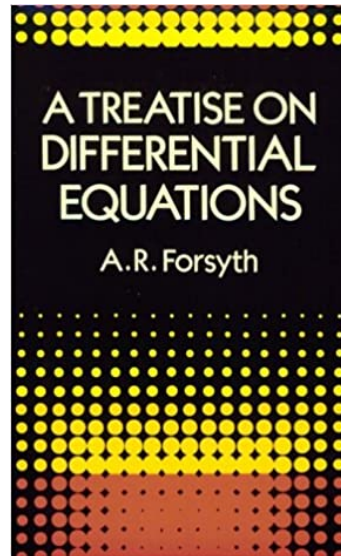**A Solution Manual For**

A treatise on Differential Equations by A. R. Forsyth. 6th edition. 1929. Macmillan Co. ltd. New York, reprinted 1956

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October 12, 2023

Contents

1 Chapter VI. Note I. Integration of linear equations in series by the method of Frobenius. page 243

1.1 problem Ex. 5, page 256

Internal problem ID [4717]

Book: A treatise on Differential Equations by A. R. Forsyth. 6th edition. 1929. Macmillan Co. ltd. New York, reprinted 1956
Section: Chapter VI. Note I. Integration of linear equations in series by the method of Frobenius. page 243
Problem number: Ex. 5, page 256.
ODE order: 2.
ODE degree: 1.

CAS Maple gives this as type [[_2nd_order, _with_linear_symmetries]]

$$x\left(-x^2 + 2\right)y'' - \left(x^2 + 4x + 2\right)\left((1 - x)y' + y\right) = 0$$

With the expansion point for the power series method at $x = 0$.

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 44

```
Order:=6;
dsolve(x*(2-x^2)*diff(y(x),x$2)-(x^2+4*x+2)*((1-x)*diff(y(x),x)+y(x))=0,y(x),type='series',x=
```

$$y(x) = c_1 x^2 \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + O\left(x^6\right)\right)$$
$$+ c_2 \left(-2 + 2x + 4x^2 + 4x^3 + 2x^4 + \frac{2}{3}x^5 + O\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 64

```
AsymptoticDSolveValue[x*(2-x^2)*y''[x]-(x^2+4*x+2)*((1-x)*y'[x]+y[x])==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(-\frac{5x^4}{4} - \frac{5x^3}{2} - \frac{5x^2}{2} - x + 1\right) + c_2 \left(\frac{x^6}{24} + \frac{x^5}{6} + \frac{x^4}{2} + x^3 + x^2\right)$$

1.2 problem Ex. 6(i), page 257

Internal problem ID [4718]

Book: A treatise on Differential Equations by A. R. Forsyth. 6th edition. 1929. Macmillan Co. ltd. New York, reprinted 1956
Section: Chapter VI. Note I. Integration of linear equations in series by the method of Frobenius. page 243
Problem number: Ex. 6(i), page 257.
ODE order: 2.
ODE degree: 1.

CAS Maple gives this as type `[[_2nd_order, _with_linear_symmetries]]`

$$x^2(x+1)\,y'' - (1+2x)\,(-y + y'x) = 0$$

With the expansion point for the power series method at $x = 0$.

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*(1+x)*diff(y(x),x$2)-(1+2*x)*(x*diff(y(x),x)-y(x))=0,y(x),type='series',x=0);
```

$$y(x) = \left((\ln(x)\,c_2 + c_1)\left(1 + O(x^6)\right) + \left(x + O(x^6)\right)c_2\right)x$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 2760

```
AsymptoticDSolveValue[x^2*(1+x)*y''[x]-(1+2*x)*(x*y'[x]+y[x])==0,y[x],{x,0,5}]
```

Too large to display

1.3 problem Ex. 6(ii), page 257

Internal problem ID [4719]

Book: A treatise on Differential Equations by A. R. Forsyth. 6th edition. 1929. Macmillan Co. ltd. New York, reprinted 1956
Section: Chapter VI. Note I. Integration of linear equations in series by the method of Frobenius. page 243
Problem number: Ex. 6(ii), page 257.
ODE order: 3.
ODE degree: 1.

CAS Maple gives this as type [[_3rd_order, _with_linear_symmetries]]

Solve

$$x^3(1+x)\,y''' - (2+4x)\,x^2y'' + (4+10x)\,xy' - (4+12x)\,y = 0$$

With the expansion point for the power series method at $x = 0$.

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 79

```
Order:=6;
dsolve(x^3*(1+x)*diff(y(x),x$3)-(2+4*x)*x^2*diff(y(x),x$2)+(4+10*x)*x*diff(y(x),x)-(4+12*x)*y
```

$$y(x) = x\Big(\big(2 + O\left(x^6\right)\big)\ln(x)\,c_2 x + \big(2x + O\left(x^6\right)\big)\ln(x)^2\,c_3 + \big(5 + O\left(x^6\right)\big)c_2 x$$
$$+ 2\big((-4)x + O\left(x^6\right)\big)\ln(x)\,c_3 + c_1 x\big(1 + O\left(x^6\right)\big) + \big(2 + 4x + 2x^2 + O\left(x^6\right)\big)c_3\Big)$$

✓ Solution by Mathematica

Time used: 0.443 (sec). Leaf size: 49

```
AsymptoticDSolveValue[x^3*(1+x)*y'''[x]-(2+4*x)*x^2*y''[x]+(4+10*x)*x*y'[x]-(4+12*x)*y[x]==0,
```

$$y(x) \to c_2 x^2 + c_1\left(2\left(x^2 + 11x + 1\right)x + 2x^2\log^2(x) - 14x^2\log(x)\right) + c_3 x^2\log(x)$$

1.4 problem Ex. 6(iii), page 257

Internal problem ID [4720]

Book: A treatise on Differential Equations by A. R. Forsyth. 6th edition. 1929. Macmillan Co.
ltd. New York, reprinted 1956
Section: Chapter VI. Note I. Integration of linear equations in series by the method of Frobenius.
page 243
Problem number: Ex. 6(iii), page 257.
ODE order: 3.
ODE degree: 1.

