x]^2)*Csc[ArcCsc[a + b*x]/2]^2 + 3*ArcCsc[a + b*x]^2*Csc[ArcCsc[a + b*x]/2]^4 - (2*ArcCsc[a + b*x]*(-1 + 6*a*ArcCsc[a + b*x])*Csc[ArcCsc[a + b*x]/2]^4)/(a + b*x) - 64*(1 + 9*a^2)*Log[(a + b*x)^(-1)] + 192*(a + 2*a^3)*(ArcCsc[a + b*x]*(Log[1 - E^(I*ArcCsc[a + b*x])] - Log[1 + E^(I*ArcCsc[a + b*x])])) + I*(PolyLog[2, -E^(I*ArcCsc[a + b*x])] - PolyLog[2, E^(I*ArcCsc[a + b*x])])) + 2*(2 + 24*a*ArcCsc[a + b*x] + (3 + 36*a^2)*ArcCsc[a + b*x]^2)*Sec[ArcCsc[a + b*x]/2]^2 + 3*ArcCsc[a + b*x]^2*Sec[ArcCsc[a + b*x]/2]^4 - 32*(a + b*x)^3*ArcCsc[a + b*x]*(1 + 6*a*ArcCsc[a + b*x])*Sin[ArcCsc[a + b*x]/2]^4 - 16*(6*a + 2*(1 + 9*a^2)*ArcCsc[a + b*x] + 3*(a + 2*a^3)*ArcCsc[a + b*x]^2)*Tan[ArcCsc[a + b*x]/2]/(192*b^4)
```

**Rubi [A] (verified)**Time = 0.70 (sec) , antiderivative size = 342, normalized size of antiderivative = 0.93, number of steps used = 7, number of rules used = 6,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {5782, 25, 4927, 3042, 4678, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x^3 \csc^{-1}(a + bx)^2 dx$$

$$\downarrow 5782$$

$$\begin{aligned}
 & \frac{\int b^3 x^3 (a + bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \operatorname{csc}^{-1}(a + bx)^2 d \operatorname{csc}^{-1}(a + bx)}{b^4} \\
 & \quad \downarrow 25 \\
 & \frac{\int -b^3 x^3 (a + bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \operatorname{csc}^{-1}(a + bx)^2 d \operatorname{csc}^{-1}(a + bx)}{b^4} \\
 & \quad \downarrow 4927 \\
 & \frac{\frac{1}{2} \int b^4 x^4 \operatorname{csc}^{-1}(a + bx) d \operatorname{csc}^{-1}(a + bx) - \frac{1}{4} b^4 x^4 \operatorname{csc}^{-1}(a + bx)^2}{b^4} \\
 & \quad \downarrow 3042 \\
 & \frac{\frac{1}{2} \int \operatorname{csc}^{-1}(a + bx) (a - \operatorname{csc}(\operatorname{csc}^{-1}(a + bx)))^4 d \operatorname{csc}^{-1}(a + bx) - \frac{1}{4} b^4 x^4 \operatorname{csc}^{-1}(a + bx)^2}{b^4} \\
 & \quad \downarrow 4678 \\
 & \frac{\frac{1}{2} \int (\operatorname{csc}^{-1}(a + bx) a^4 - 4(a + bx) \operatorname{csc}^{-1}(a + bx) a^3 + 6(a + bx)^2 \operatorname{csc}^{-1}(a + bx) a^2 - 4(a + bx)^3 \operatorname{csc}^{-1}(a + bx) a + \dots)}{b^4} \\
 & \quad \downarrow 2009 \\
 & \frac{-\frac{1}{4} b^4 x^4 \operatorname{csc}^{-1}(a + bx)^2 + \frac{1}{2} \left( \frac{1}{2} a^4 \operatorname{csc}^{-1}(a + bx)^2 + 8a^3 \operatorname{csc}^{-1}(a + bx) \operatorname{arctanh} \left( e^{i \operatorname{csc}^{-1}(a + bx)} \right) - 4ia^3 \operatorname{PolyLog} \left( 2, \dots \right) \right)}{b^4}
 \end{aligned}$$

input `Int[x^3*ArcCsc[a + b*x]^2,x]`

output `-((-1/4*(b^4*x^4*ArcCsc[a + b*x]^2) + (2*a*(a + b*x) - (a + b*x)^2/6 - (2*(a + b*x)*Sqrt[1 - (a + b*x)^(-2)]*ArcCsc[a + b*x])/3 - 6*a^2*(a + b*x)*Sqrt[1 - (a + b*x)^(-2)]*ArcCsc[a + b*x] + 2*a*(a + b*x)^2*Sqrt[1 - (a + b*x)^(-2)]*ArcCsc[a + b*x] - ((a + b*x)^3*Sqrt[1 - (a + b*x)^(-2)]*ArcCsc[a + b*x])/3 + (a^4*ArcCsc[a + b*x]^2)/2 + 4*a*ArcCsc[a + b*x]*ArcTanh[E^(I*ArcCsc[a + b*x])]) + 8*a^3*ArcCsc[a + b*x]*ArcTanh[E^(I*ArcCsc[a + b*x])]) + (2*Log[(a + b*x)^(-1)])/3 + 6*a^2*Log[(a + b*x)^(-1)] - (2*I)*a*PolyLog[2, -E^(I*ArcCsc[a + b*x])] - (4*I)*a^3*PolyLog[2, -E^(I*ArcCsc[a + b*x])] + (2*I)*a*PolyLog[2, E^(I*ArcCsc[a + b*x])] + (4*I)*a^3*PolyLog[2, E^(I*ArcCsc[a + b*x])])]/2)/b^4`

## Definitions of rubi rules used

rule 25	$\text{Int}[-(\text{Fx}_), \text{x\_Symbol}] \rightarrow \text{Simp}[\text{Identity}[-1] \quad \text{Int}[\text{Fx}, \text{x}], \text{x}]$
rule 2009	$\text{Int}[\text{u}_, \text{x\_Symbol}] \rightarrow \text{Simp}[\text{IntSum}[\text{u}, \text{x}], \text{x}] \text{ /; SumQ}[\text{u}]$
rule 3042	$\text{Int}[\text{u}_, \text{x\_Symbol}] \rightarrow \text{Int}[\text{DeactivateTrig}[\text{u}, \text{x}], \text{x}] \text{ /; FunctionOfTrigOfLinearQ}[\text{u}, \text{x}]$
rule 4678	$\text{Int}[(\text{csc}[(\text{e}_.) + (\text{f}_.) * (\text{x}_)] * (\text{b}_.) + (\text{a}_.)^{\text{(n}_.)} * ((\text{c}_.) + (\text{d}_.) * (\text{x}_))^{\text{(m}_.)}], \text{x\_Symbol}] \rightarrow \text{Int}[\text{ExpandIntegrand}[(\text{c} + \text{d} * \text{x})^{\text{m}}, (\text{a} + \text{b} * \text{Csc}[\text{e} + \text{f} * \text{x}])^{\text{n}}, \text{x}], \text{x}] \text{ /; FreeQ}[\{\text{a}, \text{b}, \text{c}, \text{d}, \text{e}, \text{f}, \text{m}\}, \text{x}] \&\& \text{IGtQ}[\text{m}, 0] \&\& \text{IGtQ}[\text{n}, 0]$
rule 4927	$\text{Int}[\text{Cot}[(\text{c}_.) + (\text{d}_.) * (\text{x}_)] * \text{Csc}[(\text{c}_.) + (\text{d}_.) * (\text{x}_)] * (\text{Csc}[(\text{c}_.) + (\text{d}_.) * (\text{x}_)] * (\text{b}_.) + (\text{a}_.)^{\text{(n}_.)} * ((\text{e}_.) + (\text{f}_.) * (\text{x}_))^{\text{(m}_.)}], \text{x\_Symbol}] \rightarrow \text{Simp}[(-(\text{e} + \text{f} * \text{x})^{\text{m}} * ((\text{a} + \text{b} * \text{Csc}[\text{c} + \text{d} * \text{x}])^{\text{(n} + 1)} / (\text{b} * \text{d} * (\text{n} + 1))))], \text{x}] + \text{Simp}[\text{f} * (\text{m} / (\text{b} * \text{d} * (\text{n} + 1))) \quad \text{Int}[(\text{e} + \text{f} * \text{x})^{\text{(m} - 1)} * (\text{a} + \text{b} * \text{Csc}[\text{c} + \text{d} * \text{x}])^{\text{(n} + 1)}, \text{x}], \text{x}] \text{ /; FreeQ}[\{\text{a}, \text{b}, \text{c}, \text{d}, \text{e}, \text{f}, \text{n}\}, \text{x}] \&\& \text{IGtQ}[\text{m}, 0] \&\& \text{NeQ}[\text{n}, -1]$
rule 5782	$\text{Int}[(\text{a}_.) + \text{ArcCsc}[(\text{c}_.) + (\text{d}_.) * (\text{x}_)] * (\text{b}_.)]^{\text{(p}_.)} * ((\text{e}_.) + (\text{f}_.) * (\text{x}_))^{\text{(m}_.)}], \text{x\_Symbol}] \rightarrow \text{Simp}[-(\text{d}^{\text{(m} + 1)})^{\text{(-1)}} \quad \text{Subst}[\text{Int}[(\text{a} + \text{b} * \text{x})^{\text{p}} * \text{Csc}[\text{x}] * \text{Cot}[\text{x}] * (\text{d} * \text{e} - \text{c} * \text{f} + \text{f} * \text{Csc}[\text{x}])^{\text{m}}, \text{x}], \text{x}, \text{ArcCsc}[\text{c} + \text{d} * \text{x}]], \text{x}] \text{ /; FreeQ}[\{\text{a}, \text{b}, \text{c}, \text{d}, \text{e}, \text{f}\}, \text{x}] \&\& \text{IGtQ}[\text{p}, 0] \&\& \text{IntegerQ}[\text{m}]$

## Maple [A] (verified)

Time = 1.14 (sec) , antiderivative size = 703, normalized size of antiderivative = 1.92

method	result
derivativedivides	$-(bx+a)a + \frac{(bx+a)^2}{12} - \frac{\ln\left(1 + \frac{i}{bx+a} + \sqrt{1 - \frac{1}{(bx+a)^2}}\right)}{3} + 3 \operatorname{arccsc}(bx+a) \sqrt{\frac{(bx+a)^2-1}{(bx+a)^2}} a^2 (bx+a) - \operatorname{arccsc}(bx+a) \sqrt{\frac{(bx+a)^2-1}{(bx+a)^2}}$
default	$-(bx+a)a + \frac{(bx+a)^2}{12} - \frac{\ln\left(1 + \frac{i}{bx+a} + \sqrt{1 - \frac{1}{(bx+a)^2}}\right)}{3} + 3 \operatorname{arccsc}(bx+a) \sqrt{\frac{(bx+a)^2-1}{(bx+a)^2}} a^2 (bx+a) - \operatorname{arccsc}(bx+a) \sqrt{\frac{(bx+a)^2-1}{(bx+a)^2}}$

input `int(x^3*arccsc(b*x+a)^2,x,method=_RETURNVERBOSE)`

output

$$\begin{aligned} & 1/b^4*(-(b*x+a)*a-1/3*\ln(I/(b*x+a)+(1-1/(b*x+a)^2)^{(1/2)}-1)-1/3*\ln(1+I/(b*x+a)+(1-1/(b*x+a)^2)^{(1/2}))+1/12*(b*x+a)^2+2/3*\ln(I/(b*x+a)+(1-1/(b*x+a)^2)^{(1/2}))+3*arccsc(b*x+a)*(((b*x+a)^2-1)/(b*x+a)^2)^{(1/2)}*a^2*(b*x+a)-arccsc(b*x+a)*(((b*x+a)^2-1)/(b*x+a)^2)^{(1/2)}*a*(b*x+a)^2+1/4*arccsc(b*x+a)^2*(b*x+a)^4-1/3*I*arccsc(b*x+a)+6*\ln(I/(b*x+a)+(1-1/(b*x+a)^2)^{(1/2)})*a^2-3*\ln(1+I/(b*x+a)+(1-1/(b*x+a)^2)^{(1/2)})*a^2-3*\ln(I/(b*x+a)+(1-1/(b*x+a)^2)^{(1/2)}-1)*a^2-2*\ln(1+I/(b*x+a)+(1-1/(b*x+a)^2)^{(1/2)})*a^3*arccsc(b*x+a)+2*\ln(1-I/(b*x+a)-(1-1/(b*x+a)^2)^{(1/2)})*a^3*arccsc(b*x+a)+2*I*polylog(2,-I/(b*x+a)-(1-1/(b*x+a)^2)^{(1/2)})*a^3-2*I*polylog(2,I/(b*x+a)+(1-1/(b*x+a)^2)^{(1/2)})*a^3-\ln(1+I/(b*x+a)+(1-1/(b*x+a)^2)^{(1/2)})*a*arccsc(b*x+a)-arccsc(b*x+a)^2*a^3*(b*x+a)+3/2*arccsc(b*x+a)^2*a^2*(b*x+a)^2-arccsc(b*x+a)^2*a*(b*x+a)^3+1/6*arccsc(b*x+a)*(((b*x+a)^2-1)/(b*x+a)^2)^{(1/2)}*(b*x+a)^3-3*I*a^2*arccsc(b*x+a)+1/3*arccsc(b*x+a)*(((b*x+a)^2-1)/(b*x+a)^2)^{(1/2)}*(b*x+a)+\ln(1-I/(b*x+a)-(1-1/(b*x+a)^2)^{(1/2)})*a*arccsc(b*x+a)+I*polylog(2,-I/(b*x+a)-(1-1/(b*x+a)^2)^{(1/2)})*a-I*polylog(2,I/(b*x+a)+(1-1/(b*x+a)^2)^{(1/2)})*a \end{aligned}$$

### Fricas [F]

$$\int x^3 \csc^{-1}(a + bx)^2 dx = \int x^3 \operatorname{arccsc}(bx + a)^2 dx$$

input `integrate(x^3*arccsc(b*x+a)^2,x, algorithm="fricas")`

output `integral(x^3*arccsc(b*x + a)^2, x)`

**Sympy [F]**

$$\int x^3 \csc^{-1}(a + bx)^2 dx = \int x^3 \operatorname{acsc}^2(a + bx) dx$$

input `integrate(x**3*acsc(b*x+a)**2,x)`

output `Integral(x**3*acsc(a + b*x)**2, x)`

**Maxima [F]**

$$\int x^3 \csc^{-1}(a + bx)^2 dx = \int x^3 \operatorname{arccsc}(bx + a)^2 dx$$

input `integrate(x^3*arccsc(b*x+a)^2,x, algorithm="maxima")`

output `1/4*x^4*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1))^2 - 1/16*x^4*log(b^2*x^2 + 2*a*b*x + a^2)^2 + integrate(1/4*(2*sqrt(b*x + a + 1)*sqrt(b*x + a - 1)*b*x^4*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) - 4*(b^3*x^6 + 3*a*b^2*x^5 + (3*a^2 - 1)*b*x^4 + (a^3 - a)*x^3)*log(b*x + a)^2 + (b^3*x^6 + 2*a*b^2*x^5 + (a^2 - 1)*b*x^4 + 4*(b^3*x^6 + 3*a*b^2*x^5 + (3*a^2 - 1)*b*x^4 + (a^3 - a)*x^3)*log(b*x + a))*log(b^2*x^2 + 2*a*b*x + a^2))/(b^3*x^3 + 3*a*b^2*x^2 + a^3 + (3*a^2 - 1)*b*x - a), x)`

**Giac [F(-2)]**

Exception generated.

$$\int x^3 \csc^{-1}(a + bx)^2 dx = \text{Exception raised: RuntimeError}$$

input `integrate(x^3*arccsc(b*x+a)^2,x, algorithm="giac")`

output

```
Exception raised: RuntimeError >> an error occurred running a Giac command
:INPUT:sage2OUTPUT:sym2poly/r2sym(const gen & e,const index_m & i,const ve
cteur & l) Error: Bad Argument Value
```

**Mupad [F(-1)]**

Timed out.

$$\int x^3 \csc^{-1}(a + bx)^2 dx = \int x^3 \operatorname{asin}\left(\frac{1}{a + bx}\right)^2 dx$$

input

```
int(x^3*asin(1/(a + b*x))^2,x)
```

output

```
int(x^3*asin(1/(a + b*x))^2, x)
```

**Reduce [F]**

$$\int x^3 \csc^{-1}(a + bx)^2 dx = \int \operatorname{acsc}(bx + a)^2 x^3 dx$$

input

```
int(x^3*acsc(b*x+a)^2,x)
```

output

```
int(acsc(a + b*x)**2*x**3,x)
```

### 3.28 $\int x^2 \csc^{-1}(a + bx)^2 dx$

Optimal result	238
Mathematica [A] (warning: unable to verify)	239
Rubi [A] (verified)	240
Maple [A] (verified)	242
Fricas [F]	242
Sympy [F]	243
Maxima [F]	243
Giac [F]	243
Mupad [F(-1)]	244
Reduce [F]	244

#### Optimal result

Integrand size = 12, antiderivative size = 272

$$\begin{aligned}
 \int x^2 \csc^{-1}(a + bx)^2 dx = & \frac{x}{3b^2} - \frac{2a(a + bx) \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx)}{b^3} \\
 & + \frac{(a + bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx)}{3b^3} + \frac{a^3 \csc^{-1}(a + bx)^2}{3b^3} \\
 & + \frac{1}{3} x^3 \csc^{-1}(a + bx)^2 + \frac{2 \csc^{-1}(a + bx) \operatorname{arctanh}\left(e^{i \csc^{-1}(a+bx)}\right)}{3b^3} \\
 & + \frac{4a^2 \csc^{-1}(a + bx) \operatorname{arctanh}\left(e^{i \csc^{-1}(a+bx)}\right)}{b^3} \\
 & - \frac{2a \log(a + bx)}{b^3} - \frac{i \operatorname{PolyLog}\left(2, -e^{i \csc^{-1}(a+bx)}\right)}{3b^3} \\
 & - \frac{2ia^2 \operatorname{PolyLog}\left(2, -e^{i \csc^{-1}(a+bx)}\right)}{b^3} \\
 & + \frac{i \operatorname{PolyLog}\left(2, e^{i \csc^{-1}(a+bx)}\right)}{3b^3} + \frac{2ia^2 \operatorname{PolyLog}\left(2, e^{i \csc^{-1}(a+bx)}\right)}{b^3}
 \end{aligned}$$

output

```
1/3*x/b^2-2*a*(b*x+a)*(1-1/(b*x+a)^2)^(1/2)*arccsc(b*x+a)/b^3+1/3*(b*x+a)^
2*(1-1/(b*x+a)^2)^(1/2)*arccsc(b*x+a)/b^3+1/3*a^3*arccsc(b*x+a)^2/b^3+1/3*
x^3*arccsc(b*x+a)^2+2/3*arccsc(b*x+a)*arctanh(I/(b*x+a)+(1-1/(b*x+a)^2)^(1
/2))/b^3+4*a^2*arccsc(b*x+a)*arctanh(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/b^3-
2*a*ln(b*x+a)/b^3-1/3*I*polylog(2,-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))/b^3-2*
I*a^2*polylog(2,-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))/b^3+1/3*I*polylog(2,I/(b
*x+a)+(1-1/(b*x+a)^2)^(1/2))/b^3+2*I*a^2*polylog(2,I/(b*x+a)+(1-1/(b*x+a)^
2)^(1/2))/b^3
```

**Mathematica [A] (warning: unable to verify)**

Time = 2.94 (sec) , antiderivative size = 314, normalized size of antiderivative = 1.15

$$\int x^2 \csc^{-1}(a + bx)^2 dx =$$

$$\frac{-2(2 - 12a \csc^{-1}(a + bx) + (1 + 6a^2) \csc^{-1}(a + bx)^2) \cot\left(\frac{1}{2} \csc^{-1}(a + bx)\right) + 2 \csc^{-1}(a + bx) (-1 + \dots)}{\dots}$$

input

```
Integrate[x^2*ArcCsc[a + b*x]^2,x]
```

output

```
-1/24*(-2*(2 - 12*a*ArcCsc[a + b*x] + (1 + 6*a^2)*ArcCsc[a + b*x]^2)*Cot[ArcCsc[a + b*x]/2] + 2*ArcCsc[a + b*x]*(-1 + 3*a*ArcCsc[a + b*x])*Csc[ArcCsc[a + b*x]/2]^2 - (ArcCsc[a + b*x]^2*Csc[ArcCsc[a + b*x]/2]^4)/(2*(a + b*x)) - 48*a*Log[(a + b*x)^(-1)] + 8*(1 + 6*a^2)*(ArcCsc[a + b*x]*(Log[1 - E^(I*ArcCsc[a + b*x]]) - Log[1 + E^(I*ArcCsc[a + b*x]])]) + I*(PolyLog[2, -E^(I*ArcCsc[a + b*x])] - PolyLog[2, E^(I*ArcCsc[a + b*x])])) + 2*ArcCsc[a + b*x]*(1 + 3*a*ArcCsc[a + b*x])*Sec[ArcCsc[a + b*x]/2]^2 - 8*(a + b*x)^3*ArcCsc[a + b*x]^2*Sin[ArcCsc[a + b*x]/2]^4 - 2*(2 + 12*a*ArcCsc[a + b*x] + (1 + 6*a^2)*ArcCsc[a + b*x]^2)*Tan[ArcCsc[a + b*x]/2])/b^3
```



**Rubi [A] (verified)**

Time = 0.60 (sec) , antiderivative size = 255, normalized size of antiderivative = 0.94, number of steps used = 6, number of rules used = 5,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.417$ , Rules used = {5782, 4927, 3042, 4678, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int x^2 \csc^{-1}(a+bx)^2 dx \\
 & \quad \downarrow 5782 \\
 & \frac{\int b^2 x^2 (a+bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)^2 d \csc^{-1}(a+bx)}{b^3} \\
 & \quad \downarrow 4927 \\
 & \frac{-\frac{2}{3} \int -b^3 x^3 \csc^{-1}(a+bx) d \csc^{-1}(a+bx) - \frac{1}{3} b^3 x^3 \csc^{-1}(a+bx)^2}{b^3} \\
 & \quad \downarrow 3042 \\
 & \frac{-\frac{2}{3} \int \csc^{-1}(a+bx) (a - \csc(\csc^{-1}(a+bx)))^3 d \csc^{-1}(a+bx) - \frac{1}{3} b^3 x^3 \csc^{-1}(a+bx)^2}{b^3} \\
 & \quad \downarrow 4678 \\
 & \frac{-\frac{2}{3} \int (\csc^{-1}(a+bx)a^3 - 3(a+bx) \csc^{-1}(a+bx)a^2 + 3(a+bx)^2 \csc^{-1}(a+bx)a - (a+bx)^3 \csc^{-1}(a+bx)) d \csc^{-1}(a+bx)}{b^3} \\
 & \quad \downarrow 2009 \\
 & \frac{-\frac{1}{3} b^3 x^3 \csc^{-1}(a+bx)^2 - \frac{2}{3} \left( \frac{1}{2} a^3 \csc^{-1}(a+bx)^2 + 6a^2 \csc^{-1}(a+bx) \operatorname{arctanh}\left(e^{i \csc^{-1}(a+bx)}\right) - 3ia^2 \operatorname{PolyLog}\left(2, e^{i \csc^{-1}(a+bx)}\right)\right)}{b^3}
 \end{aligned}$$

input `Int[x^2*ArcCsc[a + b*x]^2,x]`

output

```

-((-1/3*(b^3*x^3*ArcCsc[a + b*x]^2) - (2*((a + b*x)/2 - 3*a*(a + b*x)*Sqrt
[1 - (a + b*x)^(-2)]*ArcCsc[a + b*x] + ((a + b*x)^2*Sqrt[1 - (a + b*x)^(-2
)]*ArcCsc[a + b*x])/2 + (a^3*ArcCsc[a + b*x]^2)/2 + ArcCsc[a + b*x]*ArcTan
h[E^(I*ArcCsc[a + b*x])]) + 6*a^2*ArcCsc[a + b*x]*ArcTanh[E^(I*ArcCsc[a + b
*x])] + 3*a*Log[(a + b*x)^(-1)] - (I/2)*PolyLog[2, -E^(I*ArcCsc[a + b*x])]
- (3*I)*a^2*PolyLog[2, -E^(I*ArcCsc[a + b*x])] + (I/2)*PolyLog[2, E^(I*Ar
cCsc[a + b*x])] + (3*I)*a^2*PolyLog[2, E^(I*ArcCsc[a + b*x])]))/3/b^3)

```

**Defintions of rubi rules used**

rule 2009

```
Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 3042

```
Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4678

```
Int[(csc[(e_.) + (f_.)*(x_.)]*(b_.) + (a_.))^(n_.)*((c_.) + (d_.)*(x_.))^(m_.)
, x_Symbol] := Int[ExpandIntegrand[(c + d*x)^m, (a + b*Csc[e + f*x])^n, x],
x] /; FreeQ[{a, b, c, d, e, f, m}, x] && IGtQ[m, 0] && IGtQ[n, 0]
```

rule 4927

```
Int[Cot[(c_.) + (d_.)*(x_.)]*Csc[(c_.) + (d_.)*(x_.)]*(Csc[(c_.) + (d_.)*(x_)
]*(b_.) + (a_.))^(n_.)*((e_.) + (f_.)*(x_.))^(m_.), x_Symbol] := Simp[(-e +
f*x)^m*((a + b*Csc[c + d*x])^(n + 1)/(b*d*(n + 1))), x] + Simp[f*(m/(b*d*(
n + 1))) Int[(e + f*x)^(m - 1)*(a + b*Csc[c + d*x])^(n + 1), x], x] /; Fr
eeQ[{a, b, c, d, e, f, n}, x] && IGtQ[m, 0] && NeQ[n, -1]
```

rule 5782

```
Int[((a_.) + ArcCsc[(c_) + (d_.)*(x_.)]*(b_.))^(p_.)*((e_.) + (f_.)*(x_.))^(m
_.), x_Symbol] := Simp[-(d^(m + 1))^(-1) Subst[Int[(a + b*x)^p*Csc[x]*Cot
[x]*(d*e - c*f + f*Csc[x])^m, x], x, ArcCsc[c + d*x]], x] /; FreeQ[{a, b, c
, d, e, f}, x] && IGtQ[p, 0] && IntegerQ[m]
```

**Maple [A] (verified)**

Time = 1.03 (sec) , antiderivative size = 500, normalized size of antiderivative = 1.84

method	result
derivativedivides	$\frac{\operatorname{arccsc}(bx+a)^2 a^2 (bx+a) - \operatorname{arccsc}(bx+a)^2 a (bx+a)^2 + \frac{\operatorname{arccsc}(bx+a)^2 (bx+a)^3}{3} - 2 \operatorname{arccsc}(bx+a) \sqrt{\frac{(bx+a)^2 - 1}{(bx+a)^2}} a (bx+a) + \dots}{\dots}$
default	$\frac{\operatorname{arccsc}(bx+a)^2 a^2 (bx+a) - \operatorname{arccsc}(bx+a)^2 a (bx+a)^2 + \frac{\operatorname{arccsc}(bx+a)^2 (bx+a)^3}{3} - 2 \operatorname{arccsc}(bx+a) \sqrt{\frac{(bx+a)^2 - 1}{(bx+a)^2}} a (bx+a) + \dots}{\dots}$

input `int(arccsc(b*x+a)^2*x^2,x,method=_RETURNVERBOSE)`

output

$$\begin{aligned} & 1/b^3 * (\operatorname{arccsc}(b*x+a)^2 * a^2 * (b*x+a) - \operatorname{arccsc}(b*x+a)^2 * a * (b*x+a)^2 + 1/3 * \operatorname{arccsc}(b*x+a)^2 * (b*x+a)^3 - 2 * \operatorname{arccsc}(b*x+a) * \sqrt{\frac{(b*x+a)^2 - 1}{(b*x+a)^2}} * a * (b*x+a) + \dots) \\ & + 1/3 * \operatorname{arccsc}(b*x+a) * \left( \left( \frac{(b*x+a)^2 - 1}{(b*x+a)^2} \right)^{1/2} * (b*x+a)^2 - 2 * I * \operatorname{polylog}(2, -I/(b*x+a) - (1 - 1/(b*x+a)^2)^{1/2}) * a^2 + 1/3 * b*x + 1/3 * a + 1/3 * \operatorname{arccsc}(b*x+a) * \ln(1 + I/(b*x+a) + (1 - 1/(b*x+a)^2)^{1/2}) + 1/3 * I * \operatorname{polylog}(2, I/(b*x+a) + (1 - 1/(b*x+a)^2)^{1/2}) - 1/3 * \operatorname{arccsc}(b*x+a) * \ln(1 - I/(b*x+a) - (1 - 1/(b*x+a)^2)^{1/2}) + 2 * I * \operatorname{polylog}(2, I/(b*x+a) + (1 - 1/(b*x+a)^2)^{1/2}) * a^2 + 2 * \ln(1 + I/(b*x+a) + (1 - 1/(b*x+a)^2)^{1/2}) * a - 4 * \ln(I/(b*x+a) + (1 - 1/(b*x+a)^2)^{1/2}) * a + 2 * \ln(I/(b*x+a) + (1 - 1/(b*x+a)^2)^{1/2}) - 1 * a + 2 * \ln(1 + I/(b*x+a) + (1 - 1/(b*x+a)^2)^{1/2}) * a^2 * \operatorname{arccsc}(b*x+a) - 2 * \ln(1 - I/(b*x+a) - (1 - 1/(b*x+a)^2)^{1/2}) * a^2 * \operatorname{arccsc}(b*x+a) + 2 * I * a * \operatorname{arccsc}(b*x+a) - 1/3 * I * \operatorname{polylog}(2, -I/(b*x+a) - (1 - 1/(b*x+a)^2)^{1/2}) \right) \end{aligned}$$
**Fricas [F]**

$$\int x^2 \csc^{-1}(a + bx)^2 dx = \int x^2 \operatorname{arccsc}(bx + a)^2 dx$$

input `integrate(x^2*arccsc(b*x+a)^2,x, algorithm="fricas")`

output `integral(x^2*arccsc(b*x + a)^2, x)`

**Sympy [F]**

$$\int x^2 \csc^{-1}(a + bx)^2 dx = \int x^2 \operatorname{acsc}^2(a + bx) dx$$

input `integrate(x**2*acsc(b*x+a)**2,x)`

output `Integral(x**2*acsc(a + b*x)**2, x)`

**Maxima [F]**

$$\int x^2 \csc^{-1}(a + bx)^2 dx = \int x^2 \operatorname{arccsc}(bx + a)^2 dx$$

input `integrate(x^2*arccsc(b*x+a)^2,x, algorithm="maxima")`

output `1/3*x^3*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1))^2 - 1/12*x^3*log(b^2*x^2 + 2*a*b*x + a^2)^2 + integrate(1/3*(2*sqrt(b*x + a + 1)*sqrt(b*x + a - 1)*b*x^3*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) - 3*(b^3*x^5 + 3*a*b^2*x^4 + (3*a^2 - 1)*b*x^3 + (a^3 - a)*x^2)*log(b*x + a)^2 + (b^3*x^5 + 2*a*b^2*x^4 + (a^2 - 1)*b*x^3 + 3*(b^3*x^5 + 3*a*b^2*x^4 + (3*a^2 - 1)*b*x^3 + (a^3 - a)*x^2)*log(b*x + a))*log(b^2*x^2 + 2*a*b*x + a^2))/(b^3*x^3 + 3*a*b^2*x^2 + a^3 + (3*a^2 - 1)*b*x - a), x)`

**Giac [F]**

$$\int x^2 \csc^{-1}(a + bx)^2 dx = \int x^2 \operatorname{arccsc}(bx + a)^2 dx$$

input `integrate(x^2*arccsc(b*x+a)^2,x, algorithm="giac")`

output `integrate(x^2*arccsc(b*x + a)^2, x)`

**Mupad [F(-1)]**

Timed out.

$$\int x^2 \csc^{-1}(a + bx)^2 dx = \int x^2 \operatorname{asin}\left(\frac{1}{a + bx}\right)^2 dx$$

input `int(x^2*asin(1/(a + b*x))^2,x)`output `int(x^2*asin(1/(a + b*x))^2, x)`**Reduce [F]**

$$\int x^2 \csc^{-1}(a + bx)^2 dx = \int \operatorname{acsc}(bx + a)^2 x^2 dx$$

input `int(x^2*acsc(b*x+a)^2,x)`output `int(acsc(a + b*x)**2*x**2,x)`

### 3.29 $\int x \csc^{-1}(a + bx)^2 dx$

Optimal result	245
Mathematica [A] (verified)	246
Rubi [A] (verified)	246
Maple [A] (verified)	248
Fricas [F]	249
Sympy [F]	249
Maxima [F]	249
Giac [F]	250
Mupad [F(-1)]	250
Reduce [F]	250

#### Optimal result

Integrand size = 10, antiderivative size = 145

$$\int x \csc^{-1}(a + bx)^2 dx = \frac{(a + bx)\sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx)}{b^2} - \frac{a^2 \csc^{-1}(a + bx)^2}{2b^2} + \frac{1}{2}x^2 \csc^{-1}(a + bx)^2 - \frac{4a \csc^{-1}(a + bx)\operatorname{arctanh}\left(e^{i \csc^{-1}(a+bx)}\right)}{b^2} + \frac{\log(a + bx)}{b^2} + \frac{2ia \operatorname{PolyLog}\left(2, -e^{i \csc^{-1}(a+bx)}\right)}{b^2} - \frac{2ia \operatorname{PolyLog}\left(2, e^{i \csc^{-1}(a+bx)}\right)}{b^2}$$

output

```
(b*x+a)*(1-1/(b*x+a)^2)^(1/2)*arccsc(b*x+a)/b^2-1/2*a^2*arccsc(b*x+a)^2/b^2+1/2*x^2*arccsc(b*x+a)^2-4*a*arccsc(b*x+a)*arctanh(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/b^2+ln(b*x+a)/b^2+2*I*a*polylog(2,-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))/b^2-2*I*a*polylog(2,I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/b^2
```

**Mathematica [A] (verified)**

Time = 0.61 (sec) , antiderivative size = 213, normalized size of antiderivative = 1.47

$$\int x \csc^{-1}(a + bx)^2 dx$$

$$= \frac{2a \sqrt{\frac{-1+a^2+2abx+b^2x^2}{(a+bx)^2}} \csc^{-1}(a + bx) + 2bx \sqrt{\frac{-1+a^2+2abx+b^2x^2}{(a+bx)^2}} \csc^{-1}(a + bx) - a^2 \csc^{-1}(a + bx)^2 + b^2 x^2 \csc^{-1}(a + bx)^2}{b^2}$$

input

```
Integrate[x*ArcCsc[a + b*x]^2,x]
```

output

```
(2*a*Sqrt[(-1 + a^2 + 2*a*b*x + b^2*x^2)/(a + b*x)^2]*ArcCsc[a + b*x] + 2*
b*x*Sqrt[(-1 + a^2 + 2*a*b*x + b^2*x^2)/(a + b*x)^2]*ArcCsc[a + b*x] - a^2
*ArcCsc[a + b*x]^2 + b^2*x^2*ArcCsc[a + b*x]^2 + 4*a*ArcCsc[a + b*x]*Log[1
- E^(I*ArcCsc[a + b*x])] - 4*a*ArcCsc[a + b*x]*Log[1 + E^(I*ArcCsc[a + b*
x])] - 2*Log[(a + b*x)^(-1)] + (4*I)*a*PolyLog[2, -E^(I*ArcCsc[a + b*x])]
- (4*I)*a*PolyLog[2, E^(I*ArcCsc[a + b*x])])/(2*b^2)
```

**Rubi [A] (verified)**

Time = 0.47 (sec) , antiderivative size = 137, normalized size of antiderivative = 0.94, number of steps used = 7, number of rules used = 6,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.600$ , Rules used = {5782, 25, 4927, 3042, 4678, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x \csc^{-1}(a + bx)^2 dx$$

$$\downarrow \text{5782}$$

$$-\frac{\int bx(a + bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx)^2 d \csc^{-1}(a + bx)}{b^2}$$

$$\downarrow \text{25}$$

$$\frac{\int -bx(a + bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx)^2 d \csc^{-1}(a + bx)}{b^2}$$

$$\begin{aligned}
& \int \frac{b^2 x^2 \csc^{-1}(a+bx) d \csc^{-1}(a+bx) - \frac{1}{2} b^2 x^2 \csc^{-1}(a+bx)^2}{b^2} \\
& \quad \downarrow 4927 \\
& \int \frac{\csc^{-1}(a+bx) (a - \csc(\csc^{-1}(a+bx)))^2 d \csc^{-1}(a+bx) - \frac{1}{2} b^2 x^2 \csc^{-1}(a+bx)^2}{b^2} \\
& \quad \downarrow 3042 \\
& \int \frac{(\csc^{-1}(a+bx)a^2 - 2(a+bx)\csc^{-1}(a+bx)a + (a+bx)^2 \csc^{-1}(a+bx)) d \csc^{-1}(a+bx) - \frac{1}{2} b^2 x^2 \csc^{-1}(a+bx)^2}{b^2} \\
& \quad \downarrow 4678 \\
& \int \frac{\csc^{-1}(a+bx)a^2 - 2(a+bx)\csc^{-1}(a+bx)a + (a+bx)^2 \csc^{-1}(a+bx) d \csc^{-1}(a+bx) - \frac{1}{2} b^2 x^2 \csc^{-1}(a+bx)^2}{b^2} \\
& \quad \downarrow 2009 \\
& \frac{\frac{1}{2} a^2 \csc^{-1}(a+bx)^2 + 4a \csc^{-1}(a+bx) \operatorname{arctanh}\left(e^{i \csc^{-1}(a+bx)}\right) - \frac{1}{2} b^2 x^2 \csc^{-1}(a+bx)^2 - 2ia \operatorname{PolyLog}\left(2, -e^{i \csc^{-1}(a+bx)}\right)}{b^2}
\end{aligned}$$

input `Int[x*ArcCsc[a + b*x]^2,x]`

output `-((-((a + b*x)*Sqrt[1 - (a + b*x)^(-2)]*ArcCsc[a + b*x]) + (a^2*ArcCsc[a + b*x]^2)/2 - (b^2*x^2*ArcCsc[a + b*x]^2)/2 + 4*a*ArcCsc[a + b*x]*ArcTanh[E^(I*ArcCsc[a + b*x])]) + Log[(a + b*x)^(-1)] - (2*I)*a*PolyLog[2, -E^(I*ArcCsc[a + b*x])]) + (2*I)*a*PolyLog[2, E^(I*ArcCsc[a + b*x])])/b^2`

### Defintions of rubi rules used

rule 25 `Int[-(Fx_), x_Symbol] :> Simp[Identity[-1] Int[Fx, x], x]`

rule 2009 `Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]`

rule 3042 `Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`



rule 4678

```
Int[(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_)^(n_.))*((c_.) + (d_.)*(x_))^(m_.)
, x_Symbol] := Int[ExpandIntegrand[(c + d*x)^m, (a + b*Csc[e + f*x])^n, x],
x] /; FreeQ[{a, b, c, d, e, f, m}, x] && IGtQ[m, 0] && IGtQ[n, 0]
```

rule 4927

```
Int[Cot[(c_.) + (d_.)*(x_)]*Csc[(c_.) + (d_.)*(x_)]*(Csc[(c_.) + (d_.)*(x_)]
)*(b_.) + (a_)^(n_.))*((e_.) + (f_.)*(x_))^(m_.), x_Symbol] := Simp[(-e +
f*x)^m*((a + b*Csc[c + d*x])^(n + 1)/(b*d*(n + 1))), x] + Simp[f*m/(b*d*(
n + 1)) Int[(e + f*x)^(m - 1)*(a + b*Csc[c + d*x])^(n + 1), x], x] /; Fr
eeQ[{a, b, c, d, e, f, n}, x] && IGtQ[m, 0] && NeQ[n, -1]
```

rule 5782

```
Int[((a_.) + ArcCsc[(c_.) + (d_.)*(x_)]*(b_.))^(p_.))*((e_.) + (f_.)*(x_))^(m
_.), x_Symbol] := Simp[-(d^(m + 1))^(-1) Subst[Int[(a + b*x)^p*Csc[x]*Cot
[x]*(d*e - c*f + f*Csc[x])^m, x], x, ArcCsc[c + d*x]], x] /; FreeQ[{a, b, c
, d, e, f}, x] && IGtQ[p, 0] && IntegerQ[m]
```

## Maple [A] (verified)

Time = 0.68 (sec) , antiderivative size = 212, normalized size of antiderivative = 1.46

method	result
derivativedivides	$\frac{-a \left( \operatorname{arccsc}(bx+a)^2 (bx+a) - 2 \operatorname{arccsc}(bx+a) \ln \left( 1 - \frac{i}{bx+a} - \sqrt{1 - \frac{1}{(bx+a)^2}} \right) + 2 \operatorname{arccsc}(bx+a) \ln \left( 1 + \frac{i}{bx+a} + \sqrt{1 - \frac{1}{(bx+a)^2}} \right) \right)}{1}$
default	$\frac{-a \left( \operatorname{arccsc}(bx+a)^2 (bx+a) - 2 \operatorname{arccsc}(bx+a) \ln \left( 1 - \frac{i}{bx+a} - \sqrt{1 - \frac{1}{(bx+a)^2}} \right) + 2 \operatorname{arccsc}(bx+a) \ln \left( 1 + \frac{i}{bx+a} + \sqrt{1 - \frac{1}{(bx+a)^2}} \right) \right)}{1}$

input

