Computer Algebra Independent Integration Tests

Summer 2024

7-Inverse-hyperbolic-functions/7.4-Inverse-hyperbolic-cotangent/342-7.4.2

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May 18, 2024 Compiled on May 18, 2024 at 9:50am

Contents

CHAPTER 1

Introduction

This report gives the result of running the computer algebra independent integration test.The download section in on the main webpage contains links to download the problems in plain text format used for all CAS systems. The number of integrals in this report is [**49**]. This is test number [342].

1.1 Listing of CAS systems tested

The following are the CAS systems tested:

1. Mathematica 14 (January 9, 2024) on windows 10 pro.
2. Rubi 4.17.3 (Sept 25, 2023) on Mathematica 14 on windows 10m pro.
3. Maple 2024 (March 1, 2024) on windows 10 pro.
4. Maxima 5.47 (June 1, 2023) using Lisp SBCL 2.4.0 on Linux Manjaro 23.1.2 KDE via sagemath 10.3.
5. FriCAS 1.3.10 built with sbcl 2.3.11 (January 10, 2024) on Linux Manjaro 23.1.2 KDE via sagemath 10.3.
6. Giac/Xcas 1.9.0-99 on Linux via sagemath 10.3.
7. Sympy 1.12 using Python 3.11.6 (Nov 14 2023, 09:36:21) [GCC 13.2.1 20230801] on Linux Manjaro 23.1.2 KDE.
8. Mupad using Matlab 2021a with Symbolic Math Toolbox Version 8.7 on windows 10.
9. Reduce CSL rev 6687 (January 9, 2024) on Linux Manjaro 23.1.2 KDE.

Maxima and Fricas and Giac are called using Sagemath. This was done using Sagemath `integrate` command by changing the name of the algorithm to use the different CAS systems.

Sympy was run directly in Python not via sagemath.

Reduce was called directly.

1.2 Results

Important note: A number of problems in this test suite have no antiderivative in closed form. This means the antiderivative of these integrals can not be expressed in terms of elementary, special functions or `Hypergeometric2F1` functions. `RootSum` and `RootOf` are not allowed. If a CAS returns the above integral unevaluated within the time limit, then the result is counted as passed and assigned an A grade.

However, if CAS times out, then it is assigned an F grade even if the integral is not integrable, as this implies CAS could not determine that the integral is not integrable in the time limit.

If a CAS returns an antiderivative to such an integral, it is assigned an A grade automatically and this special result is listed in the introduction section of each individual test report to make it easy to identify as this can be important result to investigate.

The results given in in the table below reflects the above.

System	% solved	% Failed
Rubi	100.00 (49)	0.00 (0)
Mathematica	100.00 (49)	0.00 (0)
Maple	97.96 (48)	2.04 (1)
Maxima	83.67 (41)	16.33 (8)
Fricas	57.14 (28)	42.86 (21)
Giac	55.10 (27)	44.90 (22)
Reduce	55.10 (27)	44.90 (22)
Mupad	53.06 (26)	46.94 (23)
Sympy	46.94 (23)	53.06 (26)

Table 1.1: Percentage solved for each CAS

The table below gives additional break down of the grading of quality of the antiderivatives generated by each CAS. The grading is given using the letters A,B,C and F with A being the best quality. The grading is accomplished by comparing the antiderivative generated with the optimal antiderivatives included in the test suite. The following table describes the meaning of these grades.

grade	description
A	Integral was solved and antiderivative is optimal in quality and leaf size.
B	Integral was solved and antiderivative is optimal in quality but leaf size is larger than twice the optimal antiderivatives leaf size.
C	Integral was solved and antiderivative is non-optimal in quality. This can be due to one or more of the following reasons 1. antiderivative contains a hypergeometric function and the optimal antiderivative does not. 2. antiderivative contains a special function and the optimal antiderivative does not. 3. antiderivative contains the imaginary unit and the optimal antiderivative does not.
F	Integral was not solved. Either the integral was returned unevaluated within the time limit, or it timed out, or CAS hanged or crashed or an exception was raised.

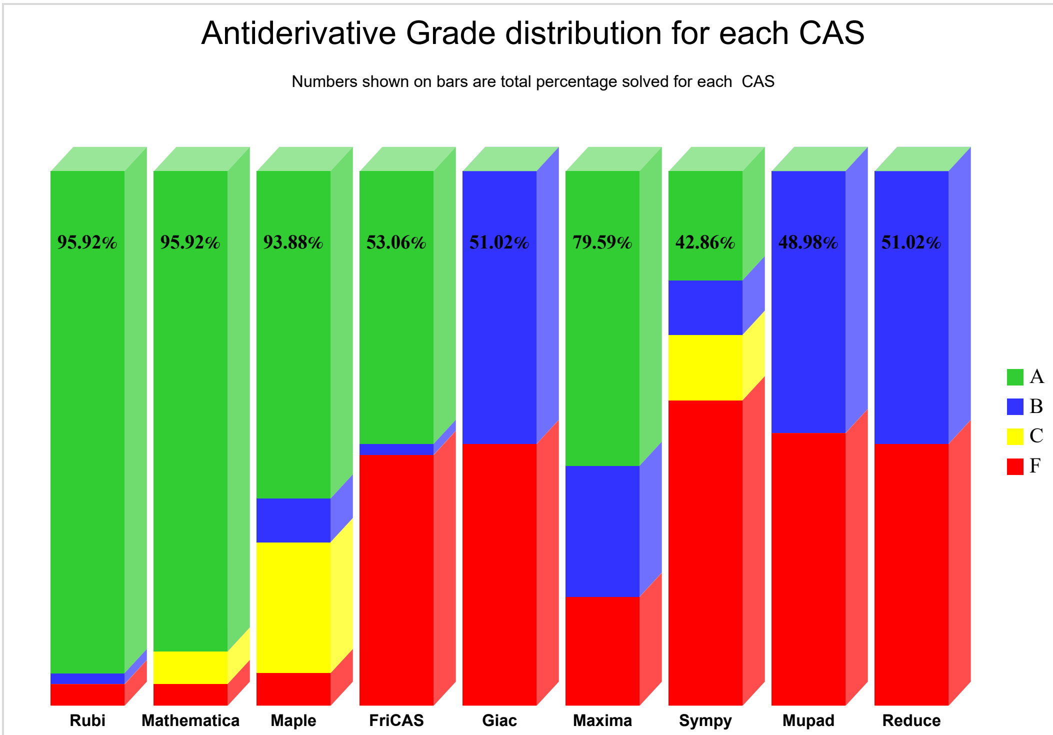
Table 1.2: Description of grading applied to integration result

Grading is implemented for all CAS systems. Based on the above, the following table summarizes the grading for this test suite.

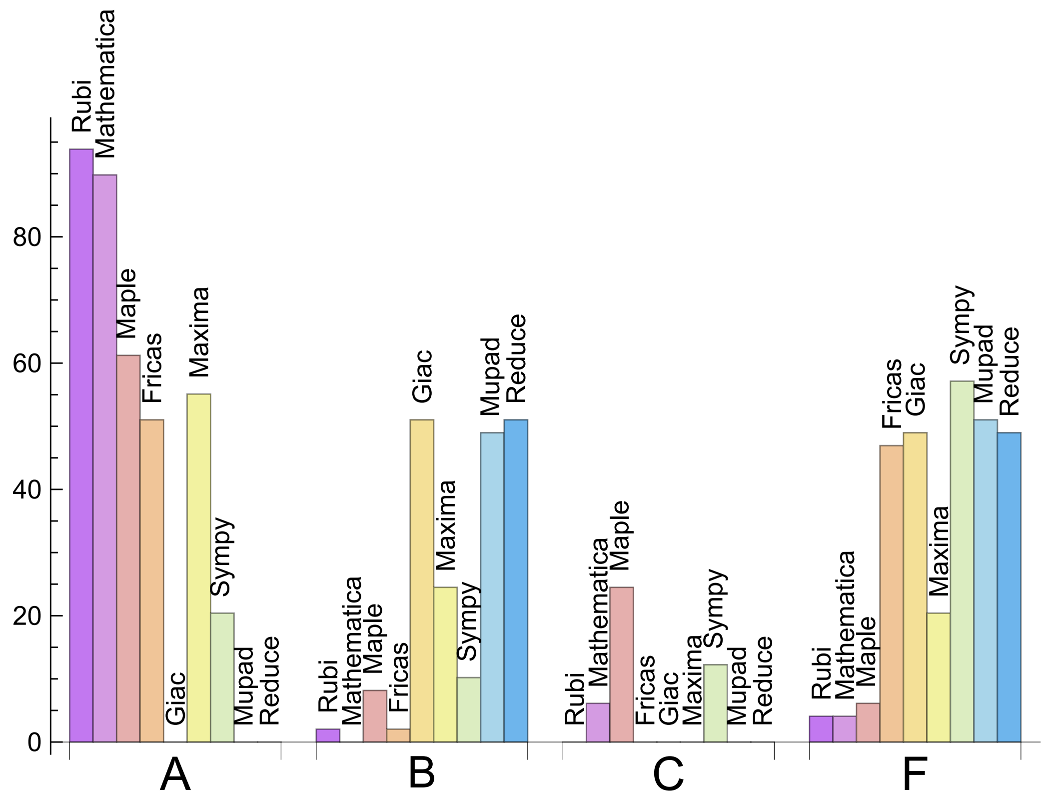
System	% A grade	% B grade	% C grade	% F grade
Rubi	93.878	2.041	0.000	4.082
Mathematica	89.796	0.000	6.122	4.082
Maple	61.224	8.163	24.490	6.122
Maxima	55.102	24.490	0.000	20.408
Fricas	51.020	2.041	0.000	46.939
Sympy	20.408	10.204	12.245	57.143
Giac	0.000	51.020	0.000	48.980
Mupad	0.000	48.980	0.000	51.020
Reduce	0.000	51.020	0.000	48.980

Table 1.3: Antiderivative Grade distribution of each CAS

The following is a Bar chart illustration of the data in the above table.



The figure below compares the grades of the CAS systems.



The following table shows the distribution of the different types of failures for each CAS. There are 3 types failures. The first is when CAS returns the input within the time limit, which means it could not solve it. This is the typical failure and given as **F**.

The second failure is due to time out. CAS could not solve the integral within the 3 minutes time limit which is assigned. This is assigned **F(-1)**.

The third is due to an exception generated, indicated as **F(-2)**. This most likely indicates an interface problem between sagemath and the CAS (applicable only to FriCAS, Maxima and Giac) or it could be an indication of an internal error in the CAS itself. This type of error requires more investigation to determine the cause.

System	Number failed	Percentage normal failure	Percentage timeout failure	Percentage exception failure
Rubi	0	0.00	0.00	0.00
Mathematica	0	0.00	0.00	0.00
Maple	1	100.00	0.00	0.00
Maxima	8	100.00	0.00	0.00
Fricas	21	100.00	0.00	0.00
Giac	22	100.00	0.00	0.00
Reduce	22	100.00	0.00	0.00
Mupad	23	0.00	100.00	0.00
Sympy	26	96.15	3.85	0.00

Table 1.4: Failure statistics for each CAS

1.3 Time and leaf size Performance

The table below summarizes the performance of each CAS system in terms of time used and leaf size of results.

Mean size is the average leaf size produced by the CAS (before any normalization). The Normalized mean is relative to the mean size of the optimal anti-derivative given in the input files.

For example, if CAS has **Normalized mean** of 3, then the mean size of its leaf size is 3 times as large as the mean size of the optimal leaf size.

Median size is value of leaf size where half the values are larger than this and half are smaller (before any normalization). i.e. The Middle value.

Similarly the **Normalized median** is relative to the median leaf size of the optimal.

For example, if a CAS has Normalized median of 1.2, then its median is 1.2 as large as the median leaf size of the optimal.

System	Mean time (sec)
Maxima	0.08
Fricas	0.09
Mathematica	0.09
Giac	0.12
Reduce	0.17
Sympy	0.45
Rubi	0.60
Maple	1.17
Mupad	3.59

Table 1.5: Time performance for each CAS

System	Mean size	Normalized mean	Median size	Normalized median
Reduce	38.41	0.94	31.00	0.90
Mupad	44.12	1.01	32.00	0.85
Fricas	46.61	1.18	38.00	1.10
Sympy	61.22	1.65	49.00	1.03
Mathematica	61.76	1.00	52.00	0.93
Rubi	84.90	1.15	52.00	1.04
Maxima	97.68	2.02	66.00	1.29
Giac	164.85	3.85	144.00	4.14
Maple	350.15	3.51	47.50	1.17

Table 1.6: Leaf size performance for each CAS

1.4 Performance based on number of rules Rubi used

This section shows how each CAS performed based on the number of rules Rubi needed to solve the same integral. One diagram is given for each CAS.

On the y axis is the percentage solved which Rubi itself needed the number of rules given the x axis. These plots show that as more rules are needed then most CAS system percentage of solving decreases which indicates the integral is becoming more complicated to solve.

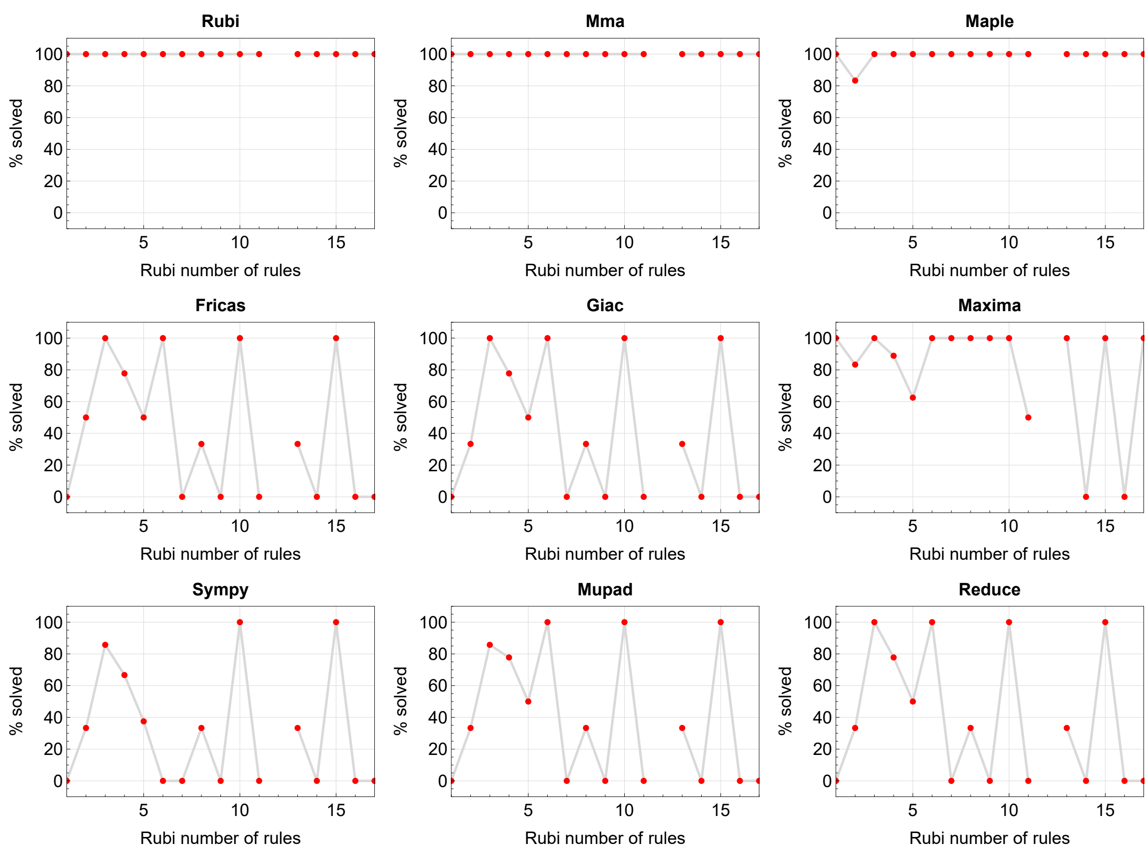


Figure 1.1: Solving statistics per number of Rubi rules used

1.5 Performance based on number of steps Rubi used

This section shows how each CAS performed based on the number of steps Rubi needed to solve the same integral. Note that the number of steps Rubi needed can be much higher than the number of rules, as the same rule could be used more than once.

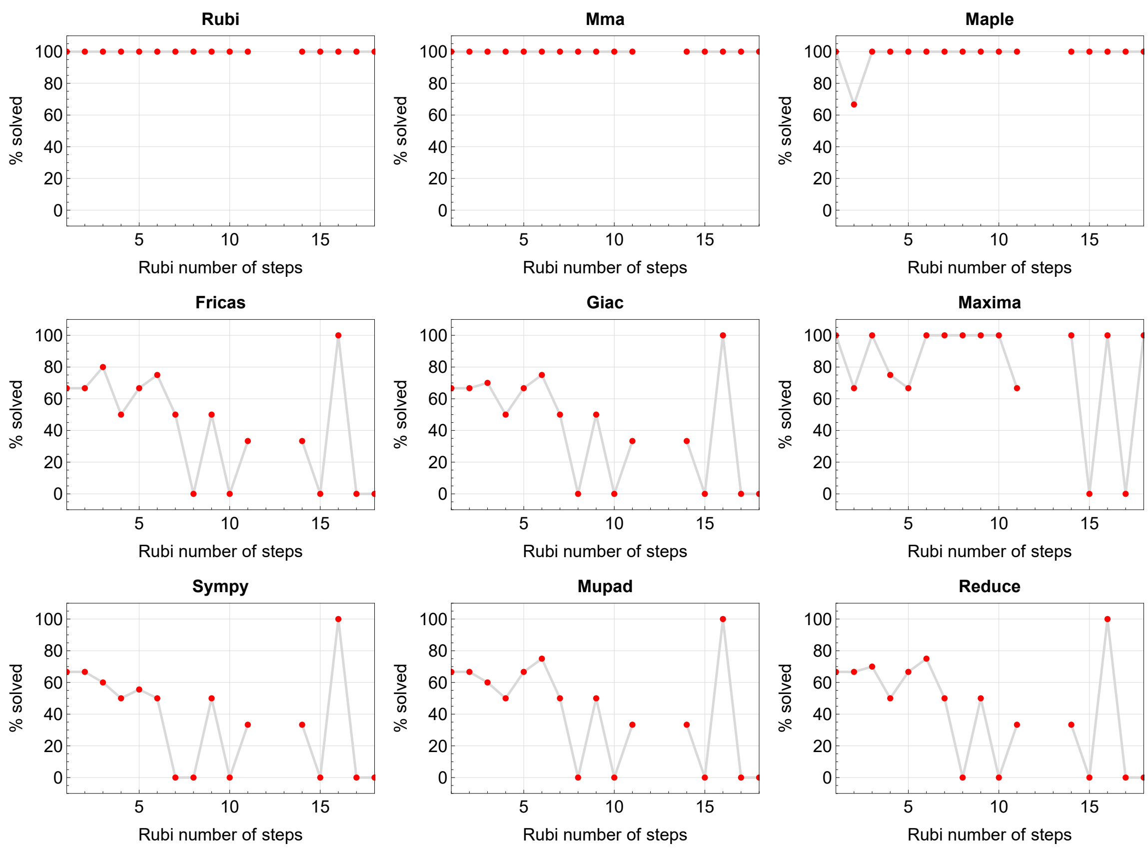


Figure 1.2: Solving statistics per number of Rubi steps used

The above diagram show that the precentage of solved intergals decreases for most CAS systems as the number of steps increases. As expected, for integrals that required less steps by Rubi, CAS systems had more success which indicates the integral was not as hard to solve. As Rubi needed more steps to solve the integral, the solved percentage decreased for most CAS systems which indicates the integral is becoming harder to solve.

1.6 Solved integrals histogram based on leaf size of result

The following shows the distribution of solved integrals for each CAS system based on leaf size of the antiderivatives produced by each CAS. It shows that most integrals solved produced leaf size less than about 100 to 150. The bin size used is 40.

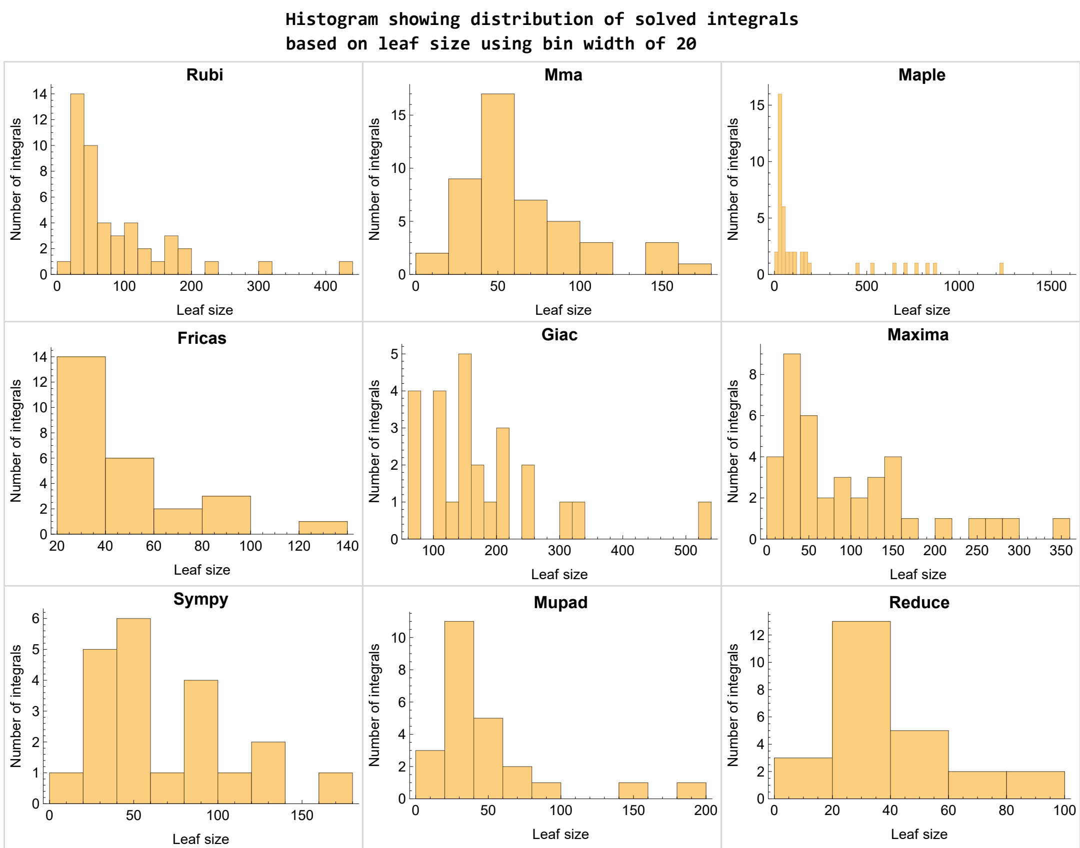


Figure 1.3: Solved integrals based on leaf size distribution

1.7 Solved integrals histogram based on CPU time used

The following shows the distribution of solved integrals for each CAS system based on CPU time used in seconds. The bin size used is 0.1 second.

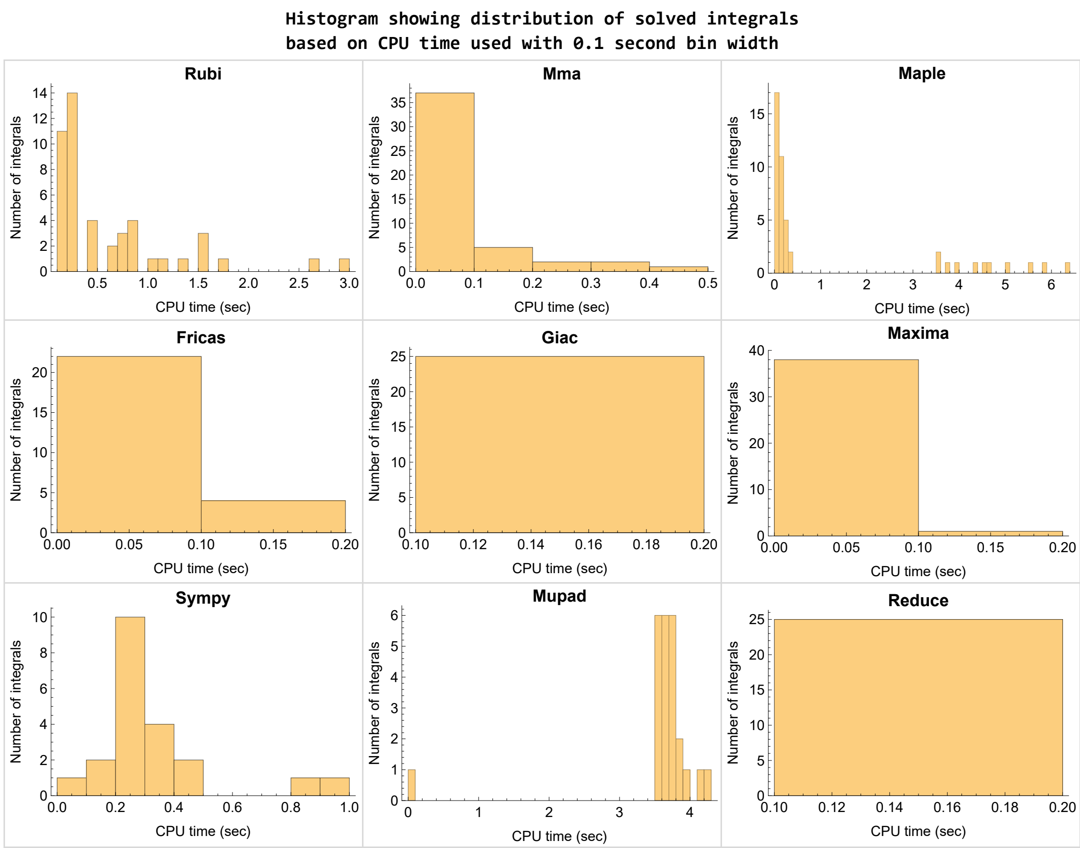


Figure 1.4: Solved integrals histogram based on CPU time used

1.8 Leaf size vs. CPU time used

The following shows the relation between the CPU time used to solve an integral and the leaf size of the antiderivative.

The result for Fricas, Maxima and Giac is shifted more to the right than the other CAS system due to the use of sagemath to call them, which causes an initial slight delay in the timing to start the integration due to overhead of starting a new process each time. This should also be taken into account when looking at the timing of these three CAS systems. Direct calls not using sagemath would result in faster timings, but current implementation uses sagemath as this makes testing much easier to do.

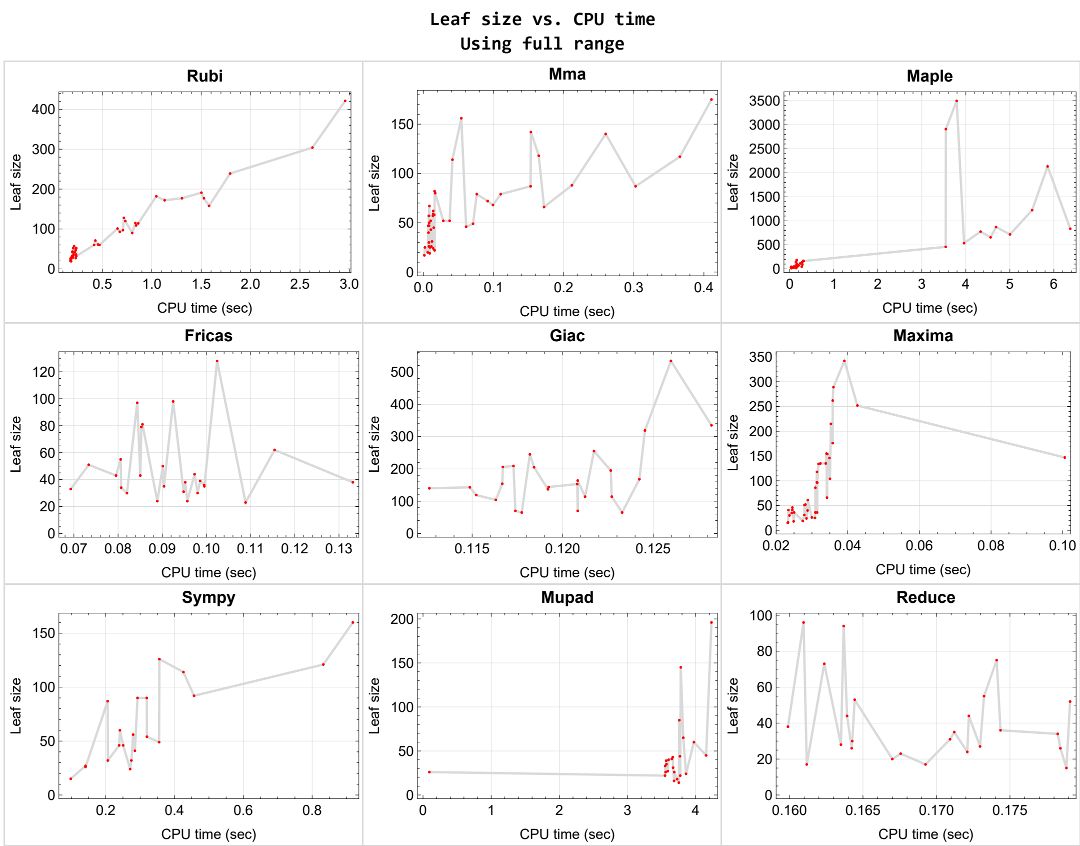


Figure 1.5: Leaf size vs. CPU time. Full range

1.9 list of integrals with no known antiderivative

$\{34, 35\}$

1.10 List of integrals solved by CAS but has no known antiderivative

Rubi $\{\}$

Mathematica $\{\}$

Maple $\{\}$

Maxima $\{\}$

Fricas $\{\}$

Sympy $\{\}$

Giac $\{\}$

Reduce $\{\}$

Mupad $\{\}$

1.11 list of integrals solved by CAS but failed verification

The following are integrals solved by CAS but the verification phase failed to verify the anti-derivative produced is correct. This does not necessarily mean that the anti-derivative is wrong as additional methods of verification might be needed, or more time is needed (3 minutes time limit was used). These integrals are listed here to make it possible to do further investigation to determine why the result could not be verified.

Rubi $\{\}$

Mathematica $\{\}$

Maple $\{18, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33\}$

Maxima Verification phase not currently implemented.

Fricas Verification phase not currently implemented.

Sympy Verification phase not currently implemented.

Giac Verification phase not currently implemented.

Reduce Verification phase not currently implemented.

Mupad Verification phase not currently implemented.

1.12 Timing

The command `AbsoluteTiming[]` was used in Mathematica to obtain the elapsed time for each integrate call. In Maple, the command `Usage` was used as in the following example

```
cpu_time := Usage(assign ('result_of_int',int(expr,x)),output='realtime'
```

For all other CAS systems, the elapsed time to complete each integral was found by taking the difference between the time after the call completed from the time before the call was made. This was done using Python's `time.time()` call.

All elapsed times shown are in seconds. A time limit of 3 CPU minutes was used for each integral. If the integrate command did not complete within this time limit, the integral was aborted and considered to have failed and assigned an F grade. The time used by failed integrals due to time out was not counted in the final statistics.

1.13 Verification

A verification phase was applied on the result of integration for `Rubi` and `Mathematica`.

Future version of this report will implement verification for the other CAS systems. For the integrals whose result was not run through a verification phase, it is assumed that the antiderivative was correct.

Verification phase also had 3 minutes time out. An integral whose result was not verified could still be correct, but further investigation is needed on those integrals. These integrals were marked in the summary table below and also in each integral separate section so they are easy to identify and locate.

1.14 Important notes about some of the results

Important note about Maxima results

Since tests were run in a batch mode, and using an automated script, then any integral where Maxima needed an interactive response from the user to answer a question during the evaluation of the integral will fail.

The exception raised is `ValueError`. Therefore Maxima results is lower than what would result if Maxima was run directly and each question was answered correctly.

The percentage of such failures were not counted for each test file, but for an example, for the `Timofeev` test file, there were about 14 such integrals out of total 705, or about 2 percent. This percentage can be higher or lower depending on the specific input test file.

Such integrals can be identified by looking at the output of the integration in each section for Maxima. The exception message will indicate the cause of error.

Maxima integrate was run using SageMath with the following settings set by default

```
'besselexpand : true'
'display2d : false'
'domain : complex'
'keepfloat : true'
'load(to_poly_solve)'
'load(simplify_sum)'
'load(abs_integrate)' 'load(diag)'
```

SageMath automatic loading of Maxima `abs_integrate` was found to cause some problems. So the following code was added to disable this effect.

```
from sage.interfaces.maxima_lib import maxima_lib
maxima_lib.set('extra_definite_integration_methods', '[]')
maxima_lib.set('extra_integration_methods', '[]')
```

See `https://ask.sagemath.org/question/43088/integrate-results-that-are-different-from-using-maxima/` for reference.

Important note about FriCAS result

There were few integrals which failed due to SageMath interface and not because FriCAS system could not do the integration.

These will fail With error `Exception raised: NotImplementedError`.

The number of such cases seems to be very small. About 1 or 2 percent of all integrals. These can be identified by looking at the exception message given in the result.

Important note about finding leaf size of antiderivative

For Mathematica, Rubi, and Maple, the builtin system function `LeafSize` was used to find the leaf size of each antiderivative.

The other CAS systems (SageMath and Sympy) do not have special builtin function for this purpose at this time. Therefore the leaf size for Fricas and Sympy antiderivative was determined using the following function, thanks to user `slelievre` at `https://ask.sagemath.org/question/57123/could-we-have-a-leaf_count-function-in-base-sagemath/`

```
def tree_size(expr):
    r"""
    Return the tree size of this expression.
    """
    if expr not in SR:
        # deal with lists, tuples, vectors
        return 1 + sum(tree_size(a) for a in expr)
    expr = SR(expr)
    x, aa = expr.operator(), expr.operands()
    if x is None:
        return 1
    else:
        return 1 + sum(tree_size(a) for a in aa)
```

For Sympy, which was called directly from Python, the following code was used to obtain the leafsize of its result

```
try:
  # 1.7 is a fudge factor since it is low side from actual leaf count
  leafCount = round(1.7*count_ops(anti))

  except Exception as ee:
        leafCount =1
```

Important note about Mupad results

Matlab's symbolic toolbox does not have a leaf count function to measure the size of the antiderivative. Maple was used to determine the leaf size of Mupad output by post processing Mupad result.

Currently no grading of the antiderivative for Mupad is implemented. If it can integrate the problem, it was assigned a B grade automatically as a placeholder. In the future, when grading function is implemented for Mupad, the tests will be rerun again.

The following is an example of using Matlab's symbolic toolbox (Mupad) to solve an integral

```
integrand = evalin(symengine,'cos(x)*sin(x)')
the_variable = evalin(symengine,'x')
anti = int(integrand,the_variable)
```

Which gives `sin(x)^2/2`

1.15 Current tree layout of integration tests

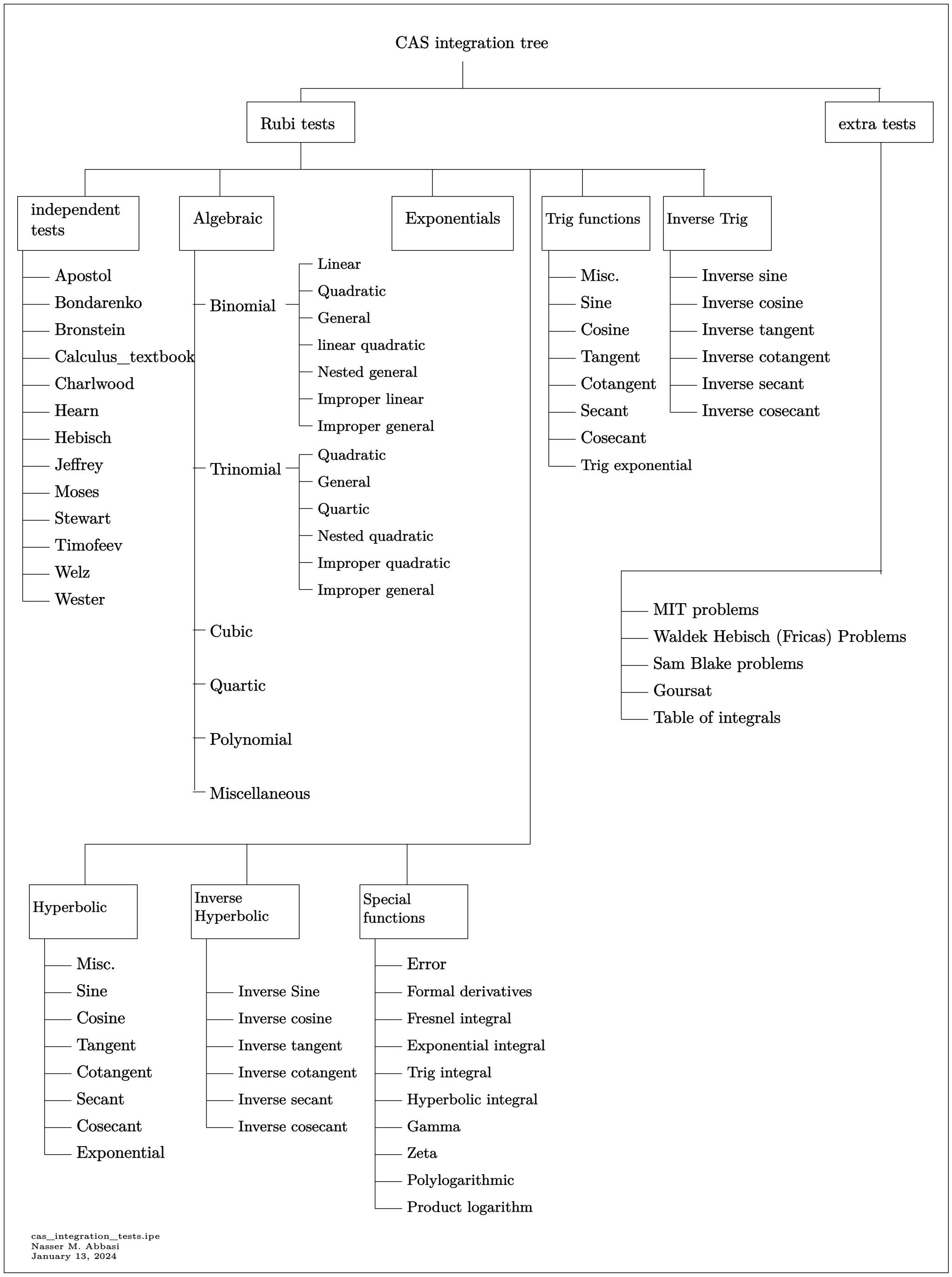
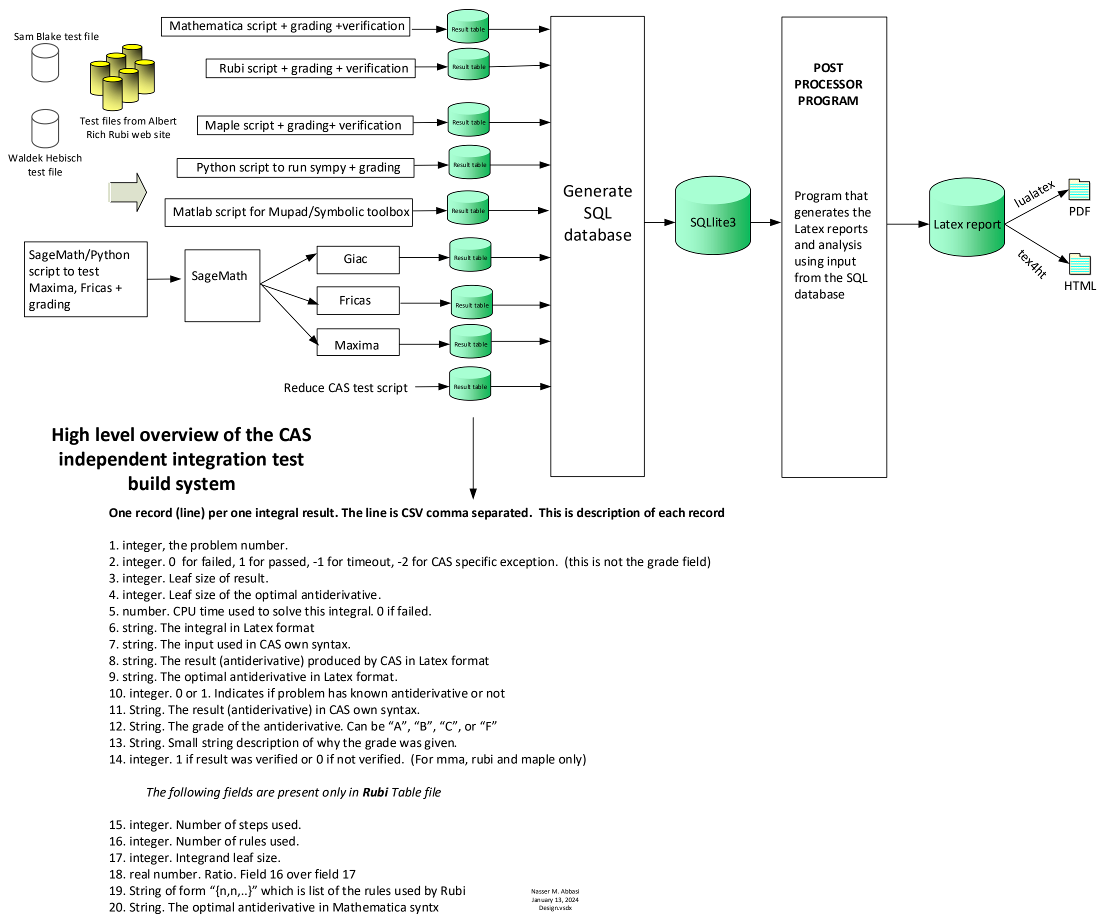


Figure 1.6: CAS integration tests tree

1.16 Design of the test system

The following diagram gives a high level view of the current test build system.



CHAPTER 2

DETAILED SUMMARY TABLES OF RESULTS

2.1 List of integrals sorted by grade for each CAS

Rubi

A grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49 }

B grade { 23 }

C grade { }

F normal fail { }

F(-1) timedout fail { }

F(-2) exception fail { }

Mma

A grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 25, 27, 28, 29, 30, 31, 32, 33, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48 }

B grade { }

C grade { 24, 26, 49 }

F normal fail { }

F(-1) timedout fail { }

F(-2) exception fail { }

Maple

A grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 20, 22, 38, 39, 40, 41, 43, 44, 45, 46, 47, 48, 49 }

B grade { 19, 21, 28, 42 }

C grade { 18, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 37 }

F normal fail { 36 }

F(-1) timedout fail { }

F(-2) exception fail { }

Fricas

A grade { 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 14, 16, 20, 22, 38, 39, 40, 41, 43, 44, 45, 46, 47, 48 }

B grade { 49 }

C grade { }

F normal fail { 7, 13, 15, 17, 18, 19, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 36, 37, 42 }

F(-1) timedout fail { }

F(-2) exception fail { }

Maxima

A grade { 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 20, 21, 23, 38, 39, 40, 41, 43, 44, 45, 46, 47, 48 }

B grade { 7, 16, 17, 19, 22, 25, 27, 31, 33, 37, 42, 49 }

C grade { }

F normal fail { 18, 24, 26, 28, 29, 30, 32, 36 }

F(-1) timedout fail { }

F(-2) exception fail { }

Giac

A grade { }

B grade { 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 14, 16, 20, 22, 38, 39, 40, 41, 43, 44, 45, 46, 47, 48 }

C grade { }

F normal fail { 7, 13, 15, 17, 18, 19, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 36, 37, 42, 49 }

F(-1) timedout fail { }

F(-2) exception fail { }

Mupad

A grade { }

B grade { 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 14, 16, 20, 22, 38, 39, 40, 41, 43, 44, 45, 47, 48 }

C grade { }

F normal fail { }

F(-1) timedout fail { 7, 13, 15, 17, 18, 19, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 36, 37, 42, 46, 49 }

F(-2) exception fail { }

Sympy

A grade { 8, 9, 10, 11, 12, 14, 16, 20, 22, 38 }

B grade { 43, 44, 45, 47, 48 }

C grade { 1, 2, 3, 4, 5, 6 }

F normal fail { 7, 13, 15, 17, 18, 19, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 36, 39, 40, 41, 42, 46, 49 }

F(-1) timedout fail { 37 }

F(-2) exception fail { }

Reduce

A grade { }

B grade { 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 14, 16, 20, 22, 38, 39, 40, 41, 43, 44, 45, 46, 47, 48 }

C grade { }

F normal fail { 7, 13, 15, 17, 18, 19, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 36, 37, 42, 49 }

F(-1) timedout fail { }

F(-2) exception fail { }

2.2 Detailed conclusion table per each integral for all CAS systems

Detailed conclusion table per each integral is given by the table below. The elapsed time is in seconds. For failed result it is given as **F(-1)** if the failure was due to timeout. It is given as **F(-2)** if the failure was due to an exception being raised, which could indicate a bug in the system. If the failure was due to integral not being evaluated within the time limit, then it is given as **F**.

In this table, the column `N.S.` means **normalized size** and is defined as $\frac{\text{antiderivative leaf size}}{\text{optimal antiderivative leaf size}}$. To make the table fit the page, the name `Mathematica` was abbreviated to `MMA`.

Problem 1	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	C	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	51	52	67	45	61	51	49	245	44	41
N.S.	1	1.02	1.31	0.88	1.20	1.00	0.96	4.80	0.86	0.80
time (sec)	N/A	0.236	0.008	0.152	0.029	0.073	0.355	0.118	0.164	3.651

Problem 2	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	C	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	50	52	50	47	46	55	54	255	52	43
N.S.	1	1.04	1.00	0.94	0.92	1.10	1.08	5.10	1.04	0.86
time (sec)	N/A	0.233	0.008	0.108	0.025	0.081	0.320	0.122	0.179	3.664

Problem 3	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	C	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	41	42	57	37	52	43	41	195	36	33
N.S.	1	1.02	1.39	0.90	1.27	1.05	1.00	4.76	0.88	0.80
time (sec)	N/A	0.221	0.008	0.095	0.028	0.085	0.285	0.123	0.174	3.553

Problem 4	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	C	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	40	42	40	38	35	44	46	206	44	35
N.S.	1	1.05	1.00	0.95	0.88	1.10	1.15	5.15	1.10	0.88
time (sec)	N/A	0.218	0.007	0.074	0.024	0.097	0.239	0.117	0.172	3.572

Problem 5	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	C	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	31	32	47	27	41	34	32	144	27	26
N.S.	1	1.03	1.52	0.87	1.32	1.10	1.03	4.65	0.87	0.84
time (sec)	N/A	0.192	0.007	0.088	0.023	0.081	0.206	0.119	0.173	3.560

Problem 6	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	C	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	25	25	25	23	25	33	27	153	28	22
N.S.	1	1.00	1.00	0.92	1.00	1.32	1.08	6.12	1.12	0.88
time (sec)	N/A	0.177	0.002	0.059	0.031	0.069	0.141	0.121	0.164	3.550

Problem 7	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	B	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	28	28	26	28	86	0	0	0	10	0
N.S.	1	1.00	0.93	1.00	3.07	0.00	0.00	0.00	0.36	0.00
time (sec)	N/A	0.183	0.008	0.089	0.031	0.000	0.000	0.000	0.159	0.000

Problem 8	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	A	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	30	33	30	35	30	39	26	143	38	27
N.S.	1	1.10	1.00	1.17	1.00	1.30	0.87	4.77	1.27	0.90
time (sec)	N/A	0.198	0.008	0.076	0.024	0.099	0.141	0.115	0.160	3.591

Problem 9	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	A	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	31	29	47	26	36	35	24	140	26	40
N.S.	1	0.94	1.52	0.84	1.16	1.13	0.77	4.52	0.84	1.29
time (sec)	N/A	0.193	0.007	0.090	0.031	0.090	0.271	0.113	0.164	3.602

Problem 10	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	A	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	47	47	47	48	40	50	46	209	55	39
N.S.	1	1.00	1.00	1.02	0.85	1.06	0.98	4.45	1.17	0.83
time (sec)	N/A	0.229	0.008	0.091	0.029	0.090	0.251	0.117	0.173	3.567

Problem 11	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	A	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	41	41	57	36	51	43	32	205	35	60
N.S.	1	1.00	1.39	0.88	1.24	1.05	0.78	5.00	0.85	1.46
time (sec)	N/A	0.205	0.008	0.110	0.028	0.080	0.275	0.118	0.171	3.974

Problem 12	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	A	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	105	177	80	92	135	98	114	534	94	85
N.S.	1	1.69	0.76	0.88	1.29	0.93	1.09	5.09	0.90	0.81
time (sec)	N/A	1.306	0.017	0.286	0.032	0.092	0.426	0.126	0.164	3.760

Problem 13	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	127	172	87	164	155	0	0	0	82	0
N.S.	1	1.35	0.69	1.29	1.22	0.00	0.00	0.00	0.65	0.00
time (sec)	N/A	1.130	0.303	0.316	0.034	0.000	0.000	0.000	0.173	0.000

Problem 14	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	A	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	81	115	62	72	118	81	90	335	73	65
N.S.	1	1.42	0.77	0.89	1.46	1.00	1.11	4.14	0.90	0.80
time (sec)	N/A	0.834	0.014	0.201	0.031	0.086	0.319	0.128	0.162	3.818

Problem 15	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	103	120	66	144	134	0	0	0	58	0
N.S.	1	1.17	0.64	1.40	1.30	0.00	0.00	0.00	0.56	0.00
time (sec)	N/A	0.732	0.172	0.284	0.032	0.000	0.000	0.000	0.163	0.000

Problem 16	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	B	A	A	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	54	61	43	49	97	62	60	154	53	44
N.S.	1	1.13	0.80	0.91	1.80	1.15	1.11	2.85	0.98	0.81
time (sec)	N/A	0.456	0.011	0.172	0.031	0.115	0.241	0.117	0.164	3.769

Problem 17	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	B	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	58	71	46	116	135	0	0	0	8	0
N.S.	1	1.22	0.79	2.00	2.33	0.00	0.00	0.00	0.14	0.00
time (sec)	N/A	0.429	0.061	0.240	0.034	0.000	0.000	0.000	0.169	0.000

Problem 18	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	F	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD	TBD
size	97	128	114	459	0	0	0	0	12	0
N.S.	1	1.32	1.18	4.73	0.00	0.00	0.00	0.00	0.12	0.00
time (sec)	N/A	0.717	0.041	3.542	0.000	0.000	0.000	0.000	0.176	0.000

Problem 19	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	B	B	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	55	60	49	145	146	0	0	0	37	0
N.S.	1	1.09	0.89	2.64	2.65	0.00	0.00	0.00	0.67	0.00
time (sec)	N/A	0.416	0.071	0.141	0.035	0.000	0.000	0.000	0.180	0.000

Problem 20	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	A	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	61	60	57	72	96	79	56	137	75	145
N.S.	1	0.98	0.93	1.18	1.57	1.30	0.92	2.25	1.23	2.38
time (sec)	N/A	0.471	0.013	0.146	0.032	0.085	0.280	0.119	0.174	3.781

Problem 21	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	B	A	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	103	97	87	184	176	0	0	0	69	0
N.S.	1	0.94	0.84	1.79	1.71	0.00	0.00	0.00	0.67	0.00
time (sec)	N/A	0.708	0.153	0.157	0.036	0.000	0.000	0.000	0.172	0.000

Problem 22	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	B	A	A	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	90	114	82	101	154	97	90	319	96	196
N.S.	1	1.27	0.91	1.12	1.71	1.08	1.00	3.54	1.07	2.18
time (sec)	N/A	0.863	0.016	0.158	0.034	0.084	0.292	0.125	0.161	4.232

Problem 23	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	B	A	C	A	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD	TBD
size	186	421	117	2133	289	0	0	0	132	0
N.S.	1	2.26	0.63	11.47	1.55	0.00	0.00	0.00	0.71	0.00
time (sec)	N/A	2.957	0.366	5.857	0.036	0.000	0.000	0.000	0.156	0.000

Problem 24	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	C	C	F	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD	TBD
size	196	304	175	1223	0	0	0	0	123	0
N.S.	1	1.55	0.89	6.24	0.00	0.00	0.00	0.00	0.63	0.00
time (sec)	N/A	2.627	0.411	5.506	0.000	0.000	0.000	0.000	0.157	0.000

Problem 25	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	B	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD	TBD
size	139	239	88	871	262	0	0	0	96	0
N.S.	1	1.72	0.63	6.27	1.88	0.00	0.00	0.00	0.69	0.00
time (sec)	N/A	1.793	0.212	4.686	0.036	0.000	0.000	0.000	0.158	0.000

Problem 26	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	C	C	F	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD	TBD
size	149	177	140	776	0	0	0	0	89	0
N.S.	1	1.19	0.94	5.21	0.00	0.00	0.00	0.00	0.60	0.00
time (sec)	N/A	1.528	0.260	4.335	0.000	0.000	0.000	0.000	0.161	0.000

Problem 27	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	B	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD	TBD
size	95	109	68	2910	215	0	0	0	85	0
N.S.	1	1.15	0.72	30.63	2.26	0.00	0.00	0.00	0.89	0.00
time (sec)	N/A	0.841	0.099	3.546	0.035	0.000	0.000	0.000	0.158	0.000

Problem 28	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	B	F	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	85	101	79	168	0	0	0	0	8	0
N.S.	1	1.19	0.93	1.98	0.00	0.00	0.00	0.00	0.09	0.00
time (sec)	N/A	0.653	0.076	0.308	0.000	0.000	0.000	0.000	0.163	0.000

Problem 29	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	F	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD	TBD
size	150	182	156	536	0	0	0	0	12	0
N.S.	1	1.21	1.04	3.57	0.00	0.00	0.00	0.00	0.08	0.00
time (sec)	N/A	1.046	0.054	3.959	0.000	0.000	0.000	0.000	0.160	0.000

Problem 30	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	F	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD	TBD
size	79	93	72	718	0	0	0	0	39	0
N.S.	1	1.18	0.91	9.09	0.00	0.00	0.00	0.00	0.49	0.00
time (sec)	N/A	0.676	0.091	5.003	0.000	0.000	0.000	0.000	0.157	0.000

Problem 31	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	B	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD	TBD
size	95	90	79	3498	252	0	0	0	86	0
N.S.	1	0.95	0.83	36.82	2.65	0.00	0.00	0.00	0.91	0.00
time (sec)	N/A	0.802	0.110	3.792	0.043	0.000	0.000	0.000	0.157	0.000

Problem 32	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	F	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD	TBD
size	154	158	142	836	0	0	0	0	120	0
N.S.	1	1.03	0.92	5.43	0.00	0.00	0.00	0.00	0.78	0.00
time (sec)	N/A	1.580	0.153	6.372	0.000	0.000	0.000	0.000	0.154	0.000

Problem 33	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	B	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	No	TBD	TBD	TBD	TBD	TBD	TBD
size	141	191	118	657	342	0	0	0	108	0
N.S.	1	1.35	0.84	4.66	2.43	0.00	0.00	0.00	0.77	0.00
time (sec)	N/A	1.502	0.164	4.562	0.039	0.000	0.000	0.000	0.160	0.000

Problem 34	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
verified	N/A	N/A	N/A	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	10	10	12	10	124	12	10	12	12	12
N.S.	1	1.00	1.20	1.00	12.40	1.20	1.00	1.20	1.20	1.20
time (sec)	N/A	0.184	0.720	0.285	1.102	0.089	2.331	0.139	0.166	3.525

Problem 35	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
verified	N/A	N/A	N/A	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	10	10	12	10	93	12	10	12	12	12
N.S.	1	1.00	1.20	1.00	9.30	1.20	1.00	1.20	1.20	1.20
time (sec)	N/A	0.186	0.731	0.193	0.728	0.098	1.220	0.140	0.161	3.662

Problem 36	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	F	F	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	N/A	TBD	TBD	TBD	TBD	TBD	TBD
size	57	57	52	0	0	0	0	0	99	0
N.S.	1	1.00	0.91	0.00	0.00	0.00	0.00	0.00	1.74	0.00
time (sec)	N/A	0.213	0.028	0.000	0.000	0.000	0.000	0.000	0.166	0.000

Problem 37	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	C	B	F	F(-1)	F	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	28	32	26	95	104	0	0	0	12	0
N.S.	1	1.14	0.93	3.39	3.71	0.00	0.00	0.00	0.43	0.00
time (sec)	N/A	0.236	0.011	0.170	0.035	0.000	0.000	0.000	0.163	0.000

Problem 38	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	A	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	19	19	17	16	15	23	15	104	17	26
N.S.	1	1.00	0.89	0.84	0.79	1.21	0.79	5.47	0.89	1.37
time (sec)	N/A	0.183	0.001	0.145	0.023	0.109	0.098	0.116	0.161	0.093

Problem 39	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	51	52	59	42	41	38	0	164	30	31
N.S.	1	1.02	1.16	0.82	0.80	0.75	0.00	3.22	0.59	0.61
time (sec)	N/A	0.204	0.014	0.043	0.025	0.095	0.000	0.121	0.164	3.668

Problem 40	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	42	43	52	37	36	31	0	114	23	26
N.S.	1	1.02	1.24	0.88	0.86	0.74	0.00	2.71	0.55	0.62
time (sec)	N/A	0.194	0.010	0.031	0.031	0.095	0.000	0.123	0.168	3.686

Problem 41	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	22	29	22	27	26	24	0	65	15	16
N.S.	1	1.32	1.00	1.23	1.18	1.09	0.00	2.95	0.68	0.73
time (sec)	N/A	0.189	0.016	0.029	0.030	0.089	0.000	0.118	0.179	3.683

Problem 42	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	B	B	F	F	F	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	19	27	19	33	66	0	0	0	9	0
N.S.	1	1.42	1.00	1.74	3.47	0.00	0.00	0.00	0.47	0.00
time (sec)	N/A	0.230	0.009	0.142	0.034	0.000	0.000	0.000	0.185	0.000

Problem 43	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	B	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	25	32	45	32	31	30	92	65	17	18
N.S.	1	1.28	1.80	1.28	1.24	1.20	3.68	2.60	0.68	0.72
time (sec)	N/A	0.202	0.015	0.042	0.028	0.098	0.457	0.123	0.169	3.728

Problem 44	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	B	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	42	43	58	37	36	38	160	114	26	45
N.S.	1	1.02	1.38	0.88	0.86	0.90	3.81	2.71	0.62	1.07
time (sec)	N/A	0.200	0.015	0.042	0.025	0.133	0.918	0.121	0.178	4.153

Problem 45	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	B	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	38	35	31	35	24	35	121	168	31	24
N.S.	1	0.92	0.82	0.92	0.63	0.92	3.18	4.42	0.82	0.63
time (sec)	N/A	0.208	0.012	0.036	0.028	0.100	0.833	0.124	0.171	3.859

Problem 46	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	F	B	B	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	31	28	25	30	19	30	0	119	24	0
N.S.	1	0.90	0.81	0.97	0.61	0.97	0.00	3.84	0.77	0.00
time (sec)	N/A	0.200	0.009	0.033	0.027	0.082	0.000	0.115	0.172	0.000

Problem 47	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	B	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	20	20	20	15	16	24	87	70	20	14
N.S.	1	1.00	1.00	0.75	0.80	1.20	4.35	3.50	1.00	0.70
time (sec)	N/A	0.176	0.006	0.033	0.023	0.096	0.206	0.117	0.167	3.753

Problem 48	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	A	A	A	A	B	B	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	24	24	24	29	18	36	126	70	34	22
N.S.	1	1.00	1.00	1.21	0.75	1.50	5.25	2.92	1.42	0.92
time (sec)	N/A	0.182	0.014	0.034	0.025	0.099	0.356	0.121	0.178	3.774

Problem 49	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Reduce	Mupad
grade	N/A	A	C	A	B	B	F	F	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD	TBD
size	38	36	52	45	147	128	0	0	12	0
N.S.	1	0.95	1.37	1.18	3.87	3.37	0.00	0.00	0.32	0.00
time (sec)	N/A	0.233	0.037	0.272	0.101	0.102	0.000	0.000	0.179	0.000

2.3 Detailed conclusion table specific for Rubi results

The following table is specific to Rubi only. It gives additional statistics for each integral. the column **steps** is the number of steps used by Rubi to obtain the antiderivative. The **rules** column is the number of unique rules used. The **integrand size** column is the leaf size of the integrand. Finally the ratio $\frac{\text{number of rules}}{\text{integrand size}}$ is also given. The larger this ratio is, the harder the integral is to solve. In this test file, problem number [23] had the largest ratio of [1.6999999999999996]

Table 2.1: Rubi specific breakdown of results for each integral

#	grade	number of steps used	number of unique rules	normalized antiderivative leaf size	integrand leaf size	$\frac{\text{number of rules}}{\text{integrand leaf size}}$
1	A	3	3	1.02	8	0.375
2	A	5	4	1.04	8	0.500
3	A	3	3	1.02	8	0.375
4	A	5	4	1.05	8	0.500
5	A	3	3	1.03	6	0.500
6	A	2	2	1.00	4	0.500
7	A	1	1	1.00	8	0.125
8	A	6	5	1.10	8	0.625
9	A	3	3	0.94	8	0.375
10	A	5	4	1.00	8	0.500
11	A	4	4	1.00	8	0.500
12	A	16	15	1.69	10	1.500
13	A	14	13	1.35	10	1.300
14	A	11	10	1.42	10	1.000
15	A	10	9	1.17	10	0.900
16	A	5	5	1.13	8	0.625
17	A	6	5	1.22	6	0.833
18	A	4	4	1.32	10	0.400
19	A	4	4	1.09	10	0.400
20	A	9	8	0.98	10	0.800
21	A	8	8	0.94	10	0.800

Continued on next page

Table 2.1 – continued from previous page

#	grade	number of steps used	number of unique rules	normalized antiderivative leaf size	integrand leaf size	$\frac{\text{number of rules}}{\text{integrand leaf size}}$
22	A	14	13	1.27	10	1.300
23	B	18	17	2.26	10	1.700
24	A	17	16	1.55	10	1.600
25	A	14	13	1.72	10	1.300
26	A	11	11	1.19	10	1.100
27	A	9	8	1.15	8	1.000
28	A	5	5	1.19	6	0.833
29	A	5	5	1.21	10	0.500
30	A	5	5	1.18	10	0.500
31	A	7	7	0.95	10	0.700
32	A	15	14	1.03	10	1.400
33	A	11	11	1.35	10	1.100
34	N/A	1	0	1.00	10	0.000
35	N/A	1	0	1.00	10	0.000
36	A	2	2	1.00	8	0.250
37	A	3	2	1.14	10	0.200
38	A	3	3	1.00	4	0.750
39	A	7	6	1.02	10	0.600
40	A	6	5	1.02	8	0.625
41	A	5	4	1.32	6	0.667
42	A	3	2	1.42	10	0.200
43	A	5	4	1.28	10	0.400
44	A	6	5	1.02	10	0.500
45	A	3	3	0.92	12	0.250
46	A	3	3	0.90	12	0.250
47	A	2	2	1.00	12	0.167
48	A	4	4	1.00	12	0.333
49	A	3	2	0.95	10	0.200

CHAPTER 3

Listing of integrals

3.1 $\int x^5 \coth^{-1}(ax)\,dx$

Optimal result

Integrand size = 8, antiderivative size = 51

$$\int x^5 \coth^{-1}(ax)\,dx = \frac{x}{6a^5} + \frac{x^3}{18a^3} + \frac{x^5}{30a} + \frac{1}{6}x^6 \coth^{-1}(ax) - \frac{\operatorname{arctanh}(ax)}{6a^6}$$

output
```
1/6*x/a^5+1/18*x^3/a^3+1/30*x^5/a+1/6*x^6*arccoth(a*x)-1/6*arctanh(a*x)/a^
6
```

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 67, normalized size of antiderivative = 1.31

$$\int x^5 \coth^{-1}(ax)\,dx = \frac{x}{6a^5} + \frac{x^3}{18a^3} + \frac{x^5}{30a} + \frac{1}{6}x^6 \coth^{-1}(ax) + \frac{\log(1-ax)}{12a^6} - \frac{\log(1+ax)}{12a^6}$$

input
```
Integrate[x^5*ArcCoth[a*x],x]
```

output
```
x/(6*a^5) + x^3/(18*a^3) + x^5/(30*a) + (x^6*ArcCoth[a*x])/6 + Log[1 - a*x
]/(12*a^6) - Log[1 + a*x]/(12*a^6)
```

Rubi [A] (verified)

Time = 0.24 (sec) , antiderivative size = 52, normalized size of antiderivative = 1.02, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.375$, Rules used = {6453, 254, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x^5 \coth^{-1}(ax) \, dx$$

$$\Big\downarrow \text{6453}$$

$$\frac{1}{6}x^6 \coth^{-1}(ax) - \frac{1}{6}a \int \frac{x^6}{1-a^2x^2}dx$$

$$\Big\downarrow \text{254}$$

$$\frac{1}{6}x^6 \coth^{-1}(ax) - \frac{1}{6}a \int \left(-\frac{x^4}{a^2} - \frac{x^2}{a^4} + \frac{1}{a^6\,(1-a^2x^2)} - \frac{1}{a^6}\right) dx$$

$$\Big\downarrow \text{2009}$$

$$\frac{1}{6}x^6 \coth^{-1}(ax) - \frac{1}{6}a\left(\frac{\operatorname{arctanh}(ax)}{a^7} - \frac{x}{a^6} - \frac{x^3}{3a^4} - \frac{x^5}{5a^2}\right)$$

input
```
Int[x^5*ArcCoth[a*x],x]
```

output
```
(x^6*ArcCoth[a*x])/6 - (a*(-(x/a^6) - x^3/(3*a^4) - x^5/(5*a^2) + ArcTanh[
a*x]/a^7))/6
```

Defintions of rubi rules used

rule 254
```
Int[(x_)^(m_)/((a_) + (b_.)*(x_)^2), x_Symbol] :> Int[PolynomialDivide[x^m,
 a + b*x^2, x], x] /; FreeQ[{a, b}, x] && IGtQ[m, 3]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

Maple [A] (verified)

Time = 0.15 (sec) , antiderivative size = 45, normalized size of antiderivative = 0.88

method	result	size
parallelrisch	$-\frac{-15x^6a^6\,\mathrm{arccoth}(xa)-3x^5a^5-5x^3a^3-15xa+15\,\mathrm{arccoth}(xa)}{90a^6}$	45
derivativedivides	$\frac{\frac{x^6a^6\,\mathrm{arccoth}(xa)}{6}+\frac{x^5a^5}{30}+\frac{x^3a^3}{18}+\frac{xa}{6}+\frac{\ln(xa-1)}{12}-\frac{\ln(xa+1)}{12}}{a^6}$	54
default	$\frac{\frac{x^6a^6\,\mathrm{arccoth}(xa)}{6}+\frac{x^5a^5}{30}+\frac{x^3a^3}{18}+\frac{xa}{6}+\frac{\ln(xa-1)}{12}-\frac{\ln(xa+1)}{12}}{a^6}$	54
parts	$\frac{x^6\,\mathrm{arccoth}(xa)}{6}+\frac{a\left(\frac{\frac{1}{5}a^4x^5+\frac{1}{3}a^2x^3+x}{a^6}-\frac{\ln(xa+1)}{2a^7}+\frac{\ln(xa-1)}{2a^7}\right)}{6}$	59
risch	$\frac{x^6\ln(xa+1)}{12}-\frac{x^6\ln(xa-1)}{12}+\frac{x^5}{30a}+\frac{x^3}{18a^3}+\frac{x}{6a^5}+\frac{\ln(-xa+1)}{12a^6}-\frac{\ln(xa+1)}{12a^6}$	69
orering	$\frac{(3a^6x^6+a^4x^4+5a^2x^2-9)\,\mathrm{arccoth}(xa)}{9a^6}-\frac{(3a^4x^4+5a^2x^2+15)(xa+1)(xa-1)\left(5x^4\,\mathrm{arccoth}(xa)-\frac{x^5a}{a^2x^2-1}\right)}{90x^4a^6}$	99

input
```
int(x^5*arccoth(x*a),x,method=_RETURNVERBOSE)
```

output
```
-1/90*(-15*x^6*a^6*arccoth(x*a)-3*x^5*a^5-5*x^3*a^3-15*x*a+15*arccoth(x*a)
)/a^6
```

Fricas [A] (verification not implemented)

Time = 0.07 (sec) , antiderivative size = 51, normalized size of antiderivative = 1.00

$$\int x^5\coth^{-1}(ax)\,dx=\frac{6\,a^5x^5+10\,a^3x^3+30\,ax+15\,(a^6x^6-1)\log\left(\frac{ax+1}{ax-1}\right)}{180\,a^6}$$

input
```
integrate(x^5*arccoth(a*x),x, algorithm="fricas")
```

output
```
1/180*(6*a^5*x^5 + 10*a^3*x^3 + 30*a*x + 15*(a^6*x^6 - 1)*log((a*x + 1)/(a
*x - 1)))/a^6
```

Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 0.36 (sec) , antiderivative size = 49, normalized size of antiderivative = 0.96

$$\int x^5 \coth^{-1}(ax)\,dx = \begin{cases} \frac{x^6 \operatorname{acoth}(ax)}{6} + \frac{x^5}{30a} + \frac{x^3}{18a^3} + \frac{x}{6a^5} - \frac{\operatorname{acoth}(ax)}{6a^6} & \text{for } a \neq 0 \\ \frac{i\pi x^6}{12} & \text{otherwise} \end{cases}$$

input
```
integrate(x**5*acoth(a*x),x)
```

output
```
Piecewise((x**6*acoth(a*x)/6 + x**5/(30*a) + x**3/(18*a**3) + x/(6*a**5) -
 acoth(a*x)/(6*a**6), Ne(a, 0)), (I*pi*x**6/12, True))
```

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 61, normalized size of antiderivative = 1.20

$$\int x^5 \coth^{-1}(ax)\,dx = \frac{1}{6}\,x^6 \operatorname{arcoth}(ax) + \frac{1}{180}\,a\left(\frac{2\,(3\,a^4x^5 + 5\,a^2x^3 + 15\,x)}{a^6} - \frac{15\,\log(ax+1)}{a^7} + \frac{15\,\log(ax-1)}{a^7}\right)$$

input
```
integrate(x^5*arccoth(a*x),x, algorithm="maxima")
```

output
```
1/6*x^6*arccoth(a*x) + 1/180*a*(2*(3*a^4*x^5 + 5*a^2*x^3 + 15*x)/a^6 - 15*
log(a*x + 1)/a^7 + 15*log(a*x - 1)/a^7)
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 245 vs. $2(41) = 82$.

Time = 0.12 (sec) , antiderivative size = 245, normalized size of antiderivative = 4.80

$$\int x^5 \coth^{-1}(ax)\,dx$$

$$= \frac{1}{45}a\left(\frac{\frac{45(ax+1)^4}{(ax-1)^4} - \frac{90(ax+1)^3}{(ax-1)^3} + \frac{140(ax+1)^2}{(ax-1)^2} - \frac{70(ax+1)}{ax-1} + 23}{a^7\left(\frac{ax+1}{ax-1}-1\right)^5} + \frac{15\left(\frac{3(ax+1)^5}{(ax-1)^5} + \frac{10(ax+1)^3}{(ax-1)^3} + \frac{3(ax+1)}{ax-1}\right)\log\left(-\right.}{a^7\left(\frac{ax+1}{ax-1}-1\right)^6}\right.$$

input
```
integrate(x^5*arccoth(a*x),x, algorithm="giac")
```

output
```
1/45*a*((45*(a*x + 1)^4/(a*x - 1)^4 - 90*(a*x + 1)^3/(a*x - 1)^3 + 140*(a*
x + 1)^2/(a*x - 1)^2 - 70*(a*x + 1)/(a*x - 1) + 23)/(a^7*((a*x + 1)/(a*x -
 1) - 1)^5) + 15*(3*(a*x + 1)^5/(a*x - 1)^5 + 10*(a*x + 1)^3/(a*x - 1)^3 +
 3*(a*x + 1)/(a*x - 1))*log(-(((a*x + 1)*a/(a*x - 1) - a)/(a*((a*x + 1)/(a
*x - 1) + 1)) + 1)/(((a*x + 1)*a/(a*x - 1) - a)/(a*((a*x + 1)/(a*x - 1) +
1)) - 1))/(a^7*((a*x + 1)/(a*x - 1) - 1)^6))
```

Mupad [B] (verification not implemented)

Time = 3.65 (sec) , antiderivative size = 41, normalized size of antiderivative = 0.80

$$\int x^5 \coth^{-1}(ax)\,dx = \frac{\frac{a\,x}{6} - \frac{\mathrm{acoth}(a\,x)}{6} + \frac{a^3\,x^3}{18} + \frac{a^5\,x^5}{30}}{a^6} + \frac{x^6\,\mathrm{acoth}(a\,x)}{6}$$

input
```
int(x^5*acoth(a*x),x)
```

output
```
((a*x)/6 - acoth(a*x)/6 + (a^3*x^3)/18 + (a^5*x^5)/30)/a^6 + (x^6*acoth(a*
x))/6
```

Reduce [B] (verification not implemented)

Time = 0.16 (sec) , antiderivative size = 44, normalized size of antiderivative = 0.86

$$\int x^5 \coth^{-1}(ax)\,dx = \frac{15acoth(ax)\,a^6x^6 - 15acoth(ax) - 3a^5x^5 - 5a^3x^3 - 15ax}{90a^6}$$

input

```
int(x^5*acoth(a*x),x)
```

output

```
(15*acoth(a*x)*a**6*x**6 - 15*acoth(a*x) - 3*a**5*x**5 - 5*a**3*x**3 - 15*
a*x)/(90*a**6)
```

3.2 $\int x^4 \coth^{-1}(ax)\,dx$

Optimal result

Integrand size = 8, antiderivative size = 50

$$\int x^4 \coth^{-1}(ax)\,dx = \frac{x^2}{10a^3} + \frac{x^4}{20a} + \frac{1}{5}x^5 \coth^{-1}(ax) + \frac{\log(1-a^2x^2)}{10a^5}$$

output
```
1/10*x^2/a^3+1/20*x^4/a+1/5*x^5*arccoth(a*x)+1/10*ln(-a^2*x^2+1)/a^5
```

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 50, normalized size of antiderivative = 1.00

$$\int x^4 \coth^{-1}(ax)\,dx = \frac{x^2}{10a^3} + \frac{x^4}{20a} + \frac{1}{5}x^5 \coth^{-1}(ax) + \frac{\log(1-a^2x^2)}{10a^5}$$

input
```
Integrate[x^4*ArcCoth[a*x],x]
```

output
```
x^2/(10*a^3) + x^4/(20*a) + (x^5*ArcCoth[a*x])/5 + Log[1 - a^2*x^2]/(10*a^
5)
```

Rubi [A] (verified)

Time = 0.23 (sec) , antiderivative size = 52, normalized size of antiderivative = 1.04, number of steps used = 5, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.500, Rules used = {6453, 243, 49, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x^4 \coth^{-1}(ax)\,dx$$

$$\Big\downarrow \ 6453$$

$$\frac{1}{5}x^5\coth^{-1}(ax)-\frac{1}{5}a\int \frac{x^5}{1-a^2x^2}dx$$

$$\Big\downarrow \ 243$$

$$\frac{1}{5}x^5\coth^{-1}(ax)-\frac{1}{10}a\int \frac{x^4}{1-a^2x^2}dx^2$$

$$\Big\downarrow \ 49$$

$$\frac{1}{5}x^5\coth^{-1}(ax)-\frac{1}{10}a\int \left(-\frac{x^2}{a^2}-\frac{1}{a^4\left(a^2x^2-1\right)}-\frac{1}{a^4}\right)dx^2$$

$$\Big\downarrow \ 2009$$

$$\frac{1}{5}x^5\coth^{-1}(ax)-\frac{1}{10}a\left(-\frac{x^2}{a^4}-\frac{x^4}{2a^2}-\frac{\log\left(1-a^2x^2\right)}{a^6}\right)$$

input
```
Int[x^4*ArcCoth[a*x],x]
```

output
```
(x^5*ArcCoth[a*x])/5 - (a*(-(x^2/a^4) - x^4/(2*a^2) - Log[1 - a^2*x^2]/a^6
))/10
```

Defintions of rubi rules used

rule 49
```
Int[((a_.) + (b_.)*(x_))^(m_.)*((c_.) + (d_.)*(x_))^(n_.), x_Symbol] :> Int
[ExpandIntegrand[(a + b*x)^m*(c + d*x)^n, x], x] /; FreeQ[{a, b, c, d}, x]
&& IGtQ[m, 0] && IGtQ[m + n + 2, 0]
```

rule 243
```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] :> Simp[1/2   Subst[In
t[x^((m - 1)/2)*(a + b*x)^p, x], x, x^2], x] /; FreeQ[{a, b, m, p}, x] && I
ntegerQ[(m - 1)/2]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

Maple [A] (verified)

Time = 0.11 (sec) , antiderivative size = 47, normalized size of antiderivative = 0.94

method	result	size
parts	$\frac{x^{5} \operatorname{arccoth}\left(x a \right)}{5}+\frac{a \left(\frac{\ln \left(a^{2} x^{2}-1\right)}{2 a^{6}}+\frac{\frac{1}{2} a^{2} x^{4}+x^{2}}{2 a^{4}}\right)}{5}$	47
derivativedivides	$\frac{\frac{x^{5} a^{5} \operatorname{arccoth}\left(x a \right)}{5}+\frac{a^{4} x^{4}}{20}+\frac{a^{2} x^{2}}{10}+\frac{\ln \left(x a -1\right)}{10}+\frac{\ln \left(x a +1\right)}{10}}{a^{5}}$	50
default	$\frac{\frac{x^{5} a^{5} \operatorname{arccoth}\left(x a \right)}{5}+\frac{a^{4} x^{4}}{20}+\frac{a^{2} x^{2}}{10}+\frac{\ln \left(x a -1\right)}{10}+\frac{\ln \left(x a +1\right)}{10}}{a^{5}}$	50
parallelrisch	$-\frac{-4 x^{5} a^{5} \operatorname{arccoth}\left(x a \right)-a^{4} x^{4}-2-2 a^{2} x^{2}-4 \ln \left(x a -1\right)-4 \operatorname{arccoth}\left(x a \right)}{20 a^{5}}$	50
risch	$\frac{x^{5} \ln \left(x a +1\right)}{10}-\frac{x^{5} \ln \left(x a -1\right)}{10}+\frac{x^{4}}{20 a}+\frac{x^{2}}{10 a^{3}}+\frac{\ln \left(a^{2} x^{2}-1\right)}{10 a^{5}}+\frac{1}{20 a^{5}}$	60

input
```
int(x^4*arccoth(x*a),x,method=_RETURNVERBOSE)
```

output
```
1/5*x^5*arccoth(x*a)+1/5*a*(1/2/a^6*ln(a^2*x^2-1)+1/2/a^4*(1/2*a^2*x^4+x^2
))
```

Fricas [A] (verification not implemented)

Time = 0.08 (sec) , antiderivative size = 55, normalized size of antiderivative = 1.10

$$\int x^4 \coth^{-1}(ax)\,dx = \frac{2\,a^5x^5 \log\left(\frac{ax+1}{ax-1}\right) + a^4x^4 + 2\,a^2x^2 + 2\,\log\left(a^2x^2-1\right)}{20\,a^5}$$

input
```
integrate(x^4*arccoth(a*x),x, algorithm="fricas")
```

output
```
1/20*(2*a^5*x^5*log((a*x + 1)/(a*x - 1)) + a^4*x^4 + 2*a^2*x^2 + 2*log(a^2
*x^2 - 1))/a^5
```

Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 0.32 (sec) , antiderivative size = 54, normalized size of antiderivative = 1.08

$$\int x^4 \coth^{-1}(ax)\,dx = \begin{cases} \frac{x^5 \operatorname{acoth}(ax)}{5} + \frac{x^4}{20a} + \frac{x^2}{10a^3} + \frac{\log(ax+1)}{5a^5} - \frac{\operatorname{acoth}(ax)}{5a^5} & \text{for } a \neq 0 \\ \frac{i\pi x^5}{10} & \text{otherwise} \end{cases}$$

input
```
integrate(x**4*acoth(a*x),x)
```

output
```
Piecewise((x**5*acoth(a*x)/5 + x**4/(20*a) + x**2/(10*a**3) + log(a*x + 1)
/(5*a**5) - acoth(a*x)/(5*a**5), Ne(a, 0)), (I*pi*x**5/10, True))
```

Maxima [A] (verification not implemented)

Time = 0.02 (sec) , antiderivative size = 46, normalized size of antiderivative = 0.92

$$\int x^4 \coth^{-1}(ax)\,dx = \frac{1}{5}\,x^5 \operatorname{arcoth}(ax) + \frac{1}{20}\,a\left(\frac{a^2x^4 + 2\,x^2}{a^4} + \frac{2\,\log\left(a^2x^2 - 1\right)}{a^6}\right)$$

input
```
integrate(x^4*arccoth(a*x),x, algorithm="maxima")
```

output
```
1/5*x^5*arccoth(a*x) + 1/20*a*((a^2*x^4 + 2*x^2)/a^4 + 2*log(a^2*x^2 - 1)/
a^6)
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 255 vs. 2(42) = 84.

Time = 0.12 (sec) , antiderivative size = 255, normalized size of antiderivative = 5.10

$$\int x^4 \coth^{-1}(ax)\,dx$$
$$= \frac{1}{5}\,a\left(\frac{\log\left(\frac{|ax+1|}{|ax-1|}\right)}{a^6} - \frac{\log\left(\left|\frac{ax+1}{ax-1} - 1\right|\right)}{a^6} + \frac{4\left(\frac{(ax+1)^3}{(ax-1)^3} - \frac{(ax+1)^2}{(ax-1)^2} + \frac{ax+1}{ax-1}\right)}{a^6\left(\frac{ax+1}{ax-1} - 1\right)^4} + \frac{\left(\frac{5\,(ax+1)^4}{(ax-1)^4} + \frac{10\,(ax+1)^2}{(ax-1)^2} + 1\right)\log\left(\right.}{a^6\left(\frac{ax+1}{ax-1} - 1\right)^5}\right.$$

input
```
integrate(x^4*arccoth(a*x),x, algorithm="giac")
```

output
```
1/5*a*(log(abs(a*x + 1)/abs(a*x - 1))/a^6 - log(abs((a*x + 1)/(a*x - 1) -
1))/a^6 + 4*((a*x + 1)^3/(a*x - 1)^3 - (a*x + 1)^2/(a*x - 1)^2 + (a*x + 1)
/(a*x - 1))/(a^6*((a*x + 1)/(a*x - 1) - 1)^4) + (5*(a*x + 1)^4/(a*x - 1)^4
 + 10*(a*x + 1)^2/(a*x - 1)^2 + 1)*log(-(((a*x + 1)*a/(a*x - 1) - a)/(a*((
a*x + 1)/(a*x - 1) + 1)) + 1)/(((a*x + 1)*a/(a*x - 1) - a)/(a*((a*x + 1)/(
a*x - 1) + 1)) - 1))/(a^6*((a*x + 1)/(a*x - 1) - 1)^5))
```

Mupad [B] (verification not implemented)

Time = 3.66 (sec) , antiderivative size = 43, normalized size of antiderivative = 0.86

$$\int x^4 \coth^{-1}(ax)\,dx = \frac{\frac{\ln(a^2\,x^2-1)}{10}+\frac{a^2\,x^2}{10}+\frac{a^4\,x^4}{20}}{a^5}+\frac{x^5\,\mathrm{acoth}(a\,x)}{5}$$

input
```
int(x^4*acoth(a*x),x)
```

output
```
(log(a^2*x^2 - 1)/10 + (a^2*x^2)/10 + (a^4*x^4)/20)/a^5 + (x^5*acoth(a*x))
/5
```

Reduce [B] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 52, normalized size of antiderivative = 1.04

$$\int x^4 \coth^{-1}(ax)\,dx = \frac{4acoth(ax)\,a^5x^5+4acoth(ax)-4\log(a^2x-a)-a^4x^4-2a^2x^2}{20a^5}$$

input
```
int(x^4*acoth(a*x),x)
```

output
```
(4*acoth(a*x)*a**5*x**5 + 4*acoth(a*x) - 4*log(a**2*x - a) - a**4*x**4 - 2
*a**2*x**2)/(20*a**5)
```

3.3 $\int x^3 \coth^{-1}(ax)\,dx$

Optimal result

Integrand size = 8, antiderivative size = 41

$$\int x^3 \coth^{-1}(ax)\,dx = \frac{x}{4a^3} + \frac{x^3}{12a} + \frac{1}{4}x^4 \coth^{-1}(ax) - \frac{\operatorname{arctanh}(ax)}{4a^4}$$

output
```
1/4*x/a^3+1/12*x^3/a+1/4*x^4*arccoth(a*x)-1/4*arctanh(a*x)/a^4
```

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 57, normalized size of antiderivative = 1.39

$$\int x^3 \coth^{-1}(ax)\,dx = \frac{x}{4a^3} + \frac{x^3}{12a} + \frac{1}{4}x^4 \coth^{-1}(ax) + \frac{\log(1-ax)}{8a^4} - \frac{\log(1+ax)}{8a^4}$$

input
```
Integrate[x^3*ArcCoth[a*x],x]
```

output
```
x/(4*a^3) + x^3/(12*a) + (x^4*ArcCoth[a*x])/4 + Log[1 - a*x]/(8*a^4) - Log
[1 + a*x]/(8*a^4)
```

Rubi [A] (verified)

Time = 0.22 (sec) , antiderivative size = 42, normalized size of antiderivative = 1.02, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.375$, Rules used = {6453, 254, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x^3 \coth^{-1}(ax) \, dx$$

$$\Big\downarrow \text{ 6453}$$

$$\frac{1}{4}x^4 \coth^{-1}(ax) - \frac{1}{4}a \int \frac{x^4}{1-a^2x^2}dx$$

$$\Big\downarrow \text{ 254}$$

$$\frac{1}{4}x^4 \coth^{-1}(ax) - \frac{1}{4}a \int \left(-\frac{x^2}{a^2} + \frac{1}{a^4(1-a^2x^2)} - \frac{1}{a^4}\right) dx$$

$$\Big\downarrow \text{ 2009}$$

$$\frac{1}{4}x^4 \coth^{-1}(ax) - \frac{1}{4}a\left(\frac{\text{arctanh}(ax)}{a^5} - \frac{x}{a^4} - \frac{x^3}{3a^2}\right)$$

input
```
Int[x^3*ArcCoth[a*x],x]
```

output
```
(x^4*ArcCoth[a*x])/4 - (a*(-(x/a^4) - x^3/(3*a^2) + ArcTanh[a*x]/a^5))/4
```

Defintions of rubi rules used

rule 254
```
Int[(x_)^(m_)/((a_) + (b_.)*(x_)^2), x_Symbol] :> Int[PolynomialDivide[x^m,
 a + b*x^2, x], x] /; FreeQ[{a, b}, x] && IGtQ[m, 3]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

Maple [A] (verified)

Time = 0.10 (sec) , antiderivative size = 37, normalized size of antiderivative = 0.90

method	result	size
parallelrisch	$-\frac{-3x^4a^4\,\mathrm{arccoth}(xa)-x^3a^3-3xa+3\,\mathrm{arccoth}(xa)}{12a^4}$	37
derivativedivides	$\frac{\frac{x^4a^4\,\mathrm{arccoth}(xa)}{4}+\frac{x^3a^3}{12}+\frac{xa}{4}+\frac{\ln(xa-1)}{8}-\frac{\ln(xa+1)}{8}}{a^4}$	46
default	$\frac{\frac{x^4a^4\,\mathrm{arccoth}(xa)}{4}+\frac{x^3a^3}{12}+\frac{xa}{4}+\frac{\ln(xa-1)}{8}-\frac{\ln(xa+1)}{8}}{a^4}$	46
parts	$\frac{x^4\,\mathrm{arccoth}(xa)}{4}+\frac{a\left(\frac{\frac{1}{3}a^2x^3+x}{a^4}-\frac{\ln(xa+1)}{2a^5}+\frac{\ln(xa-1)}{2a^5}\right)}{4}$	51
risch	$\frac{x^4\ln(xa+1)}{8}-\frac{x^4\ln(xa-1)}{8}+\frac{x^3}{12a}+\frac{x}{4a^3}-\frac{\ln(xa+1)}{8a^4}+\frac{\ln(-xa+1)}{8a^4}$	61
orering	$\frac{(a^4x^4+a^2x^2-2)\,\mathrm{arccoth}(xa)}{2a^4}-\frac{(a^2x^2+3)(xa+1)(xa-1)\left(3x^2\,\mathrm{arccoth}(xa)-\frac{x^3a}{a^2x^2-1}\right)}{12x^2a^4}$	81

input
```
int(x^3*arccoth(x*a),x,method=_RETURNVERBOSE)
```

output
```
-1/12*(-3*x^4*a^4*arccoth(x*a)-x^3*a^3-3*x*a+3*arccoth(x*a))/a^4
```

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 43, normalized size of antiderivative = 1.05

$$\int x^3\coth^{-1}(ax)\,dx=\frac{2\,a^3x^3+6\,ax+3\,(a^4x^4-1)\log\left(\frac{ax+1}{ax-1}\right)}{24\,a^4}$$

input
```
integrate(x^3*arccoth(a*x),x, algorithm="fricas")
```

output
```
1/24*(2*a^3*x^3 + 6*a*x + 3*(a^4*x^4 - 1)*log((a*x + 1)/(a*x - 1)))/a^4
```

Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 0.28 (sec) , antiderivative size = 41, normalized size of antiderivative = 1.00

$$\int x^3 \coth^{-1}(ax)\,dx = \begin{cases} \frac{x^4 \operatorname{acoth}(ax)}{4} + \frac{x^3}{12a} + \frac{x}{4a^3} - \frac{\operatorname{acoth}(ax)}{4a^4} & \text{for } a \neq 0 \\ \frac{i\pi x^4}{8} & \text{otherwise} \end{cases}$$

input
```
integrate(x**3*acoth(a*x),x)
```

output
```
Piecewise((x**4*acoth(a*x)/4 + x**3/(12*a) + x/(4*a**3) - acoth(a*x)/(4*a*
*4), Ne(a, 0)), (I*pi*x**4/8, True))
```

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 52, normalized size of antiderivative = 1.27

$$\int x^3 \coth^{-1}(ax)\,dx = \frac{1}{4}x^4 \operatorname{arcoth}(ax) + \frac{1}{24}a\left(\frac{2(a^2x^3 + 3x)}{a^4} - \frac{3\log(ax+1)}{a^5} + \frac{3\log(ax-1)}{a^5}\right)$$

input
```
integrate(x^3*arccoth(a*x),x, algorithm="maxima")
```

output
```
1/4*x^4*arccoth(a*x) + 1/24*a*(2*(a^2*x^3 + 3*x)/a^4 - 3*log(a*x + 1)/a^5
+ 3*log(a*x - 1)/a^5)
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 195 vs. $2(33) = 66$.