```
int(x*arccsc(b*x+a)^2,x,method=_RETURNVERBOSE)
```

output

```
1/b^2*(-a*(arccsc(b*x+a)^2*(b*x+a)-2*arccsc(b*x+a)*ln(1-I/(b*x+a)-(1-1/(b*
x+a)^2)^(1/2))+2*arccsc(b*x+a)*ln(1+I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))-2*I*d
ilog(1+I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))+2*I*dilog(1-I/(b*x+a)-(1-1/(b*x+a)
^2)^(1/2)))+1/2*arccsc(b*x+a)^2*(b*x+a)^2+arccsc(b*x+a)*(((b*x+a)^2-1)/(b*
x+a)^2)^(1/2)*(b*x+a)-ln(1/(b*x+a)))
```

**Fricas [F]**

$$\int x \csc^{-1}(a + bx)^2 dx = \int x \operatorname{arccsc}(bx + a)^2 dx$$

input `integrate(x*arccsc(b*x+a)^2,x, algorithm="fricas")`

output `integral(x*arccsc(b*x + a)^2, x)`

**Sympy [F]**

$$\int x \csc^{-1}(a + bx)^2 dx = \int x \operatorname{acsc}^2(a + bx) dx$$

input `integrate(x*acsc(b*x+a)**2,x)`

output `Integral(x*acsc(a + b*x)**2, x)`

**Maxima [F]**

$$\int x \csc^{-1}(a + bx)^2 dx = \int x \operatorname{arccsc}(bx + a)^2 dx$$

input `integrate(x*arccsc(b*x+a)^2,x, algorithm="maxima")`

output `1/2*x^2*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1))^2 - 1/8*x^2*log(b^2*x^2 + 2*a*b*x + a^2)^2 + integrate(1/2*(2*sqrt(b*x + a + 1)*sqrt(b*x + a - 1)*b*x^2*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) - 2*(b^3*x^4 + 3*a*b^2*x^3 + (3*a^2 - 1)*b*x^2 + (a^3 - a)*x)*log(b*x + a)^2 + (b^3*x^4 + 2*a*b^2*x^3 + (a^2 - 1)*b*x^2 + 2*(b^3*x^4 + 3*a*b^2*x^3 + (3*a^2 - 1)*b*x^2 + (a^3 - a)*x)*log(b*x + a))*log(b^2*x^2 + 2*a*b*x + a^2))/(b^3*x^3 + 3*a*b^2*x^2 + a^3 + (3*a^2 - 1)*b*x - a), x)`

**Giac [F]**

$$\int x \csc^{-1}(a + bx)^2 dx = \int x \operatorname{arccsc}(bx + a)^2 dx$$

input `integrate(x*arccsc(b*x+a)^2,x, algorithm="giac")`

output `integrate(x*arccsc(b*x + a)^2, x)`

**Mupad [F(-1)]**

Timed out.

$$\int x \csc^{-1}(a + bx)^2 dx = \int x \operatorname{asin}\left(\frac{1}{a + bx}\right)^2 dx$$

input `int(x*asin(1/(a + b*x))^2,x)`

output `int(x*asin(1/(a + b*x))^2, x)`

**Reduce [F]**

$$\int x \csc^{-1}(a + bx)^2 dx = \int \operatorname{acsc}(bx + a)^2 x dx$$

input `int(x*acsc(b*x+a)^2,x)`

output `int(acsc(a + b*x)**2*x,x)`

### 3.30 $\int \csc^{-1}(a + bx)^2 dx$

Optimal result	251
Mathematica [A] (verified)	251
Rubi [A] (warning: unable to verify)	252
Maple [A] (verified)	254
Fricas [F]	255
Sympy [F]	255
Maxima [F]	255
Giac [F]	256
Mupad [F(-1)]	256
Reduce [F]	256

#### Optimal result

Integrand size = 8, antiderivative size = 86

$$\int \csc^{-1}(a + bx)^2 dx = \frac{(a + bx) \csc^{-1}(a + bx)^2}{b} + \frac{4 \csc^{-1}(a + bx) \operatorname{arctanh}\left(e^{i \csc^{-1}(a + bx)}\right)}{b} - \frac{2i \operatorname{PolyLog}\left(2, -e^{i \csc^{-1}(a + bx)}\right)}{b} + \frac{2i \operatorname{PolyLog}\left(2, e^{i \csc^{-1}(a + bx)}\right)}{b}$$

output

```
(b*x+a)*arccsc(b*x+a)^2/b+4*arccsc(b*x+a)*arctanh(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/b-2*I*polylog(2,-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))/b+2*I*polylog(2,I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/b
```

#### Mathematica [A] (verified)

Time = 0.10 (sec) , antiderivative size = 99, normalized size of antiderivative = 1.15

$$\int \csc^{-1}(a + bx)^2 dx = \frac{\csc^{-1}(a + bx) \left( (a + bx) \csc^{-1}(a + bx) - 2 \log\left(1 - e^{i \csc^{-1}(a + bx)}\right) + 2 \log\left(1 + e^{i \csc^{-1}(a + bx)}\right) \right)}{b} - 2i \operatorname{PolyLog}\left(2, -e^{i \csc^{-1}(a + bx)}\right) + 2i \operatorname{PolyLog}\left(2, e^{i \csc^{-1}(a + bx)}\right)$$

input

```
Integrate[ArcCsc[a + b*x]^2,x]
```

output

```
(ArcCsc[a + b*x]*((a + b*x)*ArcCsc[a + b*x] - 2*Log[1 - E^(I*ArcCsc[a + b*x]]) + 2*Log[1 + E^(I*ArcCsc[a + b*x])]) - (2*I)*PolyLog[2, -E^(I*ArcCsc[a + b*x])] + (2*I)*PolyLog[2, E^(I*ArcCsc[a + b*x])])/b
```

**Rubi [A] (warning: unable to verify)**

Time = 0.44 (sec) , antiderivative size = 77, normalized size of antiderivative = 0.90, number of steps used = 8, number of rules used = 7,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.875$ , Rules used = {5776, 5740, 4245, 3042, 4671, 2715, 2838}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \csc^{-1}(a + bx)^2 dx \\
 & \quad \downarrow \text{5776} \\
 & \frac{\int \csc^{-1}(a + bx)^2 d(a + bx)}{b} \\
 & \quad \downarrow \text{5740} \\
 & \frac{\int (a + bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx)^2 d \csc^{-1}(a + bx)}{b} \\
 & \quad \downarrow \text{4245} \\
 & \frac{2 \int (a + bx) \csc^{-1}(a + bx) d \csc^{-1}(a + bx) - (a + bx) \csc^{-1}(a + bx)^2}{b} \\
 & \quad \downarrow \text{3042} \\
 & \frac{2 \int \csc^{-1}(a + bx) \csc(\csc^{-1}(a + bx)) d \csc^{-1}(a + bx) - (a + bx) \csc^{-1}(a + bx)^2}{b} \\
 & \quad \downarrow \text{4671} \\
 & \frac{-(a + bx) \csc^{-1}(a + bx)^2 + 2 \left( - \int \log(1 - e^{i \csc^{-1}(a+bx)}) d \csc^{-1}(a + bx) + \int \log(1 + e^{i \csc^{-1}(a+bx)}) d \csc^{-1}(a + bx) \right)}{b} \\
 & \quad \downarrow \text{2715}
 \end{aligned}$$

$$\frac{-(a+bx)\csc^{-1}(a+bx)^2 + 2\left(-i\int e^{-i\csc^{-1}(a+bx)}\log\left(1+e^{i\csc^{-1}(a+bx)}\right)de^{i\csc^{-1}(a+bx)} + i\int e^{-i\csc^{-1}(a+bx)}\log\left(1-e^{i\csc^{-1}(a+bx)}\right)de^{i\csc^{-1}(a+bx)}\right)}{b}$$

↓ 2838

$$\frac{-(a+bx)\csc^{-1}(a+bx)^2 + 2\left(-2\csc^{-1}(a+bx)\operatorname{arctanh}\left(e^{i\csc^{-1}(a+bx)}\right) + i\operatorname{PolyLog}(2, -a-bx) - i\operatorname{PolyLog}(2, -a-bx)\right)}{b}$$

input

```
Int[ArcCsc[a + b*x]^2, x]
```

output

```
-((-(a + b*x)*ArcCsc[a + b*x]^2) + 2*(-2*ArcCsc[a + b*x]*ArcTanh[E^(I*ArcCsc[a + b*x])] - I*PolyLog[2, E^(I*ArcCsc[a + b*x])] + I*PolyLog[2, -a - b*x]))/b)
```

### Defintions of rubi rules used

rule 2715

```
Int[Log[(a_) + (b_)*((F_)^((e_)*((c_) + (d_)*(x_)))^(n_))], x_Symbol]
:> Simp[1/(d*e*n*Log[F]) Subst[Int[Log[a + b*x]/x, x], x, (F^(e*(c + d*x)))^n], x] /; FreeQ[{F, a, b, c, d, e, n}, x] && GtQ[a, 0]
```

rule 2838

```
Int[Log[(c_)*((d_) + (e_)*(x_)^(n_))]/(x_), x_Symbol] :> Simp[-PolyLog[2, (-c)*e*x^n]/n, x] /; FreeQ[{c, d, e, n}, x] && EqQ[c*d, 1]
```

rule 3042

```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]
```

rule 4245

```
Int[Cot[(a_) + (b_)*(x_)^(n_)]^(q_)*Csc[(a_) + (b_)*(x_)^(n_)]^(p_)*
(x_)^(m_), x_Symbol] :> Simp[(-x^(m - n + 1))*(Csc[a + b*x^n]^p/(b*n*p)),
x] + Simp[(m - n + 1)/(b*n*p) Int[x^(m - n)*Csc[a + b*x^n]^p, x], x] /;
FreeQ[{a, b, p}, x] && IntegerQ[n] && GeQ[m, n] && EqQ[q, 1]
```

rule 4671 `Int[csc[(e_.) + (f_.)*(x_)]*((c_.) + (d_.)*(x_))^(m_.), x_Symbol] := Simp[-2*(c + d*x)^m*(ArcTanh[E^(I*(e + f*x))]/f), x] + (-Simp[d*(m/f) Int[(c + d*x)^(m - 1)*Log[1 - E^(I*(e + f*x))], x], x] + Simp[d*(m/f) Int[(c + d*x)^(m - 1)*Log[1 + E^(I*(e + f*x))], x], x]) /; FreeQ[{c, d, e, f}, x] && IGtQ[m, 0]`

rule 5740 `Int[((a_.) + ArcCsc[(c_.)*(x_)]*(b_.))^(n_), x_Symbol] := Simp[-c^(-1) Subst[Int[(a + b*x)^n*Csc[x]*Cot[x], x], x, ArcCsc[c*x]], x] /; FreeQ[{a, b, c, n}, x] && IGtQ[n, 0]`

rule 5776 `Int[((a_.) + ArcCsc[(c_) + (d_.)*(x_)]*(b_.))^(p_.), x_Symbol] := Simp[1/d Subst[Int[(a + b*ArcCsc[x])^p, x], x, c + d*x], x] /; FreeQ[{a, b, c, d}, x] && IGtQ[p, 0]`

## Maple [A] (verified)

Time = 0.40 (sec) , antiderivative size = 150, normalized size of antiderivative = 1.74

method	result
derivativedivides	$\frac{\operatorname{arccsc}(bx+a)^2(bx+a)-2 \operatorname{arccsc}(bx+a) \ln\left(1-\frac{i}{bx+a}-\sqrt{1-\frac{1}{(bx+a)^2}}\right)+2 \operatorname{arccsc}(bx+a) \ln\left(1+\frac{i}{bx+a}+\sqrt{1-\frac{1}{(bx+a)^2}}\right)}{b}$
default	$\frac{\operatorname{arccsc}(bx+a)^2(bx+a)-2 \operatorname{arccsc}(bx+a) \ln\left(1-\frac{i}{bx+a}-\sqrt{1-\frac{1}{(bx+a)^2}}\right)+2 \operatorname{arccsc}(bx+a) \ln\left(1+\frac{i}{bx+a}+\sqrt{1-\frac{1}{(bx+a)^2}}\right)}{b}$

input `int(arccsc(b*x+a)^2,x,method=_RETURNVERBOSE)`

output `1/b*(arccsc(b*x+a)^2*(b*x+a)-2*arccsc(b*x+a)*ln(1-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))+2*arccsc(b*x+a)*ln(1+I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))-2*I*dilog(1+I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))+2*I*dilog(1-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2)))`

**Fricas [F]**

$$\int \csc^{-1}(a + bx)^2 dx = \int \operatorname{arccsc}(bx + a)^2 dx$$

input `integrate(arccsc(b*x+a)^2,x, algorithm="fricas")`

output `integral(arccsc(b*x + a)^2, x)`

**Sympy [F]**

$$\int \csc^{-1}(a + bx)^2 dx = \int \operatorname{acsc}^2(a + bx) dx$$

input `integrate(acsc(b*x+a)**2,x)`

output `Integral(acsc(a + b*x)**2, x)`

**Maxima [F]**

$$\int \csc^{-1}(a + bx)^2 dx = \int \operatorname{arccsc}(bx + a)^2 dx$$

input `integrate(arccsc(b*x+a)^2,x, algorithm="maxima")`

output `x*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1))^2 - 1/4*x*log(b^2*x^2 + 2*a*b*x + a^2)^2 + integrate((2*sqrt(b*x + a + 1)*sqrt(b*x + a - 1)*b*x*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) - (b^3*x^3 + 3*a*b^2*x^2 + a^3 + (3*a^2 - 1)*b*x - a)*log(b*x + a)^2 + (b^3*x^3 + 2*a*b^2*x^2 + (a^2 - 1)*b*x + (b^3*x^3 + 3*a*b^2*x^2 + a^3 + (3*a^2 - 1)*b*x - a)*log(b*x + a))*log(b^2*x^2 + 2*a*b*x + a^2))/(b^3*x^3 + 3*a*b^2*x^2 + a^3 + (3*a^2 - 1)*b*x - a), x)`



**Giac [F]**

$$\int \csc^{-1}(a + bx)^2 dx = \int \operatorname{arccsc}(bx + a)^2 dx$$

input `integrate(arccsc(b*x+a)^2,x, algorithm="giac")`

output `integrate(arccsc(b*x + a)^2, x)`

**Mupad [F(-1)]**

Timed out.

$$\int \csc^{-1}(a + bx)^2 dx = \int \operatorname{asin}\left(\frac{1}{a + bx}\right)^2 dx$$

input `int(asin(1/(a + b*x))^2,x)`

output `int(asin(1/(a + b*x))^2, x)`

**Reduce [F]**

$$\int \csc^{-1}(a + bx)^2 dx = \int \operatorname{acsc}(bx + a)^2 dx$$

input `int(acsc(b*x+a)^2,x)`

output `int(acsc(a + b*x)**2,x)`

### 3.31 $\int \frac{\csc^{-1}(a+bx)^2}{x} dx$

Optimal result	257
Mathematica [A] (verified)	258
Rubi [A] (verified)	259
Maple [F]	265
Fricas [F]	265
Sympy [F]	265
Maxima [F]	266
Giac [F]	266
Mupad [F(-1)]	266
Reduce [F]	267

#### Optimal result

Integrand size = 12, antiderivative size = 324

$$\begin{aligned}
\int \frac{\csc^{-1}(a+bx)^2}{x} dx &= \csc^{-1}(a+bx)^2 \log\left(1 + \frac{iae^{i \csc^{-1}(a+bx)}}{1 - \sqrt{1-a^2}}\right) \\
&\quad + \csc^{-1}(a+bx)^2 \log\left(1 + \frac{iae^{i \csc^{-1}(a+bx)}}{1 + \sqrt{1-a^2}}\right) \\
&\quad - \csc^{-1}(a+bx)^2 \log\left(1 - e^{2i \csc^{-1}(a+bx)}\right) \\
&\quad - 2i \csc^{-1}(a+bx) \operatorname{PolyLog}\left(2, -\frac{iae^{i \csc^{-1}(a+bx)}}{1 - \sqrt{1-a^2}}\right) \\
&\quad - 2i \csc^{-1}(a+bx) \operatorname{PolyLog}\left(2, -\frac{iae^{i \csc^{-1}(a+bx)}}{1 + \sqrt{1-a^2}}\right) \\
&\quad + i \csc^{-1}(a+bx) \operatorname{PolyLog}\left(2, e^{2i \csc^{-1}(a+bx)}\right) \\
&\quad + 2 \operatorname{PolyLog}\left(3, -\frac{iae^{i \csc^{-1}(a+bx)}}{1 - \sqrt{1-a^2}}\right) \\
&\quad + 2 \operatorname{PolyLog}\left(3, -\frac{iae^{i \csc^{-1}(a+bx)}}{1 + \sqrt{1-a^2}}\right) \\
&\quad - \frac{1}{2} \operatorname{PolyLog}\left(3, e^{2i \csc^{-1}(a+bx)}\right)
\end{aligned}$$

output

```

arccsc(b*x+a)^2*ln(1+I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1-(-a^2+1)^(1/2)))
+arccsc(b*x+a)^2*ln(1+I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1+(-a^2+1)^(1/2)))
-arccsc(b*x+a)^2*ln(1-I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1-(-a^2+1)^(1/2)))
-2*I*arccsc(b*x+a)*polylog(2,-I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1-(-a^2+1)^(1/2)))
-2*I*arccsc(b*x+a)*polylog(2,-I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1+(-a^2+1)^(1/2)))
+I*arccsc(b*x+a)*polylog(2,(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1-(-a^2+1)^(1/2)))
+2*polylog(3,-I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1-(-a^2+1)^(1/2)))
+2*polylog(3,-I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1+(-a^2+1)^(1/2)))
-1/2*polylog(3,(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))^2)

```

**Mathematica [A] (verified)**

Time = 0.26 (sec) , antiderivative size = 408, normalized size of antiderivative = 1.26

$$\begin{aligned}
\int \frac{\csc^{-1}(a+bx)^2}{x} dx &= \frac{i\pi^3}{6} - \frac{1}{3}i \csc^{-1}(a+bx)^3 - \csc^{-1}(a+bx)^2 \log\left(1 - e^{-i \csc^{-1}(a+bx)}\right) \\
&\quad - \csc^{-1}(a+bx)^2 \log\left(1 + e^{i \csc^{-1}(a+bx)}\right) \\
&\quad + \csc^{-1}(a+bx)^2 \log\left(1 - \frac{iae^{i \csc^{-1}(a+bx)}}{-1 + \sqrt{1-a^2}}\right) \\
&\quad + \csc^{-1}(a+bx)^2 \log\left(1 + \frac{iae^{i \csc^{-1}(a+bx)}}{1 + \sqrt{1-a^2}}\right) \\
&\quad - 2i \csc^{-1}(a+bx) \text{PolyLog}\left(2, e^{-i \csc^{-1}(a+bx)}\right) \\
&\quad + 2i \csc^{-1}(a+bx) \text{PolyLog}\left(2, -e^{i \csc^{-1}(a+bx)}\right) \\
&\quad - 2i \csc^{-1}(a+bx) \text{PolyLog}\left(2, \frac{iae^{i \csc^{-1}(a+bx)}}{-1 + \sqrt{1-a^2}}\right) \\
&\quad - 2i \csc^{-1}(a+bx) \text{PolyLog}\left(2, -\frac{iae^{i \csc^{-1}(a+bx)}}{1 + \sqrt{1-a^2}}\right) \\
&\quad - 2 \text{PolyLog}\left(3, e^{-i \csc^{-1}(a+bx)}\right) - 2 \text{PolyLog}\left(3, -e^{i \csc^{-1}(a+bx)}\right) \\
&\quad + 2 \text{PolyLog}\left(3, \frac{iae^{i \csc^{-1}(a+bx)}}{-1 + \sqrt{1-a^2}}\right) \\
&\quad + 2 \text{PolyLog}\left(3, -\frac{iae^{i \csc^{-1}(a+bx)}}{1 + \sqrt{1-a^2}}\right)
\end{aligned}$$

input `Integrate[ArcCsc[a + b*x]^2/x,x]`

output 
$$\begin{aligned} & (I/6)*\text{Pi}^3 - (I/3)*\text{ArcCsc}[a + b*x]^3 - \text{ArcCsc}[a + b*x]^2*\text{Log}[1 - E^{((-I)*\text{ArcCsc}[a + b*x])}] - \text{ArcCsc}[a + b*x]^2*\text{Log}[1 + E^{(I*\text{ArcCsc}[a + b*x])}] + \text{ArcCsc}[a + b*x]^2*\text{Log}[1 - (I*a*E^{(I*\text{ArcCsc}[a + b*x])})/(-1 + \text{Sqrt}[1 - a^2])] + \\ & \text{ArcCsc}[a + b*x]^2*\text{Log}[1 + (I*a*E^{(I*\text{ArcCsc}[a + b*x])})/(1 + \text{Sqrt}[1 - a^2])] - (2*I)*\text{ArcCsc}[a + b*x]*\text{PolyLog}[2, E^{((-I)*\text{ArcCsc}[a + b*x])}] + (2*I)*\text{ArcCsc}[a + b*x]*\text{PolyLog}[2, -E^{(I*\text{ArcCsc}[a + b*x])}] - (2*I)*\text{ArcCsc}[a + b*x]*\text{PolyLog}[2, (I*a*E^{(I*\text{ArcCsc}[a + b*x])})/(-1 + \text{Sqrt}[1 - a^2])] - (2*I)*\text{ArcCsc}[a + b*x]*\text{PolyLog}[2, ((-I)*a*E^{(I*\text{ArcCsc}[a + b*x])})/(1 + \text{Sqrt}[1 - a^2])] - 2*\text{PolyLog}[3, E^{((-I)*\text{ArcCsc}[a + b*x])}] - 2*\text{PolyLog}[3, -E^{(I*\text{ArcCsc}[a + b*x])}] + 2*\text{PolyLog}[3, (I*a*E^{(I*\text{ArcCsc}[a + b*x])})/(-1 + \text{Sqrt}[1 - a^2])] + 2*\text{PolyLog}[3, ((-I)*a*E^{(I*\text{ArcCsc}[a + b*x])})/(1 + \text{Sqrt}[1 - a^2])] \end{aligned}$$

### Rubi [A] (verified)

Time = 1.69 (sec) , antiderivative size = 395, normalized size of antiderivative = 1.22, number of steps used = 18, number of rules used = 17,  $\frac{\text{number of rules}}{\text{integrand size}} = 1.417$ , Rules used = {5782, 25, 5063, 5040, 25, 3042, 25, 4200, 25, 2620, 3011, 2720, 5030, 2620, 3011, 2720, 7143}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \frac{\csc^{-1}(a + bx)^2}{x} dx \\ & \quad \downarrow \text{5782} \\ & - \int \frac{(a + bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx)^2}{bx} d \csc^{-1}(a + bx) \\ & \quad \downarrow \text{25} \\ & \int - \frac{(a + bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx)^2}{bx} d \csc^{-1}(a + bx) \\ & \quad \downarrow \text{5063} \end{aligned}$$

$$\begin{aligned}
& \int \frac{(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}\csc^{-1}(a+bx)^2}{\frac{a}{a+bx}-1}d\csc^{-1}(a+bx) \\
& \quad \downarrow 5040 \\
& a \int -\frac{\sqrt{1-\frac{1}{(a+bx)^2}}\csc^{-1}(a+bx)^2}{1-\frac{a}{a+bx}}d\csc^{-1}(a+bx) - \int (a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}\csc^{-1}(a+bx)^2 d\csc^{-1}(a+bx) \\
& \quad \downarrow 25 \\
& - \int (a+bx)\sqrt{1-\frac{1}{(a+bx)^2}}\csc^{-1}(a+bx)^2 d\csc^{-1}(a+bx) - \\
& \quad a \int \frac{\sqrt{1-\frac{1}{(a+bx)^2}}\csc^{-1}(a+bx)^2}{1-\frac{a}{a+bx}}d\csc^{-1}(a+bx) \\
& \quad \downarrow 3042 \\
& -a \int \frac{\sqrt{1-\frac{1}{(a+bx)^2}}\csc^{-1}(a+bx)^2}{1-\frac{a}{a+bx}}d\csc^{-1}(a+bx) - \int -\csc^{-1}(a+bx)^2 \tan\left(\csc^{-1}(a+bx) + \frac{\pi}{2}\right) d\csc^{-1}(a+bx) \\
& \quad \downarrow 25 \\
& \int \csc^{-1}(a+bx)^2 \tan\left(\csc^{-1}(a+bx) + \frac{\pi}{2}\right) d\csc^{-1}(a+bx) - \\
& \quad a \int \frac{\sqrt{1-\frac{1}{(a+bx)^2}}\csc^{-1}(a+bx)^2}{1-\frac{a}{a+bx}}d\csc^{-1}(a+bx) \\
& \quad \downarrow 4200 \\
& -2i \int -\frac{e^{2i\csc^{-1}(a+bx)}\csc^{-1}(a+bx)^2}{1-e^{2i\csc^{-1}(a+bx)}}d\csc^{-1}(a+bx) - \\
& \quad a \int \frac{\sqrt{1-\frac{1}{(a+bx)^2}}\csc^{-1}(a+bx)^2}{1-\frac{a}{a+bx}}d\csc^{-1}(a+bx) + \frac{1}{3}i \csc^{-1}(a+bx)^3 \\
& \quad \downarrow 25 \\
& 2i \int \frac{e^{2i\csc^{-1}(a+bx)}\csc^{-1}(a+bx)^2}{1-e^{2i\csc^{-1}(a+bx)}}d\csc^{-1}(a+bx) - \\
& \quad a \int \frac{\sqrt{1-\frac{1}{(a+bx)^2}}\csc^{-1}(a+bx)^2}{1-\frac{a}{a+bx}}d\csc^{-1}(a+bx) + \frac{1}{3}i \csc^{-1}(a+bx)^3
\end{aligned}$$

$$\begin{aligned} & \downarrow 2620 \\ & -a \int \frac{\sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)^2}{1 - \frac{a}{a+bx}} d \csc^{-1}(a+bx) + \\ & 2i \left( \frac{1}{2} i \csc^{-1}(a+bx)^2 \log \left( 1 - e^{2i \csc^{-1}(a+bx)} \right) - i \int \csc^{-1}(a+bx) \log \left( 1 - e^{2i \csc^{-1}(a+bx)} \right) d \csc^{-1}(a+bx) \right) + \\ & \frac{1}{3} i \csc^{-1}(a+bx)^3 \end{aligned}$$

↓ 3011

$$\begin{aligned} & 2i \left( \frac{1}{2} i \csc^{-1}(a+bx)^2 \log \left( 1 - e^{2i \csc^{-1}(a+bx)} \right) - i \left( \frac{1}{2} i \csc^{-1}(a+bx) \text{PolyLog} \left( 2, e^{2i \csc^{-1}(a+bx)} \right) - \frac{1}{2} i \int \text{PolyLog} \right. \right. \\ & \left. \left. a \int \frac{\sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)^2}{1 - \frac{a}{a+bx}} d \csc^{-1}(a+bx) + \frac{1}{3} i \csc^{-1}(a+bx)^3 \right) \right. \end{aligned}$$

↓ 2720

$$\begin{aligned} & 2i \left( \frac{1}{2} i \csc^{-1}(a+bx)^2 \log \left( 1 - e^{2i \csc^{-1}(a+bx)} \right) - i \left( \frac{1}{2} i \csc^{-1}(a+bx) \text{PolyLog} \left( 2, e^{2i \csc^{-1}(a+bx)} \right) - \frac{1}{4} \int e^{-2i \csc^{-1}(a+bx)} \right. \right. \\ & \left. \left. a \int \frac{\sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)^2}{1 - \frac{a}{a+bx}} d \csc^{-1}(a+bx) + \frac{1}{3} i \csc^{-1}(a+bx)^3 \right) \right. \end{aligned}$$

↓ 5030

$$\begin{aligned} & -a \left( \int \frac{e^{i \csc^{-1}(a+bx)} \csc^{-1}(a+bx)^2}{ie^{i \csc^{-1}(a+bx)} a - \sqrt{1-a^2} + 1} d \csc^{-1}(a+bx) + \int \frac{e^{i \csc^{-1}(a+bx)} \csc^{-1}(a+bx)^2}{ie^{i \csc^{-1}(a+bx)} a + \sqrt{1-a^2} + 1} d \csc^{-1}(a+bx) + \frac{i \csc^{-1}(a+bx)^3}{3} \right) \\ & 2i \left( \frac{1}{2} i \csc^{-1}(a+bx)^2 \log \left( 1 - e^{2i \csc^{-1}(a+bx)} \right) - i \left( \frac{1}{2} i \csc^{-1}(a+bx) \text{PolyLog} \left( 2, e^{2i \csc^{-1}(a+bx)} \right) - \frac{1}{4} \int e^{-2i \csc^{-1}(a+bx)} \right. \right. \\ & \left. \left. \frac{1}{3} i \csc^{-1}(a+bx)^3 \right) \right. \end{aligned}$$

↓ 2620

$$\begin{aligned} & -a \left( \frac{2 \int \csc^{-1}(a+bx) \log \left( \frac{ie^{i \csc^{-1}(a+bx)} a}{1 - \sqrt{1-a^2}} + 1 \right) d \csc^{-1}(a+bx)}{a} + \frac{2 \int \csc^{-1}(a+bx) \log \left( \frac{ie^{i \csc^{-1}(a+bx)} a}{\sqrt{1-a^2} + 1} + 1 \right) d \csc^{-1}(a+bx)}{a} \right) \\ & 2i \left( \frac{1}{2} i \csc^{-1}(a+bx)^2 \log \left( 1 - e^{2i \csc^{-1}(a+bx)} \right) - i \left( \frac{1}{2} i \csc^{-1}(a+bx) \text{PolyLog} \left( 2, e^{2i \csc^{-1}(a+bx)} \right) - \frac{1}{4} \int e^{-2i \csc^{-1}(a+bx)} \right. \right. \\ & \left. \left. \frac{1}{3} i \csc^{-1}(a+bx)^3 \right) \right. \end{aligned}$$

↓ 3011

$$-a \left( \frac{2 \left( i \operatorname{csc}^{-1}(a+bx) \operatorname{PolyLog} \left( 2, -\frac{iae^{i \operatorname{csc}^{-1}(a+bx)}}{1-\sqrt{1-a^2}} \right) - i \int \operatorname{PolyLog} \left( 2, -\frac{iae^{i \operatorname{csc}^{-1}(a+bx)}}{1-\sqrt{1-a^2}} \right) d \operatorname{csc}^{-1}(a+bx) \right)}{a} + \frac{2 \left( i \operatorname{csc}^{-1}(a+bx) \right)}{a} \right) + \frac{2i \left( \frac{1}{2} i \operatorname{csc}^{-1}(a+bx)^2 \log \left( 1 - e^{2i \operatorname{csc}^{-1}(a+bx)} \right) - i \left( \frac{1}{2} i \operatorname{csc}^{-1}(a+bx) \operatorname{PolyLog} \left( 2, e^{2i \operatorname{csc}^{-1}(a+bx)} \right) - \frac{1}{4} \int e^{-2i \operatorname{csc}^{-1}(a+bx)} \right) - \frac{1}{3} i \operatorname{csc}^{-1}(a+bx)^3 \right)}{a}$$

↓ 2720

$$-a \left( \frac{2 \left( i \operatorname{csc}^{-1}(a+bx) \operatorname{PolyLog} \left( 2, -\frac{iae^{i \operatorname{csc}^{-1}(a+bx)}}{1-\sqrt{1-a^2}} \right) - \int e^{-i \operatorname{csc}^{-1}(a+bx)} \operatorname{PolyLog} \left( 2, -\frac{iae^{i \operatorname{csc}^{-1}(a+bx)}}{1-\sqrt{1-a^2}} \right) de^{i \operatorname{csc}^{-1}(a+bx)} \right)}{a} + \frac{2 \left( i \operatorname{csc}^{-1}(a+bx) \right)}{a} \right) + \frac{2i \left( \frac{1}{2} i \operatorname{csc}^{-1}(a+bx)^2 \log \left( 1 - e^{2i \operatorname{csc}^{-1}(a+bx)} \right) - i \left( \frac{1}{2} i \operatorname{csc}^{-1}(a+bx) \operatorname{PolyLog} \left( 2, e^{2i \operatorname{csc}^{-1}(a+bx)} \right) - \frac{1}{4} \int e^{-2i \operatorname{csc}^{-1}(a+bx)} \right) - \frac{1}{3} i \operatorname{csc}^{-1}(a+bx)^3 \right)}{a}$$

↓ 7143

$$-a \left( \frac{2 \left( i \operatorname{csc}^{-1}(a+bx) \operatorname{PolyLog} \left( 2, -\frac{iae^{i \operatorname{csc}^{-1}(a+bx)}}{1-\sqrt{1-a^2}} \right) - \operatorname{PolyLog} \left( 3, -\frac{iae^{i \operatorname{csc}^{-1}(a+bx)}}{1-\sqrt{1-a^2}} \right) \right)}{a} + \frac{2 \left( i \operatorname{csc}^{-1}(a+bx) \operatorname{PolyLog} \left( 2, -\frac{iae^{i \operatorname{csc}^{-1}(a+bx)}}{1-\sqrt{1-a^2}} \right) \right)}{a} \right) + \frac{2i \left( \frac{1}{2} i \operatorname{csc}^{-1}(a+bx)^2 \log \left( 1 - e^{2i \operatorname{csc}^{-1}(a+bx)} \right) - i \left( \frac{1}{2} i \operatorname{csc}^{-1}(a+bx) \operatorname{PolyLog} \left( 2, e^{2i \operatorname{csc}^{-1}(a+bx)} \right) - \frac{1}{4} \operatorname{PolyLog} \left( 3, e^{2i \operatorname{csc}^{-1}(a+bx)} \right) - \frac{1}{3} i \operatorname{csc}^{-1}(a+bx)^3 \right)}{a}$$

input

```
Int[ArcCsc[a + b*x]^2/x,x]
```

output

```
(I/3)*ArcCsc[a + b*x]^3 - a*((I/3)*ArcCsc[a + b*x]^3)/a - (ArcCsc[a + b*x]^2*Log[1 + (I*a*E^(I*ArcCsc[a + b*x]))/(1 - Sqrt[1 - a^2])])/a - (ArcCsc[a + b*x]^2*Log[1 + (I*a*E^(I*ArcCsc[a + b*x]))/(1 + Sqrt[1 - a^2])])/a + (2*(I*ArcCsc[a + b*x]*PolyLog[2, ((-I)*a*E^(I*ArcCsc[a + b*x]))/(1 - Sqrt[1 - a^2])]) - PolyLog[3, ((-I)*a*E^(I*ArcCsc[a + b*x]))/(1 - Sqrt[1 - a^2])])/a + (2*(I*ArcCsc[a + b*x]*PolyLog[2, ((-I)*a*E^(I*ArcCsc[a + b*x]))/(1 + Sqrt[1 - a^2])]) - PolyLog[3, ((-I)*a*E^(I*ArcCsc[a + b*x]))/(1 + Sqrt[1 - a^2])])/a + (2*I)*((I/2)*ArcCsc[a + b*x]^2*Log[1 - E^((2*I)*ArcCsc[a + b*x])] - I*((I/2)*ArcCsc[a + b*x]*PolyLog[2, E^((2*I)*ArcCsc[a + b*x])] - PolyLog[3, E^((2*I)*ArcCsc[a + b*x])]/4))
```

## Definitions of rubi rules used

- rule 25 `Int[-(Fx), x_Symbol] := Simp[Identity[-1] Int[Fx, x], x]`
- rule 2620 `Int[(((F-)((g-)*(e-) + (f-)*(x-)))(n-)((c-) + (d-)*(x-)(m-)))/  
((a-) + (b-)*((F-)((g-)*(e-) + (f-)*(x-)))(n-))], x_Symbol] := Simp  
[((c + d*x)m/(b*f*g*n*Log[F]))*Log[1 + b*((F(g*(e + f*x)))n/a], x] - Si  
mp[d*(m/(b*f*g*n*Log[F])) Int[(c + d*x)(m - 1)*Log[1 + b*((F(g*(e + f*x  
)n/a], x], x] /; FreeQ[{F, a, b, c, d, e, f, g, n}, x] && IGtQ[m, 0]`
- rule 2720 `Int[u-, x_Symbol] := With[{v = FunctionOfExponential[u, x]}, Simp[v/D[v, x]  
Subst[Int[FunctionOfExponentialFunction[u, x]/x, x], x, v], x]] /; Funct  
ionOfExponentialQ[u, x] && !MatchQ[u, (w-)*((a-)*(v-)(n-))(m-) /; FreeQ  
[{a, m, n}, x] && IntegerQ[m*n] && !MatchQ[u, E((c-)*(a-) + (b-)*x)  
*(F-)[v-] /; FreeQ[{a, b, c}, x] && InverseFunctionQ[F[x]]]`
- rule 3011 `Int[Log[1 + (e-)*((F-)((c-)*(a-) + (b-)*(x-)))(n-)]*((f-) + (g-)  
*(x-)(m-)]], x_Symbol] := Simp[(-f + g*x)m*(PolyLog[2, (-e)*(F(c*(a +  
b*x)))n]/(b*c*n*Log[F])), x] + Simp[g*(m/(b*c*n*Log[F])) Int[(f + g*x)(  
m - 1)*PolyLog[2, (-e)*(F(c*(a + b*x)))n], x], x] /; FreeQ[{F, a, b, c, e  
, f, g, n}, x] && GtQ[m, 0]`
- rule 3042 `Int[u-, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear  
Q[u, x]`
- rule 4200 `Int[(((c-) + (d-)*(x-)(m-))*tan[(e-) + Pi*(k-) + (f-)*(x-)], x_Symbol  
] := Simp[I*((c + d*x)(m + 1)/(d*(m + 1))), x] - Simp[2*I Int[(c + d*x)  
m*E(2*I*k*Pi)*(E(2*I*(e + f*x))/(1 + E(2*I*k*Pi)*E(2*I*(e + f*x)))), x]  
, x] /; FreeQ[{c, d, e, f}, x] && IntegerQ[4*k] && IGtQ[m, 0]`



rule 5030

```
Int[(Cos[(c_.) + (d_.)*(x_)]*((e_.) + (f_.)*(x_)^(m_.)))/((a_) + (b_.)*Sin[
(c_.) + (d_.)*(x_)]), x_Symbol] := Simp[(-I)*((e + f*x)^(m + 1)/(b*f*(m + 1
))), x] + (Int[(e + f*x)^m*(E^(I*(c + d*x)))/(a - Rt[a^2 - b^2, 2] - I*b*E^(
I*(c + d*x)))]), x] + Int[(e + f*x)^m*(E^(I*(c + d*x)))/(a + Rt[a^2 - b^2, 2]
- I*b*E^(I*(c + d*x)))]), x] /; FreeQ[{a, b, c, d, e, f}, x] && IGtQ[m, 0]
&& PosQ[a^2 - b^2]
```

rule 5040

```
Int[(Cot[(c_.) + (d_.)*(x_)]^(n_.)*((e_.) + (f_.)*(x_)^(m_.)))/((a_) + (b_.
)*Sin[(c_.) + (d_.)*(x_)]), x_Symbol] := Simp[1/a Int[(e + f*x)^m*Cot[c +
d*x]^n, x], x] - Simp[b/a Int[(e + f*x)^m*Cos[c + d*x]*(Cot[c + d*x]^(n
- 1)/(a + b*Sin[c + d*x]))], x], x] /; FreeQ[{a, b, c, d, e, f}, x] && IGtQ[
m, 0] && IGtQ[n, 0]
```

rule 5063

```
Int[(((e_.) + (f_.)*(x_)^(m_.))*(F_)[(c_.) + (d_.)*(x_)]^(n_.)*(G_)[(c_.) +
(d_.)*(x_)]^(p_.))/(Csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_.)), x_Symbol] := In
t[(e + f*x)^m*Sin[c + d*x]*F[c + d*x]^n*(G[c + d*x]^p/(b + a*Sin[c + d*x]))
, x] /; FreeQ[{a, b, c, d, e, f}, x] && TrigQ[F] && TrigQ[G] && IntegersQ[m
, n, p]
```

rule 5782

```
Int[((a_.) + ArcCsc[(c_) + (d_.)*(x_)]*(b_.))^(p_.)*((e_.) + (f_.)*(x_)^(m
_.), x_Symbol] := Simp[-(d^(m + 1))^(-1) Subst[Int[(a + b*x)^p*Csc[x]*Cot
[x]*(d*e - c*f + f*Csc[x])^m, x], x, ArcCsc[c + d*x]], x] /; FreeQ[{a, b, c
, d, e, f}, x] && IGtQ[p, 0] && IntegerQ[m]
```

rule 7143

```
Int[PolyLog[n_, (c_.)*((a_.) + (b_.)*(x_)^(p_.)]/((d_.) + (e_.)*(x_)), x_S
ymbol] := Simp[PolyLog[n + 1, c*(a + b*x)^p/(e*p), x] /; FreeQ[{a, b, c, d
, e, n, p}, x] && EqQ[b*d, a*e]
```

**Maple [F]**

$$\int \frac{\operatorname{arccsc}(bx+a)^2}{x} dx$$

input `int(arccsc(b*x+a)^2/x,x)`

output `int(arccsc(b*x+a)^2/x,x)`

**Fricas [F]**

$$\int \frac{\csc^{-1}(a+bx)^2}{x} dx = \int \frac{\operatorname{arccsc}(bx+a)^2}{x} dx$$

input `integrate(arccsc(b*x+a)^2/x,x, algorithm="fricas")`

output `integral(arccsc(b*x + a)^2/x, x)`

**Sympy [F]**

$$\int \frac{\csc^{-1}(a+bx)^2}{x} dx = \int \frac{\operatorname{acsc}^2(a+bx)}{x} dx$$

input `integrate(acsc(b*x+a)**2/x,x)`

output `Integral(acsc(a + b*x)**2/x, x)`

**Maxima [F]**

$$\int \frac{\csc^{-1}(a + bx)^2}{x} dx = \int \frac{\operatorname{arccsc}(bx + a)^2}{x} dx$$

input `integrate(arccsc(b*x+a)^2/x,x, algorithm="maxima")`

output `integrate(arccsc(b*x + a)^2/x, x)`

**Giac [F]**

$$\int \frac{\csc^{-1}(a + bx)^2}{x} dx = \int \frac{\operatorname{arccsc}(bx + a)^2}{x} dx$$

input `integrate(arccsc(b*x+a)^2/x,x, algorithm="giac")`

output `integrate(arccsc(b*x + a)^2/x, x)`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{\csc^{-1}(a + bx)^2}{x} dx = \int \frac{\operatorname{asin}\left(\frac{1}{a+bx}\right)^2}{x} dx$$

input `int(asin(1/(a + b*x))^2/x,x)`

output `int(asin(1/(a + b*x))^2/x, x)`

**Reduce [F]**

$$\int \frac{\csc^{-1}(a + bx)^2}{x} dx = \int \frac{\operatorname{acsc}(bx + a)^2}{x} dx$$

input `int(acsc(b*x+a)^2/x,x)`

output `int(acsc(a + b*x)**2/x,x)`

### 3.32 $\int \frac{\csc^{-1}(a+bx)^2}{x^2} dx$

Optimal result	268
Mathematica [B] (warning: unable to verify)	269
Rubi [A] (verified)	270
Maple [A] (verified)	272
Fricas [F]	272
Sympy [F]	273
Maxima [F]	273
Giac [F]	273
Mupad [F(-1)]	274
Reduce [F]	274

#### Optimal result

Integrand size = 12, antiderivative size = 254

$$\int \frac{\csc^{-1}(a+bx)^2}{x^2} dx = -\frac{b \csc^{-1}(a+bx)^2}{a} - \frac{\csc^{-1}(a+bx)^2}{x} - \frac{2ib \csc^{-1}(a+bx) \log\left(1 + \frac{iae^{i \csc^{-1}(a+bx)}}{1-\sqrt{1-a^2}}\right)}{a\sqrt{1-a^2}} + \frac{2ib \csc^{-1}(a+bx) \log\left(1 + \frac{iae^{i \csc^{-1}(a+bx)}}{1+\sqrt{1-a^2}}\right)}{a\sqrt{1-a^2}} - \frac{2b \text{PolyLog}\left(2, -\frac{iae^{i \csc^{-1}(a+bx)}}{1-\sqrt{1-a^2}}\right)}{a\sqrt{1-a^2}} + \frac{2b \text{PolyLog}\left(2, -\frac{iae^{i \csc^{-1}(a+bx)}}{1+\sqrt{1-a^2}}\right)}{a\sqrt{1-a^2}}$$

output