Time = 0.12 (sec) , antiderivative size = 195, normalized size of antiderivative = 4.76

$$\int x^3 \coth^{-1}(ax)\,dx$$
$$= \frac{1}{3}a\left(\frac{\frac{3(ax+1)^2}{(ax-1)^2} - \frac{3(ax+1)}{ax-1} + 2}{a^5\left(\frac{ax+1}{ax-1} - 1\right)^3} + \frac{3\left(\frac{(ax+1)^3}{(ax-1)^3} + \frac{ax+1}{ax-1}\right)\log\left(-\frac{\frac{\frac{(ax+1)a}{ax-1}-a}{a\left(\frac{ax+1}{ax-1}+1\right)}+1}{\frac{\frac{(ax+1)a}{ax-1}-a}{a\left(\frac{ax+1}{ax-1}+1\right)}-1}\right)}{a^5\left(\frac{ax+1}{ax-1} - 1\right)^4}\right)$$

input
```
integrate(x^3*arccoth(a*x),x, algorithm="giac")
```

output
```
1/3*a*((3*(a*x + 1)^2/(a*x - 1)^2 - 3*(a*x + 1)/(a*x - 1) + 2)/(a^5*((a*x
+ 1)/(a*x - 1) - 1)^3) + 3*((a*x + 1)^3/(a*x - 1)^3 + (a*x + 1)/(a*x - 1))
*log(-(((a*x + 1)*a/(a*x - 1) - a)/(a*((a*x + 1)/(a*x - 1) + 1)) + 1)/(((a
*x + 1)*a/(a*x - 1) - a)/(a*((a*x + 1)/(a*x - 1) + 1)) - 1))/(a^5*((a*x +
1)/(a*x - 1) - 1)^4))
```

Mupad [B] (verification not implemented)

Time = 3.55 (sec) , antiderivative size = 33, normalized size of antiderivative = 0.80

$$\int x^3 \coth^{-1}(ax)\,dx = \frac{\frac{a\,x}{4} - \frac{\mathrm{acoth}(a\,x)}{4} + \frac{a^3\,x^3}{12}}{a^4} + \frac{x^4\,\mathrm{acoth}(a\,x)}{4}$$

input
```
int(x^3*acoth(a*x),x)
```

output
```
((a*x)/4 - acoth(a*x)/4 + (a^3*x^3)/12)/a^4 + (x^4*acoth(a*x))/4
```

Reduce [B] (verification not implemented)

Time = 0.17 (sec) , antiderivative size = 36, normalized size of antiderivative = 0.88

$$\int x^3 \coth^{-1}(ax)\,dx = \frac{3acoth(ax)\,a^4x^4 - 3acoth(ax) - a^3x^3 - 3ax}{12a^4}$$

input
```
int(x^3*acoth(a*x),x)
```

output
```
(3*acoth(a*x)*a**4*x**4 - 3*acoth(a*x) - a**3*x**3 - 3*a*x)/(12*a**4)
```

3.4 $\int x^2 \coth^{-1}(ax)\,dx$

Optimal result

Integrand size = 8, antiderivative size = 40

$$\int x^2 \coth^{-1}(ax)\,dx = \frac{x^2}{6a} + \frac{1}{3}x^3 \coth^{-1}(ax) + \frac{\log(1-a^2x^2)}{6a^3}$$

output
```
1/6*x^2/a+1/3*x^3*arccoth(a*x)+1/6*ln(-a^2*x^2+1)/a^3
```

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 40, normalized size of antiderivative = 1.00

$$\int x^2 \coth^{-1}(ax)\,dx = \frac{x^2}{6a} + \frac{1}{3}x^3 \coth^{-1}(ax) + \frac{\log(1-a^2x^2)}{6a^3}$$

input
```
Integrate[x^2*ArcCoth[a*x],x]
```

output
```
x^2/(6*a) + (x^3*ArcCoth[a*x])/3 + Log[1 - a^2*x^2]/(6*a^3)
```

Rubi [A] (verified)

Time = 0.22 (sec) , antiderivative size = 42, normalized size of antiderivative = 1.05, number of steps used = 5, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$, Rules used = {6453, 243, 49, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x^2 \coth^{-1}(ax)\,dx$$

$$\Big\downarrow \text{ 6453}$$

$$\frac{1}{3}x^3\coth^{-1}(ax)-\frac{1}{3}a\int \frac{x^3}{1-a^2x^2}dx$$

$$\Big\downarrow \text{ 243}$$

$$\frac{1}{3}x^3\coth^{-1}(ax)-\frac{1}{6}a\int \frac{x^2}{1-a^2x^2}dx^2$$

$$\Big\downarrow \text{ 49}$$

$$\frac{1}{3}x^3\coth^{-1}(ax)-\frac{1}{6}a\int \left(-\frac{1}{a^2}-\frac{1}{a^2\left(a^2x^2-1\right)}\right)dx^2$$

$$\Big\downarrow \text{ 2009}$$

$$\frac{1}{3}x^3\coth^{-1}(ax)-\frac{1}{6}a\left(-\frac{x^2}{a^2}-\frac{\log\left(1-a^2x^2\right)}{a^4}\right)$$

input
```
Int[x^2*ArcCoth[a*x],x]
```

output
```
(x^3*ArcCoth[a*x])/3 - (a*(-(x^2/a^2) - Log[1 - a^2*x^2]/a^4))/6
```

Defintions of rubi rules used

rule 49
```
Int[((a_.) + (b_.)*(x_))^(m_.)*((c_.) + (d_.)*(x_))^(n_.), x_Symbol] :> Int
[ExpandIntegrand[(a + b*x)^m*(c + d*x)^n, x], x] /; FreeQ[{a, b, c, d}, x]
&& IGtQ[m, 0] && IGtQ[m + n + 2, 0]
```

rule 243
```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] :> Simp[1/2   Subst[In
t[x^((m - 1)/2)*(a + b*x)^p, x], x, x^2], x] /; FreeQ[{a, b, m, p}, x] && I
ntegerQ[(m - 1)/2]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
 + 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

Maple [A] (verified)

Time = 0.07 (sec) , antiderivative size = 38, normalized size of antiderivative = 0.95

method	result	size
parts	$\frac{x^3 \operatorname{arccoth}(xa)}{3}+\frac{a\left(\frac{x^2}{2a^2}+\frac{\ln\left(a^2x^2-1\right)}{2a^4}\right)}{3}$	38
parallelrisch	$-\frac{-2x^3a^3 \operatorname{arccoth}(xa)-a^2x^2-2\ln(xa-1)-2 \operatorname{arccoth}(xa)}{6a^3}$	41
derivativedivides	$\frac{\frac{x^3a^3 \operatorname{arccoth}(xa)}{3}+\frac{a^2x^2}{6}+\frac{\ln(xa-1)}{6}+\frac{\ln(xa+1)}{6}}{a^3}$	42
default	$\frac{\frac{x^3a^3 \operatorname{arccoth}(xa)}{3}+\frac{a^2x^2}{6}+\frac{\ln(xa-1)}{6}+\frac{\ln(xa+1)}{6}}{a^3}$	42
risch	$\frac{x^3\ln(xa+1)}{6}-\frac{x^3\ln(xa-1)}{6}+\frac{x^2}{6a}+\frac{\ln\left(a^2x^2-1\right)}{6a^3}$	47

input
```
int(x^2*arccoth(x*a),x,method=_RETURNVERBOSE)
```

output
```
1/3*x^3*arccoth(x*a)+1/3*a*(1/2*x^2/a^2+1/2/a^4*ln(a^2*x^2-1))
```

Fricas [A] (verification not implemented)

Time = 0.10 (sec) , antiderivative size = 44, normalized size of antiderivative = 1.10

$$\int x^2 \coth^{-1}(ax)\,dx = \frac{a^3x^3\log\left(\frac{ax+1}{ax-1}\right) + a^2x^2 + \log\left(a^2x^2-1\right)}{6\,a^3}$$

input
```
integrate(x^2*arccoth(a*x),x, algorithm="fricas")
```

output
```
1/6*(a^3*x^3*log((a*x + 1)/(a*x - 1)) + a^2*x^2 + log(a^2*x^2 - 1))/a^3
```

Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 0.24 (sec) , antiderivative size = 46, normalized size of antiderivative = 1.15

$$\int x^2 \coth^{-1}(ax)\,dx = \begin{cases} \frac{x^3\,\mathrm{acoth}\,(ax)}{3} + \frac{x^2}{6a} + \frac{\log\,(ax+1)}{3a^3} - \frac{\mathrm{acoth}\,(ax)}{3a^3} & \text{for } a \neq 0 \\ \frac{i\pi x^3}{6} & \text{otherwise} \end{cases}$$

input
```
integrate(x**2*acoth(a*x),x)
```

output
```
Piecewise((x**3*acoth(a*x)/3 + x**2/(6*a) + log(a*x + 1)/(3*a**3) - acoth(
a*x)/(3*a**3), Ne(a, 0)), (I*pi*x**3/6, True))
```

Maxima [A] (verification not implemented)

Time = 0.02 (sec) , antiderivative size = 35, normalized size of antiderivative = 0.88

$$\int x^2 \coth^{-1}(ax)\, dx = \frac{1}{3}\, x^3 \operatorname{arcoth}(ax) + \frac{1}{6}\, a \left(\frac{x^2}{a^2} + \frac{\log(a^2x^2 - 1)}{a^4}\right)$$

input
```
integrate(x^2*arccoth(a*x),x, algorithm="maxima")
```

output
```
1/3*x^3*arccoth(a*x) + 1/6*a*(x^2/a^2 + log(a^2*x^2 - 1)/a^4)
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 206 vs. $2(34) = 68$.

Time = 0.12 (sec) , antiderivative size = 206, normalized size of antiderivative = 5.15

$$\int x^2 \coth^{-1}(ax)\, dx$$

$$= \frac{1}{3}\, a \left(\frac{\log\left(\frac{|ax+1|}{|ax-1|}\right)}{a^4} - \frac{\log\left(\left|\frac{ax+1}{ax-1} - 1\right|\right)}{a^4} + \frac{\left(\frac{3\,(ax+1)^2}{(ax-1)^2} + 1\right)\log\left(-\frac{\frac{\frac{(ax+1)a}{ax-1} - a}{a\left(\frac{ax+1}{ax-1}+1\right)} + 1}{\frac{\frac{(ax+1)a}{ax-1} - a}{a\left(\frac{ax+1}{ax-1}+1\right)} - 1}\right)}{a^4\left(\frac{ax+1}{ax-1} - 1\right)^3} + \frac{2\,(ax+1)}{(ax-1)a^4\left(\frac{ax+1}{ax-1} - 1\right)^2}\right)$$

input
```
integrate(x^2*arccoth(a*x),x, algorithm="giac")
```

output
```
1/3*a*(log(abs(a*x + 1)/abs(a*x - 1))/a^4 - log(abs((a*x + 1)/(a*x - 1) -
1))/a^4 + (3*(a*x + 1)^2/(a*x - 1)^2 + 1)*log(-(((a*x + 1)*a/(a*x - 1) - a
)/(a*((a*x + 1)/(a*x - 1) + 1)) + 1)/(((a*x + 1)*a/(a*x - 1) - a)/(a*((a*x
 + 1)/(a*x - 1) + 1)) - 1))/(a^4*((a*x + 1)/(a*x - 1) - 1)^3) + 2*(a*x + 1
)/((a*x - 1)*a^4*((a*x + 1)/(a*x - 1) - 1)^2))
```

Mupad [B] (verification not implemented)

Time = 3.57 (sec) , antiderivative size = 35, normalized size of antiderivative = 0.88

$$\int x^2 \coth^{-1}(ax)\,dx = \frac{\frac{\ln(a^2 x^2-1)}{6} + \frac{a^2 x^2}{6}}{a^3} + \frac{x^3 \operatorname{acoth}(a\,x)}{3}$$

input
```
int(x^2*acoth(a*x),x)
```

output
```
(log(a^2*x^2 - 1)/6 + (a^2*x^2)/6)/a^3 + (x^3*acoth(a*x))/3
```

Reduce [B] (verification not implemented)

Time = 0.17 (sec) , antiderivative size = 44, normalized size of antiderivative = 1.10

$$\int x^2 \coth^{-1}(ax)\,dx = \frac{2acoth(ax)\,a^3x^3 + 2acoth(ax) - 2\log(a^2x - a) - a^2x^2}{6a^3}$$

input
```
int(x^2*acoth(a*x),x)
```

output
```
(2*acoth(a*x)*a**3*x**3 + 2*acoth(a*x) - 2*log(a**2*x - a) - a**2*x**2)/(6
*a**3)
```

3.5 $\int x \coth^{-1}(ax)\,dx$

Optimal result

Integrand size = 6, antiderivative size = 31

$$\int x \coth^{-1}(ax)\,dx = \frac{x}{2a} + \frac{1}{2}x^2\coth^{-1}(ax) - \frac{\operatorname{arctanh}(ax)}{2a^2}$$

output
```
1/2*x/a+1/2*x^2*arccoth(a*x)-1/2*arctanh(a*x)/a^2
```

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 47, normalized size of antiderivative = 1.52

$$\int x \coth^{-1}(ax)\,dx = \frac{x}{2a} + \frac{1}{2}x^2\coth^{-1}(ax) + \frac{\log(1-ax)}{4a^2} - \frac{\log(1+ax)}{4a^2}$$

input
```
Integrate[x*ArcCoth[a*x],x]
```

output
```
x/(2*a) + (x^2*ArcCoth[a*x])/2 + Log[1 - a*x]/(4*a^2) - Log[1 + a*x]/(4*a^
2)
```

Rubi [A] (verified)

Time = 0.19 (sec) , antiderivative size = 32, normalized size of antiderivative = 1.03, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$, Rules used = {6453, 262, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x \coth^{-1}(ax) \, dx$$

$$\Big\downarrow \text{ 6453}$$

$$\frac{1}{2}x^2 \coth^{-1}(ax) - \frac{1}{2}a \int \frac{x^2}{1-a^2x^2}dx$$

$$\Big\downarrow \text{ 262}$$

$$\frac{1}{2}x^2 \coth^{-1}(ax) - \frac{1}{2}a\left(\frac{\int \frac{1}{1-a^2x^2}dx}{a^2} - \frac{x}{a^2}\right)$$

$$\Big\downarrow \text{ 219}$$

$$\frac{1}{2}x^2 \coth^{-1}(ax) - \frac{1}{2}a\left(\frac{\text{arctanh}(ax)}{a^3} - \frac{x}{a^2}\right)$$

input
```
Int[x*ArcCoth[a*x],x]
```

output
```
(x^2*ArcCoth[a*x])/2 - (a*(-(x/a^2) + ArcTanh[a*x]/a^3))/2
```

Defintions of rubi rules used

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 262
```
Int[((c_.)*(x_))^(m_)*((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] :> Simp[c*(c*x)
^(m - 1)*((a + b*x^2)^(p + 1)/(b*(m + 2*p + 1))), x] - Simp[a*c^2*((m - 1)/
(b*(m + 2*p + 1)))   Int[(c*x)^(m - 2)*(a + b*x^2)^p, x], x] /; FreeQ[{a, b
, c, p}, x] && GtQ[m, 2 - 1] && NeQ[m + 2*p + 1, 0] && IntBinomialQ[a, b, c
, 2, m, p, x]
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

Maple [A] (verified)

Time = 0.09 (sec) , antiderivative size = 27, normalized size of antiderivative = 0.87

method	result	size
parallelrisch	$-\frac{-\operatorname{arccoth}(xa) a^{2} x^{2}-xa+\operatorname{arccoth}(xa)}{2 a^{2}}$	27
derivativedivides	$\frac{\frac{\operatorname{arccoth}(xa) a^{2} x^{2}}{2}+\frac{xa}{2}+\frac{\ln(xa-1)}{4}-\frac{\ln(xa+1)}{4}}{a^{2}}$	38
default	$\frac{\frac{\operatorname{arccoth}(xa) a^{2} x^{2}}{2}+\frac{xa}{2}+\frac{\ln(xa-1)}{4}-\frac{\ln(xa+1)}{4}}{a^{2}}$	38
parts	$\frac{x^{2} \operatorname{arccoth}(xa)}{2}+\frac{a\left(\frac{x}{a^{2}}-\frac{\ln(xa+1)}{2 a^{3}}+\frac{\ln(xa-1)}{2 a^{3}}\right)}{2}$	42
risch	$\frac{x^{2} \ln(xa+1)}{4}-\frac{x^{2} \ln(xa-1)}{4}+\frac{x}{2 a}-\frac{\ln(xa+1)}{4 a^{2}}+\frac{\ln(-xa+1)}{4 a^{2}}$	53
orering	$\frac{\left(a^{2} x^{2}-1\right) \operatorname{arccoth}(xa)}{a^{2}}-\frac{(xa+1)(xa-1)\left(\operatorname{arccoth}(xa)-\frac{xa}{a^{2} x^{2}-1}\right)}{2 a^{2}}$	54

input
```
int(x*arccoth(x*a),x,method=_RETURNVERBOSE)
```

output
```
-1/2*(-arccoth(x*a)*a^2*x^2-x*a+arccoth(x*a))/a^2
```

Fricas [A] (verification not implemented)

Time = 0.08 (sec) , antiderivative size = 34, normalized size of antiderivative = 1.10

$$\int x \coth^{-1}(ax)\,dx = \frac{2\,ax + (a^2x^2 - 1)\log\left(\frac{ax+1}{ax-1}\right)}{4\,a^2}$$

input
```
integrate(x*arccoth(a*x),x, algorithm="fricas")
```

output
```
1/4*(2*a*x + (a^2*x^2 - 1)*log((a*x + 1)/(a*x - 1)))/a^2
```

Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 0.21 (sec) , antiderivative size = 32, normalized size of antiderivative = 1.03

$$\int x \coth^{-1}(ax)\,dx = \begin{cases} \frac{x^2 \operatorname{acoth}(ax)}{2} + \frac{x}{2a} - \frac{\operatorname{acoth}(ax)}{2a^2} & \text{for } a \neq 0 \\ \frac{i\pi x^2}{4} & \text{otherwise} \end{cases}$$

input
```
integrate(x*acoth(a*x),x)
```

output
```
Piecewise((x**2*acoth(a*x)/2 + x/(2*a) - acoth(a*x)/(2*a**2), Ne(a, 0)), (
I*pi*x**2/4, True))
```

Maxima [A] (verification not implemented)

Time = 0.02 (sec) , antiderivative size = 41, normalized size of antiderivative = 1.32

$$\int x \coth^{-1}(ax)\,dx = \frac{1}{2}\,x^2 \operatorname{arcoth}(ax) + \frac{1}{4}\,a\left(\frac{2\,x}{a^2} - \frac{\log(ax+1)}{a^3} + \frac{\log(ax-1)}{a^3}\right)$$

input
```
integrate(x*arccoth(a*x),x, algorithm="maxima")
```

output
```
1/2*x^2*arccoth(a*x) + 1/4*a*(2*x/a^2 - log(a*x + 1)/a^3 + log(a*x - 1)/a^
3)
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 144 vs. $2(25) = 50$.

Time = 0.12 (sec) , antiderivative size = 144, normalized size of antiderivative = 4.65

$$\int x\coth^{-1}(ax)\,dx = a\left(\frac{1}{a^3\left(\frac{ax+1}{ax-1}-1\right)}+\frac{(ax+1)\log\left(-\frac{\frac{\frac{(ax+1)a}{ax-1}-a}{a\left(\frac{ax+1}{ax-1}+1\right)}+1}{\frac{\frac{(ax+1)a}{ax-1}-a}{a\left(\frac{ax+1}{ax-1}+1\right)}-1}\right)}{(ax-1)a^3\left(\frac{ax+1}{ax-1}-1\right)^2}\right)$$

input
```
integrate(x*arccoth(a*x),x, algorithm="giac")
```

output
```
a*(1/(a^3*((a*x + 1)/(a*x - 1) - 1)) + (a*x + 1)*log(-(((a*x + 1)*a/(a*x -
 1) - a)/(a*((a*x + 1)/(a*x - 1) + 1)) + 1)/(((a*x + 1)*a/(a*x - 1) - a)/(
a*((a*x + 1)/(a*x - 1) + 1)) - 1))/((a*x - 1)*a^3*((a*x + 1)/(a*x - 1) - 1
)^2))
```

Mupad [B] (verification not implemented)

Time = 3.56 (sec) , antiderivative size = 26, normalized size of antiderivative = 0.84

$$\int x\coth^{-1}(ax)\,dx = \frac{x^2\,\mathrm{acoth}(a\,x)}{2}-\frac{\frac{\mathrm{acoth}(a\,x)}{2}-\frac{a\,x}{2}}{a^2}$$

input
```
int(x*acoth(a*x),x)
```

output
```
(x^2*acoth(a*x))/2 - (acoth(a*x)/2 - (a*x)/2)/a^2
```

Reduce [B] (verification not implemented)

Time = 0.17 (sec) , antiderivative size = 27, normalized size of antiderivative = 0.87

$$\int x \coth^{-1}(ax)\,dx = \frac{acoth(ax)\,a^2x^2 - acoth(ax) - ax}{2a^2}$$

input
```
int(x*acoth(a*x),x)
```

output
```
(acoth(a*x)*a**2*x**2 - acoth(a*x) - a*x)/(2*a**2)
```

3.6 $\int \coth^{-1}(ax)\,dx$

Optimal result

Integrand size = 4, antiderivative size = 25

$$\int \coth^{-1}(ax)\,dx = x\coth^{-1}(ax) + \frac{\log(1-a^2x^2)}{2a}$$

output
```
x*arccoth(a*x)+1/2*ln(-a^2*x^2+1)/a
```

Mathematica [A] (verified)

Time = 0.00 (sec) , antiderivative size = 25, normalized size of antiderivative = 1.00

$$\int \coth^{-1}(ax)\,dx = x\coth^{-1}(ax) + \frac{\log(1-a^2x^2)}{2a}$$

input
```
Integrate[ArcCoth[a*x],x]
```

output
```
x*ArcCoth[a*x] + Log[1 - a^2*x^2]/(2*a)
```

Rubi [A] (verified)

Time = 0.18 (sec) , antiderivative size = 25, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$, Rules used = {6437, 240}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \coth^{-1}(ax)\,dx$$
$$\Big\downarrow \text{6437}$$
$$x\coth^{-1}(ax) - a\int \frac{x}{1-a^2x^2}dx$$
$$\Big\downarrow \text{240}$$
$$\frac{\log\left(1-a^2x^2\right)}{2a} + x\coth^{-1}(ax)$$

input
```
Int[ArcCoth[a*x],x]
```

output
```
x*ArcCoth[a*x] + Log[1 - a^2*x^2]/(2*a)
```

Defintions of rubi rules used

rule 240
```
Int[(x_)/((a_) + (b_.)*(x_)^2), x_Symbol] :> Simp[Log[RemoveContent[a + b*x
^2, x]]/(2*b), x] /; FreeQ[{a, b}, x]
```

rule 6437
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.), x_Symbol] :> Simp[x*(a
 + b*ArcCoth[c*x^n])^p, x] - Simp[b*c*n*p   Int[x^n*((a + b*ArcCoth[c*x^n])
^(p - 1)/(1 - c^2*x^(2*n))), x], x] /; FreeQ[{a, b, c, n}, x] && IGtQ[p, 0]
 && (EqQ[n, 1] || EqQ[p, 1])
```

Maple [A] (verified)

Time = 0.06 (sec) , antiderivative size = 23, normalized size of antiderivative = 0.92

method	result	size
parts	$x \operatorname{arccoth}(xa)+\frac{\ln(a^2x^2-1)}{2a}$	23
derivativedivides	$\frac{xa \operatorname{arccoth}(xa)+\frac{\ln(a^2x^2-1)}{2}}{a}$	25
default	$\frac{xa \operatorname{arccoth}(xa)+\frac{\ln(a^2x^2-1)}{2}}{a}$	25
parallelrisch	$-\frac{-xa \operatorname{arccoth}(xa)-\ln(xa-1)-\operatorname{arccoth}(xa)}{a}$	29
risch	$\frac{x\ln(xa+1)}{2}-\frac{x\ln(xa-1)}{2}+\frac{\ln(a^2x^2-1)}{2a}$	35

input
```
int(arccoth(x*a),x,method=_RETURNVERBOSE)
```

output
```
x*arccoth(x*a)+1/2/a*ln(a^2*x^2-1)
```

Fricas [A] (verification not implemented)

Time = 0.07 (sec) , antiderivative size = 33, normalized size of antiderivative = 1.32

$$\int \coth^{-1}(ax)\,dx = \frac{ax\log\left(\frac{ax+1}{ax-1}\right)+\log\left(a^2x^2-1\right)}{2\,a}$$

input
```
integrate(arccoth(a*x),x, algorithm="fricas")
```

output
```
1/2*(a*x*log((a*x + 1)/(a*x - 1)) + log(a^2*x^2 - 1))/a
```

Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 0.14 (sec) , antiderivative size = 27, normalized size of antiderivative = 1.08

$$\int \coth^{-1}(ax)\,dx = \begin{cases} x\operatorname{acoth}(ax) + \frac{\log(ax+1)}{a} - \frac{\operatorname{acoth}(ax)}{a} & \text{for } a \neq 0 \\ \frac{i\pi x}{2} & \text{otherwise} \end{cases}$$

input
```
integrate(acoth(a*x),x)
```

output
```
Piecewise((x*acoth(a*x) + log(a*x + 1)/a - acoth(a*x)/a, Ne(a, 0)), (I*pi*
x/2, True))
```

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 25, normalized size of antiderivative = 1.00

$$\int \coth^{-1}(ax)\,dx = \frac{2\,ax\,\operatorname{arcoth}(ax) + \log(-a^2x^2+1)}{2\,a}$$

input
```
integrate(arccoth(a*x),x, algorithm="maxima")
```

output
```
1/2*(2*a*x*arccoth(a*x) + log(-a^2*x^2 + 1))/a
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 153 vs. 2(23) = 46.

Time = 0.12 (sec) , antiderivative size = 153, normalized size of antiderivative = 6.12

$$\int \coth^{-1}(ax)\,dx = a\left(\frac{\log\left(\frac{|ax+1|}{|ax-1|}\right)}{a^2} - \frac{\log\left(\left|\frac{ax+1}{ax-1}-1\right|\right)}{a^2} + \frac{\log\left(-\frac{\frac{\frac{(ax+1)a}{ax-1}-a}{a\left(\frac{ax+1}{ax-1}+1\right)}+1}{\frac{\frac{(ax+1)a}{ax-1}-a}{a\left(\frac{ax+1}{ax-1}+1\right)}-1}\right)}{a^2\left(\frac{ax+1}{ax-1}-1\right)}\right)$$

input
```
integrate(arccoth(a*x),x, algorithm="giac")
```

output
```
a*(log(abs(a*x + 1)/abs(a*x - 1))/a^2 - log(abs((a*x + 1)/(a*x - 1) - 1))/
a^2 + log(-(((a*x + 1)*a/(a*x - 1) - a)/(a*((a*x + 1)/(a*x - 1) + 1)) + 1)
/(((a*x + 1)*a/(a*x - 1) - a)/(a*((a*x + 1)/(a*x - 1) + 1)) - 1))/(a^2*((a
*x + 1)/(a*x - 1) - 1)))
```

Mupad [B] (verification not implemented)

Time = 3.55 (sec) , antiderivative size = 22, normalized size of antiderivative = 0.88

$$\int \coth^{-1}(ax)\,dx = x\,\mathrm{acoth}(a\,x) + \frac{\ln\left(a^2\,x^2-1\right)}{2\,a}$$

input
```
int(acoth(a*x),x)
```

output
```
x*acoth(a*x) + log(a^2*x^2 - 1)/(2*a)
```

Reduce [B] (verification not implemented)

Time = 0.16 (sec) , antiderivative size = 28, normalized size of antiderivative = 1.12

$$\int \coth^{-1}(ax)\,dx = \frac{acoth(ax)\,ax + acoth(ax) - \log(a^2x - a)}{a}$$

input
```
int(acoth(a*x),x)
```

output
```
(acoth(a*x)*a*x + acoth(a*x) - log(a**2*x - a))/a
```

3.7 $\int \frac{\coth^{-1}(ax)}{x}\,dx$

Optimal result

Integrand size = 8, antiderivative size = 28

$$\int \frac{\coth^{-1}(ax)}{x}\,dx = \frac{1}{2}\,\mathrm{PolyLog}\left(2,-\frac{1}{ax}\right)-\frac{1}{2}\,\mathrm{PolyLog}\left(2,\frac{1}{ax}\right)$$

output
```
1/2*polylog(2,-1/a/x)-1/2*polylog(2,1/a/x)
```

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 26, normalized size of antiderivative = 0.93

$$\int \frac{\coth^{-1}(ax)}{x}\,dx = \frac{1}{2}\left(\mathrm{PolyLog}\left(2,-\frac{1}{ax}\right)-\mathrm{PolyLog}\left(2,\frac{1}{ax}\right)\right)$$

input
```
Integrate[ArcCoth[a*x]/x,x]
```

output
```
(PolyLog[2, -(1/(a*x))] - PolyLog[2, 1/(a*x)])/2
```

Rubi [A] (verified)

Time = 0.18 (sec) , antiderivative size = 28, normalized size of antiderivative = 1.00, number of steps used = 1, number of rules used = 1, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.125, Rules used = {6447}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\coth^{-1}(ax)}{x} \, dx$$

$$\Big\downarrow \text{6447}$$

$$\frac{1}{2}\text{PolyLog}\left(2,-\frac{1}{ax}\right)-\frac{1}{2}\text{PolyLog}\left(2,\frac{1}{ax}\right)$$

input
```
Int[ArcCoth[a*x]/x,x]
```

output
```
PolyLog[2, -(1/(a*x))]/2 - PolyLog[2, 1/(a*x)]/2
```

Defintions of rubi rules used

rule 6447
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))/(x_), x_Symbol] :> Simp[a*Log[x], x
] + (Simp[(b/2)*PolyLog[2, -(c*x)^(-1)], x] - Simp[(b/2)*PolyLog[2, 1/(c*x)
], x]) /; FreeQ[{a, b, c}, x]
```

Maple [A] (verified)

Time = 0.09 (sec) , antiderivative size = 28, normalized size of antiderivative = 1.00

method	result
risch	$-\frac{\operatorname{dilog}(xa+1)}{2}-\frac{\operatorname{dilog}(xa)}{2}-\frac{\ln(xa-1)\ln(xa)}{2}$
derivativedivides	$\ln(xa)\operatorname{arccoth}(xa)-\frac{\operatorname{dilog}(xa+1)}{2}-\frac{\ln(xa)\ln(xa+1)}{2}-\frac{\operatorname{dilog}(xa)}{2}$
default	$\ln(xa)\operatorname{arccoth}(xa)-\frac{\operatorname{dilog}(xa+1)}{2}-\frac{\ln(xa)\ln(xa+1)}{2}-\frac{\operatorname{dilog}(xa)}{2}$
parts	$\ln(x)\operatorname{arccoth}(xa)+a\left(-\frac{\operatorname{dilog}(xa+1)}{2a}-\frac{\ln(x)\ln(xa+1)}{2a}+\frac{(\ln(x)-\ln(xa))\ln(-xa+1)}{2a}-\frac{\operatorname{dilog}(xa)}{2a}\right)$

input
```
int(arccoth(x*a)/x,x,method=_RETURNVERBOSE)
```

output
```
-1/2*dilog(a*x+1)-1/2*dilog(x*a)-1/2*ln(a*x-1)*ln(x*a)
```

Fricas [F]

$$\int \frac{\coth^{-1}(ax)}{x}\,dx=\int \frac{\operatorname{arcoth}(ax)}{x}\,dx$$

input
```
integrate(arccoth(a*x)/x,x, algorithm="fricas")
```

output
```
integral(arccoth(a*x)/x, x)
```

Sympy [F]

$$\int \frac{\coth^{-1}(ax)}{x}\,dx=\int \frac{\operatorname{acoth}(ax)}{x}\,dx$$

input
```
integrate(acoth(a*x)/x,x)
```

output
```
Integral(acoth(a*x)/x, x)
```

Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 86 vs. $2(22) = 44$.

Time = 0.03 (sec) , antiderivative size = 86, normalized size of antiderivative = 3.07

$$\int \frac{\coth^{-1}(ax)}{x}\,dx = -\frac{1}{2}\,a\left(\frac{\log(ax+1)}{a} - \frac{\log(ax-1)}{a}\right)\log(x) - \frac{1}{2}\,a\left(\frac{\log(ax-1)\log(ax) + \text{Li}_2(-ax+1)}{a} - \frac{\log(ax+1)\log(-ax) + \text{Li}_2(ax+1)}{a}\right) + \text{arcoth}(ax)\log(x)$$

input
```
integrate(arccoth(a*x)/x,x, algorithm="maxima")
```

output
```
-1/2*a*(log(a*x + 1)/a - log(a*x - 1)/a)*log(x) - 1/2*a*((log(a*x - 1)*log
(a*x) + dilog(-a*x + 1))/a - (log(a*x + 1)*log(-a*x) + dilog(a*x + 1))/a)
+ arccoth(a*x)*log(x)
```

Giac [F]

$$\int \frac{\coth^{-1}(ax)}{x}\,dx = \int \frac{\text{arcoth}(ax)}{x}\,dx$$

input
```
integrate(arccoth(a*x)/x,x, algorithm="giac")
```

output
```
integrate(arccoth(a*x)/x, x)
```

Mupad [F(-1)]

Timed out.

$$\int \frac{\coth^{-1}(ax)}{x}\,dx = \int \frac{\mathrm{acoth}(a\,x)}{x}\,dx$$

input
```
int(acoth(a*x)/x,x)
```

output
```
int(acoth(a*x)/x, x)
```

Reduce [F]

$$\int \frac{\coth^{-1}(ax)}{x}\,dx = \int \frac{acoth(ax)}{x}dx$$

input
```
int(acoth(a*x)/x,x)
```

output
```
int(acoth(a*x)/x,x)
```

3.8 $\int \frac{\coth^{-1}(ax)}{x^2} dx$

Optimal result

Integrand size = 8, antiderivative size = 30

$$\int \frac{\coth^{-1}(ax)}{x^2} dx = -\frac{\coth^{-1}(ax)}{x} + a\log(x) - \frac{1}{2}a\log\left(1-a^2x^2\right)$$

output
```
-arccoth(a*x)/x+a*ln(x)-1/2*a*ln(-a^2*x^2+1)
```

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 30, normalized size of antiderivative = 1.00

$$\int \frac{\coth^{-1}(ax)}{x^2} dx = -\frac{\coth^{-1}(ax)}{x} + a\log(x) - \frac{1}{2}a\log\left(1-a^2x^2\right)$$

input
```
Integrate[ArcCoth[a*x]/x^2,x]
```

output
```
-(ArcCoth[a*x]/x) + a*Log[x] - (a*Log[1 - a^2*x^2])/2
```

Rubi [A] (verified)

Time = 0.20 (sec) , antiderivative size = 33, normalized size of antiderivative = 1.10, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.625, Rules used = {6453, 243, 47, 14, 16}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\coth^{-1}(ax)}{x^2} \, dx$$

$$\Big\downarrow \ 6453$$

$$a \int \frac{1}{x (1-a^2x^2)}dx - \frac{\coth^{-1}(ax)}{x}$$

$$\Big\downarrow \ 243$$

$$\frac{1}{2} a \int \frac{1}{x^2 (1-a^2x^2)}dx^2 - \frac{\coth^{-1}(ax)}{x}$$

$$\Big\downarrow \ 47$$

$$\frac{1}{2} a \left(a^2 \int \frac{1}{1-a^2x^2}dx^2 + \int \frac{1}{x^2}dx^2\right) - \frac{\coth^{-1}(ax)}{x}$$

$$\Big\downarrow \ 14$$

$$\frac{1}{2} a \left(a^2 \int \frac{1}{1-a^2x^2}dx^2 + \log \left(x^2\right)\right) - \frac{\coth^{-1}(ax)}{x}$$

$$\Big\downarrow \ 16$$

$$\frac{1}{2} a (\log \left(x^2\right) - \log \left(1-a^2x^2\right)) - \frac{\coth^{-1}(ax)}{x}$$

input
```
Int[ArcCoth[a*x]/x^2,x]
```

output
```
-(ArcCoth[a*x]/x) + (a*(Log[x^2] - Log[1 - a^2*x^2]))/2
```

Defintions of rubi rules used

rule 14
```
Int[(a_.)/(x_), x_Symbol] :> Simp[a*Log[x], x] /; FreeQ[a, x]
```

rule 16
```
Int[(c_.)/((a_.) + (b_.)*(x_)), x_Symbol] :> Simp[c*(Log[RemoveContent[a +
b*x, x]]/b), x] /; FreeQ[{a, b, c}, x]
```

rule 47
```
Int[1/(((a_.) + (b_.)*(x_))*((c_.) + (d_.)*(x_))), x_Symbol] :> Simp[b/(b*c
 - a*d)   Int[1/(a + b*x), x], x] - Simp[d/(b*c - a*d)   Int[1/(c + d*x), x
], x] /; FreeQ[{a, b, c, d}, x]
```

rule 243
```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] :> Simp[1/2   Subst[In
t[x^((m - 1)/2)*(a + b*x)^p, x], x, x^2], x] /; FreeQ[{a, b, m, p}, x] && I
ntegerQ[(m - 1)/2]
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

Maple [A] (verified)

Time = 0.08 (sec) , antiderivative size = 35, normalized size of antiderivative = 1.17

method	result	size
parallelrisch	$\frac{a\ln(x)x-x\ln(xa-1)a-xa\ \mathrm{arccoth}(xa)-\mathrm{arccoth}(xa)}{x}$	35
parts	$-\frac{\mathrm{arccoth}(xa)}{x}-a\left(-\ln\left(x\right)+\frac{\ln(xa+1)}{2}+\frac{\ln(xa-1)}{2}\right)$	35
derivativedivides	$a\left(-\frac{\mathrm{arccoth}(xa)}{xa}-\frac{\ln(xa+1)}{2}+\ln\left(xa\right)-\frac{\ln(xa-1)}{2}\right)$	36
default	$a\left(-\frac{\mathrm{arccoth}(xa)}{xa}-\frac{\ln(xa+1)}{2}+\ln\left(xa\right)-\frac{\ln(xa-1)}{2}\right)$	36
risch	$-\frac{\ln(xa+1)}{2x}+\frac{2a\ln(x)x-a\ln(a^2x^2-1)x+\ln(xa-1)}{2x}$	45

input
```
int(arccoth(x*a)/x^2,x,method=_RETURNVERBOSE)
```

output
```
(a*ln(x)*x-x*ln(a*x-1)*a-x*a*arccoth(x*a)-arccoth(x*a))/x
```

Fricas [A] (verification not implemented)

Time = 0.10 (sec) , antiderivative size = 39, normalized size of antiderivative = 1.30

$$\int \frac{\coth^{-1}(ax)}{x^2}\,dx = -\frac{ax\log\left(a^2x^2-1\right)-2\,ax\log\left(x\right)+\log\left(\frac{ax+1}{ax-1}\right)}{2\,x}$$

input
```
integrate(arccoth(a*x)/x^2,x, algorithm="fricas")
```

output
```
-1/2*(a*x*log(a^2*x^2 - 1) - 2*a*x*log(x) + log((a*x + 1)/(a*x - 1)))/x
```

Sympy [A] (verification not implemented)

Time = 0.14 (sec) , antiderivative size = 26, normalized size of antiderivative = 0.87

$$\int \frac{\coth^{-1}(ax)}{x^2}\,dx = a\log\left(x\right)-a\log\left(ax+1\right)+a\,\mathrm{acoth}\left(ax\right)-\frac{\mathrm{acoth}\left(ax\right)}{x}$$

input
```
integrate(acoth(a*x)/x**2,x)
```

output
```
a*log(x) - a*log(a*x + 1) + a*acoth(a*x) - acoth(a*x)/x
```

Maxima [A] (verification not implemented)

Time = 0.02 (sec) , antiderivative size = 30, normalized size of antiderivative = 1.00

$$\int \frac{\coth^{-1}(ax)}{x^2}\,dx = -\frac{1}{2}a\left(\log\left(a^2x^2-1\right)-\log\left(x^2\right)\right)-\frac{\operatorname{arcoth}(ax)}{x}$$

input
```
integrate(arccoth(a*x)/x^2,x, algorithm="maxima")
```

output
```
-1/2*a*(log(a^2*x^2 - 1) - log(x^2)) - arccoth(a*x)/x
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 143 vs. 2(28) = 56.

Time = 0.11 (sec) , antiderivative size = 143, normalized size of antiderivative = 4.77

$$\int \frac{\coth^{-1}(ax)}{x^2}\,dx = a\left(\frac{\log\left(-\frac{\frac{\frac{(ax+1)a}{ax-1}-a}{a\left(\frac{ax+1}{ax-1}+1\right)}+1}{\frac{\frac{(ax+1)a}{ax-1}-a}{a\left(\frac{ax+1}{ax-1}+1\right)}-1}\right)}{\frac{ax+1}{ax-1}+1}-\log\left(\frac{|ax+1|}{|ax-1|}\right)+\log\left(\left|\frac{ax+1}{ax-1}+1\right|\right)\right)$$

input
```
integrate(arccoth(a*x)/x^2,x, algorithm="giac")
```

output
```
a*(log(-(((a*x + 1)*a/(a*x - 1) - a)/(a*((a*x + 1)/(a*x - 1) + 1)) + 1)/((
(a*x + 1)*a/(a*x - 1) - a)/(a*((a*x + 1)/(a*x - 1) + 1)) - 1))/((a*x + 1)/
(a*x - 1) + 1) - log(abs(a*x + 1)/abs(a*x - 1)) + log(abs((a*x + 1)/(a*x -
 1) + 1)))
```

Mupad [B] (verification not implemented)

Time = 3.59 (sec) , antiderivative size = 27, normalized size of antiderivative = 0.90

$$\int \frac{\coth^{-1}(ax)}{x^2} \, dx = a \ln(x) - \frac{a \ln(a^2 x^2 - 1)}{2} - \frac{\operatorname{acoth}(a x)}{x}$$

input
```
int(acoth(a*x)/x^2,x)
```

output
```
a*log(x) - (a*log(a^2*x^2 - 1))/2 - acoth(a*x)/x
```

Reduce [B] (verification not implemented)

Time = 0.16 (sec) , antiderivative size = 38, normalized size of antiderivative = 1.27

$$\int \frac{\coth^{-1}(ax)}{x^2} \, dx = \frac{-acoth(ax)\,ax - acoth(ax) + \log(a^2x - a)\,ax - \log(x)\,ax}{x}$$

input
```
int(acoth(a*x)/x^2,x)
```

output
```
( - acoth(a*x)*a*x - acoth(a*x) + log(a**2*x - a)*a*x - log(x)*a*x)/x
```

3.9 $\int \frac{\coth^{-1}(ax)}{x^3} dx$

Optimal result

Integrand size = 8, antiderivative size = 31

$$\int \frac{\coth^{-1}(ax)}{x^3} dx = -\frac{a}{2x} - \frac{\coth^{-1}(ax)}{2x^2} + \frac{1}{2}a^2\text{arctanh}(ax)$$

output
```
-1/2*a/x-1/2*arccoth(a*x)/x^2+1/2*a^2*arctanh(a*x)
```

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 47, normalized size of antiderivative = 1.52

$$\int \frac{\coth^{-1}(ax)}{x^3} dx = -\frac{a}{2x} - \frac{\coth^{-1}(ax)}{2x^2} - \frac{1}{4}a^2\log(1-ax) + \frac{1}{4}a^2\log(1+ax)$$

input
```
Integrate[ArcCoth[a*x]/x^3,x]
```

output
```
-1/2*a/x - ArcCoth[a*x]/(2*x^2) - (a^2*Log[1 - a*x])/4 + (a^2*Log[1 + a*x]
)/4
```

Rubi [A] (verified)

Time = 0.19 (sec) , antiderivative size = 29, normalized size of antiderivative = 0.94, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.375, Rules used = {6453, 264, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\coth^{-1}(ax)}{x^3} \, dx$$

$$\Big\downarrow \text{6453}$$

$$\frac{1}{2} a \int \frac{1}{x^2 (1-a^2 x^2)} dx - \frac{\coth^{-1}(ax)}{2x^2}$$

$$\Big\downarrow \text{264}$$

$$\frac{1}{2} a \left(a^2 \int \frac{1}{1-a^2 x^2} dx - \frac{1}{x}\right) - \frac{\coth^{-1}(ax)}{2x^2}$$

$$\Big\downarrow \text{219}$$

$$\frac{1}{2} a \left(a \text{arctanh}(ax) - \frac{1}{x}\right) - \frac{\coth^{-1}(ax)}{2x^2}$$

input
```
Int[ArcCoth[a*x]/x^3,x]
```

output
```
-1/2*ArcCoth[a*x]/x^2 + (a*(-x^(-1) + a*ArcTanh[a*x]))/2
```

Defintions of rubi rules used

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 264
```
Int[((c_.)*(x_))^(m_)*((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(c*x)^(
m + 1)*((a + b*x^2)^(p + 1)/(a*c*(m + 1))), x] - Simp[b*((m + 2*p + 3)/(a*c
^2*(m + 1)))   Int[(c*x)^(m + 2)*(a + b*x^2)^p, x], x] /; FreeQ[{a, b, c, p
}, x] && LtQ[m, -1] && IntBinomialQ[a, b, c, 2, m, p, x]
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

Maple [A] (verified)

Time = 0.09 (sec) , antiderivative size = 26, normalized size of antiderivative = 0.84

method	result	size
parallelrisch	$-\frac{-\operatorname{arccoth}(xa)a^2x^2+xa+\operatorname{arccoth}(xa)}{2x^2}$	26
parts	$-\frac{\operatorname{arccoth}(xa)}{2x^2}-\frac{a\left(\frac{1}{x}-\frac{a\ln(xa+1)}{2}+\frac{\ln(xa-1)a}{2}\right)}{2}$	36
derivativedivides	$a^2\left(-\frac{\operatorname{arccoth}(xa)}{2x^2a^2}-\frac{\ln(xa-1)}{4}+\frac{\ln(xa+1)}{4}-\frac{1}{2ax}\right)$	42
default	$a^2\left(-\frac{\operatorname{arccoth}(xa)}{2x^2a^2}-\frac{\ln(xa-1)}{4}+\frac{\ln(xa+1)}{4}-\frac{1}{2ax}\right)$	42
risch	$-\frac{\ln(xa+1)}{4x^2}-\frac{\ln(-xa+1)a^2x^2-\ln(-xa-1)a^2x^2+2xa-\ln(xa-1)}{4x^2}$	60
orering	$\frac{(2a^2x^3-2x)\operatorname{arccoth}(xa)}{x^3}+\frac{(xa-1)(xa+1)x^2\left(-\frac{a}{(a^2x^2-1)x^3}-\frac{3\operatorname{arccoth}(xa)}{x^4}\right)}{2}$	64

input
```
int(arccoth(x*a)/x^3,x,method=_RETURNVERBOSE)
```

output
```
-1/2*(-arccoth(x*a)*a^2*x^2+x*a+arccoth(x*a))/x^2
```

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 35, normalized size of antiderivative = 1.13

$$\int \frac{\coth^{-1}(ax)}{x^3}\,dx = -\frac{2\,ax - (a^2x^2 - 1)\log\left(\frac{ax+1}{ax-1}\right)}{4\,x^2}$$

input
```
integrate(arccoth(a*x)/x^3,x, algorithm="fricas")
```

output
```
-1/4*(2*a*x - (a^2*x^2 - 1)*log((a*x + 1)/(a*x - 1)))/x^2
```

Sympy [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 24, normalized size of antiderivative = 0.77

$$\int \frac{\coth^{-1}(ax)}{x^3}\,dx = \frac{a^2\operatorname{acoth}(ax)}{2} - \frac{a}{2x} - \frac{\operatorname{acoth}(ax)}{2x^2}$$

input
```
integrate(acoth(a*x)/x**3,x)
```

output
```
a**2*acoth(a*x)/2 - a/(2*x) - acoth(a*x)/(2*x**2)
```

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 36, normalized size of antiderivative = 1.16

$$\int \frac{\coth^{-1}(ax)}{x^3}\,dx = \frac{1}{4}\left(a\log(ax+1) - a\log(ax-1) - \frac{2}{x}\right)a - \frac{\operatorname{arcoth}(ax)}{2\,x^2}$$

input
```
integrate(arccoth(a*x)/x^3,x, algorithm="maxima")
```

output
```
1/4*(a*log(a*x + 1) - a*log(a*x - 1) - 2/x)*a - 1/2*arccoth(a*x)/x^2
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 140 vs. $2(25) = 50$.

Time = 0.11 (sec) , antiderivative size = 140, normalized size of antiderivative = 4.52

$$\int \frac{\coth^{-1}(ax)}{x^3}\,dx = a\left(\frac{a}{\frac{ax+1}{ax-1}+1}+\frac{(ax+1)a\log\left(-\frac{\frac{\frac{(ax+1)a}{ax-1}-a}{a\left(\frac{ax+1}{ax-1}+1\right)}+1}{\frac{\frac{(ax+1)a}{ax-1}-a}{a\left(\frac{ax+1}{ax-1}+1\right)}-1}\right)}{(ax-1)\left(\frac{ax+1}{ax-1}+1\right)^2}\right)$$

input
```
integrate(arccoth(a*x)/x^3,x, algorithm="giac")
```

output
```
a*(a/((a*x + 1)/(a*x - 1) + 1) + (a*x + 1)*a*log(-(((a*x + 1)*a/(a*x - 1)
- a)/(a*((a*x + 1)/(a*x - 1) + 1)) + 1)/(((a*x + 1)*a/(a*x - 1) - a)/(a*((
a*x + 1)/(a*x - 1) + 1)) - 1))/((a*x - 1)*((a*x + 1)/(a*x - 1) + 1)^2))
```

Mupad [B] (verification not implemented)

Time = 3.60 (sec) , antiderivative size = 40, normalized size of antiderivative = 1.29

$$\int \frac{\coth^{-1}(ax)}{x^3}\,dx = \frac{a\,\mathrm{atan}\left(\frac{a^2\,x}{\sqrt{-a^2}}\right)\,\sqrt{-a^2}}{2}-\frac{\frac{\mathrm{acoth}(a\,x)}{2}+\frac{a\,x}{2}}{x^2}$$

input
```
int(acoth(a*x)/x^3,x)
```

output
```
(a*atan((a^2*x)/(-a^2)^(1/2))*(-a^2)^(1/2))/2 - (acoth(a*x)/2 + (a*x)/2)/x
^2
```

Reduce [B] (verification not implemented)

Time = 0.16 (sec) , antiderivative size = 26, normalized size of antiderivative = 0.84

$$\int \frac{\coth^{-1}(ax)}{x^3}\,dx = \frac{acoth(ax)\,a^2x^2 - acoth(ax) + ax}{2x^2}$$

input
```
int(acoth(a*x)/x^3,x)
```

output
```
(acoth(a*x)*a**2*x**2 - acoth(a*x) + a*x)/(2*x**2)
```

3.10 $\int \frac{\coth^{-1}(ax)}{x^4} \, dx$

Optimal result

Integrand size = 8, antiderivative size = 47

$$\int \frac{\coth^{-1}(ax)}{x^4} \, dx = -\frac{a}{6x^2} - \frac{\coth^{-1}(ax)}{3x^3} + \frac{1}{3}a^3 \log(x) - \frac{1}{6}a^3 \log\left(1 - a^2x^2\right)$$

output
```
-1/6*a/x^2-1/3*arccoth(a*x)/x^3+1/3*a^3*ln(x)-1/6*a^3*ln(-a^2*x^2+1)
```

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 47, normalized size of antiderivative = 1.00

$$\int \frac{\coth^{-1}(ax)}{x^4} \, dx = -\frac{a}{6x^2} - \frac{\coth^{-1}(ax)}{3x^3} + \frac{1}{3}a^3 \log(x) - \frac{1}{6}a^3 \log\left(1 - a^2x^2\right)$$

input
```
Integrate[ArcCoth[a*x]/x^4,x]
```

output
```
-1/6*a/x^2 - ArcCoth[a*x]/(3*x^3) + (a^3*Log[x])/3 - (a^3*Log[1 - a^2*x^2]
)/6
```

Rubi [A] (verified)

Time = 0.23 (sec) , antiderivative size = 47, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.500, Rules used = {6453, 243, 54, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\coth^{-1}(ax)}{x^4} \, dx$$

$$\Big\downarrow \; 6453$$

$$\frac{1}{3}a\int \frac{1}{x^3(1-a^2x^2)}dx-\frac{\coth^{-1}(ax)}{3x^3}$$

$$\Big\downarrow \; 243$$

$$\frac{1}{6}a\int \frac{1}{x^4(1-a^2x^2)}dx^2-\frac{\coth^{-1}(ax)}{3x^3}$$

$$\Big\downarrow \; 54$$

$$\frac{1}{6}a\int \left(-\frac{a^4}{a^2x^2-1}+\frac{a^2}{x^2}+\frac{1}{x^4}\right)dx^2-\frac{\coth^{-1}(ax)}{3x^3}$$

$$\Big\downarrow \; 2009$$

$$\frac{1}{6}a\left(a^2\log\left(x^2\right)-a^2\log\left(1-a^2x^2\right)-\frac{1}{x^2}\right)-\frac{\coth^{-1}(ax)}{3x^3}$$

input
```
Int[ArcCoth[a*x]/x^4,x]
```

output
```
-1/3*ArcCoth[a*x]/x^3 + (a*(-x^(-2) + a^2*Log[x^2] - a^2*Log[1 - a^2*x^2])
)/6
```

Defintions of rubi rules used

rule 54
```
Int[((a_) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_.), x_Symbol] :> Int[E
xpandIntegrand[(a + b*x)^m*(c + d*x)^n, x], x] /; FreeQ[{a, b, c, d}, x] &&
 ILtQ[m, 0] && IntegerQ[n] &&  !(IGtQ[n, 0] && LtQ[m + n + 2, 0])
```

rule 243
```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] :> Simp[1/2   Subst[In
t[x^((m - 1)/2)*(a + b*x)^p, x], x, x^2], x] /; FreeQ[{a, b, m, p}, x] && I
ntegerQ[(m - 1)/2]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

Maple [A] (verified)

Time = 0.09 (sec) , antiderivative size = 48, normalized size of antiderivative = 1.02

method	result	size
derivativedivides	$a^3\left(-\frac{\operatorname{arccoth}(xa)}{3x^3a^3}-\frac{\ln(xa-1)}{6}-\frac{\ln(xa+1)}{6}-\frac{1}{6x^2a^2}+\frac{\ln(xa)}{3}\right)$	48
default	$a^3\left(-\frac{\operatorname{arccoth}(xa)}{3x^3a^3}-\frac{\ln(xa-1)}{6}-\frac{\ln(xa+1)}{6}-\frac{1}{6x^2a^2}+\frac{\ln(xa)}{3}\right)$	48
parts	$-\frac{\operatorname{arccoth}(xa)}{3x^3}-\frac{a\left(\frac{1}{2x^2}-a^2\ln(x)+\frac{a^2\ln(xa+1)}{2}+\frac{a^2\ln(xa-1)}{2}\right)}{3}$	49
risch	$-\frac{\ln(xa+1)}{6x^3}+\frac{2\ln(x)a^3x^3-\ln(a^2x^2-1)a^3x^3-xa+\ln(xa-1)}{6x^3}$	57
parallelrisch	$\frac{2\ln(x)a^3x^3-2x^3\ln(xa-1)a^3-2x^3a^3\operatorname{arccoth}(xa)-x^3a^3-xa-2\operatorname{arccoth}(xa)}{6x^3}$	61

input
```
int(arccoth(x*a)/x^4,x,method=_RETURNVERBOSE)
```

output
```
a^3*(-1/3/x^3/a^3*arccoth(x*a)-1/6*ln(a*x-1)-1/6*ln(a*x+1)-1/6/x^2/a^2+1/3
*ln(x*a))
```

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 50, normalized size of antiderivative = 1.06

$$\int \frac{\coth^{-1}(ax)}{x^4}\,dx = -\frac{a^3x^3\log\left(a^2x^2-1\right) - 2\,a^3x^3\log\left(x\right) + ax + \log\left(\frac{ax+1}{ax-1}\right)}{6\,x^3}$$

input
```
integrate(arccoth(a*x)/x^4,x, algorithm="fricas")
```

output
```
-1/6*(a^3*x^3*log(a^2*x^2 - 1) - 2*a^3*x^3*log(x) + a*x + log((a*x + 1)/(a
*x - 1)))/x^3
```

Sympy [A] (verification not implemented)

Time = 0.25 (sec) , antiderivative size = 46, normalized size of antiderivative = 0.98

$$\int \frac{\coth^{-1}(ax)}{x^4}\,dx = \frac{a^3\log\left(x\right)}{3} - \frac{a^3\log\left(ax+1\right)}{3} + \frac{a^3\operatorname{acoth}\left(ax\right)}{3} - \frac{a}{6x^2} - \frac{\operatorname{acoth}\left(ax\right)}{3x^3}$$

input
```
integrate(acoth(a*x)/x**4,x)
```

output
```
a**3*log(x)/3 - a**3*log(a*x + 1)/3 + a**3*acoth(a*x)/3 - a/(6*x**2) - aco
th(a*x)/(3*x**3)
```

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 40, normalized size of antiderivative = 0.85

$$\int \frac{\coth^{-1}(ax)}{x^4}\,dx = -\frac{1}{6}\left(a^2\log\left(a^2x^2-1\right) - a^2\log\left(x^2\right) + \frac{1}{x^2}\right)a - \frac{\operatorname{arcoth}(ax)}{3\,x^3}$$

input
```
integrate(arccoth(a*x)/x^4,x, algorithm="maxima")
```

output
```
-1/6*(a^2*log(a^2*x^2 - 1) - a^2*log(x^2) + 1/x^2)*a - 1/3*arccoth(a*x)/x^
3
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 209 vs. 2(39) = 78.

Time = 0.12 (sec) , antiderivative size = 209, normalized size of antiderivative = 4.45

$$\int \frac{\coth^{-1}(ax)}{x^4}\,dx = -\frac{1}{3}\left(a^2\log\left(\frac{|ax+1|}{|ax-1|}\right) - a^2\log\left(\left|\frac{ax+1}{ax-1}+1\right|\right) - \frac{2\,(ax+1)a^2}{(ax-1)\left(\frac{ax+1}{ax-1}+1\right)^2} - \frac{\left(\frac{3\,(ax+1)^2a^2}{(ax-1)^2}+a^2\right)\log\left(-\cdots\right)}{\left(\frac{ax+1}{ax-1}+1\right)^3}\cdots\right.$$

input
```
integrate(arccoth(a*x)/x^4,x, algorithm="giac")
```

output
```
-1/3*(a^2*log(abs(a*x + 1)/abs(a*x - 1)) - a^2*log(abs((a*x + 1)/(a*x - 1)
 + 1)) - 2*(a*x + 1)*a^2/((a*x - 1)*((a*x + 1)/(a*x - 1) + 1)^2) - (3*(a*x
 + 1)^2*a^2/(a*x - 1)^2 + a^2)*log(-(((a*x + 1)*a/(a*x - 1) - a)/(a*((a*x
+ 1)/(a*x - 1) + 1)) + 1)/(((a*x + 1)*a/(a*x - 1) - a)/(a*((a*x + 1)/(a*x
- 1) + 1)) - 1))/((a*x + 1)/(a*x - 1) + 1)^3)*a
```

Mupad [B] (verification not implemented)

Time = 3.57 (sec) , antiderivative size = 39, normalized size of antiderivative = 0.83

$$\int \frac{\coth^{-1}(ax)}{x^4}\,dx = \frac{a^3\,\ln(x)}{3} - \frac{\frac{\mathrm{acoth}(a\,x)}{3} + \frac{a\,x}{6}}{x^3} - \frac{a^3\,\ln(a^2\,x^2-1)}{6}$$

input
```
int(acoth(a*x)/x^4,x)
```

output
```
(a^3*log(x))/3 - (acoth(a*x)/3 + (a*x)/6)/x^3 - (a^3*log(a^2*x^2 - 1))/6
```

Reduce [B] (verification not implemented)

Time = 0.17 (sec) , antiderivative size = 55, normalized size of antiderivative = 1.17

$$\int \frac{\coth^{-1}(ax)}{x^4}\,dx = \frac{-2acoth(ax)\,a^3x^3 - 2acoth(ax) + 2\log(a^2x-a)\,a^3x^3 - 2\log(x)\,a^3x^3 + ax}{6x^3}$$

input
```
int(acoth(a*x)/x^4,x)
```

output
```
( - 2*acoth(a*x)*a**3*x**3 - 2*acoth(a*x) + 2*log(a**2*x - a)*a**3*x**3 -
2*log(x)*a**3*x**3 + a*x)/(6*x**3)
```

3.11 $\int \frac{\coth^{-1}(ax)}{x^5} \, dx$

Optimal result

Integrand size = 8, antiderivative size = 41

$$\int \frac{\coth^{-1}(ax)}{x^5} \, dx = -\frac{a}{12x^3} - \frac{a^3}{4x} - \frac{\coth^{-1}(ax)}{4x^4} + \frac{1}{4}a^4\text{arctanh}(ax)$$

output
```
-1/12*a/x^3-1/4*a^3/x-1/4*arccoth(a*x)/x^4+1/4*a^4*arctanh(a*x)
```

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 57, normalized size of antiderivative = 1.39

$$\int \frac{\coth^{-1}(ax)}{x^5} \, dx = -\frac{a}{12x^3} - \frac{a^3}{4x} - \frac{\coth^{-1}(ax)}{4x^4} - \frac{1}{8}a^4\log(1-ax) + \frac{1}{8}a^4\log(1+ax)$$

input
```
Integrate[ArcCoth[a*x]/x^5,x]
```

output
```
-1/12*a/x^3 - a^3/(4*x) - ArcCoth[a*x]/(4*x^4) - (a^4*Log[1 - a*x])/8 + (a
^4*Log[1 + a*x])/8
```

Rubi [A] (verified)

Time = 0.20 (sec) , antiderivative size = 41, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$, Rules used = {6453, 264, 264, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\coth^{-1}(ax)}{x^5} \, dx$$

$$\Big\downarrow \text{6453}$$

$$\frac{1}{4}a\int \frac{1}{x^4\left(1-a^2x^2\right)}dx-\frac{\coth^{-1}(ax)}{4x^4}$$

$$\Big\downarrow \text{264}$$

$$\frac{1}{4}a\left(a^2\int \frac{1}{x^2\left(1-a^2x^2\right)}dx-\frac{1}{3x^3}\right)-\frac{\coth^{-1}(ax)}{4x^4}$$

$$\Big\downarrow \text{264}$$

$$\frac{1}{4}a\left(a^2\left(a^2\int \frac{1}{1-a^2x^2}dx-\frac{1}{x}\right)-\frac{1}{3x^3}\right)-\frac{\coth^{-1}(ax)}{4x^4}$$

$$\Big\downarrow \text{219}$$

$$\frac{1}{4}a\left(a^2\left(a\text{arctanh}(ax)-\frac{1}{x}\right)-\frac{1}{3x^3}\right)-\frac{\coth^{-1}(ax)}{4x^4}$$

input
```
Int[ArcCoth[a*x]/x^5,x]
```

output
```
-1/4*ArcCoth[a*x]/x^4 + (a*(-1/3*1/x^3 + a^2*(-x^(-1) + a*ArcTanh[a*x])))/
4
```

Defintions of rubi rules used

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 264
```
Int[((c_.)*(x_))^(m_)*((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(c*x)^(
m + 1)*((a + b*x^2)^(p + 1)/(a*c*(m + 1))), x] - Simp[b*((m + 2*p + 3)/(a*c
^2*(m + 1)))   Int[(c*x)^(m + 2)*(a + b*x^2)^p, x], x] /; FreeQ[{a, b, c, p
}, x] && LtQ[m, -1] && IntBinomialQ[a, b, c, 2, m, p, x]
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

Maple [A] (verified)

Time = 0.11 (sec) , antiderivative size = 36, normalized size of antiderivative = 0.88

method	result	size
parallelrisch	$-\frac{-3x^4a^4\operatorname{arccoth}(xa)+3x^3a^3+xa+3\operatorname{arccoth}(xa)}{12x^4}$	36
parts	$-\frac{\operatorname{arccoth}(xa)}{4x^4}-\frac{a\left(\frac{1}{3x^3}+\frac{a^2}{x}-\frac{a^3\ln(xa+1)}{2}+\frac{a^3\ln(xa-1)}{2}\right)}{4}$	49
derivativedivides	$a^4\left(-\frac{\operatorname{arccoth}(xa)}{4x^4a^4}-\frac{1}{12x^3a^3}-\frac{1}{4ax}+\frac{\ln(xa+1)}{8}-\frac{\ln(xa-1)}{8}\right)$	50
default	$a^4\left(-\frac{\operatorname{arccoth}(xa)}{4x^4a^4}-\frac{1}{12x^3a^3}-\frac{1}{4ax}+\frac{\ln(xa+1)}{8}-\frac{\ln(xa-1)}{8}\right)$	50
risch	$-\frac{\ln(xa+1)}{8x^4}+\frac{3\ln(-xa-1)a^4x^4-3\ln(-xa+1)a^4x^4-6x^3a^3-2xa+3\ln(xa-1)}{24x^4}$	69
orering	$\frac{\left(\frac{3}{2}a^4x^5-\frac{5}{6}a^2x^3-\frac{2}{3}x\right)\operatorname{arccoth}(xa)}{x^5}+\frac{(3a^2x^2+1)(xa-1)(xa+1)x^2\left(-\frac{a}{(a^2x^2-1)x^5}-\frac{5\operatorname{arccoth}(xa)}{x^6}\right)}{12}$	82

input
```
int(arccoth(x*a)/x^5,x,method=_RETURNVERBOSE)
```

output
```
-1/12*(-3*x^4*a^4*arccoth(x*a)+3*x^3*a^3+x*a+3*arccoth(x*a))/x^4
```

Fricas [A] (verification not implemented)

Time = 0.08 (sec) , antiderivative size = 43, normalized size of antiderivative = 1.05

$$\int \frac{\coth^{-1}(ax)}{x^5}\,dx = -\frac{6\,a^3x^3+2\,ax-3\,(a^4x^4-1)\log\left(\frac{ax+1}{ax-1}\right)}{24\,x^4}$$

input
```
integrate(arccoth(a*x)/x^5,x, algorithm="fricas")
```

output
```
-1/24*(6*a^3*x^3 + 2*a*x - 3*(a^4*x^4 - 1)*log((a*x + 1)/(a*x - 1)))/x^4
```

Sympy [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 32, normalized size of antiderivative = 0.78

$$\int \frac{\coth^{-1}(ax)}{x^5}\,dx = \frac{a^4\operatorname{acoth}(ax)}{4}-\frac{a^3}{4x}-\frac{a}{12x^3}-\frac{\operatorname{acoth}(ax)}{4x^4}$$

input
```
integrate(acoth(a*x)/x**5,x)
```

output
```
a**4*acoth(a*x)/4 - a**3/(4*x) - a/(12*x**3) - acoth(a*x)/(4*x**4)
```

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 51, normalized size of antiderivative = 1.24

$$\int \frac{\coth^{-1}(ax)}{x^5}\,dx = \frac{1}{24}\left(3\,a^3\log(ax+1)-3\,a^3\log(ax-1)-\frac{2\,(3\,a^2x^2+1)}{x^3}\right)a - \frac{\operatorname{arcoth}(ax)}{4\,x^4}$$

input
```
integrate(arccoth(a*x)/x^5,x, algorithm="maxima")
```

output
```
1/24*(3*a^3*log(a*x + 1) - 3*a^3*log(a*x - 1) - 2*(3*a^2*x^2 + 1)/x^3)*a -
 1/4*arccoth(a*x)/x^4
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 205 vs. $2(33) = 66$.

Time = 0.12 (sec) , antiderivative size = 205, normalized size of antiderivative = 5.00

$$\int \frac{\coth^{-1}(ax)}{x^5}\,dx = \frac{1}{3}a\left(\frac{\frac{3(ax+1)^2a^3}{(ax-1)^2}+\frac{3(ax+1)a^3}{ax-1}+2a^3}{\left(\frac{ax+1}{ax-1}+1\right)^3}+\frac{3\left(\frac{(ax+1)^3a^3}{(ax-1)^3}+\frac{(ax+1)a^3}{ax-1}\right)\log\left(-\frac{\frac{\frac{(ax+1)a}{ax-1}-a}{a\left(\frac{ax+1}{ax-1}+1\right)}+1}{\frac{\frac{(ax+1)a}{ax-1}-a}{a\left(\frac{ax+1}{ax-1}+1\right)}-1}\right)}{\left(\frac{ax+1}{ax-1}+1\right)^4}\right)$$

input
```
integrate(arccoth(a*x)/x^5,x, algorithm="giac")
```

output
```
1/3*a*((3*(a*x + 1)^2*a^3/(a*x - 1)^2 + 3*(a*x + 1)*a^3/(a*x - 1) + 2*a^3)
/((a*x + 1)/(a*x - 1) + 1)^3 + 3*((a*x + 1)^3*a^3/(a*x - 1)^3 + (a*x + 1)*
a^3/(a*x - 1))*log(-(((a*x + 1)*a/(a*x - 1) - a)/(a*((a*x + 1)/(a*x - 1) +
 1)) + 1)/(((a*x + 1)*a/(a*x - 1) - a)/(a*((a*x + 1)/(a*x - 1) + 1)) - 1))
/((a*x + 1)/(a*x - 1) + 1)^4)
```

Mupad [B] (verification not implemented)

Time = 3.97 (sec) , antiderivative size = 60, normalized size of antiderivative = 1.46

$$\int \frac{\coth^{-1}(ax)}{x^5}\,dx = \frac{\ln\left(1-\frac{1}{a\,x}\right)}{8\,x^4}-\frac{\ln\left(\frac{1}{a\,x}+1\right)}{8\,x^4}-\frac{a^3\,x^2+\frac{a}{3}}{4\,x^3}-\frac{a^4\,\mathrm{atan}(a\,x\,1\mathrm{i})\,1\mathrm{i}}{4}$$

input
```
int(acoth(a*x)/x^5,x)
```

output
```
log(1 - 1/(a*x))/(8*x^4) - (a^4*atan(a*x*1i)*1i)/4 - log(1/(a*x) + 1)/(8*x
^4) - (a/3 + a^3*x^2)/(4*x^3)
```

Reduce [B] (verification not implemented)

Time = 0.17 (sec) , antiderivative size = 35, normalized size of antiderivative = 0.85

$$\int \frac{\coth^{-1}(ax)}{x^5} \, dx = \frac{3acoth(ax) a^4 x^4 - 3acoth(ax) + 3a^3 x^3 + ax}{12x^4}$$

input
```
int(acoth(a*x)/x^5,x)
```

output
```
(3*acoth(a*x)*a**4*x**4 - 3*acoth(a*x) + 3*a**3*x**3 + a*x)/(12*x**4)
```

3.12 $\int x^5 \coth^{-1}(ax)^2 \, dx$

Optimal result

Integrand size = 10, antiderivative size = 105

$$\int x^5 \coth^{-1}(ax)^2 \, dx = \frac{4x^2}{45a^4} + \frac{x^4}{60a^2} + \frac{x \coth^{-1}(ax)}{3a^5} + \frac{x^3 \coth^{-1}(ax)}{9a^3} + \frac{x^5 \coth^{-1}(ax)}{15a} - \frac{\coth^{-1}(ax)^2}{6a^6} + \frac{1}{6}x^6 \coth^{-1}(ax)^2 + \frac{23 \log(1-a^2x^2)}{90a^6}$$

output
```
4/45*x^2/a^4+1/60*x^4/a^2+1/3*x*arccoth(a*x)/a^5+1/9*x^3*arccoth(a*x)/a^3+
1/15*x^5*arccoth(a*x)/a-1/6*arccoth(a*x)^2/a^6+1/6*x^6*arccoth(a*x)^2+23/9
0*ln(-a^2*x^2+1)/a^6
```

Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 80, normalized size of antiderivative = 0.76

$$\int x^5 \coth^{-1}(ax)^2 \, dx = \frac{16a^2x^2 + 3a^4x^4 + 4ax(15 + 5a^2x^2 + 3a^4x^4) \coth^{-1}(ax) + 30(-1 + a^6x^6) \coth^{-1}(ax)^2 + 46 \log(1 - a^2x^2}{180a^6}$$

input
```
Integrate[x^5*ArcCoth[a*x]^2,x]
```

output
```
(16*a^2*x^2 + 3*a^4*x^4 + 4*a*x*(15 + 5*a^2*x^2 + 3*a^4*x^4)*ArcCoth[a*x]
+ 30*(-1 + a^6*x^6)*ArcCoth[a*x]^2 + 46*Log[1 - a^2*x^2])/(180*a^6)
```

Rubi [A] (verified)

Time = 1.31 (sec) , antiderivative size = 177, normalized size of antiderivative = 1.69, number of steps used = 16, number of rules used = 15, $\frac{\text{number of rules}}{\text{integrand size}}$ = 1.500, Rules used = {6453, 6543, 6453, 243, 49, 2009, 6543, 6453, 243, 49, 2009, 6543, 6437, 240, 6511}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x^5 \coth^{-1}(ax)^2 \, dx$$

$$\Big\downarrow \text{6453}$$

$$\frac{1}{6}x^6 \coth^{-1}(ax)^2 - \frac{1}{3}a \int \frac{x^6 \coth^{-1}(ax)}{1-a^2x^2}dx$$

$$\Big\downarrow \text{6543}$$

$$\frac{1}{6}x^6 \coth^{-1}(ax)^2 - \frac{1}{3}a \left(\frac{\int \frac{x^4 \coth^{-1}(ax)}{1-a^2x^2}dx}{a^2} - \frac{\int x^4 \coth^{-1}(ax)dx}{a^2}\right)$$

$$\Big\downarrow \text{6453}$$

$$\frac{1}{6}x^6 \coth^{-1}(ax)^2 - \frac{1}{3}a \left(\frac{\int \frac{x^4 \coth^{-1}(ax)}{1-a^2x^2}dx}{a^2} - \frac{\frac{1}{5}x^5 \coth^{-1}(ax) - \frac{1}{5}a \int \frac{x^5}{1-a^2x^2}dx}{a^2}\right)$$

$$\Big\downarrow \text{243}$$

$$\frac{1}{6}x^6 \coth^{-1}(ax)^2 - \frac{1}{3}a \left(\frac{\int \frac{x^4 \coth^{-1}(ax)}{1-a^2x^2}dx}{a^2} - \frac{\frac{1}{5}x^5 \coth^{-1}(ax) - \frac{1}{10}a \int \frac{x^4}{1-a^2x^2}dx^2}{a^2}\right)$$

$$\Big\downarrow \text{49}$$

$$\frac{1}{6}x^6 \coth^{-1}(ax)^2 - \frac{1}{3}a \left(\frac{\int \frac{x^4 \coth^{-1}(ax)}{1-a^2x^2}dx}{a^2} - \frac{\frac{1}{5}x^5 \coth^{-1}(ax) - \frac{1}{10}a \int \left(-\frac{x^2}{a^2} - \frac{1}{a^4(a^2x^2-1)} - \frac{1}{a^4}\right)dx^2}{a^2}\right)$$

$$\Big\downarrow \text{2009}$$

$$\frac{1}{6}x^6\coth^{-1}(ax)^2-\frac{1}{3}a\left(\frac{\int \frac{x^4\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{5}x^5\coth^{-1}(ax)-\frac{1}{10}a\left(-\frac{x^2}{a^4}-\frac{x^4}{2a^2}-\frac{\log(1-a^2x^2)}{a^6}\right)}{a^2}\right)$$

$$\Big\downarrow \text{6543}$$

$$\frac{1}{6}x^6\coth^{-1}(ax)^2-\frac{1}{3}a\left(\frac{\frac{\int \frac{x^2\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\int x^2\coth^{-1}(ax)dx}{a^2}}{a^2}-\frac{\frac{1}{5}x^5\coth^{-1}(ax)-\frac{1}{10}a\left(-\frac{x^2}{a^4}-\frac{x^4}{2a^2}-\frac{\log(1-a^2x^2)}{a^6}\right)}{a^2}\right)$$

$$\Big\downarrow \text{6453}$$

$$\frac{1}{6}x^6\coth^{-1}(ax)^2-\frac{1}{3}a\left(\frac{\frac{\int \frac{x^2\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)-\frac{1}{3}a\int \frac{x^3}{1-a^2x^2}dx}{a^2}}{a^2}-\frac{\frac{1}{5}x^5\coth^{-1}(ax)-\frac{1}{10}a\left(-\frac{x^2}{a^4}-\frac{x^4}{2a^2}-\frac{\log(1-a^2x^2)}{a^6}\right)}{a^2}\right)$$

$$\Big\downarrow \text{243}$$

$$\frac{1}{6}x^6\coth^{-1}(ax)^2-\frac{1}{3}a\left(\frac{\frac{\int \frac{x^2\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)-\frac{1}{6}a\int \frac{x^2}{1-a^2x^2}dx^2}{a^2}}{a^2}-\frac{\frac{1}{5}x^5\coth^{-1}(ax)-\frac{1}{10}a\left(-\frac{x^2}{a^4}-\frac{x^4}{2a^2}-\frac{\log(1-a^2x^2)}{a^6}\right)}{a^2}\right)$$

$$\Big\downarrow \text{49}$$

$$\frac{1}{6}x^6\coth^{-1}(ax)^2-\frac{1}{3}a\left(\frac{\frac{\int \frac{x^2\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)-\frac{1}{6}a\int \left(-\frac{1}{a^2}-\frac{1}{a^2\left(a^2x^2-1\right)}\right)dx^2}{a^2}}{a^2}-\frac{\frac{1}{5}x^5\coth^{-1}(ax)-\frac{1}{10}a\left(-\frac{x^2}{a^4}-\frac{x^4}{2a^2}-\frac{\log(1-a^2x^2)}{a^6}\right.}{a^2}\right.$$

$$\Big\downarrow \text{2009}$$

$$\frac{1}{6}x^6\coth^{-1}(ax)^2-\frac{1}{3}a\left(\frac{\frac{\int \frac{x^2\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)-\frac{1}{6}a\left(-\frac{x^2}{a^2}-\frac{\log\left(1-a^2x^2\right)}{a^4}\right)}{a^2}}{a^2}-\frac{\frac{1}{5}x^5\coth^{-1}(ax)-\frac{1}{10}a\left(-\frac{x^2}{a^4}-\frac{x^4}{2a^2}-\frac{\log(1-a^2x^2)}{a^6}\right)}{a^2}\right)$$

$$\Big\downarrow \text{6543}$$

$$\frac{1}{6}x^6\coth^{-1}(ax)^2-\frac{1}{3}a\left(\frac{\frac{\frac{\int\frac{\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\int\coth^{-1}(ax)dx}{a^2}}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)-\frac{1}{6}a\left(-\frac{x^2}{a^2}-\frac{\log(1-a^2x^2)}{a^4}\right)}{a^2}}{a^2}-\frac{\frac{1}{5}x^5\coth^{-1}(ax)-\frac{1}{10}a\left(-\frac{x^2}{a^4}-\frac{x^4}{2a^2}-\frac{\log(\ldots}{\ldots}\right.}{a^2}\right.$$

$$\Big\downarrow \text{6437}$$

$$\frac{1}{6}x^6\coth^{-1}(ax)^2-\frac{1}{3}a\left(\frac{\frac{\frac{\int\frac{\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{x\coth^{-1}(ax)-a\int\frac{x}{1-a^2x^2}dx}{a^2}}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)-\frac{1}{6}a\left(-\frac{x^2}{a^2}-\frac{\log(1-a^2x^2)}{a^4}\right)}{a^2}}{a^2}-\frac{\frac{1}{5}x^5\coth^{-1}(ax)-\frac{1}{10}a\left(-\frac{x^2}{a^4}-\ldots\right.}{a^2}\right.$$

$$\Big\downarrow \text{240}$$

$$\frac{1}{6}x^6\coth^{-1}(ax)^2-\frac{1}{3}a\left(\frac{\frac{\frac{\int\frac{\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{\log(1-a^2x^2)}{2a}+x\coth^{-1}(ax)}{a^2}}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)-\frac{1}{6}a\left(-\frac{x^2}{a^2}-\frac{\log(1-a^2x^2)}{a^4}\right)}{a^2}}{a^2}-\frac{\frac{1}{5}x^5\coth^{-1}(ax)-\frac{1}{10}a\left(-\frac{x^2}{a^4}-\ldots\right.}{a^2}\right.$$

$$\Big\downarrow \text{6511}$$

$$\frac{1}{6}x^6\coth^{-1}(ax)^2-\frac{1}{3}a\left(\frac{\frac{\frac{\coth^{-1}(ax)^2}{2a^3}-\frac{\frac{\log(1-a^2x^2)}{2a}+x\coth^{-1}(ax)}{a^2}}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)-\frac{1}{6}a\left(-\frac{x^2}{a^2}-\frac{\log(1-a^2x^2)}{a^4}\right)}{a^2}}{a^2}-\frac{\frac{1}{5}x^5\coth^{-1}(ax)-\frac{1}{10}a\left(-\frac{x^2}{a^4}-\frac{x^4}{2a^2}\ldots\right.}{a^2}\right.$$

input
```
Int[x^5*ArcCoth[a*x]^2,x]
```

output
```
(x^6*ArcCoth[a*x]^2)/6 - (a*(-(((x^5*ArcCoth[a*x])/5 - (a*(-(x^2/a^4) - x^
4/(2*a^2) - Log[1 - a^2*x^2]/a^6))/10)/a^2) + (-(((x^3*ArcCoth[a*x])/3 - (
a*(-(x^2/a^2) - Log[1 - a^2*x^2]/a^4))/6)/a^2) + (ArcCoth[a*x]^2/(2*a^3) -
 (x*ArcCoth[a*x] + Log[1 - a^2*x^2]/(2*a))/a^2)/a^2)/a^2))/3
```

Defintions of rubi rules used

rule 49
```
Int[((a_.) + (b_.)*(x_))^(m_.)*((c_.) + (d_.)*(x_))^(n_.), x_Symbol] :> Int
[ExpandIntegrand[(a + b*x)^m*(c + d*x)^n, x], x] /; FreeQ[{a, b, c, d}, x]
&& IGtQ[m, 0] && IGtQ[m + n + 2, 0]
```

rule 240
```
Int[(x_)/((a_) + (b_.)*(x_)^2), x_Symbol] :> Simp[Log[RemoveContent[a + b*x
^2, x]]/(2*b), x] /; FreeQ[{a, b}, x]
```

rule 243
```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] :> Simp[1/2   Subst[In
t[x^((m - 1)/2)*(a + b*x)^p, x], x, x^2], x] /; FreeQ[{a, b, m, p}, x] && I
ntegerQ[(m - 1)/2]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 6437
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.), x_Symbol] :> Simp[x*(a
 + b*ArcCoth[c*x^n])^p, x] - Simp[b*c*n*p   Int[x^n*((a + b*ArcCoth[c*x^n])
^(p - 1)/(1 - c^2*x^(2*n))), x], x] /; FreeQ[{a, b, c, n}, x] && IGtQ[p, 0]
 && (EqQ[n, 1] || EqQ[p, 1])
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

rule 6511
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((d_) + (e_.)*(x_)^2), x_Symb
ol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*c*d*(p + 1)), x] /; FreeQ[{a, b
, c, d, e, p}, x] && EqQ[c^2*d + e, 0] && NeQ[p, -1]
```

rule 6543
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*((f_.)*(x_))^(m_))/((d_) + (
e_.)*(x_)^2), x_Symbol] :> Simp[f^2/e   Int[(f*x)^(m - 2)*(a + b*ArcCoth[c*
x])^p, x], x] - Simp[d*(f^2/e)   Int[(f*x)^(m - 2)*((a + b*ArcCoth[c*x])^p/
(d + e*x^2)), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && GtQ[p, 0] && GtQ[m,
 1]
```

Maple [A] (verified)

Time = 0.29 (sec) , antiderivative size = 92, normalized size of antiderivative = 0.88

method	result
parallelrisch	$-\frac{-30x^6a^6\operatorname{arccoth}(xa)^2-12x^5a^5\operatorname{arccoth}(xa)-16-3a^4x^4-20x^3a^3\operatorname{arccoth}(xa)-16a^2x^2-60xa\operatorname{arccoth}(xa)+30\operatorname{arcc}}{180a^6}$
parts	$\frac{x^6\operatorname{arccoth}(xa)^2}{6}+\frac{\frac{x^5a^5\operatorname{arccoth}(xa)}{5}+\frac{x^3a^3\operatorname{arccoth}(xa)}{3}+xa\operatorname{arccoth}(xa)+\frac{\operatorname{arccoth}(xa)\ln(xa-1)}{2}-\frac{\operatorname{arccoth}(xa)\ln(xa+1)}{2}+}{}$
derivativedivides	$\frac{\frac{x^6a^6\operatorname{arccoth}(xa)^2}{6}+\frac{x^5a^5\operatorname{arccoth}(xa)}{15}+\frac{x^3a^3\operatorname{arccoth}(xa)}{9}+\frac{xa\operatorname{arccoth}(xa)}{3}+\frac{\operatorname{arccoth}(xa)\ln(xa-1)}{6}-\frac{\operatorname{arccoth}(xa)\ln(xa+1)}{6}+\frac{\ln(}{}}{a}$
default	$\frac{\frac{x^6a^6\operatorname{arccoth}(xa)^2}{6}+\frac{x^5a^5\operatorname{arccoth}(xa)}{15}+\frac{x^3a^3\operatorname{arccoth}(xa)}{9}+\frac{xa\operatorname{arccoth}(xa)}{3}+\frac{\operatorname{arccoth}(xa)\ln(xa-1)}{6}-\frac{\operatorname{arccoth}(xa)\ln(xa+1)}{6}+\frac{\ln(}{}}{a}$
risch	$\frac{(a^6x^6-1)\ln(xa+1)^2}{24a^6}-\frac{(15x^6\ln(xa-1)a^6-6x^5a^5-10x^3a^3-30xa-15\ln(xa-1))\ln(xa+1)}{180a^6}+\frac{x^6\ln(xa-1)^2}{24}-\frac{x^5}{}$

input
```
int(x^5*arccoth(x*a)^2,x,method=_RETURNVERBOSE)
```

output
```
-1/180*(-30*x^6*a^6*arccoth(x*a)^2-12*x^5*a^5*arccoth(x*a)-16-3*a^4*x^4-20
*x^3*a^3*arccoth(x*a)-16*a^2*x^2-60*x*a*arccoth(x*a)+30*arccoth(x*a)^2-92*
ln(a*x-1)-92*arccoth(x*a))/a^6
```

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 98, normalized size of antiderivative = 0.93

$$\int x^5\coth^{-1}(ax)^2\,dx$$
$$=\frac{6\,a^4x^4+32\,a^2x^2+15\,(a^6x^6-1)\log\left(\frac{ax+1}{ax-1}\right)^2+4\,(3\,a^5x^5+5\,a^3x^3+15\,ax)\log\left(\frac{ax+1}{ax-1}\right)+92\,\log\left(a^2x^2-1\right)}{360\,a^6}$$

input
```
integrate(x^5*arccoth(a*x)^2,x, algorithm="fricas")
```

output
```
1/360*(6*a^4*x^4 + 32*a^2*x^2 + 15*(a^6*x^6 - 1)*log((a*x + 1)/(a*x - 1))^
2 + 4*(3*a^5*x^5 + 5*a^3*x^3 + 15*a*x)*log((a*x + 1)/(a*x - 1)) + 92*log(a
^2*x^2 - 1))/a^6
```

Sympy [A] (verification not implemented)

Time = 0.43 (sec) , antiderivative size = 114, normalized size of antiderivative = 1.09

$$\int x^5 \coth^{-1}(ax)^2\,dx$$
$$= \begin{cases} \frac{x^6\,\mathrm{acoth}^2\,(ax)}{6} + \frac{x^5\,\mathrm{acoth}\,(ax)}{15a} + \frac{x^4}{60a^2} + \frac{x^3\,\mathrm{acoth}\,(ax)}{9a^3} + \frac{4x^2}{45a^4} + \frac{x\,\mathrm{acoth}\,(ax)}{3a^5} + \frac{23\log\,(ax+1)}{45a^6} - \frac{\mathrm{acoth}^2\,(ax)}{6a^6} - \frac{23\,\mathrm{acoth}\,(ax)}{45a^6} \\ -\frac{\pi^2x^6}{24} \end{cases}$$

input
```
integrate(x**5*acoth(a*x)**2,x)
```

output
```
Piecewise((x**6*acoth(a*x)**2/6 + x**5*acoth(a*x)/(15*a) + x**4/(60*a**2)
+ x**3*acoth(a*x)/(9*a**3) + 4*x**2/(45*a**4) + x*acoth(a*x)/(3*a**5) + 23
*log(a*x + 1)/(45*a**6) - acoth(a*x)**2/(6*a**6) - 23*acoth(a*x)/(45*a**6)
, Ne(a, 0)), (-pi**2*x**6/24, True))
```

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 135, normalized size of antiderivative = 1.29

$$\int x^5 \coth^{-1}(ax)^2\,dx = \frac{1}{6}\,x^6\,\mathrm{arcoth}\,(ax)^2$$
$$+ \frac{1}{90}\,a\left(\frac{2\,(3\,a^4x^5 + 5\,a^2x^3 + 15\,x)}{a^6} - \frac{15\,\log\,(ax+1)}{a^7} + \frac{15\,\log\,(ax-1)}{a^7}\right)\mathrm{arcoth}\,(ax)$$
$$+ \frac{6\,a^4x^4 + 32\,a^2x^2 - 2\,(15\,\log\,(ax-1) - 46)\log\,(ax+1) + 15\,\log\,(ax+1)^2 + 15\,\log\,(ax-1)^2 + 92\,\mathrm{lo}}{360\,a^6}$$

input
```
integrate(x^5*arccoth(a*x)^2,x, algorithm="maxima")
```

output
```
1/6*x^6*arccoth(a*x)^2 + 1/90*a*(2*(3*a^4*x^5 + 5*a^2*x^3 + 15*x)/a^6 - 15
*log(a*x + 1)/a^7 + 15*log(a*x - 1)/a^7)*arccoth(a*x) + 1/360*(6*a^4*x^4 +
 32*a^2*x^2 - 2*(15*log(a*x - 1) - 46)*log(a*x + 1) + 15*log(a*x + 1)^2 +
15*log(a*x - 1)^2 + 92*log(a*x - 1))/a^6
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 534 vs. $2(89) = 178$.

Time = 0.13 (sec) , antiderivative size = 534, normalized size of antiderivative = 5.09

$$\int x^5 \coth^{-1}(ax)^2 \, dx = \text{Too large to display}$$

input
```
integrate(x^5*arccoth(a*x)^2,x, algorithm="giac")
```

output
```
1/90*(15*(3*(a*x + 1)^5/(a*x - 1)^5 + 10*(a*x + 1)^3/(a*x - 1)^3 + 3*(a*x
+ 1)/(a*x - 1))*log((a*x + 1)/(a*x - 1))^2/((a*x + 1)^6*a^7/(a*x - 1)^6 -
6*(a*x + 1)^5*a^7/(a*x - 1)^5 + 15*(a*x + 1)^4*a^7/(a*x - 1)^4 - 20*(a*x +
 1)^3*a^7/(a*x - 1)^3 + 15*(a*x + 1)^2*a^7/(a*x - 1)^2 - 6*(a*x + 1)*a^7/(
a*x - 1) + a^7) + 2*(45*(a*x + 1)^4/(a*x - 1)^4 - 90*(a*x + 1)^3/(a*x - 1)
^3 + 140*(a*x + 1)^2/(a*x - 1)^2 - 70*(a*x + 1)/(a*x - 1) + 23)*log((a*x +
 1)/(a*x - 1))/((a*x + 1)^5*a^7/(a*x - 1)^5 - 5*(a*x + 1)^4*a^7/(a*x - 1)^
4 + 10*(a*x + 1)^3*a^7/(a*x - 1)^3 - 10*(a*x + 1)^2*a^7/(a*x - 1)^2 + 5*(a
*x + 1)*a^7/(a*x - 1) - a^7) + 4*(11*(a*x + 1)^3/(a*x - 1)^3 - 16*(a*x + 1
)^2/(a*x - 1)^2 + 11*(a*x + 1)/(a*x - 1))/((a*x + 1)^4*a^7/(a*x - 1)^4 - 4
*(a*x + 1)^3*a^7/(a*x - 1)^3 + 6*(a*x + 1)^2*a^7/(a*x - 1)^2 - 4*(a*x + 1)
*a^7/(a*x - 1) + a^7) - 46*log((a*x + 1)/(a*x - 1) - 1)/a^7 + 46*log((a*x
+ 1)/(a*x - 1))/a^7)*a
```

Mupad [B] (verification not implemented)

Time = 3.76 (sec) , antiderivative size = 85, normalized size of antiderivative = 0.81

$$\int x^5 \coth^{-1}(ax)^2 \, dx$$
$$= \frac{x^6 \operatorname{acoth}(a\,x)^2}{6} + \frac{\frac{23\,\ln(a^2\,x^2-1)}{90} + \frac{4\,a^2\,x^2}{45} + \frac{a^4\,x^4}{60} - \frac{\operatorname{acoth}(a\,x)^2}{6} + \frac{a^3\,x^3\operatorname{acoth}(a\,x)}{9} + \frac{a^5\,x^5\operatorname{acoth}(a\,x)}{15} + \frac{a\,x\operatorname{acoth}(a\,x)}{3}}{a^6}$$

input
```
int(x^5*acoth(a*x)^2,x)
```

output
```
(x^6*acoth(a*x)^2)/6 + ((23*log(a^2*x^2 - 1))/90 + (4*a^2*x^2)/45 + (a^4*x
^4)/60 - acoth(a*x)^2/6 + (a^3*x^3*acoth(a*x))/9 + (a^5*x^5*acoth(a*x))/15
 + (a*x*acoth(a*x))/3)/a^6
```

Reduce [B] (verification not implemented)

Time = 0.16 (sec) , antiderivative size = 94, normalized size of antiderivative = 0.90

$$\int x^5 \coth^{-1}(ax)^2 \, dx$$
$$= \frac{30acoth(ax)^2 a^6x^6 - 30acoth(ax)^2 - 12acoth(ax)\,a^5x^5 - 20acoth(ax)\,a^3x^3 - 60acoth(ax)\,ax - 92acoth}{180a^6}$$

input
```
int(x^5*acoth(a*x)^2,x)
```

output
```
(30*acoth(a*x)**2*a**6*x**6 - 30*acoth(a*x)**2 - 12*acoth(a*x)*a**5*x**5 -
 20*acoth(a*x)*a**3*x**3 - 60*acoth(a*x)*a*x - 92*acoth(a*x) + 92*log(a**2
*x - a) + 3*a**4*x**4 + 16*a**2*x**2)/(180*a**6)
```

3.13 $\int x^4 \coth^{-1}(ax)^2 \, dx$

Optimal result

Integrand size = 10, antiderivative size = 127

$$\int x^4 \coth^{-1}(ax)^2 \, dx = \frac{3x}{10a^4} + \frac{x^3}{30a^2} + \frac{x^2\coth^{-1}(ax)}{5a^3} + \frac{x^4\coth^{-1}(ax)}{10a} + \frac{\coth^{-1}(ax)^2}{5a^5} + \frac{1}{5}x^5\coth^{-1}(ax)^2 - \frac{3\text{arctanh}(ax)}{10a^5} - \frac{2\coth^{-1}(ax)\log\left(\frac{2}{1-ax}\right)}{5a^5} - \frac{\text{PolyLog}\left(2,1-\frac{2}{1-ax}\right)}{5a^5}$$

output
```
3/10*x/a^4+1/30*x^3/a^2+1/5*x^2*arccoth(a*x)/a^3+1/10*x^4*arccoth(a*x)/a+1
/5*arccoth(a*x)^2/a^5+1/5*x^5*arccoth(a*x)^2-3/10*arctanh(a*x)/a^5-2/5*arc
coth(a*x)*ln(2/(-a*x+1))/a^5-1/5*polylog(2,1-2/(-a*x+1))/a^5
```

Mathematica [A] (verified)

Time = 0.30 (sec) , antiderivative size = 87, normalized size of antiderivative = 0.69

$$\int x^4 \coth^{-1}(ax)^2 \, dx = \frac{ax(9+a^2x^2)+6(-1+a^5x^5)\coth^{-1}(ax)^2+3\coth^{-1}(ax)\left(-3+2a^2x^2+a^4x^4-4\log\left(1-e^{-2\coth^{-1}(ax)}\right.\right.}{30a^5}$$

input
```
Integrate[x^4*ArcCoth[a*x]^2,x]
```

output
```
(a*x*(9 + a^2*x^2) + 6*(-1 + a^5*x^5)*ArcCoth[a*x]^2 + 3*ArcCoth[a*x]*(-3
+ 2*a^2*x^2 + a^4*x^4 - 4*Log[1 - E^(-2*ArcCoth[a*x])]) + 6*PolyLog[2, E^(
-2*ArcCoth[a*x])])/(30*a^5)
```

Rubi [A] (verified)

Time = 1.13 (sec) , antiderivative size = 172, normalized size of antiderivative = 1.35, number of steps used = 14, number of rules used = 13, $\frac{\text{number of rules}}{\text{integrand size}}$ = 1.300, Rules used = {6453, 6543, 6453, 254, 2009, 6543, 6453, 262, 219, 6547, 6471, 2849, 2752}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x^4 \coth^{-1}(ax)^2 \, dx$$

$$\Big\downarrow \text{6453}$$

$$\frac{1}{5}x^5 \coth^{-1}(ax)^2 - \frac{2}{5}a \int \frac{x^5 \coth^{-1}(ax)}{1-a^2x^2}dx$$

$$\Big\downarrow \text{6543}$$

$$\frac{1}{5}x^5 \coth^{-1}(ax)^2 - \frac{2}{5}a \left(\frac{\int \frac{x^3 \coth^{-1}(ax)}{1-a^2x^2}dx}{a^2} - \frac{\int x^3 \coth^{-1}(ax)dx}{a^2}\right)$$

$$\Big\downarrow \text{6453}$$

$$\frac{1}{5}x^5 \coth^{-1}(ax)^2 - \frac{2}{5}a \left(\frac{\int \frac{x^3 \coth^{-1}(ax)}{1-a^2x^2}dx}{a^2} - \frac{\frac{1}{4}x^4 \coth^{-1}(ax) - \frac{1}{4}a \int \frac{x^4}{1-a^2x^2}dx}{a^2}\right)$$

$$\Big\downarrow \text{254}$$

$$\frac{1}{5}x^5 \coth^{-1}(ax)^2 - \frac{2}{5}a \left(\frac{\int \frac{x^3 \coth^{-1}(ax)}{1-a^2x^2}dx}{a^2} - \frac{\frac{1}{4}x^4 \coth^{-1}(ax) - \frac{1}{4}a \int \left(-\frac{x^2}{a^2} + \frac{1}{a^4(1-a^2x^2)} - \frac{1}{a^4}\right)dx}{a^2}\right)$$

$$\Big\downarrow \text{2009}$$

$$\frac{1}{5}x^5\coth^{-1}(ax)^2-\frac{2}{5}a\left(\frac{\int\frac{x^3\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{4}x^4\coth^{-1}(ax)-\frac{1}{4}a\left(\frac{\text{arctanh}(ax)}{a^5}-\frac{x}{a^4}-\frac{x^3}{3a^2}\right)}{a^2}\right)$$

$\big\downarrow$ 6543

$$\frac{1}{5}x^5\coth^{-1}(ax)^2-\frac{2}{5}a\left(\frac{\frac{\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\int x\coth^{-1}(ax)dx}{a^2}}{a^2}-\frac{\frac{1}{4}x^4\coth^{-1}(ax)-\frac{1}{4}a\left(\frac{\text{arctanh}(ax)}{a^5}-\frac{x}{a^4}-\frac{x^3}{3a^2}\right)}{a^2}\right)$$

$\big\downarrow$ 6453

$$\frac{1}{5}x^5\coth^{-1}(ax)^2-\frac{2}{5}a\left(\frac{\frac{\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)-\frac{1}{2}a\int\frac{x^2}{1-a^2x^2}dx}{a^2}}{a^2}-\frac{\frac{1}{4}x^4\coth^{-1}(ax)-\frac{1}{4}a\left(\frac{\text{arctanh}(ax)}{a^5}-\frac{x}{a^4}-\frac{x^3}{3a^2}\right)}{a^2}\right)$$

$\big\downarrow$ 262

$$\frac{1}{5}x^5\coth^{-1}(ax)^2-\frac{2}{5}a\left(\frac{\frac{\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)-\frac{1}{2}a\left(\frac{\int\frac{1}{1-a^2x^2}dx}{a^2}-\frac{x}{a^2}\right)}{a^2}}{a^2}-\frac{\frac{1}{4}x^4\coth^{-1}(ax)-\frac{1}{4}a\left(\frac{\text{arctanh}(ax)}{a^5}-\frac{x}{a^4}-\frac{x^3}{3a^2}\right)}{a^2}\right)$$

$\big\downarrow$ 219

$$\frac{1}{5}x^5\coth^{-1}(ax)^2-\frac{2}{5}a\left(\frac{\frac{\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)-\frac{1}{2}a\left(\frac{\text{arctanh}(ax)}{a^3}-\frac{x}{a^2}\right)}{a^2}}{a^2}-\frac{\frac{1}{4}x^4\coth^{-1}(ax)-\frac{1}{4}a\left(\frac{\text{arctanh}(ax)}{a^5}-\frac{x}{a^4}-\frac{x^3}{3a^2}\right)}{a^2}\right)$$

$\big\downarrow$ 6547

$$\frac{1}{5}x^5\coth^{-1}(ax)^2-\frac{2}{5}a\left(\frac{\frac{\frac{\int\frac{\coth^{-1}(ax)}{1-ax}dx}{a}-\frac{\coth^{-1}(ax)^2}{2a^2}}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)-\frac{1}{2}a\left(\frac{\text{arctanh}(ax)}{a^3}-\frac{x}{a^2}\right)}{a^2}}{a^2}-\frac{\frac{1}{4}x^4\coth^{-1}(ax)-\frac{1}{4}a\left(\frac{\text{arctanh}(ax)}{a^5}-\frac{x}{a^4}-\right.}{a^2}\right.$$

$\big\downarrow$ 6471

$$\frac{1}{5}x^5\coth^{-1}(ax)^2-$$
$$\frac{2}{5}a\left(\frac{\frac{\frac{\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)}{a}-\int\frac{\log\left(\frac{2}{1-ax}\right)}{1-a^2x^2}dx}{a}-\frac{\coth^{-1}(ax)^2}{2a^2}}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)-\frac{1}{2}a\left(\frac{\text{arctanh}(ax)}{a^3}-\frac{x}{a^2}\right)}{a^2}}{a^2}-\frac{\frac{1}{4}x^4\coth^{-1}(ax)-\frac{1}{4}a\left(\text{a}\right.}{a^2}\right.$$

$\Big\downarrow$ 2849

$$\frac{1}{5}x^5\coth^{-1}(ax)^2-$$
$$\frac{2}{5}a\left(\frac{\frac{\frac{\frac{\int\frac{\log\left(\frac{2}{1-ax}\right)}{1-\frac{2}{1-ax}}d\frac{1}{1-ax}}{a}+\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)}{a}}{a}-\frac{\coth^{-1}(ax)^2}{2a^2}}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)-\frac{1}{2}a\left(\frac{\text{arctanh}(ax)}{a^3}-\frac{x}{a^2}\right)}{a^2}}{a^2}-\frac{\frac{1}{4}x^4\coth^{-1}(ax)-\frac{1}{4}c}{}\right.$$

$\Big\downarrow$ 2752

$$\frac{1}{5}x^5\coth^{-1}(ax)^2-$$
$$\frac{2}{5}a\left(\frac{\frac{\frac{\frac{\text{PolyLog}\left(2,1-\frac{2}{1-ax}\right)}{2a}+\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)}{a}}{a}-\frac{\coth^{-1}(ax)^2}{2a^2}}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)-\frac{1}{2}a\left(\frac{\text{arctanh}(ax)}{a^3}-\frac{x}{a^2}\right)}{a^2}}{a^2}-\frac{\frac{1}{4}x^4\coth^{-1}(ax)-\frac{1}{4}c}{}\right.$$

input
```
Int[x^4*ArcCoth[a*x]^2,x]
```

output
```
(x^5*ArcCoth[a*x]^2)/5 - (2*a*(-((((x^4*ArcCoth[a*x])/4 - (a*(-(x/a^4) - x^
3/(3*a^2) + ArcTanh[a*x]/a^5))/4)/a^2) + (-(((x^2*ArcCoth[a*x])/2 - (a*(-(
x/a^2) + ArcTanh[a*x]/a^3))/2)/a^2) + (-1/2*ArcCoth[a*x]^2/a^2 + ((ArcCoth
[a*x]*Log[2/(1 - a*x)])/a + PolyLog[2, 1 - 2/(1 - a*x)]/(2*a))/a)/a^2)/a^2
))/5
```

Defintions of rubi rules used

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 254
```
Int[(x_)^(m_)/((a_) + (b_.)*(x_)^2), x_Symbol] :> Int[PolynomialDivide[x^m,
 a + b*x^2, x], x] /; FreeQ[{a, b}, x] && IGtQ[m, 3]
```

rule 262
```
Int[((c_.)*(x_))^(m_)*((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] :> Simp[c*(c*x)
^(m - 1)*((a + b*x^2)^(p + 1)/(b*(m + 2*p + 1))), x] - Simp[a*c^2*((m - 1)/
(b*(m + 2*p + 1)))   Int[(c*x)^(m - 2)*(a + b*x^2)^p, x], x] /; FreeQ[{a, b
, c, p}, x] && GtQ[m, 2 - 1] && NeQ[m + 2*p + 1, 0] && IntBinomialQ[a, b, c
, 2, m, p, x]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2752
```
Int[Log[(c_.)*(x_)]/((d_) + (e_.)*(x_)), x_Symbol] :> Simp[(-e^(-1))*PolyLo
g[2, 1 - c*x], x] /; FreeQ[{c, d, e}, x] && EqQ[e + c*d, 0]
```

rule 2849
```
Int[Log[(c_.)/((d_) + (e_.)*(x_))]/((f_) + (g_.)*(x_)^2), x_Symbol] :> Simp
[-e/g   Subst[Int[Log[2*d*x]/(1 - 2*d*x), x], x, 1/(d + e*x)], x] /; FreeQ[
{c, d, e, f, g}, x] && EqQ[c, 2*d] && EqQ[e^2*f + d^2*g, 0]
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

rule 6471
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((d_) + (e_.)*(x_)), x_Symbol
] :> Simp[(-(a + b*ArcCoth[c*x])^p)*(Log[2/(1 + e*(x/d))]/e), x] + Simp[b*c
*(p/e)   Int[(a + b*ArcCoth[c*x])^(p - 1)*(Log[2/(1 + e*(x/d))]/(1 - c^2*x^
2)), x], x] /; FreeQ[{a, b, c, d, e}, x] && IGtQ[p, 0] && EqQ[c^2*d^2 - e^2
, 0]
```

rule 6543
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*((f_.)*(x_))^(m_))/((d_) + (
e_.)*(x_)^2), x_Symbol] :> Simp[f^2/e   Int[(f*x)^(m - 2)*(a + b*ArcCoth[c*
x])^p, x], x] - Simp[d*(f^2/e)   Int[(f*x)^(m - 2)*((a + b*ArcCoth[c*x])^p/
(d + e*x^2)), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && GtQ[p, 0] && GtQ[m,
 1]
```

rule 6547
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*(x_))/((d_) + (e_.)*(x_)^2),
 x_Symbol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*e*(p + 1)), x] + Simp[1/
(c*d)   Int[(a + b*ArcCoth[c*x])^p/(1 - c*x), x], x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[c^2*d + e, 0] && IGtQ[p, 0]
```

Maple [A] (verified)

Time = 0.32 (sec) , antiderivative size = 164, normalized size of antiderivative = 1.29

method	result
parts	$\frac{x^5 \operatorname{arccoth}(xa)^2}{5} + \frac{x^4 a^4 \operatorname{arccoth}(xa)}{10} + \frac{\operatorname{arccoth}(xa) a^2 x^2}{5} + \frac{\operatorname{arccoth}(xa) \ln(xa-1)}{5} + \frac{\operatorname{arccoth}(xa) \ln(xa+1)}{5} + \frac{x^3 a^3}{30} + \frac{3xa}{10} +$
derivativedivides	$\frac{x^5 a^5 \operatorname{arccoth}(xa)^2}{5} + \frac{x^4 a^4 \operatorname{arccoth}(xa)}{10} + \frac{\operatorname{arccoth}(xa) a^2 x^2}{5} + \frac{\operatorname{arccoth}(xa) \ln(xa-1)}{5} + \frac{\operatorname{arccoth}(xa) \ln(xa+1)}{5} + \frac{x^3 a^3}{30} + \frac{3xa}{10} +$
default	$\frac{x^5 a^5 \operatorname{arccoth}(xa)^2}{5} + \frac{x^4 a^4 \operatorname{arccoth}(xa)}{10} + \frac{\operatorname{arccoth}(xa) a^2 x^2}{5} + \frac{\operatorname{arccoth}(xa) \ln(xa-1)}{5} + \frac{\operatorname{arccoth}(xa) \ln(xa+1)}{5} + \frac{x^3 a^3}{30} + \frac{3xa}{10} +$
risch	$\frac{413}{2250a^5} - \frac{x^5 \ln(xa+1)}{50} + \frac{\ln(xa+1) x^4}{40a} - \frac{\ln(xa+1) x^3}{30a^2} + \frac{\ln(xa+1) x^2}{20a^3} - \frac{\ln(xa+1) x}{10a^4} - \frac{\ln(xa-1) x^4}{40a} -$

input
```
int(x^4*arccoth(x*a)^2,x,method=_RETURNVERBOSE)
```

output
```
1/5*x^5*arccoth(x*a)^2+2/5/a^5*(1/4*x^4*a^4*arccoth(x*a)+1/2*arccoth(x*a)*
a^2*x^2+1/2*arccoth(x*a)*ln(a*x-1)+1/2*arccoth(x*a)*ln(a*x+1)+1/12*x^3*a^3
+3/4*x*a+3/8*ln(a*x-1)-3/8*ln(a*x+1)-1/2*dilog(1/2*x*a+1/2)-1/4*ln(a*x-1)*
ln(1/2*x*a+1/2)+1/8*ln(a*x-1)^2-1/8*ln(a*x+1)^2+1/4*(ln(a*x+1)-ln(1/2*x*a+
1/2))*ln(-1/2*x*a+1/2))
```

Fricas [F]

$$\int x^4 \coth^{-1}(ax)^2\,dx = \int x^4 \operatorname{arcoth}\left(ax\right)^2\,dx$$

input
```
integrate(x^4*arccoth(a*x)^2,x, algorithm="fricas")
```

output
```
integral(x^4*arccoth(a*x)^2, x)
```

Sympy [F]

$$\int x^4 \coth^{-1}(ax)^2\,dx = \int x^4 \operatorname{acoth}^2\left(ax\right)dx$$

input
```
integrate(x**4*acoth(a*x)**2,x)
```

output
```
Integral(x**4*acoth(a*x)**2, x)
```

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 155, normalized size of antiderivative = 1.22

$$\int x^4 \coth^{-1}(ax)^2\,dx = \frac{1}{5}\,x^5 \operatorname{arcoth}(ax)^2 + \frac{1}{60}\,a^2\left(\frac{2\,a^3x^3 + 18\,ax - 3\,\log(ax+1)^2 + 6\,\log(ax+1)\log(ax-1) + 3\,\log(ax-1)^2 + 9\,\log(ax-\right.}{a^7} + \frac{1}{10}\,a\left(\frac{a^2x^4 + 2\,x^2}{a^4} + \frac{2\,\log(a^2x^2-1)}{a^6}\right)\operatorname{arcoth}(ax)$$

input
```
integrate(x^4*arccoth(a*x)^2,x, algorithm="maxima")
```

output
```
1/5*x^5*arccoth(a*x)^2 + 1/60*a^2*((2*a^3*x^3 + 18*a*x - 3*log(a*x + 1)^2
+ 6*log(a*x + 1)*log(a*x - 1) + 3*log(a*x - 1)^2 + 9*log(a*x - 1))/a^7 - 1
2*(log(a*x - 1)*log(1/2*a*x + 1/2) + dilog(-1/2*a*x + 1/2))/a^7 - 9*log(a*
x + 1)/a^7) + 1/10*a*((a^2*x^4 + 2*x^2)/a^4 + 2*log(a^2*x^2 - 1)/a^6)*arcc
oth(a*x)
```

Giac [F]

$$\int x^4 \coth^{-1}(ax)^2\,dx = \int x^4 \operatorname{arcoth}(ax)^2\,dx$$

input
```
integrate(x^4*arccoth(a*x)^2,x, algorithm="giac")
```

output
```
integrate(x^4*arccoth(a*x)^2, x)
```

Mupad [F(-1)]

Timed out.

$$\int x^4 \coth^{-1}(ax)^2 \, dx = \int x^4 \operatorname{acoth}(a\,x)^2 \, dx$$

input
```
int(x^4*acoth(a*x)^2,x)
```

output
```
int(x^4*acoth(a*x)^2, x)
```

Reduce [F]

$$\int x^4 \coth^{-1}(ax)^2 \, dx$$
$$= \frac{6acoth(ax)^2 a^5x^5 - 6acoth(ax)^2 ax - 3acoth(ax)\, a^4x^4 - 6acoth(ax)\, a^2x^2 + 9acoth(ax) + 6\left(\int acoth(ax)\right.}{30a^5}$$

input
```
int(x^4*acoth(a*x)^2,x)
```

output
```
(6*acoth(a*x)**2*a**5*x**5 - 6*acoth(a*x)**2*a*x - 3*acoth(a*x)*a**4*x**4
- 6*acoth(a*x)*a**2*x**2 + 9*acoth(a*x) + 6*int(acoth(a*x)**2,x)*a + a**3*
x**3 + 9*a*x)/(30*a**5)
```

3.14 $\int x^3 \coth^{-1}(ax)^2 \, dx$

Optimal result

Integrand size = 10, antiderivative size = 81

$$\int x^3 \coth^{-1}(ax)^2 \, dx = \frac{x^2}{12a^2} + \frac{x \coth^{-1}(ax)}{2a^3} + \frac{x^3 \coth^{-1}(ax)}{6a} - \frac{\coth^{-1}(ax)^2}{4a^4} + \frac{1}{4} x^4 \coth^{-1}(ax)^2 + \frac{\log(1-a^2x^2)}{3a^4}$$

output
```
1/12*x^2/a^2+1/2*x*arccoth(a*x)/a^3+1/6*x^3*arccoth(a*x)/a-1/4*arccoth(a*x
)^2/a^4+1/4*x^4*arccoth(a*x)^2+1/3*ln(-a^2*x^2+1)/a^4
```

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 62, normalized size of antiderivative = 0.77

$$\int x^3 \coth^{-1}(ax)^2 \, dx = \frac{a^2x^2 + 2ax(3+a^2x^2)\coth^{-1}(ax) + 3(-1+a^4x^4)\coth^{-1}(ax)^2 + 4\log(1-a^2x^2)}{12a^4}$$

input
```
Integrate[x^3*ArcCoth[a*x]^2,x]
```

output
```
(a^2*x^2 + 2*a*x*(3 + a^2*x^2)*ArcCoth[a*x] + 3*(-1 + a^4*x^4)*ArcCoth[a*x
]^2 + 4*Log[1 - a^2*x^2])/(12*a^4)
```

Rubi [A] (verified)

Time = 0.83 (sec) , antiderivative size = 115, normalized size of antiderivative = 1.42, number of steps used = 11, number of rules used = 10, $\frac{\text{number of rules}}{\text{integrand size}}$ = 1.000, Rules used = {6453, 6543, 6453, 243, 49, 2009, 6543, 6437, 240, 6511}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x^3 \coth^{-1}(ax)^2 \, dx$$

$$\Big\downarrow \text{6453}$$

$$\frac{1}{4}x^4 \coth^{-1}(ax)^2 - \frac{1}{2}a\int \frac{x^4 \coth^{-1}(ax)}{1-a^2x^2}dx$$

$$\Big\downarrow \text{6543}$$

$$\frac{1}{4}x^4 \coth^{-1}(ax)^2 - \frac{1}{2}a\left(\frac{\int \frac{x^2 \coth^{-1}(ax)}{1-a^2x^2}dx}{a^2} - \frac{\int x^2 \coth^{-1}(ax)dx}{a^2}\right)$$

$$\Big\downarrow \text{6453}$$

$$\frac{1}{4}x^4 \coth^{-1}(ax)^2 - \frac{1}{2}a\left(\frac{\int \frac{x^2 \coth^{-1}(ax)}{1-a^2x^2}dx}{a^2} - \frac{\frac{1}{3}x^3 \coth^{-1}(ax) - \frac{1}{3}a\int \frac{x^3}{1-a^2x^2}dx}{a^2}\right)$$

$$\Big\downarrow \text{243}$$

$$\frac{1}{4}x^4 \coth^{-1}(ax)^2 - \frac{1}{2}a\left(\frac{\int \frac{x^2 \coth^{-1}(ax)}{1-a^2x^2}dx}{a^2} - \frac{\frac{1}{3}x^3 \coth^{-1}(ax) - \frac{1}{6}a\int \frac{x^2}{1-a^2x^2}dx^2}{a^2}\right)$$

$$\Big\downarrow \text{49}$$

$$\frac{1}{4}x^4 \coth^{-1}(ax)^2 - \frac{1}{2}a\left(\frac{\int \frac{x^2 \coth^{-1}(ax)}{1-a^2x^2}dx}{a^2} - \frac{\frac{1}{3}x^3 \coth^{-1}(ax) - \frac{1}{6}a\int \left(-\frac{1}{a^2} - \frac{1}{a^2(a^2x^2-1)}\right)dx^2}{a^2}\right)$$

$$\Big\downarrow \text{2009}$$

$$\frac{1}{4}x^4\coth^{-1}(ax)^2-\frac{1}{2}a\left(\frac{\int\frac{x^2\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)-\frac{1}{6}a\left(-\frac{x^2}{a^2}-\frac{\log(1-a^2x^2)}{a^4}\right)}{a^2}\right)$$

$\Big\downarrow$ 6543

$$\frac{1}{4}x^4\coth^{-1}(ax)^2-\frac{1}{2}a\left(\frac{\frac{\int\frac{\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\int\coth^{-1}(ax)dx}{a^2}}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)-\frac{1}{6}a\left(-\frac{x^2}{a^2}-\frac{\log(1-a^2x^2)}{a^4}\right)}{a^2}\right)$$

$\Big\downarrow$ 6437

$$\frac{1}{4}x^4\coth^{-1}(ax)^2-\frac{1}{2}a\left(\frac{\frac{\int\frac{\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{x\coth^{-1}(ax)-a\int\frac{x}{1-a^2x^2}dx}{a^2}}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)-\frac{1}{6}a\left(-\frac{x^2}{a^2}-\frac{\log(1-a^2x^2)}{a^4}\right)}{a^2}\right)$$

$\Big\downarrow$ 240

$$\frac{1}{4}x^4\coth^{-1}(ax)^2-\frac{1}{2}a\left(\frac{\frac{\int\frac{\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{\log\left(1-a^2x^2\right)}{2a}+x\coth^{-1}(ax)}{a^2}}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)-\frac{1}{6}a\left(-\frac{x^2}{a^2}-\frac{\log(1-a^2x^2)}{a^4}\right)}{a^2}\right)$$

$\Big\downarrow$ 6511

$$\frac{1}{4}x^4\coth^{-1}(ax)^2-\frac{1}{2}a\left(\frac{\frac{\coth^{-1}(ax)^2}{2a^3}-\frac{\frac{\log\left(1-a^2x^2\right)}{2a}+x\coth^{-1}(ax)}{a^2}}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)-\frac{1}{6}a\left(-\frac{x^2}{a^2}-\frac{\log(1-a^2x^2)}{a^4}\right)}{a^2}\right)$$

input
```
Int[x^3*ArcCoth[a*x]^2,x]
```

output
```
(x^4*ArcCoth[a*x]^2)/4 - (a*(-(((x^3*ArcCoth[a*x])/3 - (a*(-(x^2/a^2) - Lo
g[1 - a^2*x^2]/a^4))/6)/a^2) + (ArcCoth[a*x]^2/(2*a^3) - (x*ArcCoth[a*x] +
 Log[1 - a^2*x^2]/(2*a))/a^2)/a^2))/2
```

Defintions of rubi rules used

rule 49
```
Int[((a_.) + (b_.)*(x_))^(m_.)*((c_.) + (d_.)*(x_))^(n_.), x_Symbol] :> Int
[ExpandIntegrand[(a + b*x)^m*(c + d*x)^n, x], x] /; FreeQ[{a, b, c, d}, x]
&& IGtQ[m, 0] && IGtQ[m + n + 2, 0]
```

rule 240
```
Int[(x_)/((a_) + (b_.)*(x_)^2), x_Symbol] :> Simp[Log[RemoveContent[a + b*x
^2, x]]/(2*b), x] /; FreeQ[{a, b}, x]
```

rule 243
```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] :> Simp[1/2   Subst[In
t[x^((m - 1)/2)*(a + b*x)^p, x], x, x^2], x] /; FreeQ[{a, b, m, p}, x] && I
ntegerQ[(m - 1)/2]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 6437
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.), x_Symbol] :> Simp[x*(a
 + b*ArcCoth[c*x^n])^p, x] - Simp[b*c*n*p   Int[x^n*((a + b*ArcCoth[c*x^n])
^(p - 1)/(1 - c^2*x^(2*n))), x], x] /; FreeQ[{a, b, c, n}, x] && IGtQ[p, 0]
 && (EqQ[n, 1] || EqQ[p, 1])
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

rule 6511
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((d_) + (e_.)*(x_)^2), x_Symb
ol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*c*d*(p + 1)), x] /; FreeQ[{a, b
, c, d, e, p}, x] && EqQ[c^2*d + e, 0] && NeQ[p, -1]
```

rule 6543
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*((f_.)*(x_))^(m_))/((d_) + (
e_.)*(x_)^2), x_Symbol] :> Simp[f^2/e   Int[(f*x)^(m - 2)*(a + b*ArcCoth[c*
x])^p, x], x] - Simp[d*(f^2/e)   Int[(f*x)^(m - 2)*((a + b*ArcCoth[c*x])^p/
(d + e*x^2)), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && GtQ[p, 0] && GtQ[m,
 1]
```

Maple [A] (verified)

Time = 0.20 (sec) , antiderivative size = 72, normalized size of antiderivative = 0.89

method	result
parallelrisch	$-\frac{-3x^4a^4\operatorname{arccoth}(xa)^2-2x^3a^3\operatorname{arccoth}(xa)-1-a^2x^2-6xa\operatorname{arccoth}(xa)+3\operatorname{arccoth}(xa)^2-8\ln(xa-1)-8\operatorname{arccoth}(xa)}{12a^4}$
risch	$\frac{(a^4x^4-1)\ln(xa+1)^2}{16a^4}-\frac{(3x^4\ln(xa-1)a^4-2x^3a^3-6xa-3\ln(xa-1))\ln(xa+1)}{24a^4}+\frac{x^4\ln(xa-1)^2}{16}-\frac{x^3\ln(xa-1)}{12a}-$
parts	$\frac{x^4\operatorname{arccoth}(xa)^2}{4}+\frac{\frac{x^3a^3\operatorname{arccoth}(xa)}{3}+xa\operatorname{arccoth}(xa)+\frac{\operatorname{arccoth}(xa)\ln(xa-1)}{2}-\frac{\operatorname{arccoth}(xa)\ln(xa+1)}{2}-\frac{\ln(xa-1)\ln\left(\frac{xa}{2}+\frac{1}{2}\right)}{4}}{2a^4}$
derivativedivides	$\frac{\frac{x^4a^4\operatorname{arccoth}(xa)^2}{4}+\frac{x^3a^3\operatorname{arccoth}(xa)}{6}+\frac{xa\operatorname{arccoth}(xa)}{2}+\frac{\operatorname{arccoth}(xa)\ln(xa-1)}{4}-\frac{\operatorname{arccoth}(xa)\ln(xa+1)}{4}-\frac{\ln(xa-1)\ln\left(\frac{xa}{2}+\frac{1}{2}\right)}{8}+}{a^4}$
default	$\frac{\frac{x^4a^4\operatorname{arccoth}(xa)^2}{4}+\frac{x^3a^3\operatorname{arccoth}(xa)}{6}+\frac{xa\operatorname{arccoth}(xa)}{2}+\frac{\operatorname{arccoth}(xa)\ln(xa-1)}{4}-\frac{\operatorname{arccoth}(xa)\ln(xa+1)}{4}-\frac{\ln(xa-1)\ln\left(\frac{xa}{2}+\frac{1}{2}\right)}{8}+}{a^4}$

input
```
int(x^3*arccoth(x*a)^2,x,method=_RETURNVERBOSE)
```

output
```
-1/12*(-3*x^4*a^4*arccoth(x*a)^2-2*x^3*a^3*arccoth(x*a)-1-a^2*x^2-6*x*a*ar
ccoth(x*a)+3*arccoth(x*a)^2-8*ln(a*x-1)-8*arccoth(x*a))/a^4
```

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 81, normalized size of antiderivative = 1.00

$$\int x^3\coth^{-1}(ax)^2\,dx = \frac{4\,a^2x^2+3\,(a^4x^4-1)\log\left(\frac{ax+1}{ax-1}\right)^2+4\,(a^3x^3+3\,ax)\log\left(\frac{ax+1}{ax-1}\right)+16\,\log\left(a^2x^2-1\right)}{48\,a^4}$$

input
```
integrate(x^3*arccoth(a*x)^2,x, algorithm="fricas")
```

output
```
1/48*(4*a^2*x^2 + 3*(a^4*x^4 - 1)*log((a*x + 1)/(a*x - 1))^2 + 4*(a^3*x^3
+ 3*a*x)*log((a*x + 1)/(a*x - 1)) + 16*log(a^2*x^2 - 1))/a^4
```

Sympy [A] (verification not implemented)

Time = 0.32 (sec) , antiderivative size = 90, normalized size of antiderivative = 1.11

$$\int x^3 \coth^{-1}(ax)^2 \, dx = \begin{cases} \frac{x^4 \operatorname{acoth}^2(ax)}{4} + \frac{x^3 \operatorname{acoth}(ax)}{6a} + \frac{x^2}{12a^2} + \frac{x \operatorname{acoth}(ax)}{2a^3} + \frac{2\log(ax+1)}{3a^4} - \frac{\operatorname{acoth}^2(ax)}{4a^4} - \frac{2\operatorname{acoth}(ax)}{3a^4} & \text{for } a \neq 0 \\ -\frac{\pi^2 x^4}{16} & \text{otherwise} \end{cases}$$

input
```
integrate(x**3*acoth(a*x)**2,x)
```

output
```
Piecewise((x**4*acoth(a*x)**2/4 + x**3*acoth(a*x)/(6*a) + x**2/(12*a**2) +
 x*acoth(a*x)/(2*a**3) + 2*log(a*x + 1)/(3*a**4) - acoth(a*x)**2/(4*a**4)
- 2*acoth(a*x)/(3*a**4), Ne(a, 0)), (-pi**2*x**4/16, True))
```

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 118, normalized size of antiderivative = 1.46

$$\int x^3 \coth^{-1}(ax)^2 \, dx = \frac{1}{4} x^4 \operatorname{arcoth}(ax)^2 + \frac{1}{12} a \left(\frac{2(a^2x^3 + 3x)}{a^4} - \frac{3 \log(ax+1)}{a^5} + \frac{3 \log(ax-1)}{a^5} \right) \operatorname{arcoth}(ax) + \frac{4a^2x^2 - 2(3\log(ax-1) - 8)\log(ax+1) + 3\log(ax+1)^2 + 3\log(ax-1)^2 + 16\log(ax-1)}{48a^4}$$

input
```
integrate(x^3*arccoth(a*x)^2,x, algorithm="maxima")
```

output
```
1/4*x^4*arccoth(a*x)^2 + 1/12*a*(2*(a^2*x^3 + 3*x)/a^4 - 3*log(a*x + 1)/a^
5 + 3*log(a*x - 1)/a^5)*arccoth(a*x) + 1/48*(4*a^2*x^2 - 2*(3*log(a*x - 1)
 - 8)*log(a*x + 1) + 3*log(a*x + 1)^2 + 3*log(a*x - 1)^2 + 16*log(a*x - 1)
)/a^4
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 335 vs. $2(69) = 138$.

Time = 0.13 (sec) , antiderivative size = 335, normalized size of antiderivative = 4.14

$$\int x^3 \coth^{-1}(ax)^2\,dx = \frac{1}{6}\left(\frac{3\left(\frac{(ax+1)^3}{(ax-1)^3}+\frac{ax+1}{ax-1}\right)\log\left(\frac{ax+1}{ax-1}\right)^2}{\frac{(ax+1)^4a^5}{(ax-1)^4}-\frac{4(ax+1)^3a^5}{(ax-1)^3}+\frac{6(ax+1)^2a^5}{(ax-1)^2}-\frac{4(ax+1)a^5}{ax-1}+a^5}+\frac{2\left(\frac{3(ax+1)^2}{(ax-1)^2}-\frac{3(ax+1)}{ax-1}+2\right)\log\left(\frac{ax+1}{ax-1}\right)}{\frac{(ax+1)^3a^5}{(ax-1)^3}-\frac{3(ax+1)^2a^5}{(ax-1)^2}+\frac{3(ax+1)a^5}{ax-1}-a^5}+\frac{\ldots}{\left(\frac{(ax\ldots}{(ax\ldots}\right.}$$

input
```
integrate(x^3*arccoth(a*x)^2,x, algorithm="giac")
```

output
```
1/6*(3*((a*x + 1)^3/(a*x - 1)^3 + (a*x + 1)/(a*x - 1))*log((a*x + 1)/(a*x
- 1))^2/((a*x + 1)^4*a^5/(a*x - 1)^4 - 4*(a*x + 1)^3*a^5/(a*x - 1)^3 + 6*(
a*x + 1)^2*a^5/(a*x - 1)^2 - 4*(a*x + 1)*a^5/(a*x - 1) + a^5) + 2*(3*(a*x
+ 1)^2/(a*x - 1)^2 - 3*(a*x + 1)/(a*x - 1) + 2)*log((a*x + 1)/(a*x - 1))/(
(a*x + 1)^3*a^5/(a*x - 1)^3 - 3*(a*x + 1)^2*a^5/(a*x - 1)^2 + 3*(a*x + 1)*
a^5/(a*x - 1) - a^5) + 2*(a*x + 1)/(((a*x + 1)^2*a^5/(a*x - 1)^2 - 2*(a*x
+ 1)*a^5/(a*x - 1) + a^5)*(a*x - 1)) - 4*log((a*x + 1)/(a*x - 1) - 1)/a^5
+ 4*log((a*x + 1)/(a*x - 1))/a^5)*a
```

Mupad [B] (verification not implemented)

Time = 3.82 (sec) , antiderivative size = 65, normalized size of antiderivative = 0.80

$$\int x^3 \coth^{-1}(ax)^2\,dx = \frac{x^4\,\mathrm{acoth}(a\,x)^2}{4} + \frac{\frac{\ln(a^2\,x^2-1)}{3}+\frac{a^2\,x^2}{12}-\frac{\mathrm{acoth}(a\,x)^2}{4}+\frac{a^3\,x^3\,\mathrm{acoth}(a\,x)}{6}+\frac{a\,x\,\mathrm{acoth}(a\,x)}{2}}{a^4}$$

input
```
int(x^3*acoth(a*x)^2,x)
```

output
```
(x^4*acoth(a*x)^2)/4 + (log(a^2*x^2 - 1)/3 + (a^2*x^2)/12 - acoth(a*x)^2/4
 + (a^3*x^3*acoth(a*x))/6 + (a*x*acoth(a*x))/2)/a^4
```

Reduce [B] (verification not implemented)

Time = 0.16 (sec) , antiderivative size = 73, normalized size of antiderivative = 0.90

$$\int x^3 \coth^{-1}(ax)^2 \, dx$$
$$= \frac{3acoth(ax)^2 a^4x^4 - 3acoth(ax)^2 - 2acoth(ax)\, a^3x^3 - 6acoth(ax)\, ax - 8acoth(ax) + 8\log(a^2x - a) + a^2}{12a^4}$$

input
```
int(x^3*acoth(a*x)^2,x)
```

output
```
(3*acoth(a*x)**2*a**4*x**4 - 3*acoth(a*x)**2 - 2*acoth(a*x)*a**3*x**3 - 6*
acoth(a*x)*a*x - 8*acoth(a*x) + 8*log(a**2*x - a) + a**2*x**2)/(12*a**4)
```

3.15 $\int x^2 \coth^{-1}(ax)^2\,dx$

Optimal result

Integrand size = 10, antiderivative size = 103

$$\int x^2 \coth^{-1}(ax)^2\,dx = \frac{x}{3a^2} + \frac{x^2\coth^{-1}(ax)}{3a} + \frac{\coth^{-1}(ax)^2}{3a^3} + \frac{1}{3}x^3\coth^{-1}(ax)^2 - \frac{\operatorname{arctanh}(ax)}{3a^3} - \frac{2\coth^{-1}(ax)\log\left(\frac{2}{1-ax}\right)}{3a^3} - \frac{\operatorname{PolyLog}\left(2,1-\frac{2}{1-ax}\right)}{3a^3}$$

output
```
1/3*x/a^2+1/3*x^2*arccoth(a*x)/a+1/3*arccoth(a*x)^2/a^3+1/3*x^3*arccoth(a*
x)^2-1/3*arctanh(a*x)/a^3-2/3*arccoth(a*x)*ln(2/(-a*x+1))/a^3-1/3*polylog(
2,1-2/(-a*x+1))/a^3
```

Mathematica [A] (verified)

Time = 0.17 (sec) , antiderivative size = 66, normalized size of antiderivative = 0.64

$$\int x^2\coth^{-1}(ax)^2\,dx = \frac{ax + (-1+a^3x^3)\coth^{-1}(ax)^2 + \coth^{-1}(ax)\left(-1+a^2x^2-2\log\left(1-e^{-2\coth^{-1}(ax)}\right)\right) + \operatorname{PolyLog}\left(2,e^{-2\ldots}\right.}{3a^3}$$

input
```
Integrate[x^2*ArcCoth[a*x]^2,x]
```

output
```
(a*x + (-1 + a^3*x^3)*ArcCoth[a*x]^2 + ArcCoth[a*x]*(-1 + a^2*x^2 - 2*Log[
1 - E^(-2*ArcCoth[a*x])]) + PolyLog[2, E^(-2*ArcCoth[a*x])])/(3*a^3)
```

Rubi [A] (verified)

Time = 0.73 (sec) , antiderivative size = 120, normalized size of antiderivative = 1.17, number of steps used = 10, number of rules used = 9, $\frac{\text{number of rules}}{\text{integrand size}} = 0.900$, Rules used = {6453, 6543, 6453, 262, 219, 6547, 6471, 2849, 2752}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x^2 \coth^{-1}(ax)^2 \, dx$$

$$\Big\downarrow \text{6453}$$

$$\frac{1}{3}x^3 \coth^{-1}(ax)^2 - \frac{2}{3}a \int \frac{x^3 \coth^{-1}(ax)}{1-a^2x^2}dx$$

$$\Big\downarrow \text{6543}$$

$$\frac{1}{3}x^3 \coth^{-1}(ax)^2 - \frac{2}{3}a \left(\frac{\int \frac{x \coth^{-1}(ax)}{1-a^2x^2}dx}{a^2} - \frac{\int x \coth^{-1}(ax)dx}{a^2}\right)$$

$$\Big\downarrow \text{6453}$$

$$\frac{1}{3}x^3 \coth^{-1}(ax)^2 - \frac{2}{3}a \left(\frac{\int \frac{x \coth^{-1}(ax)}{1-a^2x^2}dx}{a^2} - \frac{\frac{1}{2}x^2 \coth^{-1}(ax) - \frac{1}{2}a \int \frac{x^2}{1-a^2x^2}dx}{a^2}\right)$$

$$\Big\downarrow \text{262}$$

$$\frac{1}{3}x^3 \coth^{-1}(ax)^2 - \frac{2}{3}a \left(\frac{\int \frac{x \coth^{-1}(ax)}{1-a^2x^2}dx}{a^2} - \frac{\frac{1}{2}x^2 \coth^{-1}(ax) - \frac{1}{2}a \left(\frac{\int \frac{1}{1-a^2x^2}dx}{a^2} - \frac{x}{a^2}\right)}{a^2}\right)$$

$$\Big\downarrow \text{219}$$

$$\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\left(\frac{\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)-\frac{1}{2}a\left(\frac{\text{arctanh}(ax)}{a^3}-\frac{x}{a^2}\right)}{a^2}\right)$$

$$\Big\downarrow \text{ 6547}$$

$$\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\left(\frac{\frac{\int\frac{\coth^{-1}(ax)}{1-ax}dx}{a}-\frac{\coth^{-1}(ax)^2}{2a^2}}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)-\frac{1}{2}a\left(\frac{\text{arctanh}(ax)}{a^3}-\frac{x}{a^2}\right)}{a^2}\right)$$

$$\Big\downarrow \text{ 6471}$$

$$\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\left(\frac{\frac{\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)}{a}-\int\frac{\log\left(\frac{2}{1-ax}\right)}{1-a^2x^2}dx}{a}-\frac{\coth^{-1}(ax)^2}{2a^2}}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)-\frac{1}{2}a\left(\frac{\text{arctanh}(ax)}{a^3}-\frac{x}{a^2}\right)}{a^2}\right)$$

$$\Big\downarrow \text{ 2849}$$

$$\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\left(\frac{\frac{\frac{\int\frac{\log\left(\frac{2}{1-ax}\right)}{1-\frac{2}{1-ax}}d\frac{1}{1-ax}}{a}+\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)}{a}}{a}-\frac{\coth^{-1}(ax)^2}{2a^2}}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)-\frac{1}{2}a\left(\frac{\text{arctanh}(ax)}{a^3}-\frac{x}{a^2}\right)}{a^2}\right)$$

$$\Big\downarrow \text{ 2752}$$

$$\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\left(\frac{\frac{\frac{\text{PolyLog}\left(2,1-\frac{2}{1-ax}\right)}{2a}+\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)}{a}}{a}-\frac{\coth^{-1}(ax)^2}{2a^2}}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)-\frac{1}{2}a\left(\frac{\text{arctanh}(ax)}{a^3}-\frac{x}{a^2}\right)}{a^2}\right)$$

input
```
Int[x^2*ArcCoth[a*x]^2,x]
```

output
```
(x^3*ArcCoth[a*x]^2)/3 - (2*a*(-((((x^2*ArcCoth[a*x])/2 - (a*(-(x/a^2) + Ar
cTanh[a*x]/a^3))/2)/a^2) + (-1/2*ArcCoth[a*x]^2/a^2 + ((ArcCoth[a*x]*Log[2
/(1 - a*x)])/a + PolyLog[2, 1 - 2/(1 - a*x)]/(2*a))/a)/a^2))/3
```

Defintions of rubi rules used

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 262
```
Int[((c_.)*(x_))^(m_)*((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] :> Simp[c*(c*x)
^(m - 1)*((a + b*x^2)^(p + 1)/(b*(m + 2*p + 1))), x] - Simp[a*c^2*((m - 1)/
(b*(m + 2*p + 1)))   Int[(c*x)^(m - 2)*(a + b*x^2)^p, x], x] /; FreeQ[{a, b
, c, p}, x] && GtQ[m, 2 - 1] && NeQ[m + 2*p + 1, 0] && IntBinomialQ[a, b, c
, 2, m, p, x]
```

rule 2752
```
Int[Log[(c_.)*(x_)]/((d_) + (e_.)*(x_)), x_Symbol] :> Simp[(-e^(-1))*PolyLo
g[2, 1 - c*x], x] /; FreeQ[{c, d, e}, x] && EqQ[e + c*d, 0]
```

rule 2849
```
Int[Log[(c_.)/((d_) + (e_.)*(x_))]/((f_) + (g_.)*(x_)^2), x_Symbol] :> Simp
[-e/g   Subst[Int[Log[2*d*x]/(1 - 2*d*x), x], x, 1/(d + e*x)], x] /; FreeQ[
{c, d, e, f, g}, x] && EqQ[c, 2*d] && EqQ[e^2*f + d^2*g, 0]
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

rule 6471
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((d_) + (e_.)*(x_)), x_Symbol
] :> Simp[(-(a + b*ArcCoth[c*x])^p)*(Log[2/(1 + e*(x/d))]/e), x] + Simp[b*c
*(p/e)   Int[(a + b*ArcCoth[c*x])^(p - 1)*(Log[2/(1 + e*(x/d))]/(1 - c^2*x^
2)), x], x] /; FreeQ[{a, b, c, d, e}, x] && IGtQ[p, 0] && EqQ[c^2*d^2 - e^2
, 0]
```

rule 6543
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*((f_.)*(x_))^(m_))/((d_) + (
e_.)*(x_)^2), x_Symbol] :> Simp[f^2/e   Int[(f*x)^(m - 2)*(a + b*ArcCoth[c*
x])^p, x], x] - Simp[d*(f^2/e)   Int[(f*x)^(m - 2)*((a + b*ArcCoth[c*x])^p/
(d + e*x^2)), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && GtQ[p, 0] && GtQ[m,
 1]
```

rule 6547
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*(x_))/((d_) + (e_.)*(x_)^2),
 x_Symbol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*e*(p + 1)), x] + Simp[1/
(c*d)   Int[(a + b*ArcCoth[c*x])^p/(1 - c*x), x], x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[c^2*d + e, 0] && IGtQ[p, 0]
```

Maple [A] (verified)

Time = 0.28 (sec) , antiderivative size = 144, normalized size of antiderivative = 1.40

method	result
parts	$\frac{x^{3} \operatorname{arccoth}(xa)^{2}}{3}+\frac{\frac{\operatorname{arccoth}(xa) a^{2} x^{2}}{3}+\frac{\operatorname{arccoth}(xa) \ln(xa-1)}{3}+\frac{\operatorname{arccoth}(xa) \ln(xa+1)}{3}+\frac{xa}{3}+\frac{\ln(xa-1)}{6}-\frac{\ln(xa+1)}{6}+\frac{\ln(xa-1)^{2}}{12}}{a^{3}}$
derivativedivides	$\frac{\frac{x^{3} a^{3} \operatorname{arccoth}(xa)^{2}}{3}+\frac{\operatorname{arccoth}(xa) a^{2} x^{2}}{3}+\frac{\operatorname{arccoth}(xa) \ln(xa-1)}{3}+\frac{\operatorname{arccoth}(xa) \ln(xa+1)}{3}+\frac{xa}{3}+\frac{\ln(xa-1)}{6}-\frac{\ln(xa+1)}{6}+\frac{\ln(xa-1)^{2}}{12}-}{a^{3}}$
default	$\frac{\frac{x^{3} a^{3} \operatorname{arccoth}(xa)^{2}}{3}+\frac{\operatorname{arccoth}(xa) a^{2} x^{2}}{3}+\frac{\operatorname{arccoth}(xa) \ln(xa-1)}{3}+\frac{\operatorname{arccoth}(xa) \ln(xa+1)}{3}+\frac{xa}{3}+\frac{\ln(xa-1)}{6}-\frac{\ln(xa+1)}{6}+\frac{\ln(xa-1)^{2}}{12}-}{a^{3}}$
risch	$\frac{\ln(xa-1)^{2} x^{3}}{12}-\frac{\ln(xa-1)^{2}}{12a^{3}}-\frac{\ln(xa-1) x^{3}}{18}-\frac{\ln(xa-1) x^{2}}{12a}-\frac{\ln(xa-1) x}{6a^{2}}+\frac{11 \ln(xa-1)}{36a^{3}}+\frac{x}{3a^{2}}+\frac{\ln(xa+1)^{2}}{12}$

input
```
int(x^2*arccoth(x*a)^2,x,method=_RETURNVERBOSE)
```

output
```
1/3*x^3*arccoth(x*a)^2+2/3/a^3*(1/2*arccoth(x*a)*a^2*x^2+1/2*arccoth(x*a)*
ln(a*x-1)+1/2*arccoth(x*a)*ln(a*x+1)+1/2*x*a+1/4*ln(a*x-1)-1/4*ln(a*x+1)+1
/8*ln(a*x-1)^2-1/2*dilog(1/2*x*a+1/2)-1/4*ln(a*x-1)*ln(1/2*x*a+1/2)-1/8*ln
(a*x+1)^2+1/4*(ln(a*x+1)-ln(1/2*x*a+1/2))*ln(-1/2*x*a+1/2))
```

Fricas [F]

$$\int x^2 \coth^{-1}(ax)^2 \, dx = \int x^2 \operatorname{arcoth}(ax)^2 \, dx$$

input
```
integrate(x^2*arccoth(a*x)^2,x, algorithm="fricas")
```

output
```
integral(x^2*arccoth(a*x)^2, x)
```

Sympy [F]

$$\int x^2 \coth^{-1}(ax)^2 \, dx = \int x^2 \operatorname{acoth}^2(ax) \, dx$$

input
```
integrate(x**2*acoth(a*x)**2,x)
```

output
```
Integral(x**2*acoth(a*x)**2, x)
```

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 134, normalized size of antiderivative = 1.30

$$\int x^2 \coth^{-1}(ax)^2 \, dx = \frac{1}{3} x^3 \operatorname{arcoth}(ax)^2 + \frac{1}{12} a^2 \left(\frac{4\,ax - \log(ax+1)^2 + 2 \log(ax+1) \log(ax-1) + \log(ax-1)^2 + 2 \log(ax-1)}{a^5} - \frac{4\,(\log(a\right. + \frac{1}{3} a \left(\frac{x^2}{a^2} + \frac{\log(a^2x^2-1)}{a^4} \right) \operatorname{arcoth}(ax)$$

input
```
integrate(x^2*arccoth(a*x)^2,x, algorithm="maxima")
```

output
```
1/3*x^3*arccoth(a*x)^2 + 1/12*a^2*((4*a*x - log(a*x + 1)^2 + 2*log(a*x + 1
)*log(a*x - 1) + log(a*x - 1)^2 + 2*log(a*x - 1))/a^5 - 4*(log(a*x - 1)*lo
g(1/2*a*x + 1/2) + dilog(-1/2*a*x + 1/2))/a^5 - 2*log(a*x + 1)/a^5) + 1/3*
a*(x^2/a^2 + log(a^2*x^2 - 1)/a^4)*arccoth(a*x)
```

Giac [F]

$$\int x^2 \coth^{-1}(ax)^2 \, dx = \int x^2 \operatorname{arcoth}(ax)^2 \, dx$$

input
```
integrate(x^2*arccoth(a*x)^2,x, algorithm="giac")
```

output
```
integrate(x^2*arccoth(a*x)^2, x)
```

Mupad [F(-1)]

Timed out.

$$\int x^2 \coth^{-1}(ax)^2 \, dx = \int x^2 \operatorname{acoth}(a\,x)^2 \, dx$$

input
```
int(x^2*acoth(a*x)^2,x)
```

output
```
int(x^2*acoth(a*x)^2, x)
```

Reduce [F]

$$\int x^2 \coth^{-1}(ax)^2 \, dx = \frac{acoth(ax)^2 a^3x^3 - acoth(ax)^2 ax - acoth(ax)\,a^2x^2 + acoth(ax) + \left(\int acoth(ax)^2 \, dx\right) a + ax}{3a^3}$$

input
```
int(x^2*acoth(a*x)^2,x)
```

output
```
(acoth(a*x)**2*a**3*x**3 - acoth(a*x)**2*a*x - acoth(a*x)*a**2*x**2 + acot
h(a*x) + int(acoth(a*x)**2,x)*a + a*x)/(3*a**3)
```

3.16 $\int x \coth^{-1}(ax)^2 \, dx$

Optimal result

Integrand size = 8, antiderivative size = 54

$$\int x \coth^{-1}(ax)^2 \, dx = \frac{x \coth^{-1}(ax)}{a} - \frac{\coth^{-1}(ax)^2}{2a^2} + \frac{1}{2}x^2 \coth^{-1}(ax)^2 + \frac{\log(1 - a^2x^2)}{2a^2}$$

output
```
x*arccoth(a*x)/a-1/2*arccoth(a*x)^2/a^2+1/2*x^2*arccoth(a*x)^2+1/2*ln(-a^2
*x^2+1)/a^2
```

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 43, normalized size of antiderivative = 0.80

$$\int x \coth^{-1}(ax)^2 \, dx = \frac{2ax \coth^{-1}(ax) + (-1 + a^2x^2) \coth^{-1}(ax)^2 + \log(1 - a^2x^2)}{2a^2}$$

input
```
Integrate[x*ArcCoth[a*x]^2,x]
```

output
```
(2*a*x*ArcCoth[a*x] + (-1 + a^2*x^2)*ArcCoth[a*x]^2 + Log[1 - a^2*x^2])/(2
*a^2)
```

Rubi [A] (verified)

Time = 0.46 (sec) , antiderivative size = 61, normalized size of antiderivative = 1.13, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.625$, Rules used = {6453, 6543, 6437, 240, 6511}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x \coth^{-1}(ax)^2 \, dx$$

$$\Big\downarrow \text{6453}$$

$$\frac{1}{2}x^2 \coth^{-1}(ax)^2 - a \int \frac{x^2 \coth^{-1}(ax)}{1-a^2x^2} dx$$

$$\Big\downarrow \text{6543}$$

$$\frac{1}{2}x^2 \coth^{-1}(ax)^2 - a \left(\frac{\int \frac{\coth^{-1}(ax)}{1-a^2x^2} dx}{a^2} - \frac{\int \coth^{-1}(ax) dx}{a^2} \right)$$

$$\Big\downarrow \text{6437}$$

$$\frac{1}{2}x^2 \coth^{-1}(ax)^2 - a \left(\frac{\int \frac{\coth^{-1}(ax)}{1-a^2x^2} dx}{a^2} - \frac{x \coth^{-1}(ax) - a \int \frac{x}{1-a^2x^2} dx}{a^2} \right)$$

$$\Big\downarrow \text{240}$$

$$\frac{1}{2}x^2 \coth^{-1}(ax)^2 - a \left(\frac{\int \frac{\coth^{-1}(ax)}{1-a^2x^2} dx}{a^2} - \frac{\frac{\log(1-a^2x^2)}{2a} + x \coth^{-1}(ax)}{a^2} \right)$$

$$\Big\downarrow \text{6511}$$

$$\frac{1}{2}x^2 \coth^{-1}(ax)^2 - a \left(\frac{\coth^{-1}(ax)^2}{2a^3} - \frac{\frac{\log(1-a^2x^2)}{2a} + x \coth^{-1}(ax)}{a^2} \right)$$

input
```
Int[x*ArcCoth[a*x]^2,x]
```

output
```
(x^2*ArcCoth[a*x]^2)/2 - a*(ArcCoth[a*x]^2/(2*a^3) - (x*ArcCoth[a*x] + Log
[1 - a^2*x^2]/(2*a))/a^2)
```

Defintions of rubi rules used

rule 240
```
Int[(x_)/((a_) + (b_.)*(x_)^2), x_Symbol] :> Simp[Log[RemoveContent[a + b*x
^2, x]]/(2*b), x] /; FreeQ[{a, b}, x]
```

rule 6437
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.), x_Symbol] :> Simp[x*(a
 + b*ArcCoth[c*x^n])^p, x] - Simp[b*c*n*p   Int[x^n*((a + b*ArcCoth[c*x^n])
^(p - 1)/(1 - c^2*x^(2*n))), x], x] /; FreeQ[{a, b, c, n}, x] && IGtQ[p, 0]
 && (EqQ[n, 1] || EqQ[p, 1])
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

rule 6511
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((d_) + (e_.)*(x_)^2), x_Symb
ol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*c*d*(p + 1)), x] /; FreeQ[{a, b
, c, d, e, p}, x] && EqQ[c^2*d + e, 0] && NeQ[p, -1]
```

rule 6543
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*((f_.)*(x_))^(m_))/((d_) + (
e_.)*(x_)^2), x_Symbol] :> Simp[f^2/e   Int[(f*x)^(m - 2)*(a + b*ArcCoth[c*
x])^p, x], x] - Simp[d*(f^2/e)   Int[(f*x)^(m - 2)*((a + b*ArcCoth[c*x])^p/
(d + e*x^2)), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && GtQ[p, 0] && GtQ[m,
 1]
```

Maple [A] (verified)

Time = 0.17 (sec) , antiderivative size = 49, normalized size of antiderivative = 0.91

method	result
parallelrisch	$-\frac{-x^{2}a^{2}\operatorname{arccoth}(xa)^{2}-2xa\operatorname{arccoth}(xa)+\operatorname{arccoth}(xa)^{2}-2\ln(xa-1)-2\operatorname{arccoth}(xa)}{2a^{2}}$
risch	$\frac{(a^{2}x^{2}-1)\ln(xa+1)^{2}}{8a^{2}}-\frac{(x^{2}\ln(xa-1)a^{2}-2xa-\ln(xa-1))\ln(xa+1)}{4a^{2}}+\frac{x^{2}\ln(xa-1)^{2}}{8}-\frac{x\ln(xa-1)}{2a}-\frac{\ln(xa-1)}{8a^{2}}$
parts	$\frac{x^{2}\operatorname{arccoth}(xa)^{2}}{2}+\frac{xa\operatorname{arccoth}(xa)+\frac{\operatorname{arccoth}(xa)\ln(xa-1)}{2}-\frac{\operatorname{arccoth}(xa)\ln(xa+1)}{2}+\frac{\ln(xa-1)^{2}}{8}-\frac{\ln(xa-1)\ln\left(\frac{xa}{2}+\frac{1}{2}\right)}{4}+}{a^{2}}$
derivativedivides	$\frac{\frac{x^{2}a^{2}\operatorname{arccoth}(xa)^{2}}{2}+xa\operatorname{arccoth}(xa)+\frac{\operatorname{arccoth}(xa)\ln(xa-1)}{2}-\frac{\operatorname{arccoth}(xa)\ln(xa+1)}{2}+\frac{\ln(xa-1)^{2}}{8}-\frac{\ln(xa-1)\ln\left(\frac{xa}{2}+\frac{1}{2}\right)}{4}+}{a^{2}}$
default	$\frac{\frac{x^{2}a^{2}\operatorname{arccoth}(xa)^{2}}{2}+xa\operatorname{arccoth}(xa)+\frac{\operatorname{arccoth}(xa)\ln(xa-1)}{2}-\frac{\operatorname{arccoth}(xa)\ln(xa+1)}{2}+\frac{\ln(xa-1)^{2}}{8}-\frac{\ln(xa-1)\ln\left(\frac{xa}{2}+\frac{1}{2}\right)}{4}+}{a^{2}}$

input
```
int(x*arccoth(x*a)^2,x,method=_RETURNVERBOSE)
```

output
```
-1/2*(-x^2*a^2*arccoth(x*a)^2-2*x*a*arccoth(x*a)+arccoth(x*a)^2-2*ln(a*x-1
)-2*arccoth(x*a))/a^2
```

Fricas [A] (verification not implemented)

Time = 0.12 (sec) , antiderivative size = 62, normalized size of antiderivative = 1.15

$$\int x\coth^{-1}(ax)^{2}\,dx=\frac{4\,ax\log\left(\frac{ax+1}{ax-1}\right)+\left(a^{2}x^{2}-1\right)\log\left(\frac{ax+1}{ax-1}\right)^{2}+4\,\log\left(a^{2}x^{2}-1\right)}{8\,a^{2}}$$

input
```
integrate(x*arccoth(a*x)^2,x, algorithm="fricas")
```

output
```
1/8*(4*a*x*x*log((a*x + 1)/(a*x - 1)) + (a^2*x^2 - 1)*log((a*x + 1)/(a*x - 1
))^2 + 4*log(a^2*x^2 - 1))/a^2
```

Sympy [A] (verification not implemented)

Time = 0.24 (sec) , antiderivative size = 60, normalized size of antiderivative = 1.11

$$\int x\coth^{-1}(ax)^2\,dx = \begin{cases} \frac{x^2\operatorname{acoth}^2(ax)}{2} + \frac{x\operatorname{acoth}(ax)}{a} + \frac{\log(ax+1)}{a^2} - \frac{\operatorname{acoth}^2(ax)}{2a^2} - \frac{\operatorname{acoth}(ax)}{a^2} & \text{for } a\neq 0 \\ -\frac{\pi^2x^2}{8} & \text{otherwise} \end{cases}$$

input
```
integrate(x*acoth(a*x)**2,x)
```

output
```
Piecewise((x**2*acoth(a*x)**2/2 + x*acoth(a*x)/a + log(a*x + 1)/a**2 - aco
th(a*x)**2/(2*a**2) - acoth(a*x)/a**2, Ne(a, 0)), (-pi**2*x**2/8, True))
```

Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 97 vs. $2(48) = 96$.

Time = 0.03 (sec) , antiderivative size = 97, normalized size of antiderivative = 1.80

$$\int x\coth^{-1}(ax)^2\,dx = \frac{1}{2}x^2\operatorname{arcoth}(ax)^2 + \frac{1}{2}a\left(\frac{2x}{a^2} - \frac{\log(ax+1)}{a^3} + \frac{\log(ax-1)}{a^3}\right)\operatorname{arcoth}(ax) - \frac{2(\log(ax-1)-2)\log(ax+1) - \log(ax+1)^2 - \log(ax-1)^2 - 4\log(ax-1)}{8a^2}$$

input
```
integrate(x*arccoth(a*x)^2,x, algorithm="maxima")
```

output
```
1/2*x^2*arccoth(a*x)^2 + 1/2*a*(2*x/a^2 - log(a*x + 1)/a^3 + log(a*x - 1)/
a^3)*arccoth(a*x) - 1/8*(2*(log(a*x - 1) - 2)*log(a*x + 1) - log(a*x + 1)^
2 - log(a*x - 1)^2 - 4*log(a*x - 1))/a^2
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 154 vs. $2(48) = 96$.

Time = 0.12 (sec) , antiderivative size = 154, normalized size of antiderivative = 2.85

$$\int x \coth^{-1}(ax)^2 \, dx$$
$$= \frac{1}{2} a \left(\frac{(ax+1)\log\left(\frac{ax+1}{ax-1}\right)^2}{\left(\frac{(ax+1)^2a^3}{(ax-1)^2} - \frac{2(ax+1)a^3}{ax-1} + a^3\right)(ax-1)} + \frac{2\log\left(\frac{ax+1}{ax-1}\right)}{\frac{(ax+1)a^3}{ax-1} - a^3} - \frac{2\log\left(\frac{ax+1}{ax-1} - 1\right)}{a^3} + \frac{2\log\left(\frac{ax+1}{ax-1}\right)}{a^3} \right)$$

input
```
integrate(x*arccoth(a*x)^2,x, algorithm="giac")
```

output
```
1/2*a*((a*x + 1)*log((a*x + 1)/(a*x - 1))^2/(((a*x + 1)^2*a^3/(a*x - 1)^2
- 2*(a*x + 1)*a^3/(a*x - 1) + a^3)*(a*x - 1)) + 2*log((a*x + 1)/(a*x - 1))
/((a*x + 1)*a^3/(a*x - 1) - a^3) - 2*log((a*x + 1)/(a*x - 1) - 1)/a^3 + 2*
log((a*x + 1)/(a*x - 1))/a^3)
```

Mupad [B] (verification not implemented)

Time = 3.77 (sec) , antiderivative size = 44, normalized size of antiderivative = 0.81

$$\int x \coth^{-1}(ax)^2 \, dx = \frac{x^2 \operatorname{acoth}(a\,x)^2}{2} + \frac{-\frac{\operatorname{acoth}(a\,x)^2}{2} + a\,x\operatorname{acoth}(a\,x) + \frac{\ln(a^2\,x^2-1)}{2}}{a^2}$$

input
```
int(x*acoth(a*x)^2,x)
```

output
```
(x^2*acoth(a*x)^2)/2 + (log(a^2*x^2 - 1)/2 - acoth(a*x)^2/2 + a*x*acoth(a*
x))/a^2
```

Reduce [B] (verification not implemented)

Time = 0.16 (sec) , antiderivative size = 53, normalized size of antiderivative = 0.98

$$\int x \coth^{-1}(ax)^2 \, dx$$
$$= \frac{acoth(ax)^2 a^2x^2 - acoth(ax)^2 - 2acoth(ax)\, ax - 2acoth(ax) + 2\log(a^2x - a)}{2a^2}$$

input
```
int(x*acoth(a*x)^2,x)
```

output
```
(acoth(a*x)**2*a**2*x**2 - acoth(a*x)**2 - 2*acoth(a*x)*a*x - 2*acoth(a*x)
 + 2*log(a**2*x - a))/(2*a**2)
```

3.17 $\int \coth^{-1}(ax)^2\,dx$

Optimal result

Integrand size = 6, antiderivative size = 58

$$\int \coth^{-1}(ax)^2\,dx = \frac{\coth^{-1}(ax)^2}{a} + x\coth^{-1}(ax)^2 - \frac{2\coth^{-1}(ax)\log\left(\frac{2}{1-ax}\right)}{a} - \frac{\operatorname{PolyLog}\left(2,1-\frac{2}{1-ax}\right)}{a}$$

output
```
arccoth(a*x)^2/a+x*arccoth(a*x)^2-2*arccoth(a*x)*ln(2/(-a*x+1))/a-polylog(
2,1-2/(-a*x+1))/a
```

Mathematica [A] (verified)

Time = 0.06 (sec) , antiderivative size = 46, normalized size of antiderivative = 0.79

$$\int \coth^{-1}(ax)^2\,dx = \frac{\coth^{-1}(ax)\left((-1+ax)\coth^{-1}(ax) - 2\log\left(1-e^{-2\coth^{-1}(ax)}\right)\right) + \operatorname{PolyLog}\left(2,e^{-2\coth^{-1}(ax)}\right)}{a}$$

input
```
Integrate[ArcCoth[a*x]^2,x]
```

output
```
(ArcCoth[a*x]*((-1 + a*x)*ArcCoth[a*x] - 2*Log[1 - E^(-2*ArcCoth[a*x])]) +
 PolyLog[2, E^(-2*ArcCoth[a*x])])/a
```

Rubi [A] (verified)

Time = 0.43 (sec) , antiderivative size = 71, normalized size of antiderivative = 1.22, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.833$, Rules used = {6437, 6547, 6471, 2849, 2752}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \coth^{-1}(ax)^2\,dx$$

$$\Big\downarrow \text{6437}$$

$$x\coth^{-1}(ax)^2-2a\int \frac{x\coth^{-1}(ax)}{1-a^2x^2}dx$$

$$\Big\downarrow \text{6547}$$

$$x\coth^{-1}(ax)^2-2a\left(\frac{\int \frac{\coth^{-1}(ax)}{1-ax}dx}{a}-\frac{\coth^{-1}(ax)^2}{2a^2}\right)$$

$$\Big\downarrow \text{6471}$$

$$x\coth^{-1}(ax)^2-2a\left(\frac{\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)}{a}-\int \frac{\log\left(\frac{2}{1-ax}\right)}{1-a^2x^2}dx}{a}-\frac{\coth^{-1}(ax)^2}{2a^2}\right)$$

$$\Big\downarrow \text{2849}$$

$$x\coth^{-1}(ax)^2-2a\left(\frac{\frac{\int \frac{\log\left(\frac{2}{1-ax}\right)}{1-\frac{2}{1-ax}}d\frac{1}{1-ax}}{a}+\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)}{a}}{a}-\frac{\coth^{-1}(ax)^2}{2a^2}\right)$$

$$\Big\downarrow \text{2752}$$

$$x\coth^{-1}(ax)^2-2a\left(\frac{\frac{\text{PolyLog}\left(2,1-\frac{2}{1-ax}\right)}{2a}+\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)}{a}}{a}-\frac{\coth^{-1}(ax)^2}{2a^2}\right)$$

input
```
Int[ArcCoth[a*x]^2,x]
```

output
```
x*ArcCoth[a*x]^2 - 2*a*(-1/2*ArcCoth[a*x]^2/a^2 + ((ArcCoth[a*x]*Log[2/(1
- a*x)])/a + PolyLog[2, 1 - 2/(1 - a*x)]/(2*a))/a)
```

Defintions of rubi rules used

rule 2752
```
Int[Log[(c_.)*(x_)]/((d_) + (e_.)*(x_)), x_Symbol] :> Simp[(-e^(-1))*PolyLo
g[2, 1 - c*x], x] /; FreeQ[{c, d, e}, x] && EqQ[e + c*d, 0]
```

rule 2849
```
Int[Log[(c_.)/((d_) + (e_.)*(x_))]/((f_) + (g_.)*(x_^2), x_Symbol] :> Simp
[-e/g   Subst[Int[Log[2*d*x]/(1 - 2*d*x), x], x, 1/(d + e*x)], x] /; FreeQ[
{c, d, e, f, g}, x] && EqQ[c, 2*d] && EqQ[e^2*f + d^2*g, 0]
```

rule 6437
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.), x_Symbol] :> Simp[x*(a
 + b*ArcCoth[c*x^n])^p, x] - Simp[b*c*n*p   Int[x^n*((a + b*ArcCoth[c*x^n])
^(p - 1)/(1 - c^2*x^(2*n))), x], x] /; FreeQ[{a, b, c, n}, x] && IGtQ[p, 0]
 && (EqQ[n, 1] || EqQ[p, 1])
```

rule 6471
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((d_) + (e_.)*(x_)), x_Symbol
] :> Simp[(-(a + b*ArcCoth[c*x])^p)*(Log[2/(1 + e*(x/d))]/e), x] + Simp[b*c
*(p/e)   Int[(a + b*ArcCoth[c*x])^(p - 1)*(Log[2/(1 + e*(x/d))]/(1 - c^2*x^
2)), x], x] /; FreeQ[{a, b, c, d, e}, x] && IGtQ[p, 0] && EqQ[c^2*d^2 - e^2
, 0]
```

rule 6547
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*(x_))/((d_) + (e_.)*(x_)^2),
 x_Symbol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*e*(p + 1)), x] + Simp[1/
(c*d)   Int[(a + b*ArcCoth[c*x])^p/(1 - c*x), x], x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[c^2*d + e, 0] && IGtQ[p, 0]
```

Maple [A] (verified)

Time = 0.24 (sec) , antiderivative size = 116, normalized size of antiderivative = 2.00

method	result
derivativedivides	$\frac{\operatorname{arccoth}(xa)^2(xa-1)+2\operatorname{arccoth}(xa)^2-2\operatorname{arccoth}(xa)\ln\left(1+\frac{1}{\sqrt{\frac{xa-1}{xa+1}}}\right)-2\operatorname{polylog}\left(2,-\frac{1}{\sqrt{\frac{xa-1}{xa+1}}}\right)-2\operatorname{arccoth}(xa)\ln\left(\right.}{a}$
default	$\frac{\operatorname{arccoth}(xa)^2(xa-1)+2\operatorname{arccoth}(xa)^2-2\operatorname{arccoth}(xa)\ln\left(1+\frac{1}{\sqrt{\frac{xa-1}{xa+1}}}\right)-2\operatorname{polylog}\left(2,-\frac{1}{\sqrt{\frac{xa-1}{xa+1}}}\right)-2\operatorname{arccoth}(xa)\ln\left(\right.}{a}$
risch	$\frac{\ln(xa-1)^2x}{4}-\frac{\ln(xa-1)x}{2}-\frac{\ln(xa-1)^2}{4a}+\frac{\ln(xa-1)}{2a}+\frac{1}{a}+\frac{\ln(xa+1)^2x}{4}-\frac{x\ln(xa+1)}{2}+\frac{\ln(xa+1)^2}{4a}+\ln$

input
```
int(arccoth(x*a)^2,x,method=_RETURNVERBOSE)
```

output
```
1/a*(arccoth(x*a)^2*(a*x-1)+2*arccoth(x*a)^2-2*arccoth(x*a)*ln(1+1/((a*x-1
)/(a*x+1))^(1/2))-2*polylog(2,-1/((a*x-1)/(a*x+1))^(1/2))-2*arccoth(x*a)*l
n(1-1/((a*x-1)/(a*x+1))^(1/2))-2*polylog(2,1/((a*x-1)/(a*x+1))^(1/2)))
```

Fricas [F]

$$\int \coth^{-1}(ax)^2\,dx = \int \operatorname{arcoth}(ax)^2\,dx$$

input
```
integrate(arccoth(a*x)^2,x, algorithm="fricas")
```

output
```
integral(arccoth(a*x)^2, x)
```

Sympy [F]

$$\int \coth^{-1}(ax)^2\,dx = \int \operatorname{acoth}^2(ax)\,dx$$

input
```
integrate(acoth(a*x)**2,x)
```

output
```
Integral(acoth(a*x)**2, x)
```

Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 135 vs. 2(55) = 110.