```
-b*arccsc(b*x+a)^2/a-arccsc(b*x+a)^2/x-2*I*b*arccsc(b*x+a)*ln(1+I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1-(-a^2+1)^(1/2)))/a/(-a^2+1)^(1/2)+2*I*b*arccsc(b*x+a)*ln(1+I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1+(-a^2+1)^(1/2)))/a/(-a^2+1)^(1/2)-2*b*polylog(2,-I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1-(-a^2+1)^(1/2)))/a/(-a^2+1)^(1/2)+2*b*polylog(2,-I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1+(-a^2+1)^(1/2)))/a/(-a^2+1)^(1/2)
```

**Mathematica [B] (warning: unable to verify)**

Both result and optimal contain complex but leaf count is larger than twice the leaf count of optimal. 802 vs.  $2(254) = 508$ .

Time = 2.18 (sec) , antiderivative size = 802, normalized size of antiderivative = 3.16

$$\int \frac{\csc^{-1}(a + bx)^2}{x^2} dx = \text{Too large to display}$$

input

```
Integrate[ArcCsc[a + b*x]^2/x^2,x]
```

output

```

-((b*((a + b*x)*ArcCsc[a + b*x]^2)/(b*x) + (2*Pi*ArcTan[(a - Tan[ArcCsc[a
+ b*x]/2])/Sqrt[1 - a^2]])/Sqrt[1 - a^2] + (2*(-2*ArcCos[a^(-1)]*ArcTanh[
((1 + a)*Cot[(Pi + 2*ArcCsc[a + b*x])/4])/Sqrt[-1 + a^2]] + (Pi - 2*ArcCsc
[a + b*x])*ArcTanh[((-1 + a)*Tan[(Pi + 2*ArcCsc[a + b*x])/4])/Sqrt[-1 + a^
2]] + (ArcCos[a^(-1)] + (2*I)*(-ArcTanh[((1 + a)*Cot[(Pi + 2*ArcCsc[a + b*
x])/4])/Sqrt[-1 + a^2]] + ArcTanh[((-1 + a)*Tan[(Pi + 2*ArcCsc[a + b*x])/4
])/Sqrt[-1 + a^2]]))*Log[((1/2 + I/2)*Sqrt[-1 + a^2])/(Sqrt[a]*E^((I/2)*Ar
cCsc[a + b*x])*Sqrt[-((b*x)/(a + b*x))]) + (ArcCos[a^(-1)] + (2*I)*ArcTan
h[((1 + a)*Cot[(Pi + 2*ArcCsc[a + b*x])/4])/Sqrt[-1 + a^2]] - (2*I)*ArcTan
h[((-1 + a)*Tan[(Pi + 2*ArcCsc[a + b*x])/4])/Sqrt[-1 + a^2]])*Log[((1/2 -
I/2)*Sqrt[-1 + a^2]*E^((I/2)*ArcCsc[a + b*x])/(Sqrt[a]*Sqrt[-((b*x)/(a +
b*x))])) - (ArcCos[a^(-1)] - (2*I)*ArcTanh[((1 + a)*Cot[(Pi + 2*ArcCsc[a +
b*x])/4])/Sqrt[-1 + a^2]])*Log[((-1 + a)*(I + I*a + Sqrt[-1 + a^2])*(-I +
Cot[(Pi + 2*ArcCsc[a + b*x])/4]))/(a*(-1 + a + Sqrt[-1 + a^2])*Cot[(Pi + 2
*ArcCsc[a + b*x])/4])) - (ArcCos[a^(-1)] + (2*I)*ArcTanh[((1 + a)*Cot[(Pi
+ 2*ArcCsc[a + b*x])/4])/Sqrt[-1 + a^2]])*Log[((-1 + a)*(-I - I*a + Sqrt[
-1 + a^2])*(I + Cot[(Pi + 2*ArcCsc[a + b*x])/4]))/(a*(-1 + a + Sqrt[-1 + a
^2])*Cot[(Pi + 2*ArcCsc[a + b*x])/4])) + I*(-PolyLog[2, ((1 - I*Sqrt[-1 +
a^2])*(1 - a + Sqrt[-1 + a^2])*Cot[(Pi + 2*ArcCsc[a + b*x])/4]))/(a*(-1 + a
+ Sqrt[-1 + a^2])*Cot[(Pi + 2*ArcCsc[a + b*x])/4])) + PolyLog[2, ((1 + ...

```

**Rubi [A] (verified)**

Time = 0.75 (sec) , antiderivative size = 258, normalized size of antiderivative = 1.02, number of steps used = 6, number of rules used = 5,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.417$ , Rules used = {5782, 4927, 3042, 4679, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{\csc^{-1}(a+bx)^2}{x^2} dx \\
 & \quad \downarrow \text{5782} \\
 & -b \int \frac{(a+bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)^2}{b^2 x^2} d \csc^{-1}(a+bx) \\
 & \quad \downarrow \text{4927} \\
 & -b \left( 2 \int -\frac{\csc^{-1}(a+bx)}{bx} d \csc^{-1}(a+bx) + \frac{\csc^{-1}(a+bx)^2}{bx} \right) \\
 & \quad \downarrow \text{3042} \\
 & -b \left( 2 \int \frac{\csc^{-1}(a+bx)}{a - \csc(\csc^{-1}(a+bx))} d \csc^{-1}(a+bx) + \frac{\csc^{-1}(a+bx)^2}{bx} \right) \\
 & \quad \downarrow \text{4679} \\
 & -b \left( 2 \int \left( \frac{\csc^{-1}(a+bx)}{a} + \frac{\csc^{-1}(a+bx)}{a \left( \frac{a}{a+bx} - 1 \right)} \right) d \csc^{-1}(a+bx) + \frac{\csc^{-1}(a+bx)^2}{bx} \right) \\
 & \quad \downarrow \text{2009} \\
 & -b \left( \frac{\csc^{-1}(a+bx)^2}{bx} + 2 \left( \frac{\text{PolyLog} \left( 2, -\frac{iae^{i \csc^{-1}(a+bx)}}{1-\sqrt{1-a^2}} \right)}{a\sqrt{1-a^2}} - \frac{\text{PolyLog} \left( 2, -\frac{iae^{i \csc^{-1}(a+bx)}}{\sqrt{1-a^2}+1} \right)}{a\sqrt{1-a^2}} + \frac{i \csc^{-1}(a+bx) \log(1)}{a\sqrt{1-a^2}} \right) \right)
 \end{aligned}$$

input `Int[ArcCsc[a + b*x]^2/x^2,x]`

output

```

-(b*(ArcCsc[a + b*x]^2/(b*x) + 2*(ArcCsc[a + b*x]^2/(2*a) + (I*ArcCsc[a +
b*x]*Log[1 + (I*a*E^(I*ArcCsc[a + b*x]))/(1 - Sqrt[1 - a^2])])/(a*Sqrt[1 -
a^2]) - (I*ArcCsc[a + b*x]*Log[1 + (I*a*E^(I*ArcCsc[a + b*x]))/(1 + Sqrt[
1 - a^2])])/(a*Sqrt[1 - a^2]) + PolyLog[2, ((-I)*a*E^(I*ArcCsc[a + b*x]))/
(1 - Sqrt[1 - a^2])]/(a*Sqrt[1 - a^2]) - PolyLog[2, ((-I)*a*E^(I*ArcCsc[a
+ b*x]))/(1 + Sqrt[1 - a^2])]/(a*Sqrt[1 - a^2])))

```

### Defintions of rubi rules used

rule 2009

```
Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 3042

```
Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4679

```
Int[(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_.))^(n_.)*((c_.) + (d_.)*(x_))^(m_.)
, x_Symbol] := Int[ExpandIntegrand[(c + d*x)^m, 1/(Sin[e + f*x]^n/(b + a*Si
n[e + f*x])^n), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && ILtQ[n, 0] && IGt
Q[m, 0]
```

rule 4927

```
Int[Cot[(c_.) + (d_.)*(x_)]*Csc[(c_.) + (d_.)*(x_)]*(Csc[(c_.) + (d_.)*(x_
)]*(b_.) + (a_.))^(n_.)*((e_.) + (f_.)*(x_))^(m_.), x_Symbol] := Simp[(-e +
f*x)^m)*((a + b*Csc[c + d*x])^(n + 1)/(b*d*(n + 1))), x] + Simp[f*(m/(b*d*(
n + 1))) Int[(e + f*x)^(m - 1)*(a + b*Csc[c + d*x])^(n + 1), x], x] /; Fr
eeQ[{a, b, c, d, e, f, n}, x] && IGtQ[m, 0] && NeQ[n, -1]
```

rule 5782

```
Int[((a_.) + ArcCsc[(c_) + (d_.)*(x_)]*(b_.))^(p_.)*((e_.) + (f_.)*(x_))^(m
_.), x_Symbol] := Simp[-(d^(m + 1))^(-1) Subst[Int[(a + b*x)^p*Csc[x]*Cot
[x]*(d*e - c*f + f*Csc[x])^m, x], x, ArcCsc[c + d*x]], x] /; FreeQ[{a, b, c
, d, e, f}, x] && IGtQ[p, 0] && IntegerQ[m]
```



### Maple [A] (verified)

Time = 1.24 (sec) , antiderivative size = 302, normalized size of antiderivative = 1.19

method	result
derivativedivides	$b \left( -\frac{(bx+a) \operatorname{arccsc}(bx+a)^2}{abx} - \frac{2 \operatorname{arccsc}(bx+a) \ln \left( \frac{-\left(\frac{i}{bx+a} + \sqrt{1 - \frac{1}{(bx+a)^2}}\right) a + i + \sqrt{a^2 - 1}}{i + \sqrt{a^2 - 1}} \right)}{a\sqrt{a^2 - 1}} \right) + \frac{2 \operatorname{arccsc}(bx+a) \ln \left( \dots \right)}{a\sqrt{a^2 - 1}}$
default	$b \left( -\frac{(bx+a) \operatorname{arccsc}(bx+a)^2}{abx} - \frac{2 \operatorname{arccsc}(bx+a) \ln \left( \frac{-\left(\frac{i}{bx+a} + \sqrt{1 - \frac{1}{(bx+a)^2}}\right) a + i + \sqrt{a^2 - 1}}{i + \sqrt{a^2 - 1}} \right)}{a\sqrt{a^2 - 1}} \right) + \frac{2 \operatorname{arccsc}(bx+a) \ln \left( \dots \right)}{a\sqrt{a^2 - 1}}$

```
input int(arccsc(b*x+a)^2/x^2,x,method=_RETURNVERBOSE)
```

```
output b*(-(b*x+a)*arccsc(b*x+a)^2/a/b/x-2/a*arccsc(b*x+a)/(a^2-1)^(1/2)*ln((-I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))*a+I+(a^2-1)^(1/2))/(I+(a^2-1)^(1/2)))+2/a*arccsc(b*x+a)/(a^2-1)^(1/2)*ln((-I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))*a+I-(a^2-1)^(1/2))/(I-(a^2-1)^(1/2)))+2*I/a/(a^2-1)^(1/2)*dilog((-I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))*a+I+(a^2-1)^(1/2))/(I+(a^2-1)^(1/2))-2*I/a/(a^2-1)^(1/2)*dilog((-I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))*a+I-(a^2-1)^(1/2))/(I-(a^2-1)^(1/2)))
```

### Fricas [F]

$$\int \frac{\operatorname{csc}^{-1}(a + bx)^2}{x^2} dx = \int \frac{\operatorname{arccsc}(bx + a)^2}{x^2} dx$$

```
input integrate(arccsc(b*x+a)^2/x^2,x, algorithm="fricas")
```

```
output integral(arccsc(b*x + a)^2/x^2, x)
```

**Sympy [F]**

$$\int \frac{\csc^{-1}(a + bx)^2}{x^2} dx = \int \frac{\operatorname{acsc}^2(a + bx)}{x^2} dx$$

input `integrate(acsc(b*x+a)**2/x**2,x)`

output `Integral(acsc(a + b*x)**2/x**2, x)`

**Maxima [F]**

$$\int \frac{\csc^{-1}(a + bx)^2}{x^2} dx = \int \frac{\operatorname{arccsc}(bx + a)^2}{x^2} dx$$

input `integrate(arccsc(b*x+a)^2/x^2,x, algorithm="maxima")`

output `-1/4*(4*arctan2(1, sqrt(b*x + a + 1))*sqrt(b*x + a - 1))^2 + 4*x*integrate(
(2*sqrt(b*x + a + 1)*sqrt(b*x + a - 1)*b*x*arctan2(1, sqrt(b*x + a + 1)*sq
rt(b*x + a - 1)) + (b^3*x^3 + 3*a*b^2*x^2 + a^3 + (3*a^2 - 1)*b*x - a)*log
(b*x + a)^2 + (b^3*x^3 + 2*a*b^2*x^2 + (a^2 - 1)*b*x - (b^3*x^3 + 3*a*b^2*
x^2 + a^3 + (3*a^2 - 1)*b*x - a)*log(b*x + a))*log(b^2*x^2 + 2*a*b*x + a^2
))/(b^3*x^5 + 3*a*b^2*x^4 + (3*a^2 - 1)*b*x^3 + (a^3 - a)*x^2), x) - log(b
^2*x^2 + 2*a*b*x + a^2)^2)/x`

**Giac [F]**

$$\int \frac{\csc^{-1}(a + bx)^2}{x^2} dx = \int \frac{\operatorname{arccsc}(bx + a)^2}{x^2} dx$$

input `integrate(arccsc(b*x+a)^2/x^2,x, algorithm="giac")`

output `integrate(arccsc(b*x + a)^2/x^2, x)`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{\csc^{-1}(a + bx)^2}{x^2} dx = \int \frac{\operatorname{asin}\left(\frac{1}{a+bx}\right)^2}{x^2} dx$$

input `int(asin(1/(a + b*x))^2/x^2,x)`output `int(asin(1/(a + b*x))^2/x^2, x)`**Reduce [F]**

$$\int \frac{\csc^{-1}(a + bx)^2}{x^2} dx = \int \frac{\operatorname{acsc}(bx + a)^2}{x^2} dx$$

input `int(acsc(b*x+a)^2/x^2,x)`output `int(acsc(a + b*x)**2/x**2,x)`

**3.33**  $\int x^2 \csc^{-1}(a + bx)^3 dx$ 

Optimal result	276
Mathematica [A] (warning: unable to verify)	277
Rubi [A] (verified)	278
Maple [A] (verified)	280
Fricas [F]	281
Sympy [F]	281
Maxima [F]	282
Giac [F]	283
Mupad [F(-1)]	283
Reduce [F]	283

**Optimal result**

Integrand size = 12, antiderivative size = 464

$$\begin{aligned}
\int x^2 \csc^{-1}(a+bx)^3 dx = & \frac{(a+bx) \csc^{-1}(a+bx)}{b^3} - \frac{3ia \csc^{-1}(a+bx)^2}{b^3} \\
& - \frac{3a(a+bx) \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)^2}{b^3} \\
& + \frac{(a+bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)^2}{2b^3} + \frac{a^3 \csc^{-1}(a+bx)^3}{3b^3} \\
& + \frac{1}{3} x^3 \csc^{-1}(a+bx)^3 + \frac{\csc^{-1}(a+bx)^2 \operatorname{arctanh}\left(e^{i \csc^{-1}(a+bx)}\right)}{b^3} \\
& + \frac{6a^2 \csc^{-1}(a+bx)^2 \operatorname{arctanh}\left(e^{i \csc^{-1}(a+bx)}\right)}{b^3} \\
& + \frac{\operatorname{arctanh}\left(\sqrt{1 - \frac{1}{(a+bx)^2}}\right)}{b^3} \\
& + \frac{6a \csc^{-1}(a+bx) \log\left(1 - e^{2i \csc^{-1}(a+bx)}\right)}{b^3} \\
& - \frac{i \csc^{-1}(a+bx) \operatorname{PolyLog}\left(2, -e^{i \csc^{-1}(a+bx)}\right)}{b^3} \\
& - \frac{6ia^2 \csc^{-1}(a+bx) \operatorname{PolyLog}\left(2, -e^{i \csc^{-1}(a+bx)}\right)}{b^3} \\
& + \frac{i \csc^{-1}(a+bx) \operatorname{PolyLog}\left(2, e^{i \csc^{-1}(a+bx)}\right)}{b^3} \\
& + \frac{6ia^2 \csc^{-1}(a+bx) \operatorname{PolyLog}\left(2, e^{i \csc^{-1}(a+bx)}\right)}{b^3} \\
& - \frac{3ia \operatorname{PolyLog}\left(2, e^{2i \csc^{-1}(a+bx)}\right)}{b^3} \\
& + \frac{\operatorname{PolyLog}\left(3, -e^{i \csc^{-1}(a+bx)}\right)}{b^3} \\
& + \frac{6a^2 \operatorname{PolyLog}\left(3, -e^{i \csc^{-1}(a+bx)}\right)}{b^3} \\
& - \frac{\operatorname{PolyLog}\left(3, e^{i \csc^{-1}(a+bx)}\right)}{b^3} - \frac{6a^2 \operatorname{PolyLog}\left(3, e^{i \csc^{-1}(a+bx)}\right)}{b^3}
\end{aligned}$$

output

```
(b*x+a)*arccsc(b*x+a)/b^3-3*I*a*arccsc(b*x+a)^2/b^3-3*a*(b*x+a)*(1-1/(b*x+a)^2)^(1/2)*arccsc(b*x+a)^2/b^3+1/2*(b*x+a)^2*(1-1/(b*x+a)^2)^(1/2)*arccsc(b*x+a)^2/b^3+1/3*a^3*arccsc(b*x+a)^3/b^3+1/3*x^3*arccsc(b*x+a)^3+arccsc(b*x+a)^2*arctanh(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/b^3+6*a^2*arccsc(b*x+a)^2*arctanh(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/b^3+arctanh((1-1/(b*x+a)^2)^(1/2))/b^3+6*a*arccsc(b*x+a)*ln(1-(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))^2)/b^3+I*arccsc(b*x+a)*polylog(2,I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/b^3+6*I*a^2*arccsc(b*x+a)*polylog(2,I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/b^3-6*I*a^2*arccsc(b*x+a)*polylog(2,-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))/b^3-I*arccsc(b*x+a)*polylog(2,-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))/b^3-3*I*a*polylog(2,(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))^2)/b^3+polylog(3,-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))/b^3+6*a^2*polylog(3,-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))/b^3-polylog(3,I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/b^3-6*a^2*polylog(3,I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/b^3
```

**Mathematica [A] (warning: unable to verify)**

Time = 6.42 (sec) , antiderivative size = 656, normalized size of antiderivative = 1.41

$$\int x^2 \csc^{-1}(a + bx)^3 dx = \text{Too large to display}$$

input

```
Integrate[x^2*ArcCsc[a + b*x]^3,x]
```

output

```

-1/24*((72*I)*a*ArcCsc[a + b*x]^2 - 12*ArcCsc[a + b*x]*Cot[ArcCsc[a + b*x]
/2] + 36*a*ArcCsc[a + b*x]^2*Cot[ArcCsc[a + b*x]/2] - 2*ArcCsc[a + b*x]^3*
Cot[ArcCsc[a + b*x]/2] - 12*a^2*ArcCsc[a + b*x]^3*Cot[ArcCsc[a + b*x]/2] -
3*ArcCsc[a + b*x]^2*Csc[ArcCsc[a + b*x]/2]^2 + 6*a*ArcCsc[a + b*x]^3*Csc[
ArcCsc[a + b*x]/2]^2 - (ArcCsc[a + b*x]^3*Csc[ArcCsc[a + b*x]/2]^4)/(2*(a
+ b*x)) + 12*ArcCsc[a + b*x]^2*Log[1 - E^(I*ArcCsc[a + b*x])] + 72*a^2*Arc
Csc[a + b*x]^2*Log[1 - E^(I*ArcCsc[a + b*x])] - 12*ArcCsc[a + b*x]^2*Log[1
+ E^(I*ArcCsc[a + b*x])] - 72*a^2*ArcCsc[a + b*x]^2*Log[1 + E^(I*ArcCsc[a
+ b*x])] - 144*a*ArcCsc[a + b*x]*Log[1 - E^((2*I)*ArcCsc[a + b*x])] + 24*
Log[Tan[ArcCsc[a + b*x]/2]] + (24*I)*(1 + 6*a^2)*ArcCsc[a + b*x]*PolyLog[2
, -E^(I*ArcCsc[a + b*x])] - (24*I)*(1 + 6*a^2)*ArcCsc[a + b*x]*PolyLog[2,
E^(I*ArcCsc[a + b*x])] + (72*I)*a*PolyLog[2, E^((2*I)*ArcCsc[a + b*x])] -
24*PolyLog[3, -E^(I*ArcCsc[a + b*x])] - 144*a^2*PolyLog[3, -E^(I*ArcCsc[a
+ b*x])] + 24*PolyLog[3, E^(I*ArcCsc[a + b*x])] + 144*a^2*PolyLog[3, E^(I*
ArcCsc[a + b*x])] + 3*ArcCsc[a + b*x]^2*Sec[ArcCsc[a + b*x]/2]^2 + 6*a*Arc
Csc[a + b*x]^3*Sec[ArcCsc[a + b*x]/2]^2 - 8*(a + b*x)^3*ArcCsc[a + b*x]^3*
Sin[ArcCsc[a + b*x]/2]^4 - 12*ArcCsc[a + b*x]*Tan[ArcCsc[a + b*x]/2] - 36*
a*ArcCsc[a + b*x]^2*Tan[ArcCsc[a + b*x]/2] - 2*ArcCsc[a + b*x]^3*Tan[ArcCs
c[a + b*x]/2] - 12*a^2*ArcCsc[a + b*x]^3*Tan[ArcCsc[a + b*x]/2])/b^3

```

**Rubi [A] (verified)**

Time = 0.79 (sec) , antiderivative size = 420, normalized size of antiderivative = 0.91, number of steps used = 6, number of rules used = 5,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.417$ , Rules used = {5782, 4927, 3042, 4678, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int x^2 \csc^{-1}(a + bx)^3 dx \\
 & \quad \downarrow \text{5782} \\
 & \frac{\int b^2 x^2 (a + bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx)^3 d \csc^{-1}(a + bx)}{b^3} \\
 & \quad \downarrow \text{4927} \\
 & \frac{-\int -b^3 x^3 \csc^{-1}(a + bx)^2 d \csc^{-1}(a + bx) - \frac{1}{3} b^3 x^3 \csc^{-1}(a + bx)^3}{b^3}
 \end{aligned}$$

$$\begin{array}{c}
 \downarrow 3042 \\
 \frac{-\int \csc^{-1}(a+bx)^2 (a - \csc(\csc^{-1}(a+bx)))^3 d \csc^{-1}(a+bx) - \frac{1}{3}b^3 x^3 \csc^{-1}(a+bx)^3}{b^3} \\
 \downarrow 4678 \\
 \frac{-\int (\csc^{-1}(a+bx)^2 a^3 - 3(a+bx) \csc^{-1}(a+bx)^2 a^2 + 3(a+bx)^2 \csc^{-1}(a+bx)^2 a - (a+bx)^3 \csc^{-1}(a+bx)^2)}{b^3} \\
 \downarrow 2009 \\
 \frac{-\frac{1}{3}a^3 \csc^{-1}(a+bx)^3 - 6a^2 \csc^{-1}(a+bx)^2 \operatorname{arctanh}(e^{i \csc^{-1}(a+bx)}) + 6ia^2 \csc^{-1}(a+bx) \operatorname{PolyLog}(2, -e^{i \csc^{-1}(a+bx)})}{b^3}
 \end{array}$$

input `Int[x^2*ArcCsc[a + b*x]^3,x]`

output

```

-(((a + b*x)*ArcCsc[a + b*x]) + (3*I)*a*ArcCsc[a + b*x]^2 + 3*a*(a + b*x)
)*Sqrt[1 - (a + b*x)^(-2)]*ArcCsc[a + b*x]^2 - ((a + b*x)^2*Sqrt[1 - (a +
b*x)^(-2)]*ArcCsc[a + b*x]^2)/2 - (a^3*ArcCsc[a + b*x]^3)/3 - (b^3*x^3*Arc
Csc[a + b*x]^3)/3 - ArcCsc[a + b*x]^2*ArcTanh[E^(I*ArcCsc[a + b*x])] - 6*a
^2*ArcCsc[a + b*x]^2*ArcTanh[E^(I*ArcCsc[a + b*x])] - ArcTanh[Sqrt[1 - (a
+ b*x)^(-2)]] - 6*a*ArcCsc[a + b*x]*Log[1 - E^((2*I)*ArcCsc[a + b*x])] + I
*ArcCsc[a + b*x]*PolyLog[2, -E^(I*ArcCsc[a + b*x])] + (6*I)*a^2*ArcCsc[a +
b*x]*PolyLog[2, -E^(I*ArcCsc[a + b*x])] - I*ArcCsc[a + b*x]*PolyLog[2, E^
(I*ArcCsc[a + b*x])] - (6*I)*a^2*ArcCsc[a + b*x]*PolyLog[2, E^(I*ArcCsc[a
+ b*x])] + (3*I)*a*PolyLog[2, E^((2*I)*ArcCsc[a + b*x])] - PolyLog[3, -E^
(I*ArcCsc[a + b*x])] - 6*a^2*PolyLog[3, -E^(I*ArcCsc[a + b*x])] + PolyLog[3
, E^(I*ArcCsc[a + b*x])] + 6*a^2*PolyLog[3, E^(I*ArcCsc[a + b*x])]/b^3

```

### Defintions of rubi rules used

rule 2009 `Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`



rule 4678 `Int[(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_.))^(n_.)*((c_.) + (d_.)*(x_))^(m_.), x_Symbol] := Int[ExpandIntegrand[(c + d*x)^m, (a + b*Csc[e + f*x])^n, x], x] /; FreeQ[{a, b, c, d, e, f, m}, x] && IGtQ[m, 0] && IGtQ[n, 0]`

rule 4927 `Int[Cot[(c_.) + (d_.)*(x_)]*Csc[(c_.) + (d_.)*(x_)]*(Csc[(c_.) + (d_.)*(x_)]*(b_.) + (a_.))^(n_.)*((e_.) + (f_.)*(x_))^(m_.), x_Symbol] := Simp[(-(e + f*x)^m)*((a + b*Csc[c + d*x])^(n + 1)/(b*d*(n + 1))), x] + Simp[f*(m/(b*d*(n + 1))) Int[(e + f*x)^(m - 1)*(a + b*Csc[c + d*x])^(n + 1), x], x] /; FreeQ[{a, b, c, d, e, f, n}, x] && IGtQ[m, 0] && NeQ[n, -1]`

rule 5782 `Int[((a_.) + ArcCsc[(c_.) + (d_.)*(x_)]*(b_.))^(p_.)*((e_.) + (f_.)*(x_))^(m_.), x_Symbol] := Simp[-(d^(m + 1))^(-1) Subst[Int[(a + b*x)^p*Csc[x]*Cot[x]*(d*e - c*f + f*Csc[x])^m, x], x, ArcCsc[c + d*x]], x] /; FreeQ[{a, b, c, d, e, f}, x] && IGtQ[p, 0] && IntegerQ[m]`

## Maple [A] (verified)

Time = 1.11 (sec) , antiderivative size = 749, normalized size of antiderivative = 1.61

method	result
derivativedivides	$\frac{-6ia \operatorname{arccsc}(bx+a)^2 + \frac{\operatorname{arccsc}(bx+a)^2 \ln\left(1 + \frac{i}{bx+a} + \sqrt{1 - \frac{1}{(bx+a)^2}}\right)}{2} + 6i \operatorname{polylog}\left(2, \frac{i}{bx+a} + \sqrt{1 - \frac{1}{(bx+a)^2}}\right) a^2 \operatorname{arccsc}(bx+a)}{1}$
default	$\frac{-6ia \operatorname{arccsc}(bx+a)^2 + \frac{\operatorname{arccsc}(bx+a)^2 \ln\left(1 + \frac{i}{bx+a} + \sqrt{1 - \frac{1}{(bx+a)^2}}\right)}{2} + 6i \operatorname{polylog}\left(2, \frac{i}{bx+a} + \sqrt{1 - \frac{1}{(bx+a)^2}}\right) a^2 \operatorname{arccsc}(bx+a)}{1}$

input `int(x^2*arccsc(b*x+a)^3,x,method=_RETURNVERBOSE)`

output

```

1/b^3*(-6*I*a*arccsc(b*x+a)^2+1/2*arccsc(b*x+a)^2*ln(1+I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))+6*I*polylog(2,I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))*a^2*arccsc(b*x+a)+polylog(3,-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))-1/2*arccsc(b*x+a)^2*ln(1-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))+1/6*arccsc(b*x+a)*(6*arccsc(b*x+a)^2*a^2*(b*x+a)-6*arccsc(b*x+a)^2*a*(b*x+a)^2+2*arccsc(b*x+a)^2*(b*x+a)^3-18*arccsc(b*x+a)*(((b*x+a)^2-1)/(b*x+a)^2)^(1/2)*a*(b*x+a)+3*arccsc(b*x+a)*(((b*x+a)^2-1)/(b*x+a)^2)^(1/2)*(b*x+a)^2+18*I*a*arccsc(b*x+a)+6*b*x+6*a)-polylog(3,I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))+2*arctanh(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))+6*ln(1+I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))*a*arccsc(b*x+a)+6*ln(1-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))*a*arccsc(b*x+a)-6*I*polylog(2,-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))*a^2*arccsc(b*x+a)-6*I*polylog(2,-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))*a-I*arccsc(b*x+a)*polylog(2,-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))+I*arccsc(b*x+a)*polylog(2,I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))-6*I*polylog(2,I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))*a+3*ln(1+I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))*a^2*arccsc(b*x+a)^2-3*ln(1-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))*a^2*arccsc(b*x+a)^2+6*polylog(3,-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))*a^2-6*polylog(3,I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))*a^2)

```

**Fricas [F]**

$$\int x^2 \csc^{-1}(a + bx)^3 dx = \int x^2 \operatorname{arccsc}(bx + a)^3 dx$$

input

```
integrate(x^2*arccsc(b*x+a)^3,x, algorithm="fricas")
```

output

```
integral(x^2*arccsc(b*x + a)^3, x)
```

**Sympy [F]**

$$\int x^2 \csc^{-1}(a + bx)^3 dx = \int x^2 \operatorname{acsc}^3(a + bx) dx$$

input

```
integrate(x**2*acsc(b*x+a)**3,x)
```

output `Integral(x**2*acsc(a + b*x)**3, x)`

### Maxima [F]

$$\int x^2 \csc^{-1}(a + bx)^3 dx = \int x^2 \operatorname{arccsc}(bx + a)^3 dx$$

input `integrate(x^2*arccsc(b*x+a)^3,x, algorithm="maxima")`

output

```

1/3*x^3*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1))^3 - 1/4*x^3*arctan
2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1))*log(b^2*x^2 + 2*a*b*x + a^2)^2 -
integrate(1/4*(12*(b^3*x^5*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)
) + 3*a*b^2*x^4*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) + (3*a^2*a
rctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) - arctan2(1, sqrt(b*x + a +
1)*sqrt(b*x + a - 1)))*b*x^3 + (a^3*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x
+ a - 1)) - a*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)))*x^2)*log(b
*x + a)^2 - (4*b*x^3*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1))^2 - b
*x^3*log(b^2*x^2 + 2*a*b*x + a^2)^2)*sqrt(b*x + a + 1)*sqrt(b*x + a - 1) -
4*(b^3*x^5*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) + 2*a*b^2*x^4*
arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) + (a^2*arctan2(1, sqrt(b*x
+ a + 1)*sqrt(b*x + a - 1)) - arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a -
1)))*b*x^3 + 3*(b^3*x^5*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) +
3*a*b^2*x^4*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) + (3*a^2*arct
an2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) - arctan2(1, sqrt(b*x + a + 1)
*sqrt(b*x + a - 1)))*b*x^3 + (a^3*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x +
a - 1)) - a*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)))*x^2)*log(b*x
+ a))*log(b^2*x^2 + 2*a*b*x + a^2))/(b^3*x^3 + 3*a*b^2*x^2 + a^3 + (3*a^2
- 1)*b*x - a), x)

```

**Giac [F]**

$$\int x^2 \csc^{-1}(a + bx)^3 dx = \int x^2 \operatorname{arccsc}(bx + a)^3 dx$$

input `integrate(x^2*arccsc(b*x+a)^3,x, algorithm="giac")`

output `integrate(x^2*arccsc(b*x + a)^3, x)`

**Mupad [F(-1)]**

Timed out.

$$\int x^2 \csc^{-1}(a + bx)^3 dx = \int x^2 \operatorname{asin}\left(\frac{1}{a + bx}\right)^3 dx$$

input `int(x^2*asin(1/(a + b*x))^3,x)`

output `int(x^2*asin(1/(a + b*x))^3, x)`

**Reduce [F]**

$$\int x^2 \csc^{-1}(a + bx)^3 dx = \int \operatorname{acsc}(bx + a)^3 x^2 dx$$

input `int(x^2*acsc(b*x+a)^3,x)`

output `int(acsc(a + b*x)**3*x**2,x)`

### 3.34 $\int x \csc^{-1}(a + bx)^3 dx$

Optimal result	284
Mathematica [A] (verified)	285
Rubi [A] (verified)	286
Maple [A] (verified)	288
Fricas [F]	288
Sympy [F]	289
Maxima [F]	289
Giac [F]	290
Mupad [F(-1)]	290
Reduce [F]	290

#### Optimal result

Integrand size = 10, antiderivative size = 264

$$\begin{aligned}
 \int x \csc^{-1}(a + bx)^3 dx = & \frac{3i \csc^{-1}(a + bx)^2}{2b^2} + \frac{3(a + bx) \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx)^2}{2b^2} \\
 & - \frac{a^2 \csc^{-1}(a + bx)^3}{2b^2} + \frac{1}{2} x^2 \csc^{-1}(a + bx)^3 \\
 & - \frac{6a \csc^{-1}(a + bx)^2 \operatorname{arctanh}\left(e^{i \csc^{-1}(a+bx)}\right)}{b^2} \\
 & - \frac{3 \csc^{-1}(a + bx) \log\left(1 - e^{2i \csc^{-1}(a+bx)}\right)}{b^2} \\
 & + \frac{6ia \csc^{-1}(a + bx) \operatorname{PolyLog}\left(2, -e^{i \csc^{-1}(a+bx)}\right)}{b^2} \\
 & - \frac{6ia \csc^{-1}(a + bx) \operatorname{PolyLog}\left(2, e^{i \csc^{-1}(a+bx)}\right)}{b^2} \\
 & + \frac{3i \operatorname{PolyLog}\left(2, e^{2i \csc^{-1}(a+bx)}\right)}{2b^2} \\
 & - \frac{6a \operatorname{PolyLog}\left(3, -e^{i \csc^{-1}(a+bx)}\right)}{b^2} \\
 & + \frac{6a \operatorname{PolyLog}\left(3, e^{i \csc^{-1}(a+bx)}\right)}{b^2}
 \end{aligned}$$

output

$$\begin{aligned} & 3/2*I*\operatorname{arccsc}(b*x+a)^2/b^2+3/2*(b*x+a)*(1-1/(b*x+a)^2)^{(1/2)}*\operatorname{arccsc}(b*x+a)^2/b^2-1/2*a^2*\operatorname{arccsc}(b*x+a)^3/b^2+1/2*x^2*\operatorname{arccsc}(b*x+a)^3-6*a*\operatorname{arccsc}(b*x+a)^2*\operatorname{arctanh}(I/(b*x+a)+(1-1/(b*x+a)^2)^{(1/2)})/b^2-3*\operatorname{arccsc}(b*x+a)*\ln(1-(I/(b*x+a)+(1-1/(b*x+a)^2)^{(1/2)})^2)/b^2+6*I*a*\operatorname{arccsc}(b*x+a)*\operatorname{polylog}(2,-I/(b*x+a)-(1-1/(b*x+a)^2)^{(1/2)})/b^2-6*I*a*\operatorname{arccsc}(b*x+a)*\operatorname{polylog}(2,I/(b*x+a)+(1-1/(b*x+a)^2)^{(1/2)})/b^2+3/2*I*\operatorname{polylog}(2,(I/(b*x+a)+(1-1/(b*x+a)^2)^{(1/2)})^2)/b^2-6*a*\operatorname{polylog}(3,-I/(b*x+a)-(1-1/(b*x+a)^2)^{(1/2)})/b^2+6*a*\operatorname{polylog}(3,I/(b*x+a)+(1-1/(b*x+a)^2)^{(1/2)})/b^2 \end{aligned}$$
**Mathematica [A] (verified)**

Time = 0.57 (sec) , antiderivative size = 314, normalized size of antiderivative = 1.19

$$\int x \operatorname{csc}^{-1}(a + bx)^3 dx$$

$$= \frac{3i \operatorname{csc}^{-1}(a + bx)^2 + 3a \sqrt{\frac{-1+a^2+2abx+b^2x^2}{(a+bx)^2}} \operatorname{csc}^{-1}(a + bx)^2 + 3bx \sqrt{\frac{-1+a^2+2abx+b^2x^2}{(a+bx)^2}} \operatorname{csc}^{-1}(a + bx)^2 - a^2 \operatorname{csc}^{-1}(a + bx)}{b^2}$$

input

Integrate[x\*ArcCsc[a + b\*x]^3,x]

output

$$\begin{aligned} & ((3*I)*\operatorname{ArcCsc}[a + b*x]^2 + 3*a*\operatorname{Sqrt}[(-1 + a^2 + 2*a*b*x + b^2*x^2)/(a + b*x)^2]*\operatorname{ArcCsc}[a + b*x]^2 + 3*b*x*\operatorname{Sqrt}[(-1 + a^2 + 2*a*b*x + b^2*x^2)/(a + b*x)^2]*\operatorname{ArcCsc}[a + b*x]^2 - a^2*\operatorname{ArcCsc}[a + b*x]^3 + b^2*x^2*\operatorname{ArcCsc}[a + b*x]^3 + 6*a*\operatorname{ArcCsc}[a + b*x]^2*\operatorname{Log}[1 - E^(I*\operatorname{ArcCsc}[a + b*x])] - 6*a*\operatorname{ArcCsc}[a + b*x]^2*\operatorname{Log}[1 + E^(I*\operatorname{ArcCsc}[a + b*x])] - 6*\operatorname{ArcCsc}[a + b*x]*\operatorname{Log}[1 - E^((2*I)*\operatorname{ArcCsc}[a + b*x])] + (12*I)*a*\operatorname{ArcCsc}[a + b*x]*\operatorname{PolyLog}[2, -E^(I*\operatorname{ArcCsc}[a + b*x])] - (12*I)*a*\operatorname{ArcCsc}[a + b*x]*\operatorname{PolyLog}[2, E^(I*\operatorname{ArcCsc}[a + b*x])] + (3*I)*\operatorname{PolyLog}[2, E^((2*I)*\operatorname{ArcCsc}[a + b*x])] - 12*a*\operatorname{PolyLog}[3, -E^(I*\operatorname{ArcCsc}[a + b*x])] + 12*a*\operatorname{PolyLog}[3, E^(I*\operatorname{ArcCsc}[a + b*x])])/(2*b^2) \end{aligned}$$

**Rubi [A] (verified)**

Time = 0.62 (sec) , antiderivative size = 241, normalized size of antiderivative = 0.91, number of steps used = 7, number of rules used = 6,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.600$ , Rules used = {5782, 25, 4927, 3042, 4678, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int x \csc^{-1}(a + bx)^3 dx \\
 & \quad \downarrow \text{5782} \\
 & - \frac{\int bx(a + bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx)^3 d \csc^{-1}(a + bx)}{b^2} \\
 & \quad \downarrow \text{25} \\
 & - \frac{\int -bx(a + bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx)^3 d \csc^{-1}(a + bx)}{b^2} \\
 & \quad \downarrow \text{4927} \\
 & - \frac{\frac{3}{2} \int b^2 x^2 \csc^{-1}(a + bx)^2 d \csc^{-1}(a + bx) - \frac{1}{2} b^2 x^2 \csc^{-1}(a + bx)^3}{b^2} \\
 & \quad \downarrow \text{3042} \\
 & - \frac{\frac{3}{2} \int \csc^{-1}(a + bx)^2 (a - \csc(\csc^{-1}(a + bx)))^2 d \csc^{-1}(a + bx) - \frac{1}{2} b^2 x^2 \csc^{-1}(a + bx)^3}{b^2} \\
 & \quad \downarrow \text{4678} \\
 & - \frac{\frac{3}{2} \int (a^2 \csc^{-1}(a + bx)^2 + (a + bx)^2 \csc^{-1}(a + bx)^2 - 2a(a + bx) \csc^{-1}(a + bx)^2) d \csc^{-1}(a + bx) - \frac{1}{2} b^2 x^2 \csc^{-1}(a + bx)^3}{b^2} \\
 & \quad \downarrow \text{2009} \\
 & - \frac{\frac{1}{2} b^2 x^2 \csc^{-1}(a + bx)^3 + \frac{3}{2} \left( \frac{1}{3} a^2 \csc^{-1}(a + bx)^3 + 4a \csc^{-1}(a + bx)^2 \operatorname{arctanh}\left(e^{i \csc^{-1}(a+bx)}\right) - 4ia \csc^{-1}(a + bx) \right)}{b^2}
 \end{aligned}$$

input `Int[x*ArcCsc[a + b*x]^3,x]`

output

```

-((-1/2*(b^2*x^2*ArcCsc[a + b*x]^3) + (3*((-I)*ArcCsc[a + b*x]^2 - (a + b*
x)*Sqrt[1 - (a + b*x)^(-2)]*ArcCsc[a + b*x]^2 + (a^2*ArcCsc[a + b*x]^3)/3
+ 4*a*ArcCsc[a + b*x]^2*ArcTanh[E^(I*ArcCsc[a + b*x])]) + 2*ArcCsc[a + b*x]
*Log[1 - E^((2*I)*ArcCsc[a + b*x])] - (4*I)*a*ArcCsc[a + b*x]*PolyLog[2, -
E^(I*ArcCsc[a + b*x])] + (4*I)*a*ArcCsc[a + b*x]*PolyLog[2, E^(I*ArcCsc[a
+ b*x])] - I*PolyLog[2, E^((2*I)*ArcCsc[a + b*x])] + 4*a*PolyLog[3, -E^(I*
ArcCsc[a + b*x])] - 4*a*PolyLog[3, E^(I*ArcCsc[a + b*x])]))/2)/b^2

```

### Defintions of rubi rules used

rule 25