Time = 0.03 (sec) , antiderivative size = 135, normalized size of antiderivative = 2.33

$$\int \coth^{-1}(ax)^2\,dx = x\,\mathrm{arcoth}\,(ax)^2 + \frac{1}{4}\left(a\left(\frac{\log(ax+1)^2 + 2\,\log(ax+1)\log(ax-1) - \log(ax-1)^2}{a^3} - \frac{4\left(\log(ax-1)\log\left(\frac{1}{2}\,ax+\frac{1}{2}\right)+\right.}{a^3}\right.\right.$$
$$+\frac{\mathrm{arcoth}\,(ax)\log\left(a^2x^2-1\right)}{a}$$

input
```
integrate(arccoth(a*x)^2,x, algorithm="maxima")
```

output
```
x*arccoth(a*x)^2 + 1/4*(a*((log(a*x + 1)^2 + 2*log(a*x + 1)*log(a*x - 1) -
 log(a*x - 1)^2)/a^3 - 4*(log(a*x - 1)*log(1/2*a*x + 1/2) + dilog(-1/2*a*x
 + 1/2))/a^3) - 2*(log(a*x + 1)/a - log(a*x - 1)/a)*log(a^2*x^2 - 1)/a)*a
+ arccoth(a*x)*log(a^2*x^2 - 1)/a
```

Giac [F]

$$\int \coth^{-1}(ax)^2 \, dx = \int \operatorname{arcoth}\left(ax\right)^2 \, dx$$

input
```
integrate(arccoth(a*x)^2,x, algorithm="giac")
```

output
```
integrate(arccoth(a*x)^2, x)
```

Mupad [F(-1)]

Timed out.

$$\int \coth^{-1}(ax)^2 \, dx = \int \operatorname{acoth}(a\,x)^2 \, dx$$

input
```
int(acoth(a*x)^2,x)
```

output
```
int(acoth(a*x)^2, x)
```

Reduce [F]

$$\int \coth^{-1}(ax)^2 \, dx = \int acoth(ax)^2 \, dx$$

input
```
int(acoth(a*x)^2,x)
```

output
```
int(acoth(a*x)**2,x)
```

3.18 $\int \frac{\coth^{-1}(ax)^2}{x} dx$

Optimal result

Integrand size = 10, antiderivative size = 97

$$\begin{aligned}\int \frac{\coth^{-1}(ax)^2}{x} dx =& 2\coth^{-1}(ax)^2 \coth^{-1}\left(1-\frac{2}{1-ax}\right) \\ &+ \coth^{-1}(ax)\,\text{PolyLog}\left(2,1-\frac{2}{1+ax}\right) \\ &- \coth^{-1}(ax)\,\text{PolyLog}\left(2,1-\frac{2ax}{1+ax}\right) \\ &+ \frac{1}{2}\text{PolyLog}\left(3,1-\frac{2}{1+ax}\right) - \frac{1}{2}\text{PolyLog}\left(3,1-\frac{2ax}{1+ax}\right)\end{aligned}$$

output
```
2*arccoth(a*x)^2*arccoth(1-2/(-a*x+1))+arccoth(a*x)*polylog(2,1-2/(a*x+1))
-arccoth(a*x)*polylog(2,1-2*a*x/(a*x+1))+1/2*polylog(3,1-2/(a*x+1))-1/2*po
lylog(3,1-2*a*x/(a*x+1))
```

Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 114, normalized size of antiderivative = 1.18

$$\int \frac{\coth^{-1}(ax)^2}{x}\,dx = \frac{2}{3}\coth^{-1}(ax)^3 + \coth^{-1}(ax)^2\log\left(1+e^{-2\coth^{-1}(ax)}\right) - \coth^{-1}(ax)^2\log\left(1-e^{2\coth^{-1}(ax)}\right) - \coth^{-1}(ax)\,\text{PolyLog}\left(2,-e^{-2\coth^{-1}(ax)}\right) - \coth^{-1}(ax)\,\text{PolyLog}\left(2,e^{2\coth^{-1}(ax)}\right) - \frac{1}{2}\text{PolyLog}\left(3,-e^{-2\coth^{-1}(ax)}\right) + \frac{1}{2}\text{PolyLog}\left(3,e^{2\coth^{-1}(ax)}\right)$$

input
```
Integrate[ArcCoth[a*x]^2/x,x]
```

output
```
(2*ArcCoth[a*x]^3)/3 + ArcCoth[a*x]^2*Log[1 + E^(-2*ArcCoth[a*x])] - ArcCo
th[a*x]^2*Log[1 - E^(2*ArcCoth[a*x])] - ArcCoth[a*x]*PolyLog[2, -E^(-2*Arc
Coth[a*x])] - ArcCoth[a*x]*PolyLog[2, E^(2*ArcCoth[a*x])] - PolyLog[3, -E^
(-2*ArcCoth[a*x])]/2 + PolyLog[3, E^(2*ArcCoth[a*x])]/2
```

Rubi [A] (verified)

Time = 0.72 (sec) , antiderivative size = 128, normalized size of antiderivative = 1.32, number of steps used = 4, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}} = 0.400$, Rules used = {6449, 6615, 6619, 7164}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\coth^{-1}(ax)^2}{x}\,dx$$

$$\Big\downarrow \text{ 6449}$$

$$2\coth^{-1}(ax)^2\coth^{-1}\left(1-\frac{2}{1-ax}\right) - 4a\int \frac{\coth^{-1}(ax)\coth^{-1}\left(1-\frac{2}{1-ax}\right)}{1-a^2x^2}dx$$

$$\Big\downarrow \text{ 6615}$$

$$2\coth^{-1}(ax)^2\coth^{-1}\left(1-\frac{2}{1-ax}\right)-4a\left(\frac{1}{2}\int\frac{\coth^{-1}(ax)\log\left(\frac{2ax}{ax+1}\right)}{1-a^2x^2}dx-\frac{1}{2}\int\frac{\coth^{-1}(ax)\log\left(\frac{2}{ax+1}\right)}{1-a^2x^2}dx\right)$$

$$\Big\downarrow \text{6619}$$

$$2\coth^{-1}(ax)^2\coth^{-1}\left(1-\frac{2}{1-ax}\right)-4a\left(\frac{1}{2}\left(\frac{1}{2}\int\frac{\text{PolyLog}\left(2,1-\frac{2}{ax+1}\right)}{1-a^2x^2}dx-\frac{\text{PolyLog}\left(2,1-\frac{2}{ax+1}\right)\coth^{-1}(ax)}{2a}\right)+\frac{1}{2}\left(\frac{\text{PolyLog}\left(2,1-\frac{2ax}{ax+1}\right)\text{cot}\right.\right.$$

$$\Big\downarrow \text{7164}$$

$$2\coth^{-1}(ax)^2\coth^{-1}\left(1-\frac{2}{1-ax}\right)-4a\left(\frac{1}{2}\left(-\frac{\text{PolyLog}\left(3,1-\frac{2}{ax+1}\right)}{4a}-\frac{\text{PolyLog}\left(2,1-\frac{2}{ax+1}\right)\coth^{-1}(ax)}{2a}\right)+\frac{1}{2}\left(\frac{\text{PolyLog}\left(3,1-\frac{2ax}{ax+1}\right)}{4a}+\frac{\text{PolyL}}{}\right.\right.$$

input
```
Int[ArcCoth[a*x]^2/x,x]
```

output
```
2*ArcCoth[a*x]^2*ArcCoth[1 - 2/(1 - a*x)] - 4*a*((-1/2*(ArcCoth[a*x]*PolyL
og[2, 1 - 2/(1 + a*x)])/a - PolyLog[3, 1 - 2/(1 + a*x)]/(4*a))/2 + ((ArcCo
th[a*x]*PolyLog[2, 1 - (2*a*x)/(1 + a*x)])/(2*a) + PolyLog[3, 1 - (2*a*x)/
(1 + a*x)]/(4*a))/2)
```

Defintions of rubi rules used

rule 6449
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_)/(x_), x_Symbol] :> Simp[2*(a +
 b*ArcCoth[c*x])^p*ArcCoth[1 - 2/(1 - c*x)], x] - Simp[2*b*c*p   Int[(a + b
*ArcCoth[c*x])^(p - 1)*(ArcCoth[1 - 2/(1 - c*x)]/(1 - c^2*x^2)), x], x] /;
FreeQ[{a, b, c}, x] && IGtQ[p, 1]
```

rule 6615
```
Int[(ArcCoth[u_]*((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.))/((d_) + (e_.)*(
x_)^2), x_Symbol] :> Simp[1/2   Int[Log[SimplifyIntegrand[1 + 1/u, x]]*((a
+ b*ArcCoth[c*x])^p/(d + e*x^2)), x], x] - Simp[1/2   Int[Log[SimplifyInteg
rand[1 - 1/u, x]]*((a + b*ArcCoth[c*x])^p/(d + e*x^2)), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && IGtQ[p, 0] && EqQ[c^2*d + e, 0] && EqQ[u^2 - (1 - 2/(1
- c*x))^2, 0]
```

rule 6619
```
Int[(Log[u_]*((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.))/((d_) + (e_.)*(x_)^
2), x_Symbol] :> Simp[(a + b*ArcCoth[c*x])^p*(PolyLog[2, 1 - u]/(2*c*d)), x
] - Simp[b*(p/2)   Int[(a + b*ArcCoth[c*x])^(p - 1)*(PolyLog[2, 1 - u]/(d +
 e*x^2)), x], x] /; FreeQ[{a, b, c, d, e}, x] && IGtQ[p, 0] && EqQ[c^2*d +
e, 0] && EqQ[(1 - u)^2 - (1 - 2/(1 + c*x))^2, 0]
```

rule 7164
```
Int[(u_)*PolyLog[n_, v_], x_Symbol] :> With[{w = DerivativeDivides[v, u*v,
x]}, Simp[w*PolyLog[n + 1, v], x] /;  !FalseQ[w]] /; FreeQ[n, x]
```

Maple [C] (warning: unable to verify)

Result contains higher order function than in optimal. Order 9 vs. order 4.

Time = 3.54 (sec) , antiderivative size = 459, normalized size of antiderivative = 4.73

method	result
derivativedivides	$\ln(xa)\operatorname{arccoth}(xa)^2+\frac{i\pi\operatorname{csgn}\left(\frac{i\left(1+\frac{xa+1}{xa-1}\right)}{\frac{xa+1}{xa-1}-1}\right)\left(\operatorname{csgn}\left(\frac{i}{\frac{xa+1}{xa-1}-1}\right)\operatorname{csgn}\left(i\left(1+\frac{xa+1}{xa-1}\right)\right)-\operatorname{csgn}\left(\frac{i}{\frac{xa+1}{xa-1}-1}\right)\operatorname{csg}\right.}{}$
default	$\ln(xa)\operatorname{arccoth}(xa)^2+\frac{i\pi\operatorname{csgn}\left(\frac{i\left(1+\frac{xa+1}{xa-1}\right)}{\frac{xa+1}{xa-1}-1}\right)\left(\operatorname{csgn}\left(\frac{i}{\frac{xa+1}{xa-1}-1}\right)\operatorname{csgn}\left(i\left(1+\frac{xa+1}{xa-1}\right)\right)-\operatorname{csgn}\left(\frac{i}{\frac{xa+1}{xa-1}-1}\right)\operatorname{csg}\right.}{}$
parts	$\ln(x)\operatorname{arccoth}(xa)^2+2a\left(\frac{\left(i\pi\operatorname{csgn}\left(\frac{i}{a}\right)\operatorname{csgn}\left(\frac{i\left(1+\frac{xa+1}{xa-1}\right)}{\frac{xa+1}{xa-1}-1}\right)\operatorname{csgn}\left(\frac{i\left(1+\frac{xa+1}{xa-1}\right)}{a\left(\frac{xa+1}{xa-1}-1\right)}\right)-i\pi\operatorname{csgn}\left(\frac{i}{a}\right)\operatorname{csgn}\left(\frac{i(1+}{a(\frac{xa}{xa}}\right.\right.}{}\right.$

input
```
int(arccoth(x*a)^2/x,x,method=_RETURNVERBOSE)
```

output

```
ln(x*a)*arccoth(x*a)^2+1/2*I*Pi*csgn(I/((a*x+1)/(a*x-1)-1)*(1+(a*x+1)/(a*x
-1)))*(csgn(I/((a*x+1)/(a*x-1)-1))*csgn(I*(1+(a*x+1)/(a*x-1)))-csgn(I/((a*
x+1)/(a*x-1)-1))*csgn(I/((a*x+1)/(a*x-1)-1)*(1+(a*x+1)/(a*x-1)))-csgn(I/((
a*x+1)/(a*x-1)-1)*(1+(a*x+1)/(a*x-1)))*csgn(I*(1+(a*x+1)/(a*x-1)))+csgn(I/
((a*x+1)/(a*x-1)-1)*(1+(a*x+1)/(a*x-1)))^2)*arccoth(x*a)^2+arccoth(x*a)^2*
ln((a*x+1)/(a*x-1)-1)-arccoth(x*a)^2*ln(1-1/((a*x-1)/(a*x+1))^(1/2))-2*arc
coth(x*a)*polylog(2,1/((a*x-1)/(a*x+1))^(1/2))+2*polylog(3,1/((a*x-1)/(a*x
+1))^(1/2))-arccoth(x*a)^2*ln(1+1/((a*x-1)/(a*x+1))^(1/2))-2*arccoth(x*a)*
polylog(2,-1/((a*x-1)/(a*x+1))^(1/2))+2*polylog(3,-1/((a*x-1)/(a*x+1))^(1/
2))+arccoth(x*a)*polylog(2,-(a*x+1)/(a*x-1))-1/2*polylog(3,-(a*x+1)/(a*x-1
))
```

Fricas [F]

$$\int \frac{\coth^{-1}(ax)^2}{x}\,dx = \int \frac{\operatorname{arcoth}(ax)^2}{x}\,dx$$

input

```
integrate(arccoth(a*x)^2/x,x, algorithm="fricas")
```

output

```
integral(arccoth(a*x)^2/x, x)
```

Sympy [F]

$$\int \frac{\coth^{-1}(ax)^2}{x}\,dx = \int \frac{\operatorname{acoth}^2(ax)}{x}\,dx$$

input

```
integrate(acoth(a*x)**2/x,x)
```

output

```
Integral(acoth(a*x)**2/x, x)
```

Maxima [F]

$$\int \frac{\coth^{-1}(ax)^2}{x} \, dx = \int \frac{\operatorname{arcoth}(ax)^2}{x} \, dx$$

input
```
integrate(arccoth(a*x)^2/x,x, algorithm="maxima")
```

output
```
integrate(arccoth(a*x)^2/x, x)
```

Giac [F]

$$\int \frac{\coth^{-1}(ax)^2}{x} \, dx = \int \frac{\operatorname{arcoth}(ax)^2}{x} \, dx$$

input
```
integrate(arccoth(a*x)^2/x,x, algorithm="giac")
```

output
```
integrate(arccoth(a*x)^2/x, x)
```

Mupad [F(-1)]

Timed out.

$$\int \frac{\coth^{-1}(ax)^2}{x} \, dx = \int \frac{\operatorname{acoth}(a\,x)^2}{x} \, dx$$

input
```
int(acoth(a*x)^2/x,x)
```

output
```
int(acoth(a*x)^2/x, x)
```

Reduce [F]

$$\int \frac{\coth^{-1}(ax)^2}{x}\,dx = \int \frac{acoth(ax)^2}{x}dx$$

input
```
int(acoth(a*x)^2/x,x)
```

output
```
int(acoth(a*x)**2/x,x)
```

3.19 $\int \frac{\coth^{-1}(ax)^2}{x^2}\,dx$

Optimal result

Integrand size = 10, antiderivative size = 55

$$\int \frac{\coth^{-1}(ax)^2}{x^2}\,dx = a\coth^{-1}(ax)^2 - \frac{\coth^{-1}(ax)^2}{x} + 2a\coth^{-1}(ax)\log\left(2-\frac{2}{1+ax}\right) - a\,\text{PolyLog}\left(2,-1+\frac{2}{1+ax}\right)$$

output
```
a*arccoth(a*x)^2-arccoth(a*x)^2/x+2*a*arccoth(a*x)*ln(2-2/(a*x+1))-a*polyl
og(2,-1+2/(a*x+1))
```

Mathematica [A] (verified)

Time = 0.07 (sec) , antiderivative size = 49, normalized size of antiderivative = 0.89

$$\int \frac{\coth^{-1}(ax)^2}{x^2}\,dx = \frac{(-1+ax)\coth^{-1}(ax)^2}{x} + 2a\coth^{-1}(ax)\log\left(1+e^{-2\coth^{-1}(ax)}\right) - a\,\text{PolyLog}\left(2,-e^{-2\coth^{-1}(ax)}\right)$$

input
```
Integrate[ArcCoth[a*x]^2/x^2,x]
```

output
```
((-1 + a*x)*ArcCoth[a*x]^2)/x + 2*a*ArcCoth[a*x]*Log[1 + E^(-2*ArcCoth[a*x
])] - a*PolyLog[2, -E^(-2*ArcCoth[a*x])]
```

Rubi [A] (verified)

Time = 0.42 (sec) , antiderivative size = 60, normalized size of antiderivative = 1.09, number of steps used = 4, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}} = 0.400$, Rules used = {6453, 6551, 6495, 2897}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\coth^{-1}(ax)^2}{x^2} \, dx$$

$$\Big\downarrow \text{6453}$$

$$2a \int \frac{\coth^{-1}(ax)}{x(1-a^2x^2)}dx - \frac{\coth^{-1}(ax)^2}{x}$$

$$\Big\downarrow \text{6551}$$

$$2a\left(\int \frac{\coth^{-1}(ax)}{x(ax+1)}dx + \frac{1}{2}\coth^{-1}(ax)^2\right) - \frac{\coth^{-1}(ax)^2}{x}$$

$$\Big\downarrow \text{6495}$$

$$2a\left(-a\int \frac{\log\left(2-\frac{2}{ax+1}\right)}{1-a^2x^2}dx + \frac{1}{2}\coth^{-1}(ax)^2 + \log\left(2-\frac{2}{ax+1}\right)\coth^{-1}(ax)\right) - \frac{\coth^{-1}(ax)^2}{x}$$

$$\Big\downarrow \text{2897}$$

$$2a\left(-\frac{1}{2}\text{PolyLog}\left(2,\frac{2}{ax+1}-1\right) + \frac{1}{2}\coth^{-1}(ax)^2 + \log\left(2-\frac{2}{ax+1}\right)\coth^{-1}(ax)\right) - \frac{\coth^{-1}(ax)^2}{x}$$

input
```
Int[ArcCoth[a*x]^2/x^2,x]
```

output
```
-(ArcCoth[a*x]^2/x) + 2*a*(ArcCoth[a*x]^2/2 + ArcCoth[a*x]*Log[2 - 2/(1 +
a*x)] - PolyLog[2, -1 + 2/(1 + a*x)]/2)
```

Defintions of rubi rules used

rule 2897
```
Int[Log[u_]*(Pq_)^(m_.), x_Symbol] :> With[{C = FullSimplify[Pq^m*((1 - u)/
D[u, x])]}, Simp[C*PolyLog[2, 1 - u], x] /; FreeQ[C, x]] /; IntegerQ[m] &&
PolyQ[Pq, x] && RationalFunctionQ[u, x] && LeQ[RationalFunctionExponents[u,
 x][[2]], Expon[Pq, x]]
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

rule 6495
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((x_)*((d_) + (e_.)*(x_))), x
_Symbol] :> Simp[(a + b*ArcCoth[c*x])^p*(Log[2 - 2/(1 + e*(x/d))]/d), x] -
Simp[b*c*(p/d)   Int[(a + b*ArcCoth[c*x])^(p - 1)*(Log[2 - 2/(1 + e*(x/d))]
/(1 - c^2*x^2)), x], x] /; FreeQ[{a, b, c, d, e}, x] && IGtQ[p, 0] && EqQ[c
^2*d^2 - e^2, 0]
```

rule 6551
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((x_)*((d_) + (e_.)*(x_)^2)),
 x_Symbol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*d*(p + 1)), x] + Simp[1/
d   Int[(a + b*ArcCoth[c*x])^p/(x*(1 + c*x)), x], x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[c^2*d + e, 0] && GtQ[p, 0]
```

Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 144 vs. $2(55) = 110$.

Time = 0.14 (sec) , antiderivative size = 145, normalized size of antiderivative = 2.64

method	result
derivativedivides	$a\left(-\frac{\operatorname{arccoth}(xa)^2}{xa} - \operatorname{arccoth}(xa)\ln(xa+1) + 2\ln(xa)\operatorname{arccoth}(xa) - \operatorname{arccoth}(xa)\ln(x$
default	$a\left(-\frac{\operatorname{arccoth}(xa)^2}{xa} - \operatorname{arccoth}(xa)\ln(xa+1) + 2\ln(xa)\operatorname{arccoth}(xa) - \operatorname{arccoth}(xa)\ln(x$
parts	$-\frac{\operatorname{arccoth}(xa)^2}{x} - 2a\left(\frac{\operatorname{arccoth}(xa)\ln(xa+1)}{2} - \ln(xa)\operatorname{arccoth}(xa) + \frac{\operatorname{arccoth}(xa)\ln(xa-1)}{2} + \frac{\ln(xa-}{8}$

input
```
int(arccoth(x*a)^2/x^2,x,method=_RETURNVERBOSE)
```

output
```
a*(-1/x/a*arccoth(x*a)^2-arccoth(x*a)*ln(a*x+1)+2*ln(x*a)*arccoth(x*a)-arc
coth(x*a)*ln(a*x-1)-1/4*ln(a*x-1)^2+dilog(1/2*x*a+1/2)+1/2*ln(a*x-1)*ln(1/
2*x*a+1/2)+1/4*ln(a*x+1)^2-1/2*(ln(a*x+1)-ln(1/2*x*a+1/2))*ln(-1/2*x*a+1/2
)-dilog(a*x+1)-ln(x*a)*ln(a*x+1)-dilog(x*a))
```

Fricas [F]

$$\int \frac{\coth^{-1}(ax)^2}{x^2}\,dx = \int \frac{\operatorname{arcoth}(ax)^2}{x^2}\,dx$$

input
```
integrate(arccoth(a*x)^2/x^2,x, algorithm="fricas")
```

output
```
integral(arccoth(a*x)^2/x^2, x)
```

Sympy [F]

$$\int \frac{\coth^{-1}(ax)^2}{x^2}\,dx = \int \frac{\operatorname{acoth}^2(ax)}{x^2}\,dx$$

input
```
integrate(acoth(a*x)**2/x**2,x)
```

output
```
Integral(acoth(a*x)**2/x**2, x)
```

Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 146 vs. 2(54) = 108.

Time = 0.03 (sec) , antiderivative size = 146, normalized size of antiderivative = 2.65

$$\int \frac{\coth^{-1}(ax)^2}{x^2}\,dx$$
$$= \frac{1}{4}a^2\left(\frac{\log(ax+1)^2 - 2\log(ax+1)\log(ax-1) - \log(ax-1)^2}{a} + \frac{4\left(\log(ax-1)\log\left(\frac{1}{2}ax+\frac{1}{2}\right) + \mathrm{Li}_2(\right.}{a}\right.$$
$$- a\left(\log\left(a^2x^2-1\right) - \log\left(x^2\right)\right)\operatorname{arcoth}(ax) - \frac{\operatorname{arcoth}(ax)^2}{x}$$

input
```
integrate(arccoth(a*x)^2/x^2,x, algorithm="maxima")
```

output
```
1/4*a^2*((log(a*x + 1)^2 - 2*log(a*x + 1)*log(a*x - 1) - log(a*x - 1)^2)/a
 + 4*(log(a*x - 1)*log(1/2*a*x + 1/2) + dilog(-1/2*a*x + 1/2))/a - 4*(log(
a*x + 1)*log(x) + dilog(-a*x))/a + 4*(log(-a*x + 1)*log(x) + dilog(a*x))/a
) - a*(log(a^2*x^2 - 1) - log(x^2))*arccoth(a*x) - arccoth(a*x)^2/x
```

Giac [F]

$$\int \frac{\coth^{-1}(ax)^2}{x^2} \, dx = \int \frac{\operatorname{arcoth}(ax)^2}{x^2} \, dx$$

input
```
integrate(arccoth(a*x)^2/x^2,x, algorithm="giac")
```

output
```
integrate(arccoth(a*x)^2/x^2, x)
```

Mupad [F(-1)]

Timed out.

$$\int \frac{\coth^{-1}(ax)^2}{x^2} \, dx = \int \frac{\operatorname{acoth}(a\,x)^2}{x^2} \, dx$$

input
```
int(acoth(a*x)^2/x^2,x)
```

output
```
int(acoth(a*x)^2/x^2, x)
```

Reduce [F]

$$\int \frac{\coth^{-1}(ax)^2}{x^2} \, dx = \frac{-acoth(ax)^2 + 2\left(\int \frac{acoth(ax)}{a^2x^3-x}dx\right) ax}{x}$$

input
```
int(acoth(a*x)^2/x^2,x)
```

output
```
( - acoth(a*x)**2 + 2*int(acoth(a*x)/(a**2*x**3 - x),x)*a*x)/x
```

3.20 $\int \frac{\coth^{-1}(ax)^2}{x^3} \, dx$

Optimal result

Integrand size = 10, antiderivative size = 61

$$\int \frac{\coth^{-1}(ax)^2}{x^3} \, dx = -\frac{a\coth^{-1}(ax)}{x} + \frac{1}{2}a^2\coth^{-1}(ax)^2 - \frac{\coth^{-1}(ax)^2}{2x^2} + a^2\log(x) - \frac{1}{2}a^2\log\left(1-a^2x^2\right)$$

output
```
-a*arccoth(a*x)/x+1/2*a^2*arccoth(a*x)^2-1/2*arccoth(a*x)^2/x^2+a^2*ln(x)-
1/2*a^2*ln(-a^2*x^2+1)
```

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 57, normalized size of antiderivative = 0.93

$$\int \frac{\coth^{-1}(ax)^2}{x^3} \, dx = -\frac{a\coth^{-1}(ax)}{x} + \frac{(-1+a^2x^2)\coth^{-1}(ax)^2}{2x^2} + a^2\log(x) - \frac{1}{2}a^2\log\left(1-a^2x^2\right)$$

input
```
Integrate[ArcCoth[a*x]^2/x^3,x]
```

output
```
-((a*ArcCoth[a*x])/x) + ((-1 + a^2*x^2)*ArcCoth[a*x]^2)/(2*x^2) + a^2*Log[
x] - (a^2*Log[1 - a^2*x^2])/2
```

Rubi [A] (verified)

Time = 0.47 (sec) , antiderivative size = 60, normalized size of antiderivative = 0.98, number of steps used = 9, number of rules used = 8, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.800, Rules used = {6453, 6545, 6453, 243, 47, 14, 16, 6511}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\coth^{-1}(ax)^2}{x^3}dx$$

$$\Big\downarrow \text{ 6453}$$

$$a\int \frac{\coth^{-1}(ax)}{x^2(1-a^2x^2)}dx-\frac{\coth^{-1}(ax)^2}{2x^2}$$

$$\Big\downarrow \text{ 6545}$$

$$a\left(a^2\int \frac{\coth^{-1}(ax)}{1-a^2x^2}dx+\int \frac{\coth^{-1}(ax)}{x^2}dx\right)-\frac{\coth^{-1}(ax)^2}{2x^2}$$

$$\Big\downarrow \text{ 6453}$$

$$a\left(a\int \frac{1}{x(1-a^2x^2)}dx+a^2\int \frac{\coth^{-1}(ax)}{1-a^2x^2}dx-\frac{\coth^{-1}(ax)}{x}\right)-\frac{\coth^{-1}(ax)^2}{2x^2}$$

$$\Big\downarrow \text{ 243}$$

$$a\left(\frac{1}{2}a\int \frac{1}{x^2(1-a^2x^2)}dx^2+a^2\int \frac{\coth^{-1}(ax)}{1-a^2x^2}dx-\frac{\coth^{-1}(ax)}{x}\right)-\frac{\coth^{-1}(ax)^2}{2x^2}$$

$$\Big\downarrow \text{ 47}$$

$$a\left(\frac{1}{2}a\left(a^2\int \frac{1}{1-a^2x^2}dx^2+\int \frac{1}{x^2}dx^2\right)+a^2\int \frac{\coth^{-1}(ax)}{1-a^2x^2}dx-\frac{\coth^{-1}(ax)}{x}\right)-\frac{\coth^{-1}(ax)^2}{2x^2}$$

$$\Big\downarrow \text{ 14}$$

$$a\left(\frac{1}{2}a\left(a^2\int \frac{1}{1-a^2x^2}dx^2+\log\left(x^2\right)\right)+a^2\int \frac{\coth^{-1}(ax)}{1-a^2x^2}dx-\frac{\coth^{-1}(ax)}{x}\right)-\frac{\coth^{-1}(ax)^2}{2x^2}$$

$$\Big\downarrow \text{ 16}$$

$$a\left(a^2\int \frac{\coth^{-1}(ax)}{1-a^2x^2}dx+\frac{1}{2}a\left(\log\left(x^2\right)-\log\left(1-a^2x^2\right)\right)-\frac{\coth^{-1}(ax)}{x}\right)-\frac{\coth^{-1}(ax)^2}{2x^2}$$

$$\Big\downarrow \text{6511}$$

$$a\left(\frac{1}{2}a\left(\log\left(x^2\right)-\log\left(1-a^2x^2\right)\right)+\frac{1}{2}a\coth^{-1}(ax)^2-\frac{\coth^{-1}(ax)}{x}\right)-\frac{\coth^{-1}(ax)^2}{2x^2}$$

input
```
Int[ArcCoth[a*x]^2/x^3,x]
```

output
```
-1/2*ArcCoth[a*x]^2/x^2 + a*(-(ArcCoth[a*x]/x) + (a*ArcCoth[a*x]^2)/2 + (a
*(Log[x^2] - Log[1 - a^2*x^2]))/2)
```

Defintions of rubi rules used

rule 14
```
Int[(a_.)/(x_), x_Symbol] :> Simp[a*Log[x], x] /; FreeQ[a, x]
```

rule 16
```
Int[(c_.)/((a_.) + (b_.)*(x_)), x_Symbol] :> Simp[c*(Log[RemoveContent[a +
b*x, x]]/b), x] /; FreeQ[{a, b, c}, x]
```

rule 47
```
Int[1/(((a_.) + (b_.)*(x_))*((c_.) + (d_.)*(x_))), x_Symbol] :> Simp[b/(b*c
 - a*d)   Int[1/(a + b*x), x], x] - Simp[d/(b*c - a*d)   Int[1/(c + d*x), x
], x] /; FreeQ[{a, b, c, d}, x]
```

rule 243
```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] :> Simp[1/2   Subst[In
t[x^((m - 1)/2)*(a + b*x)^p, x], x, x^2], x] /; FreeQ[{a, b, m, p}, x] && I
ntegerQ[(m - 1)/2]
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

rule 6511
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((d_) + (e_.)*(x_)^2), x_Symb
ol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*c*d*(p + 1)), x] /; FreeQ[{a, b
, c, d, e, p}, x] && EqQ[c^2*d + e, 0] && NeQ[p, -1]
```

rule 6545
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*((f_.)*(x_))^(m_))/((d_) + (
e_.)*(x_)^2), x_Symbol] :> Simp[1/d   Int[(f*x)^m*(a + b*ArcCoth[c*x])^p, x
], x] - Simp[e/(d*f^2)   Int[(f*x)^(m + 2)*((a + b*ArcCoth[c*x])^p/(d + e*x
^2)), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && GtQ[p, 0] && LtQ[m, -1]
```

Maple [A] (verified)

Time = 0.15 (sec) , antiderivative size = 72, normalized size of antiderivative = 1.18

method	result
parallelrisch	$\frac{x^2a^2\operatorname{arccoth}(xa)^2+2a^2\ln(x)x^2-2x^2\ln(xa-1)a^2-2\operatorname{arccoth}(xa)a^2x^2-2xa\operatorname{arccoth}(xa)-\operatorname{arccoth}(xa)^2}{2x^2}$
risch	$\frac{(a^2x^2-1)\ln(xa+1)^2}{8x^2}-\frac{(x^2\ln(xa-1)a^2+2xa-\ln(xa-1))\ln(xa+1)}{4x^2}+\frac{a^2x^2\ln(xa-1)^2+8a^2\ln(x)x^2-4a^2\ln(a^2x^2}{8x^2}$
derivativedivides	$a^2\left(-\frac{\operatorname{arccoth}(xa)^2}{2x^2a^2}-\frac{\operatorname{arccoth}(xa)\ln(xa-1)}{2}+\frac{\operatorname{arccoth}(xa)\ln(xa+1)}{2}-\frac{\operatorname{arccoth}(xa)}{xa}-\frac{\ln(xa-1)^2}{8}+\frac{\ln(xa-}{}\right.$
default	$a^2\left(-\frac{\operatorname{arccoth}(xa)^2}{2x^2a^2}-\frac{\operatorname{arccoth}(xa)\ln(xa-1)}{2}+\frac{\operatorname{arccoth}(xa)\ln(xa+1)}{2}-\frac{\operatorname{arccoth}(xa)}{xa}-\frac{\ln(xa-1)^2}{8}+\frac{\ln(xa-}{}\right.$
parts	$-\frac{\operatorname{arccoth}(xa)^2}{2x^2}-a^2\left(\frac{\operatorname{arccoth}(xa)\ln(xa-1)}{2}-\frac{\operatorname{arccoth}(xa)\ln(xa+1)}{2}+\frac{\operatorname{arccoth}(xa)}{xa}+\frac{\ln(xa-1)^2}{8}-\frac{\ln(xa-}{}\right.$

input
```
int(arccoth(x*a)^2/x^3,x,method=_RETURNVERBOSE)
```

output
```
1/2*(x^2*a^2*arccoth(x*a)^2+2*a^2*ln(x)*x^2-2*x^2*ln(a*x-1)*a^2-2*arccoth(
x*a)*a^2*x^2-2*x*a*arccoth(x*a)-arccoth(x*a)^2)/x^2
```

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 79, normalized size of antiderivative = 1.30

$$\int \frac{\coth^{-1}(ax)^2}{x^3}\,dx = -\frac{4\,a^2x^2\log\left(a^2x^2-1\right)-8\,a^2x^2\log\left(x\right)+4\,ax\log\left(\frac{ax+1}{ax-1}\right)-\left(a^2x^2-1\right)\log\left(\frac{ax+1}{ax-1}\right)^2}{8\,x^2}$$

input
```
integrate(arccoth(a*x)^2/x^3,x, algorithm="fricas")
```

output
```
-1/8*(4*a^2*x^2*log(a^2*x^2 - 1) - 8*a^2*x^2*log(x) + 4*a*x*log((a*x + 1)/
(a*x - 1)) - (a^2*x^2 - 1)*log((a*x + 1)/(a*x - 1))^2)/x^2
```

Sympy [A] (verification not implemented)

Time = 0.28 (sec) , antiderivative size = 56, normalized size of antiderivative = 0.92

$$\int \frac{\coth^{-1}(ax)^2}{x^3}\,dx = a^2\log\left(x\right)-a^2\log\left(ax+1\right)+\frac{a^2\operatorname{acoth}^2\left(ax\right)}{2}+a^2\operatorname{acoth}\left(ax\right)-\frac{a\operatorname{acoth}\left(ax\right)}{x}-\frac{\operatorname{acoth}^2\left(ax\right)}{2x^2}$$

input
```
integrate(acoth(a*x)**2/x**3,x)
```

output
```
a**2*log(x) - a**2*log(a*x + 1) + a**2*acoth(a*x)**2/2 + a**2*acoth(a*x) -
 a*acoth(a*x)/x - acoth(a*x)**2/(2*x**2)
```

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 96, normalized size of antiderivative = 1.57

$$\int \frac{\coth^{-1}(ax)^2}{x^3} dx$$
$$= \frac{1}{8} \left(2 \left(\log \left(ax - 1\right) - 2\right) \log \left(ax + 1\right) - \log \left(ax + 1\right)^2 - \log \left(ax - 1\right)^2 - 4 \log \left(ax - 1\right) + 8 \log \left(x\right)\right) a^2$$
$$+ \frac{1}{2} \left(a \log \left(ax + 1\right) - a \log \left(ax - 1\right) - \frac{2}{x}\right) a \operatorname{arcoth}\left(ax\right) - \frac{\operatorname{arcoth}\left(ax\right)^2}{2 x^2}$$

input
```
integrate(arccoth(a*x)^2/x^3,x, algorithm="maxima")
```

output
```
1/8*(2*(log(a*x - 1) - 2)*log(a*x + 1) - log(a*x + 1)^2 - log(a*x - 1)^2 -
 4*log(a*x - 1) + 8*log(x))*a^2 + 1/2*(a*log(a*x + 1) - a*log(a*x - 1) - 2
/x)*a*arccoth(a*x) - 1/2*arccoth(a*x)^2/x^2
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 137 vs. 2(55) = 110.

Time = 0.12 (sec) , antiderivative size = 137, normalized size of antiderivative = 2.25

$$\int \frac{\coth^{-1}(ax)^2}{x^3} dx$$
$$= \frac{1}{2} \left(2 a \log \left(\frac{ax + 1}{ax - 1} + 1\right) - 2 a \log \left(\frac{ax + 1}{ax - 1}\right) + \frac{(ax + 1) a \log \left(\frac{ax+1}{ax-1}\right)^2}{(ax - 1) \left(\frac{(ax+1)^2}{(ax-1)^2} + \frac{2 (ax+1)}{ax-1} + 1\right)} + \frac{2 a \log \left(\frac{ax+1}{ax-1}\right)}{\frac{ax+1}{ax-1} + 1}\right) a$$

input
```
integrate(arccoth(a*x)^2/x^3,x, algorithm="giac")
```

output
```
1/2*(2*a*log((a*x + 1)/(a*x - 1) + 1) - 2*a*log((a*x + 1)/(a*x - 1)) + (a*
x + 1)*a*log((a*x + 1)/(a*x - 1))^2/((a*x - 1)*((a*x + 1)^2/(a*x - 1)^2 +
2*(a*x + 1)/(a*x - 1) + 1)) + 2*a*log((a*x + 1)/(a*x - 1))/((a*x + 1)/(a*x
 - 1) + 1))*a
```

Mupad [B] (verification not implemented)

Time = 3.78 (sec) , antiderivative size = 145, normalized size of antiderivative = 2.38

$$\int \frac{\coth^{-1}(ax)^2}{x^3}\,dx = a^2\,\ln(x) + \ln\left(\frac{1}{a\,x}+1\right)^2\left(\frac{a^2}{8}-\frac{1}{8\,x^2}\right) + \ln\left(1-\frac{1}{a\,x}\right)^2\left(\frac{a^2}{8}-\frac{1}{8\,x^2}\right) - \frac{a^2\,\ln\left(a^2\,x^2-1\right)}{2} + \ln\left(1-\frac{1}{a\,x}\right)\left(\frac{4\,a\,x-2}{16\,x^2}+\frac{4\,a\,x+2}{16\,x^2} - \ln\left(\frac{1}{a\,x}+1\right)\left(\frac{a^2}{4}-\frac{1}{4\,x^2}\right)\right) - \frac{a\,\ln\left(\frac{1}{a\,x}+1\right)}{2\,x}$$

input
```
int(acoth(a*x)^2/x^3,x)
```

output
```
a^2*log(x) + log(1/(a*x) + 1)^2*(a^2/8 - 1/(8*x^2)) + log(1 - 1/(a*x))^2*(
a^2/8 - 1/(8*x^2)) - (a^2*log(a^2*x^2 - 1))/2 + log(1 - 1/(a*x))*((4*a*x -
 2)/(16*x^2) + (4*a*x + 2)/(16*x^2) - log(1/(a*x) + 1)*(a^2/4 - 1/(4*x^2))
) - (a*log(1/(a*x) + 1))/(2*x)
```

Reduce [B] (verification not implemented)

Time = 0.17 (sec) , antiderivative size = 75, normalized size of antiderivative = 1.23

$$\int \frac{\coth^{-1}(ax)^2}{x^3}\,dx = \frac{acoth(ax)^2\,a^2x^2 - acoth(ax)^2 + 2acoth(ax)\,a^2x^2 + 2acoth(ax)\,ax - 2\log(a^2x-a)\,a^2x^2 + 2\log(x)\,a^2x^2}{2x^2}$$

input
```
int(acoth(a*x)^2/x^3,x)
```

output
```
(acoth(a*x)**2*a**2*x**2 - acoth(a*x)**2 + 2*acoth(a*x)*a**2*x**2 + 2*acot
h(a*x)*a*x - 2*log(a**2*x - a)*a**2*x**2 + 2*log(x)*a**2*x**2)/(2*x**2)
```

3.21 $\int \frac{\coth^{-1}(ax)^2}{x^4}\,dx$

Optimal result

Integrand size = 10, antiderivative size = 103

$$\int \frac{\coth^{-1}(ax)^2}{x^4}\,dx = -\frac{a^2}{3x} - \frac{a\coth^{-1}(ax)}{3x^2} + \frac{1}{3}a^3\coth^{-1}(ax)^2 - \frac{\coth^{-1}(ax)^2}{3x^3} + \frac{1}{3}a^3\operatorname{arctanh}(ax) + \frac{2}{3}a^3\coth^{-1}(ax)\log\left(2-\frac{2}{1+ax}\right) - \frac{1}{3}a^3\operatorname{PolyLog}\left(2,-1+\frac{2}{1+ax}\right)$$

output
```
-1/3*a^2/x-1/3*a*arccoth(a*x)/x^2+1/3*a^3*arccoth(a*x)^2-1/3*arccoth(a*x)^
2/x^3+1/3*a^3*arctanh(a*x)+2/3*a^3*arccoth(a*x)*ln(2-2/(a*x+1))-1/3*a^3*po
lylog(2,-1+2/(a*x+1))
```

Mathematica [A] (verified)

Time = 0.15 (sec) , antiderivative size = 87, normalized size of antiderivative = 0.84

$$\int \frac{\coth^{-1}(ax)^2}{x^4}\,dx = \frac{-a^2x^2+(-1+a^3x^3)\coth^{-1}(ax)^2+ax\coth^{-1}(ax)\left(-1+a^2x^2+2a^2x^2\log\left(1+e^{-2\coth^{-1}(ax)}\right)\right)-a^3x^3}{3x^3}$$

input
```
Integrate[ArcCoth[a*x]^2/x^4,x]
```

output
```
(-(a^2*x^2) + (-1 + a^3*x^3)*ArcCoth[a*x]^2 + a*x*ArcCoth[a*x]*(-1 + a^2*x
^2 + 2*a^2*x^2*Log[1 + E^(-2*ArcCoth[a*x])]) - a^3*x^3*PolyLog[2, -E^(-2*A
rcCoth[a*x])])/(3*x^3)
```

Rubi [A] (verified)

Time = 0.71 (sec) , antiderivative size = 97, normalized size of antiderivative = 0.94, number of steps used = 8, number of rules used = 8, $\frac{\text{number of rules}}{\text{integrand size}} = 0.800$, Rules used = {6453, 6545, 6453, 264, 219, 6551, 6495, 2897}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\coth^{-1}(ax)^2}{x^4} dx$$

$$\Big\downarrow \text{6453}$$

$$\frac{2}{3}a\int \frac{\coth^{-1}(ax)}{x^3(1-a^2x^2)}dx - \frac{\coth^{-1}(ax)^2}{3x^3}$$

$$\Big\downarrow \text{6545}$$

$$\frac{2}{3}a\left(a^2\int \frac{\coth^{-1}(ax)}{x(1-a^2x^2)}dx + \int \frac{\coth^{-1}(ax)}{x^3}dx\right) - \frac{\coth^{-1}(ax)^2}{3x^3}$$

$$\Big\downarrow \text{6453}$$

$$\frac{2}{3}a\left(\frac{1}{2}a\int \frac{1}{x^2(1-a^2x^2)}dx + a^2\int \frac{\coth^{-1}(ax)}{x(1-a^2x^2)}dx - \frac{\coth^{-1}(ax)}{2x^2}\right) - \frac{\coth^{-1}(ax)^2}{3x^3}$$

$$\Big\downarrow \text{264}$$

$$\frac{2}{3}a\left(\frac{1}{2}a\left(a^2\int \frac{1}{1-a^2x^2}dx - \frac{1}{x}\right) + a^2\int \frac{\coth^{-1}(ax)}{x(1-a^2x^2)}dx - \frac{\coth^{-1}(ax)}{2x^2}\right) - \frac{\coth^{-1}(ax)^2}{3x^3}$$

$$\Big\downarrow \text{219}$$

$$\frac{2}{3}a\left(a^2\int \frac{\coth^{-1}(ax)}{x(1-a^2x^2)}dx + \frac{1}{2}a\left(a\text{arctanh}(ax) - \frac{1}{x}\right) - \frac{\coth^{-1}(ax)}{2x^2}\right) - \frac{\coth^{-1}(ax)^2}{3x^3}$$

$$\Big\downarrow \text{6551}$$

$$\frac{2}{3}a\left(a^2\left(\int \frac{\coth^{-1}(ax)}{x(ax+1)}dx+\frac{1}{2}\coth^{-1}(ax)^2\right)+\frac{1}{2}a\left(a\text{arctanh}(ax)-\frac{1}{x}\right)-\frac{\coth^{-1}(ax)}{2x^2}\right)-\frac{\coth^{-1}(ax)^2}{3x^3}$$

$\Big\downarrow$ 6495

$$\frac{2}{3}a\left(a^2\left(-a\int \frac{\log\left(2-\frac{2}{ax+1}\right)}{1-a^2x^2}dx+\frac{1}{2}\coth^{-1}(ax)^2+\log\left(2-\frac{2}{ax+1}\right)\coth^{-1}(ax)\right)+\frac{1}{2}a\left(a\text{arctanh}(ax)-\frac{1}{x}\right.\right.$$
$$\frac{\coth^{-1}(ax)^2}{3x^3}$$

$\Big\downarrow$ 2897

$$\frac{2}{3}a\left(a^2\left(-\frac{1}{2}\text{PolyLog}\left(2,\frac{2}{ax+1}-1\right)+\frac{1}{2}\coth^{-1}(ax)^2+\log\left(2-\frac{2}{ax+1}\right)\coth^{-1}(ax)\right)+\frac{1}{2}a\left(a\text{arctanh}(ax)\right.\right.$$
$$\frac{\coth^{-1}(ax)^2}{3x^3}$$

input
```
Int[ArcCoth[a*x]^2/x^4,x]
```

output
```
-1/3*ArcCoth[a*x]^2/x^3 + (2*a*(-1/2*ArcCoth[a*x]/x^2 + (a*(-x^(-1) + a*Ar
cTanh[a*x]))/2 + a^2*(ArcCoth[a*x]^2/2 + ArcCoth[a*x]*Log[2 - 2/(1 + a*x)]
 - PolyLog[2, -1 + 2/(1 + a*x)]/2)))/3
```

Defintions of rubi rules used

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 264
```
Int[((c_.)*(x_))^(m_)*((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(c*x)^(
m + 1)*((a + b*x^2)^(p + 1)/(a*c*(m + 1))), x] - Simp[b*((m + 2*p + 3)/(a*c
^2*(m + 1)))   Int[(c*x)^(m + 2)*(a + b*x^2)^p, x], x] /; FreeQ[{a, b, c, p
}, x] && LtQ[m, -1] && IntBinomialQ[a, b, c, 2, m, p, x]
```

rule 2897
```
Int[Log[u_]*(Pq_)^(m_.), x_Symbol] :> With[{C = FullSimplify[Pq^m*((1 - u)/
D[u, x])]}, Simp[C*PolyLog[2, 1 - u], x] /; FreeQ[C, x]] /; IntegerQ[m] &&
PolyQ[Pq, x] && RationalFunctionQ[u, x] && LeQ[RationalFunctionExponents[u,
 x][[2]], Expon[Pq, x]]
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

rule 6495
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((x_)*((d_) + (e_.)*(x_))), x
_Symbol] :> Simp[(a + b*ArcCoth[c*x])^p*(Log[2 - 2/(1 + e*(x/d))]/d), x] -
Simp[b*c*(p/d)   Int[(a + b*ArcCoth[c*x])^(p - 1)*(Log[2 - 2/(1 + e*(x/d))]
/(1 - c^2*x^2)), x], x] /; FreeQ[{a, b, c, d, e}, x] && IGtQ[p, 0] && EqQ[c
^2*d^2 - e^2, 0]
```

rule 6545
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*((f_.)*(x_))^(m_))/((d_) + (
e_.)*(x_)^2), x_Symbol] :> Simp[1/d   Int[(f*x)^m*(a + b*ArcCoth[c*x])^p, x
], x] - Simp[e/(d*f^2)   Int[(f*x)^(m + 2)*((a + b*ArcCoth[c*x])^p/(d + e*x
^2)), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && GtQ[p, 0] && LtQ[m, -1]
```

rule 6551
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((x_)*((d_) + (e_.)*(x_)^2)),
 x_Symbol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*d*(p + 1)), x] + Simp[1/
d   Int[(a + b*ArcCoth[c*x])^p/(x*(1 + c*x)), x], x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[c^2*d + e, 0] && GtQ[p, 0]
```

Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 183 vs. $2(89) = 178$.

Time = 0.16 (sec) , antiderivative size = 184, normalized size of antiderivative = 1.79

method	result
parts	$-\frac{\operatorname{arccoth}(xa)^2}{3x^3} - \frac{2a^3\left(\frac{\operatorname{arccoth}(xa)\ln(xa+1)}{2}+\frac{\operatorname{arccoth}(xa)\ln(xa-1)}{2}+\frac{\operatorname{arccoth}(xa)}{2x^2a^2}-\ln(xa)\operatorname{arccoth}(xa)-\frac{\ln(xa+1)}{4}+\frac{\ln(x}{}\right.}{}$
derivativedivides	$a^3\left(-\frac{\operatorname{arccoth}(xa)^2}{3x^3a^3}-\frac{\operatorname{arccoth}(xa)\ln(xa+1)}{3}-\frac{\operatorname{arccoth}(xa)\ln(xa-1)}{3}-\frac{\operatorname{arccoth}(xa)}{3x^2a^2}+\frac{2\ln(xa)\operatorname{arccoth}(xa)}{3}\right.$
default	$a^3\left(-\frac{\operatorname{arccoth}(xa)^2}{3x^3a^3}-\frac{\operatorname{arccoth}(xa)\ln(xa+1)}{3}-\frac{\operatorname{arccoth}(xa)\ln(xa-1)}{3}-\frac{\operatorname{arccoth}(xa)}{3x^2a^2}+\frac{2\ln(xa)\operatorname{arccoth}(xa)}{3}\right.$

input
```
int(arccoth(x*a)^2/x^4,x,method=_RETURNVERBOSE)
```

output
```
-1/3*arccoth(x*a)^2/x^3-2/3*a^3*(1/2*arccoth(x*a)*ln(a*x+1)+1/2*arccoth(x*
a)*ln(a*x-1)+1/2/x^2/a^2*arccoth(x*a)-ln(x*a)*arccoth(x*a)-1/4*ln(a*x+1)+1
/4*ln(a*x-1)+1/2/a/x+1/8*ln(a*x-1)^2-1/2*dilog(1/2*x*a+1/2)-1/4*ln(a*x-1)*
ln(1/2*x*a+1/2)-1/8*ln(a*x+1)^2+1/4*(ln(a*x+1)-ln(1/2*x*a+1/2))*ln(-1/2*x*
a+1/2)+1/2*dilog(a*x+1)+1/2*ln(x*a)*ln(a*x+1)+1/2*dilog(x*a))
```

Fricas [F]

$$\int \frac{\coth^{-1}(ax)^2}{x^4}\,dx = \int \frac{\operatorname{arcoth}(ax)^2}{x^4}\,dx$$

input
```
integrate(arccoth(a*x)^2/x^4,x, algorithm="fricas")
```

output
```
integral(arccoth(a*x)^2/x^4, x)
```

Sympy [F]

$$\int \frac{\coth^{-1}(ax)^2}{x^4}\,dx = \int \frac{\operatorname{acoth}^2(ax)}{x^4}\,dx$$

input
```
integrate(acoth(a*x)**2/x**4,x)
```

output
```
Integral(acoth(a*x)**2/x**4, x)
```

Maxima [A] (verification not implemented)

Time = 0.04 (sec) , antiderivative size = 176, normalized size of antiderivative = 1.71

$$\int \frac{\coth^{-1}(ax)^2}{x^4}\,dx$$
$$= \frac{1}{12}\left(4\left(\log(ax-1)\log\left(\frac{1}{2}ax+\frac{1}{2}\right)+\mathrm{Li}_2\left(-\frac{1}{2}ax+\frac{1}{2}\right)\right)a-4\left(\log(ax+1)\log(x)+\mathrm{Li}_2(-ax)\right)a+4\right.$$
$$-\frac{1}{3}\left(a^2\log\left(a^2x^2-1\right)-a^2\log\left(x^2\right)+\frac{1}{x^2}\right)a\,\mathrm{arcoth}(ax)-\frac{\mathrm{arcoth}(ax)^2}{3x^3}$$

input
```
integrate(arccoth(a*x)^2/x^4,x, algorithm="maxima")
```

output
```
1/12*(4*(log(a*x - 1)*log(1/2*a*x + 1/2) + dilog(-1/2*a*x + 1/2))*a - 4*(l
og(a*x + 1)*log(x) + dilog(-a*x))*a + 4*(log(-a*x + 1)*log(x) + dilog(a*x)
)*a + 2*a*log(a*x + 1) - 2*a*log(a*x - 1) + (a*x*log(a*x + 1)^2 - 2*a*x*lo
g(a*x + 1)*log(a*x - 1) - a*x*log(a*x - 1)^2 - 4)/x)*a^2 - 1/3*(a^2*log(a^
2*x^2 - 1) - a^2*log(x^2) + 1/x^2)*a*arccoth(a*x) - 1/3*arccoth(a*x)^2/x^3
```

Giac [F]

$$\int \frac{\coth^{-1}(ax)^2}{x^4}\,dx = \int \frac{\mathrm{arcoth}(ax)^2}{x^4}\,dx$$

input
```
integrate(arccoth(a*x)^2/x^4,x, algorithm="giac")
```

output
```
integrate(arccoth(a*x)^2/x^4, x)
```

Mupad [F(-1)]

Timed out.

$$\int \frac{\coth^{-1}(ax)^2}{x^4}\,dx = \int \frac{\mathrm{acoth}(a\,x)^2}{x^4}\,dx$$

input
```
int(acoth(a*x)^2/x^4,x)
```

output
```
int(acoth(a*x)^2/x^4, x)
```

Reduce [F]

$$\int \frac{\coth^{-1}(ax)^2}{x^4}\,dx$$
$$= \frac{-acoth(ax)^2 - acoth(ax)\,a^3x^3 + acoth(ax)\,ax + 2\left(\int \frac{acoth(ax)}{a^2x^3-x}dx\right)a^3x^3 - a^2x^2}{3x^3}$$

input
```
int(acoth(a*x)^2/x^4,x)
```

output
```
( - acoth(a*x)**2 - acoth(a*x)*a**3*x**3 + acoth(a*x)*a*x + 2*int(acoth(a*
x)/(a**2*x**3 - x),x)*a**3*x**3 - a**2*x**2)/(3*x**3)
```

3.22 $\int \frac{\coth^{-1}(ax)^2}{x^5}\,dx$

Optimal result

Integrand size = 10, antiderivative size = 90

$$\int \frac{\coth^{-1}(ax)^2}{x^5}\,dx = -\frac{a^2}{12x^2} - \frac{a\coth^{-1}(ax)}{6x^3} - \frac{a^3\coth^{-1}(ax)}{2x} + \frac{1}{4}a^4\coth^{-1}(ax)^2 - \frac{\coth^{-1}(ax)^2}{4x^4} + \frac{2}{3}a^4\log(x) - \frac{1}{3}a^4\log\left(1-a^2x^2\right)$$

output
```
-1/12*a^2/x^2-1/6*a*arccoth(a*x)/x^3-1/2*a^3*arccoth(a*x)/x+1/4*a^4*arccot
h(a*x)^2-1/4*arccoth(a*x)^2/x^4+2/3*a^4*ln(x)-1/3*a^4*ln(-a^2*x^2+1)
```

Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 82, normalized size of antiderivative = 0.91

$$\int \frac{\coth^{-1}(ax)^2}{x^5}\,dx = -\frac{a^2}{12x^2} - \frac{a(1+3a^2x^2)\coth^{-1}(ax)}{6x^3} + \frac{(-1+a^4x^4)\coth^{-1}(ax)^2}{4x^4} + \frac{2}{3}a^4\log(x) - \frac{1}{3}a^4\log\left(1-a^2x^2\right)$$

input
```
Integrate[ArcCoth[a*x]^2/x^5,x]
```

output
```
-1/12*a^2/x^2 - (a*(1 + 3*a^2*x^2)*ArcCoth[a*x])/(6*x^3) + ((-1 + a^4*x^4)
*ArcCoth[a*x]^2)/(4*x^4) + (2*a^4*Log[x])/3 - (a^4*Log[1 - a^2*x^2])/3
```

Rubi [A] (verified)

Time = 0.86 (sec) , antiderivative size = 114, normalized size of antiderivative = 1.27, number of steps used = 14, number of rules used = 13, $\frac{\text{number of rules}}{\text{integrand size}}$ = 1.300, Rules used = {6453, 6545, 6453, 243, 54, 2009, 6545, 6453, 243, 47, 14, 16, 6511}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\coth^{-1}(ax)^2}{x^5}dx$$

$$\Big\downarrow \text{6453}$$

$$\frac{1}{2}a\int \frac{\coth^{-1}(ax)}{x^4(1-a^2x^2)}dx-\frac{\coth^{-1}(ax)^2}{4x^4}$$

$$\Big\downarrow \text{6545}$$

$$\frac{1}{2}a\left(a^2\int \frac{\coth^{-1}(ax)}{x^2(1-a^2x^2)}dx+\int \frac{\coth^{-1}(ax)}{x^4}dx\right)-\frac{\coth^{-1}(ax)^2}{4x^4}$$

$$\Big\downarrow \text{6453}$$

$$\frac{1}{2}a\left(a^2\int \frac{\coth^{-1}(ax)}{x^2(1-a^2x^2)}dx+\frac{1}{3}a\int \frac{1}{x^3(1-a^2x^2)}dx-\frac{\coth^{-1}(ax)}{3x^3}\right)-\frac{\coth^{-1}(ax)^2}{4x^4}$$

$$\Big\downarrow \text{243}$$

$$\frac{1}{2}a\left(a^2\int \frac{\coth^{-1}(ax)}{x^2(1-a^2x^2)}dx+\frac{1}{6}a\int \frac{1}{x^4(1-a^2x^2)}dx^2-\frac{\coth^{-1}(ax)}{3x^3}\right)-\frac{\coth^{-1}(ax)^2}{4x^4}$$

$$\Big\downarrow \text{54}$$

$$\frac{1}{2}a\left(a^2\int \frac{\coth^{-1}(ax)}{x^2(1-a^2x^2)}dx+\frac{1}{6}a\int \left(-\frac{a^4}{a^2x^2-1}+\frac{a^2}{x^2}+\frac{1}{x^4}\right)dx^2-\frac{\coth^{-1}(ax)}{3x^3}\right)-\frac{\coth^{-1}(ax)^2}{4x^4}$$

$$\Big\downarrow \text{2009}$$

$$\frac{1}{2}a\left(a^2\int\frac{\coth^{-1}(ax)}{x^2(1-a^2x^2)}dx+\frac{1}{6}a\left(a^2\log\left(x^2\right)-a^2\log\left(1-a^2x^2\right)-\frac{1}{x^2}\right)-\frac{\coth^{-1}(ax)}{3x^3}\right)-\frac{\coth^{-1}(ax)^2}{4x^4}$$

$\downarrow$ 6545

$$\frac{1}{2}a\left(a^2\left(a^2\int\frac{\coth^{-1}(ax)}{1-a^2x^2}dx+\int\frac{\coth^{-1}(ax)}{x^2}dx\right)+\frac{1}{6}a\left(a^2\log\left(x^2\right)-a^2\log\left(1-a^2x^2\right)-\frac{1}{x^2}\right)-\frac{\coth^{-1}(ax)}{3x^3}\right)$$
$$\frac{\coth^{-1}(ax)^2}{4x^4}$$

$\downarrow$ 6453

$$\frac{1}{2}a\left(a^2\left(a\int\frac{1}{x(1-a^2x^2)}dx+a^2\int\frac{\coth^{-1}(ax)}{1-a^2x^2}dx-\frac{\coth^{-1}(ax)}{x}\right)+\frac{1}{6}a\left(a^2\log\left(x^2\right)-a^2\log\left(1-a^2x^2\right)-\frac{1}{x^2}\right.\right.$$
$$\frac{\coth^{-1}(ax)^2}{4x^4}$$

$\downarrow$ 243

$$\frac{1}{2}a\left(a^2\left(\frac{1}{2}a\int\frac{1}{x^2(1-a^2x^2)}dx^2+a^2\int\frac{\coth^{-1}(ax)}{1-a^2x^2}dx-\frac{\coth^{-1}(ax)}{x}\right)+\frac{1}{6}a\left(a^2\log\left(x^2\right)-a^2\log\left(1-a^2x^2\right)-\right.\right.$$
$$\frac{\coth^{-1}(ax)^2}{4x^4}$$

$\downarrow$ 47

$$\frac{1}{2}a\left(a^2\left(\frac{1}{2}a\left(a^2\int\frac{1}{1-a^2x^2}dx^2+\int\frac{1}{x^2}dx^2\right)+a^2\int\frac{\coth^{-1}(ax)}{1-a^2x^2}dx-\frac{\coth^{-1}(ax)}{x}\right)+\frac{1}{6}a\left(a^2\log\left(x^2\right)-a^2\log\right.\right.$$
$$\frac{\coth^{-1}(ax)^2}{4x^4}$$

$\downarrow$ 14

$$\frac{1}{2}a\left(a^2\left(\frac{1}{2}a\left(a^2\int\frac{1}{1-a^2x^2}dx^2+\log\left(x^2\right)\right)+a^2\int\frac{\coth^{-1}(ax)}{1-a^2x^2}dx-\frac{\coth^{-1}(ax)}{x}\right)+\frac{1}{6}a\left(a^2\log\left(x^2\right)-a^2\log\left(1\right.\right.\right.$$
$$\frac{\coth^{-1}(ax)^2}{4x^4}$$

$\downarrow$ 16

$$\frac{1}{2}a\left(a^2\left(a^2\int\frac{\coth^{-1}(ax)}{1-a^2x^2}dx+\frac{1}{2}a(\log(x^2)-\log(1-a^2x^2))-\frac{\coth^{-1}(ax)}{x}\right)+\frac{1}{6}a\left(a^2\log(x^2)-a^2\log(1-a^2\right.\right.$$
$$\frac{\coth^{-1}(ax)^2}{4x^4}$$

$$\Big\downarrow \text{6511}$$

$$\frac{1}{2}a\left(\frac{1}{6}a\left(a^2\log(x^2)-a^2\log(1-a^2x^2)-\frac{1}{x^2}\right)+a^2\left(\frac{1}{2}a(\log(x^2)-\log(1-a^2x^2))+\frac{1}{2}a\coth^{-1}(ax)^2-\frac{\coth^{-1}}{x}\right.\right.$$
$$\frac{\coth^{-1}(ax)^2}{4x^4}$$

input
```
Int[ArcCoth[a*x]^2/x^5,x]
```

output
```
-1/4*ArcCoth[a*x]^2/x^4 + (a*(-1/3*ArcCoth[a*x]/x^3 + a^2*(-(ArcCoth[a*x]/
x) + (a*ArcCoth[a*x]^2)/2 + (a*(Log[x^2] - Log[1 - a^2*x^2]))/2) + (a*(-x^
(-2) + a^2*Log[x^2] - a^2*Log[1 - a^2*x^2]))/6))/2
```

Defintions of rubi rules used

rule 14
```
Int[(a_.)/(x_), x_Symbol] :> Simp[a*Log[x], x] /; FreeQ[a, x]
```

rule 16
```
Int[(c_.)/((a_.) + (b_.)*(x_)), x_Symbol] :> Simp[c*(Log[RemoveContent[a +
b*x, x]]/b), x] /; FreeQ[{a, b, c}, x]
```

rule 47
```
Int[1/(((a_.) + (b_.)*(x_))*((c_.) + (d_.)*(x_))), x_Symbol] :> Simp[b/(b*c
 - a*d)   Int[1/(a + b*x), x], x] - Simp[d/(b*c - a*d)   Int[1/(c + d*x), x
], x] /; FreeQ[{a, b, c, d}, x]
```

rule 54
```
Int[((a_) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_.), x_Symbol] :> Int[E
xpandIntegrand[(a + b*x)^m*(c + d*x)^n, x], x] /; FreeQ[{a, b, c, d}, x] &&
 ILtQ[m, 0] && IntegerQ[n] &&  !(IGtQ[n, 0] && LtQ[m + n + 2, 0])
```

rule 243
```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] :> Simp[1/2   Subst[In
t[x^((m - 1)/2)*(a + b*x)^p, x], x, x^2], x] /; FreeQ[{a, b, m, p}, x] && I
ntegerQ[(m - 1)/2]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

rule 6511
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((d_) + (e_.)*(x_)^2), x_Symb
ol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*c*d*(p + 1)), x] /; FreeQ[{a, b
, c, d, e, p}, x] && EqQ[c^2*d + e, 0] && NeQ[p, -1]
```

rule 6545
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*((f_.)*(x_))^(m_))/((d_) + (
e_.)*(x_)^2), x_Symbol] :> Simp[1/d   Int[(f*x)^m*(a + b*ArcCoth[c*x])^p, x
], x] - Simp[e/(d*f^2)   Int[(f*x)^(m + 2)*((a + b*ArcCoth[c*x])^p/(d + e*x
^2)), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && GtQ[p, 0] && LtQ[m, -1]
```

Maple [A] (verified)

Time = 0.16 (sec) , antiderivative size = 101, normalized size of antiderivative = 1.12

method	result
parallelrisch	$\frac{3x^4a^4\operatorname{arccoth}(xa)^2+8\ln(x)a^4x^4-8x^4\ln(xa-1)a^4-8x^4a^4\operatorname{arccoth}(xa)-a^4x^4-6x^3a^3\operatorname{arccoth}(xa)-a^2x^2-2xa\operatorname{arccot}}{12x^4}$
parts	$-\frac{\operatorname{arccoth}(xa)^2}{4x^4}-\frac{a^4\left(-\frac{\operatorname{arccoth}(xa)\ln(xa+1)}{2}+\frac{\operatorname{arccoth}(xa)}{3x^3a^3}+\frac{\operatorname{arccoth}(xa)}{xa}+\frac{\operatorname{arccoth}(xa)\ln(xa-1)}{2}-\frac{\ln(xa-1)\ln\left(\frac{xa}{2}+\frac{1}{2}\right)}{4}+\right.}{}$
derivativedivides	$a^4\left(-\frac{\operatorname{arccoth}(xa)^2}{4x^4a^4}+\frac{\operatorname{arccoth}(xa)\ln(xa+1)}{4}-\frac{\operatorname{arccoth}(xa)}{6x^3a^3}-\frac{\operatorname{arccoth}(xa)}{2xa}-\frac{\operatorname{arccoth}(xa)\ln(xa-1)}{4}+\frac{\ln(xa}{}\right.$
default	$a^4\left(-\frac{\operatorname{arccoth}(xa)^2}{4x^4a^4}+\frac{\operatorname{arccoth}(xa)\ln(xa+1)}{4}-\frac{\operatorname{arccoth}(xa)}{6x^3a^3}-\frac{\operatorname{arccoth}(xa)}{2xa}-\frac{\operatorname{arccoth}(xa)\ln(xa-1)}{4}+\frac{\ln(xa}{}\right.$
risch	$\frac{(a^4x^4-1)\ln(xa+1)^2}{16x^4}-\frac{(3x^4\ln(xa-1)a^4+6x^3a^3+2xa-3\ln(xa-1))\ln(xa+1)}{24x^4}+\frac{3a^4x^4\ln(xa-1)^2+32\ln(x)a^4x^4-}{}$

input
```
int(arccoth(x*a)^2/x^5,x,method=_RETURNVERBOSE)
```

output
```
1/12*(3*x^4*a^4*arccoth(x*a)^2+8*ln(x)*a^4*x^4-8*x^4*ln(a*x-1)*a^4-8*x^4*a
^4*arccoth(x*a)-a^4*x^4-6*x^3*a^3*arccoth(x*a)-a^2*x^2-2*x*a*arccoth(x*a)-
3*arccoth(x*a)^2)/x^4
```

Fricas [A] (verification not implemented)

Time = 0.08 (sec) , antiderivative size = 97, normalized size of antiderivative = 1.08

$$\int \frac{\coth^{-1}(ax)^2}{x^5}\,dx = -\frac{16\,a^4x^4\log\left(a^2x^2-1\right)-32\,a^4x^4\log\left(x\right)+4\,a^2x^2-3\left(a^4x^4-1\right)\log\left(\frac{ax+1}{ax-1}\right)^2+4\left(3\,a^3x^3+ax\right)\log\left(\frac{ax+1}{ax-1}\right.}{48\,x^4}$$

input
```
integrate(arccoth(a*x)^2/x^5,x, algorithm="fricas")
```

output
```
-1/48*(16*a^4*x^4*log(a^2*x^2 - 1) - 32*a^4*x^4*log(x) + 4*a^2*x^2 - 3*(a^
4*x^4 - 1)*log((a*x + 1)/(a*x - 1))^2 + 4*(3*a^3*x^3 + a*x)*log((a*x + 1)/
(a*x - 1)))/x^4
```

Sympy [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 90, normalized size of antiderivative = 1.00

$$\int \frac{\coth^{-1}(ax)^2}{x^5}\,dx = \frac{2a^4\log\left(x\right)}{3}-\frac{2a^4\log\left(ax+1\right)}{3}+\frac{a^4\operatorname{acoth}^2\left(ax\right)}{4}+\frac{2a^4\operatorname{acoth}\left(ax\right)}{3}-\frac{a^3\operatorname{acoth}\left(ax\right)}{2x}-\frac{a^2}{12x^2}-\frac{a\operatorname{acoth}\left(ax\right)}{6x^3}-\frac{\operatorname{acoth}^2\left(ax\right)}{4x^4}$$

input
```
integrate(acoth(a*x)**2/x**5,x)
```

output
```
2*a**4*log(x)/3 - 2*a**4*log(a*x + 1)/3 + a**4*acoth(a*x)**2/4 + 2*a**4*ac
oth(a*x)/3 - a**3*acoth(a*x)/(2*x) - a**2/(12*x**2) - a*acoth(a*x)/(6*x**3
) - acoth(a*x)**2/(4*x**4)
```

Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 154 vs. $2(76) = 152$.

Time = 0.03 (sec) , antiderivative size = 154, normalized size of antiderivative = 1.71

$$\int \frac{\coth^{-1}(ax)^2}{x^5}\,dx$$
$$= \frac{1}{48}\left(32\,a^2\log(x) - \frac{3\,a^2x^2\log(ax+1)^2 + 3\,a^2x^2\log(ax-1)^2 + 16\,a^2x^2\log(ax-1) - 2\,(3\,a^2x^2\log(ax\right.}{x^2}$$
$$+ \frac{1}{12}\left(3\,a^3\log(ax+1) - 3\,a^3\log(ax-1) - \frac{2\,(3\,a^2x^2+1)}{x^3}\right)a\,\text{arcoth}(ax)$$
$$- \frac{\text{arcoth}(ax)^2}{4\,x^4}$$

input
```
integrate(arccoth(a*x)^2/x^5,x, algorithm="maxima")
```

output
```
1/48*(32*a^2*log(x) - (3*a^2*x^2*log(a*x + 1)^2 + 3*a^2*x^2*log(a*x - 1)^2
 + 16*a^2*x^2*log(a*x - 1) - 2*(3*a^2*x^2*log(a*x - 1) - 8*a^2*x^2)*log(a*
x + 1) + 4)/x^2)*a^2 + 1/12*(3*a^3*log(a*x + 1) - 3*a^3*log(a*x - 1) - 2*(
3*a^2*x^2 + 1)/x^3)*a*arccoth(a*x) - 1/4*arccoth(a*x)^2/x^4
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 319 vs. $2(76) = 152$.

Time = 0.12 (sec) , antiderivative size = 319, normalized size of antiderivative = 3.54

$$\int \frac{\coth^{-1}(ax)^2}{x^5}\,dx$$
$$= \frac{1}{6}\left(4\,a^3\log\left(\frac{ax+1}{ax-1}+1\right) - 4\,a^3\log\left(\frac{ax+1}{ax-1}\right) + \frac{2\,(ax+1)a^3}{(ax-1)\left(\frac{(ax+1)^2}{(ax-1)^2} + \frac{2\,(ax+1)}{ax-1} + 1\right)} + \frac{3\left(\frac{(ax+1)^3a^3}{(ax-1)^3} + \right.}{\frac{(ax+1)^4}{(ax-1)^4} + \frac{4\,(ax+1)^3}{(ax-1)^3}}\right.$$

input
```
integrate(arccoth(a*x)^2/x^5,x, algorithm="giac")
```

output
```
1/6*(4*a^3*log((a*x + 1)/(a*x - 1) + 1) - 4*a^3*log((a*x + 1)/(a*x - 1)) +
 2*(a*x + 1)*a^3/((a*x - 1)*((a*x + 1)^2/(a*x - 1)^2 + 2*(a*x + 1)/(a*x -
1) + 1)) + 3*((a*x + 1)^3*a^3/(a*x - 1)^3 + (a*x + 1)*a^3/(a*x - 1))*log((
a*x + 1)/(a*x - 1))^2/((a*x + 1)^4/(a*x - 1)^4 + 4*(a*x + 1)^3/(a*x - 1)^3
 + 6*(a*x + 1)^2/(a*x - 1)^2 + 4*(a*x + 1)/(a*x - 1) + 1) + 2*(3*(a*x + 1)
^2*a^3/(a*x - 1)^2 + 3*(a*x + 1)*a^3/(a*x - 1) + 2*a^3)*log((a*x + 1)/(a*x
 - 1))/((a*x + 1)^3/(a*x - 1)^3 + 3*(a*x + 1)^2/(a*x - 1)^2 + 3*(a*x + 1)/
(a*x - 1) + 1))*a
```

Mupad [B] (verification not implemented)

Time = 4.23 (sec) , antiderivative size = 196, normalized size of antiderivative = 2.18

$$\int \frac{\coth^{-1}(ax)^2}{x^5}\,dx = \frac{2\,a^4\,\ln(x)}{3} + \ln\left(\frac{1}{a\,x}+1\right)^2\left(\frac{a^4}{16}-\frac{1}{16\,x^4}\right) + \ln\left(1-\frac{1}{a\,x}\right)^2\left(\frac{a^4}{16}-\frac{1}{16\,x^4}\right) + \ln\left(1-\frac{1}{a\,x}\right)\left(\frac{24\,a^3\,x^3-12\,a^2\,x^2+8\,a\,x-6}{192\,x^4} + \frac{24\,a^3\,x^3+12\,a^2\,x^2+8\,a\,x+6}{192\,x^4} - \ln\left(\frac{1}{a\,x}+1\right)\left(\frac{a^4}{8}-\frac{1}{8\,x^4}\right)\right) - \frac{a^4\,\ln(a^2\,x^2-1)}{3} - \frac{a^2}{12\,x^2} - \frac{a\,\ln\left(\frac{1}{a\,x}+1\right)\left(\frac{a^2\,x^2}{4}+\frac{1}{12}\right)}{x^3}$$

input
```
int(acoth(a*x)^2/x^5,x)
```

output
```
(2*a^4*log(x))/3 + log(1/(a*x) + 1)^2*(a^4/16 - 1/(16*x^4)) + log(1 - 1/(a
*x))^2*(a^4/16 - 1/(16*x^4)) + log(1 - 1/(a*x))*((8*a*x - 12*a^2*x^2 + 24*
a^3*x^3 - 6)/(192*x^4) + (8*a*x + 12*a^2*x^2 + 24*a^3*x^3 + 6)/(192*x^4) -
 log(1/(a*x) + 1)*(a^4/8 - 1/(8*x^4))) - (a^4*log(a^2*x^2 - 1))/3 - a^2/(1
2*x^2) - (a*log(1/(a*x) + 1)*((a^2*x^2)/4 + 1/12))/x^3
```

Reduce [B] (verification not implemented)

Time = 0.16 (sec) , antiderivative size = 96, normalized size of antiderivative = 1.07

$$\int \frac{\coth^{-1}(ax)^2}{x^5}\,dx$$
$$= \frac{3acoth(ax)^2\,a^4x^4 - 3acoth(ax)^2 + 8acoth(ax)\,a^4x^4 + 6acoth(ax)\,a^3x^3 + 2acoth(ax)\,ax - 8\log(a^2x - a)}{12x^4}$$

input
```
int(acoth(a*x)^2/x^5,x)
```

output
```
(3*acoth(a*x)**2*a**4*x**4 - 3*acoth(a*x)**2 + 8*acoth(a*x)*a**4*x**4 + 6*
acoth(a*x)*a**3*x**3 + 2*acoth(a*x)*a*x - 8*log(a**2*x - a)*a**4*x**4 + 8*
log(x)*a**4*x**4 - a**2*x**2)/(12*x**4)
```

3.23 $\int x^5 \coth^{-1}(ax)^3 \, dx$

Optimal result

Integrand size = 10, antiderivative size = 186

$$\int x^5 \coth^{-1}(ax)^3 \, dx = \frac{19x}{60a^5} + \frac{x^3}{60a^3} + \frac{4x^2 \coth^{-1}(ax)}{15a^4} + \frac{x^4 \coth^{-1}(ax)}{20a^2} + \frac{23 \coth^{-1}(ax)^2}{30a^6} + \frac{x \coth^{-1}(ax)^2}{2a^5} + \frac{x^3 \coth^{-1}(ax)^2}{6a^3} + \frac{x^5 \coth^{-1}(ax)^2}{10a} - \frac{\coth^{-1}(ax)^3}{6a^6} + \frac{1}{6}x^6 \coth^{-1}(ax)^3 - \frac{19\text{arctanh}(ax)}{60a^6} - \frac{23 \coth^{-1}(ax) \log\left(\frac{2}{1-ax}\right)}{15a^6} - \frac{23 \text{PolyLog}\left(2, 1 - \frac{2}{1-ax}\right)}{30a^6}$$

output
```
19/60*x/a^5+1/60*x^3/a^3+4/15*x^2*arccoth(a*x)/a^4+1/20*x^4*arccoth(a*x)/a
^2+23/30*arccoth(a*x)^2/a^6+1/2*x*arccoth(a*x)^2/a^5+1/6*x^3*arccoth(a*x)^
2/a^3+1/10*x^5*arccoth(a*x)^2/a-1/6*arccoth(a*x)^3/a^6+1/6*x^6*arccoth(a*x
)^3-19/60*arctanh(a*x)/a^6-23/15*arccoth(a*x)*ln(2/(-a*x+1))/a^6-23/30*pol
ylog(2,1-2/(-a*x+1))/a^6
```

Mathematica [A] (verified)

Time = 0.37 (sec) , antiderivative size = 117, normalized size of antiderivative = 0.63

$$\int x^5 \coth^{-1}(ax)^3 \, dx$$
$$= \frac{ax(19+a^2x^2)+2(-23+15ax+5a^3x^3+3a^5x^5)\coth^{-1}(ax)^2+10(-1+a^6x^6)\coth^{-1}(ax)^3+\coth^{-1}(a}{60a^6}$$

input
```
Integrate[x^5*ArcCoth[a*x]^3,x]
```

output
```
(a*x*(19 + a^2*x^2) + 2*(-23 + 15*a*x + 5*a^3*x^3 + 3*a^5*x^5)*ArcCoth[a*x
]^2 + 10*(-1 + a^6*x^6)*ArcCoth[a*x]^3 + ArcCoth[a*x]*(-19 + 16*a^2*x^2 +
3*a^4*x^4 - 92*Log[1 - E^(-2*ArcCoth[a*x])]) + 46*PolyLog[2, E^(-2*ArcCoth
[a*x])])/(60*a^6)
```

Rubi [B] (verified)

Leaf count is larger than twice the leaf count of optimal. 421 vs. $2(186) = 372$.

Time = 2.96 (sec) , antiderivative size = 421, normalized size of antiderivative = 2.26, number of steps used = 18, number of rules used = 17, $\frac{\text{number of rules}}{\text{integrand size}} = 1.700$, Rules used = {6453, 6543, 6453, 6543, 6453, 254, 2009, 6543, 6437, 6453, 262, 219, 6511, 6547, 6471, 2849, 2752}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x^5 \coth^{-1}(ax)^3 \, dx$$

$$\Big\downarrow \text{6453}$$

$$\frac{1}{6}x^6 \coth^{-1}(ax)^3 - \frac{1}{2}a \int \frac{x^6 \coth^{-1}(ax)^2}{1-a^2x^2} dx$$

$$\Big\downarrow \text{6543}$$

$$\frac{1}{6}x^6\coth^{-1}(ax)^3-\frac{1}{2}a\left(\frac{\int\frac{x^4\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{\int x^4\coth^{-1}(ax)^2dx}{a^2}\right)$$

$$\Big\downarrow\ 6453$$

$$\frac{1}{6}x^6\coth^{-1}(ax)^3-\frac{1}{2}a\left(\frac{\int\frac{x^4\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{5}x^5\coth^{-1}(ax)^2-\frac{2}{5}a\int\frac{x^5\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}\right)$$

$$\Big\downarrow\ 6543$$

$$\frac{1}{6}x^6\coth^{-1}(ax)^3-\frac{1}{2}a\left(\frac{\frac{\int\frac{x^2\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{\int x^2\coth^{-1}(ax)^2dx}{a^2}}{a^2}-\frac{\frac{1}{5}x^5\coth^{-1}(ax)^2-\frac{2}{5}a\left(\frac{\int\frac{x^3\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\int x^3\coth^{-1}(ax)dx}{a^2}\right)}{a^2}\right)$$

$$\Big\downarrow\ 6453$$

$$\frac{1}{6}x^6\coth^{-1}(ax)^3-\frac{1}{2}a\left(\frac{\frac{\int\frac{x^2\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\int\frac{x^3\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}}{a^2}-\frac{\frac{1}{5}x^5\coth^{-1}(ax)^2-\frac{2}{5}a\left(\frac{\int\frac{x^3\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{4}x^4\coth^{-1}(a\ldots}{\ldots}\right.}{a^2}\right.$$

$$\Big\downarrow\ 254$$

$$\frac{1}{6}x^6\coth^{-1}(ax)^3-\frac{1}{2}a\left(\frac{\frac{\int\frac{x^2\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\int\frac{x^3\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}}{a^2}-\frac{\frac{1}{5}x^5\coth^{-1}(ax)^2-\frac{2}{5}a\left(\frac{\int\frac{x^3\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{4}x^4\coth^{-1}(a\ldots}{\ldots}\right.}{a^2}\right.$$

$$\Big\downarrow\ 2009$$

$$\frac{1}{6}x^6\coth^{-1}(ax)^3 - \frac{1}{2}a\left(\frac{\frac{\int \frac{x^2\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2} - \frac{\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\int\frac{x^3\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}}{a^2} - \frac{\frac{1}{5}x^5\coth^{-1}(ax)^2-\frac{2}{5}a\left(\frac{\int\frac{x^3\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2} - \frac{\frac{1}{4}x^4\coth^{-1}(a\cdots}{\cdots}\right.}{a^2}\right.$$

$\downarrow$ 6543

$$\frac{1}{6}x^6\coth^{-1}(ax)^3 - \frac{1}{2}a\left(\frac{\frac{\frac{\int\frac{\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{\int\coth^{-1}(ax)^2dx}{a^2}}{a^2} - \frac{\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\left(\frac{\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\int x\coth^{-1}(ax)dx}{a^2}\right)}{a^2}}{a^2} - \frac{\frac{1}{5}x^5\coth^{-1}(ax)^2-\frac{2}{5}a\left(\cdots\right.}{\cdots}\right.$$

$\downarrow$ 6437

$$\frac{1}{6}x^6\coth^{-1}(ax)^3 - \frac{1}{2}a\left(\frac{\frac{\frac{\int\frac{\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{x\coth^{-1}(ax)^2-2a\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}}{a^2} - \frac{\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\left(\frac{\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\int x\coth^{-1}(ax)dx}{a^2}\right)}{a^2}}{a^2} - \frac{\frac{1}{5}x^5\coth^{-1}\cdots}{\cdots}\right.$$

$\downarrow$ 6453

$$\frac{1}{6}x^6\coth^{-1}(ax)^3 - \frac{1}{2}a\left(\frac{\frac{\frac{\int\frac{\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{x\coth^{-1}(ax)^2-2a\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}}{a^2} - \frac{\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\left(\frac{\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)-\frac{1}{2}a\int\frac{x^2}{1-a^2x^2}dx}{a^2}\right)}{a^2}}{a^2} - \cdots\right.$$

$\downarrow$ 262

$$\frac{1}{6}x^6\coth^{-1}(ax)^3 - \frac{1}{2}a\left(\frac{\frac{\frac{\int \frac{\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2} - \frac{x\coth^{-1}(ax)^2 - 2a\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}}{a^2} - \frac{\frac{1}{3}x^3\coth^{-1}(ax)^2 - \frac{2}{3}a\left(\frac{\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2} - \frac{\frac{1}{2}x^2\coth^{-1}(ax) - \frac{1}{2}a\left(\frac{\int\frac{1}{1-a^2x^2}dx}{a^2} - \frac{x}{a^2}\right.}{a^2}\right.}{a^2}}{a^2}\right.$$

$$\Big\downarrow \; 219$$

$$\frac{1}{6}x^6\coth^{-1}(ax)^3 - \frac{1}{2}a\left(\frac{\frac{\frac{\int \frac{\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2} - \frac{x\coth^{-1}(ax)^2 - 2a\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}}{a^2} - \frac{\frac{1}{3}x^3\coth^{-1}(ax)^2 - \frac{2}{3}a\left(\frac{\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2} - \frac{\frac{1}{2}x^2\coth^{-1}(ax) - \frac{1}{2}a\left(\frac{\text{arctanh}(ax)}{a^3} - \right.}{a^2}\right.}{a^2}}{a^2}\right.$$

$$\Big\downarrow \; 6511$$

$$\frac{1}{6}x^6\coth^{-1}(ax)^3 - \frac{1}{2}a\left(\frac{\frac{\frac{\coth^{-1}(ax)^3}{3a^3} - \frac{x\coth^{-1}(ax)^2 - 2a\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}}{a^2} - \frac{\frac{1}{3}x^3\coth^{-1}(ax)^2 - \frac{2}{3}a\left(\frac{\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2} - \frac{\frac{1}{2}x^2\coth^{-1}(ax) - \frac{1}{2}a\left(\frac{\text{arctanh}(ax)}{a^3} - \frac{x}{a^2}\right)}{a^2}\right.}{a^2}}{a^2}\right.$$

$$\Big\downarrow \; 6547$$

$$\frac{1}{6}x^6\coth^{-1}(ax)^3-\frac{1}{2}a\left(\frac{\frac{\frac{\coth^{-1}(ax)^3}{3a^3}-\frac{x\coth^{-1}(ax)^2-2a\left(\frac{\int\frac{\coth^{-1}(ax)}{1-ax}dx}{a}-\frac{\coth^{-1}(ax)^2}{2a^2}\right)}{a^2}}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\left(\frac{\frac{\int\frac{\coth^{-1}(ax)}{1-ax}dx}{a}-\frac{\coth^{-1}(ax)^2}{2a^2}}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(\cdots}{\cdots}\cdots\right.}{a^2}\cdots}{a^2}\cdots\right.$$

$\downarrow$ 6471

$$\frac{1}{6}x^6\coth^{-1}(ax)^3-\frac{1}{2}a\left(\frac{\frac{\frac{\coth^{-1}(ax)^3}{3a^3}-\frac{x\coth^{-1}(ax)^2-2a\left(\frac{\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)}{a}-\int\frac{\log\left(\frac{2}{1-ax}\right)}{1-a^2x^2}dx}{a}-\frac{\coth^{-1}(ax)^2}{2a^2}\right)}{a^2}}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\left(\frac{\frac{\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(a\cdots}{a}}{a}\cdots}{\cdots}\cdots\right.}{\cdots}\cdots}{a^2}\cdots\right.$$

$\downarrow$ 2849

$$\frac{1}{6}x^6\coth^{-1}(ax)^3-\frac{1}{2}a\left(\frac{\frac{\frac{\coth^{-1}(ax)^3}{3a^3}-\frac{x\coth^{-1}(ax)^2-2a\left(\frac{\frac{\int\frac{\log\left(\frac{2}{1-ax}\right)}{1-\frac{2}{1-ax}}d\frac{1}{1-ax}}{a}+\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)}{a}}{a}-\frac{\coth^{-1}(ax)^2}{2a^2}\right)}{a^2}}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\left(\frac{\frac{\int\frac{\log\left(\frac{2}{1-ax}\right)}{1-\frac{2}{1-ax}}d\frac{1}{\cdots}}{a}\cdots}{\cdots}\cdots\right.}{\cdots}\cdots}{a^2}\cdots\right.$$

$\downarrow$ 2752

$$\frac{1}{6}x^6\coth^{-1}(ax)^3 -$$

$$\frac{1}{2}a\left(\frac{\frac{\frac{\coth^{-1}(ax)^3}{3a^3}-\frac{x\coth^{-1}(ax)^2-2a\left(\frac{\frac{\text{PolyLog}\left(2,1-\frac{2}{1-ax}\right)}{2a}+\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)}{a}}{a}-\frac{\coth^{-1}(ax)^2}{2a^2}\right)}{a^2}}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\left(\frac{\frac{\text{PolyLog}\left(2,1-\frac{2}{1-\cdots}\right)}{2a}\cdots}{\cdots}\right.}{a^2}\right.$$

input
```
Int[x^5*ArcCoth[a*x]^3,x]
```

output
```
(x^6*ArcCoth[a*x]^3)/6 - (a*(-(((x^5*ArcCoth[a*x]^2)/5 - (2*a*(-(((x^4*Arc
Coth[a*x])/4 - (a*(-(x/a^4) - x^3/(3*a^2) + ArcTanh[a*x]/a^5))/4)/a^2) + (
-(((x^2*ArcCoth[a*x])/2 - (a*(-(x/a^2) + ArcTanh[a*x]/a^3))/2)/a^2) + (-1/
2*ArcCoth[a*x]^2/a^2 + ((ArcCoth[a*x]*Log[2/(1 - a*x)])/a + PolyLog[2, 1 -
 2/(1 - a*x)]/(2*a))/a)/a^2)/a^2))/5)/a^2) + (-(((x^3*ArcCoth[a*x]^2)/3 -
(2*a*(-(((x^2*ArcCoth[a*x])/2 - (a*(-(x/a^2) + ArcTanh[a*x]/a^3))/2)/a^2)
+ (-1/2*ArcCoth[a*x]^2/a^2 + ((ArcCoth[a*x]*Log[2/(1 - a*x)])/a + PolyLog[
2, 1 - 2/(1 - a*x)]/(2*a))/a)/a^2))/3)/a^2) + (ArcCoth[a*x]^3/(3*a^3) - (x
*ArcCoth[a*x]^2 - 2*a*(-1/2*ArcCoth[a*x]^2/a^2 + ((ArcCoth[a*x]*Log[2/(1 -
 a*x)])/a + PolyLog[2, 1 - 2/(1 - a*x)]/(2*a))/a))/a^2)/a^2)/a^2))/2
```

Defintions of rubi rules used

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 254
```
Int[(x_)^(m_)/((a_) + (b_.)*(x_)^2), x_Symbol] :> Int[PolynomialDivide[x^m,
 a + b*x^2, x], x] /; FreeQ[{a, b}, x] && IGtQ[m, 3]
```

rule 262
```
Int[((c_.)*(x_))^(m_)*((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] :> Simp[c*(c*x)
^(m - 1)*((a + b*x^2)^(p + 1)/(b*(m + 2*p + 1))), x] - Simp[a*c^2*((m - 1)/
(b*(m + 2*p + 1)))   Int[(c*x)^(m - 2)*(a + b*x^2)^p, x], x] /; FreeQ[{a, b
, c, p}, x] && GtQ[m, 2 - 1] && NeQ[m + 2*p + 1, 0] && IntBinomialQ[a, b, c
, 2, m, p, x]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2752
```
Int[Log[(c_.)*(x_)]/((d_) + (e_.)*(x_)), x_Symbol] :> Simp[(-e^(-1))*PolyLo
g[2, 1 - c*x], x] /; FreeQ[{c, d, e}, x] && EqQ[e + c*d, 0]
```

rule 2849
```
Int[Log[(c_.)/((d_) + (e_.)*(x_))]/((f_) + (g_.)*(x_)^2), x_Symbol] :> Simp
[-e/g   Subst[Int[Log[2*d*x]/(1 - 2*d*x), x], x, 1/(d + e*x)], x] /; FreeQ[
{c, d, e, f, g}, x] && EqQ[c, 2*d] && EqQ[e^2*f + d^2*g, 0]
```

rule 6437
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.), x_Symbol] :> Simp[x*(a
 + b*ArcCoth[c*x^n])^p, x] - Simp[b*c*n*p   Int[x^n*((a + b*ArcCoth[c*x^n])
^(p - 1)/(1 - c^2*x^(2*n))), x], x] /; FreeQ[{a, b, c, n}, x] && IGtQ[p, 0]
 && (EqQ[n, 1] || EqQ[p, 1])
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

rule 6471
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((d_) + (e_.)*(x_)), x_Symbol
] :> Simp[(-(a + b*ArcCoth[c*x])^p)*(Log[2/(1 + e*(x/d))]/e), x] + Simp[b*c
*(p/e)   Int[(a + b*ArcCoth[c*x])^(p - 1)*(Log[2/(1 + e*(x/d))]/(1 - c^2*x^
2)), x], x] /; FreeQ[{a, b, c, d, e}, x] && IGtQ[p, 0] && EqQ[c^2*d^2 - e^2
, 0]
```

rule 6511
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((d_) + (e_.)*(x_)^2), x_Symb
ol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*c*d*(p + 1)), x] /; FreeQ[{a, b
, c, d, e, p}, x] && EqQ[c^2*d + e, 0] && NeQ[p, -1]
```

rule 6543
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*((f_.)*(x_))^(m_))/((d_) + (
e_.)*(x_)^2), x_Symbol] :> Simp[f^2/e   Int[(f*x)^(m - 2)*(a + b*ArcCoth[c*
x])^p, x], x] - Simp[d*(f^2/e)   Int[(f*x)^(m - 2)*((a + b*ArcCoth[c*x])^p/
(d + e*x^2)), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && GtQ[p, 0] && GtQ[m,
 1]
```

rule 6547
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*(x_))/((d_) + (e_.)*(x_)^2),
 x_Symbol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*e*(p + 1)), x] + Simp[1/
(c*d)   Int[(a + b*ArcCoth[c*x])^p/(1 - c*x), x], x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[c^2*d + e, 0] && IGtQ[p, 0]
```

Maple [C] (warning: unable to verify)

Result contains higher order function than in optimal. Order 9 vs. order 4.

Time = 5.86 (sec) , antiderivative size = 2133, normalized size of antiderivative = 11.47

method	result	size
parts	Expression too large to display	2133
derivativedivides	Expression too large to display	2135
default	Expression too large to display	2135

input
```
int(x^5*arccoth(x*a)^3,x,method=_RETURNVERBOSE)
```

output
```
1/6*x^6*arccoth(x*a)^3+1/2/a^6*(1/3*x^3*a^3*arccoth(x*a)^2+1/5*x^5*a^5*arc
coth(x*a)^2-46/15*arccoth(x*a)*ln(1+1/((a*x-1)/(a*x+1))^(1/2))-1/3*arccoth
(x*a)^3-1/2*arccoth(x*a)^2*ln((a*x-1)/(a*x+1))+41/120*(((a*x-1)/(a*x+1))^(
1/2)*x*a+((a*x-1)/(a*x+1))^(1/2)+x*a+1)*arccoth(x*a)-41/60/(((a*x-1)/(a*x+
1))^(1/2)+1)*((a*x-1)/(a*x+1))^(1/2)-1/40*(a*x-1)/(((a*x-1)/(a*x+1))^(1/2)
*x*a+((a*x-1)/(a*x+1))^(1/2)-x*a)+1/40*(a*x-1)/(((a*x-1)/(a*x+1))^(1/2)*x*
a+((a*x-1)/(a*x+1))^(1/2)+x*a)+1/10*(2*((a*x-1)/(a*x+1))^(1/2)*x^2*a^2+2*a
^2*x^2-((a*x-1)/(a*x+1))^(1/2)-2*x*a)*(a*x-1)*arccoth(x*a)*(a*x+1)-41/60/(
((a*x-1)/(a*x+1))^(1/2)-1)*((a*x-1)/(a*x+1))^(1/2)+47/120*(((a*x-1)/(a*x+1
))^(1/2)*x*a+((a*x-1)/(a*x+1))^(1/2)-x*a)*arccoth(x*a)*(a*x+1)+1/2*arccoth
(x*a)^2*ln(a*x-1)-1/2*arccoth(x*a)^2*ln(a*x+1)-41/120*(((a*x-1)/(a*x+1))^(
1/2)*x*a+((a*x-1)/(a*x+1))^(1/2)-x*a-1)*arccoth(x*a)+arccoth(x*a)^2*x*a-1/
20*(2*((a*x-1)/(a*x+1))^(1/2)*x^2*a^2+3*((a*x-1)/(a*x+1))^(1/2)*x*a-2*a^2*
x^2+((a*x-1)/(a*x+1))^(1/2)-x*a+1)*arccoth(x*a)*(a*x+1)-1/40*(2*((a*x-1)/(
a*x+1))^(1/2)*x^2*a^2+2*((a*x-1)/(a*x+1))^(1/2)*x*a+2*a^2*x^2-1)*(a*x+1)^2
*arccoth(x*a)-1/4*I*Pi*csgn(I/(a*x-1)*(a*x+1)/((a*x+1)/(a*x-1)-1))^3*arcco
th(x*a)^2-1/4*I*Pi*csgn(I*(a*x+1)/(a*x-1))^3*arccoth(x*a)^2-1/4*I*Pi*csgn(
I/((a*x+1)/(a*x-1)-1))*csgn(I*(a*x+1)/(a*x-1))*csgn(I/(a*x-1)*(a*x+1)/((a*
x+1)/(a*x-1)-1))*arccoth(x*a)^2+46/15*dilog(1/((a*x-1)/(a*x+1))^(1/2))-46/
15*dilog(1+1/((a*x-1)/(a*x+1))^(1/2))+23/15*arccoth(x*a)^2+1/10*(2*((a*...
```

Fricas [F]

$$\int x^5 \coth^{-1}(ax)^3 \, dx = \int x^5 \operatorname{arcoth}(ax)^3 \, dx$$

input
```
integrate(x^5*arccoth(a*x)^3,x, algorithm="fricas")
```

output
```
integral(x^5*arccoth(a*x)^3, x)
```

Sympy [F]

$$\int x^5 \coth^{-1}(ax)^3 \, dx = \int x^5 \operatorname{acoth}^3(ax) \, dx$$

input
```
integrate(x**5*acoth(a*x)**3,x)
```

output
```
Integral(x**5*acoth(a*x)**3, x)
```

Maxima [A] (verification not implemented)

Time = 0.04 (sec) , antiderivative size = 289, normalized size of antiderivative = 1.55

$$\int x^5 \coth^{-1}(ax)^3 \, dx = \frac{1}{6} x^6 \operatorname{arcoth}(ax)^3 + \frac{1}{60} a \left(\frac{2(3a^4x^5 + 5a^2x^3 + 15x)}{a^6} - \frac{15 \log(ax+1)}{a^7} + \frac{15 \log(ax-1)}{a^7} \right) \operatorname{arcoth}(ax)^2 + \frac{1}{240} a \left(\frac{\frac{4a^3x^3 + (15 \log(ax-1) - 46) \log(ax+1)^2 - 5 \log(ax+1)^3 + 5 \log(ax-1)^3 + 76ax - \left(15 \log(ax-1)^2 - 92 \log(ax-1)\right) \log(ax+1) + 46}{a}}{a^6} \right.$$

input
```
integrate(x^5*arccoth(a*x)^3,x, algorithm="maxima")
```

output
```
1/6*x^6*arccoth(a*x)^3 + 1/60*a*(2*(3*a^4*x^5 + 5*a^2*x^3 + 15*x)/a^6 - 15
*log(a*x + 1)/a^7 + 15*log(a*x - 1)/a^7)*arccoth(a*x)^2 + 1/240*a*(((4*a^3
*x^3 + (15*log(a*x - 1) - 46)*log(a*x + 1)^2 - 5*log(a*x + 1)^3 + 5*log(a*
x - 1)^3 + 76*a*x - (15*log(a*x - 1)^2 - 92*log(a*x - 1))*log(a*x + 1) + 4
6*log(a*x - 1)^2 + 38*log(a*x - 1))/a - 184*(log(a*x - 1)*log(1/2*a*x + 1/
2) + dilog(-1/2*a*x + 1/2))/a - 38*log(a*x + 1)/a)/a^6 + 2*(6*a^4*x^4 + 32
*a^2*x^2 - 2*(15*log(a*x - 1) - 46)*log(a*x + 1) + 15*log(a*x + 1)^2 + 15*
log(a*x - 1)^2 + 92*log(a*x - 1))*arccoth(a*x)/a^7)
```

Giac [F]

$$\int x^5 \coth^{-1}(ax)^3 \, dx = \int x^5 \operatorname{arcoth}(ax)^3 \, dx$$

input
```
integrate(x^5*arccoth(a*x)^3,x, algorithm="giac")
```

output
```
integrate(x^5*arccoth(a*x)^3, x)
```

Mupad [F(-1)]

Timed out.

$$\int x^5 \coth^{-1}(ax)^3 \, dx = \int x^5 \operatorname{acoth}(a\,x)^3 \, dx$$

input
```
int(x^5*acoth(a*x)^3,x)
```

output
```
int(x^5*acoth(a*x)^3, x)
```

Reduce [F]

$$\int x^5 \coth^{-1}(ax)^3 \, dx = \frac{10acoth(ax)^3 a^6x^6 - 10acoth(ax)^3 - 6acoth(ax)^2 a^5x^5 - 10acoth(ax)^2 a^3x^3 - 30acoth(ax)^2 ax + 3acoth}{60a^6}$$

input
```
int(x^5*acoth(a*x)^3,x)
```

output
```
(10*acoth(a*x)**3*a**6*x**6 - 10*acoth(a*x)**3 - 6*acoth(a*x)**2*a**5*x**5
 - 10*acoth(a*x)**2*a**3*x**3 - 30*acoth(a*x)**2*a*x + 3*acoth(a*x)*a**4*x
**4 + 16*acoth(a*x)*a**2*x**2 - 19*acoth(a*x) + 92*int((acoth(a*x)*x)/(a**
2*x**2 - 1),x)*a**2 - a**3*x**3 - 19*a*x)/(60*a**6)
```

3.24 $\int x^4 \coth^{-1}(ax)^3 \, dx$

Optimal result

Integrand size = 10, antiderivative size = 196

$$\begin{aligned}\int x^4 \coth^{-1}(ax)^3 \, dx &= \frac{x^2}{20a^3} + \frac{9x\coth^{-1}(ax)}{10a^4} + \frac{x^3\coth^{-1}(ax)}{10a^2} - \frac{9\coth^{-1}(ax)^2}{20a^5} \\ &\quad + \frac{3x^2\coth^{-1}(ax)^2}{10a^3} + \frac{3x^4\coth^{-1}(ax)^2}{20a} + \frac{\coth^{-1}(ax)^3}{5a^5} \\ &\quad + \frac{1}{5}x^5\coth^{-1}(ax)^3 - \frac{3\coth^{-1}(ax)^2\log\left(\frac{2}{1-ax}\right)}{5a^5} + \frac{\log(1-a^2x^2)}{2a^5} \\ &\quad - \frac{3\coth^{-1}(ax)\operatorname{PolyLog}\left(2,1-\frac{2}{1-ax}\right)}{5a^5} + \frac{3\operatorname{PolyLog}\left(3,1-\frac{2}{1-ax}\right)}{10a^5}\end{aligned}$$

output
```
1/20*x^2/a^3+9/10*x*arccoth(a*x)/a^4+1/10*x^3*arccoth(a*x)/a^2-9/20*arccot
h(a*x)^2/a^5+3/10*x^2*arccoth(a*x)^2/a^3+3/20*x^4*arccoth(a*x)^2/a+1/5*arc
coth(a*x)^3/a^5+1/5*x^5*arccoth(a*x)^3-3/5*arccoth(a*x)^2*ln(2/(-a*x+1))/a
^5+1/2*ln(-a^2*x^2+1)/a^5-3/5*arccoth(a*x)*polylog(2,1-2/(-a*x+1))/a^5+3/1
0*polylog(3,1-2/(-a*x+1))/a^5
```

Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 0.41 (sec) , antiderivative size = 175, normalized size of antiderivative = 0.89

$$\int x^4 \coth^{-1}(ax)^3\,dx$$
$$= \frac{-2 - i\pi^3 + 2a^2x^2 + 36ax\coth^{-1}(ax) + 4a^3x^3\coth^{-1}(ax) - 18\coth^{-1}(ax)^2 + 12a^2x^2\coth^{-1}(ax)^2 + 6a^4}{}$$

input
```
Integrate[x^4*ArcCoth[a*x]^3,x]
```

output
```
(-2 - I*Pi^3 + 2*a^2*x^2 + 36*a*x*ArcCoth[a*x] + 4*a^3*x^3*ArcCoth[a*x] -
18*ArcCoth[a*x]^2 + 12*a^2*x^2*ArcCoth[a*x]^2 + 6*a^4*x^4*ArcCoth[a*x]^2 +
 8*ArcCoth[a*x]^3 + 8*a^5*x^5*ArcCoth[a*x]^3 - 24*ArcCoth[a*x]^2*Log[1 - E
^(2*ArcCoth[a*x])] - 40*Log[1/(a*Sqrt[1 - 1/(a^2*x^2)]*x)] - 24*ArcCoth[a*
x]*PolyLog[2, E^(2*ArcCoth[a*x])] + 12*PolyLog[3, E^(2*ArcCoth[a*x])])/(40
*a^5)
```

Rubi [A] (verified)

Time = 2.63 (sec) , antiderivative size = 304, normalized size of antiderivative = 1.55, number of steps used = 17, number of rules used = 16, $\frac{\text{number of rules}}{\text{integrand size}} = 1.600$, Rules used = {6453, 6543, 6453, 6543, 6453, 243, 49, 2009, 6543, 6437, 240, 6511, 6547, 6471, 6621, 7164}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x^4 \coth^{-1}(ax)^3\,dx$$
$$\Big\downarrow \text{6453}$$
$$\frac{1}{5}x^5\coth^{-1}(ax)^3 - \frac{3}{5}a\int \frac{x^5\coth^{-1}(ax)^2}{1-a^2x^2}dx$$
$$\Big\downarrow \text{6543}$$

$$\frac{1}{5}x^5\coth^{-1}(ax)^3-\frac{3}{5}a\left(\frac{\int\frac{x^3\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{\int x^3\coth^{-1}(ax)^2dx}{a^2}\right)$$

$\downarrow$ 6453

$$\frac{1}{5}x^5\coth^{-1}(ax)^3-\frac{3}{5}a\left(\frac{\int\frac{x^3\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{4}x^4\coth^{-1}(ax)^2-\frac{1}{2}a\int\frac{x^4\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}\right)$$

$\downarrow$ 6543

$$\frac{1}{5}x^5\coth^{-1}(ax)^3-\frac{3}{5}a\left(\frac{\frac{\int\frac{x\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{\int x\coth^{-1}(ax)^2dx}{a^2}}{a^2}-\frac{\frac{1}{4}x^4\coth^{-1}(ax)^2-\frac{1}{2}a\left(\frac{\int\frac{x^2\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\int x^2\coth^{-1}(ax)dx}{a^2}\right)}{a^2}\right)$$

$\downarrow$ 6453

$$\frac{1}{5}x^5\coth^{-1}(ax)^3-\frac{3}{5}a\left(\frac{\frac{\int\frac{x\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)^2-a\int\frac{x^2\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}}{a^2}-\frac{\frac{1}{4}x^4\coth^{-1}(ax)^2-\frac{1}{2}a\left(\frac{\int\frac{x^2\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)-}{a}\right.}{a^2}\right.$$

$\downarrow$ 243

$$\frac{1}{5}x^5\coth^{-1}(ax)^3-\frac{3}{5}a\left(\frac{\frac{\int\frac{x\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)^2-a\int\frac{x^2\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}}{a^2}-\frac{\frac{1}{4}x^4\coth^{-1}(ax)^2-\frac{1}{2}a\left(\frac{\int\frac{x^2\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)-}{a}\right.}{a^2}\right.$$

$\downarrow$ 49

$$\frac{1}{5}x^5\coth^{-1}(ax)^3 - \frac{3}{5}a\left(\frac{\frac{\int \frac{x\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)^2-a\int \frac{x^2\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}}{a^2}-\frac{\frac{1}{4}x^4\coth^{-1}(ax)^2-\frac{1}{2}a\left(\frac{\int \frac{x^2\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)\cdots}{\cdots}\right.}{a^2}\right.$$

$\Big\downarrow$ 2009

$$\frac{1}{5}x^5\coth^{-1}(ax)^3 - \frac{3}{5}a\left(\frac{\frac{\int \frac{x\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)^2-a\int \frac{x^2\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}}{a^2}-\frac{\frac{1}{4}x^4\coth^{-1}(ax)^2-\frac{1}{2}a\left(\frac{\int \frac{x^2\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)\cdots}{\cdots}\right.}{a^2}\right.$$

$\Big\downarrow$ 6543

$$\frac{1}{5}x^5\coth^{-1}(ax)^3 - \frac{3}{5}a\left(\frac{\frac{\int \frac{x\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)^2-a\left(\frac{\int \frac{\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\int \coth^{-1}(ax)dx}{a^2}\right)}{a^2}}{a^2}-\frac{\frac{1}{4}x^4\coth^{-1}(ax)^2-\frac{1}{2}a\left(\frac{\frac{\int \frac{\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\int \cdots}{\cdots}}{a^2}\right.}{\cdots}\right.$$

$\Big\downarrow$ 6437

$$\frac{1}{5}x^5\coth^{-1}(ax)^3 - \frac{3}{5}a\left(\frac{\frac{\int \frac{x\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)^2-a\left(\frac{\int \frac{\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{x\coth^{-1}(ax)-a\int \frac{x}{1-a^2x^2}dx}{a^2}\right)}{a^2}}{a^2}-\frac{\frac{1}{4}x^4\coth^{-1}(ax)^2-\frac{1}{2}a\left(\frac{\frac{\int \frac{\coth^{-1}\cdots}{1-a^2\cdots}}{a^2}}{\cdots}\right.}{\cdots}\right.$$

$\Big\downarrow$ 240

$$\frac{1}{5}x^5\coth^{-1}(ax)^3-\frac{3}{5}a\left(\frac{\frac{\int\frac{x\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)^2-a\left(\frac{\int\frac{\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{\log\left(1-a^2x^2\right)}{2a}+x\coth^{-1}(ax)}{a^2}\right)}{a^2}}{a^2}-\frac{\frac{1}{4}x^4\coth^{-1}(ax)^2-\frac{1}{2}a\left(\frac{\int\frac{\coth^{-1}(\cdots}{1-a^2x\cdots}}{a^2}\cdots\right.}{\cdots}\cdots\right.$$

$\big\downarrow$ 6511

$$\frac{1}{5}x^5\coth^{-1}(ax)^3-\frac{3}{5}a\left(\frac{\frac{\int\frac{x\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)^2-a\left(\frac{\coth^{-1}(ax)^2}{2a^3}-\frac{\frac{\log\left(1-a^2x^2\right)}{2a}+x\coth^{-1}(ax)}{a^2}\right)}{a^2}}{a^2}-\frac{\frac{1}{4}x^4\coth^{-1}(ax)^2-\frac{1}{2}a\left(\frac{\coth^{-1}(ax)^2}{2a^3}\cdots\right.}{\cdots}\cdots\right.$$

$\big\downarrow$ 6547

$$\frac{1}{5}x^5\coth^{-1}(ax)^3-\frac{3}{5}a\left(\frac{\frac{\frac{\int\frac{\coth^{-1}(ax)^2}{1-ax}dx}{a}-\frac{\coth^{-1}(ax)^3}{3a^2}}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)^2-a\left(\frac{\coth^{-1}(ax)^2}{2a^3}-\frac{\frac{\log\left(1-a^2x^2\right)}{2a}+x\coth^{-1}(ax)}{a^2}\right)}{a^2}}{a^2}-\frac{\frac{1}{4}x^4\coth^{-1}(ax)^2-\frac{1}{2}a\left(\cdots\right.}{\cdots}\cdots\right.$$

$\big\downarrow$ 6471

$$\frac{1}{5}x^5\coth^{-1}(ax)^3-\frac{3}{5}a\left(\frac{\frac{\frac{\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)^2}{a}-2\int\frac{\coth^{-1}(ax)\log\left(\frac{2}{1-ax}\right)}{1-a^2x^2}dx}{a}-\frac{\coth^{-1}(ax)^3}{3a^2}}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)^2-a\left(\frac{\coth^{-1}(ax)^2}{2a^3}-\frac{\frac{\log\left(1-a^2x^2\right)}{2a}+x\coth^{-1}(ax)}{a^2}\right)}{a^2}}{a^2}\cdots\right.$$

$\big\downarrow$ 6621

$$\frac{1}{5}x^5\coth^{-1}(ax)^3 - \frac{3}{5}a\left(\frac{\frac{\frac{\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)^2}{a}-2\left(\frac{1}{2}\int\frac{\operatorname{PolyLog}\left(2,1-\frac{2}{1-ax}\right)}{1-a^2x^2}dx-\frac{\operatorname{PolyLog}\left(2,1-\frac{2}{1-ax}\right)\coth^{-1}(ax)}{2a}\right)}{a}-\frac{\coth^{-1}(ax)^3}{3a^2}}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)^2-a\left(\frac{\coth^{-1}(\right.}{2a^3}}{}}{a^2}\right.$$

$\downarrow$ 7164

$$\frac{1}{5}x^5\coth^{-1}(ax)^3 - \frac{3}{5}a\left(\frac{\frac{\frac{\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)^2}{a}-2\left(\frac{\operatorname{PolyLog}\left(3,1-\frac{2}{1-ax}\right)}{4a}-\frac{\operatorname{PolyLog}\left(2,1-\frac{2}{1-ax}\right)\coth^{-1}(ax)}{2a}\right)}{a}-\frac{\coth^{-1}(ax)^3}{3a^2}}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)^2-a\left(\frac{\coth^{-1}(ax)^2}{2a^3}-\right.}{a^2}}{a^2}\right.$$

input
```
Int[x^4*ArcCoth[a*x]^3,x]
```

output
```
(x^5*ArcCoth[a*x]^3)/5 - (3*a*(-(((x^4*ArcCoth[a*x]^2)/4 - (a*(-(((x^3*Arc
Coth[a*x])/3 - (a*(-(x^2/a^2) - Log[1 - a^2*x^2]/a^4))/6)/a^2) + (ArcCoth[
a*x]^2/(2*a^3) - (x*ArcCoth[a*x] + Log[1 - a^2*x^2]/(2*a))/a^2)/a^2))/2)/a
^2) + (-(((x^2*ArcCoth[a*x]^2)/2 - a*(ArcCoth[a*x]^2/(2*a^3) - (x*ArcCoth[
a*x] + Log[1 - a^2*x^2]/(2*a))/a^2))/a^2) + (-1/3*ArcCoth[a*x]^3/a^2 + ((A
rcCoth[a*x]^2*Log[2/(1 - a*x)])/a - 2*(-1/2*(ArcCoth[a*x]*PolyLog[2, 1 - 2
/(1 - a*x)])/a + PolyLog[3, 1 - 2/(1 - a*x)]/(4*a)))/a)/a^2)/a^2))/5
```

Defintions of rubi rules used

rule 49
```
Int[((a_.) + (b_.)*(x_))^(m_.)*((c_.) + (d_.)*(x_))^(n_.), x_Symbol] :> Int
[ExpandIntegrand[(a + b*x)^m*(c + d*x)^n, x], x] /; FreeQ[{a, b, c, d}, x]
&& IGtQ[m, 0] && IGtQ[m + n + 2, 0]
```

rule 240
```
Int[(x_)/((a_) + (b_.)*(x_)^2), x_Symbol] :> Simp[Log[RemoveContent[a + b*x
^2, x]]/(2*b), x] /; FreeQ[{a, b}, x]
```

rule 243
```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] :> Simp[1/2   Subst[In
t[x^((m - 1)/2)*(a + b*x)^p, x], x, x^2], x] /; FreeQ[{a, b, m, p}, x] && I
ntegerQ[(m - 1)/2]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 6437
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.), x_Symbol] :> Simp[x*(a
 + b*ArcCoth[c*x^n])^p, x] - Simp[b*c*n*p   Int[x^n*((a + b*ArcCoth[c*x^n])
^(p - 1)/(1 - c^2*x^(2*n))), x], x] /; FreeQ[{a, b, c, n}, x] && IGtQ[p, 0]
 && (EqQ[n, 1] || EqQ[p, 1])
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

rule 6471
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((d_) + (e_.)*(x_)), x_Symbol
] :> Simp[(-(a + b*ArcCoth[c*x])^p)*(Log[2/(1 + e*(x/d))]/e), x] + Simp[b*c
*(p/e)   Int[(a + b*ArcCoth[c*x])^(p - 1)*(Log[2/(1 + e*(x/d))]/(1 - c^2*x^
2)), x], x] /; FreeQ[{a, b, c, d, e}, x] && IGtQ[p, 0] && EqQ[c^2*d^2 - e^2
, 0]
```

rule 6511
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((d_) + (e_.)*(x_)^2), x_Symb
ol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*c*d*(p + 1)), x] /; FreeQ[{a, b
, c, d, e, p}, x] && EqQ[c^2*d + e, 0] && NeQ[p, -1]
```

rule 6543
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*((f_.)*(x_))^(m_))/((d_) + (
e_.)*(x_)^2), x_Symbol] :> Simp[f^2/e   Int[(f*x)^(m - 2)*(a + b*ArcCoth[c*
x])^p, x], x] - Simp[d*(f^2/e)   Int[(f*x)^(m - 2)*((a + b*ArcCoth[c*x])^p/
(d + e*x^2)), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && GtQ[p, 0] && GtQ[m,
 1]
```

rule 6547
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*(x_))/((d_) + (e_.)*(x_)^2),
 x_Symbol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*e*(p + 1)), x] + Simp[1/
(c*d)   Int[(a + b*ArcCoth[c*x])^p/(1 - c*x), x], x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[c^2*d + e, 0] && IGtQ[p, 0]
```

rule 6621
```
Int[(Log[u_]*((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.))/((d_) + (e_.)*(x_)^
2), x_Symbol] :> Simp[(-(a + b*ArcCoth[c*x])^p)*(PolyLog[2, 1 - u]/(2*c*d))
, x] + Simp[b*(p/2)   Int[(a + b*ArcCoth[c*x])^(p - 1)*(PolyLog[2, 1 - u]/(
d + e*x^2)), x], x] /; FreeQ[{a, b, c, d, e}, x] && IGtQ[p, 0] && EqQ[c^2*d
 + e, 0] && EqQ[(1 - u)^2 - (1 - 2/(1 - c*x))^2, 0]
```

rule 7164
```
Int[(u_)*PolyLog[n_, v_], x_Symbol] :> With[{w = DerivativeDivides[v, u*v,
x]}, Simp[w*PolyLog[n + 1, v], x] /;  !FalseQ[w]] /; FreeQ[n, x]
```

Maple [C] (warning: unable to verify)

Result contains higher order function than in optimal. Order 9 vs. order 4.

Time = 5.51 (sec) , antiderivative size = 1223, normalized size of antiderivative = 6.24

method	result	size
parts	Expression too large to display	1223
derivativedivides	Expression too large to display	1225
default	Expression too large to display	1225

input
```
int(x^4*arccoth(x*a)^3,x,method=_RETURNVERBOSE)
```

output
```
1/5*x^5*arccoth(x*a)^3+3/5/a^5*(1/2*x^2*a^2*arccoth(x*a)^2+1/4*x^4*a^4*arc
coth(x*a)^2-5/3*ln(1+1/((a*x-1)/(a*x+1))^(1/2))+1/3*arccoth(x*a)^3+1/4*I*P
i*csgn(I/((a*x+1)/(a*x-1)-1))*csgn(I*(a*x+1)/(a*x-1))*csgn(I/(a*x-1)*(a*x+
1)/((a*x+1)/(a*x-1)-1))*arccoth(x*a)^2+1/2*arccoth(x*a)^2*ln((a*x-1)/(a*x+
1))+7/8*(((a*x-1)/(a*x+1))^(1/2)*x*a+((a*x-1)/(a*x+1))^(1/2)+x*a+1)*arccot
h(x*a)-1/12/(((a*x-1)/(a*x+1))^(1/2)+1)*((a*x-1)/(a*x+1))^(1/2)-1/24*(a*x-
1)/(((a*x-1)/(a*x+1))^(1/2)*x*a+((a*x-1)/(a*x+1))^(1/2)-x*a)+1/24*(a*x-1)/
(((a*x-1)/(a*x+1))^(1/2)*x*a+((a*x-1)/(a*x+1))^(1/2)+x*a)-arccoth(x*a)^2*l
n(1-1/((a*x-1)/(a*x+1))^(1/2))-arccoth(x*a)^2*ln(1+1/((a*x-1)/(a*x+1))^(1/
2))-2*arccoth(x*a)*polylog(2,-1/((a*x-1)/(a*x+1))^(1/2))-5/3*ln(-1+1/((a*x
-1)/(a*x+1))^(1/2))+1/4*I*Pi*csgn(I*(a*x+1)/(a*x-1))^3*arccoth(x*a)^2+1/4*
I*Pi*csgn(I/(a*x-1)*(a*x+1)/((a*x+1)/(a*x-1)-1))^3*arccoth(x*a)^2-1/12/(((
a*x-1)/(a*x+1))^(1/2)-1)*((a*x-1)/(a*x+1))^(1/2)+2*polylog(3,1/((a*x-1)/(a
*x+1))^(1/2))+2*polylog(3,-1/((a*x-1)/(a*x+1))^(1/2))+1/2*arccoth(x*a)^2*l
n(a*x-1)+1/2*arccoth(x*a)^2*ln(a*x+1)-7/8*(((a*x-1)/(a*x+1))^(1/2)*x*a+((a
*x-1)/(a*x+1))^(1/2)-x*a-1)*arccoth(x*a)-2*arccoth(x*a)*polylog(2,1/((a*x-
1)/(a*x+1))^(1/2))-ln(2)*arccoth(x*a)^2+arccoth(x*a)^2*ln((a*x+1)/(a*x-1)-
1)-1/24*(a*x+1)*arccoth(x*a)*(2*((a*x-1)/(a*x+1))^(1/2)*x^2*a^2-3*((a*x-1)
/(a*x+1))^(1/2)*x*a-2*a^2*x^2+((a*x-1)/(a*x+1))^(1/2)+5*x*a-5)+1/24*(a*x+1
)*arccoth(x*a)*(2*((a*x-1)/(a*x+1))^(1/2)*x^2*a^2-3*((a*x-1)/(a*x+1))^(...
```

Fricas [F]

$$\int x^4 \coth^{-1}(ax)^3 \, dx = \int x^4 \operatorname{arcoth}(ax)^3 \, dx$$

input
```
integrate(x^4*arccoth(a*x)^3,x, algorithm="fricas")
```

output
```
integral(x^4*arccoth(a*x)^3, x)
```

Sympy [F]

$$\int x^4 \coth^{-1}(ax)^3 \, dx = \int x^4 \operatorname{acoth}^3(ax) \, dx$$

input
```
integrate(x**4*acoth(a*x)**3,x)
```

output
```
Integral(x**4*acoth(a*x)**3, x)
```

Maxima [F]

$$\int x^4 \coth^{-1}(ax)^3 \, dx = \int x^4 \operatorname{arcoth}(ax)^3 \, dx$$

input
```
integrate(x^4*arccoth(a*x)^3,x, algorithm="maxima")
```

output
```
1/80*(2*(a^5*x^5 + 1)*log(a*x + 1)^3 + 3*(a^4*x^4 + 2*a^2*x^2 - 2*(a^5*x^5
 - 1)*log(a*x - 1))*log(a*x + 1)^2)/a^5 + 1/8*integrate(-1/5*(5*(a^5*x^5 +
 a^4*x^4)*log(a*x - 1)^3 + 3*(a^4*x^4 + 2*a^2*x^2 - 5*(a^5*x^5 + a^4*x^4)*
log(a*x - 1)^2 - 2*(a^5*x^5 - 1)*log(a*x - 1))*log(a*x + 1))/(a^5*x + a^4)
, x)
```

Giac [F]

$$\int x^4 \coth^{-1}(ax)^3 \, dx = \int x^4 \operatorname{arcoth}(ax)^3 \, dx$$

input
```
integrate(x^4*arccoth(a*x)^3,x, algorithm="giac")
```

output
```
integrate(x^4*arccoth(a*x)^3, x)
```

Mupad [F(-1)]

Timed out.

$$\int x^4 \coth^{-1}(ax)^3 \, dx = \int x^4 \operatorname{acoth}(a\,x)^3 \, dx$$

input
```
int(x^4*acoth(a*x)^3,x)
```

output
```
int(x^4*acoth(a*x)^3, x)
```

Reduce [F]

$$\int x^4 \coth^{-1}(ax)^3 \, dx = \frac{4acoth(ax)^3 a^5x^5 - 4acoth(ax)^3 ax - 3acoth(ax)^2 a^4x^4 - 6acoth(ax)^2 a^2x^2 + 9acoth(ax)^2 + 2acoth(ax)}{20a^5}$$

input
```
int(x^4*acoth(a*x)^3,x)
```

output
```
(4*acoth(a*x)**3*a**5*x**5 - 4*acoth(a*x)**3*a*x - 3*acoth(a*x)**2*a**4*x*
*4 - 6*acoth(a*x)**2*a**2*x**2 + 9*acoth(a*x)**2 + 2*acoth(a*x)*a**3*x**3
+ 18*acoth(a*x)*a*x + 20*acoth(a*x) + 4*int(acoth(a*x)**3,x)*a - 20*log(a*
*2*x - a) - a**2*x**2)/(20*a**5)
```

3.25 $\int x^3 \coth^{-1}(ax)^3 \, dx$

Optimal result

Integrand size = 10, antiderivative size = 139

$$\begin{aligned}\int x^3 \coth^{-1}(ax)^3 \, dx = {} & \frac{x}{4a^3} + \frac{x^2 \coth^{-1}(ax)}{4a^2} + \frac{\coth^{-1}(ax)^2}{a^4} \\ & + \frac{3x \coth^{-1}(ax)^2}{4a^3} + \frac{x^3 \coth^{-1}(ax)^2}{4a} \\ & - \frac{\coth^{-1}(ax)^3}{4a^4} + \frac{1}{4}x^4 \coth^{-1}(ax)^3 - \frac{\operatorname{arctanh}(ax)}{4a^4} \\ & - \frac{2\coth^{-1}(ax) \log\left(\frac{2}{1-ax}\right)}{a^4} - \frac{\operatorname{PolyLog}\left(2, 1 - \frac{2}{1-ax}\right)}{a^4}\end{aligned}$$

output
```
1/4*x/a^3+1/4*x^2*arccoth(a*x)/a^2+arccoth(a*x)^2/a^4+3/4*x*arccoth(a*x)^2
/a^3+1/4*x^3*arccoth(a*x)^2/a-1/4*arccoth(a*x)^3/a^4+1/4*x^4*arccoth(a*x)^
3-1/4*arctanh(a*x)/a^4-2*arccoth(a*x)*ln(2/(-a*x+1))/a^4-polylog(2,1-2/(-a
*x+1))/a^4
```

Mathematica [A] (verified)

Time = 0.21 (sec) , antiderivative size = 88, normalized size of antiderivative = 0.63

$$\int x^3 \coth^{-1}(ax)^3 \, dx = \frac{ax + (-4 + 3ax + a^3x^3)\coth^{-1}(ax)^2 + (-1 + a^4x^4)\coth^{-1}(ax)^3 + \coth^{-1}(ax)\left(-1 + a^2x^2 - 8\log\left(1 - \right.\right.}{4a^4}$$

input
```
Integrate[x^3*ArcCoth[a*x]^3,x]
```

output
```
(a*x + (-4 + 3*a*x + a^3*x^3)*ArcCoth[a*x]^2 + (-1 + a^4*x^4)*ArcCoth[a*x]
^3 + ArcCoth[a*x]*(-1 + a^2*x^2 - 8*Log[1 - E^(-2*ArcCoth[a*x])]) + 4*Poly
Log[2, E^(-2*ArcCoth[a*x])])/(4*a^4)
```

Rubi [A] (verified)

Time = 1.79 (sec) , antiderivative size = 239, normalized size of antiderivative = 1.72, number of steps used = 14, number of rules used = 13, $\frac{\text{number of rules}}{\text{integrand size}} = 1.300$, Rules used = {6453, 6543, 6453, 6543, 6437, 6453, 262, 219, 6511, 6547, 6471, 2849, 2752}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x^3 \coth^{-1}(ax)^3 \, dx$$

$$\Big\downarrow \text{ 6453}$$

$$\frac{1}{4}x^4 \coth^{-1}(ax)^3 - \frac{3}{4}a\int \frac{x^4 \coth^{-1}(ax)^2}{1 - a^2x^2}dx$$

$$\Big\downarrow \text{ 6543}$$

$$\frac{1}{4}x^4 \coth^{-1}(ax)^3 - \frac{3}{4}a\left(\frac{\int \frac{x^2 \coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2} - \frac{\int x^2 \coth^{-1}(ax)^2dx}{a^2}\right)$$

$$\Big\downarrow \text{ 6453}$$

$$\frac{1}{4}x^4\coth^{-1}(ax)^3-\frac{3}{4}a\left(\frac{\int\frac{x^2\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\int\frac{x^3\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}\right)$$

$$\Big\downarrow \; 6543$$

$$\frac{1}{4}x^4\coth^{-1}(ax)^3-\frac{3}{4}a\left(\frac{\frac{\int\frac{\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{\int\coth^{-1}(ax)^2dx}{a^2}}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\left(\frac{\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\int x\coth^{-1}(ax)dx}{a^2}\right)}{a^2}\right)$$

$$\Big\downarrow \; 6437$$

$$\frac{1}{4}x^4\coth^{-1}(ax)^3-\frac{3}{4}a\left(\frac{\frac{\int\frac{\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{x\coth^{-1}(ax)^2-2a\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\left(\frac{\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\int x\coth^{-1}(ax)dx}{a^2}\right)}{a^2}\right)$$

$$\Big\downarrow \; 6453$$

$$\frac{1}{4}x^4\coth^{-1}(ax)^3-\frac{3}{4}a\left(\frac{\frac{\int\frac{\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{x\coth^{-1}(ax)^2-2a\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\left(\frac{\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)-\frac{1}{2}a\int}{a^2}\right.}{a^2}\right.$$

$$\Big\downarrow \; 262$$

$$\frac{1}{4}x^4\coth^{-1}(ax)^3-\frac{3}{4}a\left(\frac{\frac{\int\frac{\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{x\coth^{-1}(ax)^2-2a\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\left(\frac{\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)-\frac{1}{2}a\left(\right.}{a^2}\right.}{a^2}\right.$$

$$\Big\downarrow \; 219$$

$$\frac{1}{4}x^4\coth^{-1}(ax)^3-$$

$$\frac{3}{4}a\left(\frac{\frac{\int\frac{\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{x\coth^{-1}(ax)^2-2a\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\left(\frac{\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)-\frac{1}{2}a\left(\ldots\right.}{a^2}\ldots\right.}{a^2}\ldots\right.$$

$\downarrow$ 6511

$$\frac{1}{4}x^4\coth^{-1}(ax)^3-$$

$$\frac{3}{4}a\left(\frac{\frac{\coth^{-1}(ax)^3}{3a^3}-\frac{x\coth^{-1}(ax)^2-2a\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\left(\frac{\int\frac{x\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)-\frac{1}{2}a\left(\ldots\right.}{a^2}\ldots\right.}{a^2}\ldots\right.$$

$\downarrow$ 6547

$$\frac{1}{4}x^4\coth^{-1}(ax)^3-$$

$$\frac{3}{4}a\left(\frac{\frac{\coth^{-1}(ax)^3}{3a^3}-\frac{x\coth^{-1}(ax)^2-2a\left(\frac{\int\frac{\coth^{-1}(ax)}{1-ax}dx}{a}-\frac{\coth^{-1}(ax)^2}{2a^2}\right)}{a^2}}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\left(\frac{\frac{\int\frac{\coth^{-1}(ax)}{1-ax}dx}{a}-\frac{\coth^{-1}(ax)^2}{2a^2}}{a^2}\ldots\right.}{a^2}\ldots\right.$$

$\downarrow$ 6471

$$\frac{1}{4}x^4\coth^{-1}(ax)^3-$$

$$\frac{3}{4}a\left(\frac{\frac{\coth^{-1}(ax)^3}{3a^3}-\frac{x\coth^{-1}(ax)^2-2a\left(\frac{\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)}{a}-\int\frac{\log\left(\frac{2}{1-ax}\right)}{1-a^2x^2}dx}{a}-\frac{\coth^{-1}(ax)^2}{2a^2}\right)}{a^2}}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\left(\ldots\right.}{\ldots}\ldots\right.$$

$\downarrow$ 2849

$$\frac{1}{4}x^4\coth^{-1}(ax)^3 - \frac{3}{4}a\left(\frac{\frac{\coth^{-1}(ax)^3}{3a^3}-\frac{x\coth^{-1}(ax)^2-2a\left(\frac{\frac{\int\frac{\log\left(\frac{2}{1-ax}\right)}{1-\frac{2}{1-ax}}d\frac{1}{1-ax}}{a}+\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)}{a}}{a}-\frac{\coth^{-1}(ax)^2}{2a^2}\right)}{a^2}}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\left(\int \ldots\right.}{\ldots}\right.$$

$\downarrow$ 2752

$$\frac{1}{4}x^4\coth^{-1}(ax)^3 - \frac{3}{4}a\left(\frac{\frac{\coth^{-1}(ax)^3}{3a^3}-\frac{x\coth^{-1}(ax)^2-2a\left(\frac{\frac{\text{PolyLog}\left(2,1-\frac{2}{1-ax}\right)}{2a}+\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)}{a}}{a}-\frac{\coth^{-1}(ax)^2}{2a^2}\right)}{a^2}}{a^2}-\frac{\frac{1}{3}x^3\coth^{-1}(ax)^2-\frac{2}{3}a\left(\text{Po}\ldots\right.}{\ldots}\right.$$

input

```
Int[x^3*ArcCoth[a*x]^3,x]
```

output

```
(x^4*ArcCoth[a*x]^3)/4 - (3*a*(-(((x^3*ArcCoth[a*x]^2)/3 - (2*a*(-(((x^2*A
rcCoth[a*x])/2 - (a*(-(x/a^2) + ArcTanh[a*x]/a^3))/2)/a^2) + (-1/2*ArcCoth
[a*x]^2/a^2 + ((ArcCoth[a*x]*Log[2/(1 - a*x)])/a + PolyLog[2, 1 - 2/(1 - a
*x)]/(2*a))/a)/a^2))/3)/a^2) + (ArcCoth[a*x]^3/(3*a^3) - (x*ArcCoth[a*x]^2
 - 2*a*(-1/2*ArcCoth[a*x]^2/a^2 + ((ArcCoth[a*x]*Log[2/(1 - a*x)])/a + Pol
yLog[2, 1 - 2/(1 - a*x)]/(2*a))/a))/a^2)/a^2))/4
```

Defintions of rubi rules used

rule 219

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 262

```
Int[((c_.)*(x_))^(m_)*((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] :> Simp[c*(c*x)
^(m - 1)*((a + b*x^2)^(p + 1)/(b*(m + 2*p + 1))), x] - Simp[a*c^2*((m - 1)/
(b*(m + 2*p + 1)))   Int[(c*x)^(m - 2)*(a + b*x^2)^p, x], x] /; FreeQ[{a, b
, c, p}, x] && GtQ[m, 2 - 1] && NeQ[m + 2*p + 1, 0] && IntBinomialQ[a, b, c
, 2, m, p, x]
```

rule 2752

```
Int[Log[(c_.)*(x_)]/((d_) + (e_.)*(x_)), x_Symbol] :> Simp[(-e^(-1))*PolyLo
g[2, 1 - c*x], x] /; FreeQ[{c, d, e}, x] && EqQ[e + c*d, 0]
```

rule 2849

```
Int[Log[(c_.)/((d_) + (e_.)*(x_))]/((f_) + (g_.)*(x_)^2), x_Symbol] :> Simp
[-e/g   Subst[Int[Log[2*d*x]/(1 - 2*d*x), x], x, 1/(d + e*x)], x] /; FreeQ[
{c, d, e, f, g}, x] && EqQ[c, 2*d] && EqQ[e^2*f + d^2*g, 0]
```

rule 6437

```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.), x_Symbol] :> Simp[x*(a
 + b*ArcCoth[c*x^n])^p, x] - Simp[b*c*n*p   Int[x^n*((a + b*ArcCoth[c*x^n])
^(p - 1)/(1 - c^2*x^(2*n))), x], x] /; FreeQ[{a, b, c, n}, x] && IGtQ[p, 0]
 && (EqQ[n, 1] || EqQ[p, 1])
```

rule 6453

```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

rule 6471
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((d_) + (e_.)*(x_)), x_Symbol
] :> Simp[(-(a + b*ArcCoth[c*x])^p)*(Log[2/(1 + e*(x/d))]/e), x] + Simp[b*c
*(p/e)   Int[(a + b*ArcCoth[c*x])^(p - 1)*(Log[2/(1 + e*(x/d))]/(1 - c^2*x^
2)), x], x] /; FreeQ[{a, b, c, d, e}, x] && IGtQ[p, 0] && EqQ[c^2*d^2 - e^2
, 0]
```

rule 6511
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((d_) + (e_.)*(x_)^2), x_Symb
ol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*c*d*(p + 1)), x] /; FreeQ[{a, b
, c, d, e, p}, x] && EqQ[c^2*d + e, 0] && NeQ[p, -1]
```

rule 6543
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*((f_.)*(x_))^(m_))/((d_) + (
e_.)*(x_)^2), x_Symbol] :> Simp[f^2/e   Int[(f*x)^(m - 2)*(a + b*ArcCoth[c*
x])^p, x], x] - Simp[d*(f^2/e)   Int[(f*x)^(m - 2)*((a + b*ArcCoth[c*x])^p/
(d + e*x^2)), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && GtQ[p, 0] && GtQ[m,
 1]
```

rule 6547
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*(x_))/((d_) + (e_.)*(x_)^2),
 x_Symbol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*e*(p + 1)), x] + Simp[1/
(c*d)   Int[(a + b*ArcCoth[c*x])^p/(1 - c*x), x], x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[c^2*d + e, 0] && IGtQ[p, 0]
```

Maple [C] (warning: unable to verify)

Result contains higher order function than in optimal. Order 9 vs. order 4.

Time = 4.69 (sec) , antiderivative size = 871, normalized size of antiderivative = 6.27

method	result	size
derivativedivides	Expression too large to display	871
default	Expression too large to display	871
parts	Expression too large to display	871

input
```
int(x^3*arccoth(x*a)^3,x,method=_RETURNVERBOSE)
```

output
```
1/a^4*(1/4*x^4*a^4*arccoth(x*a)^3+1/4*x^3*a^3*arccoth(x*a)^2-3/16*I*Pi*csg
n(I/((a*x+1)/(a*x-1)-1))*csgn(I*(a*x+1)/(a*x-1))*csgn(I/(a*x-1)*(a*x+1)/((
a*x+1)/(a*x-1)-1))*arccoth(x*a)^2-2*arccoth(x*a)*ln(1+1/((a*x-1)/(a*x+1))^
(1/2))-1/4*arccoth(x*a)^3-3/8*arccoth(x*a)^2*ln((a*x-1)/(a*x+1))+1/8*(((a*
x-1)/(a*x+1))^(1/2)*x*a+((a*x-1)/(a*x+1))^(1/2)+x*a+1)*arccoth(x*a)-1/4/((
(a*x-1)/(a*x+1))^(1/2)+1)*((a*x-1)/(a*x+1))^(1/2)-1/4/(((a*x-1)/(a*x+1))^(
1/2)-1)*((a*x-1)/(a*x+1))^(1/2)+1/8*(((a*x-1)/(a*x+1))^(1/2)*x*a+((a*x-1)/
(a*x+1))^(1/2)-x*a)*arccoth(x*a)*(a*x+1)+3/8*arccoth(x*a)^2*ln(a*x-1)-3/8*
arccoth(x*a)^2*ln(a*x+1)-1/8*(((a*x-1)/(a*x+1))^(1/2)*x*a+((a*x-1)/(a*x+1)
)^(1/2)-x*a-1)*arccoth(x*a)-3/16*I*Pi*csgn(I/(a*x-1)*(a*x+1)/((a*x+1)/(a*x
-1)-1))^3*arccoth(x*a)^2-3/16*I*Pi*csgn(I*(a*x+1)/(a*x-1))^3*arccoth(x*a)^
2+3/4*arccoth(x*a)^2*x*a+2*dilog(1/((a*x-1)/(a*x+1))^(1/2))-2*dilog(1+1/((
a*x-1)/(a*x+1))^(1/2))+arccoth(x*a)^2+3/8*I*Pi*csgn(I/((a*x-1)/(a*x+1))^(1
/2))*csgn(I*(a*x+1)/(a*x-1))^2*arccoth(x*a)^2-3/16*I*Pi*csgn(I/((a*x-1)/(a
*x+1))^(1/2))^2*csgn(I*(a*x+1)/(a*x-1))*arccoth(x*a)^2-1/8*(((a*x-1)/(a*x+
1))^(1/2)*x*a+((a*x-1)/(a*x+1))^(1/2)+x*a)*arccoth(x*a)*(a*x+1)-1/4*(((a*x
-1)/(a*x+1))^(1/2)*x*a-x*a+1)*arccoth(x*a)*(a*x+1)+1/4*(((a*x-1)/(a*x+1))^
(1/2)*x*a+x*a-1)*arccoth(x*a)*(a*x+1)+3/16*I*Pi*csgn(I/((a*x+1)/(a*x-1)-1)
)*csgn(I/(a*x-1)*(a*x+1)/((a*x+1)/(a*x-1)-1))^2*arccoth(x*a)^2+3/16*I*Pi*c
sgn(I*(a*x+1)/(a*x-1))*csgn(I/(a*x-1)*(a*x+1)/((a*x+1)/(a*x-1)-1))^2*ar...
```

Fricas [F]

$$\int x^3 \coth^{-1}(ax)^3 \, dx = \int x^3 \operatorname{arcoth}(ax)^3 \, dx$$

input
```
integrate(x^3*arccoth(a*x)^3,x, algorithm="fricas")
```

output
```
integral(x^3*arccoth(a*x)^3, x)
```

Sympy [F]

$$\int x^3 \coth^{-1}(ax)^3 \, dx = \int x^3 \operatorname{acoth}^3{(ax)} \, dx$$

input
```
integrate(x**3*acoth(a*x)**3,x)
```

output
```
Integral(x**3*acoth(a*x)**3, x)
```

Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 262 vs. $2(122) = 244$.

Time = 0.04 (sec) , antiderivative size = 262, normalized size of antiderivative = 1.88

$$\int x^3 \coth^{-1}(ax)^3 \, dx = \frac{1}{4} x^4 \operatorname{arcoth}(ax)^3 + \frac{1}{8} a \left(\frac{2 (a^2 x^3 + 3x)}{a^4} - \frac{3 \log(ax+1)}{a^5} + \frac{3 \log(ax-1)}{a^5} \right) \operatorname{arcoth}(ax)^2 + \frac{1}{32} a \left(\frac{\frac{(3 \log(ax-1)-8) \log(ax+1)^2 - \log(ax+1)^3 + \log(ax-1)^3 + 8ax - \left(3 \log(ax-1)^2 - 16 \log(ax-1)\right) \log(ax+1) + 8 \log(ax-1)^2 + 4 \log(a}{a}}{a^4} \right.$$

input
```
integrate(x^3*arccoth(a*x)^3,x, algorithm="maxima")
```

output
```
1/4*x^4*arccoth(a*x)^3 + 1/8*a*(2*(a^2*x^3 + 3*x)/a^4 - 3*log(a*x + 1)/a^5
 + 3*log(a*x - 1)/a^5)*arccoth(a*x)^2 + 1/32*a*((((3*log(a*x - 1) - 8)*log
(a*x + 1)^2 - log(a*x + 1)^3 + log(a*x - 1)^3 + 8*a*x - (3*log(a*x - 1)^2
- 16*log(a*x - 1))*log(a*x + 1) + 8*log(a*x - 1)^2 + 4*log(a*x - 1))/a - 3
2*(log(a*x - 1)*log(1/2*a*x + 1/2) + dilog(-1/2*a*x + 1/2))/a - 4*log(a*x
+ 1)/a)/a^4 + 2*(4*a^2*x^2 - 2*(3*log(a*x - 1) - 8)*log(a*x + 1) + 3*log(a
*x + 1)^2 + 3*log(a*x - 1)^2 + 16*log(a*x - 1))*arccoth(a*x)/a^5)
```

Giac [F]

$$\int x^3 \coth^{-1}(ax)^3 \, dx = \int x^3 \operatorname{arcoth}(ax)^3 \, dx$$

input
```
integrate(x^3*arccoth(a*x)^3,x, algorithm="giac")
```

output
```
integrate(x^3*arccoth(a*x)^3, x)
```

Mupad [F(-1)]

Timed out.

$$\int x^3 \coth^{-1}(ax)^3 \, dx = \int x^3 \operatorname{acoth}(a\,x)^3 \, dx$$

input
```
int(x^3*acoth(a*x)^3,x)
```

output
```
int(x^3*acoth(a*x)^3, x)
```

Reduce [F]

$$\int x^3 \coth^{-1}(ax)^3 \, dx = \frac{acoth(ax)^3\, a^4x^4 - acoth(ax)^3 - acoth(ax)^2\, a^3x^3 - 3acoth(ax)^2\, ax + acoth(ax)\, a^2x^2 - acoth(ax) + 8\Big(\int}{4a^4}$$

input
```
int(x^3*acoth(a*x)^3,x)
```

output
```
(acoth(a*x)**3*a**4*x**4 - acoth(a*x)**3 - acoth(a*x)**2*a**3*x**3 - 3*aco
th(a*x)**2*a*x + acoth(a*x)*a**2*x**2 - acoth(a*x) + 8*int((acoth(a*x)*x)/
(a**2*x**2 - 1),x)*a**2 - a*x)/(4*a**4)
```

3.26 $\int x^2 \coth^{-1}(ax)^3 \, dx$

Optimal result

Integrand size = 10, antiderivative size = 149

$$\int x^2 \coth^{-1}(ax)^3 \, dx = \frac{x \coth^{-1}(ax)}{a^2} - \frac{\coth^{-1}(ax)^2}{2a^3} + \frac{x^2 \coth^{-1}(ax)^2}{2a} + \frac{\coth^{-1}(ax)^3}{3a^3} + \frac{1}{3}x^3 \coth^{-1}(ax)^3 - \frac{\coth^{-1}(ax)^2 \log\left(\frac{2}{1-ax}\right)}{a^3} + \frac{\log(1-a^2x^2)}{2a^3} - \frac{\coth^{-1}(ax) \operatorname{PolyLog}\left(2, 1-\frac{2}{1-ax}\right)}{a^3} + \frac{\operatorname{PolyLog}\left(3, 1-\frac{2}{1-ax}\right)}{2a^3}$$

output
```
x*arccoth(a*x)/a^2-1/2*arccoth(a*x)^2/a^3+1/2*x^2*arccoth(a*x)^2/a+1/3*arc
coth(a*x)^3/a^3+1/3*x^3*arccoth(a*x)^3-arccoth(a*x)^2*ln(2/(-a*x+1))/a^3+1
/2*ln(-a^2*x^2+1)/a^3-arccoth(a*x)*polylog(2,1-2/(-a*x+1))/a^3+1/2*polylog
(3,1-2/(-a*x+1))/a^3
```

Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 0.26 (sec) , antiderivative size = 140, normalized size of antiderivative = 0.94

$$\int x^2 \coth^{-1}(ax)^3 \, dx$$
$$= \frac{-i\pi^3 + 24ax \coth^{-1}(ax) - 12 \coth^{-1}(ax)^2 + 12a^2x^2 \coth^{-1}(ax)^2 + 8 \coth^{-1}(ax)^3 + 8a^3x^3 \coth^{-1}(ax)^3 -}{}$$

input
```
Integrate[x^2*ArcCoth[a*x]^3,x]
```

output
```
((-I)*Pi^3 + 24*a*x*ArcCoth[a*x] - 12*ArcCoth[a*x]^2 + 12*a^2*x^2*ArcCoth[
a*x]^2 + 8*ArcCoth[a*x]^3 + 8*a^3*x^3*ArcCoth[a*x]^3 - 24*ArcCoth[a*x]^2*L
og[1 - E^(2*ArcCoth[a*x])] - 24*Log[1/(a*Sqrt[1 - 1/(a^2*x^2)]*x)] - 24*Ar
cCoth[a*x]*PolyLog[2, E^(2*ArcCoth[a*x])] + 12*PolyLog[3, E^(2*ArcCoth[a*x
])])/(24*a^3)
```

Rubi [A] (verified)

Time = 1.53 (sec) , antiderivative size = 177, normalized size of antiderivative = 1.19, number of steps used = 11, number of rules used = 11, $\frac{\text{number of rules}}{\text{integrand size}} = 1.100$, Rules used = {6453, 6543, 6453, 6543, 6437, 240, 6511, 6547, 6471, 6621, 7164}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x^2 \coth^{-1}(ax)^3 \, dx$$

$$\Big\downarrow \text{ 6453}$$

$$\frac{1}{3}x^3 \coth^{-1}(ax)^3 - a\int \frac{x^3 \coth^{-1}(ax)^2}{1-a^2x^2}dx$$

$$\Big\downarrow \text{ 6543}$$

$$\frac{1}{3}x^3 \coth^{-1}(ax)^3 - a\left(\frac{\int \frac{x \coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2} - \frac{\int x \coth^{-1}(ax)^2dx}{a^2}\right)$$

$$\Big\downarrow \text{ 6453}$$

$$\frac{1}{3}x^3\coth^{-1}(ax)^3-a\left(\frac{\int\frac{x\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)^2-a\int\frac{x^2\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}\right)$$

$$\Big\downarrow \text{6543}$$

$$\frac{1}{3}x^3\coth^{-1}(ax)^3-a\left(\frac{\int\frac{x\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)^2-a\left(\frac{\int\frac{\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\int\coth^{-1}(ax)dx}{a^2}\right)}{a^2}\right)$$

$$\Big\downarrow \text{6437}$$

$$\frac{1}{3}x^3\coth^{-1}(ax)^3-a\left(\frac{\int\frac{x\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)^2-a\left(\frac{\int\frac{\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{x\coth^{-1}(ax)-a\int\frac{x}{1-a^2x^2}dx}{a^2}\right)}{a^2}\right)$$

$$\Big\downarrow \text{240}$$

$$\frac{1}{3}x^3\coth^{-1}(ax)^3-a\left(\frac{\int\frac{x\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)^2-a\left(\frac{\int\frac{\coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}-\frac{\frac{\log\left(1-a^2x^2\right)}{2a}+x\coth^{-1}(ax)}{a^2}\right)}{a^2}\right)$$

$$\Big\downarrow \text{6511}$$

$$\frac{1}{3}x^3\coth^{-1}(ax)^3-a\left(\frac{\int\frac{x\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)^2-a\left(\frac{\coth^{-1}(ax)^2}{2a^3}-\frac{\frac{\log\left(1-a^2x^2\right)}{2a}+x\coth^{-1}(ax)}{a^2}\right)}{a^2}\right)$$

$$\Big\downarrow \text{6547}$$

$$\frac{1}{3}x^3\coth^{-1}(ax)^3 - a\left(\frac{\frac{\int \frac{\coth^{-1}(ax)^2}{1-ax}dx}{a}-\frac{\coth^{-1}(ax)^3}{3a^2}}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)^2-a\left(\frac{\coth^{-1}(ax)^2}{2a^3}-\frac{\frac{\log(1-a^2x^2)}{2a}+x\coth^{-1}(ax)}{a^2}\right)}{a^2}\right)$$

$\big\downarrow$ 6471

$$\frac{1}{3}x^3\coth^{-1}(ax)^3 - a\left(\frac{\frac{\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)^2}{a}-2\int \frac{\coth^{-1}(ax)\log\left(\frac{2}{1-ax}\right)}{1-a^2x^2}dx}{a}-\frac{\coth^{-1}(ax)^3}{3a^2}}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)^2-a\left(\frac{\coth^{-1}(ax)^2}{2a^3}-\frac{\frac{\log(1-a^2x^2)}{2a}+x\,\mathrm{co}}{a^2}\right.}{a^2}\right.$$

$\big\downarrow$ 6621

$$\frac{1}{3}x^3\coth^{-1}(ax)^3 - a\left(\frac{\frac{\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)^2}{a}-2\left(\frac{1}{2}\int \frac{\mathrm{PolyLog}\left(2,1-\frac{2}{1-ax}\right)}{1-a^2x^2}dx-\frac{\mathrm{PolyLog}\left(2,1-\frac{2}{1-ax}\right)\coth^{-1}(ax)}{2a}\right)}{a}-\frac{\coth^{-1}(ax)^3}{3a^2}}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)^2-a\left(\right.}{}\right.$$

$\big\downarrow$ 7164

$$\frac{1}{3}x^3\coth^{-1}(ax)^3 - a\left(\frac{\frac{\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)^2}{a}-2\left(\frac{\mathrm{PolyLog}\left(3,1-\frac{2}{1-ax}\right)}{4a}-\frac{\mathrm{PolyLog}\left(2,1-\frac{2}{1-ax}\right)\coth^{-1}(ax)}{2a}\right)}{a}-\frac{\coth^{-1}(ax)^3}{3a^2}}{a^2}-\frac{\frac{1}{2}x^2\coth^{-1}(ax)^2-a\left(\frac{\mathrm{coth}^{-}}{2}\right.}{}\right.$$

input
```
Int[x^2*ArcCoth[a*x]^3,x]
```

output
```
(x^3*ArcCoth[a*x]^3)/3 - a*(-(((x^2*ArcCoth[a*x]^2)/2 - a*(ArcCoth[a*x]^2/
(2*a^3) - (x*ArcCoth[a*x] + Log[1 - a^2*x^2]/(2*a))/a^2))/a^2) + (-1/3*Arc
Coth[a*x]^3/a^2 + ((ArcCoth[a*x]^2*Log[2/(1 - a*x)])/a - 2*(-1/2*(ArcCoth[
a*x]*PolyLog[2, 1 - 2/(1 - a*x)])/a + PolyLog[3, 1 - 2/(1 - a*x)]/(4*a)))/
a)/a^2)
```

Defintions of rubi rules used

rule 240
```
Int[(x_)/((a_) + (b_.)*(x_)^2), x_Symbol] :> Simp[Log[RemoveContent[a + b*x
^2, x]]/(2*b), x] /; FreeQ[{a, b}, x]
```

rule 6437
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.), x_Symbol] :> Simp[x*(a
 + b*ArcCoth[c*x^n])^p, x] - Simp[b*c*n*p   Int[x^n*((a + b*ArcCoth[c*x^n])
^(p - 1)/(1 - c^2*x^(2*n))), x], x] /; FreeQ[{a, b, c, n}, x] && IGtQ[p, 0]
 && (EqQ[n, 1] || EqQ[p, 1])
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

rule 6471
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((d_) + (e_.)*(x_)), x_Symbol
] :> Simp[(-(a + b*ArcCoth[c*x])^p)*(Log[2/(1 + e*(x/d))]/e), x] + Simp[b*c
*(p/e)   Int[(a + b*ArcCoth[c*x])^(p - 1)*(Log[2/(1 + e*(x/d))]/(1 - c^2*x^
2)), x], x] /; FreeQ[{a, b, c, d, e}, x] && IGtQ[p, 0] && EqQ[c^2*d^2 - e^2
, 0]
```

rule 6511
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((d_) + (e_.)*(x_)^2), x_Symb
ol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*c*d*(p + 1)), x] /; FreeQ[{a, b
, c, d, e, p}, x] && EqQ[c^2*d + e, 0] && NeQ[p, -1]
```

rule 6543
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*((f_.)*(x_))^(m_))/((d_) + (
e_.)*(x_)^2), x_Symbol] :> Simp[f^2/e   Int[(f*x)^(m - 2)*(a + b*ArcCoth[c*
x])^p, x], x] - Simp[d*(f^2/e)   Int[(f*x)^(m - 2)*((a + b*ArcCoth[c*x])^p/
(d + e*x^2)), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && GtQ[p, 0] && GtQ[m,
 1]
```

rule 6547
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*(x_))/((d_) + (e_.)*(x_)^2),
 x_Symbol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*e*(p + 1)), x] + Simp[1/
(c*d)   Int[(a + b*ArcCoth[c*x])^p/(1 - c*x), x], x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[c^2*d + e, 0] && IGtQ[p, 0]
```

rule 6621
```
Int[(Log[u_]*((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.))/((d_) + (e_.)*(x_)^
2), x_Symbol] :> Simp[(-(a + b*ArcCoth[c*x])^p)*(PolyLog[2, 1 - u]/(2*c*d))
, x] + Simp[b*(p/2)   Int[(a + b*ArcCoth[c*x])^(p - 1)*(PolyLog[2, 1 - u]/(
d + e*x^2)), x], x] /; FreeQ[{a, b, c, d, e}, x] && IGtQ[p, 0] && EqQ[c^2*d
 + e, 0] && EqQ[(1 - u)^2 - (1 - 2/(1 - c*x))^2, 0]
```

rule 7164
```
Int[(u_)*PolyLog[n_, v_], x_Symbol] :> With[{w = DerivativeDivides[v, u*v,
x]}, Simp[w*PolyLog[n + 1, v], x] /;  !FalseQ[w]] /; FreeQ[n, x]
```

Maple [C] (warning: unable to verify)

Result contains higher order function than in optimal. Order 9 vs. order 4.

Time = 4.34 (sec) , antiderivative size = 776, normalized size of antiderivative = 5.21

method	result
parts	$\frac{x^3 \operatorname{arccoth}(xa)^3}{3} + \frac{\frac{x^2 a^2 \operatorname{arccoth}(xa)^2}{2} - \ln\left(1+\frac{1}{\sqrt{\frac{xa-1}{xa+1}}}\right) + \frac{\operatorname{arccoth}(xa)^3}{3} + \frac{\operatorname{arccoth}(xa)^2 \ln\left(\frac{xa-1}{xa+1}\right)}{2} + \frac{\left(\sqrt{\frac{xa-1}{xa+1}}\, xa + \sqrt{\frac{xa-1}{xa+1}}\right.}{2}}{}$
derivativedivides	$\frac{\frac{x^2 a^2 \operatorname{arccoth}(xa)^2}{2} - \ln\left(1+\frac{1}{\sqrt{\frac{xa-1}{xa+1}}}\right) + \frac{\operatorname{arccoth}(xa)^3}{3} + \frac{\operatorname{arccoth}(xa)^2 \ln\left(\frac{xa-1}{xa+1}\right)}{2} + \frac{\left(\sqrt{\frac{xa-1}{xa+1}}\, xa + \sqrt{\frac{xa-1}{xa+1}} + xa + 1\right) \operatorname{arccoth}(xa)}{2} -}{}$
default	$\frac{\frac{x^2 a^2 \operatorname{arccoth}(xa)^2}{2} - \ln\left(1+\frac{1}{\sqrt{\frac{xa-1}{xa+1}}}\right) + \frac{\operatorname{arccoth}(xa)^3}{3} + \frac{\operatorname{arccoth}(xa)^2 \ln\left(\frac{xa-1}{xa+1}\right)}{2} + \frac{\left(\sqrt{\frac{xa-1}{xa+1}}\, xa + \sqrt{\frac{xa-1}{xa+1}} + xa + 1\right) \operatorname{arccoth}(xa)}{2} -}{}$

input
```
int(x^2*arccoth(x*a)^3,x,method=_RETURNVERBOSE)
```

output
```
1/3*x^3*arccoth(x*a)^3+1/a^3*(1/2*x^2*a^2*arccoth(x*a)^2-ln(1+1/((a*x-1)/(
a*x+1))^(1/2))+1/3*arccoth(x*a)^3+1/4*I*Pi*csgn(I/((a*x+1)/(a*x-1)-1))*csg
n(I*(a*x+1)/(a*x-1))*csgn(I/(a*x-1)*(a*x+1)/((a*x+1)/(a*x-1)-1))*arccoth(x
*a)^2+1/2*arccoth(x*a)^2*ln((a*x-1)/(a*x+1))+1/2*(((a*x-1)/(a*x+1))^(1/2)*
x*a+((a*x-1)/(a*x+1))^(1/2)+x*a+1)*arccoth(x*a)-arccoth(x*a)^2*ln(1-1/((a*
x-1)/(a*x+1))^(1/2))-arccoth(x*a)^2*ln(1+1/((a*x-1)/(a*x+1))^(1/2))-2*arcc
oth(x*a)*polylog(2,-1/((a*x-1)/(a*x+1))^(1/2))-ln(-1+1/((a*x-1)/(a*x+1))^(
1/2))+1/4*I*Pi*csgn(I*(a*x+1)/(a*x-1))^3*arccoth(x*a)^2+1/4*I*Pi*csgn(I/(a
*x-1)*(a*x+1)/((a*x+1)/(a*x-1)-1))^3*arccoth(x*a)^2+2*polylog(3,1/((a*x-1)
/(a*x+1))^(1/2))+2*polylog(3,-1/((a*x-1)/(a*x+1))^(1/2))+1/2*arccoth(x*a)^
2*ln(a*x-1)+1/2*arccoth(x*a)^2*ln(a*x+1)-1/2*(((a*x-1)/(a*x+1))^(1/2)*x*a+
((a*x-1)/(a*x+1))^(1/2)-x*a-1)*arccoth(x*a)-2*arccoth(x*a)*polylog(2,1/((a
*x-1)/(a*x+1))^(1/2))-ln(2)*arccoth(x*a)^2+arccoth(x*a)^2*ln((a*x+1)/(a*x-
1)-1)-1/2*I*Pi*csgn(I/((a*x-1)/(a*x+1))^(1/2))*csgn(I*(a*x+1)/(a*x-1))^2*a
rccoth(x*a)^2-1/4*I*Pi*csgn(I*(a*x+1)/(a*x-1))*csgn(I/(a*x-1)*(a*x+1)/((a*
x+1)/(a*x-1)-1))^2*arccoth(x*a)^2-1/4*I*Pi*csgn(I/((a*x+1)/(a*x-1)-1))*csg
n(I/(a*x-1)*(a*x+1)/((a*x+1)/(a*x-1)-1))^2*arccoth(x*a)^2+1/4*I*Pi*csgn(I/
((a*x-1)/(a*x+1))^(1/2))^2*csgn(I*(a*x+1)/(a*x-1))*arccoth(x*a)^2-1/2*arcc
oth(x*a)^2)
```

Fricas [F]

$$\int x^2 \coth^{-1}(ax)^3\,dx = \int x^2 \operatorname{arcoth}(ax)^3\,dx$$

input
```
integrate(x^2*arccoth(a*x)^3,x, algorithm="fricas")
```

output
```
integral(x^2*arccoth(a*x)^3, x)
```

Sympy [F]

$$\int x^2 \coth^{-1}(ax)^3 \, dx = \int x^2 \operatorname{acoth}^3{(ax)}\, dx$$

input
```
integrate(x**2*acoth(a*x)**3,x)
```

output
```
Integral(x**2*acoth(a*x)**3, x)
```

Maxima [F]

$$\int x^2 \coth^{-1}(ax)^3 \, dx = \int x^2 \operatorname{arcoth}(ax)^3 \, dx$$

input
```
integrate(x^2*arccoth(a*x)^3,x, algorithm="maxima")
```

output
```
1/24*((a^3*x^3 + 1)*log(a*x + 1)^3 + 3*(a^2*x^2 - (a^3*x^3 - 1)*log(a*x -
1))*log(a*x + 1)^2)/a^3 + 1/8*integrate(-((a^3*x^3 + a^2*x^2)*log(a*x - 1)
^3 + (2*a^2*x^2 - 3*(a^3*x^3 + a^2*x^2)*log(a*x - 1)^2 - 2*(a^3*x^3 - 1)*l
og(a*x - 1))*log(a*x + 1))/(a^3*x + a^2), x)
```

Giac [F]

$$\int x^2 \coth^{-1}(ax)^3 \, dx = \int x^2 \operatorname{arcoth}(ax)^3 \, dx$$

input
```
integrate(x^2*arccoth(a*x)^3,x, algorithm="giac")
```

output
```
integrate(x^2*arccoth(a*x)^3, x)
```

Mupad [F(-1)]

Timed out.

$$\int x^2 \coth^{-1}(ax)^3 \, dx = \int x^2 \operatorname{acoth}(a\,x)^3 \, dx$$

input
```
int(x^2*acoth(a*x)^3,x)
```

output
```
int(x^2*acoth(a*x)^3, x)
```

Reduce [F]

$$\int x^2 \coth^{-1}(ax)^3 \, dx$$
$$= \frac{2acoth(ax)^3 a^3x^3 - 2acoth(ax)^3 ax - 3acoth(ax)^2 a^2x^2 + 3acoth(ax)^2 + 6acoth(ax) ax + 6acoth(ax) +}{6a^3}$$

input
```
int(x^2*acoth(a*x)^3,x)
```

output
```
(2*acoth(a*x)**3*a**3*x**3 - 2*acoth(a*x)**3*a*x - 3*acoth(a*x)**2*a**2*x*
*2 + 3*acoth(a*x)**2 + 6*acoth(a*x)*a*x + 6*acoth(a*x) + 2*int(acoth(a*x)*
*3,x)*a - 6*log(a**2*x - a))/(6*a**3)
```

3.27 $\int x \coth^{-1}(ax)^3 \, dx$

Optimal result

Integrand size = 8, antiderivative size = 95

$$\int x \coth^{-1}(ax)^3 \, dx = \frac{3\coth^{-1}(ax)^2}{2a^2} + \frac{3x\coth^{-1}(ax)^2}{2a} - \frac{\coth^{-1}(ax)^3}{2a^2} + \frac{1}{2}x^2\coth^{-1}(ax)^3 - \frac{3\coth^{-1}(ax)\log\left(\frac{2}{1-ax}\right)}{a^2} - \frac{3\,\text{PolyLog}\left(2, 1-\frac{2}{1-ax}\right)}{2a^2}$$

output
```
3/2*arccoth(a*x)^2/a^2+3/2*x*arccoth(a*x)^2/a-1/2*arccoth(a*x)^3/a^2+1/2*x
^2*arccoth(a*x)^3-3*arccoth(a*x)*ln(2/(-a*x+1))/a^2-3/2*polylog(2,1-2/(-a*
x+1))/a^2
```

Mathematica [A] (verified)

Time = 0.10 (sec) , antiderivative size = 68, normalized size of antiderivative = 0.72

$$\int x \coth^{-1}(ax)^3 \, dx = \frac{\coth^{-1}(ax)\left(3(-1+ax)\coth^{-1}(ax) + (-1+a^2x^2)\coth^{-1}(ax)^2 - 6\log\left(1-e^{-2\coth^{-1}(ax)}\right)\right) + 3\,\text{PolyLo}}{2a^2}$$

input
```
Integrate[x*ArcCoth[a*x]^3,x]
```

output
```
(ArcCoth[a*x]*(3*(-1 + a*x)*ArcCoth[a*x] + (-1 + a^2*x^2)*ArcCoth[a*x]^2 -
 6*Log[1 - E^(-2*ArcCoth[a*x])]) + 3*PolyLog[2, E^(-2*ArcCoth[a*x])])/(2*a
^2)
```

Rubi [A] (verified)

Time = 0.84 (sec) , antiderivative size = 109, normalized size of antiderivative = 1.15, number of steps used = 9, number of rules used = 8, $\frac{\text{number of rules}}{\text{integrand size}}$ = 1.000, Rules used = {6453, 6543, 6437, 6511, 6547, 6471, 2849, 2752}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x \coth^{-1}(ax)^3 \, dx$$

$$\Big\downarrow \text{6453}$$

$$\frac{1}{2}x^2 \coth^{-1}(ax)^3 - \frac{3}{2}a \int \frac{x^2 \coth^{-1}(ax)^2}{1-a^2x^2}dx$$

$$\Big\downarrow \text{6543}$$

$$\frac{1}{2}x^2 \coth^{-1}(ax)^3 - \frac{3}{2}a \left(\frac{\int \frac{\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2} - \frac{\int \coth^{-1}(ax)^2dx}{a^2}\right)$$

$$\Big\downarrow \text{6437}$$

$$\frac{1}{2}x^2 \coth^{-1}(ax)^3 - \frac{3}{2}a \left(\frac{\int \frac{\coth^{-1}(ax)^2}{1-a^2x^2}dx}{a^2} - \frac{x \coth^{-1}(ax)^2 - 2a \int \frac{x \coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}\right)$$

$$\Big\downarrow \text{6511}$$

$$\frac{1}{2}x^2 \coth^{-1}(ax)^3 - \frac{3}{2}a \left(\frac{\coth^{-1}(ax)^3}{3a^3} - \frac{x \coth^{-1}(ax)^2 - 2a \int \frac{x \coth^{-1}(ax)}{1-a^2x^2}dx}{a^2}\right)$$

$$\Big\downarrow \text{6547}$$

$$\frac{1}{2}x^2 \coth^{-1}(ax)^3 - \frac{3}{2}a \left(\frac{\coth^{-1}(ax)^3}{3a^3} - \frac{x \coth^{-1}(ax)^2 - 2a \left(\frac{\int \frac{\coth^{-1}(ax)}{1-ax}dx}{a} - \frac{\coth^{-1}(ax)^2}{2a^2}\right)}{a^2}\right)$$

$$\Big\downarrow \ 6471$$

$$\frac{1}{2}x^2\coth^{-1}(ax)^3-\frac{3}{2}a\left(\frac{\coth^{-1}(ax)^3}{3a^3}-\frac{x\coth^{-1}(ax)^2-2a\left(\frac{\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)}{a}-\int\frac{\log\left(\frac{2}{1-ax}\right)}{1-a^2x^2}dx}{a}-\frac{\coth^{-1}(ax)^2}{2a^2}\right)}{a^2}\right)$$

$$\Big\downarrow \ 2849$$

$$\frac{1}{2}x^2\coth^{-1}(ax)^3-\frac{3}{2}a\left(\frac{\coth^{-1}(ax)^3}{3a^3}-\frac{x\coth^{-1}(ax)^2-2a\left(\frac{\frac{\int\frac{\log\left(\frac{2}{1-ax}\right)}{1-\frac{2}{1-ax}}d\frac{1}{1-ax}}{a}+\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)}{a}}{a}-\frac{\coth^{-1}(ax)^2}{2a^2}\right)}{a^2}\right)$$

$$\Big\downarrow \ 2752$$

$$\frac{1}{2}x^2\coth^{-1}(ax)^3-\frac{3}{2}a\left(\frac{\coth^{-1}(ax)^3}{3a^3}-\frac{x\coth^{-1}(ax)^2-2a\left(\frac{\frac{\text{PolyLog}\left(2,1-\frac{2}{1-ax}\right)}{2a}+\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)}{a}}{a}-\frac{\coth^{-1}(ax)^2}{2a^2}\right)}{a^2}\right)$$

input
```
Int[x*ArcCoth[a*x]^3,x]
```

output
```
(x^2*ArcCoth[a*x]^3)/2 - (3*a*(ArcCoth[a*x]^3/(3*a^3) - (x*ArcCoth[a*x]^2
 - 2*a*(-1/2*ArcCoth[a*x]^2/a^2 + ((ArcCoth[a*x]*Log[2/(1 - a*x)])/a + Poly
Log[2, 1 - 2/(1 - a*x)]/(2*a))/a))/a^2))/2
```

Defintions of rubi rules used

rule 2752
```
Int[Log[(c_.)*(x_)]/((d_) + (e_.)*(x_)), x_Symbol] :> Simp[(-e^(-1))*PolyLo
g[2, 1 - c*x], x] /; FreeQ[{c, d, e}, x] && EqQ[e + c*d, 0]
```

rule 2849
```
Int[Log[(c_.)/((d_) + (e_.)*(x_))]/((f_) + (g_.)*(x_)^2), x_Symbol] :> Simp
[-e/g   Subst[Int[Log[2*d*x]/(1 - 2*d*x), x], x, 1/(d + e*x)], x] /; FreeQ[
{c, d, e, f, g}, x] && EqQ[c, 2*d] && EqQ[e^2*f + d^2*g, 0]
```

rule 6437
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.), x_Symbol] :> Simp[x*(a
 + b*ArcCoth[c*x^n])^p, x] - Simp[b*c*n*p   Int[x^n*((a + b*ArcCoth[c*x^n])
^(p - 1)/(1 - c^2*x^(2*n))), x], x] /; FreeQ[{a, b, c, n}, x] && IGtQ[p, 0]
 && (EqQ[n, 1] || EqQ[p, 1])
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

rule 6471
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((d_) + (e_.)*(x_)), x_Symbol
] :> Simp[(-(a + b*ArcCoth[c*x])^p)*(Log[2/(1 + e*(x/d))]/e), x] + Simp[b*c
*(p/e)   Int[(a + b*ArcCoth[c*x])^(p - 1)*(Log[2/(1 + e*(x/d))]/(1 - c^2*x^
2)), x], x] /; FreeQ[{a, b, c, d, e}, x] && IGtQ[p, 0] && EqQ[c^2*d^2 - e^2
, 0]
```

rule 6511
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((d_) + (e_.)*(x_)^2), x_Symb
ol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*c*d*(p + 1)), x] /; FreeQ[{a, b
, c, d, e, p}, x] && EqQ[c^2*d + e, 0] && NeQ[p, -1]
```

rule 6543
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*((f_.)*(x_))^(m_))/((d_) + (
e_.)*(x_)^2), x_Symbol] :> Simp[f^2/e   Int[(f*x)^(m - 2)*(a + b*ArcCoth[c*
x])^p, x], x] - Simp[d*(f^2/e)   Int[(f*x)^(m - 2)*((a + b*ArcCoth[c*x])^p/
(d + e*x^2)), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && GtQ[p, 0] && GtQ[m,
 1]
```