```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1] Int[Fx, x], x]
```

rule 2009

```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 3042

```
Int[u_, x_Symbol] :> Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4678

```
Int[(csc[(e_.) + (f_.)*(x_)]*(b_.) + (a_.))^(n_.)*((c_.) + (d_.)*(x_))^(m_.)
, x_Symbol] :> Int[ExpandIntegrand[(c + d*x)^m, (a + b*Csc[e + f*x])^n, x],
x] /; FreeQ[{a, b, c, d, e, f, m}, x] && IGtQ[m, 0] && IGtQ[n, 0]
```

rule 4927

```
Int[Cot[(c_.) + (d_.)*(x_)]*Csc[(c_.) + (d_.)*(x_)]*(Csc[(c_.) + (d_.)*(x_)]
*(b_.) + (a_.))^(n_.)*((e_.) + (f_.)*(x_))^(m_.), x_Symbol] :> Simp[(-(e +
f*x)^m)*((a + b*Csc[c + d*x])^(n + 1)/(b*d*(n + 1))), x] + Simp[f*(m/(b*d*(
n + 1))) Int[(e + f*x)^(m - 1)*(a + b*Csc[c + d*x])^(n + 1), x], x] /; Fr
eeQ[{a, b, c, d, e, f, n}, x] && IGtQ[m, 0] && NeQ[n, -1]
```

rule 5782

```
Int[((a_.) + ArcCsc[(c_) + (d_.)*(x_)]*(b_.))^(p_.)*((e_.) + (f_.)*(x_))^(m
_.), x_Symbol] :> Simp[-(d^(m + 1))^(-1) Subst[Int[(a + b*x)^p*Csc[x]*Cot
[x]*(d*e - c*f + f*Csc[x])^m, x], x, ArcCsc[c + d*x]], x] /; FreeQ[{a, b, c
, d, e, f}, x] && IGtQ[p, 0] && IntegerQ[m]
```



**Maple [A] (verified)**

Time = 0.91 (sec) , antiderivative size = 425, normalized size of antiderivative = 1.61

method	result
derivativedivides	$\frac{\operatorname{arccsc}(bx+a)^2 \left( 2 \operatorname{arccsc}(bx+a)a(bx+a) - \operatorname{arccsc}(bx+a)(bx+a)^2 - 3\sqrt{\frac{(bx+a)^2-1}{(bx+a)^2}}(bx+a)+3i \right)}{2} - 3a \operatorname{arccsc}(bx+a)^2 \ln \left( 1 + \frac{i}{bx+a} \right)$
default	$\frac{\operatorname{arccsc}(bx+a)^2 \left( 2 \operatorname{arccsc}(bx+a)a(bx+a) - \operatorname{arccsc}(bx+a)(bx+a)^2 - 3\sqrt{\frac{(bx+a)^2-1}{(bx+a)^2}}(bx+a)+3i \right)}{2} - 3a \operatorname{arccsc}(bx+a)^2 \ln \left( 1 + \frac{i}{bx+a} \right)$

input `int(x*arccsc(b*x+a)^3,x,method=_RETURNVERBOSE)`

output

```
1/b^2*(-1/2*arccsc(b*x+a)^2*(2*arccsc(b*x+a)*a*(b*x+a)-arccsc(b*x+a)*(b*x+a)^2-3*((b*x+a)^2-1)/(b*x+a)^2)^(1/2)*(b*x+a)+3*I)-3*a*arccsc(b*x+a)^2*ln(1+I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))+3*a*arccsc(b*x+a)^2*ln(1-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))-3*arccsc(b*x+a)*ln(1+I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))-6*a*polylog(3,-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))-3*arccsc(b*x+a)*ln(1-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))+6*a*polylog(3,I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))+6*I*a*arccsc(b*x+a)*polylog(2,-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))-6*I*a*arccsc(b*x+a)*polylog(2,I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))+3*I*arccsc(b*x+a)^2+3*I*polylog(2,-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))+3*I*polylog(2,I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2)))
```

**Fricas [F]**

$$\int x \csc^{-1}(a + bx)^3 dx = \int x \operatorname{arccsc}(bx + a)^3 dx$$

input `integrate(x*arccsc(b*x+a)^3,x, algorithm="fricas")`

output `integral(x*arccsc(b*x + a)^3, x)`

**Sympy [F]**

$$\int x \csc^{-1}(a + bx)^3 dx = \int x \operatorname{acsc}^3(a + bx) dx$$

input `integrate(x*acsc(b*x+a)**3,x)`

output `Integral(x*acsc(a + b*x)**3, x)`

**Maxima [F]**

$$\int x \csc^{-1}(a + bx)^3 dx = \int x \operatorname{arccsc}(bx + a)^3 dx$$

input `integrate(x*arccsc(b*x+a)^3,x, algorithm="maxima")`

output

```

1/2*x^2*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1))^3 - 3/8*x^2*arctan
2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1))*log(b^2*x^2 + 2*a*b*x + a^2)^2 -
integrate(3/8*(8*(b^3*x^4*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1))
+ 3*a*b^2*x^3*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) + (3*a^2*ar
ctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) - arctan2(1, sqrt(b*x + a +
1)*sqrt(b*x + a - 1)))*b*x^2 + (a^3*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x
+ a - 1)) - a*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)))*x)*log(b*x
+ a)^2 - (4*b*x^2*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1))^2 - b*x^
2*log(b^2*x^2 + 2*a*b*x + a^2)^2)*sqrt(b*x + a + 1)*sqrt(b*x + a - 1) - 4*
(b^3*x^4*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) + 2*a*b^2*x^3*arc
tan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) + (a^2*arctan2(1, sqrt(b*x +
a + 1)*sqrt(b*x + a - 1)) - arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)
))*b*x^2 + 2*(b^3*x^4*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) + 3*
a*b^2*x^3*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) + (3*a^2*arctan2
(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) - arctan2(1, sqrt(b*x + a + 1)*sq
rt(b*x + a - 1)))*b*x^2 + (a^3*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a -
1)) - a*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)))*x)*log(b*x + a))
*log(b^2*x^2 + 2*a*b*x + a^2))/(b^3*x^3 + 3*a*b^2*x^2 + a^3 + (3*a^2 - 1)*
b*x - a), x)

```

**Giac [F]**

$$\int x \csc^{-1}(a + bx)^3 dx = \int x \operatorname{arccsc}(bx + a)^3 dx$$

input `integrate(x*arccsc(b*x+a)^3,x, algorithm="giac")`

output `integrate(x*arccsc(b*x + a)^3, x)`

**Mupad [F(-1)]**

Timed out.

$$\int x \csc^{-1}(a + bx)^3 dx = \int x \operatorname{asin}\left(\frac{1}{a + bx}\right)^3 dx$$

input `int(x*asin(1/(a + b*x))^3,x)`

output `int(x*asin(1/(a + b*x))^3, x)`

**Reduce [F]**

$$\int x \csc^{-1}(a + bx)^3 dx = \int \operatorname{acsc}(bx + a)^3 x dx$$

input `int(x*acsc(b*x+a)^3,x)`

output `int(acsc(a + b*x)**3*x,x)`

### 3.35 $\int \csc^{-1}(a + bx)^3 dx$

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#### Optimal result

Integrand size = 8, antiderivative size = 140

$$\int \csc^{-1}(a + bx)^3 dx = \frac{(a + bx) \csc^{-1}(a + bx)^3}{b} + \frac{6 \csc^{-1}(a + bx)^2 \operatorname{arctanh}\left(e^{i \csc^{-1}(a + bx)}\right)}{b}$$

$$- \frac{6i \csc^{-1}(a + bx) \operatorname{PolyLog}\left(2, -e^{i \csc^{-1}(a + bx)}\right)}{b}$$

$$+ \frac{6i \csc^{-1}(a + bx) \operatorname{PolyLog}\left(2, e^{i \csc^{-1}(a + bx)}\right)}{b}$$

$$+ \frac{6 \operatorname{PolyLog}\left(3, -e^{i \csc^{-1}(a + bx)}\right)}{b} - \frac{6 \operatorname{PolyLog}\left(3, e^{i \csc^{-1}(a + bx)}\right)}{b}$$

output

```
(b*x+a)*arccsc(b*x+a)^3/b+6*arccsc(b*x+a)^2*arctanh(I/(b*x+a)+(1-1/(b*x+a)
^2)^(1/2))/b-6*I*arccsc(b*x+a)*polylog(2,-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))
/b+6*I*arccsc(b*x+a)*polylog(2,I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/b+6*polylo
g(3,-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))/b-6*polylog(3,I/(b*x+a)+(1-1/(b*x+a)
^2)^(1/2))/b
```

**Mathematica [A] (verified)**

Time = 0.08 (sec) , antiderivative size = 162, normalized size of antiderivative = 1.16

$$\int \csc^{-1}(a + bx)^3 dx$$

$$= \frac{a \csc^{-1}(a + bx)^3 + bx \csc^{-1}(a + bx)^3 - 3 \csc^{-1}(a + bx)^2 \log\left(1 - e^{i \csc^{-1}(a + bx)}\right) + 3 \csc^{-1}(a + bx)^2 \log\left(1 + e^{i \csc^{-1}(a + bx)}\right)}{b}$$

input

```
Integrate[ArcCsc[a + b*x]^3,x]
```

output

```
(a*ArcCsc[a + b*x]^3 + b*x*ArcCsc[a + b*x]^3 - 3*ArcCsc[a + b*x]^2*Log[1 - E^(I*ArcCsc[a + b*x])] + 3*ArcCsc[a + b*x]^2*Log[1 + E^(I*ArcCsc[a + b*x])]) - (6*I)*ArcCsc[a + b*x]*PolyLog[2, -E^(I*ArcCsc[a + b*x])] + (6*I)*ArcCsc[a + b*x]*PolyLog[2, E^(I*ArcCsc[a + b*x])] + 6*PolyLog[3, -E^(I*ArcCsc[a + b*x])] - 6*PolyLog[3, E^(I*ArcCsc[a + b*x])])/b
```

**Rubi [A] (warning: unable to verify)**

Time = 0.63 (sec) , antiderivative size = 131, normalized size of antiderivative = 0.94, number of steps used = 9, number of rules used = 8,  $\frac{\text{number of rules}}{\text{integrand size}} = 1.000$ , Rules used = {5776, 5740, 4245, 3042, 4671, 3011, 2720, 7143}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \csc^{-1}(a + bx)^3 dx$$

$$\downarrow 5776$$

$$\frac{\int \csc^{-1}(a + bx)^3 d(a + bx)}{b}$$

$$\downarrow 5740$$

$$\frac{\int (a + bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx)^3 d \csc^{-1}(a + bx)}{b}$$

$$\downarrow 4245$$

$$\begin{aligned} & \frac{3 \int (a + bx) \csc^{-1}(a + bx)^2 d \csc^{-1}(a + bx) - (a + bx) \csc^{-1}(a + bx)^3}{b} \\ & \quad \downarrow 3042 \\ & \frac{3 \int \csc^{-1}(a + bx)^2 \csc(\csc^{-1}(a + bx)) d \csc^{-1}(a + bx) - (a + bx) \csc^{-1}(a + bx)^3}{b} \\ & \quad \downarrow 4671 \\ & \frac{-(a + bx) \csc^{-1}(a + bx)^3 + 3 \left( -2 \int \csc^{-1}(a + bx) \log \left( 1 - e^{i \csc^{-1}(a + bx)} \right) d \csc^{-1}(a + bx) + 2 \int \csc^{-1}(a + bx) \log \left( 1 + e^{i \csc^{-1}(a + bx)} \right) d \csc^{-1}(a + bx) \right)}{b} \\ & \quad \downarrow 3011 \\ & \frac{-(a + bx) \csc^{-1}(a + bx)^3 + 3 \left( 2 \left( i \csc^{-1}(a + bx) \operatorname{PolyLog} \left( 2, -e^{i \csc^{-1}(a + bx)} \right) \right) - i \int \operatorname{PolyLog} \left( 2, -e^{i \csc^{-1}(a + bx)} \right) d \csc^{-1}(a + bx) \right)}{b} \\ & \quad \downarrow 2720 \\ & \frac{-(a + bx) \csc^{-1}(a + bx)^3 + 3 \left( -2 \left( i \csc^{-1}(a + bx) \operatorname{PolyLog} \left( 2, e^{i \csc^{-1}(a + bx)} \right) \right) - \int e^{-i \csc^{-1}(a + bx)} \operatorname{PolyLog} \left( 2, e^{i \csc^{-1}(a + bx)} \right) d \csc^{-1}(a + bx) \right)}{b} \\ & \quad \downarrow 7143 \\ & \frac{-(a + bx) \csc^{-1}(a + bx)^3 + 3 \left( -2 \csc^{-1}(a + bx)^2 \operatorname{arctanh} \left( e^{i \csc^{-1}(a + bx)} \right) - 2 \left( i \csc^{-1}(a + bx) \operatorname{PolyLog} \left( 2, e^{i \csc^{-1}(a + bx)} \right) \right) \right)}{b} \end{aligned}$$

input `Int[ArcCsc[a + b*x]^3,x]`

output `-((-((a + b*x)*ArcCsc[a + b*x]^3) + 3*(-2*ArcCsc[a + b*x]^2*ArcTanh[E^(I*ArcCsc[a + b*x])]) - 2*(I*ArcCsc[a + b*x]*PolyLog[2, E^(I*ArcCsc[a + b*x])]) - PolyLog[3, E^(I*ArcCsc[a + b*x])]) + 2*(I*ArcCsc[a + b*x]*PolyLog[2, -E^(I*ArcCsc[a + b*x])]) - PolyLog[3, -a - b*x]))/b)`

## Definitions of rubi rules used

rule 2720 `Int[u_, x_Symbol] := With[{v = FunctionOfExponential[u, x]}, Simp[v/D[v, x] Subst[Int[FunctionOfExponentialFunction[u, x]/x, x], x, v], x]] /; FunctionOfExponentialQ[u, x] && !MatchQ[u, (w_)*((a_.)*(v_)^(n_))^(m_)] /; FreeQ[{a, m, n}, x] && IntegerQ[m*n] && !MatchQ[u, E^((c_.)*((a_.) + (b_.)*x)) *(F_) [v_] /; FreeQ[{a, b, c}, x] && InverseFunctionQ[F[x]]]`

rule 3011 `Int[Log[1 + (e_.)*((F_)^((c_.)*((a_.) + (b_.)*(x_))))^(n_.)]*((f_.) + (g_.)*(x_))^(m_.), x_Symbol] := Simp[(-(f + g*x)^m)*(PolyLog[2, (-e)*(F^(c*(a + b*x)))^n]/(b*c*n*Log[F])), x] + Simp[g*(m/(b*c*n*Log[F])) Int[(f + g*x)^(m - 1)*PolyLog[2, (-e)*(F^(c*(a + b*x)))^n], x], x] /; FreeQ[{F, a, b, c, e, f, g, n}, x] && GtQ[m, 0]`

rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`

rule 4245 `Int[Cot[(a_.) + (b_.)*(x_)^(n_.)]^(q_.)*Csc[(a_.) + (b_.)*(x_)^(n_.)]^(p_.)*(x_)^(m_.), x_Symbol] := Simp[(-x^(m - n + 1))*(Csc[a + b*x^n]^p/(b*n*p)), x] + Simp[(m - n + 1)/(b*n*p) Int[x^(m - n)*Csc[a + b*x^n]^p, x], x] /; FreeQ[{a, b, p}, x] && IntegerQ[n] && GeQ[m, n] && EqQ[q, 1]`

rule 4671 `Int[csc[(e_.) + (f_.)*(x_)]*((c_.) + (d_.)*(x_))^(m_.), x_Symbol] := Simp[-2*(c + d*x)^m*(ArcTanh[E^(I*(e + f*x))]/f), x] + (-Simp[d*(m/f) Int[(c + d*x)^(m - 1)*Log[1 - E^(I*(e + f*x))], x], x] + Simp[d*(m/f) Int[(c + d*x)^(m - 1)*Log[1 + E^(I*(e + f*x))], x], x]) /; FreeQ[{c, d, e, f}, x] && IGtQ[m, 0]`

rule 5740 `Int[((a_.) + ArcCsc[(c_.)*(x_)]*(b_.))^(n_), x_Symbol] := Simp[-c^(-1) Subst[Int[(a + b*x)^n*Csc[x]*Cot[x], x], x, ArcCsc[c*x]], x] /; FreeQ[{a, b, c, n}, x] && IGtQ[n, 0]`

rule 5776

```
Int[((a_.) + ArcCsc[(c_) + (d_.)*(x_)]*(b_.))^(p_.), x_Symbol] := Simp[1/d
  Subst[Int[(a + b*ArcCsc[x])^p, x], x, c + d*x], x] /; FreeQ[{a, b, c, d},
  x] && IGtQ[p, 0]
```

rule 7143

```
Int[PolyLog[n_, (c_.)*((a_.) + (b_.)*(x_))^(p_.)]/((d_.) + (e_.)*(x_)), x_S
ymbol] := Simp[PolyLog[n + 1, c*(a + b*x)^p]/(e*p), x] /; FreeQ[{a, b, c, d
, e, n, p}, x] && EqQ[b*d, a*e]
```

## Maple [A] (verified)

Time = 0.58 (sec) , antiderivative size = 224, normalized size of antiderivative = 1.60

method	result
derivativedivides	$\frac{\operatorname{arccsc}(bx+a)^3(bx+a)+3\operatorname{arccsc}(bx+a)^2\ln\left(1+\frac{i}{bx+a}+\sqrt{1-\frac{1}{(bx+a)^2}}\right)-6i\operatorname{arccsc}(bx+a)\operatorname{polylog}\left(2,-\frac{i}{bx+a}-\sqrt{1-\frac{1}{(bx+a)^2}}\right)}{\dots}$
default	$\frac{\operatorname{arccsc}(bx+a)^3(bx+a)+3\operatorname{arccsc}(bx+a)^2\ln\left(1+\frac{i}{bx+a}+\sqrt{1-\frac{1}{(bx+a)^2}}\right)-6i\operatorname{arccsc}(bx+a)\operatorname{polylog}\left(2,-\frac{i}{bx+a}-\sqrt{1-\frac{1}{(bx+a)^2}}\right)}{\dots}$

input

```
int(arccsc(b*x+a)^3,x,method=_RETURNVERBOSE)
```

output

```
1/b*(arccsc(b*x+a)^3*(b*x+a)+3*arccsc(b*x+a)^2*ln(1+I/(b*x+a)+(1-1/(b*x+a)
^2)^(1/2))-6*I*arccsc(b*x+a)*polylog(2,-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))+6
*polylog(3,-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))-3*arccsc(b*x+a)^2*ln(1-I/(b*x
+a)-(1-1/(b*x+a)^2)^(1/2))+6*I*arccsc(b*x+a)*polylog(2,I/(b*x+a)+(1-1/(b*x
+a)^2)^(1/2))-6*polylog(3,I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2)))
```

## Fricas [F]

$$\int \csc^{-1}(a + bx)^3 dx = \int \operatorname{arccsc}(bx + a)^3 dx$$

input

```
integrate(arccsc(b*x+a)^3,x, algorithm="fricas")
```



output `integral(arccsc(b*x + a)^3, x)`

### Sympy [F]

$$\int \csc^{-1}(a + bx)^3 dx = \int \operatorname{acsc}^3(a + bx) dx$$

input `integrate(acsc(b*x+a)**3,x)`

output `Integral(acsc(a + b*x)**3, x)`

### Maxima [F]

$$\int \csc^{-1}(a + bx)^3 dx = \int \operatorname{arccsc}(bx + a)^3 dx$$

input `integrate(arccsc(b*x+a)^3,x, algorithm="maxima")`

output

```
x*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1))^3 - 3/4*x*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1))*log(b^2*x^2 + 2*a*b*x + a^2)^2 - integrate(3/4*(4*(b^3*x^3*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) + 3*a*b^2*x^2*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) + a^3*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) + (3*a^2*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) - arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)))*b*x - a*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)))*log(b*x + a)^2 - (4*b*x*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1))^2 - b*x*log(b^2*x^2 + 2*a*b*x + a^2)^2)*sqrt(b*x + a + 1)*sqrt(b*x + a - 1) - 4*(b^3*x^3*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) + 2*a*b^2*x^2*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) + (a^2*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) - arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)))*b*x + (b^3*x^3*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) + 3*a*b^2*x^2*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) + a^3*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) + (3*a^2*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) - arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)))*b*x - a*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)))*log(b*x + a))*log(b^2*x^2 + 2*a*b*x + a^2))/(b^3*x^3 + 3*a*b^2*x^2 + a^3 + (3*a^2 - 1)*b*x - a), x)
```

**Giac [F]**

$$\int \csc^{-1}(a + bx)^3 dx = \int \operatorname{arccsc}(bx + a)^3 dx$$

input

```
integrate(arccsc(b*x+a)^3,x, algorithm="giac")
```

output

```
integrate(arccsc(b*x + a)^3, x)
```

**Mupad [F(-1)]**

Timed out.

$$\int \csc^{-1}(a + bx)^3 dx = \int \operatorname{asin}\left(\frac{1}{a + bx}\right)^3 dx$$

input `int(asin(1/(a + b*x))^3,x)`output `int(asin(1/(a + b*x))^3, x)`**Reduce [F]**

$$\int \csc^{-1}(a + bx)^3 dx = \int \operatorname{acsc}(bx + a)^3 dx$$

input `int(acsc(b*x+a)^3,x)`output `int(acsc(a + b*x)**3,x)`

**3.36**  $\int \frac{\csc^{-1}(a+bx)^3}{x} dx$ 

Optimal result . . . . .	300
Mathematica [A] (verified) . . . . .	302
Rubi [A] (verified) . . . . .	303
Maple [F] . . . . .	309
Fricas [F] . . . . .	309
Sympy [F] . . . . .	310
Maxima [F] . . . . .	310
Giac [F] . . . . .	310
Mupad [F(-1)] . . . . .	311
Reduce [F] . . . . .	311

**Optimal result**

Integrand size = 12, antiderivative size = 448

$$\begin{aligned}
\int \frac{\csc^{-1}(a+bx)^3}{x} dx &= \csc^{-1}(a+bx)^3 \log\left(1 + \frac{iae^{i \csc^{-1}(a+bx)}}{1 - \sqrt{1-a^2}}\right) \\
&+ \csc^{-1}(a+bx)^3 \log\left(1 + \frac{iae^{i \csc^{-1}(a+bx)}}{1 + \sqrt{1-a^2}}\right) \\
&- \csc^{-1}(a+bx)^3 \log\left(1 - e^{2i \csc^{-1}(a+bx)}\right) \\
&- 3i \csc^{-1}(a+bx)^2 \text{PolyLog}\left(2, -\frac{iae^{i \csc^{-1}(a+bx)}}{1 - \sqrt{1-a^2}}\right) \\
&- 3i \csc^{-1}(a+bx)^2 \text{PolyLog}\left(2, -\frac{iae^{i \csc^{-1}(a+bx)}}{1 + \sqrt{1-a^2}}\right) \\
&+ \frac{3}{2}i \csc^{-1}(a+bx)^2 \text{PolyLog}\left(2, e^{2i \csc^{-1}(a+bx)}\right) \\
&+ 6 \csc^{-1}(a+bx) \text{PolyLog}\left(3, -\frac{iae^{i \csc^{-1}(a+bx)}}{1 - \sqrt{1-a^2}}\right) \\
&+ 6 \csc^{-1}(a+bx) \text{PolyLog}\left(3, -\frac{iae^{i \csc^{-1}(a+bx)}}{1 + \sqrt{1-a^2}}\right) \\
&- \frac{3}{2} \csc^{-1}(a+bx) \text{PolyLog}\left(3, e^{2i \csc^{-1}(a+bx)}\right) \\
&+ 6i \text{PolyLog}\left(4, -\frac{iae^{i \csc^{-1}(a+bx)}}{1 - \sqrt{1-a^2}}\right) \\
&+ 6i \text{PolyLog}\left(4, -\frac{iae^{i \csc^{-1}(a+bx)}}{1 + \sqrt{1-a^2}}\right) \\
&- \frac{3}{4}i \text{PolyLog}\left(4, e^{2i \csc^{-1}(a+bx)}\right)
\end{aligned}$$

output

```

arccsc(b*x+a)^3*ln(1+I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1-(-a^2+1)^(1/2)))
+arccsc(b*x+a)^3*ln(1+I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1+(-a^2+1)^(1/2)))
-arccsc(b*x+a)^3*ln(1-(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))^2)-3*I*arccsc(b*x+a)^2*
polylog(2,-I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1-(-a^2+1)^(1/2)))
-3*I*arccsc(b*x+a)^2*polylog(2,-I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1+(-a^2+1)^(1/2)))
+3/2*I*arccsc(b*x+a)^2*polylog(2,(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))^2)
+6*arccsc(b*x+a)*polylog(3,-I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1-(-a^2+1)^(1/2)))
+6*arccsc(b*x+a)*polylog(3,-I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1+(-a^2+1)^(1/2)))
-3/2*arccsc(b*x+a)*polylog(3,(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))^2)
+6*I*polylog(4,-I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1-(-a^2+1)^(1/2)))
+6*I*polylog(4,-I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1+(-a^2+1)^(1/2)))
-3/4*I*polylog(4,(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))^2)

```

**Mathematica [A] (verified)**

Time = 0.27 (sec) , antiderivative size = 554, normalized size of antiderivative = 1.24

$$\begin{aligned}
\int \frac{\csc^{-1}(a+bx)^3}{x} dx &= \frac{i\pi^4}{8} - \frac{1}{4}i \csc^{-1}(a+bx)^4 - \csc^{-1}(a+bx)^3 \log\left(1 - e^{-i \csc^{-1}(a+bx)}\right) \\
&\quad - \csc^{-1}(a+bx)^3 \log\left(1 + e^{i \csc^{-1}(a+bx)}\right) \\
&\quad + \csc^{-1}(a+bx)^3 \log\left(1 - \frac{iae^{i \csc^{-1}(a+bx)}}{-1 + \sqrt{1-a^2}}\right) \\
&\quad + \csc^{-1}(a+bx)^3 \log\left(1 + \frac{iae^{i \csc^{-1}(a+bx)}}{1 + \sqrt{1-a^2}}\right) \\
&\quad - 3i \csc^{-1}(a+bx)^2 \text{PolyLog}\left(2, e^{-i \csc^{-1}(a+bx)}\right) \\
&\quad + 3i \csc^{-1}(a+bx)^2 \text{PolyLog}\left(2, -e^{i \csc^{-1}(a+bx)}\right) \\
&\quad - 3i \csc^{-1}(a+bx)^2 \text{PolyLog}\left(2, \frac{iae^{i \csc^{-1}(a+bx)}}{-1 + \sqrt{1-a^2}}\right) \\
&\quad - 3i \csc^{-1}(a+bx)^2 \text{PolyLog}\left(2, -\frac{iae^{i \csc^{-1}(a+bx)}}{1 + \sqrt{1-a^2}}\right) \\
&\quad - 6 \csc^{-1}(a+bx) \text{PolyLog}\left(3, e^{-i \csc^{-1}(a+bx)}\right) \\
&\quad - 6 \csc^{-1}(a+bx) \text{PolyLog}\left(3, -e^{i \csc^{-1}(a+bx)}\right) \\
&\quad + 6 \csc^{-1}(a+bx) \text{PolyLog}\left(3, \frac{iae^{i \csc^{-1}(a+bx)}}{-1 + \sqrt{1-a^2}}\right) \\
&\quad + 6 \csc^{-1}(a+bx) \text{PolyLog}\left(3, -\frac{iae^{i \csc^{-1}(a+bx)}}{1 + \sqrt{1-a^2}}\right) \\
&\quad + 6i \text{PolyLog}\left(4, e^{-i \csc^{-1}(a+bx)}\right) - 6i \text{PolyLog}\left(4, -e^{i \csc^{-1}(a+bx)}\right) \\
&\quad + 6i \text{PolyLog}\left(4, \frac{iae^{i \csc^{-1}(a+bx)}}{-1 + \sqrt{1-a^2}}\right) \\
&\quad + 6i \text{PolyLog}\left(4, -\frac{iae^{i \csc^{-1}(a+bx)}}{1 + \sqrt{1-a^2}}\right)
\end{aligned}$$

input

Integrate[ArcCsc[a + b\*x]^3/x, x]

output

```
(I/8)*Pi^4 - (I/4)*ArcCsc[a + b*x]^4 - ArcCsc[a + b*x]^3*Log[1 - E^((-I)*ArcCsc[a + b*x])] - ArcCsc[a + b*x]^3*Log[1 + E^(I*ArcCsc[a + b*x])] + ArcCsc[a + b*x]^3*Log[1 - (I*a*E^(I*ArcCsc[a + b*x]))/(-1 + Sqrt[1 - a^2])] + ArcCsc[a + b*x]^3*Log[1 + (I*a*E^(I*ArcCsc[a + b*x]))/(1 + Sqrt[1 - a^2])] - (3*I)*ArcCsc[a + b*x]^2*PolyLog[2, E^((-I)*ArcCsc[a + b*x])] + (3*I)*ArcCsc[a + b*x]^2*PolyLog[2, -E^(I*ArcCsc[a + b*x])] - (3*I)*ArcCsc[a + b*x]^2*PolyLog[2, (I*a*E^(I*ArcCsc[a + b*x]))/(-1 + Sqrt[1 - a^2])] - (3*I)*ArcCsc[a + b*x]^2*PolyLog[2, ((-I)*a*E^(I*ArcCsc[a + b*x]))/(1 + Sqrt[1 - a^2])] - 6*ArcCsc[a + b*x]*PolyLog[3, E^((-I)*ArcCsc[a + b*x])] - 6*ArcCsc[a + b*x]*PolyLog[3, -E^(I*ArcCsc[a + b*x])] + 6*ArcCsc[a + b*x]*PolyLog[3, (I*a*E^(I*ArcCsc[a + b*x]))/(-1 + Sqrt[1 - a^2])] + 6*ArcCsc[a + b*x]*PolyLog[3, ((-I)*a*E^(I*ArcCsc[a + b*x]))/(1 + Sqrt[1 - a^2])] + (6*I)*PolyLog[4, E^((-I)*ArcCsc[a + b*x])] - (6*I)*PolyLog[4, -E^(I*ArcCsc[a + b*x])] + (6*I)*PolyLog[4, (I*a*E^(I*ArcCsc[a + b*x]))/(-1 + Sqrt[1 - a^2])] + (6*I)*PolyLog[4, ((-I)*a*E^(I*ArcCsc[a + b*x]))/(1 + Sqrt[1 - a^2])]
```

**Rubi [A] (verified)**

Time = 1.88 (sec) , antiderivative size = 530, normalized size of antiderivative = 1.18, number of steps used = 18, number of rules used = 17,  $\frac{\text{number of rules}}{\text{integrand size}} = 1.417$ , Rules used = {5782, 25, 5063, 5040, 25, 3042, 25, 4200, 25, 2620, 3011, 5030, 2620, 3011, 7163, 2720, 7143}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{\csc^{-1}(a + bx)^3}{x} dx \\
 & \quad \downarrow \text{5782} \\
 & - \int \frac{(a + bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx)^3}{bx} d \csc^{-1}(a + bx) \\
 & \quad \downarrow \text{25} \\
 & \int - \frac{(a + bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a + bx)^3}{bx} d \csc^{-1}(a + bx) \\
 & \quad \downarrow \text{5063}
 \end{aligned}$$



$$\begin{aligned}
& \int \frac{(a+bx)\sqrt{1-\frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)^3}{\frac{a}{a+bx}-1} d \csc^{-1}(a+bx) \\
& \quad \downarrow 5040 \\
& a \int -\frac{\sqrt{1-\frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)^3}{1-\frac{a}{a+bx}} d \csc^{-1}(a+bx) - \int (a+bx)\sqrt{1-\frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)^3 d \csc^{-1}(a+bx) \\
& \quad \downarrow 25 \\
& - \int (a+bx)\sqrt{1-\frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)^3 d \csc^{-1}(a+bx) - \\
& \quad a \int \frac{\sqrt{1-\frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)^3}{1-\frac{a}{a+bx}} d \csc^{-1}(a+bx) \\
& \quad \downarrow 3042 \\
& -a \int \frac{\sqrt{1-\frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)^3}{1-\frac{a}{a+bx}} d \csc^{-1}(a+bx) - \int -\csc^{-1}(a+bx)^3 \tan\left(\csc^{-1}(a+bx) + \frac{\pi}{2}\right) d \csc^{-1}(a+bx) \\
& \quad \downarrow 25 \\
& \int \csc^{-1}(a+bx)^3 \tan\left(\csc^{-1}(a+bx) + \frac{\pi}{2}\right) d \csc^{-1}(a+bx) - \\
& \quad a \int \frac{\sqrt{1-\frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)^3}{1-\frac{a}{a+bx}} d \csc^{-1}(a+bx) \\
& \quad \downarrow 4200 \\
& -2i \int -\frac{e^{2i \csc^{-1}(a+bx)} \csc^{-1}(a+bx)^3}{1-e^{2i \csc^{-1}(a+bx)}} d \csc^{-1}(a+bx) - \\
& \quad a \int \frac{\sqrt{1-\frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)^3}{1-\frac{a}{a+bx}} d \csc^{-1}(a+bx) + \frac{1}{4}i \csc^{-1}(a+bx)^4 \\
& \quad \downarrow 25 \\
& 2i \int \frac{e^{2i \csc^{-1}(a+bx)} \csc^{-1}(a+bx)^3}{1-e^{2i \csc^{-1}(a+bx)}} d \csc^{-1}(a+bx) - \\
& \quad a \int \frac{\sqrt{1-\frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)^3}{1-\frac{a}{a+bx}} d \csc^{-1}(a+bx) + \frac{1}{4}i \csc^{-1}(a+bx)^4
\end{aligned}$$

$$\begin{aligned} & \downarrow 2620 \\ & -a \int \frac{\sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)^3}{1 - \frac{a}{a+bx}} d \csc^{-1}(a+bx) + \\ & 2i \left( \frac{1}{2} i \csc^{-1}(a+bx)^3 \log \left( 1 - e^{2i \csc^{-1}(a+bx)} \right) - \frac{3}{2} i \int \csc^{-1}(a+bx)^2 \log \left( 1 - e^{2i \csc^{-1}(a+bx)} \right) d \csc^{-1}(a+bx) \right) + \\ & \frac{1}{4} i \csc^{-1}(a+bx)^4 \end{aligned}$$

$$\begin{aligned} & \downarrow 3011 \\ & 2i \left( \frac{1}{2} i \csc^{-1}(a+bx)^3 \log \left( 1 - e^{2i \csc^{-1}(a+bx)} \right) - \frac{3}{2} i \left( \frac{1}{2} i \csc^{-1}(a+bx)^2 \text{PolyLog} \left( 2, e^{2i \csc^{-1}(a+bx)} \right) - i \int \csc^{-1}(a+bx) \right. \right. \\ & \left. \left. a \int \frac{\sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)^3}{1 - \frac{a}{a+bx}} d \csc^{-1}(a+bx) + \frac{1}{4} i \csc^{-1}(a+bx)^4 \right) \right) \end{aligned}$$

$$\begin{aligned} & \downarrow 5030 \\ & -a \left( \int \frac{e^{i \csc^{-1}(a+bx)} \csc^{-1}(a+bx)^3}{ie^{i \csc^{-1}(a+bx)} a - \sqrt{1-a^2} + 1} d \csc^{-1}(a+bx) + \int \frac{e^{i \csc^{-1}(a+bx)} \csc^{-1}(a+bx)^3}{ie^{i \csc^{-1}(a+bx)} a + \sqrt{1-a^2} + 1} d \csc^{-1}(a+bx) + \frac{i \csc^{-1}(a+bx)^4}{4} \right) \\ & 2i \left( \frac{1}{2} i \csc^{-1}(a+bx)^3 \log \left( 1 - e^{2i \csc^{-1}(a+bx)} \right) - \frac{3}{2} i \left( \frac{1}{2} i \csc^{-1}(a+bx)^2 \text{PolyLog} \left( 2, e^{2i \csc^{-1}(a+bx)} \right) - i \int \csc^{-1}(a+bx) \right. \right. \\ & \left. \left. \frac{1}{4} i \csc^{-1}(a+bx)^4 \right) \right) \end{aligned}$$

$$\begin{aligned} & \downarrow 2620 \\ & -a \left( \frac{3 \int \csc^{-1}(a+bx)^2 \log \left( \frac{ie^{i \csc^{-1}(a+bx)} a}{1 - \sqrt{1-a^2}} + 1 \right) d \csc^{-1}(a+bx)}{a} + \frac{3 \int \csc^{-1}(a+bx)^2 \log \left( \frac{ie^{i \csc^{-1}(a+bx)} a}{\sqrt{1-a^2} + 1} + 1 \right) d \csc^{-1}(a+bx)}{a} \right) \\ & 2i \left( \frac{1}{2} i \csc^{-1}(a+bx)^3 \log \left( 1 - e^{2i \csc^{-1}(a+bx)} \right) - \frac{3}{2} i \left( \frac{1}{2} i \csc^{-1}(a+bx)^2 \text{PolyLog} \left( 2, e^{2i \csc^{-1}(a+bx)} \right) - i \int \csc^{-1}(a+bx) \right. \right. \\ & \left. \left. \frac{1}{4} i \csc^{-1}(a+bx)^4 \right) \right) \end{aligned}$$

$$\downarrow 3011$$

$$\begin{aligned}
 & -a \left( \frac{3 \left( i \csc^{-1}(a + bx)^2 \operatorname{PolyLog} \left( 2, -\frac{iae^{i \csc^{-1}(a+bx)}}{1-\sqrt{1-a^2}} \right) - 2i \int \csc^{-1}(a + bx) \operatorname{PolyLog} \left( 2, -\frac{iae^{i \csc^{-1}(a+bx)}}{1-\sqrt{1-a^2}} \right) d \csc^{-1}(a + bx) \right)}{a} \right. \\
 & \left. 2i \left( \frac{1}{2} i \csc^{-1}(a + bx)^3 \log \left( 1 - e^{2i \csc^{-1}(a+bx)} \right) - \frac{3}{2} i \left( \frac{1}{2} i \csc^{-1}(a + bx)^2 \operatorname{PolyLog} \left( 2, e^{2i \csc^{-1}(a+bx)} \right) - i \int \csc^{-1}(a + bx) \operatorname{PolyLog} \left( 2, e^{2i \csc^{-1}(a+bx)} \right) d \csc^{-1}(a + bx) \right) \right. \right. \\
 & \left. \left. \frac{1}{4} i \csc^{-1}(a + bx)^4 \right) \right) \\
 & \qquad \qquad \qquad \downarrow \text{7163}
 \end{aligned}$$

$$\begin{aligned}
 & -a \left( \frac{3 \left( i \csc^{-1}(a + bx)^2 \operatorname{PolyLog} \left( 2, -\frac{iae^{i \csc^{-1}(a+bx)}}{1-\sqrt{1-a^2}} \right) - 2i \left( i \int \operatorname{PolyLog} \left( 3, -\frac{iae^{i \csc^{-1}(a+bx)}}{1-\sqrt{1-a^2}} \right) d \csc^{-1}(a + bx) - i \int \csc^{-1}(a + bx) \operatorname{PolyLog} \left( 3, -\frac{iae^{i \csc^{-1}(a+bx)}}{1-\sqrt{1-a^2}} \right) d \csc^{-1}(a + bx) \right)}{a} \right) \\
 & \left. 2i \left( \frac{1}{2} i \csc^{-1}(a + bx)^3 \log \left( 1 - e^{2i \csc^{-1}(a+bx)} \right) - \frac{3}{2} i \left( \frac{1}{2} i \csc^{-1}(a + bx)^2 \operatorname{PolyLog} \left( 2, e^{2i \csc^{-1}(a+bx)} \right) - i \left( \frac{1}{2} i \int \operatorname{PolyLog} \left( 2, e^{2i \csc^{-1}(a+bx)} \right) d \csc^{-1}(a + bx) \right) \right) \right. \right. \\
 & \left. \left. \frac{1}{4} i \csc^{-1}(a + bx)^4 \right) \right) \\
 & \qquad \qquad \qquad \downarrow \text{2720}
 \end{aligned}$$

$$\begin{aligned}
 & -a \left( \frac{3 \left( i \csc^{-1}(a + bx)^2 \operatorname{PolyLog} \left( 2, -\frac{iae^{i \csc^{-1}(a+bx)}}{1-\sqrt{1-a^2}} \right) - 2i \left( \int e^{-i \csc^{-1}(a+bx)} \operatorname{PolyLog} \left( 3, -\frac{iae^{i \csc^{-1}(a+bx)}}{1-\sqrt{1-a^2}} \right) de^{i \csc^{-1}(a+bx)} \right) \right)}{a} \right) \\
 & \left. 2i \left( \frac{1}{2} i \csc^{-1}(a + bx)^3 \log \left( 1 - e^{2i \csc^{-1}(a+bx)} \right) - \frac{3}{2} i \left( \frac{1}{2} i \csc^{-1}(a + bx)^2 \operatorname{PolyLog} \left( 2, e^{2i \csc^{-1}(a+bx)} \right) - i \left( \frac{1}{4} \int e^{-2i \csc^{-1}(a+bx)} d \csc^{-1}(a + bx) \right) \right) \right. \right. \\
 & \left. \left. \frac{1}{4} i \csc^{-1}(a + bx)^4 \right) \right) \\
 & \qquad \qquad \qquad \downarrow \text{7143}
 \end{aligned}$$

$$\begin{aligned}
 & -a \left( \frac{3 \left( i \csc^{-1}(a + bx)^2 \operatorname{PolyLog} \left( 2, -\frac{iae^{i \csc^{-1}(a+bx)}}{1-\sqrt{1-a^2}} \right) - 2i \left( \operatorname{PolyLog} \left( 4, -\frac{iae^{i \csc^{-1}(a+bx)}}{1-\sqrt{1-a^2}} \right) - i \csc^{-1}(a + bx) \operatorname{PolyLog} \left( 4, -\frac{iae^{i \csc^{-1}(a+bx)}}{1-\sqrt{1-a^2}} \right) \right) \right)}{a} \right) \\
 & \left. 2i \left( \frac{1}{2} i \csc^{-1}(a + bx)^3 \log \left( 1 - e^{2i \csc^{-1}(a+bx)} \right) - \frac{3}{2} i \left( \frac{1}{2} i \csc^{-1}(a + bx)^2 \operatorname{PolyLog} \left( 2, e^{2i \csc^{-1}(a+bx)} \right) - i \left( \frac{1}{4} \operatorname{PolyLog} \left( 2, e^{2i \csc^{-1}(a+bx)} \right) \right) \right) \right. \right. \\
 & \left. \left. \frac{1}{4} i \csc^{-1}(a + bx)^4 \right) \right)
 \end{aligned}$$

input `Int[ArcCsc[a + b*x]^3/x, x]`

output

```
(I/4)*ArcCsc[a + b*x]^4 - a*(((I/4)*ArcCsc[a + b*x]^4)/a - (ArcCsc[a + b*x]^3*Log[1 + (I*a*E^(I*ArcCsc[a + b*x]))/(1 - Sqrt[1 - a^2])])/a - (ArcCsc[a + b*x]^3*Log[1 + (I*a*E^(I*ArcCsc[a + b*x]))/(1 + Sqrt[1 - a^2])])/a + (3*(I*ArcCsc[a + b*x]^2*PolyLog[2, ((-I)*a*E^(I*ArcCsc[a + b*x]))/(1 - Sqrt[1 - a^2])]) - (2*I)*((-I)*ArcCsc[a + b*x]*PolyLog[3, ((-I)*a*E^(I*ArcCsc[a + b*x]))/(1 - Sqrt[1 - a^2])]) + PolyLog[4, ((-I)*a*E^(I*ArcCsc[a + b*x]))/(1 - Sqrt[1 - a^2])])))/a + (3*(I*ArcCsc[a + b*x]^2*PolyLog[2, ((-I)*a*E^(I*ArcCsc[a + b*x]))/(1 + Sqrt[1 - a^2])]) - (2*I)*((-I)*ArcCsc[a + b*x]*PolyLog[3, ((-I)*a*E^(I*ArcCsc[a + b*x]))/(1 + Sqrt[1 - a^2])]) + PolyLog[4, ((-I)*a*E^(I*ArcCsc[a + b*x]))/(1 + Sqrt[1 - a^2])])))/a + (2*I)*((I/2)*ArcCsc[a + b*x]^3*Log[1 - E^((2*I)*ArcCsc[a + b*x])] - ((3*I)/2)*((I/2)*ArcCsc[a + b*x]^2*PolyLog[2, E^((2*I)*ArcCsc[a + b*x])] - I*((-1/2*I)*ArcCsc[a + b*x]*PolyLog[3, E^((2*I)*ArcCsc[a + b*x])] + PolyLog[4, E^((2*I)*ArcCsc[a + b*x])])]/4))
```

### Defintions of rubi rules used

rule 25

```
Int[-(Fx_), x_Symbol] := Simp[Identity[-1] Int[Fx, x], x]
```

rule 2620

```
Int[(((F_)^((g_)*((e_) + (f_)*(x_))))^(n_)*((c_) + (d_)*(x_))^(m_))/((a_) + (b_)*((F_)^((g_)*((e_) + (f_)*(x_))))^(n_)), x_Symbol] := Simp[((c + d*x)^m/(b*f*g*n*Log[F]))*Log[1 + b*((F^(g*(e + f*x)))^n/a)], x] - Simp[d*(m/(b*f*g*n*Log[F])) Int[(c + d*x)^(m - 1)*Log[1 + b*((F^(g*(e + f*x)))^n/a)], x], x] /; FreeQ[{F, a, b, c, d, e, f, g, n}, x] && IGtQ[m, 0]
```

rule 2720

```
Int[u_, x_Symbol] := With[{v = FunctionOfExponential[u, x]}, Simp[v/D[v, x] Subst[Int[FunctionOfExponentialFunction[u, x]/x, x], x, v], x]] /; FunctionOfExponentialQ[u, x] && !MatchQ[u, (w_)*((a_)*(v_)^(n_))^(m_)] /; FreeQ[{a, m, n}, x] && IntegerQ[m*n] && !MatchQ[u, E^((c_)*((a_) + (b_)*x))* (F_) [v_] /; FreeQ[{a, b, c}, x] && InverseFunctionQ[F[x]]]
```

rule 3011  $\text{Int}[\text{Log}[1 + (e\_.) * ((F\_.)^{((c\_.) * (a\_.) + (b\_.) * (x\_)))})^{(n\_.)}] * ((f\_.) + (g\_.) * (x\_))^{(m\_.)}, x\_Symbol] \rightarrow \text{Simp}[(-f + g*x)^m * (\text{PolyLog}[2, (-e) * (F^{(c*(a + b*x))})^n]) / (b*c*n*\text{Log}[F]), x] + \text{Simp}[g*(m/(b*c*n*\text{Log}[F])) \text{Int}[(f + g*x)^{m-1} * \text{PolyLog}[2, (-e) * (F^{(c*(a + b*x))})^n], x], x] /; \text{FreeQ}\{F, a, b, c, e, f, g, n\}, x] \&\& \text{GtQ}[m, 0]$

rule 3042  $\text{Int}[u_, x\_Symbol] \rightarrow \text{Int}[\text{DeactivateTrig}[u, x], x] /; \text{FunctionOfTrigOfLinearQ}[u, x]$

rule 4200  $\text{Int}[(c\_.) + (d\_.) * (x\_)]^{(m\_.)} * \tan[(e\_.) + \text{Pi} * (k\_.) + (f\_.) * (x\_)], x\_Symbol] \rightarrow \text{Simp}[I * ((c + d*x)^{(m+1}) / (d*(m+1))), x] - \text{Simp}[2*I \text{Int}[(c + d*x)^m * E^{(2*I*k*Pi)} * (E^{(2*I*(e + f*x))} / (1 + E^{(2*I*k*Pi)} * E^{(2*I*(e + f*x))}))], x] /; \text{FreeQ}\{c, d, e, f\}, x] \&\& \text{IntegerQ}[4*k] \&\& \text{IGtQ}[m, 0]$

rule 5030  $\text{Int}[(\text{Cos}[(c\_.) + (d\_.) * (x\_)] * ((e\_.) + (f\_.) * (x\_))^{(m\_.)}) / ((a\_.) + (b\_.) * \text{Sin}[(c\_.) + (d\_.) * (x\_)]), x\_Symbol] \rightarrow \text{Simp}[(-I) * ((e + f*x)^{(m+1}) / (b*f*(m+1))), x] + (\text{Int}[(e + f*x)^m * (E^{(I*(c + d*x))} / (a - \text{Rt}[a^2 - b^2, 2] - I*b * E^{(I*(c + d*x))}))], x] + \text{Int}[(e + f*x)^m * (E^{(I*(c + d*x))} / (a + \text{Rt}[a^2 - b^2, 2] - I*b * E^{(I*(c + d*x))}))], x]) /; \text{FreeQ}\{a, b, c, d, e, f\}, x] \&\& \text{IGtQ}[m, 0] \&\& \text{PosQ}[a^2 - b^2]$

rule 5040  $\text{Int}[(\text{Cot}[(c\_.) + (d\_.) * (x\_)]^{(n\_.)} * ((e\_.) + (f\_.) * (x\_))^{(m\_.)}) / ((a\_.) + (b\_.) * \text{Sin}[(c\_.) + (d\_.) * (x\_)]), x\_Symbol] \rightarrow \text{Simp}[1/a \text{Int}[(e + f*x)^m * \text{Cot}[c + d*x]^n, x], x] - \text{Simp}[b/a \text{Int}[(e + f*x)^m * \text{Cos}[c + d*x] * (\text{Cot}[c + d*x]^{(n-1)} / (a + b * \text{Sin}[c + d*x])), x], x] /; \text{FreeQ}\{a, b, c, d, e, f\}, x] \&\& \text{IGtQ}[m, 0] \&\& \text{IGtQ}[n, 0]$

rule 5063  $\text{Int}[(c\_.) + (d\_.) * (x\_)]^{(m\_.)} * (F\_.)[(c\_.) + (d\_.) * (x\_)]^{(n\_.)} * (G\_.)[(c\_.) + (d\_.) * (x\_)]^{(p\_.)} / (\text{Csc}[(c\_.) + (d\_.) * (x\_)] * (b\_.) + (a\_)), x\_Symbol] \rightarrow \text{Int}[(e + f*x)^m * \text{Sin}[c + d*x] * F[c + d*x]^n * (G[c + d*x]^p / (b + a * \text{Sin}[c + d*x])), x] /; \text{FreeQ}\{a, b, c, d, e, f\}, x] \&\& \text{TrigQ}[F] \&\& \text{TrigQ}[G] \&\& \text{IntegersQ}[m, n, p]$

rule 5782 `Int[((a_.) + ArcCsc[(c_) + (d_.)*(x_)]*(b_.))^(p_.)*((e_.) + (f_.)*(x_))^(m_.), x_Symbol] := Simp[-(d^(m + 1))^(-1) Subst[Int[(a + b*x)^p*Csc[x]*Cot[x]*(d*e - c*f + f*Csc[x])^m, x], x, ArcCsc[c + d*x]], x] /; FreeQ[{a, b, c, d, e, f}, x] && IGtQ[p, 0] && IntegerQ[m]`

rule 7143 `Int[PolyLog[n_, (c_.)*((a_.) + (b_.)*(x_))^(p_.)]/((d_.) + (e_.)*(x_)), x_Symbol] := Simp[PolyLog[n + 1, c*(a + b*x)^p]/(e*p), x] /; FreeQ[{a, b, c, d, e, n, p}, x] && EqQ[b*d, a*e]`

rule 7163 `Int[((e_.) + (f_.)*(x_))^(m_.)*PolyLog[n_, (d_.)*((F_)^((c_.)*((a_.) + (b_.)*(x_))))^(p_.)], x_Symbol] := Simp[(e + f*x)^m*(PolyLog[n + 1, d*(F^(c*(a + b*x)))^p]/(b*c*p*Log[F])), x] - Simp[f*(m/(b*c*p*Log[F])) Int[(e + f*x)^(m - 1)*PolyLog[n + 1, d*(F^(c*(a + b*x)))^p], x], x] /; FreeQ[{F, a, b, c, d, e, f, n, p}, x] && GtQ[m, 0]`

## Maple [F]

$$\int \frac{\operatorname{arccsc}(bx + a)^3}{x} dx$$

input `int(arccsc(b*x+a)^3/x,x)`

output `int(arccsc(b*x+a)^3/x,x)`

## Fricas [F]

$$\int \frac{\operatorname{csc}^{-1}(a + bx)^3}{x} dx = \int \frac{\operatorname{arccsc}(bx + a)^3}{x} dx$$

input `integrate(arccsc(b*x+a)^3/x,x, algorithm="fricas")`

output `integral(arccsc(b*x + a)^3/x, x)`

**Sympy [F]**

$$\int \frac{\csc^{-1}(a + bx)^3}{x} dx = \int \frac{\operatorname{acsc}^3(a + bx)}{x} dx$$

input `integrate(acsc(b*x+a)**3/x,x)`

output `Integral(acsc(a + b*x)**3/x, x)`

**Maxima [F]**

$$\int \frac{\csc^{-1}(a + bx)^3}{x} dx = \int \frac{\operatorname{arccsc}(bx + a)^3}{x} dx$$

input `integrate(arccsc(b*x+a)^3/x,x, algorithm="maxima")`

output `integrate(arccsc(b*x + a)^3/x, x)`

**Giac [F]**

$$\int \frac{\csc^{-1}(a + bx)^3}{x} dx = \int \frac{\operatorname{arccsc}(bx + a)^3}{x} dx$$

input `integrate(arccsc(b*x+a)^3/x,x, algorithm="giac")`

output `integrate(arccsc(b*x + a)^3/x, x)`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{\csc^{-1}(a + bx)^3}{x} dx = \int \frac{\operatorname{asin}\left(\frac{1}{a+bx}\right)^3}{x} dx$$

input `int(asin(1/(a + b*x))^3/x,x)`output `int(asin(1/(a + b*x))^3/x, x)`**Reduce [F]**

$$\int \frac{\csc^{-1}(a + bx)^3}{x} dx = \int \frac{\operatorname{acsc}(bx + a)^3}{x} dx$$

input `int(acsc(b*x+a)^3/x,x)`output `int(acsc(a + b*x)**3/x,x)`



### 3.37 $\int \frac{\csc^{-1}(a+bx)^3}{x^2} dx$

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Reduce [F]	318

#### Optimal result

Integrand size = 12, antiderivative size = 378

$$\begin{aligned}
 \int \frac{\csc^{-1}(a+bx)^3}{x^2} dx = & -\frac{b \csc^{-1}(a+bx)^3}{a} - \frac{\csc^{-1}(a+bx)^3}{x} \\
 & - \frac{3ib \csc^{-1}(a+bx)^2 \log\left(1 + \frac{iae^{i \csc^{-1}(a+bx)}}{1-\sqrt{1-a^2}}\right)}{a\sqrt{1-a^2}} \\
 & + \frac{3ib \csc^{-1}(a+bx)^2 \log\left(1 + \frac{iae^{i \csc^{-1}(a+bx)}}{1+\sqrt{1-a^2}}\right)}{a\sqrt{1-a^2}} \\
 & - \frac{6b \csc^{-1}(a+bx) \operatorname{PolyLog}\left(2, -\frac{iae^{i \csc^{-1}(a+bx)}}{1-\sqrt{1-a^2}}\right)}{a\sqrt{1-a^2}} \\
 & + \frac{6b \csc^{-1}(a+bx) \operatorname{PolyLog}\left(2, -\frac{iae^{i \csc^{-1}(a+bx)}}{1+\sqrt{1-a^2}}\right)}{a\sqrt{1-a^2}} \\
 & - \frac{6ib \operatorname{PolyLog}\left(3, -\frac{iae^{i \csc^{-1}(a+bx)}}{1-\sqrt{1-a^2}}\right)}{a\sqrt{1-a^2}} \\
 & + \frac{6ib \operatorname{PolyLog}\left(3, -\frac{iae^{i \csc^{-1}(a+bx)}}{1+\sqrt{1-a^2}}\right)}{a\sqrt{1-a^2}}
 \end{aligned}$$

output

```
-b*arccsc(b*x+a)^3/a-arccsc(b*x+a)^3/x-3*I*b*arccsc(b*x+a)^2*ln(1+I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1-(-a^2+1)^(1/2)))/a/(-a^2+1)^(1/2)+3*I*b*arccsc(b*x+a)^2*ln(1+I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1+(-a^2+1)^(1/2)))/a/(-a^2+1)^(1/2)-6*b*arccsc(b*x+a)*polylog(2,-I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1-(-a^2+1)^(1/2)))/a/(-a^2+1)^(1/2)+6*b*arccsc(b*x+a)*polylog(2,-I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1+(-a^2+1)^(1/2)))/a/(-a^2+1)^(1/2)-6*I*b*polylog(3,-I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1-(-a^2+1)^(1/2)))/a/(-a^2+1)^(1/2)+6*I*b*polylog(3,-I*a*(I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))/(1+(-a^2+1)^(1/2)))/a/(-a^2+1)^(1/2)
```

**Mathematica [A] (verified)**

Time = 0.33 (sec) , antiderivative size = 289, normalized size of antiderivative = 0.76

$$\int \frac{\csc^{-1}(a+bx)^3}{x^2} dx = \frac{(a+bx)\csc^{-1}(a+bx)^3}{x} + \frac{3ib\left(\csc^{-1}(a+bx)^2 \log\left(1 - \frac{iae^{i\csc^{-1}(a+bx)}}{-1+\sqrt{1-a^2}}\right) - \csc^{-1}(a+bx)^2 \log\left(1 + \frac{iae^{i\csc^{-1}(a+bx)}}{1+\sqrt{1-a^2}}\right) - 2i\csc^{-1}(a+bx)\text{PolyLog}\right)}{x}$$

input

```
Integrate[ArcCsc[a + b*x]^3/x^2,x]
```

output

```
-((((a + b*x)*ArcCsc[a + b*x]^3)/x + ((3*I)*b*(ArcCsc[a + b*x]^2*Log[1 - (I*a*E^(I*ArcCsc[a + b*x]))/(-1 + Sqrt[1 - a^2]]) - ArcCsc[a + b*x]^2*Log[1 + (I*a*E^(I*ArcCsc[a + b*x]))/(1 + Sqrt[1 - a^2]]) - (2*I)*ArcCsc[a + b*x]*PolyLog[2, (I*a*E^(I*ArcCsc[a + b*x]))/(-1 + Sqrt[1 - a^2]]) + (2*I)*ArcCsc[a + b*x]*PolyLog[2, ((-I)*a*E^(I*ArcCsc[a + b*x]))/(1 + Sqrt[1 - a^2]]) + 2*PolyLog[3, (I*a*E^(I*ArcCsc[a + b*x]))/(-1 + Sqrt[1 - a^2]]) - 2*PolyLog[3, ((-I)*a*E^(I*ArcCsc[a + b*x]))/(1 + Sqrt[1 - a^2]])])/Sqrt[1 - a^2])/a)
```

**Rubi [A] (verified)**

Time = 0.94 (sec) , antiderivative size = 381, normalized size of antiderivative = 1.01, number of steps used = 6, number of rules used = 5,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.417$ , Rules used = {5782, 4927, 3042, 4679, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{\csc^{-1}(a+bx)^3}{x^2} dx \\
 & \quad \downarrow \text{5782} \\
 & -b \int \frac{(a+bx)^2 \sqrt{1 - \frac{1}{(a+bx)^2}} \csc^{-1}(a+bx)^3}{b^2 x^2} d \csc^{-1}(a+bx) \\
 & \quad \downarrow \text{4927} \\
 & -b \left( 3 \int -\frac{\csc^{-1}(a+bx)^2}{bx} d \csc^{-1}(a+bx) + \frac{\csc^{-1}(a+bx)^3}{bx} \right) \\
 & \quad \downarrow \text{3042} \\
 & -b \left( 3 \int \frac{\csc^{-1}(a+bx)^2}{a - \csc(\csc^{-1}(a+bx))} d \csc^{-1}(a+bx) + \frac{\csc^{-1}(a+bx)^3}{bx} \right) \\
 & \quad \downarrow \text{4679} \\
 & -b \left( 3 \int \left( \frac{\csc^{-1}(a+bx)^2}{a} + \frac{\csc^{-1}(a+bx)^2}{a \left( \frac{a}{a+bx} - 1 \right)} \right) d \csc^{-1}(a+bx) + \frac{\csc^{-1}(a+bx)^3}{bx} \right) \\
 & \quad \downarrow \text{2009} \\
 & -b \left( \frac{\csc^{-1}(a+bx)^3}{bx} + 3 \left( \frac{2 \csc^{-1}(a+bx) \text{PolyLog} \left( 2, -\frac{ia e^{i \csc^{-1}(a+bx)}}{1 - \sqrt{1-a^2}} \right)}{a \sqrt{1-a^2}} - \frac{2 \csc^{-1}(a+bx) \text{PolyLog} \left( 2, -\frac{ia e^{i \csc^{-1}(a+bx)}}{\sqrt{1-a^2}} \right)}{a \sqrt{1-a^2}} \right) \right)
 \end{aligned}$$

input

```
Int[ArcCsc[a + b*x]^3/x^2,x]
```

output

```

-(b*(ArcCsc[a + b*x]^3/(b*x) + 3*(ArcCsc[a + b*x]^3/(3*a) + (I*ArcCsc[a +
b*x]^2*Log[1 + (I*a*E^(I*ArcCsc[a + b*x]))/(1 - Sqrt[1 - a^2])])/(a*Sqrt[1
- a^2]) - (I*ArcCsc[a + b*x]^2*Log[1 + (I*a*E^(I*ArcCsc[a + b*x]))/(1 + S
qrt[1 - a^2])])/(a*Sqrt[1 - a^2]) + (2*ArcCsc[a + b*x]*PolyLog[2, ((-I)*a*
E^(I*ArcCsc[a + b*x]))/(1 - Sqrt[1 - a^2])])/(a*Sqrt[1 - a^2]) - (2*ArcCsc
[a + b*x]*PolyLog[2, ((-I)*a*E^(I*ArcCsc[a + b*x]))/(1 + Sqrt[1 - a^2])])/(
a*Sqrt[1 - a^2]) + ((2*I)*PolyLog[3, ((-I)*a*E^(I*ArcCsc[a + b*x]))/(1 -
Sqrt[1 - a^2])])/(a*Sqrt[1 - a^2]) - ((2*I)*PolyLog[3, ((-I)*a*E^(I*ArcCsc
[a + b*x]))/(1 + Sqrt[1 - a^2])])/(a*Sqrt[1 - a^2])))

```

### Defintions of rubi rules used

rule 2009

```
Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 3042

```
Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinear
Q[u, x]
```

rule 4679

```
Int[(csc[(e_.) + (f_.)*(x_.)]*(b_.) + (a_.))^(n_.)*((c_.) + (d_.)*(x_.))^(m_.)
, x_Symbol] := Int[ExpandIntegrand[(c + d*x)^m, 1/(Sin[e + f*x]^n/(b + a*Si
n[e + f*x]^n), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && ILtQ[n, 0] && IGt
Q[m, 0]
```

rule 4927

```
Int[Cot[(c_.) + (d_.)*(x_.)]*Csc[(c_.) + (d_.)*(x_.)]*(Csc[(c_.) + (d_.)*(x_.)
]*(b_.) + (a_.))^(n_.)*((e_.) + (f_.)*(x_.))^(m_.), x_Symbol] := Simp[(-e +
f*x)^m*((a + b*Csc[c + d*x])^(n + 1)/(b*d*(n + 1))), x] + Simp[f*(m/(b*d*(
n + 1))) Int[(e + f*x)^(m - 1)*(a + b*Csc[c + d*x])^(n + 1), x], x] /; Fr
eeQ[{a, b, c, d, e, f, n}, x] && IGtQ[m, 0] && NeQ[n, -1]
```

rule 5782

```
Int[((a_.) + ArcCsc[(c_.) + (d_.)*(x_.)]*(b_.))^(p_.)*((e_.) + (f_.)*(x_.))^(m
_.), x_Symbol] := Simp[-(d^(m + 1))^(-1) Subst[Int[(a + b*x)^p*Csc[x]*Cot
[x]*(d*e - c*f + f*Csc[x])^m, x], x, ArcCsc[c + d*x]], x] /; FreeQ[{a, b, c
, d, e, f}, x] && IGtQ[p, 0] && IntegerQ[m]
```

**Maple [F]**

$$\int \frac{\operatorname{arccsc}(bx+a)^3}{x^2} dx$$

input `int(arccsc(b*x+a)^3/x^2,x)`

output `int(arccsc(b*x+a)^3/x^2,x)`

**Fricas [F]**

$$\int \frac{\csc^{-1}(a+bx)^3}{x^2} dx = \int \frac{\operatorname{arccsc}(bx+a)^3}{x^2} dx$$

input `integrate(arccsc(b*x+a)^3/x^2,x, algorithm="fricas")`

output `integral(arccsc(b*x + a)^3/x^2, x)`

**Sympy [F]**

$$\int \frac{\csc^{-1}(a+bx)^3}{x^2} dx = \int \frac{\operatorname{acsc}^3(a+bx)}{x^2} dx$$

input `integrate(acsc(b*x+a)**3/x**2,x)`

output `Integral(acsc(a + b*x)**3/x**2, x)`

**Maxima [F]**

$$\int \frac{\csc^{-1}(a + bx)^3}{x^2} dx = \int \frac{\operatorname{arccsc}(bx + a)^3}{x^2} dx$$

input `integrate(arccsc(b*x+a)^3/x^2,x, algorithm="maxima")`

output

```
-1/4*(4*arctan2(1, sqrt(b*x + a + 1))*sqrt(b*x + a - 1))^3 - 3*arctan2(1, sqrt(b*x + a + 1))*sqrt(b*x + a - 1))*log(b^2*x^2 + 2*a*b*x + a^2)^2 - 4*x*integrate(-3/4*(4*(b^3*x^3*arctan2(1, sqrt(b*x + a + 1))*sqrt(b*x + a - 1)) + 3*a*b^2*x^2*arctan2(1, sqrt(b*x + a + 1))*sqrt(b*x + a - 1)) + a^3*arctan2(1, sqrt(b*x + a + 1))*sqrt(b*x + a - 1)) + (3*a^2*arctan2(1, sqrt(b*x + a + 1))*sqrt(b*x + a - 1)) - arctan2(1, sqrt(b*x + a + 1))*sqrt(b*x + a - 1))*b*x - a*arctan2(1, sqrt(b*x + a + 1))*sqrt(b*x + a - 1))*log(b*x + a)^2 + (4*b*x*arctan2(1, sqrt(b*x + a + 1))*sqrt(b*x + a - 1))^2 - b*x*log(b^2*x^2 + 2*a*b*x + a^2)^2)*sqrt(b*x + a + 1))*sqrt(b*x + a - 1) + 4*(b^3*x^3*arctan2(1, sqrt(b*x + a + 1))*sqrt(b*x + a - 1)) + 2*a*b^2*x^2*arctan2(1, sqrt(b*x + a + 1))*sqrt(b*x + a - 1)) + (a^2*arctan2(1, sqrt(b*x + a + 1))*sqrt(b*x + a - 1)) - arctan2(1, sqrt(b*x + a + 1))*sqrt(b*x + a - 1))*b*x - (b^3*x^3*arctan2(1, sqrt(b*x + a + 1))*sqrt(b*x + a - 1)) + 3*a*b^2*x^2*arctan2(1, sqrt(b*x + a + 1))*sqrt(b*x + a - 1)) + a^3*arctan2(1, sqrt(b*x + a + 1))*sqrt(b*x + a - 1)) + (3*a^2*arctan2(1, sqrt(b*x + a + 1))*sqrt(b*x + a - 1)) - arctan2(1, sqrt(b*x + a + 1))*sqrt(b*x + a - 1))*b*x - a*arctan2(1, sqrt(b*x + a + 1))*sqrt(b*x + a - 1))*log(b*x + a))*log(b^2*x^2 + 2*a*b*x + a^2))/(b^3*x^5 + 3*a*b^2*x^4 + (3*a^2 - 1)*b*x^3 + (a^3 - a)*x^2), x)/x
```

**Giac [F]**

$$\int \frac{\csc^{-1}(a + bx)^3}{x^2} dx = \int \frac{\operatorname{arccsc}(bx + a)^3}{x^2} dx$$

input `integrate(arccsc(b*x+a)^3/x^2,x, algorithm="giac")`

output `integrate(arccsc(b*x + a)^3/x^2, x)`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{\csc^{-1}(a + bx)^3}{x^2} dx = \int \frac{\operatorname{asin}\left(\frac{1}{a+bx}\right)^3}{x^2} dx$$

input `int(asin(1/(a + b*x))^3/x^2,x)`output `int(asin(1/(a + b*x))^3/x^2, x)`**Reduce [F]**

$$\int \frac{\csc^{-1}(a + bx)^3}{x^2} dx = \int \frac{\operatorname{acsc}(bx + a)^3}{x^2} dx$$

input `int(acsc(b*x+a)^3/x^2,x)`output `int(acsc(a + b*x)**3/x**2,x)`

### 3.38 $\int x^3 \csc^{-1}(a + bx^4) dx$

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Rubi [A] (warning: unable to verify)	320
Maple [A] (verified)	322
Fricas [B] (verification not implemented)	322
Sympy [F(-1)]	323
Maxima [A] (verification not implemented)	323
Giac [B] (verification not implemented)	324
Mupad [B] (verification not implemented)	324
Reduce [F]	325

#### Optimal result

Integrand size = 12, antiderivative size = 48

$$\int x^3 \csc^{-1}(a + bx^4) dx = \frac{(a + bx^4) \csc^{-1}(a + bx^4)}{4b} + \frac{\operatorname{arctanh}\left(\sqrt{1 - \frac{1}{(a + bx^4)^2}}\right)}{4b}$$

output

$$1/4*(b*x^4+a)*\operatorname{arccsc}(b*x^4+a)/b+1/4*\operatorname{arctanh}((1-1/(b*x^4+a)^2)^{(1/2)})/b$$

#### Mathematica [B] (verified)

Leaf count is larger than twice the leaf count of optimal. 127 vs.  $2(48) = 96$ .

Time = 0.13 (sec) , antiderivative size = 127, normalized size of antiderivative = 2.65

$$\begin{aligned} & \int x^3 \csc^{-1}(a + bx^4) dx \\ &= \frac{(a + bx^4) \csc^{-1}(a + bx^4)}{4b} \\ &+ \frac{\sqrt{-1 + (a + bx^4)^2} \left( -\log\left(1 - \frac{a + bx^4}{\sqrt{-1 + (a + bx^4)^2}}\right) + \log\left(1 + \frac{a + bx^4}{\sqrt{-1 + (a + bx^4)^2}}\right) \right)}{8b(a + bx^4) \sqrt{1 - \frac{1}{(a + bx^4)^2}}} \end{aligned}$$



input `Integrate[x^3*ArcCsc[a + b*x^4],x]`

output 
$$\frac{((a + b*x^4)*ArcCsc[a + b*x^4])/(4*b) + (Sqrt[-1 + (a + b*x^4)^2]*(-Log[1 - (a + b*x^4)/Sqrt[-1 + (a + b*x^4)^2]] + Log[1 + (a + b*x^4)/Sqrt[-1 + (a + b*x^4)^2]]))/(8*b*(a + b*x^4)*Sqrt[1 - (a + b*x^4)^(-2)])}$$

### Rubi [A] (warning: unable to verify)

Time = 0.30 (sec) , antiderivative size = 44, normalized size of antiderivative = 0.92, number of steps used = 7, number of rules used = 6,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {7266, 5774, 895, 798, 73, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int x^3 \csc^{-1}(a + bx^4) dx \\ & \quad \downarrow 7266 \\ & \frac{1}{4} \int \csc^{-1}(bx^4 + a) dx^4 \\ & \quad \downarrow 5774 \\ & \frac{1}{4} \left( \int \frac{1}{(bx^4 + a) \sqrt{1 - \frac{1}{(bx^4 + a)^2}}} dx^4 + \frac{(a + bx^4) \csc^{-1}(a + bx^4)}{b} \right) \\ & \quad \downarrow 895 \\ & \frac{1}{4} \left( \frac{\int \frac{1}{\sqrt{1 - \frac{1}{x^8} x^4}} d(bx^4 + a)}{b} + \frac{(a + bx^4) \csc^{-1}(a + bx^4)}{b} \right) \\ & \quad \downarrow 798 \\ & \frac{1}{4} \left( \frac{(a + bx^4) \csc^{-1}(a + bx^4)}{b} - \frac{\int \frac{1}{x^4 \sqrt{-bx^4 - a + 1}} d\frac{1}{x^8}}{2b} \right) \\ & \quad \downarrow 73 \end{aligned}$$

$$\frac{1}{4} \left( \frac{\int \frac{1}{1-x^8} d\sqrt{-bx^4 - a + 1}}{b} + \frac{(a + bx^4) \csc^{-1}(a + bx^4)}{b} \right)$$

↓ 219

$$\frac{1}{4} \left( \frac{\operatorname{arctanh}\left(\sqrt{-a - bx^4 + 1}\right)}{b} + \frac{(a + bx^4) \csc^{-1}(a + bx^4)}{b} \right)$$

input `Int[x^3*ArcCsc[a + b*x^4],x]`

output `((a + b*x^4)*ArcCsc[a + b*x^4])/b + ArcTanh[Sqrt[1 - a - b*x^4]]/b)/4`

### Defintions of rubi rules used

rule 73 `Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] := With[  
{p = Denominator[m]}, Simp[p/b Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +  
d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x] /; FreeQ[{a, b, c, d}, x] && Lt  
Q[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Denominator[m]] && IntL  
inearQ[a, b, c, d, m, n, x]`

rule 219 `Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*  
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt  
Q[a, 0] || LtQ[b, 0])`

rule 798 `Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] := Simp[1/n Subst  
[Int[x^(Simplify[(m + 1)/n] - 1)*(a + b*x)^p, x], x, x^n], x] /; FreeQ[{a,  
b, m, n, p}, x] && IntegerQ[Simplify[(m + 1)/n]]`

rule 895 `Int[(u_)^(m_.)*((a_) + (b_.)*(v_)^(n_))^(p_.), x_Symbol] := Simp[u^m/(Coeff  
icient[v, x, 1]*v^m) Subst[Int[x^m*(a + b*x^n)^p, x], x, v], x] /; FreeQ[  
{a, b, m, n, p}, x] && LinearPairQ[u, v, x]`

rule 5774 `Int[ArcCsc[(c_) + (d_.)*(x_)], x_Symbol] := Simp[(c + d*x)*(ArcCsc[c + d*x]/d), x] + Int[1/((c + d*x)*Sqrt[1 - 1/(c + d*x)^2]), x] /; FreeQ[{c, d}, x]`

rule 7266 `Int[(u_)*(x_)^(m_), x_Symbol] := Simp[1/(m + 1) Subst[Int[SubstFor[x^(m + 1), u, x], x], x, x^(m + 1)], x] /; FreeQ[m, x] && NeQ[m, -1] && FunctionOfQ[x^(m + 1), u, x]`

### Maple [A] (verified)

Time = 0.19 (sec) , antiderivative size = 54, normalized size of antiderivative = 1.12

method	result	size
derivativedivides	$\frac{\operatorname{arccsc}(bx^4+a)(bx^4+a) + \ln\left(bx^4+a + (bx^4+a)\sqrt{1-\frac{1}{(bx^4+a)^2}}\right)}{4b}$	54
default	$\frac{\operatorname{arccsc}(bx^4+a)(bx^4+a) + \ln\left(bx^4+a + (bx^4+a)\sqrt{1-\frac{1}{(bx^4+a)^2}}\right)}{4b}$	54

input `int(x^3*arccsc(b*x^4+a),x,method=_RETURNVERBOSE)`

output `1/4/b*(arccsc(b*x^4+a)*(b*x^4+a)+ln(b*x^4+a+(b*x^4+a)*(1-1/(b*x^4+a)^2)^(1/2)))`

### Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 88 vs.  $2(42) = 84$ .

Time = 0.11 (sec) , antiderivative size = 88, normalized size of antiderivative = 1.83

$$\int x^3 \operatorname{csc}^{-1}(a + bx^4) dx$$

$$= \frac{bx^4 \operatorname{arccsc}(bx^4 + a) - 2a \arctan(-bx^4 - a + \sqrt{b^2x^8 + 2abx^4 + a^2 - 1}) - \log(-bx^4 - a + \sqrt{b^2x^8 + 2abx^4 + a^2 - 1})}{4b}$$

input `integrate(x^3*arccsc(b*x^4+a),x, algorithm="fricas")`

output  $\frac{1}{4}(b^2x^8 + 2abx^4 + a^2 - 1) \arccsc(b^2x^8 + 2abx^4 + a^2 - 1) - \log(-bx^4 - a + \sqrt{b^2x^8 + 2abx^4 + a^2 - 1})$   
/b

### Sympy [F(-1)]

Timed out.

$$\int x^3 \csc^{-1}(a + bx^4) dx = \text{Timed out}$$

input `integrate(x**3*acsc(b*x**4+a),x)`

output Timed out

### Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 63, normalized size of antiderivative = 1.31

$$\int x^3 \csc^{-1}(a + bx^4) dx$$

$$= \frac{2(bx^4 + a) \arccsc(bx^4 + a) + \log\left(\sqrt{-\frac{1}{(bx^4+a)^2} + 1} + 1\right) - \log\left(-\sqrt{-\frac{1}{(bx^4+a)^2} + 1} + 1\right)}{8b}$$

input `integrate(x^3*arccsc(b*x^4+a),x, algorithm="maxima")`

output  $\frac{1}{8}(2(bx^4 + a) \arccsc(bx^4 + a) + \log(\sqrt{-1/(bx^4 + a)^2 + 1} + 1) - \log(-\sqrt{-1/(bx^4 + a)^2 + 1} + 1))/b$

**Giac [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 91 vs.  $2(42) = 84$ .

Time = 0.16 (sec) , antiderivative size = 91, normalized size of antiderivative = 1.90

$$\int x^3 \csc^{-1}(a + bx^4) dx$$

$$= \frac{1}{8} b \left( \frac{2(bx^4 + a) \arcsin\left(-\frac{1}{(bx^4+a)\left(\frac{a}{bx^4+a}-1\right)-a}\right)}{b^2} + \frac{\log\left(\sqrt{-\frac{1}{(bx^4+a)^2}+1}+1\right) - \log\left(-\sqrt{-\frac{1}{(bx^4+a)^2}+1}+1\right)}{b^2} \right)$$

input `integrate(x^3*arccsc(b*x^4+a),x, algorithm="giac")`

output `1/8*b*(2*(b*x^4 + a)*arcsin(-1/((b*x^4 + a)*(a/(b*x^4 + a) - 1) - a))/b^2 + (log(sqrt(-1/(b*x^4 + a)^2 + 1) + 1) - log(-sqrt(-1/(b*x^4 + a)^2 + 1) + 1))/b^2)`

**Mupad [B] (verification not implemented)**

Time = 1.28 (sec) , antiderivative size = 44, normalized size of antiderivative = 0.92

$$\int x^3 \csc^{-1}(a + bx^4) dx = \frac{\operatorname{atanh}\left(\frac{1}{\sqrt{1-\frac{1}{(bx^4+a)^2}}}\right)}{4b} + \frac{\operatorname{asin}\left(\frac{1}{bx^4+a}\right)(bx^4+a)}{4b}$$

input `int(x^3*asin(1/(a + b*x^4)),x)`

output `atanh(1/(1 - 1/(a + b*x^4)^2)^(1/2))/(4*b) + (asin(1/(a + b*x^4))*(a + b*x^4))/(4*b)`

**Reduce [F]**

$$\int x^3 \csc^{-1}(a + bx^4) dx = \int \operatorname{acsc}(bx^4 + a) x^3 dx$$

input `int(x^3*acsc(b*x^4+a),x)`

output `int(acsc(a + b*x**4)*x**3,x)`

### 3.39 $\int x^{-1+n} \csc^{-1}(a + bx^n) dx$

Optimal result	326
Mathematica [B] (verified)	326
Rubi [A] (warning: unable to verify)	327
Maple [F]	329
Fricas [B] (verification not implemented)	329
Sympy [F(-1)]	330
Maxima [A] (verification not implemented)	330
Giac [A] (verification not implemented)	330
Mupad [B] (verification not implemented)	331
Reduce [F]	331

#### Optimal result

Integrand size = 14, antiderivative size = 48

$$\int x^{-1+n} \csc^{-1}(a + bx^n) dx = \frac{(a + bx^n) \csc^{-1}(a + bx^n)}{bn} + \frac{\operatorname{arctanh}\left(\sqrt{1 - \frac{1}{(a+bx^n)^2}}\right)}{bn}$$

output

```
(a+b*x^n)*arccsc(a+b*x^n)/b/n+arctanh((1-1/(a+b*x^n)^2)^(1/2))/b/n
```

#### Mathematica [B] (verified)

Leaf count is larger than twice the leaf count of optimal. 130 vs. 2(48) = 96.

Time = 0.19 (sec) , antiderivative size = 130, normalized size of antiderivative = 2.71

$$\begin{aligned} & \int x^{-1+n} \csc^{-1}(a + bx^n) dx \\ &= \frac{(a + bx^n) \csc^{-1}(a + bx^n)}{bn} \\ &+ \frac{\sqrt{-1 + (a + bx^n)^2} \left( -\log\left(1 - \frac{a+bx^n}{\sqrt{-1+(a+bx^n)^2}}\right) + \log\left(1 + \frac{a+bx^n}{\sqrt{-1+(a+bx^n)^2}}\right) \right)}{2bn(a + bx^n) \sqrt{1 - \frac{1}{(a+bx^n)^2}}} \end{aligned}$$

input `Integrate[x^(-1 + n)*ArcCsc[a + b*x^n], x]`

output 
$$\frac{((a + b*x^n)*ArcCsc[a + b*x^n])/(b*n) + (Sqrt[-1 + (a + b*x^n)^2]*(-Log[1 - (a + b*x^n)/Sqrt[-1 + (a + b*x^n)^2]] + Log[1 + (a + b*x^n)/Sqrt[-1 + (a + b*x^n)^2]]))/(2*b*n*(a + b*x^n)*Sqrt[1 - (a + b*x^n)^(-2)])}{n}$$

### Rubi [A] (warning: unable to verify)

Time = 0.34 (sec) , antiderivative size = 44, normalized size of antiderivative = 0.92, number of steps used = 7, number of rules used = 6,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.429$ , Rules used = {7266, 5774, 895, 798, 73, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int x^{n-1} \csc^{-1}(a + bx^n) dx \\ & \quad \downarrow \text{7266} \\ & \frac{\int \csc^{-1}(bx^n + a) dx^n}{n} \\ & \quad \downarrow \text{5774} \\ & \frac{\int \frac{1}{(bx^n+a)\sqrt{1-\frac{1}{(bx^n+a)^2}}} dx^n + \frac{(a+bx^n) \csc^{-1}(a+bx^n)}{b}}{n} \\ & \quad \downarrow \text{895} \\ & \frac{\int \frac{x^{-n}}{\sqrt{1-x^{-2n}}} d(bx^n+a)}{b} + \frac{(a+bx^n) \csc^{-1}(a+bx^n)}{b}}{n} \\ & \quad \downarrow \text{798} \\ & \frac{(a+bx^n) \csc^{-1}(a+bx^n)}{b} - \frac{\int \frac{x^{-n}}{\sqrt{-bx^n-a+1}} dx^{-2n}}{2b}}{n} \\ & \quad \downarrow \text{73} \\ & \frac{\int \frac{1}{1-x^{2n}} d\sqrt{-bx^n-a+1}}{b} + \frac{(a+bx^n) \csc^{-1}(a+bx^n)}{b}}{n} \end{aligned}$$



$$\frac{\frac{\operatorname{arctanh}\left(\frac{\sqrt{-a-bx^n+1}}{b}\right)}{b} + \frac{(a+bx^n) \operatorname{csc}^{-1}(a+bx^n)}{b}}{n}$$

input `Int[x^(-1 + n)*ArcCsc[a + b*x^n], x]`

output `((a + b*x^n)*ArcCsc[a + b*x^n])/b + ArcTanh[Sqrt[1 - a - b*x^n]]/b/n`

### Defintions of rubi rules used

rule 73 `Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] := With[  
 {p = Denominator[m]}, Simp[p/b Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +  
 d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && Lt  
 Q[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Denominator[m]] && IntL  
 inearQ[a, b, c, d, m, n, x]`

rule 219 `Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*  
 ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt  
 Q[a, 0] || LtQ[b, 0])`

rule 798 `Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] := Simp[1/n Subst  
 [Int[x^(Simplify[(m + 1)/n] - 1)*(a + b*x)^p, x], x, x^n], x] /; FreeQ[{a,  
 b, m, n, p}, x] && IntegerQ[Simplify[(m + 1)/n]]`

rule 895 `Int[(u_)^(m_.)*((a_) + (b_.)*(v_)^(n_))^(p_.), x_Symbol] := Simp[u^m/(Coeff  
 icient[v, x, 1]*v^m) Subst[Int[x^m*(a + b*x^n)^p, x], x, v], x] /; FreeQ[  
 {a, b, m, n, p}, x] && LinearPairQ[u, v, x]`

rule 5774 `Int[ArcCsc[(c_) + (d_.)*(x_)], x_Symbol] := Simp[(c + d*x)*(ArcCsc[c + d*x]  
 /d), x] + Int[1/((c + d*x)*Sqrt[1 - 1/(c + d*x)^2]), x] /; FreeQ[{c, d}, x]`

rule 7266

```
Int[(u_)*(x_)^(m_.), x_Symbol] := Simp[1/(m + 1) Subst[Int[SubstFor[x^(m
+ 1), u, x], x], x, x^(m + 1)], x] /; FreeQ[m, x] && NeQ[m, -1] && Function
OfQ[x^(m + 1), u, x]
```

**Maple [F]**

$$\int x^{-1+n} \operatorname{arccsc}(a + bx^n) dx$$

input `int(x^(-1+n)*arccsc(a+b*x^n),x)`

output `int(x^(-1+n)*arccsc(a+b*x^n),x)`

**Fricas [B] (verification not implemented)**

Leaf count of result is larger than twice the leaf count of optimal. 94 vs.  $2(46) = 92$ .