rule 6547
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*(x_))/((d_) + (e_.)*(x_)^2),
 x_Symbol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*e*(p + 1)), x] + Simp[1/
(c*d)   Int[(a + b*ArcCoth[c*x])^p/(1 - c*x), x], x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[c^2*d + e, 0] && IGtQ[p, 0]
```

Maple [C] (warning: unable to verify)

Result contains higher order function than in optimal. Order 9 vs. order 4.

Time = 3.55 (sec) , antiderivative size = 2910, normalized size of antiderivative = 30.63

method	result	size
parts	Expression too large to display	2910
derivativedivides	Expression too large to display	2916
default	Expression too large to display	2916

input
```
int(x*arccoth(x*a)^3,x,method=_RETURNVERBOSE)
```

output
```
1/2*x^2*arccoth(x*a)^3+3/2/a^2*(-2*arccoth(x*a)*ln(1+1/((a*x-1)/(a*x+1))^(
1/2))-arccoth(x*a)*ln(1-1/((a*x-1)/(a*x+1))^(1/2))-polylog(2,1/((a*x-1)/(a
*x+1))^(1/2))-polylog(2,-1/((a*x-1)/(a*x+1))^(1/2))-1/4*I*Pi*csgn(I*(a*x+1
)/(a*x-1))*csgn(I/(a*x-1)*(a*x+1)/((a*x+1)/(a*x-1)-1))^2*arccoth(x*a)*ln(1
-1/((a*x-1)/(a*x+1))^(1/2))+1/4*I*Pi*csgn(I/((a*x+1)/(a*x-1)-1))*csgn(I*(a
*x+1)/(a*x-1))*csgn(I/(a*x-1)*(a*x+1)/((a*x+1)/(a*x-1)-1))*dilog(1/((a*x-1
)/(a*x+1))^(1/2))-1/4*I*Pi*csgn(I/((a*x+1)/(a*x-1)-1))*csgn(I*(a*x+1)/(a*x
-1))*csgn(I/(a*x-1)*(a*x+1)/((a*x+1)/(a*x-1)-1))*dilog(1+1/((a*x-1)/(a*x+1
))^(1/2))+1/4*I*Pi*csgn(I/((a*x+1)/(a*x-1)-1))*csgn(I*(a*x+1)/(a*x-1))*csg
n(I/(a*x-1)*(a*x+1)/((a*x+1)/(a*x-1)-1))*polylog(2,-1/((a*x-1)/(a*x+1))^(1
/2))+1/4*I*Pi*csgn(I/((a*x+1)/(a*x-1)-1))*csgn(I*(a*x+1)/(a*x-1))*csgn(I/(
a*x-1)*(a*x+1)/((a*x+1)/(a*x-1)-1))*polylog(2,1/((a*x-1)/(a*x+1))^(1/2))-1
/4*I*Pi*csgn(I/((a*x+1)/(a*x-1)-1))*csgn(I/(a*x-1)*(a*x+1)/((a*x+1)/(a*x-1
)-1))^2*arccoth(x*a)*ln(1-1/((a*x-1)/(a*x+1))^(1/2))-1/2*I*Pi*csgn(I/((a*x
-1)/(a*x+1))^(1/2))*csgn(I*(a*x+1)/(a*x-1))^2*arccoth(x*a)*ln(1-1/((a*x-1)
/(a*x+1))^(1/2))+1/4*I*Pi*csgn(I/((a*x-1)/(a*x+1))^(1/2))^2*csgn(I*(a*x+1)
/(a*x-1))*arccoth(x*a)*ln(1-1/((a*x-1)/(a*x+1))^(1/2))+1/4*I*Pi*csgn(I*(a*
x+1)/(a*x-1))^3*dilog(1/((a*x-1)/(a*x+1))^(1/2))-1/4*I*Pi*csgn(I*(a*x+1)/(
a*x-1))^3*dilog(1+1/((a*x-1)/(a*x+1))^(1/2))+1/4*I*Pi*csgn(I*(a*x+1)/(a*x-
1))^3*polylog(2,-1/((a*x-1)/(a*x+1))^(1/2))+1/4*I*Pi*csgn(I*(a*x+1)/(a*...
```

Fricas [F]

$$\int x\coth^{-1}(ax)^3\,dx=\int x\,\mathrm{arcoth}\,(ax)^3\;dx$$

input
```
integrate(x*arccoth(a*x)^3,x, algorithm="fricas")
```

output
```
integral(x*arccoth(a*x)^3, x)
```

Sympy [F]

$$\int x \coth^{-1}(ax)^3\,dx = \int x \operatorname{acoth}^3(ax)\,dx$$

input
```
integrate(x*acoth(a*x)**3,x)
```

output
```
Integral(x*acoth(a*x)**3, x)
```

Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 215 vs. $2(82) = 164$.

Time = 0.04 (sec) , antiderivative size = 215, normalized size of antiderivative = 2.26

$$\int x \coth^{-1}(ax)^3\,dx$$

$$= \frac{1}{2}x^2 \operatorname{arcoth}(ax)^3 + \frac{3}{4}a\left(\frac{2x}{a^2} - \frac{\log(ax+1)}{a^3} + \frac{\log(ax-1)}{a^3}\right)\operatorname{arcoth}(ax)^2$$

$$+ \frac{1}{16}a\left(\frac{\frac{3(\log(ax-1)-2)\log(ax+1)^2 - \log(ax+1)^3 + \log(ax-1)^3 - 3\left(\log(ax-1)^2 - 4\log(ax-1)\right)\log(ax+1) + 6\log(ax-1)^2}{a} - \frac{24(\log(ax-}{}}{a^2}\right.$$

input
```
integrate(x*arccoth(a*x)^3,x, algorithm="maxima")
```

output
```
1/2*x^2*arccoth(a*x)^3 + 3/4*a*(2*x/a^2 - log(a*x + 1)/a^3 + log(a*x - 1)/
a^3)*arccoth(a*x)^2 + 1/16*a*(((3*(log(a*x - 1) - 2)*log(a*x + 1)^2 - log(
a*x + 1)^3 + log(a*x - 1)^3 - 3*(log(a*x - 1)^2 - 4*log(a*x - 1))*log(a*x
+ 1) + 6*log(a*x - 1)^2)/a - 24*(log(a*x - 1)*log(1/2*a*x + 1/2) + dilog(-
1/2*a*x + 1/2))/a)/a^2 - 6*(2*(log(a*x - 1) - 2)*log(a*x + 1) - log(a*x +
1)^2 - log(a*x - 1)^2 - 4*log(a*x - 1))*arccoth(a*x)/a^3)
```

Giac [F]

$$\int x \coth^{-1}(ax)^3 \, dx = \int x \operatorname{arcoth}(ax)^3 \, dx$$

input
```
integrate(x*arccoth(a*x)^3,x, algorithm="giac")
```

output
```
integrate(x*arccoth(a*x)^3, x)
```

Mupad [F(-1)]

Timed out.

$$\int x \coth^{-1}(ax)^3 \, dx = \int x \operatorname{acoth}(a\,x)^3 \, dx$$

input
```
int(x*acoth(a*x)^3,x)
```

output
```
int(x*acoth(a*x)^3, x)
```

Reduce [F]

$$\int x \coth^{-1}(ax)^3 \, dx$$
$$= \frac{acoth(ax)^3 a^2x^2 - acoth(ax)^3 - 3acoth(ax)^2 ax - 3acoth(ax)\, a^2x^2 + 3acoth(ax) + 6\left(\int \frac{acoth(ax)x^3}{a^2x^2-1}dx\right) a^4}{2a^2}$$

input
```
int(x*acoth(a*x)^3,x)
```

output
```
(acoth(a*x)**3*a**2*x**2 - acoth(a*x)**3 - 3*acoth(a*x)**2*a*x - 3*acoth(a
*x)*a**2*x**2 + 3*acoth(a*x) + 6*int((acoth(a*x)*x**3)/(a**2*x**2 - 1),x)*
a**4 + 3*a*x)/(2*a**2)
```

3.28 $\int \coth^{-1}(ax)^3\,dx$

Optimal result

Integrand size = 6, antiderivative size = 85

$$\int \coth^{-1}(ax)^3\,dx = \frac{\coth^{-1}(ax)^3}{a} + x\coth^{-1}(ax)^3 - \frac{3\coth^{-1}(ax)^2\log\left(\frac{2}{1-ax}\right)}{a} - \frac{3\coth^{-1}(ax)\,\text{PolyLog}\left(2,1-\frac{2}{1-ax}\right)}{a} + \frac{3\,\text{PolyLog}\left(3,1-\frac{2}{1-ax}\right)}{2a}$$

output
```
arccoth(a*x)^3/a+x*arccoth(a*x)^3-3*arccoth(a*x)^2*ln(2/(-a*x+1))/a-3*arcc
oth(a*x)*polylog(2,1-2/(-a*x+1))/a+3/2*polylog(3,1-2/(-a*x+1))/a
```

Mathematica [A] (verified)

Time = 0.08 (sec) , antiderivative size = 79, normalized size of antiderivative = 0.93

$$\int \coth^{-1}(ax)^3\,dx = \frac{\coth^{-1}(ax)^3}{a} + x\coth^{-1}(ax)^3 - \frac{3\coth^{-1}(ax)^2\log\left(1-e^{2\coth^{-1}(ax)}\right)}{a} - \frac{3\coth^{-1}(ax)\,\text{PolyLog}\left(2,e^{2\coth^{-1}(ax)}\right)}{a} + \frac{3\,\text{PolyLog}\left(3,e^{2\coth^{-1}(ax)}\right)}{2a}$$

input
```
Integrate[ArcCoth[a*x]^3,x]
```

output
```
ArcCoth[a*x]^3/a + x*ArcCoth[a*x]^3 - (3*ArcCoth[a*x]^2*Log[1 - E^(2*ArcCo
th[a*x])])/a - (3*ArcCoth[a*x]*PolyLog[2, E^(2*ArcCoth[a*x])])/a + (3*Poly
Log[3, E^(2*ArcCoth[a*x])])/(2*a)
```

Rubi [A] (verified)

Time = 0.65 (sec) , antiderivative size = 101, normalized size of antiderivative = 1.19, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.833$, Rules used = {6437, 6547, 6471, 6621, 7164}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \coth^{-1}(ax)^3 \, dx$$

$$\Big\downarrow \text{ 6437}$$

$$x\coth^{-1}(ax)^3-3a\int \frac{x\coth^{-1}(ax)^2}{1-a^2x^2}dx$$

$$\Big\downarrow \text{ 6547}$$

$$x\coth^{-1}(ax)^3-3a\left(\frac{\int \frac{\coth^{-1}(ax)^2}{1-ax}dx}{a}-\frac{\coth^{-1}(ax)^3}{3a^2}\right)$$

$$\Big\downarrow \text{ 6471}$$

$$x\coth^{-1}(ax)^3-3a\left(\frac{\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)^2}{a}-2\int \frac{\coth^{-1}(ax)\log\left(\frac{2}{1-ax}\right)}{1-a^2x^2}dx}{a}-\frac{\coth^{-1}(ax)^3}{3a^2}\right)$$

$$\Big\downarrow \text{ 6621}$$

$$x\coth^{-1}(ax)^3-3a\left(\frac{\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)^2}{a}-2\left(\frac{1}{2}\int \frac{\text{PolyLog}\left(2,1-\frac{2}{1-ax}\right)}{1-a^2x^2}dx-\frac{\text{PolyLog}\left(2,1-\frac{2}{1-ax}\right)\coth^{-1}(ax)}{2a}\right)}{a}-\frac{\coth^{-1}(ax)^3}{3a^2}\right)$$

$$\Big\downarrow \text{7164}$$

$$x\coth^{-1}(ax)^3-3a\left(\frac{\frac{\log\left(\frac{2}{1-ax}\right)\coth^{-1}(ax)^2}{a}-2\left(\frac{\text{PolyLog}\left(3,1-\frac{2}{1-ax}\right)}{4a}-\frac{\text{PolyLog}\left(2,1-\frac{2}{1-ax}\right)\coth^{-1}(ax)}{2a}\right)}{a}-\frac{\coth^{-1}(ax)^3}{3a^2}\right)$$

input
```
Int[ArcCoth[a*x]^3,x]
```

output
```
x*ArcCoth[a*x]^3 - 3*a*(-1/3*ArcCoth[a*x]^3/a^2 + ((ArcCoth[a*x]^2*Log[2/(
1 - a*x)])/a - 2*(-1/2*(ArcCoth[a*x]*PolyLog[2, 1 - 2/(1 - a*x)])/a + Poly
Log[3, 1 - 2/(1 - a*x)]/(4*a)))/a)
```

Defintions of rubi rules used

rule 6437
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.), x_Symbol] :> Simp[x*(a
 + b*ArcCoth[c*x^n])^p, x] - Simp[b*c*n*p   Int[x^n*((a + b*ArcCoth[c*x^n])
^(p - 1)/(1 - c^2*x^(2*n))), x], x] /; FreeQ[{a, b, c, n}, x] && IGtQ[p, 0]
 && (EqQ[n, 1] || EqQ[p, 1])
```

rule 6471
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((d_) + (e_.)*(x_)), x_Symbol
] :> Simp[(-(a + b*ArcCoth[c*x])^p)*(Log[2/(1 + e*(x/d))]/e), x] + Simp[b*c
*(p/e)   Int[(a + b*ArcCoth[c*x])^(p - 1)*(Log[2/(1 + e*(x/d))]/(1 - c^2*x^
2)), x], x] /; FreeQ[{a, b, c, d, e}, x] && IGtQ[p, 0] && EqQ[c^2*d^2 - e^2
, 0]
```

rule 6547
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*(x_))/((d_) + (e_.)*(x_)^2),
 x_Symbol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*e*(p + 1)), x] + Simp[1/
(c*d)   Int[(a + b*ArcCoth[c*x])^p/(1 - c*x), x], x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[c^2*d + e, 0] && IGtQ[p, 0]
```

rule 6621
```
Int[(Log[u_]*((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.))/((d_) + (e_.)*(x_)^
2), x_Symbol] :> Simp[(-(a + b*ArcCoth[c*x])^p)*(PolyLog[2, 1 - u]/(2*c*d))
, x] + Simp[b*(p/2)   Int[(a + b*ArcCoth[c*x])^(p - 1)*(PolyLog[2, 1 - u]/(
d + e*x^2)), x], x] /; FreeQ[{a, b, c, d, e}, x] && IGtQ[p, 0] && EqQ[c^2*d
 + e, 0] && EqQ[(1 - u)^2 - (1 - 2/(1 - c*x))^2, 0]
```

rule 7164
```
Int[(u_)*PolyLog[n_, v_], x_Symbol] :> With[{w = DerivativeDivides[v, u*v,
x]}, Simp[w*PolyLog[n + 1, v], x] /;  !FalseQ[w]] /; FreeQ[n, x]
```

Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 167 vs. $2(83) = 166$.

Time = 0.31 (sec) , antiderivative size = 168, normalized size of antiderivative = 1.98

method	result
derivativedivides	$\operatorname{arccoth}(xa)^3(xa-1)+2\operatorname{arccoth}(xa)^3-3\operatorname{arccoth}(xa)^2\ln\left(1+\frac{1}{\sqrt{\frac{xa-1}{xa+1}}}\right)-6\operatorname{arccoth}(xa)\operatorname{polylog}\left(2,-\frac{1}{\sqrt{\frac{xa-1}{xa+1}}}\right)+6\,\mathrm{po}$
default	$\operatorname{arccoth}(xa)^3(xa-1)+2\operatorname{arccoth}(xa)^3-3\operatorname{arccoth}(xa)^2\ln\left(1+\frac{1}{\sqrt{\frac{xa-1}{xa+1}}}\right)-6\operatorname{arccoth}(xa)\operatorname{polylog}\left(2,-\frac{1}{\sqrt{\frac{xa-1}{xa+1}}}\right)+6\,\mathrm{po}$

input
```
int(arccoth(x*a)^3,x,method=_RETURNVERBOSE)
```

output
```
1/a*(arccoth(x*a)^3*(a*x-1)+2*arccoth(x*a)^3-3*arccoth(x*a)^2*ln(1+1/((a*x
-1)/(a*x+1))^(1/2))-6*arccoth(x*a)*polylog(2,-1/((a*x-1)/(a*x+1))^(1/2))+6
*polylog(3,-1/((a*x-1)/(a*x+1))^(1/2))-3*arccoth(x*a)^2*ln(1-1/((a*x-1)/(a
*x+1))^(1/2))-6*arccoth(x*a)*polylog(2,1/((a*x-1)/(a*x+1))^(1/2))+6*polylo
g(3,1/((a*x-1)/(a*x+1))^(1/2)))
```

Fricas [F]

$$\int \coth^{-1}(ax)^3\,dx = \int \operatorname{arcoth}\left(ax\right)^3\,dx$$

input
```
integrate(arccoth(a*x)^3,x, algorithm="fricas")
```

output
```
integral(arccoth(a*x)^3, x)
```

Sympy [F]

$$\int \coth^{-1}(ax)^3\,dx = \int \operatorname{acoth}^3\left(ax\right)dx$$

input
```
integrate(acoth(a*x)**3,x)
```

output
```
Integral(acoth(a*x)**3, x)
```

Maxima [F]

$$\int \coth^{-1}(ax)^3\,dx = \int \operatorname{arcoth}\left(ax\right)^3\,dx$$

input
```
integrate(arccoth(a*x)^3,x, algorithm="maxima")
```

output
```
1/8*((a*x + 1)*log(a*x + 1)^3 - 3*(a*x - 1)*log(a*x + 1)^2*log(a*x - 1))/a
 + 1/8*integrate(-((a*x + 1)*log(a*x - 1)^3 - 3*((a*x + 1)*log(a*x - 1)^2
+ 2*(a*x - 1)*log(a*x - 1))*log(a*x + 1))/(a*x + 1), x)
```

Giac [F]

$$\int \coth^{-1}(ax)^3\,dx = \int \operatorname{arcoth}(ax)^3\,dx$$

input
```
integrate(arccoth(a*x)^3,x, algorithm="giac")
```

output
```
integrate(arccoth(a*x)^3, x)
```

Mupad [F(-1)]

Timed out.

$$\int \coth^{-1}(ax)^3\,dx = \int \operatorname{acoth}(a\,x)^3\,dx$$

input
```
int(acoth(a*x)^3,x)
```

output
```
int(acoth(a*x)^3, x)
```

Reduce [F]

$$\int \coth^{-1}(ax)^3\,dx = \int acoth(ax)^3\,dx$$

input
```
int(acoth(a*x)^3,x)
```

output
```
int(acoth(a*x)**3,x)
```

3.29 $\int \frac{\coth^{-1}(ax)^3}{x}\,dx$

Optimal result

Integrand size = 10, antiderivative size = 150

$$\begin{aligned}\int \frac{\coth^{-1}(ax)^3}{x}\,dx &= 2\coth^{-1}(ax)^3\coth^{-1}\left(1-\frac{2}{1-ax}\right)\\&\quad+\frac{3}{2}\coth^{-1}(ax)^2\operatorname{PolyLog}\left(2,1-\frac{2}{1+ax}\right)\\&\quad-\frac{3}{2}\coth^{-1}(ax)^2\operatorname{PolyLog}\left(2,1-\frac{2ax}{1+ax}\right)\\&\quad+\frac{3}{2}\coth^{-1}(ax)\operatorname{PolyLog}\left(3,1-\frac{2}{1+ax}\right)\\&\quad-\frac{3}{2}\coth^{-1}(ax)\operatorname{PolyLog}\left(3,1-\frac{2ax}{1+ax}\right)\\&\quad+\frac{3}{4}\operatorname{PolyLog}\left(4,1-\frac{2}{1+ax}\right)-\frac{3}{4}\operatorname{PolyLog}\left(4,1-\frac{2ax}{1+ax}\right)\end{aligned}$$

output
```
2*arccoth(a*x)^3*arccoth(1-2/(-a*x+1))+3/2*arccoth(a*x)^2*polylog(2,1-2/(a
*x+1))-3/2*arccoth(a*x)^2*polylog(2,1-2*a*x/(a*x+1))+3/2*arccoth(a*x)*poly
log(3,1-2/(a*x+1))-3/2*arccoth(a*x)*polylog(3,1-2*a*x/(a*x+1))+3/4*polylog
(4,1-2/(a*x+1))-3/4*polylog(4,1-2*a*x/(a*x+1))
```

Mathematica [A] (verified)

Time = 0.05 (sec) , antiderivative size = 156, normalized size of antiderivative = 1.04

$$\int \frac{\coth^{-1}(ax)^3}{x}\,dx = \frac{1}{64}\Big(-\pi^4 + 32\coth^{-1}(ax)^4 + 64\coth^{-1}(ax)^3\log\left(1+e^{-2\coth^{-1}(ax)}\right) - 64\coth^{-1}(ax)^3\log\left(1-e^{2\coth^{-1}(ax)}\right) - 96\coth^{-1}(ax)^2\,\mathrm{PolyLog}\left(2,-e^{-2\coth^{-1}(ax)}\right) - 96\coth^{-1}(ax)^2\,\mathrm{PolyLog}\left(2,e^{2\coth^{-1}(ax)}\right) - 96\coth^{-1}(ax)\,\mathrm{PolyLog}\left(3,-e^{-2\coth^{-1}(ax)}\right) + 96\coth^{-1}(ax)\,\mathrm{PolyLog}\left(3,e^{2\coth^{-1}(ax)}\right) - 48\,\mathrm{PolyLog}\left(4,-e^{-2\coth^{-1}(ax)}\right) - 48\,\mathrm{PolyLog}\left(4,e^{2\coth^{-1}(ax)}\right)\Big)$$

input
```
Integrate[ArcCoth[a*x]^3/x,x]
```

output
```
(-Pi^4 + 32*ArcCoth[a*x]^4 + 64*ArcCoth[a*x]^3*Log[1 + E^(-2*ArcCoth[a*x])
] - 64*ArcCoth[a*x]^3*Log[1 - E^(2*ArcCoth[a*x])] - 96*ArcCoth[a*x]^2*Poly
Log[2, -E^(-2*ArcCoth[a*x])] - 96*ArcCoth[a*x]^2*PolyLog[2, E^(2*ArcCoth[a
*x])] - 96*ArcCoth[a*x]*PolyLog[3, -E^(-2*ArcCoth[a*x])] + 96*ArcCoth[a*x]
*PolyLog[3, E^(2*ArcCoth[a*x])] - 48*PolyLog[4, -E^(-2*ArcCoth[a*x])] - 48
*PolyLog[4, E^(2*ArcCoth[a*x])])/64
```

Rubi [A] (verified)

Time = 1.05 (sec) , antiderivative size = 182, normalized size of antiderivative = 1.21, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$, Rules used = {6449, 6615, 6619, 6623, 7164}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\coth^{-1}(ax)^3}{x}\,dx$$

$$\Big\downarrow \text{6449}$$

$$2\coth^{-1}(ax)^3\coth^{-1}\left(1-\frac{2}{1-ax}\right)-6a\int\frac{\coth^{-1}(ax)^2\coth^{-1}\left(1-\frac{2}{1-ax}\right)}{1-a^2x^2}dx$$

$$\Big\downarrow \text{6615}$$

$$2\coth^{-1}(ax)^3\coth^{-1}\left(1-\frac{2}{1-ax}\right)-6a\left(\frac{1}{2}\int\frac{\coth^{-1}(ax)^2\log\left(\frac{2ax}{ax+1}\right)}{1-a^2x^2}dx-\frac{1}{2}\int\frac{\coth^{-1}(ax)^2\log\left(\frac{2}{ax+1}\right)}{1-a^2x^2}dx\right)$$

$$\Big\downarrow \text{6619}$$

$$2\coth^{-1}(ax)^3\coth^{-1}\left(1-\frac{2}{1-ax}\right)-6a\left(\frac{1}{2}\left(\int\frac{\coth^{-1}(ax)\,\text{PolyLog}\left(2,1-\frac{2}{ax+1}\right)}{1-a^2x^2}dx-\frac{\text{PolyLog}\left(2,1-\frac{2}{ax+1}\right)\coth^{-1}(ax)^2}{2a}\right)+\frac{1}{2}\left(\frac{\text{PolyLog}\left(2,1-\right.}{}\right.\right.$$

$$\Big\downarrow \text{6623}$$

$$2\coth^{-1}(ax)^3\coth^{-1}\left(1-\frac{2}{1-ax}\right)-6a\left(\frac{1}{2}\left(\frac{1}{2}\int\frac{\text{PolyLog}\left(3,1-\frac{2}{ax+1}\right)}{1-a^2x^2}dx-\frac{\text{PolyLog}\left(2,1-\frac{2}{ax+1}\right)\coth^{-1}(ax)^2}{2a}-\frac{\text{PolyLog}\left(3,1-\frac{2}{ax+1}\right)\coth^{-1}(\right.}{2a}\right.\right.$$

$$\Big\downarrow \text{7164}$$

$$2\coth^{-1}(ax)^3\coth^{-1}\left(1-\frac{2}{1-ax}\right)-6a\left(\frac{1}{2}\left(-\frac{\text{PolyLog}\left(4,1-\frac{2}{ax+1}\right)}{4a}-\frac{\text{PolyLog}\left(2,1-\frac{2}{ax+1}\right)\coth^{-1}(ax)^2}{2a}-\frac{\text{PolyLog}\left(3,1-\frac{2}{ax+1}\right)\coth^{-1}(ax)}{2a}\right)\right.$$

input

```
Int[ArcCoth[a*x]^3/x,x]
```

output

```
2*ArcCoth[a*x]^3*ArcCoth[1 - 2/(1 - a*x)] - 6*a*((-1/2*(ArcCoth[a*x]^2*Pol
yLog[2, 1 - 2/(1 + a*x)])/a - (ArcCoth[a*x]*PolyLog[3, 1 - 2/(1 + a*x)])/(
2*a) - PolyLog[4, 1 - 2/(1 + a*x)]/(4*a))/2 + ((ArcCoth[a*x]^2*PolyLog[2,
1 - (2*a*x)/(1 + a*x)])/(2*a) + (ArcCoth[a*x]*PolyLog[3, 1 - (2*a*x)/(1 +
a*x)])/(2*a) + PolyLog[4, 1 - (2*a*x)/(1 + a*x)]/(4*a))/2)
```

Defintions of rubi rules used

rule 6449
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_)/(x_), x_Symbol] :> Simp[2*(a +
 b*ArcCoth[c*x])^p*ArcCoth[1 - 2/(1 - c*x)], x] - Simp[2*b*c*p   Int[(a + b
*ArcCoth[c*x])^(p - 1)*(ArcCoth[1 - 2/(1 - c*x)]/(1 - c^2*x^2)), x], x] /;
FreeQ[{a, b, c}, x] && IGtQ[p, 1]
```

rule 6615
```
Int[(ArcCoth[u_]*((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.))/((d_) + (e_.)*(
x_)^2), x_Symbol] :> Simp[1/2   Int[Log[SimplifyIntegrand[1 + 1/u, x]]*((a
+ b*ArcCoth[c*x])^p/(d + e*x^2)), x], x] - Simp[1/2   Int[Log[SimplifyInteg
rand[1 - 1/u, x]]*((a + b*ArcCoth[c*x])^p/(d + e*x^2)), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && IGtQ[p, 0] && EqQ[c^2*d + e, 0] && EqQ[u^2 - (1 - 2/(1
- c*x))^2, 0]
```

rule 6619
```
Int[(Log[u_]*((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.))/((d_) + (e_.)*(x_)^
2), x_Symbol] :> Simp[(a + b*ArcCoth[c*x])^p*(PolyLog[2, 1 - u]/(2*c*d)), x
] - Simp[b*(p/2)   Int[(a + b*ArcCoth[c*x])^(p - 1)*(PolyLog[2, 1 - u]/(d +
 e*x^2)), x], x] /; FreeQ[{a, b, c, d, e}, x] && IGtQ[p, 0] && EqQ[c^2*d +
e, 0] && EqQ[(1 - u)^2 - (1 - 2/(1 + c*x))^2, 0]
```

rule 6623
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*PolyLog[k_, u_])/((d_) + (e_
.)*(x_)^2), x_Symbol] :> Simp[(-(a + b*ArcCoth[c*x])^p)*(PolyLog[k + 1, u]/
(2*c*d)), x] + Simp[b*(p/2)   Int[(a + b*ArcCoth[c*x])^(p - 1)*(PolyLog[k +
 1, u]/(d + e*x^2)), x], x] /; FreeQ[{a, b, c, d, e, k}, x] && IGtQ[p, 0] &
& EqQ[c^2*d + e, 0] && EqQ[u^2 - (1 - 2/(1 + c*x))^2, 0]
```

rule 7164
```
Int[(u_)*PolyLog[n_, v_], x_Symbol] :> With[{w = DerivativeDivides[v, u*v,
x]}, Simp[w*PolyLog[n + 1, v], x] /;  !FalseQ[w]] /; FreeQ[n, x]
```

Maple [C] (warning: unable to verify)

Result contains higher order function than in optimal. Order 9 vs. order 4.

Time = 3.96 (sec) , antiderivative size = 536, normalized size of antiderivative = 3.57

method	result
derivativedivides	$\ln(xa)\operatorname{arccoth}(xa)^3+\frac{i\pi\operatorname{csgn}\left(\frac{i\left(1+\frac{xa+1}{xa-1}\right)}{\frac{xa+1}{xa-1}-1}\right)\left(\operatorname{csgn}\left(\frac{i}{\frac{xa+1}{xa-1}-1}\right)\operatorname{csgn}\left(i\left(1+\frac{xa+1}{xa-1}\right)\right)-\operatorname{csgn}\left(\frac{i}{\frac{xa+1}{xa-1}-1}\right)\operatorname{csg}\cdots\right.}{\cdots}$
default	$\ln(xa)\operatorname{arccoth}(xa)^3+\frac{i\pi\operatorname{csgn}\left(\frac{i\left(1+\frac{xa+1}{xa-1}\right)}{\frac{xa+1}{xa-1}-1}\right)\left(\operatorname{csgn}\left(\frac{i}{\frac{xa+1}{xa-1}-1}\right)\operatorname{csgn}\left(i\left(1+\frac{xa+1}{xa-1}\right)\right)-\operatorname{csgn}\left(\frac{i}{\frac{xa+1}{xa-1}-1}\right)\operatorname{csg}\cdots\right.}{\cdots}$
parts	$\ln(x)\operatorname{arccoth}(xa)^3+3a\left(\frac{\left(i\pi\operatorname{csgn}\left(\frac{i}{a}\right)\operatorname{csgn}\left(\frac{i\left(1+\frac{xa+1}{xa-1}\right)}{\frac{xa+1}{xa-1}-1}\right)\operatorname{csgn}\left(\frac{i\left(1+\frac{xa+1}{xa-1}\right)}{a\left(\frac{xa+1}{xa-1}-1\right)}\right)-i\pi\operatorname{csgn}\left(\frac{i}{a}\right)\operatorname{csgn}\left(\frac{i(1+\cdots}{a\left(\frac{xa}{xa}\cdots\right.}\right.\right.}{\cdots}\right.$

input
```
int(arccoth(x*a)^3/x,x,method=_RETURNVERBOSE)
```

output
```
ln(x*a)*arccoth(x*a)^3+1/2*I*Pi*csgn(I/((a*x+1)/(a*x-1)-1)*(1+(a*x+1)/(a*x
-1)))*(csgn(I/((a*x+1)/(a*x-1)-1))*csgn(I*(1+(a*x+1)/(a*x-1)))-csgn(I/((a*
x+1)/(a*x-1)-1))*csgn(I/((a*x+1)/(a*x-1)-1)*(1+(a*x+1)/(a*x-1)))-csgn(I/((
a*x+1)/(a*x-1)-1)*(1+(a*x+1)/(a*x-1)))*csgn(I*(1+(a*x+1)/(a*x-1)))+csgn(I/
((a*x+1)/(a*x-1)-1)*(1+(a*x+1)/(a*x-1)))^2)*arccoth(x*a)^3+arccoth(x*a)^3*
ln((a*x+1)/(a*x-1)-1)-arccoth(x*a)^3*ln(1-1/((a*x-1)/(a*x+1))^(1/2))-3*arc
coth(x*a)^2*polylog(2,1/((a*x-1)/(a*x+1))^(1/2))+6*arccoth(x*a)*polylog(3,
1/((a*x-1)/(a*x+1))^(1/2))-6*polylog(4,1/((a*x-1)/(a*x+1))^(1/2))-arccoth(
x*a)^3*ln(1+1/((a*x-1)/(a*x+1))^(1/2))-3*arccoth(x*a)^2*polylog(2,-1/((a*x
-1)/(a*x+1))^(1/2))+6*arccoth(x*a)*polylog(3,-1/((a*x-1)/(a*x+1))^(1/2))-6
*polylog(4,-1/((a*x-1)/(a*x+1))^(1/2))+3/2*arccoth(x*a)^2*polylog(2,-(a*x+
1)/(a*x-1))-3/2*arccoth(x*a)*polylog(3,-(a*x+1)/(a*x-1))+3/4*polylog(4,-(a
*x+1)/(a*x-1))
```

Fricas [F]

$$\int \frac{\coth^{-1}(ax)^3}{x} \, dx = \int \frac{\operatorname{arcoth}(ax)^3}{x} \, dx$$

input
```
integrate(arccoth(a*x)^3/x,x, algorithm="fricas")
```

output
```
integral(arccoth(a*x)^3/x, x)
```

Sympy [F]

$$\int \frac{\coth^{-1}(ax)^3}{x} \, dx = \int \frac{\operatorname{acoth}^3(ax)}{x} \, dx$$

input
```
integrate(acoth(a*x)**3/x,x)
```

output
```
Integral(acoth(a*x)**3/x, x)
```

Maxima [F]

$$\int \frac{\coth^{-1}(ax)^3}{x} \, dx = \int \frac{\operatorname{arcoth}(ax)^3}{x} \, dx$$

input
```
integrate(arccoth(a*x)^3/x,x, algorithm="maxima")
```

output
```
integrate(arccoth(a*x)^3/x, x)
```

Giac [F]

$$\int \frac{\coth^{-1}(ax)^3}{x} \, dx = \int \frac{\operatorname{arcoth}(ax)^3}{x} \, dx$$

input
```
integrate(arccoth(a*x)^3/x,x, algorithm="giac")
```

output
```
integrate(arccoth(a*x)^3/x, x)
```

Mupad [F(-1)]

Timed out.

$$\int \frac{\coth^{-1}(ax)^3}{x} \, dx = \int \frac{\operatorname{acoth}(a\,x)^3}{x} \, dx$$

input
```
int(acoth(a*x)^3/x,x)
```

output
```
int(acoth(a*x)^3/x, x)
```

Reduce [F]

$$\int \frac{\coth^{-1}(ax)^3}{x} \, dx = \int \frac{acoth(ax)^3}{x} dx$$

input
```
int(acoth(a*x)^3/x,x)
```

output
```
int(acoth(a*x)**3/x,x)
```

3.30 $\int \frac{\coth^{-1}(ax)^3}{x^2}\,dx$

Optimal result

Integrand size = 10, antiderivative size = 79

$$\int \frac{\coth^{-1}(ax)^3}{x^2}\,dx = a\coth^{-1}(ax)^3 - \frac{\coth^{-1}(ax)^3}{x} + 3a\coth^{-1}(ax)^2\log\left(2-\frac{2}{1+ax}\right) - 3a\coth^{-1}(ax)\,\mathrm{PolyLog}\left(2,-1+\frac{2}{1+ax}\right) - \frac{3}{2}a\,\mathrm{PolyLog}\left(3,-1+\frac{2}{1+ax}\right)$$

output
```
a*arccoth(a*x)^3-arccoth(a*x)^3/x+3*a*arccoth(a*x)^2*ln(2-2/(a*x+1))-3*a*a
rccoth(a*x)*polylog(2,-1+2/(a*x+1))-3/2*a*polylog(3,-1+2/(a*x+1))
```

Mathematica [A] (verified)

Time = 0.09 (sec) , antiderivative size = 72, normalized size of antiderivative = 0.91

$$\int \frac{\coth^{-1}(ax)^3}{x^2}\,dx = \frac{(-1+ax)\coth^{-1}(ax)^3}{x} + 3a\coth^{-1}(ax)^2\log\left(1+e^{-2\coth^{-1}(ax)}\right) - 3a\coth^{-1}(ax)\,\mathrm{PolyLog}\left(2,-e^{-2\coth^{-1}(ax)}\right) - \frac{3}{2}a\,\mathrm{PolyLog}\left(3,-e^{-2\coth^{-1}(ax)}\right)$$

input
```
Integrate[ArcCoth[a*x]^3/x^2,x]
```

output
```
((-1 + a*x)*ArcCoth[a*x]^3)/x + 3*a*ArcCoth[a*x]^2*Log[1 + E^(-2*ArcCoth[a
*x])] - 3*a*ArcCoth[a*x]*PolyLog[2, -E^(-2*ArcCoth[a*x])] - (3*a*PolyLog[3
, -E^(-2*ArcCoth[a*x])])/2
```

Rubi [A] (verified)

Time = 0.68 (sec) , antiderivative size = 93, normalized size of antiderivative = 1.18, number of steps used = 5, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$, Rules used = {6453, 6551, 6495, 6619, 7164}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\coth^{-1}(ax)^3}{x^2}dx$$

$$\Big\downarrow \text{6453}$$

$$3a\int \frac{\coth^{-1}(ax)^2}{x(1-a^2x^2)}dx-\frac{\coth^{-1}(ax)^3}{x}$$

$$\Big\downarrow \text{6551}$$

$$3a\left(\int \frac{\coth^{-1}(ax)^2}{x(ax+1)}dx+\frac{1}{3}\coth^{-1}(ax)^3\right)-\frac{\coth^{-1}(ax)^3}{x}$$

$$\Big\downarrow \text{6495}$$

$$3a\left(-2a\int \frac{\coth^{-1}(ax)\log\left(2-\frac{2}{ax+1}\right)}{1-a^2x^2}dx+\frac{1}{3}\coth^{-1}(ax)^3+\log\left(2-\frac{2}{ax+1}\right)\coth^{-1}(ax)^2\right)-\frac{\coth^{-1}(ax)^3}{x}$$

$$\Big\downarrow \text{6619}$$

$$3a\left(-2a\left(\frac{\text{PolyLog}\left(2,\frac{2}{ax+1}-1\right)\coth^{-1}(ax)}{2a}-\frac{1}{2}\int\frac{\text{PolyLog}\left(2,\frac{2}{ax+1}-1\right)}{1-a^2x^2}dx\right)+\frac{1}{3}\coth^{-1}(ax)^3+\log\left(2-\frac{}{a}\right.\right.$$
$$\frac{\coth^{-1}(ax)^3}{x}$$

$$\Big\downarrow \text{7164}$$

$$3a\left(-2a\left(\frac{\text{PolyLog}\left(3,\frac{2}{ax+1}-1\right)}{4a}+\frac{\text{PolyLog}\left(2,\frac{2}{ax+1}-1\right)\coth^{-1}(ax)}{2a}\right)+\frac{1}{3}\coth^{-1}(ax)^3+\log\left(2-\frac{2}{ax+1}\right)\right.$$
$$\frac{\coth^{-1}(ax)^3}{x}$$

input
```
Int[ArcCoth[a*x]^3/x^2,x]
```

output
```
-(ArcCoth[a*x]^3/x) + 3*a*(ArcCoth[a*x]^3/3 + ArcCoth[a*x]^2*Log[2 - 2/(1
+ a*x)] - 2*a*((ArcCoth[a*x]*PolyLog[2, -1 + 2/(1 + a*x)])/(2*a) + PolyLog
[3, -1 + 2/(1 + a*x)]/(4*a)))
```

Defintions of rubi rules used

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

rule 6495
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((x_)*((d_) + (e_.)*(x_))), x
_Symbol] :> Simp[(a + b*ArcCoth[c*x])^p*(Log[2 - 2/(1 + e*(x/d))]/d), x] -
Simp[b*c*(p/d)   Int[(a + b*ArcCoth[c*x])^(p - 1)*(Log[2 - 2/(1 + e*(x/d))]
/(1 - c^2*x^2)), x], x] /; FreeQ[{a, b, c, d, e}, x] && IGtQ[p, 0] && EqQ[c
^2*d^2 - e^2, 0]
```

rule 6551
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((x_)*((d_) + (e_.)*(x_)^2)),
 x_Symbol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*d*(p + 1)), x] + Simp[1/
d   Int[(a + b*ArcCoth[c*x])^p/(x*(1 + c*x)), x], x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[c^2*d + e, 0] && GtQ[p, 0]
```

rule 6619
```
Int[(Log[u_]*((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.))/((d_) + (e_.)*(x_)^
2), x_Symbol] :> Simp[(a + b*ArcCoth[c*x])^p*(PolyLog[2, 1 - u]/(2*c*d)), x
] - Simp[b*(p/2)   Int[(a + b*ArcCoth[c*x])^(p - 1)*(PolyLog[2, 1 - u]/(d +
 e*x^2)), x], x] /; FreeQ[{a, b, c, d, e}, x] && IGtQ[p, 0] && EqQ[c^2*d +
e, 0] && EqQ[(1 - u)^2 - (1 - 2/(1 + c*x))^2, 0]
```

rule 7164
```
Int[(u_)*PolyLog[n_, v_], x_Symbol] :> With[{w = DerivativeDivides[v, u*v,
x]}, Simp[w*PolyLog[n + 1, v], x] /;  !FalseQ[w]] /; FreeQ[n, x]
```

Maple [C] (warning: unable to verify)

Result contains higher order function than in optimal. Order 9 vs. order 4.

Time = 5.00 (sec) , antiderivative size = 718, normalized size of antiderivative = 9.09

method	result
parts	$-\frac{\operatorname{arccoth}(xa)^{3}}{x}-3a\left(\frac{\operatorname{arccoth}(xa)^{2}\ln(xa+1)}{2}-\ln(xa)\operatorname{arccoth}(xa)^{2}+\frac{\operatorname{arccoth}(xa)^{2}\ln(xa-1)}{2}+\right.$
derivativedivides	$a\left(-\frac{\operatorname{arccoth}(xa)^{3}}{xa}-\frac{3\operatorname{arccoth}(xa)^{2}\ln(xa+1)}{2}+3\ln(xa)\operatorname{arccoth}(xa)^{2}-\frac{3\operatorname{arccoth}(xa)^{2}\ln(xa-1)}{2}-\right.$
default	$a\left(-\frac{\operatorname{arccoth}(xa)^{3}}{xa}-\frac{3\operatorname{arccoth}(xa)^{2}\ln(xa+1)}{2}+3\ln(xa)\operatorname{arccoth}(xa)^{2}-\frac{3\operatorname{arccoth}(xa)^{2}\ln(xa-1)}{2}-\right.$

input
```
int(arccoth(x*a)^3/x^2,x,method=_RETURNVERBOSE)
```

output
```
-arccoth(x*a)^3/x-3*a*(1/2*arccoth(x*a)^2*ln(a*x+1)-ln(x*a)*arccoth(x*a)^2
+1/2*arccoth(x*a)^2*ln(a*x-1)+1/2*arccoth(x*a)^2*ln((a*x-1)/(a*x+1))+1/3*a
rccoth(x*a)^3-1/4*(-I*Pi*csgn(I/((a*x-1)/(a*x+1))^(1/2))^2*csgn(I*(a*x+1)/
(a*x-1))+2*I*Pi*csgn(I/((a*x+1)/(a*x-1)-1)*(1+(a*x+1)/(a*x-1)))^3+I*Pi*csg
n(I*(a*x+1)/(a*x-1))*csgn(I/(a*x-1)*(a*x+1)/((a*x+1)/(a*x-1)-1))^2-2*I*Pi*
csgn(I/((a*x+1)/(a*x-1)-1))*csgn(I/((a*x+1)/(a*x-1)-1)*(1+(a*x+1)/(a*x-1))
)^2-I*Pi*csgn(I/(a*x-1)*(a*x+1)/((a*x+1)/(a*x-1)-1))^3+I*Pi*csgn(I/((a*x+1
)/(a*x-1)-1))*csgn(I/(a*x-1)*(a*x+1)/((a*x+1)/(a*x-1)-1))^2-I*Pi*csgn(I*(a
*x+1)/(a*x-1))^3-2*I*Pi*csgn(I*(1+(a*x+1)/(a*x-1)))*csgn(I/((a*x+1)/(a*x-1
)-1)*(1+(a*x+1)/(a*x-1)))^2+2*I*Pi*csgn(I/((a*x-1)/(a*x+1))^(1/2))*csgn(I*
(a*x+1)/(a*x-1))^2+2*I*Pi*csgn(I/((a*x+1)/(a*x-1)-1))*csgn(I*(1+(a*x+1)/(a
*x-1)))*csgn(I/((a*x+1)/(a*x-1)-1)*(1+(a*x+1)/(a*x-1)))-I*Pi*csgn(I/((a*x+
1)/(a*x-1)-1))*csgn(I*(a*x+1)/(a*x-1))*csgn(I/(a*x-1)*(a*x+1)/((a*x+1)/(a*
x-1)-1))+4*ln(2))*arccoth(x*a)^2-arccoth(x*a)*polylog(2,-(a*x+1)/(a*x-1))+
1/2*polylog(3,-(a*x+1)/(a*x-1)))
```

Fricas [F]

$$\int \frac{\coth^{-1}(ax)^3}{x^2}\,dx = \int \frac{\operatorname{arcoth}(ax)^3}{x^2}\,dx$$

input
```
integrate(arccoth(a*x)^3/x^2,x, algorithm="fricas")
```

output
```
integral(arccoth(a*x)^3/x^2, x)
```

Sympy [F]

$$\int \frac{\coth^{-1}(ax)^3}{x^2}\,dx = \int \frac{\operatorname{acoth}^3(ax)}{x^2}\,dx$$

input
```
integrate(acoth(a*x)**3/x**2,x)
```

output
```
Integral(acoth(a*x)**3/x**2, x)
```

Maxima [F]

$$\int \frac{\coth^{-1}(ax)^3}{x^2}\,dx = \int \frac{\operatorname{arcoth}(ax)^3}{x^2}\,dx$$

input
```
integrate(arccoth(a*x)^3/x^2,x, algorithm="maxima")
```

output
```
1/8*a*(log(a*x + 1) - log(x))*log(a)^3 + 3/8*a*integrate(x*log(a*x - 1)/(a
*x^3 + x^2), x)*log(a)^2 - 3/8*a*integrate(x*log(x)/(a*x^3 + x^2), x)*log(
a)^2 - 1/8*(a*log(a*x + 1) - a*log(x) - 1/x)*log(a)^3 + 3/4*a^2*integrate(
x^2*log(a*x + 1)*log(a*x - 1)/(a*x^3 + x^2), x) - 3/2*a^2*integrate(x^2*lo
g(a*x + 1)*log(x)/(a*x^3 + x^2), x) + 3/4*a*integrate(x*log(a*x - 1)*log(x
)/(a*x^3 + x^2), x)*log(a) - 3/8*a*integrate(x*log(x)^2/(a*x^3 + x^2), x)*
log(a) + 3/8*integrate(log(a*x - 1)/(a*x^3 + x^2), x)*log(a)^2 - 3/8*integ
rate(log(x)/(a*x^3 + x^2), x)*log(a)^2 + 3/8*a*integrate(x*log(a*x + 1)*lo
g(a*x - 1)^2/(a*x^3 + x^2), x) - 3/8*a*integrate(x*log(a*x - 1)^2*log(x)/(
a*x^3 + x^2), x) + 3/8*a*integrate(x*log(a*x - 1)*log(x)^2/(a*x^3 + x^2),
x) - 1/8*a*integrate(x*log(x)^3/(a*x^3 + x^2), x) - 3/4*a*integrate(x*log(
a*x + 1)*log(a*x - 1)/(a*x^3 + x^2), x) - 3/8*integrate(a*x*log(a*x - 1)^2
/(a*x^3 + x^2), x)*log(a) - 3/8*integrate(log(a*x - 1)^2/(a*x^3 + x^2), x)
*log(a) + 3/4*integrate(log(a*x - 1)*log(x)/(a*x^3 + x^2), x)*log(a) - 3/8
*integrate(log(x)^2/(a*x^3 + x^2), x)*log(a) - 3/8*(a^2*log(a*x - 1) - a^2
*log(x) + a/x)*log(-1/(a*x) + 1)^2/a + 1/8*log(-1/(a*x) + 1)^3/x - 1/8*((a
*x + 1)*log(a*x + 1)^3 - 3*(2*a*x*log(x) - (a*x - 1)*log(a*x - 1))*log(a*x
 + 1)^2)/x + 1/8*(3*(a^3*x*log(a*x - 1)^2 + a^3*x*log(x)^2 - 2*a^3*x*log(x
) + 2*a^2 - 2*(a^3*x*log(x) - a^3*x)*log(a*x - 1))*log(-1/(a*x) + 1)/(a*x)
 - (a^4*x*log(a*x - 1)^3 - a^4*x*log(x)^3 + 3*a^4*x*log(x)^2 - 6*a^4*x*...
```

Giac [F]

$$\int \frac{\coth^{-1}(ax)^3}{x^2}\,dx = \int \frac{\operatorname{arcoth}(ax)^3}{x^2}\,dx$$

input
```
integrate(arccoth(a*x)^3/x^2,x, algorithm="giac")
```

output
```
integrate(arccoth(a*x)^3/x^2, x)
```

Mupad [F(-1)]

Timed out.

$$\int \frac{\coth^{-1}(ax)^3}{x^2}\,dx = \int \frac{\mathrm{acoth}(a\,x)^3}{x^2}\,dx$$

input
```
int(acoth(a*x)^3/x^2,x)
```

output
```
int(acoth(a*x)^3/x^2, x)
```

Reduce [F]

$$\int \frac{\coth^{-1}(ax)^3}{x^2}\,dx = \frac{-acoth(ax)^3 + 3\left(\int \frac{acoth(ax)^2}{a^2x^3-x}dx\right)ax}{x}$$

input
```
int(acoth(a*x)^3/x^2,x)
```

output
```
( - acoth(a*x)**3 + 3*int(acoth(a*x)**2/(a**2*x**3 - x),x)*a*x)/x
```

3.31 $\int \frac{\coth^{-1}(ax)^3}{x^3}\,dx$

Optimal result

Integrand size = 10, antiderivative size = 95

$$\int \frac{\coth^{-1}(ax)^3}{x^3}\,dx = \frac{3}{2}a^2\coth^{-1}(ax)^2 - \frac{3a\coth^{-1}(ax)^2}{2x} + \frac{1}{2}a^2\coth^{-1}(ax)^3 - \frac{\coth^{-1}(ax)^3}{2x^2} + 3a^2\coth^{-1}(ax)\log\left(2-\frac{2}{1+ax}\right) - \frac{3}{2}a^2\,\mathrm{PolyLog}\left(2,-1+\frac{2}{1+ax}\right)$$

output

```
3/2*a^2*arccoth(a*x)^2-3/2*a*arccoth(a*x)^2/x+1/2*a^2*arccoth(a*x)^3-1/2*a
rccoth(a*x)^3/x^2+3*a^2*arccoth(a*x)*ln(2-2/(a*x+1))-3/2*a^2*polylog(2,-1+
2/(a*x+1))
```

Mathematica [A] (verified)

Time = 0.11 (sec) , antiderivative size = 79, normalized size of antiderivative = 0.83

$$\int \frac{\coth^{-1}(ax)^3}{x^3}\,dx$$
$$=\frac{1}{2}\left(\frac{\coth^{-1}(ax)\left(3ax(-1+ax)\coth^{-1}(ax)+(-1+a^2x^2)\coth^{-1}(ax)^2+6a^2x^2\log\left(1+e^{-2\coth^{-1}(ax)}\right)\right)}{x^2}\right.$$
$$\left.-3a^2\operatorname{PolyLog}\left(2,-e^{-2\coth^{-1}(ax)}\right)\right)$$

input
```
Integrate[ArcCoth[a*x]^3/x^3,x]
```

output
```
((ArcCoth[a*x]*(3*a*x*(-1 + a*x)*ArcCoth[a*x] + (-1 + a^2*x^2)*ArcCoth[a*x
]^2 + 6*a^2*x^2*Log[1 + E^(-2*ArcCoth[a*x])]))/x^2 - 3*a^2*PolyLog[2, -E^(
-2*ArcCoth[a*x])])/2
```

Rubi [A] (verified)

Time = 0.80 (sec) , antiderivative size = 90, normalized size of antiderivative = 0.95, number of steps used = 7, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.700, Rules used = {6453, 6545, 6453, 6511, 6551, 6495, 2897}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\coth^{-1}(ax)^3}{x^3}\,dx$$
$$\Big\downarrow \text{ 6453}$$
$$\frac{3}{2}a\int \frac{\coth^{-1}(ax)^2}{x^2(1-a^2x^2)}dx-\frac{\coth^{-1}(ax)^3}{2x^2}$$
$$\Big\downarrow \text{ 6545}$$
$$\frac{3}{2}a\left(a^2\int \frac{\coth^{-1}(ax)^2}{1-a^2x^2}dx+\int \frac{\coth^{-1}(ax)^2}{x^2}dx\right)-\frac{\coth^{-1}(ax)^3}{2x^2}$$

$$\Big\downarrow \; 6453$$

$$\frac{3}{2}a\left(a^2\int \frac{\coth^{-1}(ax)^2}{1-a^2x^2}dx+2a\int \frac{\coth^{-1}(ax)}{x\left(1-a^2x^2\right)}dx-\frac{\coth^{-1}(ax)^2}{x}\right)-\frac{\coth^{-1}(ax)^3}{2x^2}$$

$$\Big\downarrow \; 6511$$

$$\frac{3}{2}a\left(2a\int \frac{\coth^{-1}(ax)}{x\left(1-a^2x^2\right)}dx+\frac{1}{3}a\coth^{-1}(ax)^3-\frac{\coth^{-1}(ax)^2}{x}\right)-\frac{\coth^{-1}(ax)^3}{2x^2}$$

$$\Big\downarrow \; 6551$$

$$\frac{3}{2}a\left(2a\left(\int \frac{\coth^{-1}(ax)}{x(ax+1)}dx+\frac{1}{2}\coth^{-1}(ax)^2\right)+\frac{1}{3}a\coth^{-1}(ax)^3-\frac{\coth^{-1}(ax)^2}{x}\right)-\frac{\coth^{-1}(ax)^3}{2x^2}$$

$$\Big\downarrow \; 6495$$

$$\frac{3}{2}a\left(2a\left(-a\int \frac{\log\left(2-\frac{2}{ax+1}\right)}{1-a^2x^2}dx+\frac{1}{2}\coth^{-1}(ax)^2+\log\left(2-\frac{2}{ax+1}\right)\coth^{-1}(ax)\right)+\frac{1}{3}a\coth^{-1}(ax)^3-\frac{\coth}{}\right.$$
$$\frac{\coth^{-1}(ax)^3}{2x^2}$$

$$\Big\downarrow \; 2897$$

$$\frac{3}{2}a\left(2a\left(-\frac{1}{2}\text{PolyLog}\left(2,\frac{2}{ax+1}-1\right)+\frac{1}{2}\coth^{-1}(ax)^2+\log\left(2-\frac{2}{ax+1}\right)\coth^{-1}(ax)\right)+\frac{1}{3}a\coth^{-1}(ax)^3-\right.$$
$$\frac{\coth^{-1}(ax)^3}{2x^2}$$

input
```
Int[ArcCoth[a*x]^3/x^3,x]
```

output
```
-1/2*ArcCoth[a*x]^3/x^2 + (3*a*(-(ArcCoth[a*x]^2/x) + (a*ArcCoth[a*x]^3)/3
 + 2*a*(ArcCoth[a*x]^2/2 + ArcCoth[a*x]*Log[2 - 2/(1 + a*x)] - PolyLog[2,
-1 + 2/(1 + a*x)]/2)))/2
```

Defintions of rubi rules used

rule 2897
```
Int[Log[u_]*(Pq_)^(m_.), x_Symbol] :> With[{C = FullSimplify[Pq^m*((1 - u)/
D[u, x])]}, Simp[C*PolyLog[2, 1 - u], x] /; FreeQ[C, x]] /; IntegerQ[m] &&
PolyQ[Pq, x] && RationalFunctionQ[u, x] && LeQ[RationalFunctionExponents[u,
 x][[2]], Expon[Pq, x]]
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

rule 6495
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((x_)*((d_) + (e_.)*(x_))), x
_Symbol] :> Simp[(a + b*ArcCoth[c*x])^p*(Log[2 - 2/(1 + e*(x/d))]/d), x] -
Simp[b*c*(p/d)   Int[(a + b*ArcCoth[c*x])^(p - 1)*(Log[2 - 2/(1 + e*(x/d))]
/(1 - c^2*x^2)), x], x] /; FreeQ[{a, b, c, d, e}, x] && IGtQ[p, 0] && EqQ[c
^2*d^2 - e^2, 0]
```

rule 6511
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((d_) + (e_.)*(x_)^2), x_Symb
ol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*c*d*(p + 1)), x] /; FreeQ[{a, b
, c, d, e, p}, x] && EqQ[c^2*d + e, 0] && NeQ[p, -1]
```

rule 6545
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*((f_.)*(x_))^(m_))/((d_) + (
e_.)*(x_)^2), x_Symbol] :> Simp[1/d   Int[(f*x)^m*(a + b*ArcCoth[c*x])^p, x
], x] - Simp[e/(d*f^2)   Int[(f*x)^(m + 2)*((a + b*ArcCoth[c*x])^p/(d + e*x
^2)), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && GtQ[p, 0] && LtQ[m, -1]
```

rule 6551
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((x_)*((d_) + (e_.)*(x_)^2)),
 x_Symbol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*d*(p + 1)), x] + Simp[1/
d   Int[(a + b*ArcCoth[c*x])^p/(x*(1 + c*x)), x], x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[c^2*d + e, 0] && GtQ[p, 0]
```

Maple [C] (warning: unable to verify)

Result contains higher order function than in optimal. Order 9 vs. order 4.

Time = 3.79 (sec) , antiderivative size = 3498, normalized size of antiderivative = 36.82

method	result	size
parts	Expression too large to display	3498
derivativedivides	Expression too large to display	3502
default	Expression too large to display	3502

input
```
int(arccoth(x*a)^3/x^3,x,method=_RETURNVERBOSE)
```

output
```
-1/2*arccoth(x*a)^3/x^2-3/2*a^2*(1/8*I*Pi*csgn(I/(a*x-1)*(a*x+1)/((a*x+1)/
(a*x-1)-1))^3*polylog(2,-(a*x+1)/(a*x-1))-1/4*I*Pi*csgn(I/(a*x-1)*(a*x+1)/
((a*x+1)/(a*x-1)-1))^3*dilog(1+I/((a*x-1)/(a*x+1))^(1/2))-1/4*I*Pi*csgn(I/
(a*x-1)*(a*x+1)/((a*x+1)/(a*x-1)-1))^3*dilog(1-I/((a*x-1)/(a*x+1))^(1/2))-
1/4*I*Pi*csgn(I*(a*x+1)/(a*x-1))^3*dilog(1+I/((a*x-1)/(a*x+1))^(1/2))-1/4*
I*Pi*csgn(I*(a*x+1)/(a*x-1))^3*dilog(1-I/((a*x-1)/(a*x+1))^(1/2))+1/8*I*Pi
*csgn(I*(a*x+1)/(a*x-1))^3*polylog(2,-(a*x+1)/(a*x-1))+1/x/a*arccoth(x*a)^
2-1/4*I*Pi*csgn(I/((a*x+1)/(a*x-1)-1))*csgn(I*(a*x+1)/(a*x-1))*csgn(I/(a*x
-1)*(a*x+1)/((a*x+1)/(a*x-1)-1))*arccoth(x*a)*ln(1+I/((a*x-1)/(a*x+1))^(1/
2))-1/4*I*Pi*csgn(I/((a*x+1)/(a*x-1)-1))*csgn(I*(a*x+1)/(a*x-1))*csgn(I/(a
*x-1)*(a*x+1)/((a*x+1)/(a*x-1)-1))*arccoth(x*a)*ln(1-I/((a*x-1)/(a*x+1))^(
1/2))+1/4*I*Pi*csgn(I/((a*x+1)/(a*x-1)-1))*csgn(I*(a*x+1)/(a*x-1))*csgn(I/
(a*x-1)*(a*x+1)/((a*x+1)/(a*x-1)-1))*arccoth(x*a)*ln(1+(a*x+1)/(a*x-1))-1/
2*I*Pi*csgn(I/((a*x-1)/(a*x+1))^(1/2))*csgn(I*(a*x+1)/(a*x-1))^2*arccoth(x
*a)*ln(1+(a*x+1)/(a*x-1))+1/4*I*Pi*csgn(I/((a*x-1)/(a*x+1))^(1/2))^2*csgn(
I*(a*x+1)/(a*x-1))*arccoth(x*a)*ln(1+(a*x+1)/(a*x-1))+1/8*I*Pi*csgn(I/((a*
x+1)/(a*x-1)-1))*csgn(I*(a*x+1)/(a*x-1))*csgn(I/(a*x-1)*(a*x+1)/((a*x+1)/(
a*x-1)-1))*polylog(2,-(a*x+1)/(a*x-1))-1/4*I*Pi*csgn(I/((a*x+1)/(a*x-1)-1)
)*csgn(I*(a*x+1)/(a*x-1))*csgn(I/(a*x-1)*(a*x+1)/((a*x+1)/(a*x-1)-1))*dilo
g(1+I/((a*x-1)/(a*x+1))^(1/2))-1/4*I*Pi*csgn(I/((a*x+1)/(a*x-1)-1))*csg...
```

Fricas [F]

$$\int \frac{\coth^{-1}(ax)^3}{x^3}\,dx = \int \frac{\operatorname{arcoth}(ax)^3}{x^3}\,dx$$

input
```
integrate(arccoth(a*x)^3/x^3,x, algorithm="fricas")
```

output
```
integral(arccoth(a*x)^3/x^3, x)
```

Sympy [F]

$$\int \frac{\coth^{-1}(ax)^3}{x^3}\,dx = \int \frac{\operatorname{acoth}^3(ax)}{x^3}\,dx$$

input
```
integrate(acoth(a*x)**3/x**3,x)
```

output
```
Integral(acoth(a*x)**3/x**3, x)
```

Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 252 vs. 2(84) = 168.

Time = 0.04 (sec) , antiderivative size = 252, normalized size of antiderivative = 2.65

$$\int \frac{\coth^{-1}(ax)^3}{x^3}\,dx = \frac{3}{4}\left(a\log(ax+1) - a\log(ax-1) - \frac{2}{x}\right)a\operatorname{arcoth}(ax)^2 - \frac{1}{16}\left(a^2\left(\frac{3(\log(ax-1)-2)\log(ax+1)^2 - \log(ax+1)^3 + \log(ax-1)^3 - 3\left(\log(ax-1)^2 - 4\log(\right.}{a}\right.\right. - \frac{\operatorname{arcoth}(ax)^3}{2x^2}$$

input
```
integrate(arccoth(a*x)^3/x^3,x, algorithm="maxima")
```

output
```
3/4*(a*log(a*x + 1) - a*log(a*x - 1) - 2/x)*a*arccoth(a*x)^2 - 1/16*(a^2*(
(3*(log(a*x - 1) - 2)*log(a*x + 1)^2 - log(a*x + 1)^3 + log(a*x - 1)^3 - 3
*(log(a*x - 1)^2 - 4*log(a*x - 1))*log(a*x + 1) + 6*log(a*x - 1)^2)/a - 24
*(log(a*x - 1)*log(1/2*a*x + 1/2) + dilog(-1/2*a*x + 1/2))/a + 24*(log(a*x
 + 1)*log(x) + dilog(-a*x))/a - 24*(log(-a*x + 1)*log(x) + dilog(a*x))/a)
- 6*(2*(log(a*x - 1) - 2)*log(a*x + 1) - log(a*x + 1)^2 - log(a*x - 1)^2 -
 4*log(a*x - 1) + 8*log(x))*a*arccoth(a*x))*a - 1/2*arccoth(a*x)^3/x^2
```

Giac [F]

$$\int \frac{\coth^{-1}(ax)^3}{x^3}\,dx = \int \frac{\operatorname{arcoth}(ax)^3}{x^3}\,dx$$

input
```
integrate(arccoth(a*x)^3/x^3,x, algorithm="giac")
```

output
```
integrate(arccoth(a*x)^3/x^3, x)
```

Mupad [F(-1)]

Timed out.

$$\int \frac{\coth^{-1}(ax)^3}{x^3}\,dx = \int \frac{\operatorname{acoth}(a\,x)^3}{x^3}\,dx$$

input
```
int(acoth(a*x)^3/x^3,x)
```

output
```
int(acoth(a*x)^3/x^3, x)
```

Reduce [F]

$$\int \frac{\coth^{-1}(ax)^3}{x^3}\,dx$$
$$= \frac{acoth(ax)^3 a^2x^2 - acoth(ax)^3 + 3acoth(ax)^2 ax - 3acoth(ax)\,a^2x^2 + 3acoth(ax) - 6\left(\int \frac{acoth(ax)}{a^2x^5-x^3}dx\right)x^2 -}{2x^2}$$

input
```
int(acoth(a*x)^3/x^3,x)
```

output
```
(acoth(a*x)**3*a**2*x**2 - acoth(a*x)**3 + 3*acoth(a*x)**2*a*x - 3*acoth(a
*x)*a**2*x**2 + 3*acoth(a*x) - 6*int(acoth(a*x)/(a**2*x**5 - x**3),x)*x**2
 - 3*a*x)/(2*x**2)
```

3.32 $\int \frac{\coth^{-1}(ax)^3}{x^4} \, dx$

Optimal result

Integrand size = 10, antiderivative size = 154

$$\begin{aligned}\int \frac{\coth^{-1}(ax)^3}{x^4} \, dx = & -\frac{a^2 \coth^{-1}(ax)}{x} + \frac{1}{2} a^3 \coth^{-1}(ax)^2 - \frac{a \coth^{-1}(ax)^2}{2x^2} \\ & + \frac{1}{3} a^3 \coth^{-1}(ax)^3 - \frac{\coth^{-1}(ax)^3}{3x^3} + a^3 \log(x) \\ & - \frac{1}{2} a^3 \log\left(1 - a^2 x^2\right) + a^3 \coth^{-1}(ax)^2 \log\left(2 - \frac{2}{1 + ax}\right) \\ & - a^3 \coth^{-1}(ax) \operatorname{PolyLog}\left(2, -1 + \frac{2}{1 + ax}\right) \\ & - \frac{1}{2} a^3 \operatorname{PolyLog}\left(3, -1 + \frac{2}{1 + ax}\right)\end{aligned}$$

output
```
-a^2*arccoth(a*x)/x+1/2*a^3*arccoth(a*x)^2-1/2*a*arccoth(a*x)^2/x^2+1/3*a^
3*arccoth(a*x)^3-1/3*arccoth(a*x)^3/x^3+a^3*ln(x)-1/2*a^3*ln(-a^2*x^2+1)+a
^3*arccoth(a*x)^2*ln(2-2/(a*x+1))-a^3*arccoth(a*x)*polylog(2,-1+2/(a*x+1))
-1/2*a^3*polylog(3,-1+2/(a*x+1))
```

Mathematica [A] (verified)

Time = 0.15 (sec) , antiderivative size = 142, normalized size of antiderivative = 0.92

$$\int \frac{\coth^{-1}(ax)^3}{x^4}\,dx = \frac{1}{6}\left(-\frac{6a^2\coth^{-1}(ax)}{x}+3a^3\coth^{-1}(ax)^2-\frac{3a\coth^{-1}(ax)^2}{x^2}+2a^3\coth^{-1}(ax)^3-\frac{2\coth^{-1}(ax)^3}{x^3}+6a^3\coth^{-1}(ax)^2\log\left(1+e^{-2\coth^{-1}(ax)}\right)+6a^3\log\left(\frac{1}{\sqrt{1-\frac{1}{a^2x^2}}}\right)-6a^3\coth^{-1}(ax)\,\mathrm{PolyLog}\left(2,-e^{-2\coth^{-1}(ax)}\right)-3a^3\,\mathrm{PolyLog}\left(3,-e^{-2\coth^{-1}(ax)}\right)\right)$$

input
```
Integrate[ArcCoth[a*x]^3/x^4,x]
```

output
```
((-6*a^2*ArcCoth[a*x])/x + 3*a^3*ArcCoth[a*x]^2 - (3*a*ArcCoth[a*x]^2)/x^2
 + 2*a^3*ArcCoth[a*x]^3 - (2*ArcCoth[a*x]^3)/x^3 + 6*a^3*ArcCoth[a*x]^2*Lo
g[1 + E^(-2*ArcCoth[a*x])] + 6*a^3*Log[1/Sqrt[1 - 1/(a^2*x^2)]] - 6*a^3*Ar
cCoth[a*x]*PolyLog[2, -E^(-2*ArcCoth[a*x])] - 3*a^3*PolyLog[3, -E^(-2*ArcC
oth[a*x])])/6
```

Rubi [A] (verified)

Time = 1.58 (sec) , antiderivative size = 158, normalized size of antiderivative = 1.03, number of steps used = 15, number of rules used = 14, $\frac{\text{number of rules}}{\text{integrand size}}$ = 1.400, Rules used = {6453, 6545, 6453, 6545, 6453, 243, 47, 14, 16, 6511, 6551, 6495, 6619, 7164}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\coth^{-1}(ax)^3}{x^4}\,dx$$

$$\Big\downarrow \text{6453}$$

$$a\int \frac{\coth^{-1}(ax)^2}{x^3\left(1-a^2x^2\right)}dx-\frac{\coth^{-1}(ax)^3}{3x^3}$$

$$\Big\downarrow \text{6545}$$

$$a\left(a^2\int \frac{\coth^{-1}(ax)^2}{x\left(1-a^2x^2\right)}dx+\int \frac{\coth^{-1}(ax)^2}{x^3}dx\right)-\frac{\coth^{-1}(ax)^3}{3x^3}$$

$$\Big\downarrow \text{6453}$$

$$a\left(a^2\int \frac{\coth^{-1}(ax)^2}{x\left(1-a^2x^2\right)}dx+a\int \frac{\coth^{-1}(ax)}{x^2\left(1-a^2x^2\right)}dx-\frac{\coth^{-1}(ax)^2}{2x^2}\right)-\frac{\coth^{-1}(ax)^3}{3x^3}$$

$$\Big\downarrow \text{6545}$$

$$a\left(a^2\int \frac{\coth^{-1}(ax)^2}{x\left(1-a^2x^2\right)}dx+a\left(a^2\int \frac{\coth^{-1}(ax)}{1-a^2x^2}dx+\int \frac{\coth^{-1}(ax)}{x^2}dx\right)-\frac{\coth^{-1}(ax)^2}{2x^2}\right)-\frac{\coth^{-1}(ax)^3}{3x^3}$$

$$\Big\downarrow \text{6453}$$

$$a\left(a^2\int \frac{\coth^{-1}(ax)^2}{x\left(1-a^2x^2\right)}dx+a\left(a\int \frac{1}{x\left(1-a^2x^2\right)}dx+a^2\int \frac{\coth^{-1}(ax)}{1-a^2x^2}dx-\frac{\coth^{-1}(ax)}{x}\right)-\frac{\coth^{-1}(ax)^2}{2x^2}\right)-\frac{\coth^{-1}(ax)^3}{3x^3}$$

$$\Big\downarrow \text{243}$$

$$a\left(a^2\int \frac{\coth^{-1}(ax)^2}{x\left(1-a^2x^2\right)}dx+a\left(\frac{1}{2}a\int \frac{1}{x^2\left(1-a^2x^2\right)}dx^2+a^2\int \frac{\coth^{-1}(ax)}{1-a^2x^2}dx-\frac{\coth^{-1}(ax)}{x}\right)-\frac{\coth^{-1}(ax)^2}{2x^2}\right)-\frac{\coth^{-1}(ax)^3}{3x^3}$$

$$\Big\downarrow \text{47}$$

$$a\left(a^2\int \frac{\coth^{-1}(ax)^2}{x\left(1-a^2x^2\right)}dx+a\left(\frac{1}{2}a\left(a^2\int \frac{1}{1-a^2x^2}dx^2+\int \frac{1}{x^2}dx^2\right)+a^2\int \frac{\coth^{-1}(ax)}{1-a^2x^2}dx-\frac{\coth^{-1}(ax)}{x}\right)-\text{cotl}\ldots\right.\quad \frac{\coth^{-1}(ax)^3}{3x^3}$$

$$\Big\downarrow \text{14}$$

$$a\left(a^2\int \frac{\coth^{-1}(ax)^2}{x\left(1-a^2x^2\right)}dx+a\left(\frac{1}{2}a\left(a^2\int \frac{1}{1-a^2x^2}dx^2+\log\left(x^2\right)\right)+a^2\int \frac{\coth^{-1}(ax)}{1-a^2x^2}dx-\frac{\coth^{-1}(ax)}{x}\right)-\frac{\coth}{2}$$

$$\frac{\coth^{-1}(ax)^3}{3x^3}$$

$\downarrow$ 16

$$a\left(a^2\int \frac{\coth^{-1}(ax)^2}{x\left(1-a^2x^2\right)}dx+a\left(a^2\int \frac{\coth^{-1}(ax)}{1-a^2x^2}dx+\frac{1}{2}a\left(\log\left(x^2\right)-\log\left(1-a^2x^2\right)\right)-\frac{\coth^{-1}(ax)}{x}\right)-\frac{\coth^{-1}(ax}{2x^2}$$

$$\frac{\coth^{-1}(ax)^3}{3x^3}$$

$\downarrow$ 6511

$$a\left(a^2\int \frac{\coth^{-1}(ax)^2}{x\left(1-a^2x^2\right)}dx+a\left(\frac{1}{2}a\left(\log\left(x^2\right)-\log\left(1-a^2x^2\right)\right)+\frac{1}{2}a\coth^{-1}(ax)^2-\frac{\coth^{-1}(ax)}{x}\right)-\frac{\coth^{-1}(ax)^2}{2x^2}\right)$$

$$\frac{\coth^{-1}(ax)^3}{3x^3}$$

$\downarrow$ 6551

$$a\left(a^2\left(\int \frac{\coth^{-1}(ax)^2}{x(ax+1)}dx+\frac{1}{3}\coth^{-1}(ax)^3\right)+a\left(\frac{1}{2}a\left(\log\left(x^2\right)-\log\left(1-a^2x^2\right)\right)+\frac{1}{2}a\coth^{-1}(ax)^2-\frac{\coth^{-1}(ax}{x}$$

$$\frac{\coth^{-1}(ax)^3}{3x^3}$$

$\downarrow$ 6495

$$a\left(a^2\left(-2a\int \frac{\coth^{-1}(ax)\log\left(2-\frac{2}{ax+1}\right)}{1-a^2x^2}dx+\frac{1}{3}\coth^{-1}(ax)^3+\log\left(2-\frac{2}{ax+1}\right)\coth^{-1}(ax)^2\right)+a\left(\frac{1}{2}a(\log($$

$$\frac{\coth^{-1}(ax)^3}{3x^3}$$

$\downarrow$ 6619

$$a\left(a^2\left(-2a\left(\frac{\text{PolyLog}\left(2,\frac{2}{ax+1}-1\right)\coth^{-1}(ax)}{2a}-\frac{1}{2}\int \frac{\text{PolyLog}\left(2,\frac{2}{ax+1}-1\right)}{1-a^2x^2}dx\right)+\frac{1}{3}\coth^{-1}(ax)^3+\log\left(2-\right.\right.\right.$$

$$\frac{\coth^{-1}(ax)^3}{3x^3}$$

$\downarrow$ 7164

$$a\left(a^2\left(-2a\left(\frac{\text{PolyLog}\left(3,\frac{2}{ax+1}-1\right)}{4a}+\frac{\text{PolyLog}\left(2,\frac{2}{ax+1}-1\right)\coth^{-1}(ax)}{2a}\right)+\frac{1}{3}\coth^{-1}(ax)^3+\log\left(2-\frac{2}{ax+\right.\right.\right.$$

$$\frac{\coth^{-1}(ax)^3}{3x^3}$$

input
```
Int[ArcCoth[a*x]^3/x^4,x]
```

output
```
-1/3*ArcCoth[a*x]^3/x^3 + a*(-1/2*ArcCoth[a*x]^2/x^2 + a*(-(ArcCoth[a*x]/x
) + (a*ArcCoth[a*x]^2)/2 + (a*(Log[x^2] - Log[1 - a^2*x^2]))/2) + a^2*(Arc
Coth[a*x]^3/3 + ArcCoth[a*x]^2*Log[2 - 2/(1 + a*x)] - 2*a*((ArcCoth[a*x]*P
olyLog[2, -1 + 2/(1 + a*x)])/(2*a) + PolyLog[3, -1 + 2/(1 + a*x)]/(4*a)))
```

Defintions of rubi rules used

rule 14
```
Int[(a_.)/(x_), x_Symbol] :> Simp[a*Log[x], x] /; FreeQ[a, x]
```

rule 16
```
Int[(c_.)/((a_.) + (b_.)*(x_)), x_Symbol] :> Simp[c*(Log[RemoveContent[a +
b*x, x]]/b), x] /; FreeQ[{a, b, c}, x]
```

rule 47
```
Int[1/(((a_.) + (b_.)*(x_))*((c_.) + (d_.)*(x_))), x_Symbol] :> Simp[b/(b*c
 - a*d)   Int[1/(a + b*x), x], x] - Simp[d/(b*c - a*d)   Int[1/(c + d*x), x
], x] /; FreeQ[{a, b, c, d}, x]
```

rule 243
```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] :> Simp[1/2   Subst[In
t[x^((m - 1)/2)*(a + b*x)^p, x], x, x^2], x] /; FreeQ[{a, b, m, p}, x] && I
ntegerQ[(m - 1)/2]
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

rule 6495
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((x_)*((d_) + (e_.)*(x_))), x
_Symbol] :> Simp[(a + b*ArcCoth[c*x])^p*(Log[2 - 2/(1 + e*(x/d))]/d), x] -
Simp[b*c*(p/d)   Int[(a + b*ArcCoth[c*x])^(p - 1)*(Log[2 - 2/(1 + e*(x/d))]
/(1 - c^2*x^2)), x], x] /; FreeQ[{a, b, c, d, e}, x] && IGtQ[p, 0] && EqQ[c
^2*d^2 - e^2, 0]
```

rule 6511
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((d_) + (e_.)*(x_)^2), x_Symb
ol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*c*d*(p + 1)), x] /; FreeQ[{a, b
, c, d, e, p}, x] && EqQ[c^2*d + e, 0] && NeQ[p, -1]
```

rule 6545
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*((f_.)*(x_))^(m_))/((d_) + (
e_.)*(x_)^2), x_Symbol] :> Simp[1/d   Int[(f*x)^m*(a + b*ArcCoth[c*x])^p, x
], x] - Simp[e/(d*f^2)   Int[(f*x)^(m + 2)*((a + b*ArcCoth[c*x])^p/(d + e*x
^2)), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && GtQ[p, 0] && LtQ[m, -1]
```

rule 6551
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((x_)*((d_) + (e_.)*(x_)^2)),
 x_Symbol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*d*(p + 1)), x] + Simp[1/
d   Int[(a + b*ArcCoth[c*x])^p/(x*(1 + c*x)), x], x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[c^2*d + e, 0] && GtQ[p, 0]
```

rule 6619
```
Int[(Log[u_]*((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.))/((d_) + (e_.)*(x_)^
2), x_Symbol] :> Simp[(a + b*ArcCoth[c*x])^p*(PolyLog[2, 1 - u]/(2*c*d)), x
] - Simp[b*(p/2)   Int[(a + b*ArcCoth[c*x])^(p - 1)*(PolyLog[2, 1 - u]/(d +
 e*x^2)), x], x] /; FreeQ[{a, b, c, d, e}, x] && IGtQ[p, 0] && EqQ[c^2*d +
e, 0] && EqQ[(1 - u)^2 - (1 - 2/(1 + c*x))^2, 0]
```

rule 7164
```
Int[(u_)*PolyLog[n_, v_], x_Symbol] :> With[{w = DerivativeDivides[v, u*v,
x]}, Simp[w*PolyLog[n + 1, v], x] /;  !FalseQ[w]] /; FreeQ[n, x]
```

Maple [C] (warning: unable to verify)

Result contains higher order function than in optimal. Order 9 vs. order 4.