Time = 0.14 (sec) , antiderivative size = 94, normalized size of antiderivative = 1.96

$$\int x^{-1+n} \operatorname{csc}^{-1}(a + bx^n) dx$$

$$= \frac{bx^n \operatorname{arccsc}(bx^n + a) - 2a \arctan(-bx^n - a + \sqrt{b^2x^{2n} + 2abx^n + a^2 - 1}) - \log(-bx^n - a + \sqrt{b^2x^{2n} + 2abx^n + a^2 - 1})}{bn}$$

input `integrate(x^(-1+n)*arccsc(a+b*x^n),x, algorithm="fricas")`

output `(b*x^n*arccsc(b*x^n + a) - 2*a*arctan(-b*x^n - a + sqrt(b^2*x^(2*n) + 2*a*
b*x^n + a^2 - 1)) - log(-b*x^n - a + sqrt(b^2*x^(2*n) + 2*a*b*x^n + a^2 -
1)))/(b*n)`

**Sympy [F(-1)]**

Timed out.

$$\int x^{-1+n} \csc^{-1}(a + bx^n) dx = \text{Timed out}$$

input `integrate(x**(-1+n)*acsc(a+b*x**n),x)`output `Timed out`**Maxima [A] (verification not implemented)**

Time = 0.05 (sec) , antiderivative size = 66, normalized size of antiderivative = 1.38

$$\int x^{-1+n} \csc^{-1}(a + bx^n) dx$$

$$= \frac{2(bx^n + a) \operatorname{arccsc}(bx^n + a) + \log\left(\sqrt{-\frac{1}{(bx^n+a)^2} + 1} + 1\right) - \log\left(-\sqrt{-\frac{1}{(bx^n+a)^2} + 1} + 1\right)}{2bn}$$

input `integrate(x^(-1+n)*arccsc(a+b*x^n),x, algorithm="maxima")`output `1/2*(2*(b*x^n + a)*arccsc(b*x^n + a) + log(sqrt(-1/(b*x^n + a)^2 + 1) + 1) - log(-sqrt(-1/(b*x^n + a)^2 + 1) + 1))/(b*n)`**Giac [A] (verification not implemented)**

Time = 0.18 (sec) , antiderivative size = 74, normalized size of antiderivative = 1.54

$$\int x^{-1+n} \csc^{-1}(a + bx^n) dx$$

$$= \frac{b \left( \frac{2(bx^n+a) \arcsin\left(\frac{1}{bx^n+a}\right)}{b^2} + \frac{\log\left(\sqrt{-\frac{1}{(bx^n+a)^2} + 1} + 1\right) - \log\left(-\sqrt{-\frac{1}{(bx^n+a)^2} + 1} + 1\right)}{b^2} \right)}{2n}$$

input `integrate(x^(-1+n)*arccsc(a+b*x^n),x, algorithm="giac")`

output  $\frac{1}{2}b(2(bx^n + a)\arcsin(1/(bx^n + a))/b^2 + (\log(\sqrt{-1/(bx^n + a)^2 + 1}) + 1) - \log(-\sqrt{-1/(bx^n + a)^2 + 1})/b^2)/n$

### Mupad [B] (verification not implemented)

Time = 0.97 (sec) , antiderivative size = 42, normalized size of antiderivative = 0.88

$$\int x^{-1+n} \csc^{-1}(a + bx^n) dx = \frac{\operatorname{atanh}\left(\frac{1}{\sqrt{1 - \frac{1}{(a+bx^n)^2}}}\right) + \operatorname{asin}\left(\frac{1}{a+bx^n}\right) (a + bx^n)}{bn}$$

input `int(x^(n - 1)*asin(1/(a + b*x^n)),x)`

output  $(\operatorname{atanh}(1/(1 - 1/(a + bx^n)^2)^{1/2}) + \operatorname{asin}(1/(a + bx^n))*(a + bx^n))/(b*n)$

### Reduce [F]

$$\int x^{-1+n} \csc^{-1}(a + bx^n) dx = \int \frac{x^n \operatorname{acsc}(x^n b + a)}{x} dx$$

input `int(x^(-1+n)*acsc(a+b*x^n),x)`

output `int((x**n*acsc(x**n*b + a))/x,x)`

### 3.40 $\int \csc^{-1} (ce^{a+bx}) dx$

Optimal result	332
Mathematica [B] (verified)	332
Rubi [A] (warning: unable to verify)	333
Maple [A] (verified)	336
Fricas [F(-2)]	336
Sympy [F]	337
Maxima [F]	337
Giac [F]	338
Mupad [B] (verification not implemented)	338
Reduce [F]	338

#### Optimal result

Integrand size = 10, antiderivative size = 85

$$\int \csc^{-1} (ce^{a+bx}) dx = \frac{i \csc^{-1} (ce^{a+bx})^2}{2b} - \frac{\csc^{-1} (ce^{a+bx}) \log (1 - e^{2i \csc^{-1} (ce^{a+bx})})}{b} + \frac{i \text{PolyLog} (2, e^{2i \csc^{-1} (ce^{a+bx})})}{2b}$$

output

```
1/2*I*arccsc(c*exp(b*x+a))^2/b-arccsc(c*exp(b*x+a))*ln(1-(I/c/(exp(1)^(b*x+a)))+(1-1/c^2/(exp(1)^(b*x+a))^2)^(1/2))/b+1/2*I*polylog(2,(I/c/(exp(1)^(b*x+a)))+(1-1/c^2/(exp(1)^(b*x+a))^2)^(1/2))/b
```

#### Mathematica [B] (verified)

Leaf count is larger than twice the leaf count of optimal. 280 vs. 2(85) = 170.

Time = 0.39 (sec) , antiderivative size = 280, normalized size of antiderivative = 3.29

$$\int \csc^{-1} (ce^{a+bx}) dx = x \csc^{-1} (ce^{a+bx}) + \frac{e^{-a-bx} \left( 4\sqrt{-1 + c^2 e^{2(a+bx)}} \arctan \left( \sqrt{-1 + c^2 e^{2(a+bx)}} \right) (2bx - \log (c^2 e^{2(a+bx)})) + \sqrt{1 - c^2 e^{2(a+bx)}} \left( \log^2 \right) \right)}{\dots}$$

input `Integrate[ArcCsc[c*E^(a + b*x)],x]`

output `x*ArcCsc[c*E^(a + b*x)] + (E^(-a - b*x)*(4*Sqrt[-1 + c^2*E^(2*(a + b*x))]*ArcTan[Sqrt[-1 + c^2*E^(2*(a + b*x))]]*(2*b*x - Log[c^2*E^(2*(a + b*x))]) + Sqrt[1 - c^2*E^(2*(a + b*x))]*(Log[c^2*E^(2*(a + b*x))]^2 - 4*Log[c^2*E^(2*(a + b*x))]*Log[(1 + Sqrt[1 - c^2*E^(2*(a + b*x))])/2] + 2*Log[(1 + Sqrt[1 - c^2*E^(2*(a + b*x))])/2]^2) - 4*Sqrt[1 - c^2*E^(2*(a + b*x))]*PolyLog[2, (1 - Sqrt[1 - c^2*E^(2*(a + b*x))])/2])/(8*b*c*Sqrt[1 - 1/(c^2*E^(2*(a + b*x)))]])`

### Rubi [A] (warning: unable to verify)

Time = 0.47 (sec) , antiderivative size = 101, normalized size of antiderivative = 1.19, number of steps used = 11, number of rules used = 10,  $\frac{\text{number of rules}}{\text{integrand size}} = 1.000$ , Rules used = {2720, 5742, 5136, 3042, 25, 4200, 25, 2620, 2715, 2838}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \csc^{-1}(ce^{a+bx}) dx \\
 & \quad \downarrow 2720 \\
 & \frac{\int e^{-a-bx} \csc^{-1}(ce^{a+bx}) de^{a+bx}}{b} \\
 & \quad \downarrow 5742 \\
 & \frac{\int e^{-a-bx} \arcsin\left(\frac{e^{-a-bx}}{c}\right) de^{-a-bx}}{b} \\
 & \quad \downarrow 5136 \\
 & \frac{\int ce^{a+bx} \sqrt{1 - \frac{e^{-2a-2bx}}{c^2}} \arcsin\left(\frac{e^{-a-bx}}{c}\right) d \arcsin\left(\frac{e^{-a-bx}}{c}\right)}{b} \\
 & \quad \downarrow 3042 \\
 & \frac{\int -\arcsin\left(\frac{e^{-a-bx}}{c}\right) \tan\left(\arcsin\left(\frac{e^{-a-bx}}{c}\right) + \frac{\pi}{2}\right) d \arcsin\left(\frac{e^{-a-bx}}{c}\right)}{b}
 \end{aligned}$$

$$\begin{array}{c}
\int \arcsin\left(\frac{e^{-a-bx}}{c}\right) \tan\left(\arcsin\left(\frac{e^{-a-bx}}{c}\right) + \frac{\pi}{2}\right) d\arcsin\left(\frac{e^{-a-bx}}{c}\right) \\
\hline
b \\
\downarrow 25 \\
2i \int \frac{e^{a+bx+2i \arcsin\left(\frac{e^{-a-bx}}{c}\right)}}{1-e^{2i \arcsin\left(\frac{e^{-a-bx}}{c}\right)}} d\arcsin\left(\frac{e^{-a-bx}}{c}\right) - \frac{1}{2}ie^{2a+2bx} \\
\hline
b \\
\downarrow 25 \\
-2i \int \frac{e^{a+bx+2i \arcsin\left(\frac{e^{-a-bx}}{c}\right)}}{1-e^{2i \arcsin\left(\frac{e^{-a-bx}}{c}\right)}} d\arcsin\left(\frac{e^{-a-bx}}{c}\right) - \frac{1}{2}ie^{2a+2bx} \\
\hline
b \\
\downarrow 2620 \\
-2i \left( \frac{1}{2}i \arcsin\left(\frac{e^{-a-bx}}{c}\right) \log\left(1 - e^{2i \arcsin\left(\frac{e^{-a-bx}}{c}\right)}\right) - \frac{1}{2}i \int \log\left(1 - e^{2i \arcsin\left(\frac{e^{-a-bx}}{c}\right)}\right) d\arcsin\left(\frac{e^{-a-bx}}{c}\right) \right) - \frac{1}{2}ie^{2a+2bx} \\
\hline
b \\
\downarrow 2715 \\
-2i \left( \frac{1}{2}i \arcsin\left(\frac{e^{-a-bx}}{c}\right) \log\left(1 - e^{2i \arcsin\left(\frac{e^{-a-bx}}{c}\right)}\right) - \frac{1}{4} \int e^{2i \arcsin\left(\frac{e^{-a-bx}}{c}\right)} \log\left(1 - e^{2i \arcsin\left(\frac{e^{-a-bx}}{c}\right)}\right) de^{2i \arcsin\left(\frac{e^{-a-bx}}{c}\right)} \right) \\
\hline
b \\
\downarrow 2838 \\
-2i \left( \frac{1}{4} \text{PolyLog}\left(2, e^{2i \arcsin\left(\frac{e^{-a-bx}}{c}\right)}\right) + \frac{1}{2}i \arcsin\left(\frac{e^{-a-bx}}{c}\right) \log\left(1 - e^{2i \arcsin\left(\frac{e^{-a-bx}}{c}\right)}\right) \right) - \frac{1}{2}ie^{2a+2bx} \\
\hline
b
\end{array}$$

input

```
Int[ArcCsc[c*E^(a + b*x)],x]
```

output

```
-((( (-1/2*I)*E^(2*a + 2*b*x) - (2*I)*((1/2)*ArcSin[E^(-a - b*x)/c]*Log[1 - E^((2*I)*ArcSin[E^(-a - b*x)/c]]) + PolyLog[2, E^((2*I)*ArcSin[E^(-a - b*x)/c]])/4))/b)
```

## Definitions of rubi rules used

- rule 25  $\text{Int}[-(\text{Fx}_), \text{x\_Symbol}] \rightarrow \text{Simp}[\text{Identity}[-1] \quad \text{Int}[\text{Fx}, \text{x}], \text{x}]$
- rule 2620  $\text{Int}[(((\text{F}_)^{((\text{g}_)*((\text{e}_.) + (\text{f}_.)(\text{x}_))))^{(\text{n}_.)}*((\text{c}_.) + (\text{d}_.)(\text{x}_))^{(\text{m}_.)})/((\text{a}_.) + (\text{b}_.)*((\text{F}_)^{((\text{g}_)*((\text{e}_.) + (\text{f}_.)(\text{x}_))))^{(\text{n}_.)}), \text{x\_Symbol}] \rightarrow \text{Simp}[\text{Simp}[(\text{c} + \text{d}*\text{x})^{\text{m}}/(\text{b}*f*g*n*\text{Log}[\text{F}]))*\text{Log}[1 + \text{b}*((\text{F}^{\text{g}}(\text{e} + \text{f}*\text{x}))^{\text{n}/\text{a}})], \text{x}] - \text{Simp}[\text{d}*(\text{m}/(\text{b}*f*g*n*\text{Log}[\text{F}])) \quad \text{Int}[(\text{c} + \text{d}*\text{x})^{(\text{m} - 1)}*\text{Log}[1 + \text{b}*((\text{F}^{\text{g}}(\text{e} + \text{f}*\text{x}))^{\text{n}/\text{a}})], \text{x}], \text{x}] /; \text{FreeQ}\{\text{F}, \text{a}, \text{b}, \text{c}, \text{d}, \text{e}, \text{f}, \text{g}, \text{n}\}, \text{x}\} \&\& \text{IGtQ}[\text{m}, 0]$
- rule 2715  $\text{Int}[\text{Log}[(\text{a}_.) + (\text{b}_.)*((\text{F}_)^{((\text{e}_.)*((\text{c}_.) + (\text{d}_.)(\text{x}_))))^{(\text{n}_.)}], \text{x\_Symbol}] \rightarrow \text{Simp}[1/(\text{d}*e*n*\text{Log}[\text{F}]) \quad \text{Subst}[\text{Int}[\text{Log}[\text{a} + \text{b}*\text{x}]/\text{x}, \text{x}], \text{x}, (\text{F}^{\text{e}}(\text{c} + \text{d}*\text{x}))^{\text{n}}], \text{x}] /; \text{FreeQ}\{\text{F}, \text{a}, \text{b}, \text{c}, \text{d}, \text{e}, \text{n}\}, \text{x}\} \&\& \text{GtQ}[\text{a}, 0]$
- rule 2720  $\text{Int}[\text{u}_, \text{x\_Symbol}] \rightarrow \text{With}\{\text{v} = \text{FunctionOfExponential}[\text{u}, \text{x}]\}, \text{Simp}[\text{v}/\text{D}[\text{v}, \text{x}] \quad \text{Subst}[\text{Int}[\text{FunctionOfExponentialFunction}[\text{u}, \text{x}]/\text{x}, \text{x}], \text{x}, \text{v}], \text{x}] /; \text{FunctionOfExponentialQ}[\text{u}, \text{x}] \&\& \text{!MatchQ}[\text{u}, (\text{w}_)*((\text{a}_.)*(\text{v}_)^{\text{n}})^{\text{m}}] /; \text{FreeQ}\{\text{a}, \text{m}, \text{n}\}, \text{x}\} \&\& \text{IntegerQ}[\text{m}*n] \&\& \text{!MatchQ}[\text{u}, \text{E}^{((\text{c}_.)*((\text{a}_.) + (\text{b}_.)*\text{x}))}*(\text{F}_)[\text{v}_] /; \text{FreeQ}\{\text{a}, \text{b}, \text{c}\}, \text{x}\} \&\& \text{InverseFunctionQ}[\text{F}[\text{x}]]$
- rule 2838  $\text{Int}[\text{Log}[(\text{c}_.)*((\text{d}_.) + (\text{e}_.)(\text{x}_)^{\text{n}})]/(\text{x}_), \text{x\_Symbol}] \rightarrow \text{Simp}[-\text{PolyLog}[2, (-\text{c})*\text{e}*\text{x}^{\text{n}}/\text{n}, \text{x}] /; \text{FreeQ}\{\text{c}, \text{d}, \text{e}, \text{n}\}, \text{x}\} \&\& \text{EqQ}[\text{c}*\text{d}, 1]$
- rule 3042  $\text{Int}[\text{u}_, \text{x\_Symbol}] \rightarrow \text{Int}[\text{DeactivateTrig}[\text{u}, \text{x}], \text{x}] /; \text{FunctionOfTrigOfLinearQ}[\text{u}, \text{x}]$
- rule 4200  $\text{Int}[(\text{c}_.) + (\text{d}_.)(\text{x}_)]^{\text{m}}*\text{tan}[(\text{e}_.) + \text{Pi}*(\text{k}_.) + (\text{f}_.)(\text{x}_)], \text{x\_Symbol}] \rightarrow \text{Simp}[\text{I}*((\text{c} + \text{d}*\text{x})^{\text{m} + 1}/(\text{d}*(\text{m} + 1))), \text{x}] - \text{Simp}[2*\text{I} \quad \text{Int}[(\text{c} + \text{d}*\text{x})^{\text{m}}*\text{E}^{(2*\text{I}*k*\text{Pi})}*(\text{E}^{(2*\text{I}*(\text{e} + \text{f}*\text{x}))}/(1 + \text{E}^{(2*\text{I}*k*\text{Pi})}*\text{E}^{(2*\text{I}*(\text{e} + \text{f}*\text{x}))))], \text{x}], \text{x}] /; \text{FreeQ}\{\text{c}, \text{d}, \text{e}, \text{f}\}, \text{x}\} \&\& \text{IntegerQ}[4*k] \&\& \text{IGtQ}[\text{m}, 0]$
- rule 5136  $\text{Int}[(\text{a}_.) + \text{ArcSin}[(\text{c}_.)(\text{x}_)]*(\text{b}_.)^{\text{n}}/(\text{x}_), \text{x\_Symbol}] \rightarrow \text{Subst}[\text{Int}[(\text{a} + \text{b}*\text{x})^{\text{n}}*\text{Cot}[\text{x}], \text{x}], \text{x}, \text{ArcSin}[\text{c}*\text{x}]] /; \text{FreeQ}\{\text{a}, \text{b}, \text{c}\}, \text{x}\} \&\& \text{IGtQ}[\text{n}, 0]$



rule 5742

```
Int[((a_.) + ArcCsc[(c_.)*(x_)]*(b_.))/(x_), x_Symbol] :> -Subst[Int[(a + b
*ArcSin[x/c])/x, x], x, 1/x] /; FreeQ[{a, b, c}, x]
```

**Maple [A] (verified)**

Time = 1.00 (sec) , antiderivative size = 188, normalized size of antiderivative = 2.21

method	result
derivativedivides	$\frac{i \operatorname{arccsc}\left(\frac{e^{bx+a}c}{2}\right)^2}{2} - \operatorname{arccsc}\left(e^{bx+a}c\right) \ln\left(1 + \frac{ie^{-bx-a}}{c} + \sqrt{1 - \frac{e^{-2bx-2a}}{c^2}}\right) + i \operatorname{polylog}\left(2, -\frac{ie^{-bx-a}}{c} - \sqrt{1 - \frac{e^{-2bx-2a}}{c^2}}\right) - \frac{\operatorname{arccsc}\left(e^{bx+a}c\right)}{b}$
default	$\frac{i \operatorname{arccsc}\left(\frac{e^{bx+a}c}{2}\right)^2}{2} - \operatorname{arccsc}\left(e^{bx+a}c\right) \ln\left(1 + \frac{ie^{-bx-a}}{c} + \sqrt{1 - \frac{e^{-2bx-2a}}{c^2}}\right) + i \operatorname{polylog}\left(2, -\frac{ie^{-bx-a}}{c} - \sqrt{1 - \frac{e^{-2bx-2a}}{c^2}}\right) - \frac{\operatorname{arccsc}\left(e^{bx+a}c\right)}{b}$

input

```
int(arccsc(exp(b*x+a)*c),x,method=_RETURNVERBOSE)
```

output

```
1/b*(1/2*I*arccsc(exp(b*x+a)*c)^2-arccsc(exp(b*x+a)*c)*ln(1+I/exp(b*x+a)/c
+(1-1/exp(b*x+a)^2/c^2)^(1/2))+I*polylog(2,-I/exp(b*x+a)/c-(1-1/exp(b*x+a)
^2/c^2)^(1/2))-arccsc(exp(b*x+a)*c)*ln(1-I/exp(b*x+a)/c-(1-1/exp(b*x+a)^2/
c^2)^(1/2))+I*polylog(2,I/exp(b*x+a)/c+(1-1/exp(b*x+a)^2/c^2)^(1/2)))
```

**Fricas [F(-2)]**

Exception generated.

$$\int \operatorname{csc}^{-1}(ce^{a+bx}) dx = \text{Exception raised: TypeError}$$

input

```
integrate(arccsc(c*exp(b*x+a)),x, algorithm="fricas")
```

output

```
Exception raised: TypeError >> Error detected within library code: inte
grate: implementation incomplete (constant residues)
```

**Sympy [F]**

$$\int \csc^{-1}(ce^{a+bx}) dx = \int \operatorname{acsc}(ce^{a+bx}) dx$$

input `integrate(acsc(c*exp(b*x+a)),x)`

output `Integral(acsc(c*exp(a + b*x)), x)`

**Maxima [F]**

$$\int \csc^{-1}(ce^{a+bx}) dx = \int \operatorname{arccsc}(ce^{(bx+a)}) dx$$

input `integrate(arccsc(c*exp(b*x+a)),x, algorithm="maxima")`

output `1/2*(2*b^2*c^2*integrate(x*e^(2*b*x + 2*a + 1/2*log(c*e^(b*x + a) + 1) + 1/2*log(c*e^(b*x + a) - 1))/(c^2*e^(2*b*x + 2*a) + (c^2*e^(2*b*x + 2*a) - 1)*e^(log(c*e^(b*x + a) + 1) + log(c*e^(b*x + a) - 1)) - 1), x) - 2*I*b^2*c^2*integrate(x*e^(2*b*x + 2*a)/(c^2*e^(2*b*x + 2*a) + (c^2*e^(2*b*x + 2*a) - 1)*e^(log(c*e^(b*x + a) + 1) + log(c*e^(b*x + a) - 1)) - 1), x) + I*b^2*x^2 - I*b*x*log(c^2*e^(2*b*x + 2*a)) + I*b*x*log(c*e^(b*x + a) + 1) + I*b*x*log(-c*e^(b*x + a) + 1) - 2*((-I*a - arctan2(1, sqrt(c*e^(b*x + a) + 1)*sqrt(c*e^(b*x + a) - 1)))*b - I*b*log(c))*x + I*dilog(c*e^(b*x + a)) + I*dilog(-c*e^(b*x + a)))/b`

**Giac [F]**

$$\int \csc^{-1}(ce^{a+bx}) dx = \int \operatorname{arccsc}(ce^{(bx+a)}) dx$$

input `integrate(arccsc(c*exp(b*x+a)),x, algorithm="giac")`

output `integrate(arccsc(c*e^(b*x + a)), x)`

**Mupad [B] (verification not implemented)**

Time = 1.54 (sec) , antiderivative size = 91, normalized size of antiderivative = 1.07

$$\int \csc^{-1}(ce^{a+bx}) dx = \frac{\operatorname{polylog}\left(2, e^{\operatorname{asin}\left(\frac{e^{-a-bx}}{c}\right)2i}\right) \operatorname{li}}{2b} + \frac{\operatorname{asin}\left(\frac{e^{-a-bx}}{c}\right)^2 \operatorname{li}}{2b} - \frac{\ln\left(1 - e^{\operatorname{asin}\left(\frac{e^{-a-bx}}{c}\right)2i}\right) \operatorname{asin}\left(\frac{e^{-a-bx}}{c}\right)}{b}$$

input `int(asin(exp(- a - b*x)/c),x)`

output `(polylog(2, exp(asin(exp(- a - b*x)/c)*2i))*1i)/(2*b) + (asin(exp(- a - b*x)/c)^2*1i)/(2*b) - (log(1 - exp(asin(exp(- a - b*x)/c)*2i))*asin(exp(- a - b*x)/c))/b`

**Reduce [F]**

$$\int \csc^{-1}(ce^{a+bx}) dx = \int \operatorname{acsc}(e^{bx+a}c) dx$$

input `int(acsc(c*exp(b*x+a)),x)`

output `int(acsc(e**(a + b*x)*c),x)`

### 3.41 $\int e^{\csc^{-1}(ax)} x^2 dx$

Optimal result	340
Mathematica [A] (verified)	340
Rubi [A] (verified)	341
Maple [F]	342
Fricas [F]	343
Sympy [F]	343
Maxima [F]	343
Giac [F]	344
Mupad [F(-1)]	344
Reduce [F]	344

#### Optimal result

Integrand size = 10, antiderivative size = 95

$$\int e^{\csc^{-1}(ax)} x^2 dx = \frac{\left(\frac{4}{5} - \frac{12i}{5}\right) e^{(1+3i)\csc^{-1}(ax)} \operatorname{Hypergeometric2F1}\left(\frac{3}{2} - \frac{i}{2}, 3, \frac{5}{2} - \frac{i}{2}, e^{2i\csc^{-1}(ax)}\right)}{a^3} - \frac{\left(\frac{8}{5} - \frac{24i}{5}\right) e^{(1+3i)\csc^{-1}(ax)} \operatorname{Hypergeometric2F1}\left(\frac{3}{2} - \frac{i}{2}, 4, \frac{5}{2} - \frac{i}{2}, e^{2i\csc^{-1}(ax)}\right)}{a^3}$$

output

```
(4/5-12/5*I)*exp((1+3*I)*arccsc(a*x))*hypergeom([3, 3/2-1/2*I], [5/2-1/2*I], (I/a/x+(1-1/a^2/x^2)^(1/2))^2)/a^3+(-8/5+24/5*I)*exp((1+3*I)*arccsc(a*x))*hypergeom([4, 3/2-1/2*I], [5/2-1/2*I], (I/a/x+(1-1/a^2/x^2)^(1/2))^2)/a^3
```

#### Mathematica [A] (verified)

Time = 0.32 (sec) , antiderivative size = 79, normalized size of antiderivative = 0.83

$$\int e^{\csc^{-1}(ax)} x^2 dx = \frac{e^{\csc^{-1}(ax)} \left( (4 + 4i) e^{i\csc^{-1}(ax)} \operatorname{Hypergeometric2F1}\left(\frac{1}{2} - \frac{i}{2}, 1, \frac{3}{2} - \frac{i}{2}, e^{2i\csc^{-1}(ax)}\right) + a^3 x^3 (5 - \cos(2\csc^{-1}(ax))) \right)}{12a^3}$$

input `Integrate[E^ArcCsc[a*x]*x^2,x]`

output `(E^ArcCsc[a*x]*((4 + 4*I)*E^(I*ArcCsc[a*x])*Hypergeometric2F1[1/2 - I/2, 1, 3/2 - I/2, E^((2*I)*ArcCsc[a*x])]) + a^3*x^3*(5 - Cos[2*ArcCsc[a*x]] + Sin[2*ArcCsc[a*x]]))/((12*a^3)`

### Rubi [A] (verified)

Time = 0.35 (sec) , antiderivative size = 94, normalized size of antiderivative = 0.99, number of steps used = 5, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.400$ , Rules used = {5790, 27, 4974, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int x^2 e^{\csc^{-1}(ax)} dx \\
 & \quad \downarrow \text{5790} \\
 & \frac{\int a^2 e^{\csc^{-1}(ax)} \sqrt{1 - \frac{1}{a^2 x^2}} x^4 d \csc^{-1}(ax)}{a} \\
 & \quad \downarrow \text{27} \\
 & \frac{\int a^4 e^{\csc^{-1}(ax)} \sqrt{1 - \frac{1}{a^2 x^2}} x^4 d \csc^{-1}(ax)}{a^3} \\
 & \quad \downarrow \text{4974} \\
 & \frac{\int \left( \frac{8e^{(1+3i) \csc^{-1}(ax)}}{(-1+e^{2i \csc^{-1}(ax)})^3} + \frac{16e^{(1+3i) \csc^{-1}(ax)}}{(-1+e^{2i \csc^{-1}(ax)})^4} \right) d \csc^{-1}(ax)}{a^3} \\
 & \quad \downarrow \text{2009} \\
 & \frac{\left( \frac{8}{5} - \frac{24i}{5} \right) e^{(1+3i) \csc^{-1}(ax)} \text{Hypergeometric2F1} \left( \frac{3}{2} - \frac{i}{2}, 4, \frac{5}{2} - \frac{i}{2}, e^{2i \csc^{-1}(ax)} \right) - \left( \frac{4}{5} - \frac{12i}{5} \right) e^{(1+3i) \csc^{-1}(ax)} \text{Hypergeometric2F1} \left( \frac{3}{2} - \frac{i}{2}, 4, \frac{5}{2} - \frac{i}{2}, e^{2i \csc^{-1}(ax)} \right)}{a^3}
 \end{aligned}$$

input `Int[E^ArcCsc[a*x]*x^2,x]`

output `-((( -4/5 + (12*I)/5)*E^((1 + 3*I)*ArcCsc[a*x])*Hypergeometric2F1[3/2 - I/2, 3, 5/2 - I/2, E^((2*I)*ArcCsc[a*x])] + (8/5 - (24*I)/5)*E^((1 + 3*I)*ArcCsc[a*x])*Hypergeometric2F1[3/2 - I/2, 4, 5/2 - I/2, E^((2*I)*ArcCsc[a*x])])/a^3)`

### Defintions of rubi rules used

rule 27 `Int[(a_)*(F_x_), x_Symbol] :> Simp[a Int[F_x, x], x] /; FreeQ[a, x] && !MatchQ[F_x, (b_)*(G_x_)] /; FreeQ[b, x]`

rule 2009 `Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]`

rule 4974 `Int[(F_)^((c_)*((a_) + (b_)*(x_)))*(G_)[(d_) + (e_)*(x_)]^(m_)*(H_)[(d_) + (e_)*(x_)]^(n_), x_Symbol] :> Int[ExpandTrigToExp[F^(c*(a + b*x)), G[d + e*x]^m*H[d + e*x]^n, x], x] /; FreeQ[{F, a, b, c, d, e}, x] && IGtQ[m, 0] && IGtQ[n, 0] && TrigQ[G] && TrigQ[H]`

rule 5790 `Int[(u_)*(f_)^(ArcCsc[(a_) + (b_)*(x_)]^(n_)*(c_)), x_Symbol] :> Simp[-b^(-1) Subst[Int[(u /. x -> -a/b + Csc[x]/b)*f^(c*x^n)*Csc[x]*Cot[x], x], x, ArcCsc[a + b*x]], x] /; FreeQ[{a, b, c, f}, x] && IGtQ[n, 0]`

### Maple [F]

$$\int e^{\operatorname{arccsc}(ax)} x^2 dx$$

input `int(exp(arccsc(a*x))*x^2,x)`

output `int(exp(arccsc(a*x))*x^2,x)`

**Fricas [F]**

$$\int e^{\csc^{-1}(ax)} x^2 dx = \int x^2 e^{\operatorname{arccsc}(ax)} dx$$

input `integrate(exp(arccsc(a*x))*x^2,x, algorithm="fricas")`

output `integral(x^2*e^(arccsc(a*x)), x)`

**Sympy [F]**

$$\int e^{\csc^{-1}(ax)} x^2 dx = \int x^2 e^{\operatorname{acsc}(ax)} dx$$

input `integrate(exp(acsc(a*x))*x**2,x)`

output `Integral(x**2*exp(acsc(a*x)), x)`

**Maxima [F]**

$$\int e^{\csc^{-1}(ax)} x^2 dx = \int x^2 e^{\operatorname{arccsc}(ax)} dx$$

input `integrate(exp(arccsc(a*x))*x^2,x, algorithm="maxima")`

output `integrate(x^2*e^(arccsc(a*x)), x)`



**Giac [F]**

$$\int e^{\csc^{-1}(ax)} x^2 dx = \int x^2 e^{\operatorname{arccsc}(ax)} dx$$

input `integrate(exp(arccsc(a*x))*x^2,x, algorithm="giac")`

output `integrate(x^2*e^(arccsc(a*x)), x)`

**Mupad [F(-1)]**

Timed out.

$$\int e^{\csc^{-1}(ax)} x^2 dx = \int x^2 e^{\operatorname{asin}(\frac{1}{ax})} dx$$

input `int(x^2*exp(asin(1/(a*x))),x)`

output `int(x^2*exp(asin(1/(a*x))), x)`

**Reduce [F]**

$$\int e^{\csc^{-1}(ax)} x^2 dx = \int e^{\operatorname{acsc}(ax)} x^2 dx$$

input `int(exp(acsc(a*x))*x^2,x)`

output `int(e**acsc(a*x)*x**2,x)`

### 3.42 $\int e^{\csc^{-1}(ax)} x dx$

Optimal result	345
Mathematica [A] (verified)	345
Rubi [A] (verified)	346
Maple [F]	347
Fricas [F]	348
Sympy [F]	348
Maxima [F]	348
Giac [F]	349
Mupad [F(-1)]	349
Reduce [F]	349

#### Optimal result

Integrand size = 8, antiderivative size = 87

$$\int e^{\csc^{-1}(ax)} x dx = \frac{\left(\frac{8}{5} + \frac{4i}{5}\right) e^{(1+2i)\csc^{-1}(ax)} \operatorname{Hypergeometric2F1}\left(1 - \frac{i}{2}, 2, 2 - \frac{i}{2}, e^{2i\csc^{-1}(ax)}\right)}{a^2} - \frac{\left(\frac{16}{5} + \frac{8i}{5}\right) e^{(1+2i)\csc^{-1}(ax)} \operatorname{Hypergeometric2F1}\left(1 - \frac{i}{2}, 3, 2 - \frac{i}{2}, e^{2i\csc^{-1}(ax)}\right)}{a^2}$$

output

```
(8/5+4/5*I)*exp((1+2*I)*arccsc(a*x))*hypergeom([2, 1-1/2*I],[2-1/2*I],(I/a/x+(1-1/a^2/x^2)^(1/2))^2)/a^2-(16/5+8/5*I)*exp((1+2*I)*arccsc(a*x))*hypergeom([3, 1-1/2*I],[2-1/2*I],(I/a/x+(1-1/a^2/x^2)^(1/2))^2)/a^2
```

#### Mathematica [A] (verified)

Time = 0.18 (sec) , antiderivative size = 101, normalized size of antiderivative = 1.16

$$\int e^{\csc^{-1}(ax)} x dx = \frac{\left(\frac{1}{5} + \frac{i}{10}\right) e^{\csc^{-1}(ax)} \left( (2-i)ax \left( \sqrt{1 - \frac{1}{a^2x^2}} + ax \right) + (1+2i) \operatorname{Hypergeometric2F1}\left(-\frac{i}{2}, 1, 1 - \frac{i}{2}, e^{2i\csc^{-1}(ax)}\right) \right)}{a^2}$$

input `Integrate[E^ArcCsc[a*x]*x,x]`

output  $((1/5 + I/10)*E^{\text{ArcCsc}[a*x]}*((2 - I)*a*x*\sqrt{1 - 1/(a^2*x^2)} + a*x) + (1 + 2*I)*\text{Hypergeometric2F1}[-1/2*I, 1, 1 - I/2, E^{((2*I)*\text{ArcCsc}[a*x])}] + E^{((2*I)*\text{ArcCsc}[a*x])}*\text{Hypergeometric2F1}[1, 1 - I/2, 2 - I/2, E^{((2*I)*\text{ArcCsc}[a*x])}]))/a^2$

### Rubi [A] (verified)

Time = 0.36 (sec) , antiderivative size = 86, normalized size of antiderivative = 0.99, number of steps used = 5, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {5790, 27, 4974, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int x e^{\csc^{-1}(ax)} dx \\
 & \quad \downarrow \text{5790} \\
 & - \frac{\int a^2 e^{\csc^{-1}(ax)} \sqrt{1 - \frac{1}{a^2 x^2}} x^3 d \csc^{-1}(ax)}{a} \\
 & \quad \downarrow \text{27} \\
 & - \frac{\int a^3 e^{\csc^{-1}(ax)} \sqrt{1 - \frac{1}{a^2 x^2}} x^3 d \csc^{-1}(ax)}{a^2} \\
 & \quad \downarrow \text{4974} \\
 & - \frac{\int \left( -\frac{4ie^{(1+2i)\csc^{-1}(ax)}}{(-1+e^{2i\csc^{-1}(ax)})^2} - \frac{8ie^{(1+2i)\csc^{-1}(ax)}}{(-1+e^{2i\csc^{-1}(ax)})^3} \right) d \csc^{-1}(ax)}{a^2} \\
 & \quad \downarrow \text{2009} \\
 & - \frac{\left( \frac{16}{5} + \frac{8i}{5} \right) e^{(1+2i)\csc^{-1}(ax)} \text{Hypergeometric2F1} \left( 1 - \frac{i}{2}, 3, 2 - \frac{i}{2}, e^{2i\csc^{-1}(ax)} \right) - \left( \frac{8}{5} + \frac{4i}{5} \right) e^{(1+2i)\csc^{-1}(ax)} \text{Hypergeometric2F1} \left( 1 - \frac{i}{2}, 3, 2 - \frac{i}{2}, e^{2i\csc^{-1}(ax)} \right)}{a^2}
 \end{aligned}$$

input `Int [E^ArcCsc[a*x]*x,x]`

output `-((( -8/5 - (4*I)/5)*E^((1 + 2*I)*ArcCsc[a*x])*Hypergeometric2F1[1 - I/2, 2, 2 - I/2, E^((2*I)*ArcCsc[a*x])]) + (16/5 + (8*I)/5)*E^((1 + 2*I)*ArcCsc[a*x])*Hypergeometric2F1[1 - I/2, 3, 2 - I/2, E^((2*I)*ArcCsc[a*x])])/a^2`

### Defintions of rubi rules used

rule 27 `Int[(a_)*(F_x_), x_Symbol] := Simp[a Int[F_x, x], x] /; FreeQ[a, x] && !MatchQ[F_x, (b_)*(G_x_)] /; FreeQ[b, x]`

rule 2009 `Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]`

rule 4974 `Int[(F_)^((c_)*((a_) + (b_)*(x_)))*(G_)[(d_) + (e_)*(x_)]^(m_)*(H_)[(d_) + (e_)*(x_)]^(n_), x_Symbol] := Int[ExpandTrigToExp[F^(c*(a + b*x)), G[d + e*x]^m*H[d + e*x]^n, x], x] /; FreeQ[{F, a, b, c, d, e}, x] && IGtQ[m, 0] && IGtQ[n, 0] && TrigQ[G] && TrigQ[H]`

rule 5790 `Int[(u_)*(f_)^(ArcCsc[(a_) + (b_)*(x_)]^(n_)*(c_)), x_Symbol] := Simp[-b^(-1) Subst[Int[(u /. x -> -a/b + Csc[x]/b)*f^(c*x^n)*Csc[x]*Cot[x], x], x, ArcCsc[a + b*x]], x] /; FreeQ[{a, b, c, f}, x] && IGtQ[n, 0]`

### Maple [F]

$$\int e^{\operatorname{arccsc}(ax)} x dx$$

input `int(exp(arccsc(a*x))*x,x)`

output `int(exp(arccsc(a*x))*x,x)`

**Fricas [F]**

$$\int e^{\csc^{-1}(ax)} x dx = \int x e^{\operatorname{arccsc}(ax)} dx$$

input `integrate(exp(arccsc(a*x))*x,x, algorithm="fricas")`

output `integral(x*e^(arccsc(a*x)), x)`

**Sympy [F]**

$$\int e^{\csc^{-1}(ax)} x dx = \int x e^{\operatorname{acsc}(ax)} dx$$

input `integrate(exp(acsc(a*x))*x,x)`

output `Integral(x*exp(acsc(a*x)), x)`

**Maxima [F]**

$$\int e^{\csc^{-1}(ax)} x dx = \int x e^{\operatorname{arccsc}(ax)} dx$$

input `integrate(exp(arccsc(a*x))*x,x, algorithm="maxima")`

output `integrate(x*e^(arccsc(a*x)), x)`

**Giac [F]**

$$\int e^{\csc^{-1}(ax)} x dx = \int x e^{\operatorname{arccsc}(ax)} dx$$

input `integrate(exp(arccsc(a*x))*x,x, algorithm="giac")`

output `integrate(x*e^(arccsc(a*x)), x)`

**Mupad [F(-1)]**

Timed out.

$$\int e^{\csc^{-1}(ax)} x dx = \int x e^{\operatorname{asin}(\frac{1}{ax})} dx$$

input `int(x*exp(asin(1/(a*x))),x)`

output `int(x*exp(asin(1/(a*x))), x)`

**Reduce [F]**

$$\int e^{\csc^{-1}(ax)} x dx = \int e^{\operatorname{acsc}(ax)} x dx$$

input `int(exp(acsc(a*x))*x,x)`

output `int(e**acsc(a*x)*x,x)`

### 3.43 $\int e^{\csc^{-1}(ax)} dx$

Optimal result	350
Mathematica [A] (verified)	350
Rubi [A] (verified)	351
Maple [F]	352
Fricas [F]	352
Sympy [F]	353
Maxima [F]	353
Giac [F]	353
Mupad [F(-1)]	354
Reduce [F]	354

#### Optimal result

Integrand size = 6, antiderivative size = 87

$$\int e^{\csc^{-1}(ax)} dx = -\frac{(1-i)e^{(1+i)\csc^{-1}(ax)} \operatorname{Hypergeometric2F1}\left(\frac{1}{2}-\frac{i}{2}, 1, \frac{3}{2}-\frac{i}{2}, e^{2i\csc^{-1}(ax)}\right)}{a} + \frac{(2-2i)e^{(1+i)\csc^{-1}(ax)} \operatorname{Hypergeometric2F1}\left(\frac{1}{2}-\frac{i}{2}, 2, \frac{3}{2}-\frac{i}{2}, e^{2i\csc^{-1}(ax)}\right)}{a}$$

output

```
(-1+I)*exp((1+I)*arccsc(a*x))*hypergeom([1, 1/2-1/2*I], [3/2-1/2*I], (I/a/x+(1-1/a^2/x^2)^(1/2))^2)/a+(2-2*I)*exp((1+I)*arccsc(a*x))*hypergeom([2, 1/2-1/2*I], [3/2-1/2*I], (I/a/x+(1-1/a^2/x^2)^(1/2))^2)/a
```

#### Mathematica [A] (verified)

Time = 0.07 (sec) , antiderivative size = 54, normalized size of antiderivative = 0.62

$$\int e^{\csc^{-1}(ax)} dx = \frac{e^{\csc^{-1}(ax)} \left( ax + (1+i)e^{i\csc^{-1}(ax)} \operatorname{Hypergeometric2F1}\left(\frac{1}{2}-\frac{i}{2}, 1, \frac{3}{2}-\frac{i}{2}, e^{2i\csc^{-1}(ax)}\right) \right)}{a}$$

input `Integrate[E^ArcCsc[a*x], x]`

output `(E^ArcCsc[a*x]*(a*x + (1 + I)*E^(I*ArcCsc[a*x]))*Hypergeometric2F1[1/2 - I/2, 1, 3/2 - I/2, E^((2*I)*ArcCsc[a*x])])/a`

### Rubi [A] (verified)

Time = 0.32 (sec) , antiderivative size = 86, normalized size of antiderivative = 0.99, number of steps used = 4, number of rules used = 3,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {5790, 4974, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int e^{\csc^{-1}(ax)} dx \\
 & \quad \downarrow \text{5790} \\
 & - \frac{\int a^2 e^{\csc^{-1}(ax)} \sqrt{1 - \frac{1}{a^2 x^2}} x^2 d \csc^{-1}(ax)}{a} \\
 & \quad \downarrow \text{4974} \\
 & - \frac{\int \left( \frac{2e^{(1+i) \csc^{-1}(ax)}}{1 - e^{2i \csc^{-1}(ax)}} - \frac{4e^{(1+i) \csc^{-1}(ax)}}{(-1 + e^{2i \csc^{-1}(ax)})^2} \right) d \csc^{-1}(ax)}{a} \\
 & \quad \downarrow \text{2009} \\
 & - \frac{(1 - i)e^{(1+i) \csc^{-1}(ax)} \text{Hypergeometric2F1} \left( \frac{1}{2} - \frac{i}{2}, 1, \frac{3}{2} - \frac{i}{2}, e^{2i \csc^{-1}(ax)} \right) - (2 - 2i)e^{(1+i) \csc^{-1}(ax)} \text{Hypergeometric2F1} \left( \frac{1}{2} - \frac{i}{2}, 2, \frac{3}{2} - \frac{i}{2}, e^{2i \csc^{-1}(ax)} \right)}{a}
 \end{aligned}$$

input `Int[E^ArcCsc[a*x], x]`

output `-(((1 - I)*E^((1 + I)*ArcCsc[a*x]))*Hypergeometric2F1[1/2 - I/2, 1, 3/2 - I/2, E^((2*I)*ArcCsc[a*x])]) - (2 - 2*I)*E^((1 + I)*ArcCsc[a*x])*Hypergeometric2F1[1/2 - I/2, 2, 3/2 - I/2, E^((2*I)*ArcCsc[a*x])])/a`



**Defintions of rubi rules used**

rule 2009 `Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]`

rule 4974 `Int[(F_)^((c_)*((a_) + (b_)*(x_)))*(G_)[(d_) + (e_)*(x_)]^(m_)*(H_)[(d_) + (e_)*(x_)]^(n_), x_Symbol] := Int[ExpandTrigToExp[F^(c*(a + b*x)), G[d + e*x]^m*H[d + e*x]^n, x], x] /; FreeQ[{F, a, b, c, d, e}, x] && IGtQ[m, 0] && IGtQ[n, 0] && TrigQ[G] && TrigQ[H]`

rule 5790 `Int[(u_)*(f_)^(ArcCsc[(a_) + (b_)*(x_)]^(n_)*(c_.)), x_Symbol] := Simp[-b^(-1) Subst[Int[(u /. x -> -a/b + Csc[x]/b)*f^(c*x^n)*Csc[x]*Cot[x], x], x, ArcCsc[a + b*x]], x] /; FreeQ[{a, b, c, f}, x] && IGtQ[n, 0]`

**Maple [F]**

$$\int e^{\operatorname{arccsc}(ax)} dx$$

input `int(exp(arccsc(a*x)),x)`

output `int(exp(arccsc(a*x)),x)`

**Fricas [F]**

$$\int e^{\operatorname{csc}^{-1}(ax)} dx = \int e^{(\operatorname{arccsc}(ax))} dx$$

input `integrate(exp(arccsc(a*x)),x, algorithm="fricas")`

output `integral(e^(arccsc(a*x)), x)`

**Sympy [F]**

$$\int e^{\csc^{-1}(ax)} dx = \int e^{\operatorname{acsc}(ax)} dx$$

input `integrate(exp(acsc(a*x)),x)`

output `Integral(exp(acsc(a*x)), x)`

**Maxima [F]**

$$\int e^{\csc^{-1}(ax)} dx = \int e^{(\operatorname{arccsc}(ax))} dx$$

input `integrate(exp(arccsc(a*x)),x, algorithm="maxima")`

output `integrate(e^(arccsc(a*x)), x)`

**Giac [F]**

$$\int e^{\csc^{-1}(ax)} dx = \int e^{(\operatorname{arccsc}(ax))} dx$$

input `integrate(exp(arccsc(a*x)),x, algorithm="giac")`

output `integrate(e^(arccsc(a*x)), x)`

**Mupad [F(-1)]**

Timed out.

$$\int e^{\csc^{-1}(ax)} dx = \int e^{\operatorname{asin}(\frac{1}{ax})} dx$$

input `int(exp(asin(1/(a*x))),x)`output `int(exp(asin(1/(a*x))), x)`**Reduce [F]**

$$\int e^{\csc^{-1}(ax)} dx = \int e^{\operatorname{acsc}(ax)} dx$$

input `int(exp(acsc(a*x)),x)`output `int(e**acsc(a*x),x)`

### 3.44 $\int \frac{e^{\csc^{-1}(ax)}}{x} dx$

Optimal result	355
Mathematica [A] (verified)	355
Rubi [A] (verified)	356
Maple [F]	357
Fricas [F]	357
Sympy [F]	358
Maxima [F]	358
Giac [F]	358
Mupad [F(-1)]	359
Reduce [F]	359

#### Optimal result

Integrand size = 10, antiderivative size = 43

$$\int \frac{e^{\csc^{-1}(ax)}}{x} dx = -ie^{\csc^{-1}(ax)} + 2ie^{\csc^{-1}(ax)} \operatorname{Hypergeometric2F1} \left( -\frac{i}{2}, 1, 1 - \frac{i}{2}, e^{2i \csc^{-1}(ax)} \right)$$

output

```
-I*exp(arccsc(a*x))+2*I*exp(arccsc(a*x))*hypergeom([1, -1/2*I],[1-1/2*I],(I/a/x+(1-1/a^2/x^2)^(1/2))^2)
```

#### Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 75, normalized size of antiderivative = 1.74

$$\int \frac{e^{\csc^{-1}(ax)}}{x} dx = -i \left( -e^{\csc^{-1}(ax)} \operatorname{Hypergeometric2F1} \left( -\frac{i}{2}, 1, 1 - \frac{i}{2}, e^{2i \csc^{-1}(ax)} \right) - \left( \frac{1}{5} - \frac{2i}{5} \right) e^{(1+2i) \csc^{-1}(ax)} \operatorname{Hypergeometric2F1} \left( 1, 1 - \frac{i}{2}, 2 - \frac{i}{2}, e^{2i \csc^{-1}(ax)} \right) \right)$$

input

```
Integrate[E^ArcCsc[a*x]/x,x]
```

output

```
(-I)*(-(E^ArcCsc[a*x]*Hypergeometric2F1[-1/2*I, 1, 1 - I/2, E^((2*I)*ArcCs
c[a*x])]) - (1/5 - (2*I)/5)*E^((1 + 2*I)*ArcCsc[a*x])*Hypergeometric2F1[1,
1 - I/2, 2 - I/2, E^((2*I)*ArcCsc[a*x])])
```

**Rubi [A] (verified)**

Time = 0.28 (sec) , antiderivative size = 43, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.400$ , Rules used = {5790, 27, 4943, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{e^{\csc^{-1}(ax)}}{x} dx \\
 & \quad \downarrow \text{5790} \\
 & - \frac{\int a^2 e^{\csc^{-1}(ax)} \sqrt{1 - \frac{1}{a^2 x^2}} x d \csc^{-1}(ax)}{a} \\
 & \quad \downarrow \text{27} \\
 & - \int a e^{\csc^{-1}(ax)} \sqrt{1 - \frac{1}{a^2 x^2}} x d \csc^{-1}(ax) \\
 & \quad \downarrow \text{4943} \\
 & i \int \left( \frac{2e^{\csc^{-1}(ax)}}{1 - e^{2i \csc^{-1}(ax)}} - e^{\csc^{-1}(ax)} \right) d \csc^{-1}(ax) \\
 & \quad \downarrow \text{2009} \\
 & i \left( -e^{\csc^{-1}(ax)} + 2e^{\csc^{-1}(ax)} \text{Hypergeometric2F1} \left( -\frac{i}{2}, 1, 1 - \frac{i}{2}, e^{2i \csc^{-1}(ax)} \right) \right)
 \end{aligned}$$

input

```
Int [E^ArcCsc[a*x]/x, x]
```

output

```
I*(-E^ArcCsc[a*x] + 2*E^ArcCsc[a*x]*Hypergeometric2F1[-1/2*I, 1, 1 - I/2,
E^((2*I)*ArcCsc[a*x])])
```

## Definitions of rubi rules used

rule 27 `Int[(a_)*(F_x_), x_Symbol] := Simp[a Int[F_x, x], x] /; FreeQ[a, x] && !MatchQ[F_x, (b_)*(G_x_)] /; FreeQ[b, x]`

rule 2009 `Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]`

rule 4943 `Int[Cot[(d_.) + (e_.)*(x_)]^(n_.)*(F_)^((c_.)*((a_.) + (b_.)*(x_))), x_Symbol] := Simp[(-I)^n Int[ExpandIntegrand[F^(c*(a + b*x))*((1 + E^(2*I*(d + e*x)))^n/(1 - E^(2*I*(d + e*x)))^n), x], x] /; FreeQ[{F, a, b, c, d, e}, x] && IntegerQ[n]`

rule 5790 `Int[(u_.)*(f_)^(ArcCsc[(a_.) + (b_.)*(x_)]^(n_.)*(c_)), x_Symbol] := Simp[-b^(-1) Subst[Int[(u /. x -> -a/b + Csc[x]/b)*f^(c*x^n)*Csc[x]*Cot[x], x], x, ArcCsc[a + b*x]], x] /; FreeQ[{a, b, c, f}, x] && IGtQ[n, 0]`

## Maple [F]

$$\int \frac{e^{\operatorname{arccsc}(ax)}}{x} dx$$

input `int(exp(arccsc(a*x))/x,x)`

output `int(exp(arccsc(a*x))/x,x)`

## Fricas [F]

$$\int \frac{e^{\operatorname{csc}^{-1}(ax)}}{x} dx = \int \frac{e^{\operatorname{arccsc}(ax)}}{x} dx$$

input `integrate(exp(arccsc(a*x))/x,x, algorithm="fricas")`

output `integral(e^(arccsc(a*x))/x, x)`

### Sympy [F]

$$\int \frac{e^{\csc^{-1}(ax)}}{x} dx = \int \frac{e^{\operatorname{acsc}(ax)}}{x} dx$$

input `integrate(exp(acsc(a*x))/x,x)`

output `Integral(exp(acsc(a*x))/x, x)`

### Maxima [F]

$$\int \frac{e^{\csc^{-1}(ax)}}{x} dx = \int \frac{e^{\operatorname{arccsc}(ax)}}{x} dx$$

input `integrate(exp(arccsc(a*x))/x,x, algorithm="maxima")`

output `integrate(e^(arccsc(a*x))/x, x)`

### Giac [F]

$$\int \frac{e^{\csc^{-1}(ax)}}{x} dx = \int \frac{e^{\operatorname{arccsc}(ax)}}{x} dx$$

input `integrate(exp(arccsc(a*x))/x,x, algorithm="giac")`

output `integrate(e^(arccsc(a*x))/x, x)`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{e^{\csc^{-1}(ax)}}{x} dx = \int \frac{e^{\operatorname{asin}(\frac{1}{ax})}}{x} dx$$

input `int(exp(asin(1/(a*x)))/x,x)`output `int(exp(asin(1/(a*x)))/x, x)`**Reduce [F]**

$$\int \frac{e^{\csc^{-1}(ax)}}{x} dx = \int \frac{e^{\operatorname{acsc}(ax)}}{x} dx$$

input `int(exp(acsc(a*x))/x,x)`output `int(e**acsc(a*x)/x,x)`



### 3.45 $\int \frac{e^{\csc^{-1}(ax)}}{x^2} dx$

Optimal result	360
Mathematica [A] (verified)	360
Rubi [A] (verified)	361
Maple [F]	362
Fricas [A] (verification not implemented)	362
Sympy [F]	363
Maxima [F]	363
Giac [A] (verification not implemented)	363
Mupad [F(-1)]	364
Reduce [F]	364

#### Optimal result

Integrand size = 10, antiderivative size = 39

$$\int \frac{e^{\csc^{-1}(ax)}}{x^2} dx = -\frac{1}{2}ae^{\csc^{-1}(ax)}\sqrt{1 - \frac{1}{a^2x^2}} - \frac{e^{\csc^{-1}(ax)}}{2x}$$

output `-1/2*a*exp(arccsc(a*x))*(1-1/a^2/x^2)^(1/2)-1/2*exp(arccsc(a*x))/x`

#### Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 33, normalized size of antiderivative = 0.85

$$\int \frac{e^{\csc^{-1}(ax)}}{x^2} dx = -\frac{1}{2}ae^{\csc^{-1}(ax)}\left(\sqrt{1 - \frac{1}{a^2x^2}} + \frac{1}{ax}\right)$$

input `Integrate[E^ArcCsc[a*x]/x^2,x]`

output `-1/2*(a*E^ArcCsc[a*x]*(Sqrt[1 - 1/(a^2*x^2)] + 1/(a*x)))`

**Rubi [A] (verified)**

Time = 0.26 (sec) , antiderivative size = 44, normalized size of antiderivative = 1.13, number of steps used = 4, number of rules used = 3,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.300$ , Rules used = {5790, 27, 4933}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{e^{\csc^{-1}(ax)}}{x^2} dx$$

$$\downarrow \text{5790}$$

$$-\frac{\int a^2 e^{\csc^{-1}(ax)} \sqrt{1 - \frac{1}{a^2 x^2}} d \csc^{-1}(ax)}{a}$$

$$\downarrow \text{27}$$

$$-a \int e^{\csc^{-1}(ax)} \sqrt{1 - \frac{1}{a^2 x^2}} d \csc^{-1}(ax)$$

$$\downarrow \text{4933}$$

$$-a \left( \frac{1}{2} \sqrt{1 - \frac{1}{a^2 x^2}} e^{\csc^{-1}(ax)} + \frac{e^{\csc^{-1}(ax)}}{2ax} \right)$$

input `Int [E^ArcCsc [a*x] /x^2, x]`

output `-(a*((E^ArcCsc [a*x] *Sqrt [1 - 1/(a^2*x^2)])/2 + E^ArcCsc [a*x] / (2*a*x)))`

**Defintions of rubi rules used**

rule 27 `Int [(a_)*(Fx_), x_Symbol] :> Simp[a Int[Fx, x], x] /; FreeQ[a, x] && !MatchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]`

rule 4933

```
Int[Cos[(d_.) + (e_.)*(x_)]*(F_)^((c_.)*((a_.) + (b_.)*(x_))), x_Symbol] :=
Simp[b*c*Log[F]*F^(c*(a + b*x))*(Cos[d + e*x]/(e^2 + b^2*c^2*Log[F]^2)), x
] + Simp[e*F^(c*(a + b*x))*(Sin[d + e*x]/(e^2 + b^2*c^2*Log[F]^2)), x] /; F
reeQ[{F, a, b, c, d, e}, x] && NeQ[e^2 + b^2*c^2*Log[F]^2, 0]
```

rule 5790

```
Int[(u_.)*(f_)^(ArcCsc[(a_.) + (b_.)*(x_)]^(n_.)*(c_.)), x_Symbol] := Simp[
-b^(-1) Subst[Int[(u /. x -> -a/b + Csc[x]/b)*f^(c*x^n)*Csc[x]*Cot[x], x]
, x, ArcCsc[a + b*x]], x] /; FreeQ[{a, b, c, f}, x] && IGtQ[n, 0]
```

**Maple [F]**

$$\int \frac{e^{\operatorname{arccsc}(ax)}}{x^2} dx$$

input

```
int(exp(arccsc(a*x))/x^2,x)
```

output

```
int(exp(arccsc(a*x))/x^2,x)
```

**Fricas [A] (verification not implemented)**

Time = 0.14 (sec) , antiderivative size = 23, normalized size of antiderivative = 0.59

$$\int \frac{e^{\operatorname{csc}^{-1}(ax)}}{x^2} dx = -\frac{(\sqrt{a^2x^2 - 1} + 1)e^{\operatorname{arccsc}(ax)}}{2x}$$

input

```
integrate(exp(arccsc(a*x))/x^2,x, algorithm="fricas")
```

output

```
-1/2*(sqrt(a^2*x^2 - 1) + 1)*e^(arccsc(a*x))/x
```

**Sympy [F]**

$$\int \frac{e^{\csc^{-1}(ax)}}{x^2} dx = \int \frac{e^{\operatorname{acsc}(ax)}}{x^2} dx$$

input `integrate(exp(acsc(a*x))/x**2,x)`

output `Integral(exp(acsc(a*x))/x**2, x)`

**Maxima [F]**

$$\int \frac{e^{\csc^{-1}(ax)}}{x^2} dx = \int \frac{e^{\operatorname{arccsc}(ax)}}{x^2} dx$$

input `integrate(exp(arccsc(a*x))/x^2,x, algorithm="maxima")`

output `integrate(e^(arccsc(a*x))/x^2, x)`

**Giac [A] (verification not implemented)**

Time = 0.13 (sec) , antiderivative size = 42, normalized size of antiderivative = 1.08

$$\int \frac{e^{\csc^{-1}(ax)}}{x^2} dx = -\frac{1}{2} \left( \sqrt{-\frac{1}{a^2 x^2} + 1} e^{\operatorname{arcsin}(\frac{1}{ax})} + \frac{e^{\operatorname{arcsin}(\frac{1}{ax})}}{ax} \right) a$$

input `integrate(exp(arccsc(a*x))/x^2,x, algorithm="giac")`

output `-1/2*(sqrt(-1/(a^2*x^2) + 1)*e^(arcsin(1/(a*x))) + e^(arcsin(1/(a*x)))/(a*x))*a`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{e^{\csc^{-1}(ax)}}{x^2} dx = \int \frac{e^{\operatorname{asin}(\frac{1}{ax})}}{x^2} dx$$

input `int(exp(asin(1/(a*x)))/x^2,x)`output `int(exp(asin(1/(a*x)))/x^2, x)`**Reduce [F]**

$$\int \frac{e^{\csc^{-1}(ax)}}{x^2} dx = \int \frac{e^{\operatorname{acsc}(ax)}}{x^2} dx$$

input `int(exp(acsc(a*x))/x^2,x)`output `int(e**acsc(a*x)/x**2,x)`

### 3.46 $\int \frac{e^{\csc^{-1}(ax)}}{x^3} dx$

Optimal result	365
Mathematica [A] (verified)	365
Rubi [A] (verified)	366
Maple [F]	367
Fricas [A] (verification not implemented)	368
Sympy [F]	368
Maxima [F]	368
Giac [F]	369
Mupad [F(-1)]	369
Reduce [F]	369

#### Optimal result

Integrand size = 10, antiderivative size = 41

$$\int \frac{e^{\csc^{-1}(ax)}}{x^3} dx = \frac{1}{5}a^2 e^{\csc^{-1}(ax)} \cos(2 \csc^{-1}(ax)) - \frac{1}{10}a^2 e^{\csc^{-1}(ax)} \sin(2 \csc^{-1}(ax))$$

output

$1/5*a^2*\exp(\operatorname{arccsc}(a*x))*\cos(2*\operatorname{arccsc}(a*x))-1/10*a^2*\exp(\operatorname{arccsc}(a*x))*\sin(2*\operatorname{arccsc}(a*x))$

#### Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 30, normalized size of antiderivative = 0.73

$$\int \frac{e^{\csc^{-1}(ax)}}{x^3} dx = -\frac{1}{10}a^2 e^{\csc^{-1}(ax)} (-2 \cos(2 \csc^{-1}(ax)) + \sin(2 \csc^{-1}(ax)))$$

input

`Integrate[E^ArcCsc[a*x]/x^3,x]`

output

$-1/10*(a^2*E^{\operatorname{ArcCsc}[a*x]}*(-2*\operatorname{Cos}[2*\operatorname{ArcCsc}[a*x]] + \operatorname{Sin}[2*\operatorname{ArcCsc}[a*x]]))$

**Rubi [A] (verified)**

Time = 0.30 (sec) , antiderivative size = 42, normalized size of antiderivative = 1.02, number of steps used = 6, number of rules used = 5,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$ , Rules used = {5790, 27, 4972, 27, 4932}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{e^{\csc^{-1}(ax)}}{x^3} dx \\
 & \quad \downarrow \text{5790} \\
 & - \int \frac{a^2 e^{\csc^{-1}(ax)} \sqrt{1 - \frac{1}{a^2 x^2}}}{x} d \csc^{-1}(ax) \\
 & \quad \downarrow \text{27} \\
 & -a^2 \int \frac{e^{\csc^{-1}(ax)} \sqrt{1 - \frac{1}{a^2 x^2}}}{ax} d \csc^{-1}(ax) \\
 & \quad \downarrow \text{4972} \\
 & -a^2 \int \frac{1}{2} e^{\csc^{-1}(ax)} \sin(2 \csc^{-1}(ax)) d \csc^{-1}(ax) \\
 & \quad \downarrow \text{27} \\
 & -\frac{1}{2} a^2 \int e^{\csc^{-1}(ax)} \sin(2 \csc^{-1}(ax)) d \csc^{-1}(ax) \\
 & \quad \downarrow \text{4932} \\
 & -\frac{1}{2} a^2 \left( \frac{1}{5} e^{\csc^{-1}(ax)} \sin(2 \csc^{-1}(ax)) - \frac{2}{5} e^{\csc^{-1}(ax)} \cos(2 \csc^{-1}(ax)) \right)
 \end{aligned}$$

input

```
Int [E^ArcCsc [a*x] /x^3, x]
```

output

```
-1/2*(a^2*((-2*E^ArcCsc [a*x] *Cos [2*ArcCsc [a*x]])/5 + (E^ArcCsc [a*x] *Sin [2*ArcCsc [a*x]])/5))
```

## Definitions of rubi rules used

rule 27 `Int[(a_)*(F x_), x_Symbol] := Simp[a Int[F x, x], x] /; FreeQ[a, x] && !MatchQ[F x, (b_)*(G x_)] /; FreeQ[b, x]`

rule 4932 `Int[(F_)^((c_)*((a_) + (b_)*(x_)))*Sin[(d_) + (e_)*(x_)], x_Symbol] := Simp[b*c*Log[F]*F^(c*(a + b*x))*(Sin[d + e*x]/(e^2 + b^2*c^2*Log[F]^2)), x] - Simp[e*F^(c*(a + b*x))*(Cos[d + e*x]/(e^2 + b^2*c^2*Log[F]^2)), x] /; FreeQ[{F, a, b, c, d, e}, x] && NeQ[e^2 + b^2*c^2*Log[F]^2, 0]`

rule 4972 `Int[Cos[(f_) + (g_)*(x_)]^(n_)*(F_)^((c_)*((a_) + (b_)*(x_)))*Sin[(d_) + (e_)*(x_)]^(m_), x_Symbol] := Int[ExpandTrigReduce[F^(c*(a + b*x)), Sin[d + e*x]^m*Cos[f + g*x]^n, x], x] /; FreeQ[{F, a, b, c, d, e, f, g}, x] && IGtQ[m, 0] && IGtQ[n, 0]`

rule 5790 `Int[(u_)*(f_)^(ArcCsc[(a_) + (b_)*(x_)]^(n_)*(c_)), x_Symbol] := Simp[-b^(-1) Subst[Int[(u /. x -> -a/b + Csc[x]/b)*f^(c*x^n)*Csc[x]*Cot[x], x], x, ArcCsc[a + b*x]], x] /; FreeQ[{a, b, c, f}, x] && IGtQ[n, 0]`

## Maple [F]

$$\int \frac{e^{\operatorname{arccsc}(ax)}}{x^3} dx$$

input `int(exp(arccsc(a*x))/x^3,x)`

output `int(exp(arccsc(a*x))/x^3,x)`



**Fricas [A] (verification not implemented)**

Time = 0.14 (sec) , antiderivative size = 32, normalized size of antiderivative = 0.78

$$\int \frac{e^{\operatorname{csc}^{-1}(ax)}}{x^3} dx = \frac{(a^2x^2 - \sqrt{a^2x^2 - 1} - 2)e^{\operatorname{arccsc}(ax)}}{5x^2}$$

input `integrate(exp(arccsc(a*x))/x^3,x, algorithm="fricas")`output `1/5*(a^2*x^2 - sqrt(a^2*x^2 - 1) - 2)*e^(arccsc(a*x))/x^2`**Sympy [F]**

$$\int \frac{e^{\operatorname{csc}^{-1}(ax)}}{x^3} dx = \int \frac{e^{\operatorname{acsc}(ax)}}{x^3} dx$$

input `integrate(exp(acsc(a*x))/x**3,x)`output `Integral(exp(acsc(a*x))/x**3, x)`**Maxima [F]**

$$\int \frac{e^{\operatorname{csc}^{-1}(ax)}}{x^3} dx = \int \frac{e^{\operatorname{arccsc}(ax)}}{x^3} dx$$

input `integrate(exp(arccsc(a*x))/x^3,x, algorithm="maxima")`output `integrate(e^(arccsc(a*x))/x^3, x)`

**Giac [F]**

$$\int \frac{e^{\csc^{-1}(ax)}}{x^3} dx = \int \frac{e^{\operatorname{arccsc}(ax)}}{x^3} dx$$

input `integrate(exp(arccsc(a*x))/x^3,x, algorithm="giac")`

output `integrate(e^(arccsc(a*x))/x^3, x)`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{e^{\csc^{-1}(ax)}}{x^3} dx = \int \frac{e^{\operatorname{asin}(\frac{1}{ax})}}{x^3} dx$$

input `int(exp(asin(1/(a*x)))/x^3,x)`

output `int(exp(asin(1/(a*x)))/x^3, x)`

**Reduce [F]**

$$\int \frac{e^{\csc^{-1}(ax)}}{x^3} dx = \int \frac{e^{\operatorname{acsc}(ax)}}{x^3} dx$$

input `int(exp(acsc(a*x))/x^3,x)`

output `int(e**acsc(a*x)/x**3,x)`

### 3.47 $\int \frac{e^{\csc^{-1}(ax)}}{x^4} dx$

Optimal result	370
Mathematica [A] (verified)	370
Rubi [A] (verified)	371
Maple [F]	372
Fricas [A] (verification not implemented)	373
Sympy [F]	373
Maxima [F]	373
Giac [F]	374
Mupad [F(-1)]	374
Reduce [F]	374

#### Optimal result

Integrand size = 10, antiderivative size = 84

$$\int \frac{e^{\csc^{-1}(ax)}}{x^4} dx = -\frac{1}{8}a^3 e^{\csc^{-1}(ax)} \sqrt{1 - \frac{1}{a^2 x^2}} - \frac{a^2 e^{\csc^{-1}(ax)}}{8x} + \frac{1}{40}a^3 e^{\csc^{-1}(ax)} \cos(3 \csc^{-1}(ax)) + \frac{3}{40}a^3 e^{\csc^{-1}(ax)} \sin(3 \csc^{-1}(ax))$$

output

```
-1/8*a^3*exp(arccsc(a*x))*(1-1/a^2/x^2)^(1/2)-1/8*a^2*exp(arccsc(a*x))/x+1/40*a^3*exp(arccsc(a*x))*cos(3*arccsc(a*x))+3/40*a^3*exp(arccsc(a*x))*sin(3*arccsc(a*x))
```

#### Mathematica [A] (verified)

Time = 0.10 (sec) , antiderivative size = 54, normalized size of antiderivative = 0.64

$$\int \frac{e^{\csc^{-1}(ax)}}{x^4} dx = \frac{1}{40}a^3 e^{\csc^{-1}(ax)} \left( -5\sqrt{1 - \frac{1}{a^2 x^2}} - \frac{5}{ax} + \cos(3 \csc^{-1}(ax)) + 3 \sin(3 \csc^{-1}(ax)) \right)$$

input

```
Integrate[E^ArcCsc[a*x]/x^4,x]
```

output

```
(a^3*E^ArcCsc[a*x]*(-5*Sqrt[1 - 1/(a^2*x^2)] - 5/(a*x) + Cos[3*ArcCsc[a*x]] + 3*Sin[3*ArcCsc[a*x]]))/40
```

**Rubi [A] (verified)**

Time = 0.30 (sec) , antiderivative size = 80, normalized size of antiderivative = 0.95, number of steps used = 5, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.400$ , Rules used = {5790, 27, 4972, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{e^{\csc^{-1}(ax)}}{x^4} dx$$

$$\downarrow \text{5790}$$

$$-\int \frac{a^2 e^{\csc^{-1}(ax)} \sqrt{1 - \frac{1}{a^2 x^2}}}{x^2} d \csc^{-1}(ax)$$

$$\downarrow \text{27}$$

$$-a^3 \int \frac{e^{\csc^{-1}(ax)} \sqrt{1 - \frac{1}{a^2 x^2}}}{a^2 x^2} d \csc^{-1}(ax)$$

$$\downarrow \text{4972}$$

$$-a^3 \int \left( \frac{1}{4} e^{\csc^{-1}(ax)} \sqrt{1 - \frac{1}{a^2 x^2}} - \frac{1}{4} e^{\csc^{-1}(ax)} \cos(3 \csc^{-1}(ax)) \right) d \csc^{-1}(ax)$$

$$\downarrow \text{2009}$$

$$-a^3 \left( \frac{1}{8} \sqrt{1 - \frac{1}{a^2 x^2}} e^{\csc^{-1}(ax)} + \frac{e^{\csc^{-1}(ax)}}{8ax} - \frac{1}{40} e^{\csc^{-1}(ax)} \cos(3 \csc^{-1}(ax)) - \frac{3}{40} e^{\csc^{-1}(ax)} \sin(3 \csc^{-1}(ax)) \right)$$

input

```
Int[E^ArcCsc[a*x]/x^4,x]
```

output

```
-(a^3*((E^ArcCsc[a*x]*Sqrt[1 - 1/(a^2*x^2)])/8 + E^ArcCsc[a*x]/(8*a*x) - (
E^ArcCsc[a*x]*Cos[3*ArcCsc[a*x]])/40 - (3*E^ArcCsc[a*x]*Sin[3*ArcCsc[a*x]]
)/40))
```

### Defintions of rubi rules used

rule 27

```
Int[(a_)*(F_x_), x_Symbol] := Simp[a Int[F_x, x], x] /; FreeQ[a, x] && !Ma
tchQ[F_x, (b_)*(G_x_)] /; FreeQ[b, x]
```

rule 2009

```
Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 4972

```
Int[Cos[(f_.) + (g_.)*(x_)]^(n_.)*(F_)^((c_.)*((a_.) + (b_.)*(x_)))*Sin[(d_
.) + (e_.)*(x_)]^(m_.), x_Symbol] := Int[ExpandTrigReduce[F^(c*(a + b*x)),
Sin[d + e*x]^m*cos[f + g*x]^n, x], x] /; FreeQ[{F, a, b, c, d, e, f, g}, x]
&& IGtQ[m, 0] && IGtQ[n, 0]
```

rule 5790

```
Int[(u_.)*(f_)^(ArcCsc[(a_.) + (b_.)*(x_)]^(n_.)*(c_.)), x_Symbol] := Simp[
-b^(-1) Subst[Int[(u /. x -> -a/b + Csc[x]/b)*f^(c*x^n)*Csc[x]*Cot[x], x]
, x, ArcCsc[a + b*x]], x] /; FreeQ[{a, b, c, f}, x] && IGtQ[n, 0]
```

### Maple [F]

$$\int \frac{e^{\operatorname{arccsc}(ax)}}{x^4} dx$$

input

```
int(exp(arccsc(a*x))/x^4,x)
```

output

```
int(exp(arccsc(a*x))/x^4,x)
```

**Fricas [A] (verification not implemented)**

Time = 0.13 (sec) , antiderivative size = 41, normalized size of antiderivative = 0.49

$$\int \frac{e^{\csc^{-1}(ax)}}{x^4} dx = \frac{(a^2x^2 - (a^2x^2 + 1)\sqrt{a^2x^2 - 1} - 3)e^{\operatorname{arccsc}(ax)}}{10x^3}$$

input `integrate(exp(arccsc(a*x))/x^4,x, algorithm="fricas")`output `1/10*(a^2*x^2 - (a^2*x^2 + 1)*sqrt(a^2*x^2 - 1) - 3)*e^(arccsc(a*x))/x^3`**Sympy [F]**

$$\int \frac{e^{\csc^{-1}(ax)}}{x^4} dx = \int \frac{e^{\operatorname{acsc}(ax)}}{x^4} dx$$

input `integrate(exp(acsc(a*x))/x**4,x)`output `Integral(exp(acsc(a*x))/x**4, x)`**Maxima [F]**

$$\int \frac{e^{\csc^{-1}(ax)}}{x^4} dx = \int \frac{e^{\operatorname{arccsc}(ax)}}{x^4} dx$$

input `integrate(exp(arccsc(a*x))/x^4,x, algorithm="maxima")`output `integrate(e^(arccsc(a*x))/x^4, x)`

**Giac [F]**

$$\int \frac{e^{\csc^{-1}(ax)}}{x^4} dx = \int \frac{e^{\operatorname{arccsc}(ax)}}{x^4} dx$$

input `integrate(exp(arccsc(a*x))/x^4,x, algorithm="giac")`

output `integrate(e^(arccsc(a*x))/x^4, x)`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{e^{\csc^{-1}(ax)}}{x^4} dx = \int \frac{e^{\operatorname{asin}(\frac{1}{ax})}}{x^4} dx$$

input `int(exp(asin(1/(a*x)))/x^4,x)`

output `int(exp(asin(1/(a*x)))/x^4, x)`

**Reduce [F]**

$$\int \frac{e^{\csc^{-1}(ax)}}{x^4} dx = \int \frac{e^{\operatorname{acsc}(ax)}}{x^4} dx$$

input `int(exp(acsc(a*x))/x^4,x)`

output `int(e**acsc(a*x)/x**4,x)`

### 3.48 $\int \frac{e^{\csc^{-1}(ax)}}{x^5} dx$

Optimal result	375
Mathematica [A] (verified)	375
Rubi [A] (verified)	376
Maple [F]	377
Fricas [A] (verification not implemented)	377
Sympy [F]	378
Maxima [F]	378
Giac [F]	378
Mupad [F(-1)]	379
Reduce [F]	379

#### Optimal result

Integrand size = 10, antiderivative size = 81

$$\int \frac{e^{\csc^{-1}(ax)}}{x^5} dx = \frac{1}{10} a^4 e^{\csc^{-1}(ax)} \cos(2 \csc^{-1}(ax)) - \frac{1}{34} a^4 e^{\csc^{-1}(ax)} \cos(4 \csc^{-1}(ax)) - \frac{1}{20} a^4 e^{\csc^{-1}(ax)} \sin(2 \csc^{-1}(ax)) + \frac{1}{136} a^4 e^{\csc^{-1}(ax)} \sin(4 \csc^{-1}(ax))$$

output `1/10*a^4*exp(arccsc(a*x))*cos(2*arccsc(a*x))-1/34*a^4*exp(arccsc(a*x))*cos(4*arccsc(a*x))-1/20*a^4*exp(arccsc(a*x))*sin(2*arccsc(a*x))+1/136*a^4*exp(arccsc(a*x))*sin(4*arccsc(a*x))`

#### Mathematica [A] (verified)

Time = 0.11 (sec) , antiderivative size = 50, normalized size of antiderivative = 0.62

$$\int \frac{e^{\csc^{-1}(ax)}}{x^5} dx = -\frac{1}{680} a^4 e^{\csc^{-1}(ax)} (-68 \cos(2 \csc^{-1}(ax)) + 20 \cos(4 \csc^{-1}(ax)) + 34 \sin(2 \csc^{-1}(ax)) - 5 \sin(4 \csc^{-1}(ax)))$$

input `Integrate[E^ArcCsc[a*x]/x^5,x]`



output

```
-1/680*(a^4*E^ArcCsc[a*x]*(-68*Cos[2*ArcCsc[a*x]] + 20*Cos[4*ArcCsc[a*x]]
+ 34*Sin[2*ArcCsc[a*x]] - 5*Sin[4*ArcCsc[a*x]]))
```

**Rubi [A] (verified)**

Time = 0.31 (sec) , antiderivative size = 74, normalized size of antiderivative = 0.91, number of steps used = 5, number of rules used = 4,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.400$ , Rules used = {5790, 27, 4972, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned}
 & \int \frac{e^{\csc^{-1}(ax)}}{x^5} dx \\
 & \quad \downarrow \text{5790} \\
 & - \frac{\int \frac{a^2 e^{\csc^{-1}(ax)} \sqrt{1 - \frac{1}{a^2 x^2}}}{x^3} d \csc^{-1}(ax)}{a} \\
 & \quad \downarrow \text{27} \\
 & -a^4 \int \frac{e^{\csc^{-1}(ax)} \sqrt{1 - \frac{1}{a^2 x^2}}}{a^3 x^3} d \csc^{-1}(ax) \\
 & \quad \downarrow \text{4972} \\
 & -a^4 \int \left( \frac{1}{4} e^{\csc^{-1}(ax)} \sin(2 \csc^{-1}(ax)) - \frac{1}{8} e^{\csc^{-1}(ax)} \sin(4 \csc^{-1}(ax)) \right) d \csc^{-1}(ax) \\
 & \quad \downarrow \text{2009} \\
 & -a^4 \left( -\frac{1}{10} e^{\csc^{-1}(ax)} \cos(2 \csc^{-1}(ax)) + \frac{1}{34} e^{\csc^{-1}(ax)} \cos(4 \csc^{-1}(ax)) + \frac{1}{20} e^{\csc^{-1}(ax)} \sin(2 \csc^{-1}(ax)) - \frac{1}{136} e^{\csc^{-1}(ax)} \sin(4 \csc^{-1}(ax)) \right)
 \end{aligned}$$

input

```
Int[E^ArcCsc[a*x]/x^5,x]
```

output

```
-(a^4*(-1/10*(E^ArcCsc[a*x]*Cos[2*ArcCsc[a*x]]) + (E^ArcCsc[a*x]*Cos[4*Arc
Csc[a*x]])/34 + (E^ArcCsc[a*x]*Sin[2*ArcCsc[a*x]])/20 - (E^ArcCsc[a*x]*Sin
[4*ArcCsc[a*x]]/136))
```

**Defintions of rubi rules used**

rule 27 `Int[(a_)*(F_x_), x_Symbol] := Simp[a Int[F_x, x], x] /; FreeQ[a, x] && !MatchQ[F_x, (b_)*(G_x_) /; FreeQ[b, x]]`

rule 2009 `Int[u_, x_Symbol] := Simp[IntSum[u, x], x] /; SumQ[u]`

rule 4972 `Int[Cos[(f_.) + (g_.)*(x_)]^(n_.)*(F_)^((c_.)*((a_.) + (b_.)*(x_)))*Sin[(d_.) + (e_.)*(x_)]^(m_.), x_Symbol] := Int[ExpandTrigReduce[F^(c*(a + b*x)), Sin[d + e*x]^m*cos[f + g*x]^n, x], x] /; FreeQ[{F, a, b, c, d, e, f, g}, x] && IGtQ[m, 0] && IGtQ[n, 0]`

rule 5790 `Int[(u_.)*(f_)^(ArcCsc[(a_.) + (b_.)*(x_)]^(n_.)*(c_.)), x_Symbol] := Simp[-b^(-1) Subst[Int[(u /. x -> -a/b + Csc[x]/b)*f^(c*x^n)*Csc[x]*Cot[x], x], x, ArcCsc[a + b*x]], x] /; FreeQ[{a, b, c, f}, x] && IGtQ[n, 0]`

**Maple [F]**

$$\int \frac{e^{\operatorname{arccsc}(ax)}}{x^5} dx$$

input `int(exp(arccsc(a*x))/x^5,x)`

output `int(exp(arccsc(a*x))/x^5,x)`

**Fricas [A] (verification not implemented)**

Time = 0.15 (sec) , antiderivative size = 51, normalized size of antiderivative = 0.63

$$\int \frac{e^{\operatorname{csc}^{-1}(ax)}}{x^5} dx = \frac{(6a^4x^4 + 3a^2x^2 - (6a^2x^2 + 5)\sqrt{a^2x^2 - 1} - 20)e^{\operatorname{arccsc}(ax)}}{85x^4}$$

input `integrate(exp(arccsc(a*x))/x^5,x, algorithm="fricas")`

output

```
1/85*(6*a^4*x^4 + 3*a^2*x^2 - (6*a^2*x^2 + 5)*sqrt(a^2*x^2 - 1) - 20)*e^(a
rccsc(a*x))/x^4
```

**Sympy [F]**

$$\int \frac{e^{\csc^{-1}(ax)}}{x^5} dx = \int \frac{e^{\operatorname{acsc}(ax)}}{x^5} dx$$

input

```
integrate(exp(acsc(a*x))/x**5,x)
```

output

```
Integral(exp(acsc(a*x))/x**5, x)
```

**Maxima [F]**

$$\int \frac{e^{\csc^{-1}(ax)}}{x^5} dx = \int \frac{e^{\operatorname{arccsc}(ax)}}{x^5} dx$$

input

```
integrate(exp(arccsc(a*x))/x^5,x, algorithm="maxima")
```

output

```
integrate(e^(arccsc(a*x))/x^5, x)
```

**Giac [F]**

$$\int \frac{e^{\csc^{-1}(ax)}}{x^5} dx = \int \frac{e^{\operatorname{arccsc}(ax)}}{x^5} dx$$

input