Time = 6.37 (sec) , antiderivative size = 836, normalized size of antiderivative = 5.43

method	result	size
derivativedivides	Expression too large to display	836
default	Expression too large to display	836
parts	Expression too large to display	839

input
```
int(arccoth(x*a)^3/x^4,x,method=_RETURNVERBOSE)
```

output
```
a^3*(-1/3/x^3/a^3*arccoth(x*a)^3-1/2*arccoth(x*a)^2*ln(a*x+1)-1/2*arccoth(
x*a)^2/x^2/a^2+ln(x*a)*arccoth(x*a)^2-1/2*arccoth(x*a)^2*ln(a*x-1)-arccoth
(x*a)*(a*x+1)/x/a+1/2*arccoth(x*a)^2+ln(2)*arccoth(x*a)^2+1/2*I*Pi*csgn(I/
((a*x+1)/(a*x-1)-1)*(1+(a*x+1)/(a*x-1)))^3*arccoth(x*a)^2+ln(1+(a*x+1)/(a*
x-1))+1/2*I*Pi*csgn(I/((a*x+1)/(a*x-1)-1))*csgn(I*(1+(a*x+1)/(a*x-1)))*csg
n(I/((a*x+1)/(a*x-1)-1)*(1+(a*x+1)/(a*x-1)))*arccoth(x*a)^2-1/3*arccoth(x*
a)^3+1/4*I*Pi*csgn(I*(a*x+1)/(a*x-1))*csgn(I/(a*x-1)*(a*x+1)/((a*x+1)/(a*x
-1)-1))^2*arccoth(x*a)^2+1/4*I*Pi*csgn(I/((a*x+1)/(a*x-1)-1))*csgn(I/(a*x-
1)*(a*x+1)/((a*x+1)/(a*x-1)-1))^2*arccoth(x*a)^2-1/2*polylog(3,-(a*x+1)/(a
*x-1))-1/2*I*Pi*csgn(I/((a*x+1)/(a*x-1)-1))*csgn(I/((a*x+1)/(a*x-1)-1)*(1+
(a*x+1)/(a*x-1)))^2*arccoth(x*a)^2-1/4*I*Pi*csgn(I/((a*x+1)/(a*x-1)-1))*cs
gn(I*(a*x+1)/(a*x-1))*csgn(I/(a*x-1)*(a*x+1)/((a*x+1)/(a*x-1)-1))*arccoth(
x*a)^2-1/2*I*Pi*csgn(I*(1+(a*x+1)/(a*x-1)))*csgn(I/((a*x+1)/(a*x-1)-1)*(1+
(a*x+1)/(a*x-1)))^2*arccoth(x*a)^2-1/4*I*Pi*csgn(I/((a*x-1)/(a*x+1))^(1/2)
)^2*csgn(I*(a*x+1)/(a*x-1))*arccoth(x*a)^2+arccoth(x*a)*polylog(2,-(a*x+1)
/(a*x-1))-1/2*arccoth(x*a)^2*ln((a*x-1)/(a*x+1))-1/4*I*Pi*csgn(I*(a*x+1)/(
a*x-1))^3*arccoth(x*a)^2-1/4*I*Pi*csgn(I/(a*x-1)*(a*x+1)/((a*x+1)/(a*x-1)-
1))^3*arccoth(x*a)^2+1/2*I*Pi*csgn(I/((a*x-1)/(a*x+1))^(1/2))*csgn(I*(a*x+
1)/(a*x-1))^2*arccoth(x*a)^2)
```

Fricas [F]

$$\int \frac{\coth^{-1}(ax)^3}{x^4} \, dx = \int \frac{\operatorname{arcoth}(ax)^3}{x^4} \, dx$$

input
```
integrate(arccoth(a*x)^3/x^4,x, algorithm="fricas")
```

output
```
integral(arccoth(a*x)^3/x^4, x)
```

Sympy [F]

$$\int \frac{\coth^{-1}(ax)^3}{x^4} \, dx = \int \frac{\operatorname{acoth}^3(ax)}{x^4} \, dx$$

input
```
integrate(acoth(a*x)**3/x**4,x)
```

output
```
Integral(acoth(a*x)**3/x**4, x)
```

Maxima [F]

$$\int \frac{\coth^{-1}(ax)^3}{x^4} \, dx = \int \frac{\operatorname{arcoth}(ax)^3}{x^4} \, dx$$

input
```
integrate(arccoth(a*x)^3/x^4,x, algorithm="maxima")
```

output
```
1/4*a^4*integrate(x^4*log(a*x + 1)*log(a*x - 1)/(a*x^5 + x^4), x) - 1/2*a^
4*integrate(x^4*log(a*x + 1)*log(x)/(a*x^5 + x^4), x) + 1/16*(2*a^2*log(a*
x + 1) - 2*a^2*log(x) - (2*a*x - 1)/x^2)*a*log(a)^3 + 3/8*a*integrate(x*lo
g(a*x - 1)/(a*x^5 + x^4), x)*log(a)^2 - 3/8*a*integrate(x*log(x)/(a*x^5 +
x^4), x)*log(a)^2 - 1/48*(6*a^3*log(a*x + 1) - 6*a^3*log(x) - (6*a^2*x^2 -
 3*a*x + 2)/x^3)*log(a)^3 + 1/4*a^2*integrate(x^2*log(a*x + 1)/(a*x^5 + x^
4), x) + 3/4*a*integrate(x*log(a*x - 1)*log(x)/(a*x^5 + x^4), x)*log(a) -
3/8*a*integrate(x*log(x)^2/(a*x^5 + x^4), x)*log(a) + 3/8*integrate(log(a*
x - 1)/(a*x^5 + x^4), x)*log(a)^2 - 3/8*integrate(log(x)/(a*x^5 + x^4), x)
*log(a)^2 + 3/8*a*integrate(x*log(a*x + 1)*log(a*x - 1)^2/(a*x^5 + x^4), x
) - 3/8*a*integrate(x*log(a*x - 1)^2*log(x)/(a*x^5 + x^4), x) + 3/8*a*inte
grate(x*log(a*x - 1)*log(x)^2/(a*x^5 + x^4), x) - 1/8*a*integrate(x*log(x)
^3/(a*x^5 + x^4), x) - 1/4*a*integrate(x*log(a*x + 1)*log(a*x - 1)/(a*x^5
+ x^4), x) - 3/8*integrate(a*x*log(a*x - 1)^2/(a*x^5 + x^4), x)*log(a) - 3
/8*integrate(log(a*x - 1)^2/(a*x^5 + x^4), x)*log(a) + 3/4*integrate(log(a
*x - 1)*log(x)/(a*x^5 + x^4), x)*log(a) - 3/8*integrate(log(x)^2/(a*x^5 +
x^4), x)*log(a) - 1/48*(6*a^4*log(a*x - 1) - 6*a^4*log(x) + (6*a^3*x^2 + 3
*a^2*x + 2*a)/x^3)*log(-1/(a*x) + 1)^2/a + 1/864*(6*(18*a^5*x^3*log(a*x -
1)^2 + 18*a^5*x^3*log(x)^2 - 66*a^5*x^3*log(x) + 66*a^4*x^2 + 15*a^3*x + 4
*a^2 - 6*(6*a^5*x^3*log(x) - 11*a^5*x^3)*log(a*x - 1))*log(-1/(a*x) + 1...
```

Giac [F]

$$\int \frac{\coth^{-1}(ax)^3}{x^4}\,dx = \int \frac{\operatorname{arcoth}(ax)^3}{x^4}\,dx$$

input
```
integrate(arccoth(a*x)^3/x^4,x, algorithm="giac")
```

output
```
integrate(arccoth(a*x)^3/x^4, x)
```

Mupad [F(-1)]

Timed out.

$$\int \frac{\coth^{-1}(ax)^3}{x^4}\,dx = \int \frac{\mathrm{acoth}(a\,x)^3}{x^4}\,dx$$

input
```
int(acoth(a*x)^3/x^4,x)
```

output
```
int(acoth(a*x)^3/x^4, x)
```

Reduce [F]

$$\int \frac{\coth^{-1}(ax)^3}{x^4}\,dx = \frac{-2acoth(ax)^3 - 3acoth(ax)^2 a^3x^3 + 3acoth(ax)^2 ax - 6acoth(ax)\,a^3x^3 - 6acoth(ax)\,a^2x^2 + 6\left(\int \frac{acoth(ax}{a^2x^3-}\right.}{6x^3}$$

input
```
int(acoth(a*x)^3/x^4,x)
```

output
```
( - 2*acoth(a*x)**3 - 3*acoth(a*x)**2*a**3*x**3 + 3*acoth(a*x)**2*a*x - 6*
acoth(a*x)*a**3*x**3 - 6*acoth(a*x)*a**2*x**2 + 6*int(acoth(a*x)**2/(a**2*
x**3 - x),x)*a**3*x**3 + 6*log(a**2*x - a)*a**3*x**3 - 6*log(x)*a**3*x**3)
/(6*x**3)
```

3.33 $\int \frac{\coth^{-1}(ax)^3}{x^5} \, dx$

Optimal result

Integrand size = 10, antiderivative size = 141

$$\begin{aligned}\int \frac{\coth^{-1}(ax)^3}{x^5} \, dx = &-\frac{a^3}{4x}-\frac{a^2\coth^{-1}(ax)}{4x^2}+a^4\coth^{-1}(ax)^2-\frac{a\coth^{-1}(ax)^2}{4x^3}\\&-\frac{3a^3\coth^{-1}(ax)^2}{4x}+\frac{1}{4}a^4\coth^{-1}(ax)^3-\frac{\coth^{-1}(ax)^3}{4x^4}\\&+\frac{1}{4}a^4\text{arctanh}(ax)+2a^4\coth^{-1}(ax)\log\left(2-\frac{2}{1+ax}\right)\\&-a^4\,\text{PolyLog}\left(2,-1+\frac{2}{1+ax}\right)\end{aligned}$$

output
```
-1/4*a^3/x-1/4*a^2*arccoth(a*x)/x^2+a^4*arccoth(a*x)^2-1/4*a*arccoth(a*x)^
2/x^3-3/4*a^3*arccoth(a*x)^2/x+1/4*a^4*arccoth(a*x)^3-1/4*arccoth(a*x)^3/x
^4+1/4*a^4*arctanh(a*x)+2*a^4*arccoth(a*x)*ln(2-2/(a*x+1))-a^4*polylog(2,-
1+2/(a*x+1))
```

Mathematica [A] (verified)

Time = 0.16 (sec) , antiderivative size = 118, normalized size of antiderivative = 0.84

$$\int \frac{\coth^{-1}(ax)^3}{x^5}\,dx = \frac{-a^3x^3 + ax(-1 - 3a^2x^2 + 4a^3x^3)\coth^{-1}(ax)^2 + (-1 + a^4x^4)\coth^{-1}(ax)^3 + a^2x^2\coth^{-1}(ax)\left(-1 + a^2\right.}{4x^4}$$

input
```
Integrate[ArcCoth[a*x]^3/x^5,x]
```

output
```
(-(a^3*x^3) + a*x*(-1 - 3*a^2*x^2 + 4*a^3*x^3)*ArcCoth[a*x]^2 + (-1 + a^4*
x^4)*ArcCoth[a*x]^3 + a^2*x^2*ArcCoth[a*x]*(-1 + a^2*x^2 + 8*a^2*x^2*Log[1
 + E^(-2*ArcCoth[a*x])]) - 4*a^4*x^4*PolyLog[2, -E^(-2*ArcCoth[a*x])])/(4*
x^4)
```

Rubi [A] (verified)

Time = 1.50 (sec) , antiderivative size = 191, normalized size of antiderivative = 1.35, number of steps used = 11, number of rules used = 11, $\frac{\text{number of rules}}{\text{integrand size}}$ = 1.100, Rules used = {6453, 6545, 6453, 6545, 6453, 264, 219, 6511, 6551, 6495, 2897}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\coth^{-1}(ax)^3}{x^5}\,dx$$

$$\Big\downarrow \text{ 6453}$$

$$\frac{3}{4}a\int \frac{\coth^{-1}(ax)^2}{x^4(1-a^2x^2)}dx - \frac{\coth^{-1}(ax)^3}{4x^4}$$

$$\Big\downarrow \text{ 6545}$$

$$\frac{3}{4}a\left(a^2\int \frac{\coth^{-1}(ax)^2}{x^2(1-a^2x^2)}dx + \int \frac{\coth^{-1}(ax)^2}{x^4}dx\right) - \frac{\coth^{-1}(ax)^3}{4x^4}$$

$$\Big\downarrow \text{ 6453}$$

$$\frac{3}{4}a\left(a^2\int \frac{\coth^{-1}(ax)^2}{x^2(1-a^2x^2)}dx+\frac{2}{3}a\int \frac{\coth^{-1}(ax)}{x^3(1-a^2x^2)}dx-\frac{\coth^{-1}(ax)^2}{3x^3}\right)-\frac{\coth^{-1}(ax)^3}{4x^4}$$

$$\Big\downarrow \text{6545}$$

$$\frac{3}{4}a\left(a^2\left(a^2\int \frac{\coth^{-1}(ax)^2}{1-a^2x^2}dx+\int \frac{\coth^{-1}(ax)^2}{x^2}dx\right)+\frac{2}{3}a\left(a^2\int \frac{\coth^{-1}(ax)}{x(1-a^2x^2)}dx+\int \frac{\coth^{-1}(ax)}{x^3}dx\right)-\frac{\coth^{-1}}{3x}\right.$$
$$\frac{\coth^{-1}(ax)^3}{4x^4}$$

$$\Big\downarrow \text{6453}$$

$$\frac{3}{4}a\left(a^2\left(a^2\int \frac{\coth^{-1}(ax)^2}{1-a^2x^2}dx+2a\int \frac{\coth^{-1}(ax)}{x(1-a^2x^2)}dx-\frac{\coth^{-1}(ax)^2}{x}\right)+\frac{2}{3}a\left(\frac{1}{2}a\int \frac{1}{x^2(1-a^2x^2)}dx+a^2\int \frac{\mathrm{co}}{x(}\right.\right.$$
$$\frac{\coth^{-1}(ax)^3}{4x^4}$$

$$\Big\downarrow \text{264}$$

$$\frac{3}{4}a\left(a^2\left(a^2\int \frac{\coth^{-1}(ax)^2}{1-a^2x^2}dx+2a\int \frac{\coth^{-1}(ax)}{x(1-a^2x^2)}dx-\frac{\coth^{-1}(ax)^2}{x}\right)+\frac{2}{3}a\left(\frac{1}{2}a\left(a^2\int \frac{1}{1-a^2x^2}dx-\frac{1}{x}\right)+a^2\right.\right.$$
$$\frac{\coth^{-1}(ax)^3}{4x^4}$$

$$\Big\downarrow \text{219}$$

$$\frac{3}{4}a\left(\frac{2}{3}a\left(a^2\int \frac{\coth^{-1}(ax)}{x(1-a^2x^2)}dx+\frac{1}{2}a\left(a\text{arctanh}(ax)-\frac{1}{x}\right)-\frac{\coth^{-1}(ax)}{2x^2}\right)+a^2\left(a^2\int \frac{\coth^{-1}(ax)^2}{1-a^2x^2}dx+2a\int \frac{\mathrm{c}}{x}\right.\right.$$
$$\frac{\coth^{-1}(ax)^3}{4x^4}$$

$$\Big\downarrow \text{6511}$$

$$\frac{3}{4}a\left(\frac{2}{3}a\left(a^2\int \frac{\coth^{-1}(ax)}{x(1-a^2x^2)}dx+\frac{1}{2}a\left(a\text{arctanh}(ax)-\frac{1}{x}\right)-\frac{\coth^{-1}(ax)}{2x^2}\right)+a^2\left(2a\int \frac{\coth^{-1}(ax)}{x(1-a^2x^2)}dx+\frac{1}{3}a\coth\right.\right.$$
$$\frac{\coth^{-1}(ax)^3}{4x^4}$$

$$\Big\downarrow \text{6551}$$

$$\frac{3}{4}a\left(\frac{2}{3}a\left(a^2\left(\int \frac{\coth^{-1}(ax)}{x(ax+1)}dx+\frac{1}{2}\coth^{-1}(ax)^2\right)+\frac{1}{2}a\left(a\text{arctanh}(ax)-\frac{1}{x}\right)-\frac{\coth^{-1}(ax)}{2x^2}\right)+a^2\left(2a\left(\int \frac{\coth}{x(a}\right.\right.\right.$$
$$\frac{\coth^{-1}(ax)^3}{4x^4}$$

$\downarrow$ 6495

$$\frac{3}{4}a\left(\frac{2}{3}a\left(a^2\left(-a\int\frac{\log\left(2-\frac{2}{ax+1}\right)}{1-a^2x^2}dx+\frac{1}{2}\coth^{-1}(ax)^2+\log\left(2-\frac{2}{ax+1}\right)\coth^{-1}(ax)\right)+\frac{1}{2}a\left(a\text{arctanh}(ax)\right.\right.\right.$$
$$\frac{\coth^{-1}(ax)^3}{4x^4}$$

$\downarrow$ 2897

$$\frac{3}{4}a\left(\frac{2}{3}a\left(a^2\left(-\frac{1}{2}\text{PolyLog}\left(2,\frac{2}{ax+1}-1\right)+\frac{1}{2}\coth^{-1}(ax)^2+\log\left(2-\frac{2}{ax+1}\right)\coth^{-1}(ax)\right)+\frac{1}{2}a\left(a\text{arctanh}(\right.\right.\right.$$
$$\frac{\coth^{-1}(ax)^3}{4x^4}$$

input
```
Int[ArcCoth[a*x]^3/x^5,x]
```

output
```
-1/4*ArcCoth[a*x]^3/x^4 + (3*a*(-1/3*ArcCoth[a*x]^2/x^3 + a^2*(-(ArcCoth[a
*x]^2/x) + (a*ArcCoth[a*x]^3)/3 + 2*a*(ArcCoth[a*x]^2/2 + ArcCoth[a*x]*Log
[2 - 2/(1 + a*x)] - PolyLog[2, -1 + 2/(1 + a*x)]/2)) + (2*a*(-1/2*ArcCoth[
a*x]/x^2 + (a*(-x^(-1) + a*ArcTanh[a*x]))/2 + a^2*(ArcCoth[a*x]^2/2 + ArcC
oth[a*x]*Log[2 - 2/(1 + a*x)] - PolyLog[2, -1 + 2/(1 + a*x)]/2)))/3))/4
```

Defintions of rubi rules used

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 264
```
Int[((c_.)*(x_))^(m_)*((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(c*x)^(
m + 1)*((a + b*x^2)^(p + 1)/(a*c*(m + 1))), x] - Simp[b*((m + 2*p + 3)/(a*c
^2*(m + 1)))   Int[(c*x)^(m + 2)*(a + b*x^2)^p, x], x] /; FreeQ[{a, b, c, p
}, x] && LtQ[m, -1] && IntBinomialQ[a, b, c, 2, m, p, x]
```

rule 2897
```
Int[Log[u_]*(Pq_)^(m_.), x_Symbol] :> With[{C = FullSimplify[Pq^m*((1 - u)/
D[u, x])]}, Simp[C*PolyLog[2, 1 - u], x] /; FreeQ[C, x]] /; IntegerQ[m] &&
PolyQ[Pq, x] && RationalFunctionQ[u, x] && LeQ[RationalFunctionExponents[u,
 x][[2]], Expon[Pq, x]]
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

rule 6495
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((x_)*((d_) + (e_.)*(x_))), x
_Symbol] :> Simp[(a + b*ArcCoth[c*x])^p*(Log[2 - 2/(1 + e*(x/d))]/d), x] -
Simp[b*c*(p/d)   Int[(a + b*ArcCoth[c*x])^(p - 1)*(Log[2 - 2/(1 + e*(x/d))]
/(1 - c^2*x^2)), x], x] /; FreeQ[{a, b, c, d, e}, x] && IGtQ[p, 0] && EqQ[c
^2*d^2 - e^2, 0]
```

rule 6511
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((d_) + (e_.)*(x_)^2), x_Symb
ol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*c*d*(p + 1)), x] /; FreeQ[{a, b
, c, d, e, p}, x] && EqQ[c^2*d + e, 0] && NeQ[p, -1]
```

rule 6545
```
Int[(((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)*((f_.)*(x_))^(m_))/((d_) + (
e_.)*(x_)^2), x_Symbol] :> Simp[1/d   Int[(f*x)^m*(a + b*ArcCoth[c*x])^p, x
], x] - Simp[e/(d*f^2)   Int[(f*x)^(m + 2)*((a + b*ArcCoth[c*x])^p/(d + e*x
^2)), x], x] /; FreeQ[{a, b, c, d, e, f}, x] && GtQ[p, 0] && LtQ[m, -1]
```

rule 6551
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))^(p_.)/((x_)*((d_) + (e_.)*(x_)^2)),
 x_Symbol] :> Simp[(a + b*ArcCoth[c*x])^(p + 1)/(b*d*(p + 1)), x] + Simp[1/
d   Int[(a + b*ArcCoth[c*x])^p/(x*(1 + c*x)), x], x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[c^2*d + e, 0] && GtQ[p, 0]
```

Maple [C] (warning: unable to verify)

Result contains higher order function than in optimal. Order 9 vs. order 4.

Time = 4.56 (sec) , antiderivative size = 657, normalized size of antiderivative = 4.66

method	result
parts	$-\frac{\operatorname{arccoth}(xa)^3}{4x^4}-\frac{3a^4\left(\frac{\operatorname{arccoth}(xa)^2\ln(xa-1)}{2}+\frac{\operatorname{arccoth}(xa)^2}{3x^3a^3}+\frac{\operatorname{arccoth}(xa)^2}{xa}-\frac{\operatorname{arccoth}(xa)^2\ln(xa+1)}{2}+\frac{4\operatorname{arccoth}(xa)^2}{3}-\cdots\right.}{\cdots}$
derivativedivides	$a^4\left(-\frac{\operatorname{arccoth}(xa)^3}{4x^4a^4}-\frac{3\operatorname{arccoth}(xa)^2\ln(xa-1)}{8}-\frac{\operatorname{arccoth}(xa)^2}{4x^3a^3}-\frac{3\operatorname{arccoth}(xa)^2}{4xa}+\frac{3\operatorname{arccoth}(xa)^2\ln(xa+1)}{8}\cdots\right.$
default	$a^4\left(-\frac{\operatorname{arccoth}(xa)^3}{4x^4a^4}-\frac{3\operatorname{arccoth}(xa)^2\ln(xa-1)}{8}-\frac{\operatorname{arccoth}(xa)^2}{4x^3a^3}-\frac{3\operatorname{arccoth}(xa)^2}{4xa}+\frac{3\operatorname{arccoth}(xa)^2\ln(xa+1)}{8}\cdots\right.$

input
```
int(arccoth(x*a)^3/x^5,x,method=_RETURNVERBOSE)
```

output
```
-1/4*arccoth(x*a)^3/x^4-3/4*a^4*(1/2*arccoth(x*a)^2*ln(a*x-1)+1/3/x^3/a^3*
arccoth(x*a)^2+1/x/a*arccoth(x*a)^2-1/2*arccoth(x*a)^2*ln(a*x+1)+4/3*arcco
th(x*a)^2-8/3*dilog(1+I/((a*x-1)/(a*x+1))^(1/2))-8/3*dilog(1-I/((a*x-1)/(a
*x+1))^(1/2))-1/3*arccoth(x*a)^3-1/3*(a*x-1)/x/a+2/3*arccoth(x*a)*(a*x+1)/
x/a-1/4*I*Pi*csgn(I*(a*x+1)/(a*x-1))^3*arccoth(x*a)^2-8/3*arccoth(x*a)*ln(
1+I/((a*x-1)/(a*x+1))^(1/2))-8/3*arccoth(x*a)*ln(1-I/((a*x-1)/(a*x+1))^(1/
2))-1/3*arccoth(x*a)*(a*x+1)^2/x^2/a^2-2/3*arccoth(x*a)*(a*x-1)*(a*x+1)/x^
2/a^2-1/2*arccoth(x*a)^2*ln((a*x-1)/(a*x+1))-1/4*I*Pi*csgn(I/(a*x-1)*(a*x+
1)/((a*x+1)/(a*x-1)-1))^3*arccoth(x*a)^2-1/4*I*Pi*csgn(I/((a*x-1)/(a*x+1))
^(1/2))^2*csgn(I*(a*x+1)/(a*x-1))*arccoth(x*a)^2+1/2*I*Pi*csgn(I/((a*x-1)/
(a*x+1))^(1/2))*csgn(I*(a*x+1)/(a*x-1))^2*arccoth(x*a)^2+1/4*I*Pi*csgn(I*(
a*x+1)/(a*x-1))*csgn(I/(a*x-1)*(a*x+1)/((a*x+1)/(a*x-1)-1))^2*arccoth(x*a)
^2-1/4*I*Pi*csgn(I/((a*x+1)/(a*x-1)-1))*csgn(I*(a*x+1)/(a*x-1))*csgn(I/(a*
x-1)*(a*x+1)/((a*x+1)/(a*x-1)-1))*arccoth(x*a)^2+1/4*I*Pi*csgn(I/((a*x+1)/
(a*x-1)-1))*csgn(I/(a*x-1)*(a*x+1)/((a*x+1)/(a*x-1)-1))^2*arccoth(x*a)^2)
```

Fricas [F]

$$\int \frac{\coth^{-1}(ax)^3}{x^5}\,dx = \int \frac{\operatorname{arcoth}(ax)^3}{x^5}\,dx$$

input
```
integrate(arccoth(a*x)^3/x^5,x, algorithm="fricas")
```

output
```
integral(arccoth(a*x)^3/x^5, x)
```

Sympy [F]

$$\int \frac{\coth^{-1}(ax)^3}{x^5}\,dx = \int \frac{\operatorname{acoth}^3(ax)}{x^5}\,dx$$

input
```
integrate(acoth(a*x)**3/x**5,x)
```

output
```
Integral(acoth(a*x)**3/x**5, x)
```

Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 342 vs. 2(126) = 252.

Time = 0.04 (sec) , antiderivative size = 342, normalized size of antiderivative = 2.43

$$\int \frac{\coth^{-1}(ax)^3}{x^5}\,dx$$
$$= \frac{1}{8}\left(3\,a^3\log(ax+1) - 3\,a^3\log(ax-1) - \frac{2\,(3\,a^2x^2+1)}{x^3}\right)a\operatorname{arcoth}(ax)^2$$
$$+ \frac{1}{32}\left(\left(32\left(\log(ax-1)\log\left(\frac{1}{2}\,ax+\frac{1}{2}\right) + \operatorname{Li}_2\left(-\frac{1}{2}\,ax+\frac{1}{2}\right)\right)a - 32\left(\log(ax+1)\log(x) + \operatorname{Li}_2(-ax)\right.\right.\right.$$
$$- \frac{\operatorname{arcoth}(ax)^3}{4\,x^4}$$

input
```
integrate(arccoth(a*x)^3/x^5,x, algorithm="maxima")
```

output
```
1/8*(3*a^3*log(a*x + 1) - 3*a^3*log(a*x - 1) - 2*(3*a^2*x^2 + 1)/x^3)*a*ar
ccoth(a*x)^2 + 1/32*((32*(log(a*x - 1)*log(1/2*a*x + 1/2) + dilog(-1/2*a*x
 + 1/2))*a - 32*(log(a*x + 1)*log(x) + dilog(-a*x))*a + 32*(log(-a*x + 1)*
log(x) + dilog(a*x))*a + 4*a*log(a*x + 1) - 4*a*log(a*x - 1) + (a*x*log(a*
x + 1)^3 - a*x*log(a*x - 1)^3 - 8*a*x*log(a*x - 1)^2 - (3*a*x*log(a*x - 1)
 - 8*a*x)*log(a*x + 1)^2 + (3*a*x*log(a*x - 1)^2 - 16*a*x*log(a*x - 1))*lo
g(a*x + 1) - 8)/x)*a^2 + 2*(32*a^2*log(x) - (3*a^2*x^2*log(a*x + 1)^2 + 3*
a^2*x^2*log(a*x - 1)^2 + 16*a^2*x^2*log(a*x - 1) - 2*(3*a^2*x^2*log(a*x -
1) - 8*a^2*x^2)*log(a*x + 1) + 4)/x^2)*a*arccoth(a*x))*a - 1/4*arccoth(a*x
)^3/x^4
```

Giac [F]

$$\int \frac{\coth^{-1}(ax)^3}{x^5} \, dx = \int \frac{\operatorname{arcoth}\left(ax\right)^3}{x^5} \, dx$$

input
```
integrate(arccoth(a*x)^3/x^5,x, algorithm="giac")
```

output
```
integrate(arccoth(a*x)^3/x^5, x)
```

Mupad [F(-1)]

Timed out.

$$\int \frac{\coth^{-1}(ax)^3}{x^5} \, dx = \int \frac{\operatorname{acoth}(a\,x)^3}{x^5} \, dx$$

input
```
int(acoth(a*x)^3/x^5,x)
```

output
```
int(acoth(a*x)^3/x^5, x)
```

Reduce [F]

$$\int \frac{\coth^{-1}(ax)^3}{x^5}\,dx$$
$$= \frac{acoth(ax)^3\,a^4x^4 - acoth(ax)^3 + 3acoth(ax)^2\,a^3x^3 + acoth(ax)^2\,ax + acoth(ax)\,a^4x^4 - acoth(ax)\,a^2x^2 -}{4x^4}$$

input
```
int(acoth(a*x)^3/x^5,x)
```

output
```
(acoth(a*x)**3*a**4*x**4 - acoth(a*x)**3 + 3*acoth(a*x)**2*a**3*x**3 + aco
th(a*x)**2*a*x + acoth(a*x)*a**4*x**4 - acoth(a*x)*a**2*x**2 - 8*int(acoth
(a*x)/(a**2*x**3 - x),x)*a**4*x**4 + a**3*x**3)/(4*x**4)
```

3.34 $\int x^m \coth^{-1}(ax)^3 \, dx$

Optimal result

Integrand size = 10, antiderivative size = 10

$$\int x^m \coth^{-1}(ax)^3 \, dx = \text{Int}\left(x^m \coth^{-1}(ax)^3, x\right)$$

output
```
Defer(Int)(x^m*arccoth(a*x)^3,x)
```

Mathematica [N/A]

Not integrable

Time = 0.72 (sec) , antiderivative size = 12, normalized size of antiderivative = 1.20

$$\int x^m \coth^{-1}(ax)^3 \, dx = \int x^m \coth^{-1}(ax)^3 \, dx$$

input
```
Integrate[x^m*ArcCoth[a*x]^3,x]
```

output
```
Integrate[x^m*ArcCoth[a*x]^3, x]
```

Rubi [N/A]

Not integrable

Time = 0.18 (sec) , antiderivative size = 10, normalized size of antiderivative = 1.00, number of steps used = 1, number of rules used = 0, $\frac{\text{number of rules}}{\text{integrand size}} = 0.000$, Rules used = {}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x^m \coth^{-1}(ax)^3 \, dx$$

$$\Big\downarrow \text{6469}$$

$$\int x^m \coth^{-1}(ax)^3 dx$$

input
```
Int[x^m*ArcCoth[a*x]^3,x]
```

output
```
$Aborted
```

Maple [N/A]

Not integrable

Time = 0.28 (sec) , antiderivative size = 10, normalized size of antiderivative = 1.00

$$\int x^m \operatorname{arccoth}(xa)^3 \, dx$$

input
```
int(x^m*arccoth(x*a)^3,x)
```

output
```
int(x^m*arccoth(x*a)^3,x)
```

Fricas [N/A]

Not integrable

Time = 0.09 (sec) , antiderivative size = 12, normalized size of antiderivative = 1.20

$$\int x^m \coth^{-1}(ax)^3 \, dx = \int x^m \operatorname{arcoth}(ax)^3 \, dx$$

input
```
integrate(x^m*arccoth(a*x)^3,x, algorithm="fricas")
```

output
```
integral(x^m*arccoth(a*x)^3, x)
```

Sympy [N/A]

Not integrable

Time = 2.33 (sec) , antiderivative size = 10, normalized size of antiderivative = 1.00

$$\int x^m \coth^{-1}(ax)^3 \, dx = \int x^m \operatorname{acoth}^3(ax) \, dx$$

input
```
integrate(x**m*acoth(a*x)**3,x)
```

output
```
Integral(x**m*acoth(a*x)**3, x)
```

Maxima [N/A]

Not integrable

Time = 1.10 (sec) , antiderivative size = 124, normalized size of antiderivative = 12.40

$$\int x^m \coth^{-1}(ax)^3 \, dx = \int x^m \operatorname{arcoth}(ax)^3 \, dx$$

input
```
integrate(x^m*arccoth(a*x)^3,x, algorithm="maxima")
```

output
```
1/8*x*x^m*log(a*x + 1)^3/(m + 1) - 1/8*integrate(-(3*(a*(m + 1)*x + m + 1)
*x^m*log(a*x + 1)*log(a*x - 1)^2 - (a*(m + 1)*x + m + 1)*x^m*log(a*x - 1)^
3 - 3*(a*x*x^m + (a*(m + 1)*x + m + 1)*x^m*log(a*x - 1))*log(a*x + 1)^2)/(
a*(m + 1)*x + m + 1), x)
```

Giac [N/A]

Not integrable

Time = 0.14 (sec) , antiderivative size = 12, normalized size of antiderivative = 1.20

$$\int x^m \coth^{-1}(ax)^3 \, dx = \int x^m \operatorname{arcoth}(ax)^3 \, dx$$

input
```
integrate(x^m*arccoth(a*x)^3,x, algorithm="giac")
```

output
```
integrate(x^m*arccoth(a*x)^3, x)
```

Mupad [N/A]

Not integrable

Time = 3.53 (sec) , antiderivative size = 12, normalized size of antiderivative = 1.20

$$\int x^m \coth^{-1}(ax)^3 \, dx = \int x^m \operatorname{acoth}(a\,x)^3 \, dx$$

input
```
int(x^m*acoth(a*x)^3,x)
```

output
```
int(x^m*acoth(a*x)^3, x)
```

Reduce [N/A]

Not integrable

Time = 0.17 (sec) , antiderivative size = 12, normalized size of antiderivative = 1.20

$$\int x^m \coth^{-1}(ax)^3\,dx = \int x^m acoth(ax)^3\,dx$$

input
```
int(x^m*acoth(a*x)^3,x)
```

output
```
int(x**m*acoth(a*x)**3,x)
```

3.35 $\int x^m \coth^{-1}(ax)^2 \, dx$

Optimal result

Integrand size = 10, antiderivative size = 10

$$\int x^m \coth^{-1}(ax)^2 \, dx = \text{Int}\left(x^m \coth^{-1}(ax)^2, x\right)$$

output
```
Defer(Int)(x^m*arccoth(a*x)^2,x)
```

Mathematica [N/A]

Not integrable

Time = 0.73 (sec) , antiderivative size = 12, normalized size of antiderivative = 1.20

$$\int x^m \coth^{-1}(ax)^2 \, dx = \int x^m \coth^{-1}(ax)^2 \, dx$$

input
```
Integrate[x^m*ArcCoth[a*x]^2,x]
```

output
```
Integrate[x^m*ArcCoth[a*x]^2, x]
```

Rubi [N/A]

Not integrable

Time = 0.19 (sec) , antiderivative size = 10, normalized size of antiderivative = 1.00, number of steps used = 1, number of rules used = 0, $\frac{\text{number of rules}}{\text{integrand size}} = 0.000$, Rules used = {}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x^m \coth^{-1}(ax)^2 \, dx$$

$$\Big\downarrow \text{6469}$$

$$\int x^m \coth^{-1}(ax)^2 dx$$

input
```
Int[x^m*ArcCoth[a*x]^2,x]
```

output
```
$Aborted
```

Maple [N/A]

Not integrable

Time = 0.19 (sec) , antiderivative size = 10, normalized size of antiderivative = 1.00

$$\int x^m \operatorname{arccoth}(xa)^2 \, dx$$

input
```
int(x^m*arccoth(x*a)^2,x)
```

output
```
int(x^m*arccoth(x*a)^2,x)
```

Fricas [N/A]

Not integrable

Time = 0.10 (sec) , antiderivative size = 12, normalized size of antiderivative = 1.20

$$\int x^m \coth^{-1}(ax)^2 \, dx = \int x^m \operatorname{arcoth}(ax)^2 \, dx$$

input
```
integrate(x^m*arccoth(a*x)^2,x, algorithm="fricas")
```

output
```
integral(x^m*arccoth(a*x)^2, x)
```

Sympy [N/A]

Not integrable

Time = 1.22 (sec) , antiderivative size = 10, normalized size of antiderivative = 1.00

$$\int x^m \coth^{-1}(ax)^2 \, dx = \int x^m \operatorname{acoth}^2(ax) \, dx$$

input
```
integrate(x**m*acoth(a*x)**2,x)
```

output
```
Integral(x**m*acoth(a*x)**2, x)
```

Maxima [N/A]

Not integrable

Time = 0.73 (sec) , antiderivative size = 93, normalized size of antiderivative = 9.30

$$\int x^m \coth^{-1}(ax)^2 \, dx = \int x^m \operatorname{arcoth}(ax)^2 \, dx$$

input
```
integrate(x^m*arccoth(a*x)^2,x, algorithm="maxima")
```

output
```
1/4*x*x^m*log(a*x + 1)^2/(m + 1) - 1/4*integrate(-((a*(m + 1)*x + m + 1)*x
^m*log(a*x - 1)^2 - 2*(a*x*x^m + (a*(m + 1)*x + m + 1)*x^m*log(a*x - 1))*l
og(a*x + 1))/(a*(m + 1)*x + m + 1), x)
```

Giac [N/A]

Not integrable

Time = 0.14 (sec) , antiderivative size = 12, normalized size of antiderivative = 1.20

$$\int x^m \coth^{-1}(ax)^2 \, dx = \int x^m \operatorname{arcoth}(ax)^2 \, dx$$

input
```
integrate(x^m*arccoth(a*x)^2,x, algorithm="giac")
```

output
```
integrate(x^m*arccoth(a*x)^2, x)
```

Mupad [N/A]

Not integrable

Time = 3.66 (sec) , antiderivative size = 12, normalized size of antiderivative = 1.20

$$\int x^m \coth^{-1}(ax)^2 \, dx = \int x^m \operatorname{acoth}(a\,x)^2 \,dx$$

input
```
int(x^m*acoth(a*x)^2,x)
```

output
```
int(x^m*acoth(a*x)^2, x)
```

Reduce [N/A]

Not integrable

Time = 0.16 (sec) , antiderivative size = 12, normalized size of antiderivative = 1.20

$$\int x^m \coth^{-1}(ax)^2 \, dx = \int x^m acoth(ax)^2 \, dx$$

input
```
int(x^m*acoth(a*x)^2,x)
```

output
```
int(x**m*acoth(a*x)**2,x)
```

3.36 $\int x^m \coth^{-1}(ax)\,dx$

Optimal result

Integrand size = 8, antiderivative size = 57

$$\int x^m \coth^{-1}(ax)\,dx = \frac{x^{1+m}\coth^{-1}(ax)}{1+m} - \frac{ax^{2+m}\,\text{Hypergeometric2F1}\left(1,\frac{2+m}{2},\frac{4+m}{2},a^2x^2\right)}{2+3m+m^2}$$

output
```
x^(1+m)*arccoth(a*x)/(1+m)-a*x^(2+m)*hypergeom([1, 1+1/2*m],[2+1/2*m],a^2*
x^2)/(m^2+3*m+2)
```

Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 52, normalized size of antiderivative = 0.91

$$\int x^m \coth^{-1}(ax)\,dx = \frac{x^{1+m}\left((2+m)\coth^{-1}(ax) - ax\,\text{Hypergeometric2F1}\left(1,1+\frac{m}{2},2+\frac{m}{2},a^2x^2\right)\right)}{(1+m)(2+m)}$$

input
```
Integrate[x^m*ArcCoth[a*x],x]
```

output
```
(x^(1 + m)*((2 + m)*ArcCoth[a*x] - a*x*Hypergeometric2F1[1, 1 + m/2, 2 + m
/2, a^2*x^2]))/((1 + m)*(2 + m))
```

Rubi [A] (verified)

Time = 0.21 (sec) , antiderivative size = 57, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.250, Rules used = {6453, 278}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x^m \coth^{-1}(ax) \, dx$$

$$\Big\downarrow \text{6453}$$

$$\frac{x^{m+1}\coth^{-1}(ax)}{m+1}-\frac{a\int \frac{x^{m+1}}{1-a^2x^2}dx}{m+1}$$

$$\Big\downarrow \text{278}$$

$$\frac{x^{m+1}\coth^{-1}(ax)}{m+1}-\frac{ax^{m+2}\,\text{Hypergeometric2F1}\left(1,\frac{m+2}{2},\frac{m+4}{2},a^2x^2\right)}{(m+1)(m+2)}$$

input
```
Int[x^m*ArcCoth[a*x],x]
```

output
```
(x^(1 + m)*ArcCoth[a*x])/(1 + m) - (a*x^(2 + m)*Hypergeometric2F1[1, (2 +
m)/2, (4 + m)/2, a^2*x^2])/((1 + m)*(2 + m))
```

Defintions of rubi rules used

rule 278
```
Int[((c_.)*(x_))^(m_.)*((a_) + (b_.)*(x_)^2)^(p_), x_Symbol] :> Simp[a^p*((
c*x)^(m + 1)/(c*(m + 1)))*Hypergeometric2F1[-p, (m + 1)/2, (m + 1)/2 + 1, (
-b)*(x^2/a)], x] /; FreeQ[{a, b, c, m, p}, x] &&  !IGtQ[p, 0] && (ILtQ[p, 0
] || GtQ[a, 0])
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

Maple [F]

$$\int x^m \operatorname{arccoth}(xa)\, dx$$

input
```
int(x^m*arccoth(x*a),x)
```

output
```
int(x^m*arccoth(x*a),x)
```

Fricas [F]

$$\int x^m \coth^{-1}(ax)\, dx = \int x^m \operatorname{arcoth}(ax)\, dx$$

input
```
integrate(x^m*arccoth(a*x),x, algorithm="fricas")
```

output
```
integral(x^m*arccoth(a*x), x)
```

Sympy [F]

$$\int x^m \coth^{-1}(ax)\,dx = \int x^m \operatorname{acoth}(ax)\,dx$$

input
```
integrate(x**m*acoth(a*x),x)
```

output
```
Integral(x**m*acoth(a*x), x)
```

Maxima [F]

$$\int x^m \coth^{-1}(ax)\,dx = \int x^m \operatorname{arcoth}(ax)\,dx$$

input
```
integrate(x^m*arccoth(a*x),x, algorithm="maxima")
```

output
```
a*integrate(x*x^m/(a^2*(m + 1)*x^2 - m - 1), x) + 1/2*(x*x^m*log(a*x + 1)
- x*x^m*log(a*x - 1))/(m + 1)
```

Giac [F]

$$\int x^m \coth^{-1}(ax)\,dx = \int x^m \operatorname{arcoth}(ax)\,dx$$

input
```
integrate(x^m*arccoth(a*x),x, algorithm="giac")
```

output
```
integrate(x^m*arccoth(a*x), x)
```

Mupad [F(-1)]

Timed out.

$$\int x^m \coth^{-1}(ax)\,dx = \int x^m\,\mathrm{acoth}(a\,x)\,dx$$

input
```
int(x^m*acoth(a*x),x)
```

output
```
int(x^m*acoth(a*x), x)
```

Reduce [F]

$$\int x^m \coth^{-1}(ax)\,dx$$
$$= \frac{x^m\,acoth(ax)\,amx - x^m - \left(\int \frac{x^m}{a^2m\,x^3+a^2x^3-mx-x}dx\right)m^2 - \left(\int \frac{x^m}{a^2m\,x^3+a^2x^3-mx-x}dx\right)m}{am\,(m+1)}$$

input
```
int(x^m*acoth(a*x),x)
```

output
```
(x**m*acoth(a*x)*a*m*x - x**m - int(x**m/(a**2*m*x**3 + a**2*x**3 - m*x -
x),x)*m**2 - int(x**m/(a**2*m*x**3 + a**2*x**3 - m*x - x),x)*m)/(a*m*(m +
1))
```

3.37 $\int \frac{\coth^{-1}(ax^5)}{x} dx$

Optimal result

Integrand size = 10, antiderivative size = 28

$$\int \frac{\coth^{-1}(ax^5)}{x} dx = \frac{1}{10}\text{PolyLog}\left(2, -\frac{1}{ax^5}\right) - \frac{1}{10}\text{PolyLog}\left(2, \frac{1}{ax^5}\right)$$

output
```
1/10*polylog(2,-1/a/x^5)-1/10*polylog(2,1/a/x^5)
```

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 26, normalized size of antiderivative = 0.93

$$\int \frac{\coth^{-1}(ax^5)}{x} dx = \frac{1}{10}\left(\text{PolyLog}\left(2, -\frac{1}{ax^5}\right) - \text{PolyLog}\left(2, \frac{1}{ax^5}\right)\right)$$

input
```
Integrate[ArcCoth[a*x^5]/x,x]
```

output
```
(PolyLog[2, -(1/(a*x^5))] - PolyLog[2, 1/(a*x^5)])/10
```

Rubi [A] (verified)

Time = 0.24 (sec) , antiderivative size = 32, normalized size of antiderivative = 1.14, number of steps used = 3, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}} = 0.200$, Rules used = {6451, 6447}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\coth^{-1}(ax^5)}{x} dx$$

$$\Big\downarrow \text{6451}$$

$$\frac{1}{5} \int \frac{\coth^{-1}(ax^5)}{x^5} dx^5$$

$$\Big\downarrow \text{6447}$$

$$\frac{1}{5} \left(\frac{1}{2} \text{PolyLog} \left(2, -\frac{1}{ax^5}\right) - \frac{1}{2} \text{PolyLog} \left(2, \frac{1}{ax^5}\right) \right)$$

input
```
Int[ArcCoth[a*x^5]/x,x]
```

output
```
(PolyLog[2, -(1/(a*x^5))]/2 - PolyLog[2, 1/(a*x^5)]/2)/5
```

Defintions of rubi rules used

rule 6447
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))/(x_), x_Symbol] :> Simp[a*Log[x], x
] + (Simp[(b/2)*PolyLog[2, -(c*x)^(-1)], x] - Simp[(b/2)*PolyLog[2, 1/(c*x)
], x]) /; FreeQ[{a, b, c}, x]
```

rule 6451
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_)]*(b_.))^(p_.)/(x_), x_Symbol] :> Simp[
1/n   Subst[Int[(a + b*ArcCoth[c*x])^p/x, x], x, x^n], x] /; FreeQ[{a, b, c
, n}, x] && IGtQ[p, 0]
```

Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 4.

Time = 0.17 (sec) , antiderivative size = 95, normalized size of antiderivative = 3.39

method	result
default	$\ln\left(x\right)\operatorname{arccoth}\left(a\,x^{5}\right)+5a\left(-\frac{\sum_{\_R1=\operatorname{RootOf}\left(a\_Z^{5}+1\right)}\left(\ln\left(x\right)\ln\left(\frac{\_R1-x}{\_R1}\right)+\operatorname{dilog}\left(\frac{\_R1-x}{\_R1}\right)\right)}{10a}+\frac{\sum_{\_R1=\operatorname{RootOf}\left(a\_\right.}}{}\right.$
parts	$\ln\left(x\right)\operatorname{arccoth}\left(a\,x^{5}\right)+5a\left(-\frac{\sum_{\_R1=\operatorname{RootOf}\left(a\_Z^{5}+1\right)}\left(\ln\left(x\right)\ln\left(\frac{\_R1-x}{\_R1}\right)+\operatorname{dilog}\left(\frac{\_R1-x}{\_R1}\right)\right)}{10a}+\frac{\sum_{\_R1=\operatorname{RootOf}\left(a\_\right.}}{}\right.$
risch	$\frac{\ln\left(x\right)\ln\left(a\,x^{5}+1\right)}{2}-\frac{\left(\sum_{\_R1=\operatorname{RootOf}\left(a\_Z^{5}+1\right)}\left(\ln\left(x\right)\ln\left(\frac{\_R1-x}{\_R1}\right)+\operatorname{dilog}\left(\frac{\_R1-x}{\_R1}\right)\right)\right)}{2}-\frac{\ln\left(x\right)\ln\left(a\,x^{5}-1\right)}{2}+\frac{\left(\_R1=\mathrm{R}\right.}{}$

input
```
int(arccoth(a*x^5)/x,x,method=_RETURNVERBOSE)
```

output
```
ln(x)*arccoth(a*x^5)+5*a*(-1/10/a*sum(ln(x)*ln((_R1-x)/_R1)+dilog((_R1-x)/
_R1),_R1=RootOf(_Z^5*a+1))+1/10/a*sum(ln(x)*ln((_R1-x)/_R1)+dilog((_R1-x)/
_R1),_R1=RootOf(_Z^5*a-1)))
```

Fricas [F]

$$\int \frac{\coth^{-1}\left(ax^{5}\right)}{x}\,dx=\int \frac{\operatorname{arcoth}\left(ax^{5}\right)}{x}\,dx$$

input
```
integrate(arccoth(a*x^5)/x,x, algorithm="fricas")
```

output
```
integral(arccoth(a*x^5)/x, x)
```

Sympy [F(-1)]

Timed out.

$$\int \frac{\coth^{-1}(ax^5)}{x}\,dx = \text{Timed out}$$

input
```
integrate(acoth(a*x**5)/x,x)
```

output
```
Timed out
```

Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 104 vs. 2(22) = 44.

Time = 0.04 (sec) , antiderivative size = 104, normalized size of antiderivative = 3.71

$$\int \frac{\coth^{-1}(ax^5)}{x}\,dx = -\frac{1}{2}a\left(\frac{\log(ax^5+1)}{a} - \frac{\log(ax^5-1)}{a}\right)\log(x) - \frac{1}{10}a\left(\frac{\log(ax^5-1)\log(ax^5) + \text{Li}_2(-ax^5+1)}{a} - \frac{\log(ax^5+1)\log(-ax^5) + \text{Li}_2(ax^5+1)}{a}\right) + \text{arcoth}(ax^5)\log(x)$$

input
```
integrate(arccoth(a*x^5)/x,x, algorithm="maxima")
```

output
```
-1/2*a*(log(a*x^5 + 1)/a - log(a*x^5 - 1)/a)*log(x) - 1/10*a*((log(a*x^5 -
 1)*log(a*x^5) + dilog(-a*x^5 + 1))/a - (log(a*x^5 + 1)*log(-a*x^5) + dilo
g(a*x^5 + 1))/a) + arccoth(a*x^5)*log(x)
```

Giac [F]

$$\int \frac{\coth^{-1}(ax^5)}{x}\,dx = \int \frac{\operatorname{arcoth}(ax^5)}{x}\,dx$$

input
```
integrate(arccoth(a*x^5)/x,x, algorithm="giac")
```

output
```
integrate(arccoth(a*x^5)/x, x)
```

Mupad [F(-1)]

Timed out.

$$\int \frac{\coth^{-1}(ax^5)}{x}\,dx = \int \frac{\operatorname{acoth}(a\,x^5)}{x}\,dx$$

input
```
int(acoth(a*x^5)/x,x)
```

output
```
int(acoth(a*x^5)/x, x)
```

Reduce [F]

$$\int \frac{\coth^{-1}(ax^5)}{x}\,dx = \int \frac{acoth(a\,x^5)}{x}dx$$

input
```
int(acoth(a*x^5)/x,x)
```

output
```
int(acoth(a*x**5)/x,x)
```

3.38 $\int \coth^{-1}\left(\frac{1}{x}\right) dx$

Optimal result

Integrand size = 4, antiderivative size = 19

$$\int \coth^{-1}\left(\frac{1}{x}\right) dx = x\coth^{-1}\left(\frac{1}{x}\right) + \frac{1}{2}\log\left(1-x^2\right)$$

output
```
x*arccoth(1/x)+1/2*ln(-x^2+1)
```

Mathematica [A] (verified)

Time = 0.00 (sec) , antiderivative size = 17, normalized size of antiderivative = 0.89

$$\int \coth^{-1}\left(\frac{1}{x}\right) dx = x\coth^{-1}\left(\frac{1}{x}\right) + \frac{1}{2}\log\left(-1+x^2\right)$$

input
```
Integrate[ArcCoth[x^(-1)],x]
```

output
```
x*ArcCoth[x^(-1)] + Log[-1 + x^2]/2
```

Rubi [A] (verified)

Time = 0.18 (sec) , antiderivative size = 19, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.750, Rules used = {6437, 795, 240}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \coth^{-1}\left(\frac{1}{x}\right)dx$$

$$\Big\downarrow \text{6437}$$

$$\int \frac{1}{\left(1-\frac{1}{x^2}\right)x}dx+x\coth^{-1}\left(\frac{1}{x}\right)$$

$$\Big\downarrow \text{795}$$

$$\int \frac{x}{x^2-1}dx+x\coth^{-1}\left(\frac{1}{x}\right)$$

$$\Big\downarrow \text{240}$$

$$\frac{1}{2}\log\left(1-x^2\right)+x\coth^{-1}\left(\frac{1}{x}\right)$$

input
```
Int[ArcCoth[x^(-1)],x]
```

output
```
x*ArcCoth[x^(-1)] + Log[1 - x^2]/2
```

Defintions of rubi rules used

rule 240
```
Int[(x_)/((a_) + (b_.)*(x_)^2), x_Symbol] :> Simp[Log[RemoveContent[a + b*x
^2, x]]/(2*b), x] /; FreeQ[{a, b}, x]
```

rule 795
```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Int[x^(m + n*p)*
(b + a/x^n)^p, x] /; FreeQ[{a, b, m, n}, x] && IntegerQ[p] && NegQ[n]
```

rule 6437
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.), x_Symbol] :> Simp[x*(a
 + b*ArcCoth[c*x^n])^p, x] - Simp[b*c*n*p   Int[x^n*((a + b*ArcCoth[c*x^n])
^(p - 1)/(1 - c^2*x^(2*n))), x], x] /; FreeQ[{a, b, c, n}, x] && IGtQ[p, 0]
 && (EqQ[n, 1] || EqQ[p, 1])
```

Maple [A] (verified)

Time = 0.14 (sec) , antiderivative size = 16, normalized size of antiderivative = 0.84

method	result
parallelrisch	$x \operatorname{arccoth}\left(\frac{1}{x}\right)+\ln\left(x-1\right)+\operatorname{arccoth}\left(\frac{1}{x}\right)$
parts	$x \operatorname{arccoth}\left(\frac{1}{x}\right)+\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}$
derivativedivides	$x \operatorname{arccoth}\left(\frac{1}{x}\right)-\ln\left(\frac{1}{x}\right)+\frac{\ln\left(\frac{1}{x}-1\right)}{2}+\frac{\ln\left(\frac{1}{x}+1\right)}{2}$
default	$x \operatorname{arccoth}\left(\frac{1}{x}\right)-\ln\left(\frac{1}{x}\right)+\frac{\ln\left(\frac{1}{x}-1\right)}{2}+\frac{\ln\left(\frac{1}{x}+1\right)}{2}$
risch	$\frac{x\ln(x+1)}{2}-\frac{\ln(x-1)x}{2}+\frac{i\pi \operatorname{csgn}\left(\frac{i(x-1)}{x}\right)^{2} x}{2}-\frac{i\pi x}{2}+\frac{i\pi \operatorname{csgn}\left(\frac{i}{x}\right) \operatorname{csgn}(i(x-1)) \operatorname{csgn}\left(\frac{i(x-1)}{x}\right)x}{4}-\frac{i\pi \operatorname{csgn}\left(\frac{i(\cdots}{}\right.}{4}$

input
```
int(arccoth(1/x),x,method=_RETURNVERBOSE)
```

output
```
x*arccoth(1/x)+ln(x-1)+arccoth(1/x)
```

Fricas [A] (verification not implemented)

Time = 0.11 (sec) , antiderivative size = 23, normalized size of antiderivative = 1.21

$$\int \coth^{-1}\left(\frac{1}{x}\right)\,dx=\frac{1}{2}\,x\log\left(-\frac{x+1}{x-1}\right)+\frac{1}{2}\,\log\left(x^{2}-1\right)$$

input
```
integrate(arccoth(1/x),x, algorithm="fricas")
```

output
```
1/2*x*x*log(-(x + 1)/(x - 1)) + 1/2*log(x^2 - 1)
```

Sympy [A] (verification not implemented)

Time = 0.10 (sec) , antiderivative size = 15, normalized size of antiderivative = 0.79

$$\int \coth^{-1}\left(\frac{1}{x}\right)\,dx = x\,\mathrm{acoth}\left(\frac{1}{x}\right) + \log(x+1) - \mathrm{acoth}\left(\frac{1}{x}\right)$$

input
```
integrate(acoth(1/x),x)
```

output
```
x*acoth(1/x) + log(x + 1) - acoth(1/x)
```

Maxima [A] (verification not implemented)

Time = 0.02 (sec) , antiderivative size = 15, normalized size of antiderivative = 0.79

$$\int \coth^{-1}\left(\frac{1}{x}\right)\,dx = x\,\mathrm{arcoth}\left(\frac{1}{x}\right) + \frac{1}{2}\log\left(x^2-1\right)$$

input
```
integrate(arccoth(1/x),x, algorithm="maxima")
```

output
```
x*arccoth(1/x) + 1/2*log(x^2 - 1)
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 104 vs. 2(17) = 34.

Time = 0.12 (sec) , antiderivative size = 104, normalized size of antiderivative = 5.47

$$\int \coth^{-1}\left(\frac{1}{x}\right)\,dx = \frac{\log\left(-\frac{\frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1}+1}{\frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1}-1}\right)}{\frac{x+1}{x-1}-1} + \log\left(\frac{|-x-1|}{|x-1|}\right) - \log\left(\left|-\frac{x+1}{x-1}+1\right|\right)$$

input
```
integrate(arccoth(1/x),x, algorithm="giac")
```

output
```
log(-(((x + 1)/(x - 1) + 1)/((x + 1)/(x - 1) - 1) + 1)/(((x + 1)/(x - 1) +
 1)/((x + 1)/(x - 1) - 1) - 1))/((x + 1)/(x - 1) - 1) + log(abs(-x - 1)/ab
s(x - 1)) - log(abs(-(x + 1)/(x - 1) + 1))
```

Mupad [B] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 26, normalized size of antiderivative = 1.37

$$\int \coth^{-1}\left(\frac{1}{x}\right) dx = \frac{\ln(x^2-1)}{2} + x\left(\frac{\ln(x+1)}{2} - \frac{\ln(1-x)}{2}\right)$$

input
```
int(acoth(1/x),x)
```

output
```
log(x^2 - 1)/2 + x*(log(x + 1)/2 - log(1 - x)/2)
```

Reduce [B] (verification not implemented)

Time = 0.16 (sec) , antiderivative size = 17, normalized size of antiderivative = 0.89

$$\int \coth^{-1}\left(\frac{1}{x}\right) dx = acoth\left(\frac{1}{x}\right) x + acoth\left(\frac{1}{x}\right) - \log(x-1)$$

input
```
int(acoth(1/x),x)
```

output
```
acoth(1/x)*x + acoth(1/x) - log(x - 1)
```

3.39 $\int x^2 \coth^{-1}(\sqrt{x}) \, dx$

Optimal result

Integrand size = 10, antiderivative size = 51

$$\int x^2 \coth^{-1}(\sqrt{x}) \, dx = \frac{\sqrt{x}}{3} + \frac{x^{3/2}}{9} + \frac{x^{5/2}}{15} + \frac{1}{3}x^3 \coth^{-1}(\sqrt{x}) - \frac{\operatorname{arctanh}(\sqrt{x})}{3}$$

output
```
1/3*x^(1/2)+1/9*x^(3/2)+1/15*x^(5/2)+1/3*x^3*arccoth(x^(1/2))-1/3*arctanh(
x^(1/2))
```

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 59, normalized size of antiderivative = 1.16

$$\int x^2 \coth^{-1}(\sqrt{x}) \, dx = \frac{1}{90}\big(30\sqrt{x} + 10x^{3/2} + 6x^{5/2} + 30x^3 \coth^{-1}(\sqrt{x}) + 15\log(1-\sqrt{x}) - 15\log(1+\sqrt{x})\big)$$

input
```
Integrate[x^2*ArcCoth[Sqrt[x]],x]
```

output
```
(30*Sqrt[x] + 10*x^(3/2) + 6*x^(5/2) + 30*x^3*ArcCoth[Sqrt[x]] + 15*Log[1
- Sqrt[x]] - 15*Log[1 + Sqrt[x]])/90
```

Rubi [A] (verified)

Time = 0.20 (sec) , antiderivative size = 52, normalized size of antiderivative = 1.02, number of steps used = 7, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}} = 0.600$, Rules used = {6453, 60, 60, 60, 73, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x^2 \coth^{-1}\left(\sqrt{x}\right) dx$$

$$\Big\downarrow \text{ 6453}$$

$$\frac{1}{3}x^3 \coth^{-1}\left(\sqrt{x}\right) - \frac{1}{6}\int \frac{x^{5/2}}{1-x}dx$$

$$\Big\downarrow \text{ 60}$$

$$\frac{1}{6}\left(\frac{2x^{5/2}}{5} - \int \frac{x^{3/2}}{1-x}dx\right) + \frac{1}{3}x^3 \coth^{-1}\left(\sqrt{x}\right)$$

$$\Big\downarrow \text{ 60}$$

$$\frac{1}{6}\left(-\int \frac{\sqrt{x}}{1-x}dx + \frac{2x^{5/2}}{5} + \frac{2x^{3/2}}{3}\right) + \frac{1}{3}x^3 \coth^{-1}\left(\sqrt{x}\right)$$

$$\Big\downarrow \text{ 60}$$

$$\frac{1}{6}\left(-\int \frac{1}{(1-x)\sqrt{x}}dx + \frac{2x^{5/2}}{5} + \frac{2x^{3/2}}{3} + 2\sqrt{x}\right) + \frac{1}{3}x^3 \coth^{-1}\left(\sqrt{x}\right)$$

$$\Big\downarrow \text{ 73}$$

$$\frac{1}{6}\left(-2\int \frac{1}{1-x}d\sqrt{x} + \frac{2x^{5/2}}{5} + \frac{2x^{3/2}}{3} + 2\sqrt{x}\right) + \frac{1}{3}x^3 \coth^{-1}\left(\sqrt{x}\right)$$

$$\Big\downarrow \text{ 219}$$

$$\frac{1}{6}\left(-2\text{arctanh}\left(\sqrt{x}\right) + \frac{2x^{5/2}}{5} + \frac{2x^{3/2}}{3} + 2\sqrt{x}\right) + \frac{1}{3}x^3 \coth^{-1}\left(\sqrt{x}\right)$$

input
```
Int[x^2*ArcCoth[Sqrt[x]],x]
```

output
```
(x^3*ArcCoth[Sqrt[x]])/3 + (2*Sqrt[x] + (2*x^(3/2))/3 + (2*x^(5/2))/5 - 2*
ArcTanh[Sqrt[x]])/6
```

Defintions of rubi rules used

rule 60
```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
(a + b*x)^(m + 1)*((c + d*x)^n/(b*(m + n + 1))), x] + Simp[n*((b*c - a*d)/(
b*(m + n + 1)))   Int[(a + b*x)^m*(c + d*x)^(n - 1), x], x] /; FreeQ[{a, b,
 c, d}, x] && GtQ[n, 0] && NeQ[m + n + 1, 0] &&  !(IGtQ[m, 0] && ( !Integer
Q[n] || (GtQ[m, 0] && LtQ[m - n, 0]))) &&  !ILtQ[m + n + 2, 0] && IntLinear
Q[a, b, c, d, m, n, x]
```

rule 73
```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Simp[p/b   Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
 d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && Lt
Q[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Denominator[m]] && IntL
inearQ[a, b, c, d, m, n, x]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

Maple [A] (verified)

Time = 0.04 (sec) , antiderivative size = 42, normalized size of antiderivative = 0.82

method	result	size
derivativedivides	$\frac{x^3 \operatorname{arccoth}(\sqrt{x})}{3}+\frac{x^{\frac{5}{2}}}{15}+\frac{x^{\frac{3}{2}}}{9}+\frac{\sqrt{x}}{3}+\frac{\ln(\sqrt{x}-1)}{6}-\frac{\ln(\sqrt{x}+1)}{6}$	42
default	$\frac{x^3 \operatorname{arccoth}(\sqrt{x})}{3}+\frac{x^{\frac{5}{2}}}{15}+\frac{x^{\frac{3}{2}}}{9}+\frac{\sqrt{x}}{3}+\frac{\ln(\sqrt{x}-1)}{6}-\frac{\ln(\sqrt{x}+1)}{6}$	42
parts	$\frac{x^3 \operatorname{arccoth}(\sqrt{x})}{3}+\frac{x^{\frac{5}{2}}}{15}+\frac{x^{\frac{3}{2}}}{9}+\frac{\sqrt{x}}{3}+\frac{\ln(\sqrt{x}-1)}{6}-\frac{\ln(\sqrt{x}+1)}{6}$	42

input
```
int(x^2*arccoth(x^(1/2)),x,method=_RETURNVERBOSE)
```

output
```
1/3*x^3*arccoth(x^(1/2))+1/15*x^(5/2)+1/9*x^(3/2)+1/3*x^(1/2)+1/6*ln(x^(1/
2)-1)-1/6*ln(x^(1/2)+1)
```

Fricas [A] (verification not implemented)

Time = 0.10 (sec) , antiderivative size = 38, normalized size of antiderivative = 0.75

$$\int x^2 \coth^{-1}\left(\sqrt{x}\right)\,dx=\frac{1}{6}\left(x^3-1\right)\log\left(\frac{x+2\sqrt{x}+1}{x-1}\right)+\frac{1}{45}\left(3\,x^2+5\,x+15\right)\sqrt{x}$$

input
```
integrate(x^2*arccoth(x^(1/2)),x, algorithm="fricas")
```

output
```
1/6*(x^3 - 1)*log((x + 2*sqrt(x) + 1)/(x - 1)) + 1/45*(3*x^2 + 5*x + 15)*s
qrt(x)
```

Sympy [F]

$$\int x^2 \coth^{-1}\left(\sqrt{x}\right)\,dx = \int x^2 \operatorname{acoth}\left(\sqrt{x}\right)\,dx$$

input
```
integrate(x**2*acoth(x**(1/2)),x)
```

output
```
Integral(x**2*acoth(sqrt(x)), x)
```

Maxima [A] (verification not implemented)

Time = 0.02 (sec) , antiderivative size = 41, normalized size of antiderivative = 0.80

$$\int x^2 \coth^{-1}\left(\sqrt{x}\right)\,dx = \frac{1}{3}\,x^3 \operatorname{arcoth}\left(\sqrt{x}\right) + \frac{1}{15}\,x^{\frac{5}{2}} + \frac{1}{9}\,x^{\frac{3}{2}} + \frac{1}{3}\,\sqrt{x} - \frac{1}{6}\,\log\left(\sqrt{x}+1\right) + \frac{1}{6}\,\log\left(\sqrt{x}-1\right)$$

input
```
integrate(x^2*arccoth(x^(1/2)),x, algorithm="maxima")
```

output
```
1/3*x^3*arccoth(sqrt(x)) + 1/15*x^(5/2) + 1/9*x^(3/2) + 1/3*sqrt(x) - 1/6*
log(sqrt(x) + 1) + 1/6*log(sqrt(x) - 1)
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 164 vs. 2(31) = 62.

Time = 0.12 (sec) , antiderivative size = 164, normalized size of antiderivative = 3.22

$$\int x^2 \coth^{-1}\left(\sqrt{x}\right)\,dx = \frac{2\left(\frac{45\left(\sqrt{x}+1\right)^4}{\left(\sqrt{x}-1\right)^4} - \frac{90\left(\sqrt{x}+1\right)^3}{\left(\sqrt{x}-1\right)^3} + \frac{140\left(\sqrt{x}+1\right)^2}{\left(\sqrt{x}-1\right)^2} - \frac{70\left(\sqrt{x}+1\right)}{\sqrt{x}-1} + 23\right)}{45\left(\frac{\sqrt{x}+1}{\sqrt{x}-1} - 1\right)^5} + \frac{2\left(\frac{3\left(\sqrt{x}+1\right)^5}{\left(\sqrt{x}-1\right)^5} + \frac{10\left(\sqrt{x}+1\right)^3}{\left(\sqrt{x}-1\right)^3} + \frac{3\left(\sqrt{x}+1\right)}{\sqrt{x}-1}\right)\log\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)}{3\left(\frac{\sqrt{x}+1}{\sqrt{x}-1} - 1\right)^6}$$

input
```
integrate(x^2*arccoth(x^(1/2)),x, algorithm="giac")
```

output
```
2/45*(45*(sqrt(x) + 1)^4/(sqrt(x) - 1)^4 - 90*(sqrt(x) + 1)^3/(sqrt(x) - 1
)^3 + 140*(sqrt(x) + 1)^2/(sqrt(x) - 1)^2 - 70*(sqrt(x) + 1)/(sqrt(x) - 1)
 + 23)/((sqrt(x) + 1)/(sqrt(x) - 1) - 1)^5 + 2/3*(3*(sqrt(x) + 1)^5/(sqrt(
x) - 1)^5 + 10*(sqrt(x) + 1)^3/(sqrt(x) - 1)^3 + 3*(sqrt(x) + 1)/(sqrt(x)
- 1))*log((sqrt(x) + 1)/(sqrt(x) - 1))/((sqrt(x) + 1)/(sqrt(x) - 1) - 1)^6
```

Mupad [B] (verification not implemented)

Time = 3.67 (sec) , antiderivative size = 31, normalized size of antiderivative = 0.61

$$\int x^2 \coth^{-1}\left(\sqrt{x}\right) dx = \frac{x^3 \operatorname{acoth}\left(\sqrt{x}\right)}{3} - \frac{\operatorname{acoth}\left(\sqrt{x}\right)}{3} + \frac{\sqrt{x}}{3} + \frac{x^{3/2}}{9} + \frac{x^{5/2}}{15}$$

input
```
int(x^2*acoth(x^(1/2)),x)
```

output
```
(x^3*acoth(x^(1/2)))/3 - acoth(x^(1/2))/3 + x^(1/2)/3 + x^(3/2)/9 + x^(5/2
)/15
```

Reduce [B] (verification not implemented)

Time = 0.16 (sec) , antiderivative size = 30, normalized size of antiderivative = 0.59

$$\int x^2 \coth^{-1}\left(\sqrt{x}\right) dx = \frac{acoth\left(\sqrt{x}\right) x^3}{3} - \frac{acoth\left(\sqrt{x}\right)}{3} - \frac{\sqrt{x}\, x^2}{15} - \frac{\sqrt{x}\, x}{9} - \frac{\sqrt{x}}{3}$$

input
```
int(x^2*acoth(x^(1/2)),x)
```

output
```
(15*acoth(sqrt(x))*x**3 - 15*acoth(sqrt(x)) - 3*sqrt(x)*x**2 - 5*sqrt(x)*x
 - 15*sqrt(x))/45
```

3.40 $\int x \coth^{-1}\left(\sqrt{x}\right) dx$

Optimal result

Integrand size = 8, antiderivative size = 42

$$\int x \coth^{-1}\left(\sqrt{x}\right) dx = \frac{\sqrt{x}}{2} + \frac{x^{3/2}}{6} + \frac{1}{2}x^2 \coth^{-1}\left(\sqrt{x}\right) - \frac{\operatorname{arctanh}\left(\sqrt{x}\right)}{2}$$

output
```
1/2*x^(1/2)+1/6*x^(3/2)+1/2*x^2*arccoth(x^(1/2))-1/2*arctanh(x^(1/2))
```

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 52, normalized size of antiderivative = 1.24

$$\int x \coth^{-1}\left(\sqrt{x}\right) dx = \frac{1}{12}\left(6\sqrt{x} + 2x^{3/2} + 6x^2 \coth^{-1}\left(\sqrt{x}\right) + 3\log\left(1-\sqrt{x}\right) - 3\log\left(1+\sqrt{x}\right)\right)$$

input
```
Integrate[x*ArcCoth[Sqrt[x]],x]
```

output
```
(6*Sqrt[x] + 2*x^(3/2) + 6*x^2*ArcCoth[Sqrt[x]] + 3*Log[1 - Sqrt[x]] - 3*L
og[1 + Sqrt[x]])/12
```

Rubi [A] (verified)

Time = 0.19 (sec) , antiderivative size = 43, normalized size of antiderivative = 1.02, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}} = 0.625$, Rules used = {6453, 60, 60, 73, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x \coth^{-1}\left(\sqrt{x}\right) dx$$

$$\Big\downarrow \text{6453}$$

$$\frac{1}{2}x^2 \coth^{-1}\left(\sqrt{x}\right) - \frac{1}{4}\int \frac{x^{3/2}}{1-x}dx$$

$$\Big\downarrow \text{60}$$

$$\frac{1}{4}\left(\frac{2x^{3/2}}{3} - \int \frac{\sqrt{x}}{1-x}dx\right) + \frac{1}{2}x^2 \coth^{-1}\left(\sqrt{x}\right)$$

$$\Big\downarrow \text{60}$$

$$\frac{1}{4}\left(-\int \frac{1}{(1-x)\sqrt{x}}dx + \frac{2x^{3/2}}{3} + 2\sqrt{x}\right) + \frac{1}{2}x^2 \coth^{-1}\left(\sqrt{x}\right)$$

$$\Big\downarrow \text{73}$$

$$\frac{1}{4}\left(-2\int \frac{1}{1-x}d\sqrt{x} + \frac{2x^{3/2}}{3} + 2\sqrt{x}\right) + \frac{1}{2}x^2 \coth^{-1}\left(\sqrt{x}\right)$$

$$\Big\downarrow \text{219}$$

$$\frac{1}{4}\left(-2\text{arctanh}\left(\sqrt{x}\right) + \frac{2x^{3/2}}{3} + 2\sqrt{x}\right) + \frac{1}{2}x^2 \coth^{-1}\left(\sqrt{x}\right)$$

input
```
Int[x*ArcCoth[Sqrt[x]],x]
```

output
```
(x^2*ArcCoth[Sqrt[x]])/2 + (2*Sqrt[x] + (2*x^(3/2))/3 - 2*ArcTanh[Sqrt[x]]
)/4
```

Defintions of rubi rules used

rule 60
```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
(a + b*x)^(m + 1)*((c + d*x)^n/(b*(m + n + 1))), x] + Simp[n*((b*c - a*d)/(
b*(m + n + 1)))   Int[(a + b*x)^m*(c + d*x)^(n - 1), x], x] /; FreeQ[{a, b,
 c, d}, x] && GtQ[n, 0] && NeQ[m + n + 1, 0] &&  !(IGtQ[m, 0] && ( !Integer
Q[n] || (GtQ[m, 0] && LtQ[m - n, 0]))) &&  !ILtQ[m + n + 2, 0] && IntLinear
Q[a, b, c, d, m, n, x]
```

rule 73
```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Simp[p/b   Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
 d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && Lt
Q[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Denominator[m]] && IntL
inearQ[a, b, c, d, m, n, x]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

Maple [A] (verified)

Time = 0.03 (sec) , antiderivative size = 37, normalized size of antiderivative = 0.88

method	result	size
derivativedivides	$\frac{x^{2} \operatorname{arccoth}\left(\sqrt{x}\right)}{2}+\frac{x^{\frac{3}{2}}}{6}+\frac{\sqrt{x}}{2}+\frac{\ln\left(\sqrt{x}-1\right)}{4}-\frac{\ln\left(\sqrt{x}+1\right)}{4}$	37
default	$\frac{x^{2} \operatorname{arccoth}\left(\sqrt{x}\right)}{2}+\frac{x^{\frac{3}{2}}}{6}+\frac{\sqrt{x}}{2}+\frac{\ln\left(\sqrt{x}-1\right)}{4}-\frac{\ln\left(\sqrt{x}+1\right)}{4}$	37
parts	$\frac{x^{2} \operatorname{arccoth}\left(\sqrt{x}\right)}{2}+\frac{x^{\frac{3}{2}}}{6}+\frac{\sqrt{x}}{2}+\frac{\ln\left(\sqrt{x}-1\right)}{4}-\frac{\ln\left(\sqrt{x}+1\right)}{4}$	37

input
```
int(x*arccoth(x^(1/2)),x,method=_RETURNVERBOSE)
```

output
```
1/2*x^2*arccoth(x^(1/2))+1/6*x^(3/2)+1/2*x^(1/2)+1/4*ln(x^(1/2)-1)-1/4*ln(
x^(1/2)+1)
```

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 31, normalized size of antiderivative = 0.74

$$\int x \coth^{-1}\left(\sqrt{x}\right)\,dx = \frac{1}{4}\left(x^2-1\right)\log\left(\frac{x+2\sqrt{x}+1}{x-1}\right)+\frac{1}{6}\left(x+3\right)\sqrt{x}$$

input
```
integrate(x*arccoth(x^(1/2)),x, algorithm="fricas")
```

output
```
1/4*(x^2 - 1)*log((x + 2*sqrt(x) + 1)/(x - 1)) + 1/6*(x + 3)*sqrt(x)
```

Sympy [F]

$$\int x \coth^{-1}\left(\sqrt{x}\right)\,dx = \int x \operatorname{acoth}\left(\sqrt{x}\right)dx$$

input
```
integrate(x*acoth(x**(1/2)),x)
```

output
```
Integral(x*acoth(sqrt(x)), x)
```

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 36, normalized size of antiderivative = 0.86

$$\int x \coth^{-1}\left(\sqrt{x}\right)\,dx = \frac{1}{2}\,x^2\,\text{arcoth}\left(\sqrt{x}\right) + \frac{1}{6}\,x^{\frac{3}{2}} + \frac{1}{2}\,\sqrt{x} - \frac{1}{4}\,\log\left(\sqrt{x}+1\right) + \frac{1}{4}\,\log\left(\sqrt{x} - 1\right)$$

input
```
integrate(x*arccoth(x^(1/2)),x, algorithm="maxima")
```

output
```
1/2*x^2*arccoth(sqrt(x)) + 1/6*x^(3/2) + 1/2*sqrt(x) - 1/4*log(sqrt(x) + 1
) + 1/4*log(sqrt(x) - 1)
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 114 vs. 2(26) = 52.

Time = 0.12 (sec) , antiderivative size = 114, normalized size of antiderivative = 2.71

$$\int x \coth^{-1}\left(\sqrt{x}\right)\,dx = \frac{2\left(\frac{3\left(\sqrt{x}+1\right)^2}{\left(\sqrt{x}-1\right)^2} - \frac{3\left(\sqrt{x}+1\right)}{\sqrt{x}-1} + 2\right)}{3\left(\frac{\sqrt{x}+1}{\sqrt{x}-1} - 1\right)^3} + \frac{2\left(\frac{\left(\sqrt{x}+1\right)^3}{\left(\sqrt{x}-1\right)^3} + \frac{\sqrt{x}+1}{\sqrt{x}-1}\right)\log\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)}{\left(\frac{\sqrt{x}+1}{\sqrt{x}-1} - 1\right)^4}$$

input
```
integrate(x*arccoth(x^(1/2)),x, algorithm="giac")
```

output
```
2/3*(3*(sqrt(x) + 1)^2/(sqrt(x) - 1)^2 - 3*(sqrt(x) + 1)/(sqrt(x) - 1) + 2
)/((sqrt(x) + 1)/(sqrt(x) - 1) - 1)^3 + 2*((sqrt(x) + 1)^3/(sqrt(x) - 1)^3
 + (sqrt(x) + 1)/(sqrt(x) - 1))*log((sqrt(x) + 1)/(sqrt(x) - 1))/((sqrt(x)
 + 1)/(sqrt(x) - 1) - 1)^4
```

Mupad [B] (verification not implemented)

Time = 3.69 (sec) , antiderivative size = 26, normalized size of antiderivative = 0.62

$$\int x \coth^{-1}\left(\sqrt{x}\right)\,dx = \frac{x^2\,\mathrm{acoth}\left(\sqrt{x}\right)}{2} - \frac{\mathrm{acoth}\left(\sqrt{x}\right)}{2} + \frac{\sqrt{x}}{2} + \frac{x^{3/2}}{6}$$

input
```
int(x*acoth(x^(1/2)),x)
```

output
```
(x^2*acoth(x^(1/2)))/2 - acoth(x^(1/2))/2 + x^(1/2)/2 + x^(3/2)/6
```

Reduce [B] (verification not implemented)

Time = 0.17 (sec) , antiderivative size = 23, normalized size of antiderivative = 0.55

$$\int x \coth^{-1}\left(\sqrt{x}\right)\,dx = \frac{acoth\left(\sqrt{x}\right)x^2}{2} - \frac{acoth\left(\sqrt{x}\right)}{2} - \frac{\sqrt{x}\,x}{6} - \frac{\sqrt{x}}{2}$$

input
```
int(x*acoth(x^(1/2)),x)
```

output
```
(3*acoth(sqrt(x))*x**2 - 3*acoth(sqrt(x)) - sqrt(x)*x - 3*sqrt(x))/6
```

3.41 $\int \coth^{-1}(\sqrt{x})\,dx$

Optimal result

Integrand size = 6, antiderivative size = 22

$$\int \coth^{-1}(\sqrt{x})\,dx = \sqrt{x} + x\coth^{-1}(\sqrt{x}) - \operatorname{arctanh}(\sqrt{x})$$

output
```
x^(1/2)+x*arccoth(x^(1/2))-arctanh(x^(1/2))
```

Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 22, normalized size of antiderivative = 1.00

$$\int \coth^{-1}(\sqrt{x})\,dx = \sqrt{x} + x\coth^{-1}(\sqrt{x}) - \operatorname{arctanh}(\sqrt{x})$$

input
```
Integrate[ArcCoth[Sqrt[x]],x]
```

output
```
Sqrt[x] + x*ArcCoth[Sqrt[x]] - ArcTanh[Sqrt[x]]
```

Rubi [A] (verified)

Time = 0.19 (sec) , antiderivative size = 29, normalized size of antiderivative = 1.32, number of steps used = 5, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}} = 0.667$, Rules used = {6437, 60, 73, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \coth^{-1}\left(\sqrt{x}\right) \, dx$$

$$\Big\downarrow \ 6437$$

$$x\coth^{-1}\left(\sqrt{x}\right)-\frac{1}{2}\int \frac{\sqrt{x}}{1-x}dx$$

$$\Big\downarrow \ 60$$

$$\frac{1}{2}\left(2\sqrt{x}-\int \frac{1}{(1-x)\sqrt{x}}dx\right)+x\coth^{-1}\left(\sqrt{x}\right)$$

$$\Big\downarrow \ 73$$

$$\frac{1}{2}\left(2\sqrt{x}-2\int \frac{1}{1-x}d\sqrt{x}\right)+x\coth^{-1}\left(\sqrt{x}\right)$$

$$\Big\downarrow \ 219$$

$$\frac{1}{2}\left(2\sqrt{x}-2\text{arctanh}\left(\sqrt{x}\right)\right)+x\coth^{-1}\left(\sqrt{x}\right)$$

input
```
Int[ArcCoth[Sqrt[x]],x]
```

output
```
x*ArcCoth[Sqrt[x]] + (2*Sqrt[x] - 2*ArcTanh[Sqrt[x]])/2
```

Defintions of rubi rules used

rule 60
```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
(a + b*x)^(m + 1)*((c + d*x)^n/(b*(m + n + 1))), x] + Simp[n*((b*c - a*d)/(
b*(m + n + 1)))   Int[(a + b*x)^m*(c + d*x)^(n - 1), x], x] /; FreeQ[{a, b,
 c, d}, x] && GtQ[n, 0] && NeQ[m + n + 1, 0] &&  !(IGtQ[m, 0] && ( !Integer
Q[n] || (GtQ[m, 0] && LtQ[m - n, 0]))) &&  !ILtQ[m + n + 2, 0] && IntLinear
Q[a, b, c, d, m, n, x]
```

rule 73
```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Simp[p/b   Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
 d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && Lt
Q[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Denominator[m]] && IntL
inearQ[a, b, c, d, m, n, x]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 6437
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.), x_Symbol] :> Simp[x*(a
 + b*ArcCoth[c*x^n])^p, x] - Simp[b*c*n*p   Int[x^n*((a + b*ArcCoth[c*x^n])
^(p - 1)/(1 - c^2*x^(2*n))), x], x] /; FreeQ[{a, b, c, n}, x] && IGtQ[p, 0]
 && (EqQ[n, 1] || EqQ[p, 1])
```

Maple [A] (verified)

Time = 0.03 (sec) , antiderivative size = 27, normalized size of antiderivative = 1.23

method	result	size
derivativedivides	$x \operatorname{arccoth}\left(\sqrt{x}\right)+\sqrt{x}+\frac{\ln\left(\sqrt{x}-1\right)}{2}-\frac{\ln\left(\sqrt{x}+1\right)}{2}$	27
default	$x \operatorname{arccoth}\left(\sqrt{x}\right)+\sqrt{x}+\frac{\ln\left(\sqrt{x}-1\right)}{2}-\frac{\ln\left(\sqrt{x}+1\right)}{2}$	27
parts	$x \operatorname{arccoth}\left(\sqrt{x}\right)+\sqrt{x}+\frac{\ln\left(\sqrt{x}-1\right)}{2}-\frac{\ln\left(\sqrt{x}+1\right)}{2}$	27

input
```
int(arccoth(x^(1/2)),x,method=_RETURNVERBOSE)
```

output
```
x*arccoth(x^(1/2))+x^(1/2)+1/2*ln(x^(1/2)-1)-1/2*ln(x^(1/2)+1)
```

Fricas [A] (verification not implemented)

Time = 0.09 (sec) , antiderivative size = 24, normalized size of antiderivative = 1.09

$$\int \coth^{-1}\left(\sqrt{x}\right) dx = \frac{1}{2}(x-1)\log\left(\frac{x+2\sqrt{x}+1}{x-1}\right)+\sqrt{x}$$

input
```
integrate(arccoth(x^(1/2)),x, algorithm="fricas")
```

output
```
1/2*(x - 1)*log((x + 2*sqrt(x) + 1)/(x - 1)) + sqrt(x)
```

Sympy [F]

$$\int \coth^{-1}\left(\sqrt{x}\right) dx = \int \operatorname{acoth}\left(\sqrt{x}\right) dx$$

input
```
integrate(acoth(x**(1/2)),x)
```

output
```
Integral(acoth(sqrt(x)), x)
```

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 26, normalized size of antiderivative = 1.18

$$\int \coth^{-1}\left(\sqrt{x}\right) dx = x\operatorname{arcoth}\left(\sqrt{x}\right)+\sqrt{x}-\frac{1}{2}\log\left(\sqrt{x}+1\right)+\frac{1}{2}\log\left(\sqrt{x}-1\right)$$

input
```
integrate(arccoth(x^(1/2)),x, algorithm="maxima")
```

output
```
x*arccoth(sqrt(x)) + sqrt(x) - 1/2*log(sqrt(x) + 1) + 1/2*log(sqrt(x) - 1)
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 65 vs. $2(16) = 32$.

Time = 0.12 (sec) , antiderivative size = 65, normalized size of antiderivative = 2.95

$$\int \coth^{-1}\left(\sqrt{x}\right)\,dx = \frac{2}{\frac{\sqrt{x}+1}{\sqrt{x}-1}-1} + \frac{2\left(\sqrt{x}+1\right)\log\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)}{\left(\sqrt{x}-1\right)\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}-1\right)^2}$$

input
```
integrate(arccoth(x^(1/2)),x, algorithm="giac")
```

output
```
2/((sqrt(x) + 1)/(sqrt(x) - 1) - 1) + 2*(sqrt(x) + 1)*log((sqrt(x) + 1)/(s
qrt(x) - 1))/((sqrt(x) - 1)*((sqrt(x) + 1)/(sqrt(x) - 1) - 1)^2)
```

Mupad [B] (verification not implemented)

Time = 3.68 (sec) , antiderivative size = 16, normalized size of antiderivative = 0.73

$$\int \coth^{-1}\left(\sqrt{x}\right)\,dx = x\,\mathrm{acoth}\left(\sqrt{x}\right) - \mathrm{acoth}\left(\sqrt{x}\right) + \sqrt{x}$$

input
```
int(acoth(x^(1/2)),x)
```

output
```
x*acoth(x^(1/2)) - acoth(x^(1/2)) + x^(1/2)
```

Reduce [B] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 15, normalized size of antiderivative = 0.68

$$\int \coth^{-1}\left(\sqrt{x}\right)\,dx = acoth\left(\sqrt{x}\right)x - acoth\left(\sqrt{x}\right) - \sqrt{x}$$

input
```
int(acoth(x^(1/2)),x)
```

output
```
acoth(sqrt(x))*x - acoth(sqrt(x)) - sqrt(x)
```

3.42 $\int \frac{\coth^{-1}(\sqrt{x})}{x} dx$

Optimal result

Integrand size = 10, antiderivative size = 19

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{x} dx = \text{PolyLog}\left(2,-\frac{1}{\sqrt{x}}\right) - \text{PolyLog}\left(2,\frac{1}{\sqrt{x}}\right)$$

output
```
polylog(2,-1/x^(1/2))-polylog(2,1/x^(1/2))
```

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 19, normalized size of antiderivative = 1.00

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{x} dx = \text{PolyLog}\left(2,-\frac{1}{\sqrt{x}}\right) - \text{PolyLog}\left(2,\frac{1}{\sqrt{x}}\right)$$

input
```
Integrate[ArcCoth[Sqrt[x]]/x,x]
```

output
```
PolyLog[2, -(1/Sqrt[x])] - PolyLog[2, 1/Sqrt[x]]
```

Rubi [A] (verified)

Time = 0.23 (sec) , antiderivative size = 27, normalized size of antiderivative = 1.42, number of steps used = 3, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.200, Rules used = {6451, 6447}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{x} \, dx$$

$$\Big\downarrow \text{6451}$$

$$2\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{\sqrt{x}}d\sqrt{x}$$

$$\Big\downarrow \text{6447}$$

$$2\left(\frac{1}{2}\operatorname{PolyLog}\left(2,-\frac{1}{\sqrt{x}}\right)-\frac{\operatorname{PolyLog}\left(2,\frac{1}{\sqrt{x}}\right)}{2}\right)$$

input
```
Int[ArcCoth[Sqrt[x]]/x,x]
```

output
```
2*(PolyLog[2, -(1/Sqrt[x])]/2 - PolyLog[2, 1/Sqrt[x]]/2)
```

Defintions of rubi rules used

rule 6447
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))/(x_), x_Symbol] :> Simp[a*Log[x], x
] + (Simp[(b/2)*PolyLog[2, -(c*x)^(-1)], x] - Simp[(b/2)*PolyLog[2, 1/(c*x)
], x]) /; FreeQ[{a, b, c}, x]
```

rule 6451
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_)]*(b_.))^(p_.)/(x_), x_Symbol] :> Simp[
1/n   Subst[Int[(a + b*ArcCoth[c*x])^p/x, x], x, x^n], x] /; FreeQ[{a, b, c
, n}, x] && IGtQ[p, 0]
```

Maple [B] (verified)

Leaf count of result is larger than twice the leaf count of optimal. 32 vs. 2(15) = 30.

Time = 0.14 (sec) , antiderivative size = 33, normalized size of antiderivative = 1.74

method	result	size
derivativedivides	$\ln(x)\,\mathrm{arccoth}\left(\sqrt{x}\right)-\mathrm{dilog}\left(\sqrt{x}\right)-\mathrm{dilog}\left(\sqrt{x}+1\right)-\frac{\ln(x)\ln\left(\sqrt{x}+1\right)}{2}$	33
default	$\ln(x)\,\mathrm{arccoth}\left(\sqrt{x}\right)-\mathrm{dilog}\left(\sqrt{x}\right)-\mathrm{dilog}\left(\sqrt{x}+1\right)-\frac{\ln(x)\ln\left(\sqrt{x}+1\right)}{2}$	33
parts	$\ln(x)\,\mathrm{arccoth}\left(\sqrt{x}\right)-\mathrm{dilog}\left(\sqrt{x}\right)-\mathrm{dilog}\left(\sqrt{x}+1\right)-\frac{\ln(x)\ln\left(\sqrt{x}+1\right)}{2}$	33

input
```
int(arccoth(x^(1/2))/x,x,method=_RETURNVERBOSE)
```

output
```
ln(x)*arccoth(x^(1/2))-dilog(x^(1/2))-dilog(x^(1/2)+1)-1/2*ln(x)*ln(x^(1/2
)+1)
```

Fricas [F]

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{x}\,dx=\int \frac{\mathrm{arcoth}\left(\sqrt{x}\right)}{x}\,dx$$

input
```
integrate(arccoth(x^(1/2))/x,x, algorithm="fricas")
```

output
```
integral(arccoth(sqrt(x))/x, x)
```

Sympy [F]

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{x}\,dx=\int \frac{\mathrm{acoth}\left(\sqrt{x}\right)}{x}\,dx$$

input
```
integrate(acoth(x**(1/2))/x,x)
```

output
```
Integral(acoth(sqrt(x))/x, x)
```

Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 66 vs. $2(13) = 26$.

Time = 0.03 (sec) , antiderivative size = 66, normalized size of antiderivative = 3.47

$$\int \frac{\coth^{-1}(\sqrt{x})}{x} dx = -\frac{1}{2}\left(\log\left(\sqrt{x}+1\right) - \log\left(\sqrt{x}-1\right)\right)\log(x) + \operatorname{arcoth}\left(\sqrt{x}\right)\log(x) + \log\left(-\sqrt{x}\right)\log\left(\sqrt{x}+1\right) - \frac{1}{2}\log(x)\log\left(\sqrt{x}-1\right) + \mathrm{Li}_2\left(\sqrt{x}+1\right) - \mathrm{Li}_2\left(-\sqrt{x}+1\right)$$

input
```
integrate(arccoth(x^(1/2))/x,x, algorithm="maxima")
```

output
```
-1/2*(log(sqrt(x) + 1) - log(sqrt(x) - 1))*log(x) + arccoth(sqrt(x))*log(x
) + log(-sqrt(x))*log(sqrt(x) + 1) - 1/2*log(x)*log(sqrt(x) - 1) + dilog(s
qrt(x) + 1) - dilog(-sqrt(x) + 1)
```

Giac [F]

$$\int \frac{\coth^{-1}(\sqrt{x})}{x} dx = \int \frac{\operatorname{arcoth}(\sqrt{x})}{x} dx$$

input
```
integrate(arccoth(x^(1/2))/x,x, algorithm="giac")
```

output
```
integrate(arccoth(sqrt(x))/x, x)
```

Mupad [F(-1)]

Timed out.