```
integrate(exp(arccsc(a*x))/x^5,x, algorithm="giac")
```

output

```
integrate(e^(arccsc(a*x))/x^5, x)
```

**Mupad [F(-1)]**

Timed out.

$$\int \frac{e^{\csc^{-1}(ax)}}{x^5} dx = \int \frac{e^{\operatorname{asin}(\frac{1}{ax})}}{x^5} dx$$

input `int(exp(asin(1/(a*x)))/x^5,x)`output `int(exp(asin(1/(a*x)))/x^5, x)`**Reduce [F]**

$$\int \frac{e^{\csc^{-1}(ax)}}{x^5} dx = \int \frac{e^{\operatorname{acsc}(ax)}}{x^5} dx$$

input `int(exp(acsc(a*x))/x^5,x)`output `int(e**acsc(a*x)/x**5,x)`

### 3.49 $\int \frac{\csc^{-1}(a+bx)}{\frac{ad}{b}+dx} dx$

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#### Optimal result

Integrand size = 19, antiderivative size = 69

$$\int \frac{\csc^{-1}(a+bx)}{\frac{ad}{b}+dx} dx = \frac{i \csc^{-1}(a+bx)^2}{2d} - \frac{\csc^{-1}(a+bx) \log\left(1 - e^{2i \csc^{-1}(a+bx)}\right)}{d} + \frac{i \operatorname{PolyLog}\left(2, e^{2i \csc^{-1}(a+bx)}\right)}{2d}$$

output

$1/2*I*\operatorname{arccsc}(b*x+a)^2/d - \operatorname{arccsc}(b*x+a)*\ln(1 - (I/(b*x+a) + (1 - 1/(b*x+a)^2)^{(1/2)})^2)/d + 1/2*I*\operatorname{polylog}(2, (I/(b*x+a) + (1 - 1/(b*x+a)^2)^{(1/2)})^2)/d$

#### Mathematica [A] (verified)

Time = 0.06 (sec) , antiderivative size = 59, normalized size of antiderivative = 0.86

$$\int \frac{\csc^{-1}(a+bx)}{\frac{ad}{b}+dx} dx = \frac{-\csc^{-1}(a+bx) \log\left(1 - e^{2i \csc^{-1}(a+bx)}\right) + \frac{1}{2}i\left(\csc^{-1}(a+bx)^2 + \operatorname{PolyLog}\left(2, e^{2i \csc^{-1}(a+bx)}\right)\right)}{d}$$

input

`Integrate[ArcCsc[a + b*x]/((a*d)/b + d*x), x]`

output

$$\frac{(-(\text{ArcCsc}[a + b*x]*\text{Log}[1 - E^{((2*I)*\text{ArcCsc}[a + b*x])}]) + (I/2)*(\text{ArcCsc}[a + b*x]^2 + \text{PolyLog}[2, E^{((2*I)*\text{ArcCsc}[a + b*x])}]))}{d}$$
**Rubi [A] (warning: unable to verify)**

Time = 0.52 (sec) , antiderivative size = 77, normalized size of antiderivative = 1.12, number of steps used = 12, number of rules used = 11,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.579$ , Rules used = {5780, 27, 5742, 5136, 3042, 25, 4200, 25, 2620, 2715, 2838}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\begin{aligned} & \int \frac{\csc^{-1}(a + bx)}{\frac{ad}{b} + dx} dx \\ & \quad \downarrow \text{5780} \\ & \frac{\int \frac{b \csc^{-1}(a+bx)}{d(a+bx)} d(a+bx)}{b} \\ & \quad \downarrow \text{27} \\ & \frac{\int \frac{\csc^{-1}(a+bx)}{a+bx} d(a+bx)}{d} \\ & \quad \downarrow \text{5742} \\ & \frac{\int (a + bx) \arcsin\left(\frac{1}{a+bx}\right) d\frac{1}{a+bx}}{d} \\ & \quad \downarrow \text{5136} \\ & \frac{\int (a + bx) \sqrt{1 - \frac{1}{(a+bx)^2}} \arcsin\left(\frac{1}{a+bx}\right) d \arcsin\left(\frac{1}{a+bx}\right)}{d} \\ & \quad \downarrow \text{3042} \\ & \frac{\int -\arcsin\left(\frac{1}{a+bx}\right) \tan\left(\arcsin\left(\frac{1}{a+bx}\right) + \frac{\pi}{2}\right) d \arcsin\left(\frac{1}{a+bx}\right)}{d} \\ & \quad \downarrow \text{25} \\ & \frac{\int \arcsin\left(\frac{1}{a+bx}\right) \tan\left(\arcsin\left(\frac{1}{a+bx}\right) + \frac{\pi}{2}\right) d \arcsin\left(\frac{1}{a+bx}\right)}{d} \end{aligned}$$

$$\begin{aligned}
& \downarrow 4200 \\
& \frac{2i \int -\frac{e^{2i \arcsin\left(\frac{1}{a+bx}\right)} \arcsin\left(\frac{1}{a+bx}\right)}{1-e^{2i \arcsin\left(\frac{1}{a+bx}\right)}} d \arcsin\left(\frac{1}{a+bx}\right) - \frac{i}{2(a+bx)^2}}{d} \\
& \downarrow 25 \\
& \frac{-2i \int \frac{e^{2i \arcsin\left(\frac{1}{a+bx}\right)} \arcsin\left(\frac{1}{a+bx}\right)}{1-e^{2i \arcsin\left(\frac{1}{a+bx}\right)}} d \arcsin\left(\frac{1}{a+bx}\right) - \frac{i}{2(a+bx)^2}}{d} \\
& \downarrow 2620 \\
& \frac{-2i \left( \frac{1}{2} i \arcsin\left(\frac{1}{a+bx}\right) \log\left(1 - e^{2i \arcsin\left(\frac{1}{a+bx}\right)}\right) - \frac{1}{2} i \int \log\left(1 - e^{2i \arcsin\left(\frac{1}{a+bx}\right)}\right) d \arcsin\left(\frac{1}{a+bx}\right) \right) - \frac{i}{2(a+bx)^2}}{d} \\
& \downarrow 2715 \\
& \frac{-2i \left( \frac{1}{2} i \arcsin\left(\frac{1}{a+bx}\right) \log\left(1 - e^{2i \arcsin\left(\frac{1}{a+bx}\right)}\right) - \frac{1}{4} \int (a+bx) \log(-a-bx+1) d e^{2i \arcsin\left(\frac{1}{a+bx}\right)} \right) - \frac{i}{2(a+bx)^2}}{d} \\
& \downarrow 2838 \\
& \frac{-2i \left( \frac{1}{4} \text{PolyLog}\left(2, e^{2i \arcsin\left(\frac{1}{a+bx}\right)}\right) + \frac{1}{2} i \arcsin\left(\frac{1}{a+bx}\right) \log\left(1 - e^{2i \arcsin\left(\frac{1}{a+bx}\right)}\right) \right) - \frac{i}{2(a+bx)^2}}{d}
\end{aligned}$$

input `Int[ArcCsc[a + b*x]/((a*d)/b + d*x), x]`

output `-((( -1/2*I)/(a + b*x)^2 - (2*I)*((I/2)*ArcSin[(a + b*x)^(-1)]*Log[1 - E^((2*I)*ArcSin[(a + b*x)^(-1)])] + PolyLog[2, E^((2*I)*ArcSin[(a + b*x)^(-1)])]/4))/d)`

## Definitions of rubi rules used

- rule 25 `Int[-(Fx_), x_Symbol] := Simp[Identity[-1] Int[Fx, x], x]`
- rule 27 `Int[(a_)*(Fx_), x_Symbol] := Simp[a Int[Fx, x], x] /; FreeQ[a, x] && !MatchQ[Fx, (b_)*(Gx_)] /; FreeQ[b, x]`
- rule 2620 `Int[(((F_)^((g_)*((e_) + (f_)*(x_))))^(n_))*((c_) + (d_)*(x_))^(m_)]/((a_) + (b_)*((F_)^((g_)*((e_) + (f_)*(x_))))^(n_)), x_Symbol] := Simp[((c + d*x)^m/(b*f*g*n*Log[F]))*Log[1 + b*((F^(g*(e + f*x)))^n/a)], x] - Simp[d*(m/(b*f*g*n*Log[F])) Int[(c + d*x)^(m - 1)*Log[1 + b*((F^(g*(e + f*x)))^n/a)], x], x] /; FreeQ[{F, a, b, c, d, e, f, g, n}, x] && IGtQ[m, 0]`
- rule 2715 `Int[Log[(a_) + (b_)*((F_)^((e_)*((c_) + (d_)*(x_))))^(n_)], x_Symbol] := Simp[1/(d*e*n*Log[F]) Subst[Int[Log[a + b*x]/x, x], x, (F^(e*(c + d*x)))^n], x] /; FreeQ[{F, a, b, c, d, e, n}, x] && GtQ[a, 0]`
- rule 2838 `Int[Log[(c_)*((d_) + (e_)*(x_)^(n_))]/(x_), x_Symbol] := Simp[-PolyLog[2, (-c)*e*x^n]/n, x] /; FreeQ[{c, d, e, n}, x] && EqQ[c*d, 1]`
- rule 3042 `Int[u_, x_Symbol] := Int[DeactivateTrig[u, x], x] /; FunctionOfTrigOfLinearQ[u, x]`
- rule 4200 `Int[((c_) + (d_)*(x_))^(m_)*tan[(e_) + Pi*(k_) + (f_)*(x_)], x_Symbol] := Simp[I*(c + d*x)^(m + 1)/(d*(m + 1)), x] - Simp[2*I Int[(c + d*x)^m*E^(2*I*k*Pi)*(E^(2*I*(e + f*x)))/(1 + E^(2*I*k*Pi)*E^(2*I*(e + f*x)))], x], x] /; FreeQ[{c, d, e, f}, x] && IntegerQ[4*k] && IGtQ[m, 0]`
- rule 5136 `Int[((a_) + ArcSin[(c_)*(x_)])*(b_)^(n_)/(x_), x_Symbol] := Subst[Int[(a + b*x)^n*Cot[x], x], x, ArcSin[c*x]] /; FreeQ[{a, b, c}, x] && IGtQ[n, 0]`



rule 5742 `Int[((a_.) + ArcCsc[(c_.)*(x_.)]*(b_.))/(x_), x_Symbol] := -Subst[Int[(a + b *ArcSin[x/c])/x, x], x, 1/x] /; FreeQ[{a, b, c}, x]`

rule 5780 `Int[((a_.) + ArcCsc[(c_) + (d_.)*(x_.)]*(b_.))^(p_.)*((e_.) + (f_.)*(x_.))^(m_.), x_Symbol] := Simp[1/d Subst[Int[(f*(x/d))^m*(a + b*ArcCsc[x])^p, x], x, c + d*x], x] /; FreeQ[{a, b, c, d, e, f, m}, x] && EqQ[d*e - c*f, 0] && IGtQ[p, 0]`

## Maple [A] (verified)

Time = 0.85 (sec) , antiderivative size = 167, normalized size of antiderivative = 2.42

method	result
derivativedivides	$\frac{ib \operatorname{arccsc}(bx+a)^2}{2d} - \frac{b \operatorname{arccsc}(bx+a) \ln\left(1 + \frac{i}{bx+a} + \sqrt{1 - \frac{1}{(bx+a)^2}}\right)}{d} + \frac{ib \operatorname{polylog}\left(2, -\frac{i}{bx+a} - \sqrt{1 - \frac{1}{(bx+a)^2}}\right)}{d} - \frac{b \operatorname{arccsc}(bx+a) \ln\left(1 - \frac{i}{bx+a} + \sqrt{1 - \frac{1}{(bx+a)^2}}\right)}{d}$
default	$\frac{ib \operatorname{arccsc}(bx+a)^2}{2d} - \frac{b \operatorname{arccsc}(bx+a) \ln\left(1 + \frac{i}{bx+a} + \sqrt{1 - \frac{1}{(bx+a)^2}}\right)}{d} + \frac{ib \operatorname{polylog}\left(2, -\frac{i}{bx+a} - \sqrt{1 - \frac{1}{(bx+a)^2}}\right)}{d} - \frac{b \operatorname{arccsc}(bx+a) \ln\left(1 - \frac{i}{bx+a} + \sqrt{1 - \frac{1}{(bx+a)^2}}\right)}{d}$

input `int(arccsc(b*x+a)/(a*d/b+d*x), x, method=_RETURNVERBOSE)`

output `1/b*(1/2*I/d*b*arccsc(b*x+a)^2-1/d*b*arccsc(b*x+a)*ln(1+I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2))+I/d*b*polylog(2,-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))-1/d*b*arccsc(b*x+a)*ln(1-I/(b*x+a)-(1-1/(b*x+a)^2)^(1/2))+I/d*b*polylog(2,I/(b*x+a)+(1-1/(b*x+a)^2)^(1/2)))`

## Fricas [F]

$$\int \frac{\csc^{-1}(a + bx)}{\frac{ad}{b} + dx} dx = \int \frac{\operatorname{arccsc}(bx + a)}{dx + \frac{ad}{b}} dx$$

input `integrate(arccsc(b*x+a)/(a*d/b+d*x), x, algorithm="fricas")`

output `integral(b*arccsc(b*x + a)/(b*d*x + a*d), x)`

### Sympy [F]

$$\int \frac{\csc^{-1}(a + bx)}{\frac{ad}{b} + dx} dx = \frac{b \int \frac{\operatorname{arccsc}(\frac{a+bx}{a+bx})}{a+bx} dx}{d}$$

input `integrate(arccsc(b*x+a)/(a*d/b+d*x), x)`

output `b*Integral(arccsc(a + b*x)/(a + b*x), x)/d`

### Maxima [F]

$$\int \frac{\csc^{-1}(a + bx)}{\frac{ad}{b} + dx} dx = \int \frac{\operatorname{arccsc}(bx + a)}{dx + \frac{ad}{b}} dx$$

input `integrate(arccsc(b*x+a)/(a*d/b+d*x), x, algorithm="maxima")`

output `1/2*(2*b*d*integrate(sqrt(b*x + a + 1)*sqrt(b*x + a - 1)*log(b*x + a)/(b^3*d*x^3 + 3*a*b^2*d*x^2 + (3*a^2 - 1)*b*d*x + (a^3 - a)*d), x) - 2*I*b*d*integrate(log(b*x + a)/(b^3*d*x^3 + 3*a*b^2*d*x^2 + (3*a^2 - 1)*b*d*x + (a^3 - a)*d), x) + (2*arctan2(1, sqrt(b*x + a + 1)*sqrt(b*x + a - 1)) + I*log(-b*x - a + 1))*log(b*x + a) - I*log(b^2*x^2 + 2*a*b*x + a^2)*log(b*x + a) + I*log(b*x + a + 1)*log(b*x + a) + I*log(b*x + a)^2 + I*dilog(b*x + a) + I*dilog(-b*x - a))/d`

**Giac [A] (verification not implemented)**

Time = 0.36 (sec) , antiderivative size = 114, normalized size of antiderivative = 1.65

$$\int \frac{\csc^{-1}(a + bx)}{\frac{ad}{b} + dx} dx =$$

$$-\frac{1}{4} b^2 \left( \frac{2 (bx + a)^2 \arcsin \left( \frac{1}{((bx+a)\left(\frac{a}{bx+a}-1\right)-a)\left(\frac{a}{bx+a}-1\right)+a)} \right)}{b^3 d} + \frac{(bx + a) \left( \sqrt{-\frac{1}{(bx+a)^2} + 1} - 1 \right) - \frac{1}{(bx+a) \left( \sqrt{-\frac{1}{(bx+a)^2} + 1} - 1 \right)}}{b^3 d} \right)$$

input `integrate(arccsc(b*x+a)/(a*d/b+d*x),x, algorithm="giac")`

output `-1/4*b^2*(2*(b*x + a)^2*arcsin(1/(((b*x + a)*(a/(b*x + a) - 1) - a)*(a/(b*x + a) - 1) + a)))/(b^3*d) + ((b*x + a)*(sqrt(-1/(b*x + a)^2 + 1) - 1) - 1/((b*x + a)*(sqrt(-1/(b*x + a)^2 + 1) - 1)))/(b^3*d)`

**Mupad [F(-1)]**

Timed out.

$$\int \frac{\csc^{-1}(a + bx)}{\frac{ad}{b} + dx} dx = \int \frac{\operatorname{asin}\left(\frac{1}{a+bx}\right)}{dx + \frac{ad}{b}} dx$$

input `int(asin(1/(a + b*x))/(d*x + (a*d)/b),x)`

output `int(asin(1/(a + b*x))/(d*x + (a*d)/b), x)`

**Reduce [F]**

$$\int \frac{\csc^{-1}(a + bx)}{\frac{ad}{b} + dx} dx = \frac{\left( \int \frac{\operatorname{acsc}(bx+a)}{bx+a} dx \right) b}{d}$$

input `int(acsc(b*x+a)/(a*d/b+d*x),x)`

output `(int(acsc(a + b*x)/(a + b*x),x)*b)/d`

# CHAPTER 4

## APPENDIX

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### 4.1 Listing of Grading functions

The following are the current version of the grading functions used for grading the quality of the antiderivative with reference to the optimal antiderivative included in the test suite.

There is a version for Maple and for Mathematica/Rubi. There is a version for grading Sympy and version for use with Sagemath.

The following are links to the current source code.

The following are the listings of source code of the grading functions.

#### Mathematica and Rubi grading function

```
(* Original version thanks to Albert Rich emailed on 03/21/2017 *)
(* ::Package:: *)

(* Nasser: April 7, 2022. add second output which gives reason for the grade *)
(*           Small rewrite of logic in main function to make it*)
(*           match Maple's logic. No change in functionality otherwise*)

(* ::Subsection:: *)
(*GradeAntiderivative[result,optimal]*)
```

```

(* ::Text:: *)
(*If result and optimal are mathematical expressions, *)
(*      GradeAntiderivative[result,optimal] returns*)
(* "F" if the result fails to integrate an expression that*)
(*      is integrable*)
(* "C" if result involves higher level functions than necessary*)
(* "B" if result is more than twice the size of the optimal*)
(*      antiderivative*)
(* "A" if result can be considered optimal*)

GradeAntiderivative[result_,optimal_] := Module[{expnResult,expnOptimal,leafCountResult,leafCountOptimal},
  expnResult = ExpnType[result];
  expnOptimal = ExpnType[optimal];
  leafCountResult = LeafCount[result];
  leafCountOptimal = LeafCount[optimal];

  (*Print["expnResult=",expnResult," expnOptimal=",expnOptimal];*)
  If[expnResult<=expnOptimal,
    If[Not[FreeQ[result,Complex]], (*result contains complex*)
      If[Not[FreeQ[optimal,Complex]], (*optimal contains complex*)
        If[leafCountResult<=2*leafCountOptimal,
          finalresult={"A",""}
          ,(*ELSE*)
          finalresult={"B","Both result and optimal contain complex but leaf count
          ]
        ,(*ELSE*)
        finalresult={"C","Result contains complex when optimal does not."}
      ]
    ,(*ELSE*)(*result does not contains complex*)
    If[leafCountResult<=2*leafCountOptimal,
      finalresult={"A",""}
      ,(*ELSE*)
      finalresult={"B","Leaf count is larger than twice the leaf count of optimal.
    ]
  ]
  ,(*ELSE*)(*expnResult>expnOptimal*)
  If[FreeQ[result,Integrate] && FreeQ[result,Int],
    finalresult={"C","Result contains higher order function than in optimal. Order "
  ,

```

```

        finalresult={"F","Contains unresolved integral."}
    ]
];

finalresult
]

(* ::Text:: *)
(*The following summarizes the type number assigned an *)
(*expression based on the functions it involves*)
(*1 = rational function*)
(*2 = algebraic function*)
(*3 = elementary function*)
(*4 = special function*)
(*5 = hyperpergeometric function*)
(*6 = appell function*)
(*7 = rootsum function*)
(*8 = integrate function*)
(*9 = unknown function*)

ExpnType[expn_] :=
  If[AtomQ[expn],
    1,
  If[ListQ[expn],
    Max[Map[ExpnType,expn]],
  If[Head[expn]===Power,
    If[IntegerQ[expn[[2]]],
      ExpnType[expn[[1]]],
    If[Head[expn[[2]]]===Rational,
      If[IntegerQ[expn[[1]]] || Head[expn[[1]]]===Rational,
        1,
        Max[ExpnType[expn[[1]],2]],
      Max[ExpnType[expn[[1]],ExpnType[expn[[2]],3]],
    If[Head[expn]===Plus || Head[expn]===Times,
      Max[ExpnType[First[expn]],ExpnType[Rest[expn]]],
    If[ElementaryFunctionQ[Head[expn]],
      Max[3,ExpnType[expn[[1]]]],
    If[SpecialFunctionQ[Head[expn]],
      Apply[Max,Append[Map[ExpnType,Apply[List,expn]],4]],
    If[HypergeometricFunctionQ[Head[expn]],

```

```
Apply[Max, Append[Map[ExpnType, Apply[List, expn]], 5]],
If[AppellFunctionQ[Head[expn]],
Apply[Max, Append[Map[ExpnType, Apply[List, expn]], 6]],
If[Head[expn]===RootSum,
Apply[Max, Append[Map[ExpnType, Apply[List, expn]], 7]],
If[Head[expn]===Integrate || Head[expn]===Int,
Apply[Max, Append[Map[ExpnType, Apply[List, expn]], 8]],
9]]]]]]]]]]]

ElementaryFunctionQ[func_] :=
MemberQ[{
Exp, Log,
Sin, Cos, Tan, Cot, Sec, Csc,
ArcSin, ArcCos, ArcTan, ArcCot, ArcSec, ArcCsc,
Sinh, Cosh, Tanh, Coth, Sech, Csch,
ArcSinh, ArcCosh, ArcTanh, ArcCoth, ArcSech, ArcCsch
}, func]

SpecialFunctionQ[func_] :=
MemberQ[{
Erf, Erfc, Erfi,
FresnelS, FresnelC,
ExpIntegralE, ExpIntegralEi, LogIntegral,
SinIntegral, CosIntegral, SinhIntegral, CoshIntegral,
Gamma, LogGamma, PolyGamma,
Zeta, PolyLog, ProductLog,
EllipticF, EllipticE, EllipticPi
}, func]

HypergeometricFunctionQ[func_] :=
MemberQ[{Hypergeometric1F1, Hypergeometric2F1, HypergeometricPFQ}, func]

AppellFunctionQ[func_] :=
MemberQ[{AppellF1}, func]
```



## Maple grading function

```

# File: GradeAntiderivative.mpl
# Original version thanks to Albert Rich emailed on 03/21/2017

#Nasser 03/22/2017 Use Maple leaf count instead since buildin
#Nasser 03/23/2017 missing 'ln' for ElementaryFunctionQ added
#Nasser 03/24/2017 corrected the check for complex result
#Nasser 10/27/2017 check for leafsize and do not call ExpnType()
#
#                   if leaf size is "too large". Set at 500,000
#Nasser 12/22/2019 Added debug flag, added 'dilog' to special functions
#
#                   see problem 156, file Apostol_Problems
#Nasser 4/07/2022  add second output which gives reason for the grade

GradeAntiderivative := proc(result,optimal)
local leaf_count_result,
      leaf_count_optimal,
      ExpnType_result,
      ExpnType_optimal,
      debug:=false;

      leaf_count_result:=leafcount(result);
      #do NOT call ExpnType() if leaf size is too large. Recursion problem
      if leaf_count_result > 500000 then
          return "B","result has leaf size over 500,000. Avoiding possible recursion issue";
      fi;

      leaf_count_optimal := leafcount(optimal);
      ExpnType_result   := ExpnType(result);
      ExpnType_optimal  := ExpnType(optimal);

      if debug then
          print("ExpnType_result",ExpnType_result," ExpnType_optimal=",ExpnType_optimal);
      fi;

# If result and optimal are mathematical expressions,
# GradeAntiderivative[result,optimal] returns
# "F" if the result fails to integrate an expression that
#     is integrable
# "C" if result involves higher level functions than necessary
# "B" if result is more than twice the size of the optimal

```

```

#   antiderivative
#   "A" if result can be considered optimal

#This check below actually is not needed, since I only
#call this grading only for passed integrals. i.e. I check
#for "F" before calling this. But no harm of keeping it here.
#just in case.

if not type(result,freeof('int')) then
    return "F","Result contains unresolved integral";
fi;

if ExpnType_result<=ExpnType_optimal then
    if debug then
        print("ExpnType_result<=ExpnType_optimal");
    fi;
    if is_contains_complex(result) then
        if is_contains_complex(optimal) then
            if debug then
                print("both result and optimal complex");
            fi;
            if leaf_count_result<=2*leaf_count_optimal then
                return "A"," ";
            else
                return "B",cat("Both result and optimal contain complex but leaf count of
                                convert(leaf_count_result,string)," vs. $2 (" ,
                                convert(leaf_count_optimal,string)," ) = ",convert(2*leaf

            end if
        else #result contains complex but optimal is not
            if debug then
                print("result contains complex but optimal is not");
            fi;
            return "C","Result contains complex when optimal does not.";
        fi;
    else # result do not contain complex
        # this assumes optimal do not as well. No check is needed here.
        if debug then
            print("result do not contain complex, this assumes optimal do not as well
        fi;

```

```

        if leaf_count_result<=2*leaf_count_optimal then
            if debug then
                print("leaf_count_result<=2*leaf_count_optimal");
            fi;
            return "A"," ";
        else
            if debug then
                print("leaf_count_result>2*leaf_count_optimal");
            fi;
            return "B",cat("Leaf count of result is larger than twice the leaf count of
                            convert(leaf_count_result,string)," $ vs. $2(",
                            convert(leaf_count_optimal,string),")=",convert(2*leaf_co
            fi;
        fi;
    else #ExpnType(result) > ExpnType(optimal)
        if debug then
            print("ExpnType(result) > ExpnType(optimal)");
        fi;
        return "C",cat("Result contains higher order function than in optimal. Order ",
                        convert(ExpnType_result,string)," vs. order ",
                        convert(ExpnType_optimal,string),".");
    fi;
end proc:

#
# is_contains_complex(result)
# takes expressions and returns true if it contains "I" else false
#
#Nasser 032417
is_contains_complex:= proc(expression)
    return (has(expression,I));
end proc:

# The following summarizes the type number assigned an expression
# based on the functions it involves
# 1 = rational function
# 2 = algebraic function
# 3 = elementary function
# 4 = special function
# 5 = hyperpergeometric function

```

```

# 6 = appell function
# 7 = rootsum function
# 8 = integrate function
# 9 = unknown function

ExpnType := proc(expn)
  if type(expn,'atomic') then
    1
  elif type(expn,'list') then
    apply(max,map(ExpnType,expn))
  elif type(expn,'sqrt') then
    if type(op(1,expn),'rational') then
      1
    else
      max(2,ExpnType(op(1,expn)))
    end if
  elif type(expn,'^^') then
    if type(op(2,expn),'integer') then
      ExpnType(op(1,expn))
    elif type(op(2,expn),'rational') then
      if type(op(1,expn),'rational') then
        1
      else
        max(2,ExpnType(op(1,expn)))
      end if
    else
      max(3,ExpnType(op(1,expn)),ExpnType(op(2,expn)))
    end if
  elif type(expn,'+'') or type(expn,'*') then
    max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
  elif ElementaryFunctionQ(op(0,expn)) then
    max(3,ExpnType(op(1,expn)))
  elif SpecialFunctionQ(op(0,expn)) then
    max(4,apply(max,map(ExpnType,[op(expn)])))
  elif HypergeometricFunctionQ(op(0,expn)) then
    max(5,apply(max,map(ExpnType,[op(expn)])))
  elif AppellFunctionQ(op(0,expn)) then
    max(6,apply(max,map(ExpnType,[op(expn)])))
  elif op(0,expn)='int' then
    max(8,apply(max,map(ExpnType,[op(expn)]))) else
  9
end proc

```

```

    end if
end proc:

ElementaryFunctionQ := proc(func)
  member(func, [
    exp, log, ln,
    sin, cos, tan, cot, sec, csc,
    arcsin, arccos, arctan, arccot, arcsec, arccsc,
    sinh, cosh, tanh, coth, sech, csch,
    arcsinh, arccosh, arctanh, arccoth, arcsech, arccsch])
end proc:

SpecialFunctionQ := proc(func)
  member(func, [
    erf, erfc, erfi,
    FresnelS, FresnelC,
    Ei, Ei, Li, Si, Ci, Shi, Chi,
    GAMMA, lnGAMMA, Psi, Zeta, polylog, dilog, LambertW,
    EllipticF, EllipticE, EllipticPi])
end proc:

HypergeometricFunctionQ := proc(func)
  member(func, [Hypergeometric1F1, hypergeom, HypergeometricPFQ])
end proc:

AppellFunctionQ := proc(func)
  member(func, [AppellF1])
end proc:

# u is a sum or product. rest(u) returns all but the
# first term or factor of u.
rest := proc(u) local v;
  if nops(u)=2 then
    op(2,u)
  else
    apply(op(0,u), op(2..nops(u),u))
  end if
end proc:

#leafcount(u) returns the number of nodes in u.

```

```
#Nasser 3/23/17 Replaced by build-in leafCount from package in Maple
leafcount := proc(u)
  MmaTranslator[Mma][LeafCount](u);
end proc:
```

## Sympy grading function

```
#Dec 24, 2019. Nasser M. Abbasi:
#      Port of original Maple grading function by
#      Albert Rich to use with Sympy/Python
#Dec 27, 2019 Nasser. Added `RootSum`. See problem 177, Timofeev file
#      added 'exp_polar'
from sympy import *

def leaf_count(expr):
  #sympy do not have leaf count function. This is approximation
  return round(1.7*count_ops(expr))

def is_sqrt(expr):
  if isinstance(expr,Pow):
    if expr.args[1] == Rational(1,2):
      return True
    else:
      return False
  else:
    return False

def is_elementary_function(func):
  return func in [exp,log,ln,sin,cos,tan,cot,sec,csc,
    asin,acos,atan,acot,asec,acsc,sinh,cosh,tanh,coth,sech,csch,
    asinh,acosh,atanh,acoth,asech,acsch
  ]

def is_special_function(func):
  return func in [ erf,erfc,erfi,
    fresnels,fresnelc,Ei,Ei,Li,Si,Ci,Shi,Chi,
    gamma,loggamma,digamma,zeta,polylog,LambertW,
    elliptic_f,elliptic_e,elliptic_pi,exp_polar
  ]
```

```

def is_hypergeometric_function(func):
    return func in [hyper]

def is_appell_function(func):
    return func in [appellf1]

def is_atom(expn):
    try:
        if expn.isAtom or isinstance(expn,int) or isinstance(expn,float):
            return True
        else:
            return False

    except AttributeError as error:
        return False

def expnType(expn):
    debug=False
    if debug:
        print("expn=",expn,"type(expn)=",type(expn))

    if is_atom(expn):
        return 1
    elif isinstance(expn,list):
        return max(map(expnType, expn)) #apply(max,map(ExpnType,expn))
    elif is_sqrt(expn):
        if isinstance(expn.args[0],Rational): #type(op(1,expn),'rational')
            return 1
        else:
            return max(2,expnType(expn.args[0])) #max(2,ExpnType(op(1,expn)))
    elif isinstance(expn,Pow): #type(expn,'^')
        if isinstance(expn.args[1],Integer): #type(op(2,expn),'integer')
            return expnType(expn.args[0]) #ExpnType(op(1,expn))
        elif isinstance(expn.args[1],Rational): #type(op(2,expn),'rational')
            if isinstance(expn.args[0],Rational): #type(op(1,expn),'rational')
                return 1
            else:
                return max(2,expnType(expn.args[0])) #max(2,ExpnType(op(1,expn)))
        else:
            return max(3,expnType(expn.args[0]),expnType(expn.args[1])) #max(3,ExpnType(op(1,expn)),ExpnType(op(2,expn)))
    elif isinstance(expn,Add) or isinstance(expn,Mul): #type(expn,'+') or type(expn,'*')

```

```

    m1 = expnType(expn.args[0])
    m2 = expnType(list(expn.args[1:]))
    return max(m1,m2) #max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
elif is_elementary_function(expn.func): #ElementaryFunctionQ(op(0,expn))
    return max(3,expnType(expn.args[0])) #max(3,ExpnType(op(1,expn)))
elif is_special_function(expn.func): #SpecialFunctionQ(op(0,expn))
    m1 = max(map(expnType, list(expn.args)))
    return max(4,m1) #max(4,apply(max,map(ExpnType,[op(expn)])))
elif is_hypergeometric_function(expn.func): #HypergeometricFunctionQ(op(0,expn))
    m1 = max(map(expnType, list(expn.args)))
    return max(5,m1) #max(5,apply(max,map(ExpnType,[op(expn)])))
elif is_appell_function(expn.func):
    m1 = max(map(expnType, list(expn.args)))
    return max(6,m1) #max(5,apply(max,map(ExpnType,[op(expn)])))
elif isinstance(expn,RootSum):
    m1 = max(map(expnType, list(expn.args))) #Apply[Max,Append[Map[ExpnType,Apply[List,expn]],7]],
    return max(7,m1)
elif str(expn).find("Integral") != -1:
    m1 = max(map(expnType, list(expn.args)))
    return max(8,m1) #max(5,apply(max,map(ExpnType,[op(expn)])))
else:
    return 9

#main function
def grade_antiderivative(result,optimal):

    #print("Enter grade_antiderivative for sagemath")
    #print("Enter grade_antiderivative, result=",result," optimal=",optimal)

    leaf_count_result = leaf_count(result)
    leaf_count_optimal = leaf_count(optimal)

    #print("leaf_count_result=",leaf_count_result)
    #print("leaf_count_optimal=",leaf_count_optimal)

    expnType_result = expnType(result)
    expnType_optimal = expnType(optimal)

    if str(result).find("Integral") != -1:
        grade = "F"
        grade_annotation = ""

```



```

else:
    if expnType_result <= expnType_optimal:
        if result.has(I):
            if optimal.has(I): #both result and optimal complex
                if leaf_count_result <= 2*leaf_count_optimal:
                    grade = "A"
                    grade_annotation = ""
                else:
                    grade = "B"
                    grade_annotation = "Both result and optimal contain complex but leaf count of result is lar
            else: #result contains complex but optimal is not
                grade = "C"
                grade_annotation = "Result contains complex when optimal does not."
        else: # result do not contain complex, this assumes optimal do not as well
            if leaf_count_result <= 2*leaf_count_optimal:
                grade = "A"
                grade_annotation = ""
            else:
                grade = "B"
                grade_annotation = "Leaf count of result is larger than twice the leaf count of optimal. "+str(
        else:
            grade = "C"
            grade_annotation = "Result contains higher order function than in optimal. Order "+str(ExpnType

#print("Before returning. grade=",grade, " grade_annotation=",grade_annotation)

return grade, grade_annotation

```

## SageMath grading function

```

#Dec 24, 2019. Nasser: Ported original Maple grading function by
#       Albert Rich to use with Sagemath. This is used to
#       grade Fricas, Giac and Maxima results.
#Dec 24, 2019. Nasser: Added 'exp_integral_e' and 'sng', 'sin_integral'
#       'arctan2', 'floor', 'abs', 'log_integral'
#June 4, 2022 Made default grade_annotation "none" instead of "" due
#       issue later when reading the file.
#July 14, 2022. Added ellipticF. This is until they fix sagemath, then remove it.

```

```

from sage.all import *
from sage.symbolic.operators import add_vararg, mul_vararg

debug=False;

def tree_size(expr):
    r"""
    Return the tree size of this expression.
    """
    #print("Enter tree_size, expr is ",expr)

    if expr not in SR:
        # deal with lists, tuples, vectors
        return 1 + sum(tree_size(a) for a in expr)
    expr = SR(expr)
    x, aa = expr.operator(), expr.operands()
    if x is None:
        return 1
    else:
        return 1 + sum(tree_size(a) for a in aa)

def is_sqrt(expr):
    if expr.operator() == operator.pow: #isinstance(expr,Pow):
        if expr.operands()[1]==1/2: #expr.args[1] == Rational(1,2):
            if debug: print ("expr is sqrt")
            return True
        else:
            return False
    else:
        return False

def is_elementary_function(func):
    #debug=False
    m = func.name() in ['exp','log','ln',
        'sin','cos','tan','cot','sec','csc',
        'arcsin','arccos','arctan','arccot','arcsec','arccsc',
        'sinh','cosh','tanh','coth','sech','csch',
        'arcsinh','arccosh','arctanh','arcoth','arcsech','arcsch','sgn',
        'arctan2','floor','abs'
    ]
    if debug:

```

```

    if m:
        print ("func ", func , " is elementary_function")
    else:
        print ("func ", func , " is NOT elementary_function")

    return m

def is_special_function(func):
    #debug=False
    if debug:
        print ("type(func)=", type(func))

    m= func.name() in ['erf','erfc','erfi','fresnel_sin','fresnel_cos','Ei',
        'Ei','Li','Si','sin_integral','Ci','cos_integral','Shi','sinh_integral',
        'Chi','cosh_integral','gamma','log_gamma','psi','zeta',
        'polylog','lambert_w','elliptic_f','elliptic_e','ellipticF',
        'elliptic_pi','exp_integral_e','log_integral',
        'weierstrassPInverse','weierstrass','weierstrassP','weierstrassZeta',
        'weierstrassPPrime','weierstrassSigma']

    if debug:
        print ("m=",m)
    if m:
        print ("func ", func , " is special_function")
    else:
        print ("func ", func , " is NOT special_function")

    return m

def is_hypergeometric_function(func):
    return func.name() in ['hypergeometric','hypergeometric_M','hypergeometric_U']

def is_appell_function(func):
    return func.name() in ['hypergeometric']  #[appellf1] can't find this in sagemath

def is_atom(expn):

    #debug=False

```

```

if debug:
    print ("Enter is_atom, expn=",expn)

if not hasattr(expn, 'parent'):
    return False

#thanks to answer at https://ask.sagemath.org/question/49179/what-is-sagemath-equivalent-to-atomic
try:
    if expn.parent() is SR:
        return expn.operator() is None
    if expn.parent() in (ZZ, QQ, AA, QQbar):
        return expn in expn.parent() # Should always return True
    if hasattr(expn.parent(),"base_ring") and hasattr(expn.parent(),"gens"):
        return expn in expn.parent().base_ring() or expn in expn.parent().gens()

    return False

except AttributeError as error:
    print ("Exception,AttributeError in is_atom")
    print ("caught exception" , type(error).__name__ )
    return False

def expnType(expn):

    if debug:
        print (">>>>>Enter expnType, expn=", expn)
        print (">>>>>is_atom(expn)=", is_atom(expn))

    if is_atom(expn):
        return 1
    elif type(expn)==list: #instance(expn,list):
        return max(map(expnType, expn)) #apply(max,map(ExpnType,expn))
    elif is_sqrt(expn):
        if type(expn.operands()[0])==Rational: #type(instance(expn.args[0],Rational):
            return 1
        else:
            return max(2,expnType(expn.operands()[0])) #max(2,expnType(expn.args[0]))
    elif expn.operator() == operator.pow: #instance(expn,Pow)
        if type(expn.operands()[1])==Integer: #instance(expn.args[1],Integer)

```

```

    return expnType(expn.operands()[0]) #expnType(expn.args[0])
elif type(expn.operands()[1])==Rational: #isinstance(expn.args[1],Rational)
    if type(expn.operands()[0])==Rational: #isinstance(expn.args[0],Rational)
        return 1
    else:
        return max(2,expnType(expn.operands()[0])) #max(2,expnType(expn.args[0]))
else:
    return max(3,expnType(expn.operands()[0]),expnType(expn.operands()[1])) #max(3,expnType(expn
elif expn.operator() == add_vararg or expn.operator() == mul_vararg: #isinstance(expn,Add) or isins
    m1 = expnType(expn.operands()[0]) #expnType(expn.args[0])
    m2 = expnType(expn.operands()[1:]) #expnType(list(expn.args[1:]))
    return max(m1,m2) #max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
elif is_elementary_function(expn.operator()): #is_elementary_function(expn.func)
    return max(3,expnType(expn.operands()[0]))
elif is_special_function(expn.operator()): #is_special_function(expn.func)
    m1 = max(map(expnType, expn.operands())) #max(map(expnType, list(expn.args)))
    return max(4,m1) #max(4,m1)
elif is_hypergeometric_function(expn.operator()): #is_hypergeometric_function(expn.func)
    m1 = max(map(expnType, expn.operands())) #max(map(expnType, list(expn.args)))
    return max(5,m1) #max(5,m1)
elif is_appell_function(expn.operator()):
    m1 = max(map(expnType, expn.operands())) #max(map(expnType, list(expn.args)))
    return max(6,m1) #max(6,m1)
elif str(expn).find("Integral") != -1: #this will never happen, since it
    #is checked before calling the grading function that is passed.
    #but kept it here.
    m1 = max(map(expnType, expn.operands())) #max(map(expnType, list(expn.args)))
    return max(8,m1) #max(5,apply(max,map(ExpnType,[op(expn)])))
else:
    return 9

#main function
def grade_antiderivative(result,optimal):

if debug:
    print ("Enter grade_antiderivative for sagemath")
    print ("Enter grade_antiderivative, result=",result)
    print ("Enter grade_antiderivative, optimal=",optimal)
    print ("type(anti)=", type(result))
    print ("type(optimal)=", type(optimal))

```

```

leaf_count_result = tree_size(result) #leaf_count(result)
leaf_count_optimal = tree_size(optimal) #leaf_count(optimal)

#if debug: print ("leaf_count_result=", leaf_count_result, "leaf_count_optimal=",leaf_count_optimal)

expnType_result = expnType(result)
expnType_optimal = expnType(optimal)

if debug: print ("expnType_result=", expnType_result, "expnType_optimal=",expnType_optimal)

if expnType_result <= expnType_optimal:
    if result.has(I):
        if optimal.has(I): #both result and optimal complex
            if leaf_count_result <= 2*leaf_count_optimal:
                grade = "A"
                grade_annotation = "none"
            else:
                grade = "B"
                grade_annotation = "Both result and optimal contain complex but leaf count of result is larger"
        else: #result contains complex but optimal is not
            grade = "C"
            grade_annotation = "Result contains complex when optimal does not."
    else: # result do not contain complex, this assumes optimal do not as well
        if leaf_count_result <= 2*leaf_count_optimal:
            grade = "A"
            grade_annotation = "none"
        else:
            grade = "B"
            grade_annotation = "Leaf count of result is larger than twice the leaf count of optimal. "+str(leaf_count_result - 2*leaf_count_optimal)
    else:
        grade = "C"
        grade_annotation = "Result contains higher order function than in optimal. Order "+str(expnType_result - expnType_optimal)

print("Before returning. grade=",grade, " grade_annotation=",grade_annotation)

return grade, grade_annotation

```

## 4.2 Links to plain text integration problems used in this report for each CAS

1. Mathematica integration problems as .m file
2. Maple integration problems as .txt file
3. Sagemath integration problems as .sage file
4. Reduce integration problems as .txt file
5. Mupad integration problems as .txt file
6. Sympy integration problems as .py file