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{x}\,dx = \int \frac{\operatorname{acoth}\left(\sqrt{x}\right)}{x}\,dx$$

input
```
int(acoth(x^(1/2))/x,x)
```

output
```
int(acoth(x^(1/2))/x, x)
```

Reduce [F]

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{x}\,dx = \int \frac{acoth\left(\sqrt{x}\right)}{x}dx$$

input
```
int(acoth(x^(1/2))/x,x)
```

output
```
int(acoth(sqrt(x))/x,x)
```

3.43 $\int \frac{\coth^{-1}(\sqrt{x})}{x^2}\,dx$

Optimal result

Integrand size = 10, antiderivative size = 25

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{x^2}\,dx = -\frac{1}{\sqrt{x}} - \frac{\coth^{-1}\left(\sqrt{x}\right)}{x} + \operatorname{arctanh}\left(\sqrt{x}\right)$$

output
```
-1/x^(1/2)-arccoth(x^(1/2))/x+arctanh(x^(1/2))
```

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 45, normalized size of antiderivative = 1.80

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{x^2}\,dx = -\frac{1}{\sqrt{x}} - \frac{\coth^{-1}\left(\sqrt{x}\right)}{x} - \frac{1}{2}\log\left(1-\sqrt{x}\right) + \frac{1}{2}\log\left(1+\sqrt{x}\right)$$

input
```
Integrate[ArcCoth[Sqrt[x]]/x^2,x]
```

output
```
-(1/Sqrt[x]) - ArcCoth[Sqrt[x]]/x - Log[1 - Sqrt[x]]/2 + Log[1 + Sqrt[x]]/
2
```

Rubi [A] (verified)

Time = 0.20 (sec) , antiderivative size = 32, normalized size of antiderivative = 1.28, number of steps used = 5, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.400, Rules used = {6453, 61, 73, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{x^2} \, dx$$

$$\Big\downarrow \text{6453}$$

$$\frac{1}{2} \int \frac{1}{(1-x)x^{3/2}}dx - \frac{\coth^{-1}\left(\sqrt{x}\right)}{x}$$

$$\Big\downarrow \text{61}$$

$$\frac{1}{2} \left(\int \frac{1}{(1-x)\sqrt{x}}dx - \frac{2}{\sqrt{x}} \right) - \frac{\coth^{-1}\left(\sqrt{x}\right)}{x}$$

$$\Big\downarrow \text{73}$$

$$\frac{1}{2} \left(2 \int \frac{1}{1-x}d\sqrt{x} - \frac{2}{\sqrt{x}} \right) - \frac{\coth^{-1}\left(\sqrt{x}\right)}{x}$$

$$\Big\downarrow \text{219}$$

$$\frac{1}{2} \left(2\text{arctanh}\left(\sqrt{x}\right) - \frac{2}{\sqrt{x}} \right) - \frac{\coth^{-1}\left(\sqrt{x}\right)}{x}$$

input
```
Int[ArcCoth[Sqrt[x]]/x^2,x]
```

output
```
-(ArcCoth[Sqrt[x]]/x) + (-2/Sqrt[x] + 2*ArcTanh[Sqrt[x]])/2
```

Defintions of rubi rules used

rule 61
```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
(a + b*x)^(m + 1)*((c + d*x)^(n + 1)/((b*c - a*d)*(m + 1))), x] - Simp[d*((
m + n + 2)/((b*c - a*d)*(m + 1)))   Int[(a + b*x)^(m + 1)*(c + d*x)^n, x],
x] /; FreeQ[{a, b, c, d, n}, x] && LtQ[m, -1] &&  !(LtQ[n, -1] && (EqQ[a, 0
] || (NeQ[c, 0] && LtQ[m - n, 0] && IntegerQ[n]))) && IntLinearQ[a, b, c, d
, m, n, x]
```

rule 73
```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Simp[p/b   Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
 d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && Lt
Q[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Denominator[m]] && IntL
inearQ[a, b, c, d, m, n, x]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

Maple [A] (verified)

Time = 0.04 (sec) , antiderivative size = 32, normalized size of antiderivative = 1.28

method	result	size
derivativedivides	$-\frac{\operatorname{arccoth}\left(\sqrt{x}\right)}{x}-\frac{\ln\left(\sqrt{x}-1\right)}{2}-\frac{1}{\sqrt{x}}+\frac{\ln\left(\sqrt{x}+1\right)}{2}$	32
default	$-\frac{\operatorname{arccoth}\left(\sqrt{x}\right)}{x}-\frac{\ln\left(\sqrt{x}-1\right)}{2}-\frac{1}{\sqrt{x}}+\frac{\ln\left(\sqrt{x}+1\right)}{2}$	32
parts	$-\frac{\operatorname{arccoth}\left(\sqrt{x}\right)}{x}-\frac{\ln\left(\sqrt{x}-1\right)}{2}-\frac{1}{\sqrt{x}}+\frac{\ln\left(\sqrt{x}+1\right)}{2}$	32

input
```
int(arccoth(x^(1/2))/x^2,x,method=_RETURNVERBOSE)
```

output
```
-arccoth(x^(1/2))/x-1/2*ln(x^(1/2)-1)-1/x^(1/2)+1/2*ln(x^(1/2)+1)
```

Fricas [A] (verification not implemented)

Time = 0.10 (sec) , antiderivative size = 30, normalized size of antiderivative = 1.20

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{x^2}\,dx = \frac{(x-1)\log\left(\frac{x+2\sqrt{x}+1}{x-1}\right)-2\sqrt{x}}{2\,x}$$

input
```
integrate(arccoth(x^(1/2))/x^2,x, algorithm="fricas")
```

output
```
1/2*((x - 1)*log((x + 2*sqrt(x) + 1)/(x - 1)) - 2*sqrt(x))/x
```

Sympy [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 92 vs. 2(20) = 40.

Time = 0.46 (sec) , antiderivative size = 92, normalized size of antiderivative = 3.68

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{x^2}\,dx = \frac{x^{\frac{5}{2}}\operatorname{acoth}\left(\sqrt{x}\right)}{x^{\frac{5}{2}}-x^{\frac{3}{2}}} - \frac{2x^{\frac{3}{2}}\operatorname{acoth}\left(\sqrt{x}\right)}{x^{\frac{5}{2}}-x^{\frac{3}{2}}} + \frac{\sqrt{x}\operatorname{acoth}\left(\sqrt{x}\right)}{x^{\frac{5}{2}}-x^{\frac{3}{2}}} - \frac{x^2}{x^{\frac{5}{2}}-x^{\frac{3}{2}}} + \frac{x}{x^{\frac{5}{2}}-x^{\frac{3}{2}}}$$

input
```
integrate(acoth(x**(1/2))/x**2,x)
```

output
```
x**(5/2)*acoth(sqrt(x))/(x**(5/2) - x**(3/2)) - 2*x**(3/2)*acoth(sqrt(x))/
(x**(5/2) - x**(3/2)) + sqrt(x)*acoth(sqrt(x))/(x**(5/2) - x**(3/2)) - x**
2/(x**(5/2) - x**(3/2)) + x/(x**(5/2) - x**(3/2))
```

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 31, normalized size of antiderivative = 1.24

$$\int \frac{\coth^{-1}(\sqrt{x})}{x^2} dx = -\frac{\operatorname{arcoth}(\sqrt{x})}{x} - \frac{1}{\sqrt{x}} + \frac{1}{2}\log(\sqrt{x}+1) - \frac{1}{2}\log(\sqrt{x}-1)$$

input
```
integrate(arccoth(x^(1/2))/x^2,x, algorithm="maxima")
```

output
```
-arccoth(sqrt(x))/x - 1/sqrt(x) + 1/2*log(sqrt(x) + 1) - 1/2*log(sqrt(x) -
 1)
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 65 vs. 2(19) = 38.

Time = 0.12 (sec) , antiderivative size = 65, normalized size of antiderivative = 2.60

$$\int \frac{\coth^{-1}(\sqrt{x})}{x^2} dx = \frac{2}{\frac{\sqrt{x}+1}{\sqrt{x}-1}+1} + \frac{2(\sqrt{x}+1)\log\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)}{(\sqrt{x}-1)\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}+1\right)^2}$$

input
```
integrate(arccoth(x^(1/2))/x^2,x, algorithm="giac")
```

output
```
2/((sqrt(x) + 1)/(sqrt(x) - 1) + 1) + 2*(sqrt(x) + 1)*log((sqrt(x) + 1)/(s
qrt(x) - 1))/((sqrt(x) - 1)*((sqrt(x) + 1)/(sqrt(x) - 1) + 1)^2)
```

Mupad [B] (verification not implemented)

Time = 3.73 (sec) , antiderivative size = 18, normalized size of antiderivative = 0.72

$$\int \frac{\coth^{-1}(\sqrt{x})}{x^2} dx = \operatorname{atanh}(\sqrt{x}) - \frac{\operatorname{acoth}(\sqrt{x}) + \sqrt{x}}{x}$$

input
```
int(acoth(x^(1/2))/x^2,x)
```

output
```
atanh(x^(1/2)) - (acoth(x^(1/2)) + x^(1/2))/x
```

Reduce [B] (verification not implemented)

Time = 0.17 (sec) , antiderivative size = 17, normalized size of antiderivative = 0.68

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{x^2}\,dx = \frac{acoth\left(\sqrt{x}\right)x - acoth\left(\sqrt{x}\right) + \sqrt{x}}{x}$$

input
```
int(acoth(x^(1/2))/x^2,x)
```

output
```
(acoth(sqrt(x))*x - acoth(sqrt(x)) + sqrt(x))/x
```

3.44 $\int \frac{\coth^{-1}(\sqrt{x})}{x^3}\,dx$

Optimal result

Integrand size = 10, antiderivative size = 42

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{x^3}\,dx = -\frac{1}{6x^{3/2}} - \frac{1}{2\sqrt{x}} - \frac{\coth^{-1}\left(\sqrt{x}\right)}{2x^2} + \frac{\operatorname{arctanh}\left(\sqrt{x}\right)}{2}$$

output
```
-1/6/x^(3/2)-1/2/x^(1/2)-1/2*arccoth(x^(1/2))/x^2+1/2*arctanh(x^(1/2))
```

Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 58, normalized size of antiderivative = 1.38

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{x^3}\,dx = -\frac{1}{6x^{3/2}} - \frac{1}{2\sqrt{x}} - \frac{\coth^{-1}\left(\sqrt{x}\right)}{2x^2} - \frac{1}{4}\log\left(1-\sqrt{x}\right) + \frac{1}{4}\log\left(1+\sqrt{x}\right)$$

input
```
Integrate[ArcCoth[Sqrt[x]]/x^3,x]
```

output
```
-1/6*1/x^(3/2) - 1/(2*Sqrt[x]) - ArcCoth[Sqrt[x]]/(2*x^2) - Log[1 - Sqrt[x
]]/4 + Log[1 + Sqrt[x]]/4
```

Rubi [A] (verified)

Time = 0.20 (sec) , antiderivative size = 43, normalized size of antiderivative = 1.02, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.500, Rules used = {6453, 61, 61, 73, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{x^3} \, dx$$

$$\Big\downarrow \text{6453}$$

$$\frac{1}{4} \int \frac{1}{(1-x)x^{5/2}}dx - \frac{\coth^{-1}\left(\sqrt{x}\right)}{2x^2}$$

$$\Big\downarrow \text{61}$$

$$\frac{1}{4}\left(\int \frac{1}{(1-x)x^{3/2}}dx - \frac{2}{3x^{3/2}}\right) - \frac{\coth^{-1}\left(\sqrt{x}\right)}{2x^2}$$

$$\Big\downarrow \text{61}$$

$$\frac{1}{4}\left(\int \frac{1}{(1-x)\sqrt{x}}dx - \frac{2}{3x^{3/2}} - \frac{2}{\sqrt{x}}\right) - \frac{\coth^{-1}\left(\sqrt{x}\right)}{2x^2}$$

$$\Big\downarrow \text{73}$$

$$\frac{1}{4}\left(2\int \frac{1}{1-x}d\sqrt{x} - \frac{2}{3x^{3/2}} - \frac{2}{\sqrt{x}}\right) - \frac{\coth^{-1}\left(\sqrt{x}\right)}{2x^2}$$

$$\Big\downarrow \text{219}$$

$$\frac{1}{4}\left(2\text{arctanh}\left(\sqrt{x}\right) - \frac{2}{3x^{3/2}} - \frac{2}{\sqrt{x}}\right) - \frac{\coth^{-1}\left(\sqrt{x}\right)}{2x^2}$$

input
```
Int[ArcCoth[Sqrt[x]]/x^3,x]
```

output
```
-1/2*ArcCoth[Sqrt[x]]/x^2 + (-2/(3*x^(3/2)) - 2/Sqrt[x] + 2*ArcTanh[Sqrt[x
]])/4
```

Defintions of rubi rules used

rule 61
```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
(a + b*x)^(m + 1)*((c + d*x)^(n + 1)/((b*c - a*d)*(m + 1))), x] - Simp[d*((
m + n + 2)/((b*c - a*d)*(m + 1)))   Int[(a + b*x)^(m + 1)*(c + d*x)^n, x],
x] /; FreeQ[{a, b, c, d, n}, x] && LtQ[m, -1] &&  !(LtQ[n, -1] && (EqQ[a, 0
] || (NeQ[c, 0] && LtQ[m - n, 0] && IntegerQ[n]))) && IntLinearQ[a, b, c, d
, m, n, x]
```

rule 73
```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Simp[p/b   Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
 d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && Lt
Q[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Denominator[m]] && IntL
inearQ[a, b, c, d, m, n, x]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

Maple [A] (verified)

Time = 0.04 (sec) , antiderivative size = 37, normalized size of antiderivative = 0.88

method	result	size
derivativedivides	$-\frac{\operatorname{arccoth}\left(\sqrt{x}\right)}{2x^{2}}-\frac{1}{6x^{\frac{3}{2}}}-\frac{1}{2\sqrt{x}}+\frac{\ln\left(\sqrt{x}+1\right)}{4}-\frac{\ln\left(\sqrt{x}-1\right)}{4}$	37
default	$-\frac{\operatorname{arccoth}\left(\sqrt{x}\right)}{2x^{2}}-\frac{1}{6x^{\frac{3}{2}}}-\frac{1}{2\sqrt{x}}+\frac{\ln\left(\sqrt{x}+1\right)}{4}-\frac{\ln\left(\sqrt{x}-1\right)}{4}$	37
parts	$-\frac{\operatorname{arccoth}\left(\sqrt{x}\right)}{2x^{2}}-\frac{1}{6x^{\frac{3}{2}}}-\frac{1}{2\sqrt{x}}+\frac{\ln\left(\sqrt{x}+1\right)}{4}-\frac{\ln\left(\sqrt{x}-1\right)}{4}$	37

input
```
int(arccoth(x^(1/2))/x^3,x,method=_RETURNVERBOSE)
```

output
```
-1/2*arccoth(x^(1/2))/x^2-1/6/x^(3/2)-1/2/x^(1/2)+1/4*ln(x^(1/2)+1)-1/4*ln
(x^(1/2)-1)
```

Fricas [A] (verification not implemented)

Time = 0.13 (sec) , antiderivative size = 38, normalized size of antiderivative = 0.90

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{x^3}\,dx = \frac{3\,(x^2-1)\log\left(\frac{x+2\,\sqrt{x}+1}{x-1}\right)-2\,(3\,x+1)\sqrt{x}}{12\,x^2}$$

input
```
integrate(arccoth(x^(1/2))/x^3,x, algorithm="fricas")
```

output
```
1/12*(3*(x^2 - 1)*log((x + 2*sqrt(x) + 1)/(x - 1)) - 2*(3*x + 1)*sqrt(x))/
x^2
```

Sympy [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 160 vs. 2(36) = 72.

Time = 0.92 (sec) , antiderivative size = 160, normalized size of antiderivative = 3.81

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{x^3}\,dx = \frac{3x^{\frac{7}{2}}\operatorname{acoth}\left(\sqrt{x}\right)}{6x^{\frac{7}{2}}-6x^{\frac{5}{2}}} - \frac{3x^{\frac{5}{2}}\operatorname{acoth}\left(\sqrt{x}\right)}{6x^{\frac{7}{2}}-6x^{\frac{5}{2}}} - \frac{3x^{\frac{3}{2}}\operatorname{acoth}\left(\sqrt{x}\right)}{6x^{\frac{7}{2}}-6x^{\frac{5}{2}}} + \frac{3\sqrt{x}\operatorname{acoth}\left(\sqrt{x}\right)}{6x^{\frac{7}{2}}-6x^{\frac{5}{2}}} - \frac{3x^3}{6x^{\frac{7}{2}}-6x^{\frac{5}{2}}} + \frac{2x^2}{6x^{\frac{7}{2}}-6x^{\frac{5}{2}}} + \frac{x}{6x^{\frac{7}{2}}-6x^{\frac{5}{2}}}$$

input
```
integrate(acoth(x**(1/2))/x**3,x)
```

output
```
3*x**(7/2)*acoth(sqrt(x))/(6*x**(7/2) - 6*x**(5/2)) - 3*x**(5/2)*acoth(sqr
t(x))/(6*x**(7/2) - 6*x**(5/2)) - 3*x**(3/2)*acoth(sqrt(x))/(6*x**(7/2) -
6*x**(5/2)) + 3*sqrt(x)*acoth(sqrt(x))/(6*x**(7/2) - 6*x**(5/2)) - 3*x**3/
(6*x**(7/2) - 6*x**(5/2)) + 2*x**2/(6*x**(7/2) - 6*x**(5/2)) + x/(6*x**(7/
2) - 6*x**(5/2))
```

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 36, normalized size of antiderivative = 0.86

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{x^3}\,dx = -\frac{3\,x+1}{6\,x^{\frac{3}{2}}} - \frac{\operatorname{arcoth}\left(\sqrt{x}\right)}{2\,x^2} + \frac{1}{4}\,\log\left(\sqrt{x}+1\right) - \frac{1}{4}\,\log\left(\sqrt{x}-1\right)$$

input
```
integrate(arccoth(x^(1/2))/x^3,x, algorithm="maxima")
```

output
```
-1/6*(3*x + 1)/x^(3/2) - 1/2*arccoth(sqrt(x))/x^2 + 1/4*log(sqrt(x) + 1) -
 1/4*log(sqrt(x) - 1)
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 114 vs. 2(26) = 52.

Time = 0.12 (sec) , antiderivative size = 114, normalized size of antiderivative = 2.71

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{x^3}\,dx = \frac{2\left(\frac{3\left(\sqrt{x}+1\right)^2}{\left(\sqrt{x}-1\right)^2} + \frac{3\left(\sqrt{x}+1\right)}{\sqrt{x}-1} + 2\right)}{3\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}+1\right)^3} + \frac{2\left(\frac{\left(\sqrt{x}+1\right)^3}{\left(\sqrt{x}-1\right)^3} + \frac{\sqrt{x}+1}{\sqrt{x}-1}\right)\log\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)}{\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}+1\right)^4}$$

input
```
integrate(arccoth(x^(1/2))/x^3,x, algorithm="giac")
```

output
```
2/3*(3*(sqrt(x) + 1)^2/(sqrt(x) - 1)^2 + 3*(sqrt(x) + 1)/(sqrt(x) - 1) + 2
)/((sqrt(x) + 1)/(sqrt(x) - 1) + 1)^3 + 2*((sqrt(x) + 1)^3/(sqrt(x) - 1)^3
 + (sqrt(x) + 1)/(sqrt(x) - 1))*log((sqrt(x) + 1)/(sqrt(x) - 1))/((sqrt(x)
 + 1)/(sqrt(x) - 1) + 1)^4
```

Mupad [B] (verification not implemented)

Time = 4.15 (sec) , antiderivative size = 45, normalized size of antiderivative = 1.07

$$\int \frac{\coth^{-1}(\sqrt{x})}{x^3}\,dx = \frac{\ln\left(1-\frac{1}{\sqrt{x}}\right)}{4\,x^2} - \frac{\frac{x}{2}+\frac{1}{6}}{x^{3/2}} - \frac{\ln\left(\frac{1}{\sqrt{x}}+1\right)}{4\,x^2} - \frac{\mathrm{atan}\left(\sqrt{x}\,1\mathrm{i}\right)\,1\mathrm{i}}{2}$$

input
```
int(acoth(x^(1/2))/x^3,x)
```

output
```
log(1 - 1/x^(1/2))/(4*x^2) - (atan(x^(1/2)*1i)*1i)/2 - (x/2 + 1/6)/x^(3/2)
 - log(1/x^(1/2) + 1)/(4*x^2)
```

Reduce [B] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 26, normalized size of antiderivative = 0.62

$$\int \frac{\coth^{-1}(\sqrt{x})}{x^3}\,dx = \frac{3acoth(\sqrt{x})\,x^2 - 3acoth(\sqrt{x}) + 3\sqrt{x}\,x + \sqrt{x}}{6x^2}$$

input
```
int(acoth(x^(1/2))/x^3,x)
```

output
```
(3*acoth(sqrt(x))*x**2 - 3*acoth(sqrt(x)) + 3*sqrt(x)*x + sqrt(x))/(6*x**2
)
```

3.45 $\int x^{3/2} \coth^{-1}\left(\sqrt{x}\right) dx$

Optimal result

Integrand size = 12, antiderivative size = 38

$$\int x^{3/2} \coth^{-1}\left(\sqrt{x}\right) dx = \frac{x}{5} + \frac{x^2}{10} + \frac{2}{5}x^{5/2} \coth^{-1}\left(\sqrt{x}\right) + \frac{1}{5}\log(1-x)$$

output
```
1/5*x+1/10*x^2+2/5*x^(5/2)*arccoth(x^(1/2))+1/5*ln(1-x)
```

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 31, normalized size of antiderivative = 0.82

$$\int x^{3/2} \coth^{-1}\left(\sqrt{x}\right) dx = \frac{1}{10}\left(x(2+x) + 4x^{5/2} \coth^{-1}\left(\sqrt{x}\right) + 2\log(1-x)\right)$$

input
```
Integrate[x^(3/2)*ArcCoth[Sqrt[x]],x]
```

output
```
(x*(2 + x) + 4*x^(5/2)*ArcCoth[Sqrt[x]] + 2*Log[1 - x])/10
```

Rubi [A] (verified)

Time = 0.21 (sec) , antiderivative size = 35, normalized size of antiderivative = 0.92, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.250$, Rules used = {6453, 49, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int x^{3/2}\coth^{-1}\left(\sqrt{x}\right)dx$$

$$\Big\downarrow \text{6453}$$

$$\frac{2}{5}x^{5/2}\coth^{-1}\left(\sqrt{x}\right)-\frac{1}{5}\int \frac{x^2}{1-x}dx$$

$$\Big\downarrow \text{49}$$

$$\frac{2}{5}x^{5/2}\coth^{-1}\left(\sqrt{x}\right)-\frac{1}{5}\int \left(-x+\frac{1}{1-x}-1\right)dx$$

$$\Big\downarrow \text{2009}$$

$$\frac{2}{5}x^{5/2}\coth^{-1}\left(\sqrt{x}\right)+\frac{1}{5}\left(\frac{x^2}{2}+x+\log(1-x)\right)$$

input
```
Int[x^(3/2)*ArcCoth[Sqrt[x]],x]
```

output
```
(2*x^(5/2)*ArcCoth[Sqrt[x]])/5 + (x + x^2/2 + Log[1 - x])/5
```

Defintions of rubi rules used

rule 49
```
Int[((a_.) + (b_.)*(x_))^(m_.)*((c_.) + (d_.)*(x_))^(n_.), x_Symbol] :> Int
[ExpandIntegrand[(a + b*x)^m*(c + d*x)^n, x], x] /; FreeQ[{a, b, c, d}, x]
&& IGtQ[m, 0] && IGtQ[m + n + 2, 0]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

Maple [A] (verified)

Time = 0.04 (sec) , antiderivative size = 35, normalized size of antiderivative = 0.92

method	result	size
derivativedivides	$\frac{2x^{\frac{5}{2}}\operatorname{arccoth}\left(\sqrt{x}\right)}{5}+\frac{x^{2}}{10}+\frac{x}{5}+\frac{\ln\left(\sqrt{x}-1\right)}{5}+\frac{\ln\left(\sqrt{x}+1\right)}{5}$	35
default	$\frac{2x^{\frac{5}{2}}\operatorname{arccoth}\left(\sqrt{x}\right)}{5}+\frac{x^{2}}{10}+\frac{x}{5}+\frac{\ln\left(\sqrt{x}-1\right)}{5}+\frac{\ln\left(\sqrt{x}+1\right)}{5}$	35

input
```
int(x^(3/2)*arccoth(x^(1/2)),x,method=_RETURNVERBOSE)
```

output
```
2/5*x^(5/2)*arccoth(x^(1/2))+1/10*x^2+1/5*x+1/5*ln(x^(1/2)-1)+1/5*ln(x^(1/
2)+1)
```

Fricas [A] (verification not implemented)

Time = 0.10 (sec) , antiderivative size = 35, normalized size of antiderivative = 0.92

$$\int x^{3/2}\coth^{-1}\left(\sqrt{x}\right)\,dx=\frac{1}{5}\,x^{\frac{5}{2}}\log\left(\frac{x+2\,\sqrt{x}+1}{x-1}\right)+\frac{1}{10}\,x^{2}+\frac{1}{5}\,x+\frac{1}{5}\,\log\left(x-1\right)$$

input
```
integrate(x^(3/2)*arccoth(x^(1/2)),x, algorithm="fricas")
```

output
```
1/5*x^(5/2)*log((x + 2*sqrt(x) + 1)/(x - 1)) + 1/10*x^2 + 1/5*x + 1/5*log(
x - 1)
```

Sympy [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 121 vs. 2(29) = 58.

Time = 0.83 (sec) , antiderivative size = 121, normalized size of antiderivative = 3.18

$$\int x^{3/2}\coth^{-1}\left(\sqrt{x}\right)\,dx = \frac{4x^{\frac{7}{2}}\operatorname{acoth}\left(\sqrt{x}\right)}{10x-10} - \frac{4x^{\frac{5}{2}}\operatorname{acoth}\left(\sqrt{x}\right)}{10x-10} + \frac{x^3}{10x-10} + \frac{x^2}{10x-10} + \frac{4x\log\left(\sqrt{x}+1\right)}{10x-10} - \frac{4x\operatorname{acoth}\left(\sqrt{x}\right)}{10x-10} - \frac{4\log\left(\sqrt{x}+1\right)}{10x-10} + \frac{4\operatorname{acoth}\left(\sqrt{x}\right)}{10x-10} - \frac{2}{10x-10}$$

input
```
integrate(x**(3/2)*acoth(x**(1/2)),x)
```

output
```
4*x**(7/2)*acoth(sqrt(x))/(10*x - 10) - 4*x**(5/2)*acoth(sqrt(x))/(10*x -
10) + x**3/(10*x - 10) + x**2/(10*x - 10) + 4*x*log(sqrt(x) + 1)/(10*x - 1
0) - 4*x*acoth(sqrt(x))/(10*x - 10) - 4*log(sqrt(x) + 1)/(10*x - 10) + 4*a
coth(sqrt(x))/(10*x - 10) - 2/(10*x - 10)
```

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 24, normalized size of antiderivative = 0.63

$$\int x^{3/2}\coth^{-1}\left(\sqrt{x}\right)\,dx = \frac{2}{5}\,x^{\frac{5}{2}}\operatorname{arcoth}\left(\sqrt{x}\right) + \frac{1}{10}\,x^2 + \frac{1}{5}\,x + \frac{1}{5}\,\log\left(x-1\right)$$

input
```
integrate(x^(3/2)*arccoth(x^(1/2)),x, algorithm="maxima")
```

output
```
2/5*x^(5/2)*arccoth(sqrt(x)) + 1/10*x^2 + 1/5*x + 1/5*log(x - 1)
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 168 vs. $2(26) = 52$.

Time = 0.12 (sec) , antiderivative size = 168, normalized size of antiderivative = 4.42

$$\int x^{3/2}\coth^{-1}\left(\sqrt{x}\right)\,dx = \frac{8\left(\frac{\left(\sqrt{x}+1\right)^3}{\left(\sqrt{x}-1\right)^3} - \frac{\left(\sqrt{x}+1\right)^2}{\left(\sqrt{x}-1\right)^2} + \frac{\sqrt{x}+1}{\sqrt{x}-1}\right)}{5\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}-1\right)^4} + \frac{2\left(\frac{5\left(\sqrt{x}+1\right)^4}{\left(\sqrt{x}-1\right)^4} + \frac{10\left(\sqrt{x}+1\right)^2}{\left(\sqrt{x}-1\right)^2} + 1\right)\log\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)}{5\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}-1\right)^5} + \frac{2}{5}\log\left(\frac{\sqrt{x}+1}{\left|\sqrt{x}-1\right|}\right) - \frac{2}{5}\log\left(\left|\frac{\sqrt{x}+1}{\sqrt{x}-1}-1\right|\right)$$

input
```
integrate(x^(3/2)*arccoth(x^(1/2)),x, algorithm="giac")
```

output
```
8/5*((sqrt(x) + 1)^3/(sqrt(x) - 1)^3 - (sqrt(x) + 1)^2/(sqrt(x) - 1)^2 + (
sqrt(x) + 1)/(sqrt(x) - 1))/((sqrt(x) + 1)/(sqrt(x) - 1) - 1)^4 + 2/5*(5*(
sqrt(x) + 1)^4/(sqrt(x) - 1)^4 + 10*(sqrt(x) + 1)^2/(sqrt(x) - 1)^2 + 1)*l
og((sqrt(x) + 1)/(sqrt(x) - 1))/((sqrt(x) + 1)/(sqrt(x) - 1) - 1)^5 + 2/5*
log((sqrt(x) + 1)/abs(sqrt(x) - 1)) - 2/5*log(abs((sqrt(x) + 1)/(sqrt(x) -
 1) - 1))
```

Mupad [B] (verification not implemented)

Time = 3.86 (sec) , antiderivative size = 24, normalized size of antiderivative = 0.63

$$\int x^{3/2}\coth^{-1}\left(\sqrt{x}\right)\,dx = \frac{x}{5} + \frac{\ln\left(x-1\right)}{5} + \frac{2\,x^{5/2}\,\mathrm{acoth}\left(\sqrt{x}\right)}{5} + \frac{x^2}{10}$$

input
```
int(x^(3/2)*acoth(x^(1/2)),x)
```

output
```
x/5 + log(x - 1)/5 + (2*x^(5/2)*acoth(x^(1/2)))/5 + x^2/10
```

Reduce [B] (verification not implemented)

Time = 0.17 (sec) , antiderivative size = 31, normalized size of antiderivative = 0.82

$$\int x^{3/2} \coth^{-1}\left(\sqrt{x}\right) \, dx = \frac{2\sqrt{x}\, acoth\left(\sqrt{x}\right) x^2}{5} + \frac{2acoth\left(\sqrt{x}\right)}{5} - \frac{2\log\left(\sqrt{x}-1\right)}{5} - \frac{x^2}{10} - \frac{x}{5}$$

input
```
int(x^(3/2)*acoth(x^(1/2)),x)
```

output
```
(4*sqrt(x)*acoth(sqrt(x))*x**2 + 4*acoth(sqrt(x)) - 4*log(sqrt(x) - 1) - x
**2 - 2*x)/10
```

3.46 $\int \sqrt{x}\coth^{-1}\left(\sqrt{x}\right)\,dx$

Optimal result

Integrand size = 12, antiderivative size = 31

$$\int \sqrt{x}\coth^{-1}\left(\sqrt{x}\right)\,dx=\frac{x}{3}+\frac{2}{3}x^{3/2}\coth^{-1}\left(\sqrt{x}\right)+\frac{1}{3}\log(1-x)$$

output
```
1/3*x+2/3*x^(3/2)*arccoth(x^(1/2))+1/3*ln(1-x)
```

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 25, normalized size of antiderivative = 0.81

$$\int \sqrt{x}\coth^{-1}\left(\sqrt{x}\right)\,dx=\frac{1}{3}\left(x+2x^{3/2}\coth^{-1}\left(\sqrt{x}\right)+\log(1-x)\right)$$

input
```
Integrate[Sqrt[x]*ArcCoth[Sqrt[x]],x]
```

output
```
(x + 2*x^(3/2)*ArcCoth[Sqrt[x]] + Log[1 - x])/3
```

Rubi [A] (verified)

Time = 0.20 (sec) , antiderivative size = 28, normalized size of antiderivative = 0.90, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.250$, Rules used = {6453, 49, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \sqrt{x}\coth^{-1}\left(\sqrt{x}\right)dx$$

$$\Big\downarrow \ 6453$$

$$\frac{2}{3}x^{3/2}\coth^{-1}\left(\sqrt{x}\right)-\frac{1}{3}\int \frac{x}{1-x}dx$$

$$\Big\downarrow \ 49$$

$$\frac{2}{3}x^{3/2}\coth^{-1}\left(\sqrt{x}\right)-\frac{1}{3}\int \left(\frac{1}{1-x}-1\right)dx$$

$$\Big\downarrow \ 2009$$

$$\frac{2}{3}x^{3/2}\coth^{-1}\left(\sqrt{x}\right)+\frac{1}{3}(x+\log(1-x))$$

input
```
Int[Sqrt[x]*ArcCoth[Sqrt[x]],x]
```

output
```
(2*x^(3/2)*ArcCoth[Sqrt[x]])/3 + (x + Log[1 - x])/3
```

Defintions of rubi rules used

rule 49
```
Int[((a_.) + (b_.)*(x_))^(m_.)*((c_.) + (d_.)*(x_))^(n_.), x_Symbol] :> Int
[ExpandIntegrand[(a + b*x)^m*(c + d*x)^n, x], x] /; FreeQ[{a, b, c, d}, x]
&& IGtQ[m, 0] && IGtQ[m + n + 2, 0]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

Maple [A] (verified)

Time = 0.03 (sec) , antiderivative size = 30, normalized size of antiderivative = 0.97

method	result	size
derivativedivides	$\frac{2x^{\frac{3}{2}}\operatorname{arccoth}(\sqrt{x})}{3}+\frac{x}{3}+\frac{\ln(\sqrt{x}-1)}{3}+\frac{\ln(\sqrt{x}+1)}{3}$	30
default	$\frac{2x^{\frac{3}{2}}\operatorname{arccoth}(\sqrt{x})}{3}+\frac{x}{3}+\frac{\ln(\sqrt{x}-1)}{3}+\frac{\ln(\sqrt{x}+1)}{3}$	30

input
```
int(x^(1/2)*arccoth(x^(1/2)),x,method=_RETURNVERBOSE)
```

output
```
2/3*x^(3/2)*arccoth(x^(1/2))+1/3*x+1/3*ln(x^(1/2)-1)+1/3*ln(x^(1/2)+1)
```

Fricas [A] (verification not implemented)

Time = 0.08 (sec) , antiderivative size = 30, normalized size of antiderivative = 0.97

$$\int \sqrt{x}\coth^{-1}\left(\sqrt{x}\right)\,dx=\frac{1}{3}\,x^{\frac{3}{2}}\log\left(\frac{x+2\,\sqrt{x}+1}{x-1}\right)+\frac{1}{3}\,x+\frac{1}{3}\,\log\left(x-1\right)$$

input
```
integrate(x^(1/2)*arccoth(x^(1/2)),x, algorithm="fricas")
```

output
```
1/3*x^(3/2)*log((x + 2*sqrt(x) + 1)/(x - 1)) + 1/3*x + 1/3*log(x - 1)
```

Sympy [F]

$$\int \sqrt{x}\coth^{-1}\left(\sqrt{x}\right)\,dx=\int \sqrt{x}\,\mathrm{acoth}\left(\sqrt{x}\right)dx$$

input
```
integrate(x**(1/2)*acoth(x**(1/2)),x)
```

output
```
Integral(sqrt(x)*acoth(sqrt(x)), x)
```

Maxima [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 19, normalized size of antiderivative = 0.61

$$\int \sqrt{x}\coth^{-1}\left(\sqrt{x}\right)\,dx=\frac{2}{3}\,x^{\frac{3}{2}}\,\mathrm{arcoth}\left(\sqrt{x}\right)+\frac{1}{3}\,x+\frac{1}{3}\,\log\left(x-1\right)$$

input
```
integrate(x^(1/2)*arccoth(x^(1/2)),x, algorithm="maxima")
```

output
```
2/3*x^(3/2)*arccoth(sqrt(x)) + 1/3*x + 1/3*log(x - 1)
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 119 vs. 2(21) = 42.

Time = 0.12 (sec) , antiderivative size = 119, normalized size of antiderivative = 3.84

$$\int \sqrt{x}\coth^{-1}\left(\sqrt{x}\right)\,dx=\frac{2\left(\frac{3\left(\sqrt{x}+1\right)^{2}}{\left(\sqrt{x}-1\right)^{2}}+1\right)\log\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)}{3\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}-1\right)^{3}}+\frac{4\left(\sqrt{x}+1\right)}{3\left(\sqrt{x}-1\right)\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}-1\right)^{2}}+\frac{2}{3}\log\left(\frac{\sqrt{x}+1}{\left|\sqrt{x}-1\right|}\right)-\frac{2}{3}\log\left(\left|\frac{\sqrt{x}+1}{\sqrt{x}-1}-1\right|\right)$$

input
```
integrate(x^(1/2)*arccoth(x^(1/2)),x, algorithm="giac")
```

output
```
2/3*(3*(sqrt(x) + 1)^2/(sqrt(x) - 1)^2 + 1)*log((sqrt(x) + 1)/(sqrt(x) - 1
))/((sqrt(x) + 1)/(sqrt(x) - 1) - 1)^3 + 4/3*(sqrt(x) + 1)/((sqrt(x) - 1)*
((sqrt(x) + 1)/(sqrt(x) - 1) - 1)^2) + 2/3*log((sqrt(x) + 1)/abs(sqrt(x) -
 1)) - 2/3*log(abs((sqrt(x) + 1)/(sqrt(x) - 1) - 1))
```

Mupad [F(-1)]

Timed out.

$$\int \sqrt{x}\coth^{-1}\left(\sqrt{x}\right)\,dx=\int \sqrt{x}\,\mathrm{acoth}\left(\sqrt{x}\right)\,dx$$

input
```
int(x^(1/2)*acoth(x^(1/2)),x)
```

output
```
int(x^(1/2)*acoth(x^(1/2)), x)
```

Reduce [B] (verification not implemented)

Time = 0.17 (sec) , antiderivative size = 24, normalized size of antiderivative = 0.77

$$\int \sqrt{x}\coth^{-1}\left(\sqrt{x}\right)\,dx=\frac{2\sqrt{x}\,acoth\left(\sqrt{x}\right)x}{3}+\frac{2acoth\left(\sqrt{x}\right)}{3}-\frac{2\log\left(\sqrt{x}-1\right)}{3}-\frac{x}{3}$$

input
```
int(x^(1/2)*acoth(x^(1/2)),x)
```

output
```
(2*sqrt(x)*acoth(sqrt(x))*x + 2*acoth(sqrt(x)) - 2*log(sqrt(x) - 1) - x)/3
```

3.47 $\int \frac{\coth^{-1}(\sqrt{x})}{\sqrt{x}} \, dx$

Optimal result

Integrand size = 12, antiderivative size = 20

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{\sqrt{x}} \, dx = 2\sqrt{x}\coth^{-1}\left(\sqrt{x}\right) + \log(1-x)$$

output
```
2*x^(1/2)*arccoth(x^(1/2))+ln(1-x)
```

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 20, normalized size of antiderivative = 1.00

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{\sqrt{x}} \, dx = 2\sqrt{x}\coth^{-1}\left(\sqrt{x}\right) + \log(1-x)$$

input
```
Integrate[ArcCoth[Sqrt[x]]/Sqrt[x],x]
```

output
```
2*Sqrt[x]*ArcCoth[Sqrt[x]] + Log[1 - x]
```

Rubi [A] (verified)

Time = 0.18 (sec) , antiderivative size = 20, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}} = 0.167$, Rules used = {6453, 16}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$$

$$\Big\downarrow \text{6453}$$

$$2\sqrt{x}\coth^{-1}\left(\sqrt{x}\right)-\int \frac{1}{1-x}dx$$

$$\Big\downarrow \text{16}$$

$$\log(1-x)+2\sqrt{x}\coth^{-1}\left(\sqrt{x}\right)$$

input
```
Int[ArcCoth[Sqrt[x]]/Sqrt[x],x]
```

output
```
2*Sqrt[x]*ArcCoth[Sqrt[x]] + Log[1 - x]
```

Defintions of rubi rules used

rule 16
```
Int[(c_.)/((a_.) + (b_.)*(x_)), x_Symbol] :> Simp[c*(Log[RemoveContent[a +
b*x, x]]/b), x] /; FreeQ[{a, b, c}, x]
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

Maple [A] (verified)

Time = 0.03 (sec) , antiderivative size = 15, normalized size of antiderivative = 0.75

method	result	size
derivativedivides	$2\sqrt{x}\,\operatorname{arccoth}\left(\sqrt{x}\right)+\ln\left(x-1\right)$	15
default	$2\sqrt{x}\,\operatorname{arccoth}\left(\sqrt{x}\right)+\ln\left(x-1\right)$	15

input
```
int(arccoth(x^(1/2))/x^(1/2),x,method=_RETURNVERBOSE)
```

output
```
2*x^(1/2)*arccoth(x^(1/2))+ln(x-1)
```

Fricas [A] (verification not implemented)

Time = 0.10 (sec) , antiderivative size = 24, normalized size of antiderivative = 1.20

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{\sqrt{x}}\,dx=\sqrt{x}\log\left(\frac{x+2\sqrt{x}+1}{x-1}\right)+\log\left(x-1\right)$$

input
```
integrate(arccoth(x^(1/2))/x^(1/2),x, algorithm="fricas")
```

output
```
sqrt(x)*log((x + 2*sqrt(x) + 1)/(x - 1)) + log(x - 1)
```

Sympy [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 87 vs. 2(17) = 34.

Time = 0.21 (sec) , antiderivative size = 87, normalized size of antiderivative = 4.35

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{\sqrt{x}}\,dx=\frac{2x^{\frac{3}{2}}\operatorname{acoth}\left(\sqrt{x}\right)}{x-1}-\frac{2\sqrt{x}\operatorname{acoth}\left(\sqrt{x}\right)}{x-1}+\frac{2x\log\left(\sqrt{x}+1\right)}{x-1}-\frac{2x\operatorname{acoth}\left(\sqrt{x}\right)}{x-1}-\frac{2\log\left(\sqrt{x}+1\right)}{x-1}+\frac{2\operatorname{acoth}\left(\sqrt{x}\right)}{x-1}$$

input
```
integrate(acoth(x**(1/2))/x**(1/2),x)
```

output
```
2*x**(3/2)*acoth(sqrt(x))/(x - 1) - 2*sqrt(x)*acoth(sqrt(x))/(x - 1) + 2*x
*log(sqrt(x) + 1)/(x - 1) - 2*x*acoth(sqrt(x))/(x - 1) - 2*log(sqrt(x) + 1
)/(x - 1) + 2*acoth(sqrt(x))/(x - 1)
```

Maxima [A] (verification not implemented)

Time = 0.02 (sec) , antiderivative size = 16, normalized size of antiderivative = 0.80

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{\sqrt{x}}\,dx = 2\,\sqrt{x}\,\mathrm{arcoth}\left(\sqrt{x}\right) + \log\left(-x+1\right)$$

input
```
integrate(arccoth(x^(1/2))/x^(1/2),x, algorithm="maxima")
```

output
```
2*sqrt(x)*arccoth(sqrt(x)) + log(-x + 1)
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 70 vs. 2(16) = 32.

Time = 0.12 (sec) , antiderivative size = 70, normalized size of antiderivative = 3.50

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{\sqrt{x}}\,dx = \frac{2\,\log\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)}{\frac{\sqrt{x}+1}{\sqrt{x}-1}-1} + 2\,\log\left(\frac{\sqrt{x}+1}{\left|\sqrt{x}-1\right|}\right) - 2\,\log\left(\left|\frac{\sqrt{x}+1}{\sqrt{x}-1}-1\right|\right)$$

input
```
integrate(arccoth(x^(1/2))/x^(1/2),x, algorithm="giac")
```

output
```
2*log((sqrt(x) + 1)/(sqrt(x) - 1))/((sqrt(x) + 1)/(sqrt(x) - 1) - 1) + 2*l
og((sqrt(x) + 1)/abs(sqrt(x) - 1)) - 2*log(abs((sqrt(x) + 1)/(sqrt(x) - 1)
 - 1))
```

Mupad [B] (verification not implemented)

Time = 3.75 (sec) , antiderivative size = 14, normalized size of antiderivative = 0.70

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{\sqrt{x}}\,dx = \ln\left(x-1\right) + 2\sqrt{x}\,\mathrm{acoth}\left(\sqrt{x}\right)$$

input
```
int(acoth(x^(1/2))/x^(1/2),x)
```

output
```
log(x - 1) + 2*x^(1/2)*acoth(x^(1/2))
```

Reduce [B] (verification not implemented)

Time = 0.17 (sec) , antiderivative size = 20, normalized size of antiderivative = 1.00

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{\sqrt{x}}\,dx = 2\sqrt{x}\,acoth\left(\sqrt{x}\right) + 2acoth\left(\sqrt{x}\right) - 2\log\left(\sqrt{x}-1\right)$$

input
```
int(acoth(x^(1/2))/x^(1/2),x)
```

output
```
2*(sqrt(x)*acoth(sqrt(x)) + acoth(sqrt(x)) - log(sqrt(x) - 1))
```

3.48 $\int \frac{\coth^{-1}(\sqrt{x})}{x^{3/2}}\,dx$

Optimal result

Integrand size = 12, antiderivative size = 24

$$\int \frac{\coth^{-1}(\sqrt{x})}{x^{3/2}}\,dx = -\frac{2\coth^{-1}(\sqrt{x})}{\sqrt{x}} - \log(1-x) + \log(x)$$

output
```
-2*arccoth(x^(1/2))/x^(1/2)-ln(1-x)+ln(x)
```

Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 24, normalized size of antiderivative = 1.00

$$\int \frac{\coth^{-1}(\sqrt{x})}{x^{3/2}}\,dx = -\frac{2\coth^{-1}(\sqrt{x})}{\sqrt{x}} - \log(1-x) + \log(x)$$

input
```
Integrate[ArcCoth[Sqrt[x]]/x^(3/2),x]
```

output
```
(-2*ArcCoth[Sqrt[x]])/Sqrt[x] - Log[1 - x] + Log[x]
```

Rubi [A] (verified)

Time = 0.18 (sec) , antiderivative size = 24, normalized size of antiderivative = 1.00, number of steps used = 4, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}} = 0.333$, Rules used = {6453, 47, 14, 16}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{x^{3/2}}\,dx$$

$$\Big\downarrow \text{ 6453}$$

$$\int \frac{1}{(1-x)x}dx-\frac{2\coth^{-1}\left(\sqrt{x}\right)}{\sqrt{x}}$$

$$\Big\downarrow \text{ 47}$$

$$\int \frac{1}{1-x}dx+\int \frac{1}{x}dx-\frac{2\coth^{-1}\left(\sqrt{x}\right)}{\sqrt{x}}$$

$$\Big\downarrow \text{ 14}$$

$$\int \frac{1}{1-x}dx+\log(x)-\frac{2\coth^{-1}\left(\sqrt{x}\right)}{\sqrt{x}}$$

$$\Big\downarrow \text{ 16}$$

$$-\log(1-x)+\log(x)-\frac{2\coth^{-1}\left(\sqrt{x}\right)}{\sqrt{x}}$$

input
```
Int[ArcCoth[Sqrt[x]]/x^(3/2),x]
```

output
```
(-2*ArcCoth[Sqrt[x]])/Sqrt[x] - Log[1 - x] + Log[x]
```

Defintions of rubi rules used

rule 14
```
Int[(a_.)/(x_), x_Symbol] :> Simp[a*Log[x], x] /; FreeQ[a, x]
```

rule 16
```
Int[(c_.)/((a_.) + (b_.)*(x_)), x_Symbol] :> Simp[c*(Log[RemoveContent[a +
b*x, x]]/b), x] /; FreeQ[{a, b, c}, x]
```

rule 47
```
Int[1/(((a_.) + (b_.)*(x_))*((c_.) + (d_.)*(x_))), x_Symbol] :> Simp[b/(b*c
 - a*d)   Int[1/(a + b*x), x], x] - Simp[d/(b*c - a*d)   Int[1/(c + d*x), x
], x] /; FreeQ[{a, b, c, d}, x]
```

rule 6453
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_.)]*(b_.))^(p_.)*(x_)^(m_.), x_Symbol] :
> Simp[x^(m + 1)*((a + b*ArcCoth[c*x^n])^p/(m + 1)), x] - Simp[b*c*n*(p/(m
+ 1))   Int[x^(m + n)*((a + b*ArcCoth[c*x^n])^(p - 1)/(1 - c^2*x^(2*n))), x
], x] /; FreeQ[{a, b, c, m, n}, x] && IGtQ[p, 0] && (EqQ[p, 1] || (EqQ[n, 1
] && IntegerQ[m])) && NeQ[m, -1]
```

Maple [A] (verified)

Time = 0.03 (sec) , antiderivative size = 29, normalized size of antiderivative = 1.21

method	result	size
derivativedivides	$-\frac{2\,\mathrm{arccoth}\left(\sqrt{x}\right)}{\sqrt{x}}-\ln\left(\sqrt{x}-1\right)+\ln\left(x\right)-\ln\left(\sqrt{x}+1\right)$	29
default	$-\frac{2\,\mathrm{arccoth}\left(\sqrt{x}\right)}{\sqrt{x}}-\ln\left(\sqrt{x}-1\right)+\ln\left(x\right)-\ln\left(\sqrt{x}+1\right)$	29

input
```
int(arccoth(x^(1/2))/x^(3/2),x,method=_RETURNVERBOSE)
```

output
```
-2*arccoth(x^(1/2))/x^(1/2)-ln(x^(1/2)-1)+ln(x)-ln(x^(1/2)+1)
```

Fricas [A] (verification not implemented)

Time = 0.10 (sec) , antiderivative size = 36, normalized size of antiderivative = 1.50

$$\int \frac{\coth^{-1}(\sqrt{x})}{x^{3/2}}\,dx = -\frac{x\log(x-1) - x\log(x) + \sqrt{x}\log\left(\frac{x+2\sqrt{x}+1}{x-1}\right)}{x}$$

input
```
integrate(arccoth(x^(1/2))/x^(3/2),x, algorithm="fricas")
```

output
```
-(x*log(x - 1) - x*log(x) + sqrt(x)*log((x + 2*sqrt(x) + 1)/(x - 1)))/x
```

Sympy [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 126 vs. 2(20) = 40.

Time = 0.36 (sec) , antiderivative size = 126, normalized size of antiderivative = 5.25

$$\int \frac{\coth^{-1}(\sqrt{x})}{x^{3/2}}\,dx = -\frac{2x^{\frac{3}{2}}\operatorname{acoth}(\sqrt{x})}{x^2-x} + \frac{2\sqrt{x}\operatorname{acoth}(\sqrt{x})}{x^2-x} + \frac{x^2\log(x)}{x^2-x} - \frac{2x^2\log(\sqrt{x}+1)}{x^2-x} + \frac{2x^2\operatorname{acoth}(\sqrt{x})}{x^2-x} - \frac{x\log(x)}{x^2-x} + \frac{2x\log(\sqrt{x}+1)}{x^2-x} - \frac{2x\operatorname{acoth}(\sqrt{x})}{x^2-x}$$

input
```
integrate(acoth(x**(1/2))/x**(3/2),x)
```

output
```
-2*x**(3/2)*acoth(sqrt(x))/(x**2 - x) + 2*sqrt(x)*acoth(sqrt(x))/(x**2 - x
) + x**2*log(x)/(x**2 - x) - 2*x**2*log(sqrt(x) + 1)/(x**2 - x) + 2*x**2*a
coth(sqrt(x))/(x**2 - x) - x*log(x)/(x**2 - x) + 2*x*log(sqrt(x) + 1)/(x**
2 - x) - 2*x*acoth(sqrt(x))/(x**2 - x)
```

Maxima [A] (verification not implemented)

Time = 0.02 (sec) , antiderivative size = 18, normalized size of antiderivative = 0.75

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{x^{3/2}}\,dx = -\frac{2\ \mathrm{arcoth}\left(\sqrt{x}\right)}{\sqrt{x}} - \log\left(x-1\right) + \log\left(x\right)$$

input
```
integrate(arccoth(x^(1/2))/x^(3/2),x, algorithm="maxima")
```

output
```
-2*arccoth(sqrt(x))/sqrt(x) - log(x - 1) + log(x)
```

Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 70 vs. 2(20) = 40.

Time = 0.12 (sec) , antiderivative size = 70, normalized size of antiderivative = 2.92

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{x^{3/2}}\,dx = \frac{2\ \log\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)}{\frac{\sqrt{x}+1}{\sqrt{x}-1}+1} - 2\ \log\left(\frac{\sqrt{x}+1}{\left|\sqrt{x}-1\right|}\right) + 2\ \log\left(\left|\frac{\sqrt{x}+1}{\sqrt{x}-1}+1\right|\right)$$

input
```
integrate(arccoth(x^(1/2))/x^(3/2),x, algorithm="giac")
```

output
```
2*log((sqrt(x) + 1)/(sqrt(x) - 1))/((sqrt(x) + 1)/(sqrt(x) - 1) + 1) - 2*l
og((sqrt(x) + 1)/abs(sqrt(x) - 1)) + 2*log(abs((sqrt(x) + 1)/(sqrt(x) - 1)
 + 1))
```

Mupad [B] (verification not implemented)

Time = 3.77 (sec) , antiderivative size = 22, normalized size of antiderivative = 0.92

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{x^{3/2}}\,dx = 2\ \ln\left(\sqrt{x}\right) - \ln\left(x-1\right) - \frac{2\,\mathrm{acoth}\left(\sqrt{x}\right)}{\sqrt{x}}$$

input
```
int(acoth(x^(1/2))/x^(3/2),x)
```

output
```
2*log(x^(1/2)) - log(x - 1) - (2*acoth(x^(1/2)))/x^(1/2)
```

Reduce [B] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 34, normalized size of antiderivative = 1.42

$$\int \frac{\coth^{-1}\left(\sqrt{x}\right)}{x^{3/2}}\,dx = \frac{-2\sqrt{x}\,acoth\left(\sqrt{x}\right)-2acoth\left(\sqrt{x}\right)+2\sqrt{x}\log\left(\sqrt{x}-1\right)-2\sqrt{x}\log\left(\sqrt{x}\right)}{\sqrt{x}}$$

input
```
int(acoth(x^(1/2))/x^(3/2),x)
```

output
```
(2*( - sqrt(x)*acoth(sqrt(x)) - acoth(sqrt(x)) + sqrt(x)*log(sqrt(x) - 1)
- sqrt(x)*log(sqrt(x))))/sqrt(x)
```

3.49 $\int \frac{\coth^{-1}(ax^n)}{x}\,dx$

Optimal result

Integrand size = 10, antiderivative size = 38

$$\int \frac{\coth^{-1}(ax^n)}{x}\,dx = \frac{\text{PolyLog}\left(2,-\frac{x^{-n}}{a}\right)}{2n} - \frac{\text{PolyLog}\left(2,\frac{x^{-n}}{a}\right)}{2n}$$

output
```
1/2*polylog(2,-1/a/(x^n))/n-1/2*polylog(2,1/a/(x^n))/n
```

Mathematica [C] (verified)

Result contains higher order function than in optimal. Order 5 vs. order 4 in optimal.

Time = 0.04 (sec) , antiderivative size = 52, normalized size of antiderivative = 1.37

$$\int \frac{\coth^{-1}(ax^n)}{x}\,dx = \frac{ax^n\,{}_3F_2\left(\frac{1}{2},\frac{1}{2},1;\frac{3}{2},\frac{3}{2};a^2x^{2n}\right)}{n} + \left(\coth^{-1}(ax^n) - \operatorname{arctanh}(ax^n)\right)\log(x)$$

input
```
Integrate[ArcCoth[a*x^n]/x,x]
```

output
```
(a*x^n*HypergeometricPFQ[{1/2, 1/2, 1}, {3/2, 3/2}, a^2*x^(2*n)])/n + (Arc
Coth[a*x^n] - ArcTanh[a*x^n])*Log[x]
```

Rubi [A] (verified)

Time = 0.23 (sec) , antiderivative size = 36, normalized size of antiderivative = 0.95, number of steps used = 3, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.200, Rules used = {6451, 6447}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{\coth^{-1}(ax^n)}{x} \, dx$$

$$\Big\downarrow \text{6451}$$

$$\frac{\int x^{-n}\coth^{-1}(ax^n)\,dx^n}{n}$$

$$\Big\downarrow \text{6447}$$

$$\frac{\frac{1}{2}\text{PolyLog}\left(2,-\frac{x^{-n}}{a}\right)-\frac{1}{2}\text{PolyLog}\left(2,\frac{x^{-n}}{a}\right)}{n}$$

input
```
Int[ArcCoth[a*x^n]/x,x]
```

output
```
(PolyLog[2, -(1/(a*x^n))]/2 - PolyLog[2, 1/(a*x^n)]/2)/n
```

Defintions of rubi rules used

rule 6447
```
Int[((a_.) + ArcCoth[(c_.)*(x_)]*(b_.))/(x_), x_Symbol] :> Simp[a*Log[x], x
] + (Simp[(b/2)*PolyLog[2, -(c*x)^(-1)], x] - Simp[(b/2)*PolyLog[2, 1/(c*x)
], x]) /; FreeQ[{a, b, c}, x]
```

rule 6451
```
Int[((a_.) + ArcCoth[(c_.)*(x_)^(n_)]*(b_.))^(p_.)/(x_), x_Symbol] :> Simp[
1/n   Subst[Int[(a + b*ArcCoth[c*x])^p/x, x], x, x^n], x] /; FreeQ[{a, b, c
, n}, x] && IGtQ[p, 0]
```

Maple [A] (verified)

Time = 0.27 (sec) , antiderivative size = 45, normalized size of antiderivative = 1.18

method	result	size
risch	$-\frac{\ln(a\,x^n-1)\ln(a\,x^n)}{2n}-\frac{\operatorname{dilog}(a\,x^n)}{2n}-\frac{\operatorname{dilog}(a\,x^n+1)}{2n}$	45
derivativedivides	$\frac{\ln(a\,x^n)\operatorname{arccoth}(a\,x^n)-\frac{\operatorname{dilog}(a\,x^n)}{2}-\frac{\operatorname{dilog}(a\,x^n+1)}{2}-\frac{\ln(a\,x^n)\ln(a\,x^n+1)}{2}}{n}$	53
default	$\frac{\ln(a\,x^n)\operatorname{arccoth}(a\,x^n)-\frac{\operatorname{dilog}(a\,x^n)}{2}-\frac{\operatorname{dilog}(a\,x^n+1)}{2}-\frac{\ln(a\,x^n)\ln(a\,x^n+1)}{2}}{n}$	53

input
```
int(arccoth(a*x^n)/x,x,method=_RETURNVERBOSE)
```

output
```
-1/2/n*ln(a*x^n-1)*ln(a*x^n)-1/2/n*dilog(a*x^n)-1/2/n*dilog(a*x^n+1)
```

Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 128 vs. 2(32) = 64.

Time = 0.10 (sec) , antiderivative size = 128, normalized size of antiderivative = 3.37

$$\int \frac{\coth^{-1}(ax^n)}{x}\,dx =$$

$$-\frac{n\log\left(a\cosh\left(n\log\left(x\right)\right)+a\sinh\left(n\log\left(x\right)\right)+1\right)\log\left(x\right)-n\log\left(-a\cosh\left(n\log\left(x\right)\right)-a\sinh\left(n\log\left(x\right)\right)\right.}{}$$

input
```
integrate(arccoth(a*x^n)/x,x, algorithm="fricas")
```

output
```
-1/2*(n*log(a*cosh(n*log(x)) + a*sinh(n*log(x)) + 1)*log(x) - n*log(-a*cos
h(n*log(x)) - a*sinh(n*log(x)) + 1)*log(x) - n*log(x)*log((a*cosh(n*log(x)
) + a*sinh(n*log(x)) + 1)/(a*cosh(n*log(x)) + a*sinh(n*log(x)) - 1)) - dil
og(a*cosh(n*log(x)) + a*sinh(n*log(x))) + dilog(-a*cosh(n*log(x)) - a*sinh
(n*log(x))))/n
```

Sympy [F]

$$\int \frac{\coth^{-1}(ax^n)}{x}\,dx = \int \frac{\operatorname{acoth}(ax^n)}{x}\,dx$$

input
```
integrate(acoth(a*x**n)/x,x)
```

output
```
Integral(acoth(a*x**n)/x, x)
```

Maxima [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 147 vs. 2(32) = 64.

Time = 0.10 (sec) , antiderivative size = 147, normalized size of antiderivative = 3.87

$$\int \frac{\coth^{-1}(ax^n)}{x}\,dx = -\frac{1}{2}\,an\left(\frac{\log\left(\frac{ax^n+1}{a}\right)}{an} - \frac{\log\left(\frac{ax^n-1}{a}\right)}{an}\right)\log(x) + \frac{1}{2}\,an\left(\frac{\log(ax^n+1)\log(x) - \log(ax^n-1)\log(x)}{an} - \frac{n\log(ax^n+1)\log(x) + \mathrm{Li}_2(-ax^n)}{an^2} + \frac{n\log(-a\right. + \operatorname{arcoth}(ax^n)\log(x)$$

input
```
integrate(arccoth(a*x^n)/x,x, algorithm="maxima")
```

output
```
-1/2*a*n*(log((a*x^n + 1)/a)/(a*n) - log((a*x^n - 1)/a)/(a*n))*log(x) + 1/
2*a*n*((log(a*x^n + 1)*log(x) - log(a*x^n - 1)*log(x))/(a*n) - (n*log(a*x^
n + 1)*log(x) + dilog(-a*x^n))/(a*n^2) + (n*log(-a*x^n + 1)*log(x) + dilog
(a*x^n))/(a*n^2)) + arccoth(a*x^n)*log(x)
```

Giac [F]

$$\int \frac{\coth^{-1}(ax^n)}{x}\,dx = \int \frac{\operatorname{arcoth}(ax^n)}{x}\,dx$$

input
```
integrate(arccoth(a*x^n)/x,x, algorithm="giac")
```

output
```
integrate(arccoth(a*x^n)/x, x)
```

Mupad [F(-1)]

Timed out.

$$\int \frac{\coth^{-1}(ax^n)}{x}\,dx = \int \frac{\operatorname{acoth}(a\,x^n)}{x}\,dx$$

input
```
int(acoth(a*x^n)/x,x)
```

output
```
int(acoth(a*x^n)/x, x)
```

Reduce [F]

$$\int \frac{\coth^{-1}(ax^n)}{x}\,dx = \int \frac{acoth(x^n a)}{x}dx$$

input
```
int(acoth(a*x^n)/x,x)
```

output
```
int(acoth(x**n*a)/x,x)
```

CHAPTER 4

Appendix

4.1 Listing of Grading functions

The following are the current version of the grading functions used for grading the quality of the antiderivative with reference to the optimal antiderivative included in the test suite.

There is a version for Maple and for Mathematica/Rubi. There is a version for grading Sympy and version for use with Sagemath.

The following are links to the current source code.

The following are the listings of source code of the grading functions.

Mathematica and Rubi grading function

```
(* Original version thanks to Albert Rich emailed on 03/21/2017 *)
(* ::Package:: *)

(* Nasser: April 7,2022. add second output which gives reason for the grade *)
(*                       Small rewrite of logic in main function to make it*)
(*                       match Maple's logic. No change in functionality otherwise*)

(* ::Subsection:: *)
(*GradeAntiderivative[result,optimal]*)
```

```
(* ::Text:: *)
(*If result and optimal are mathematical expressions, *)
(*         GradeAntiderivative[result,optimal] returns*)
(*  "F" if the result fails to integrate an expression that*)
(*      is integrable*)
(*  "C" if result involves higher level functions than necessary*)
(*  "B" if result is more than twice the size of the optimal*)
(*      antiderivative*)
(*  "A" if result can be considered optimal*)


GradeAntiderivative[result_,optimal_] := Module[{expnResult,expnOptimal,leafCountResult,leaf
    expnResult  = ExpnType[result];
    expnOptimal = ExpnType[optimal];
    leafCountResult = LeafCount[result];
    leafCountOptimal = LeafCount[optimal];

    (*Print["expnResult=",expnResult," expnOptimal=",expnOptimal];*)
    If[expnResult<=expnOptimal,
        If[Not[FreeQ[result,Complex]], (*result contains complex*)
            If[Not[FreeQ[optimal,Complex]], (*optimal contains complex*)
                If[leafCountResult<=2*leafCountOptimal,
                    finalresult={"A",""}
                    ,(*ELSE*)
                    finalresult={"B","Both result and optimal contain complex but leaf count
                ]
            ,(*ELSE*)
                finalresult={"C","Result contains complex when optimal does not."}
            ]
        ,(*ELSE*)(*result does not contains complex*)
            If[leafCountResult<=2*leafCountOptimal,
               finalresult={"A",""}
            ,(*ELSE*)
               finalresult={"B","Leaf count is larger than twice the leaf count of optimal.
              ]
        ]
    ,(*ELSE*) (*expnResult>expnOptimal*)
        If[FreeQ[result,Integrate] && FreeQ[result,Int],
            finalresult={"C","Result contains higher order function than in optimal. Order "
            ,
```

```
          finalresult={"F","Contains unresolved integral."}
        ]
    ];

    finalresult
]

(* ::Text:: *)
(*The following summarizes the type number assigned an *)
(*expression based on the functions it involves*)
(*1 = rational function*)
(*2 = algebraic function*)
(*3 = elementary function*)
(*4 = special function*)
(*5 = hyperpergeometric function*)
(*6 = appell function*)
(*7 = rootsum function*)
(*8 = integrate function*)
(*9 = unknown function*)


ExpnType[expn_] :=
  If[AtomQ[expn],
    1,
  If[ListQ[expn],
    Max[Map[ExpnType,expn]],
  If[Head[expn]===Power,
    If[IntegerQ[expn[[2]]],
      ExpnType[expn[[1]]],
    If[Head[expn[[2]]]===Rational,
      If[IntegerQ[expn[[1]]] || Head[expn[[1]]]===Rational,
        1,
      Max[ExpnType[expn[[1]]],2]],
    Max[ExpnType[expn[[1]]],ExpnType[expn[[2]]],3]]],
  If[Head[expn]===Plus || Head[expn]===Times,
    Max[ExpnType[First[expn]],ExpnType[Rest[expn]]],
  If[ElementaryFunctionQ[Head[expn]],
    Max[3,ExpnType[expn[[1]]]],
  If[SpecialFunctionQ[Head[expn]],
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],4]],
  If[HypergeometricFunctionQ[Head[expn]],
```

```
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],5]],
  If[AppellFunctionQ[Head[expn]],
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],6]],
  If[Head[expn]===RootSum,
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],7]],
  If[Head[expn]===Integrate || Head[expn]===Int,
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],8]],
  9]]]]]]]]]]


ElementaryFunctionQ[func_] :=
  MemberQ[{
  Exp,Log,
  Sin,Cos,Tan,Cot,Sec,Csc,
  ArcSin,ArcCos,ArcTan,ArcCot,ArcSec,ArcCsc,
  Sinh,Cosh,Tanh,Coth,Sech,Csch,
  ArcSinh,ArcCosh,ArcTanh,ArcCoth,ArcSech,ArcCsch
},func]


SpecialFunctionQ[func_] :=
  MemberQ[{
  Erf, Erfc, Erfi,
  FresnelS, FresnelC,
  ExpIntegralE, ExpIntegralEi, LogIntegral,
  SinIntegral, CosIntegral, SinhIntegral, CoshIntegral,
  Gamma, LogGamma, PolyGamma,
  Zeta, PolyLog, ProductLog,
  EllipticF, EllipticE, EllipticPi
},func]


HypergeometricFunctionQ[func_] :=
  MemberQ[{Hypergeometric1F1,Hypergeometric2F1,HypergeometricPFQ},func]


AppellFunctionQ[func_] :=
  MemberQ[{AppellF1},func]
```

Maple grading function

```
# File: GradeAntiderivative.mpl
# Original version thanks to Albert Rich emailed on 03/21/2017

#Nasser 03/22/2017  Use Maple leaf count instead since buildin
#Nasser 03/23/2017  missing 'ln' for ElementaryFunctionQ added
#Nasser 03/24/2017  corrected the check for complex result
#Nasser 10/27/2017  check for leafsize and do not call ExpnType()
#                   if leaf size is "too large". Set at 500,000
#Nasser 12/22/2019  Added debug flag, added 'dilog' to special functions
#                   see problem 156, file Apostol_Problems
#Nasser 4/07/2022   add second output which gives reason for the grade

GradeAntiderivative := proc(result,optimal)
local leaf_count_result,
        leaf_count_optimal,
        ExpnType_result,
        ExpnType_optimal,
        debug:=false;

        leaf_count_result:=leafcount(result);
        #do NOT call ExpnType() if leaf size is too large. Recursion problem
        if leaf_count_result > 500000 then
            return "B","result has leaf size over 500,000. Avoiding possible recursion issue
        fi;

        leaf_count_optimal := leafcount(optimal);
        ExpnType_result  := ExpnType(result);
        ExpnType_optimal := ExpnType(optimal);

        if debug then
            print("ExpnType_result",ExpnType_result," ExpnType_optimal=",ExpnType_optimal);
        fi;

# If result and optimal are mathematical expressions,
#  GradeAntiderivative[result,optimal] returns
#   "F" if the result fails to integrate an expression that
#       is integrable
#   "C" if result involves higher level functions than necessary
#   "B" if result is more than twice the size of the optimal
```

```
#       antiderivative
#    "A" if result can be considered optimal

    #This check below actually is not needed, since I only
    #call this grading only for passed integrals. i.e. I check
    #for "F" before calling this. But no harm of keeping it here.
    #just in case.


    if not type(result,freeof('int')) then
        return "F","Result contains unresolved integral";
    fi;


    if ExpnType_result<=ExpnType_optimal then
        if debug then
             print("ExpnType_result<=ExpnType_optimal");
        fi;
        if is_contains_complex(result) then
            if is_contains_complex(optimal) then
                if debug then
                        print("both result and optimal complex");
                fi;
                if leaf_count_result<=2*leaf_count_optimal then
                    return "A"," ";
                else
                   return "B",cat("Both result and optimal contain complex but leaf count of
                                   convert(leaf_count_result,string)," vs. $2 (",
                                   convert(leaf_count_optimal,string)," ) = ",convert(2*leaf
                end if
            else #result contains complex but optimal is not
                if debug then
                        print("result contains complex but optimal is not");
                fi;
                return "C","Result contains complex when optimal does not.";
            fi;
        else # result do not contain complex
             # this assumes optimal do not as well. No check is needed here.
            if debug then
                   print("result do not contain complex, this assumes optimal do not as well
            fi;
```

```
            if leaf_count_result<=2*leaf_count_optimal then
                if debug then
                    print("leaf_count_result<=2*leaf_count_optimal");
                fi;
                return "A"," ";
            else
                if debug then
                    print("leaf_count_result>2*leaf_count_optimal");
                fi;
                return "B",cat("Leaf count of result is larger than twice the leaf count of
                                   convert(leaf_count_result,string),"$ vs. $2(",
                                   convert(leaf_count_optimal,string),")=",convert(2*leaf_co
            fi;
        fi;
    else #ExpnType(result) > ExpnType(optimal)
        if debug then
            print("ExpnType(result) > ExpnType(optimal)");
        fi;
        return "C",cat("Result contains higher order function than in optimal. Order ",
                       convert(ExpnType_result,string)," vs. order ",
                       convert(ExpnType_optimal,string),".");
    fi;

end proc:

#
# is_contains_complex(result)
# takes expressions and returns true if it contains "I" else false
#
#Nasser 032417
is_contains_complex:= proc(expression)
  return (has(expression,I));
end proc:

# The following summarizes the type number assigned an expression
# based on the functions it involves
# 1 = rational function
# 2 = algebraic function
# 3 = elementary function
# 4 = special function
# 5 = hyperpergeometric function
```

```
# 6 = appell function
# 7 = rootsum function
# 8 = integrate function
# 9 = unknown function

ExpnType := proc(expn)
  if type(expn,'atomic') then
    1
  elif type(expn,'list') then
    apply(max,map(ExpnType,expn))
  elif type(expn,'sqrt') then
    if type(op(1,expn),'rational') then
       1
    else
       max(2,ExpnType(op(1,expn)))
    end if
  elif type(expn,'`^`') then
    if type(op(2,expn),'integer') then
      ExpnType(op(1,expn))
    elif type(op(2,expn),'rational') then
      if type(op(1,expn),'rational') then
         1
      else
         max(2,ExpnType(op(1,expn)))
      end if
    else
         max(3,ExpnType(op(1,expn)),ExpnType(op(2,expn)))
    end if
  elif type(expn,'`+`') or type(expn,'`*`') then
    max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
  elif ElementaryFunctionQ(op(0,expn)) then
    max(3,ExpnType(op(1,expn)))
  elif SpecialFunctionQ(op(0,expn)) then
    max(4,apply(max,map(ExpnType,[op(expn)])))
  elif HypergeometricFunctionQ(op(0,expn)) then
    max(5,apply(max,map(ExpnType,[op(expn)])))
  elif AppellFunctionQ(op(0,expn)) then
    max(6,apply(max,map(ExpnType,[op(expn)])))
  elif op(0,expn)='int' then
    max(8,apply(max,map(ExpnType,[op(expn)]))) else
  9
```

```
  end if
end proc:


ElementaryFunctionQ := proc(func)
  member(func,[
        exp,log,ln,
        sin,cos,tan,cot,sec,csc,
        arcsin,arccos,arctan,arccot,arcsec,arccsc,
        sinh,cosh,tanh,coth,sech,csch,
        arcsinh,arccosh,arctanh,arccoth,arcsech,arccsch])
end proc:

SpecialFunctionQ := proc(func)
  member(func,[
        erf,erfc,erfi,
        FresnelS,FresnelC,
        Ei,Ei,Li,Si,Ci,Shi,Chi,
        GAMMA,lnGAMMA,Psi,Zeta,polylog,dilog,LambertW,
        EllipticF,EllipticE,EllipticPi])
end proc:

HypergeometricFunctionQ := proc(func)
  member(func,[Hypergeometric1F1,hypergeom,HypergeometricPFQ])
end proc:

AppellFunctionQ := proc(func)
  member(func,[AppellF1])
end proc:

# u is a sum or product.  rest(u) returns all but the
# first term or factor of u.
rest := proc(u) local v;
  if nops(u)=2 then
     op(2,u)
  else
     apply(op(0,u),op(2..nops(u),u))
  end if
end proc:

#leafcount(u) returns the number of nodes in u.
```

```
#Nasser 3/23/17  Replaced by build-in leafCount from package in Maple
leafcount := proc(u)
   MmaTranslator[Mma][LeafCount](u);
end proc:
```

Sympy grading function

```
#Dec 24, 2019. Nasser M. Abbasi:
#              Port of original Maple grading function by
#              Albert Rich to use with Sympy/Python
#Dec 27, 2019  Nasser. Added `RootSum`. See problem 177, Timofeev file
#              added 'exp_polar'
from sympy import *

def leaf_count(expr):
   #sympy do not have leaf count function. This is approximation
   return round(1.7*count_ops(expr))

def is_sqrt(expr):
   if isinstance(expr,Pow):
      if expr.args[1] == Rational(1,2):
         return True
      else:
         return False
   else:
      return False

def is_elementary_function(func):
   return func in [exp,log,ln,sin,cos,tan,cot,sec,csc,
         asin,acos,atan,acot,asec,acsc,sinh,cosh,tanh,coth,sech,csch,
         asinh,acosh,atanh,acoth,asech,acsch
      ]

def is_special_function(func):
   return func in [ erf,erfc,erfi,
          fresnels,fresnelc,Ei,Ei,Li,Si,Ci,Shi,Chi,
          gamma,loggamma,digamma,zeta,polylog,LambertW,
          elliptic_f,elliptic_e,elliptic_pi,exp_polar
       ]
```

```
def is_hypergeometric_function(func):
   return func in [hyper]

def is_appell_function(func):
   return func in [appellf1]

def is_atom(expn):
   try:
      if expn.isAtom or isinstance(expn,int) or isinstance(expn,float):
        return True
      else:
        return False

   except AttributeError as error:
      return False

def expnType(expn):
   debug=False
   if debug:
      print("expn=",expn,"type(expn)=",type(expn))

   if is_atom(expn):
     return 1
   elif isinstance(expn,list):
      return max(map(expnType, expn))   #apply(max,map(ExpnType,expn))
   elif  is_sqrt(expn):
      if isinstance(expn.args[0],Rational): #type(op(1,expn),'rational')
         return 1
      else:
         return max(2,expnType(expn.args[0]))   #max(2,ExpnType(op(1,expn)))
   elif isinstance(expn,Pow):    #type(expn,'`^`')
      if isinstance(expn.args[1],Integer):   #type(op(2,expn),'integer')
         return expnType(expn.args[0])    #ExpnType(op(1,expn))
      elif isinstance(expn.args[1],Rational):   #type(op(2,expn),'rational')
         if isinstance(expn.args[0],Rational): #type(op(1,expn),'rational')
            return 1
         else:
            return max(2,expnType(expn.args[0]))   #max(2,ExpnType(op(1,expn)))
      else:
         return max(3,expnType(expn.args[0]),expnType(expn.args[1])) #max(3,ExpnType(op(1,expn)),Expn
   elif isinstance(expn,Add) or isinstance(expn,Mul): #type(expn,'`+`') or type(expn,'`*`')
```

```
        m1 = expnType(expn.args[0])
        m2 = expnType(list(expn.args[1:]))
        return max(m1,m2)  #max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
    elif is_elementary_function(expn.func):  #ElementaryFunctionQ(op(0,expn))
        return max(3,expnType(expn.args[0]))  #max(3,ExpnType(op(1,expn)))
    elif is_special_function(expn.func): #SpecialFunctionQ(op(0,expn))
        m1 = max(map(expnType, list(expn.args)))
        return max(4,m1)  #max(4,apply(max,map(ExpnType,[op(expn)])))
    elif is_hypergeometric_function(expn.func): #HypergeometricFunctionQ(op(0,expn))
        m1 = max(map(expnType, list(expn.args)))
        return max(5,m1)  #max(5,apply(max,map(ExpnType,[op(expn)])))
    elif is_appell_function(expn.func):
        m1 = max(map(expnType, list(expn.args)))
        return max(6,m1)  #max(5,apply(max,map(ExpnType,[op(expn)])))
    elif isinstance(expn,RootSum):
        m1 = max(map(expnType, list(expn.args))) #Apply[Max,Append[Map[ExpnType,Apply[List,expn]],7]],
        return max(7,m1)
    elif str(expn).find("Integral") != -1:
        m1 = max(map(expnType, list(expn.args)))
        return max(8,m1)  #max(5,apply(max,map(ExpnType,[op(expn)])))
    else:
        return 9

#main function
def grade_antiderivative(result,optimal):

    #print ("Enter grade_antiderivative for sagemath")
    #print("Enter grade_antiderivative, result=",result," optimal=",optimal)

    leaf_count_result  = leaf_count(result)
    leaf_count_optimal = leaf_count(optimal)

    #print("leaf_count_result=",leaf_count_result)
    #print("leaf_count_optimal=",leaf_count_optimal)

    expnType_result  = expnType(result)
    expnType_optimal = expnType(optimal)

    if str(result).find("Integral") != -1:
        grade = "F"
        grade_annotation =""
```

```
else:
    if expnType_result <= expnType_optimal:
        if result.has(I):
            if optimal.has(I): #both result and optimal complex
                if leaf_count_result <= 2*leaf_count_optimal:
                    grade = "A"
                    grade_annotation =""
                else:
                    grade = "B"
                    grade_annotation ="Both result and optimal contain complex but leaf count of result is lar
            else: #result contains complex but optimal is not
                grade = "C"
                grade_annotation ="Result contains complex when optimal does not."
        else: # result do not contain complex, this assumes optimal do not as well
            if leaf_count_result <= 2*leaf_count_optimal:
                grade = "A"
                grade_annotation =""
            else:
                grade = "B"
                grade_annotation ="Leaf count of result is larger than twice the leaf count of optimal. "+str(
    else:
        grade = "C"
        grade_annotation ="Result contains higher order function than in optimal. Order "+str(ExpnType(


#print("Before returning. grade=",grade, " grade_annotation=",grade_annotation)

return grade, grade_annotation
```

SageMath grading function

```
#Dec 24, 2019. Nasser: Ported original Maple grading function by
#              Albert Rich to use with Sagemath. This is used to
#              grade Fricas, Giac and Maxima results.
#Dec 24, 2019. Nasser: Added 'exp_integral_e' and 'sng', 'sin_integral'
#              'arctan2','floor','abs','log_integral'
#June 4, 2022  Made default grade_annotation "none" instead of "" due
#              issue later when reading the file.
#July 14, 2022. Added ellipticF. This is until they fix sagemath, then remove it.
```

```
from sage.all import *
from sage.symbolic.operators import add_vararg, mul_vararg

debug=False;

def tree_size(expr):
    r"""
    Return the tree size of this expression.
    """
    #print("Enter tree_size, expr is ",expr)

    if expr not in SR:
        # deal with lists, tuples, vectors
        return 1 + sum(tree_size(a) for a in expr)
    expr = SR(expr)
    x, aa = expr.operator(), expr.operands()
    if x is None:
        return 1
    else:
        return 1 + sum(tree_size(a) for a in aa)

def is_sqrt(expr):
    if expr.operator() == operator.pow:   #isinstance(expr,Pow):
        if expr.operands()[1]==1/2: #expr.args[1] == Rational(1,2):
            if debug: print ("expr is sqrt")
            return True
        else:
            return False
    else:
        return False

def is_elementary_function(func):
    #debug=False
    m = func.name() in ['exp','log','ln',
            'sin','cos','tan','cot','sec','csc',
            'arcsin','arccos','arctan','arccot','arcsec','arccsc',
            'sinh','cosh','tanh','coth','sech','csch',
            'arcsinh','arccosh','arctanh','arccoth','arcsech','arccsch','sgn',
        'arctan2','floor','abs'
        ]
    if debug:
```

```
        if m:
            print ("func ", func , " is elementary_function")
        else:
            print ("func ", func , " is NOT elementary_function")


    return m

def is_special_function(func):
    #debug=False
    if debug:
        print ("type(func)=", type(func))

    m= func.name() in ['erf','erfc','erfi','fresnel_sin','fresnel_cos','Ei',
          'Ei','Li','Si','sin_integral','Ci','cos_integral','Shi','sinh_integral'
          'Chi','cosh_integral','gamma','log_gamma','psi','zeta',
          'polylog','lambert_w','elliptic_f','elliptic_e','ellipticF',
          'elliptic_pi','exp_integral_e','log_integral',
          'weierstrassPInverse','weierstrass','weierstrassP','weierstrassZeta',
          'weierstrassPPrime','weierstrassSigma']

    if debug:
        print ("m=",m)
        if m:
            print ("func ", func ," is special_function")
        else:
            print ("func ", func ," is NOT special_function")


    return m


def is_hypergeometric_function(func):
    return func.name() in ['hypergeometric','hypergeometric_M','hypergeometric_U']

def is_appell_function(func):
    return func.name() in ['hypergeometric']   #[appellf1] can't find this in sagemath

def is_atom(expn):

    #debug=False
```

```
    if debug:
         print ("Enter is_atom, expn=",expn)

    if not hasattr(expn, 'parent'):
        return False


    #thanks to answer at https://ask.sagemath.org/question/49179/what-is-sagemath-equivalent-to-atomic
    try:
        if expn.parent() is SR:
            return expn.operator() is None
        if expn.parent() in (ZZ, QQ, AA, QQbar):
            return expn in expn.parent() # Should always return True
        if hasattr(expn.parent(),"base_ring") and hasattr(expn.parent(),"gens"):
            return expn in expn.parent().base_ring() or expn in expn.parent().gens()

        return False

    except AttributeError as error:
        print("Exception,AttributeError in  is_atom")
        print ("cought exception" , type(error).__name__ )
        return False


def expnType(expn):

    if debug:
        print (">>>>>Enter expnType, expn=", expn)
        print (">>>>>is_atom(expn)=", is_atom(expn))

    if is_atom(expn):
        return 1
    elif type(expn)==list:   #isinstance(expn,list):
        return max(map(expnType, expn))   #apply(max,map(ExpnType,expn))
    elif  is_sqrt(expn):
        if  type(expn.operands()[0])==Rational: #type(isinstance(expn.args[0],Rational):
            return 1
        else:
            return max(2,expnType(expn.operands()[0]))  #max(2,expnType(expn.args[0]))
    elif expn.operator() == operator.pow:   #isinstance(expn,Pow)
        if type(expn.operands()[1])==Integer:  #isinstance(expn.args[1],Integer)
```

```
        return expnType(expn.operands()[0])    #expnType(expn.args[0])
    elif type(expn.operands()[1])==Rational:  #isinstance(expn.args[1],Rational)
        if type(expn.operands()[0])==Rational: #isinstance(expn.args[0],Rational)
            return 1
        else:
            return max(2,expnType(expn.operands()[0]))  #max(2,expnType(expn.args[0]))
    else:
        return max(3,expnType(expn.operands()[0]),expnType(expn.operands()[1])) #max(3,expnType(expr
elif expn.operator() == add_vararg or expn.operator() == mul_vararg: #isinstance(expn,Add) or isins
    m1 = expnType(expn.operands()[0]) #expnType(expn.args[0])
    m2 = expnType(expn.operands()[1:]) #expnType(list(expn.args[1:]))
    return max(m1,m2)  #max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
elif is_elementary_function(expn.operator()):  #is_elementary_function(expn.func)
    return max(3,expnType(expn.operands()[0]))
elif is_special_function(expn.operator()): #is_special_function(expn.func)
    m1 = max(map(expnType, expn.operands()))      #max(map(expnType, list(expn.args)))
    return max(4,m1)   #max(4,m1)
elif is_hypergeometric_function(expn.operator()): #is_hypergeometric_function(expn.func)
    m1 = max(map(expnType, expn.operands()))      #max(map(expnType, list(expn.args)))
    return max(5,m1)   #max(5,m1)
elif is_appell_function(expn.operator()):
    m1 = max(map(expnType, expn.operands()))      #max(map(expnType, list(expn.args)))
    return max(6,m1)   #max(6,m1)
elif str(expn).find("Integral") != -1: #this will never happen, since it
        #is checked before calling the grading function that is passed.
        #but kept it here.
    m1 = max(map(expnType, expn.operands()))      #max(map(expnType, list(expn.args)))
    return max(8,m1)   #max(5,apply(max,map(ExpnType,[op(expn)])))
else:
    return 9

#main function
def grade_antiderivative(result,optimal):


    if debug:
        print ("Enter grade_antiderivative for sagemath")
        print("Enter grade_antiderivative, result=",result)
        print("Enter grade_antiderivative, optimal=",optimal)
        print("type(anti)=",type(result))
        print("type(optimal)=",type(optimal))
```

```
leaf_count_result  = tree_size(result) #leaf_count(result)
leaf_count_optimal = tree_size(optimal) #leaf_count(optimal)

#if debug: print ("leaf_count_result=", leaf_count_result, "leaf_count_optimal=",leaf_count_optimal)


expnType_result  = expnType(result)
expnType_optimal = expnType(optimal)

if debug: print ("expnType_result=", expnType_result, "expnType_optimal=",expnType_optimal)

if expnType_result <= expnType_optimal:
    if result.has(I):
        if optimal.has(I): #both result and optimal complex
            if leaf_count_result <= 2*leaf_count_optimal:
                grade = "A"
                grade_annotation ="none"
            else:
                grade = "B"
                grade_annotation ="Both result and optimal contain complex but leaf count of result is larger
        else: #result contains complex but optimal is not
            grade = "C"
            grade_annotation ="Result contains complex when optimal does not."
    else: # result do not contain complex, this assumes optimal do not as well
        if leaf_count_result <= 2*leaf_count_optimal:
            grade = "A"
            grade_annotation ="none"
        else:
            grade = "B"
            grade_annotation ="Leaf count of result is larger than twice the leaf count of optimal. "+str(lea
else:
    grade = "C"
    grade_annotation ="Result contains higher order function than in optimal. Order "+str(expnType_re


print("Before returning. grade=",grade, " grade_annotation=",grade_annotation)

return grade, grade_annotation
```

4.2 Links to plain text integration problems used in this report for each CAS

1. Mathematica integration problems as .m file
2. Maple integration problems as .txt file
3. Sagemath integration problems as .sage file
4. Reduce integration problems as .txt file
5. Mupad integration problems as .txt file
6. Sympy integration problems as .py file