CAS Maple gives this as type `[[_3rd_order, _with_linear_symmetries]]`

Solve

$$x^3 \left(x^2 + 1\right) y''' - \left(4x^2 + 2\right) x^2 y'' + \left(10x^2 + 4\right) xy' - \left(12x^2 + 4\right) y = 0$$

With the expansion point for the power series method at $x = 0$.

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 52

```
Order:=6;
dsolve(x^3*(1+x^2)*diff(y(x),x$3)-(2+4*x^2)*x^2*diff(y(x),x$2)+(4+10*x^2)*x*diff(y(x),x)-(4+1
```

$$y(x) = x\left(c_3\left(2 + 2x^2 + \mathrm{O}\left(x^6\right)\right) + \left(\left(1 + \mathrm{O}\left(x^6\right)\right) c_1 + c_2\left(\ln(x)\left(2 + \mathrm{O}\left(x^6\right)\right) + \left(5 + \mathrm{O}\left(x^6\right)\right)\right)\right) x\right)$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 30

```
AsymptoticDSolveValue[x^3*(1+x^2)*y'''[x]-(2+4*x^2)*x^2*y''[x]+(4+10*x^2)*x*y'[x]-(4+12*x^2)*
```

$$y(x) \to c_1\left(2x^3 + 2x\right) + c_2 x^2 + c_3 x^2 \log(x)$$

1.5 problem Ex. 6(iv), page 257

Internal problem ID [4721]

Book: A treatise on Differential Equations by A. R. Forsyth. 6th edition. 1929. Macmillan Co. ltd. New York, reprinted 1956
Section: Chapter VI. Note I. Integration of linear equations in series by the method of Frobenius. page 243
Problem number: Ex. 6(iv), page 257.
ODE order: 2.
ODE degree: 1.

CAS Maple gives this as type [[_2nd_order, _with_linear_symmetries]]

$$2(-x+2)\,x^2 y'' - (-x+4)\,xy' + (-x+3)\,y = 0$$

With the expansion point for the power series method at $x = 0$.

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
Order:=6;
dsolve(2*(2-x)*x^2*diff(y(x),x$2)-(4-x)*x*diff(y(x),x)+(3-x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \sqrt{x}\left(x\left(1 + \frac{1}{8}x + \frac{1}{32}x^2 + \frac{5}{512}x^3 + \frac{7}{2048}x^4 + \frac{21}{16384}x^5 + O\left(x^6\right)\right)c_1\right.$$
$$\left. + \left(1 + \frac{1}{4}x + \frac{1}{32}x^2 + \frac{1}{128}x^3 + \frac{5}{2048}x^4 + \frac{7}{8192}x^5 + O\left(x^6\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 94

```
AsymptoticDSolveValue[2*(2-x)*x^2*y''[x]-(4-x)*x*y'[x]+(3-x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{5x^{9/2}}{2048} - \frac{x^{7/2}}{128} - \frac{x^{5/2}}{32} - \frac{x^{3/2}}{4} + \sqrt{x}\right) + c_2\left(\frac{7x^{11/2}}{2048} + \frac{5x^{9/2}}{512} + \frac{x^{7/2}}{32} + \frac{x^{5/2}}{8} + x^{3/2}\right)$$

1.6 problem Ex. 6(v), page 257

Internal problem ID [4722]

Book: A treatise on Differential Equations by A. R. Forsyth. 6th edition. 1929. Macmillan Co. ltd. New York, reprinted 1956
Section: Chapter VI. Note I. Integration of linear equations in series by the method of Frobenius. page 243
Problem number: Ex. 6(v), page 257.
ODE order: 2.
ODE degree: 1.

CAS Maple gives this as type `[[_2nd_order, _with_linear_symmetries]]`

$$(1 - x) x^2 y'' + (5x - 4) xy' + (6 - 9x) y = 0$$

With the expansion point for the power series method at $x = 0$.

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 43

```
Order:=6;
dsolve((1-x)*x^2*diff(y(x),x$2)+(5*x-4)*x*diff(y(x),x)+(6-9*x)*y(x)=0,y(x),type='series',x=0)
```

$$y(x) = x^2 \left(c_1 x \left(1 + O\left(x^6\right)\right) + \left(x + O\left(x^6\right)\right) \ln(x) c_2 + \left(1 - x + O\left(x^6\right)\right) c_2\right)$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 30

```
AsymptoticDSolveValue[(1-x)*x^2*y''[x]+(5*x-4)*x*y'[x]+(6-9*x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 x^3 + c_1 \left(x^3 \log(x) - x^2(3x - 1)\right)$$

1.7 problem Ex. 6(vi), page 257

Internal problem ID [4723]

Book: A treatise on Differential Equations by A. R. Forsyth. 6th edition. 1929. Macmillan Co. ltd. New York, reprinted 1956

Section: Chapter VI. Note I. Integration of linear equations in series by the method of Frobenius. page 243

Problem number: Ex. 6(vi), page 257.

ODE order: 2.

ODE degree: 1.

CAS Maple gives this as type `[[_2nd_order, _with_linear_symmetries]]`

$$xy'' + \left(4x^2 + 1\right)y' + 4xy\left(x^2 + 1\right) = 0$$

With the expansion point for the power series method at $x = 0$.

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
Order:=6;
dsolve(x*diff(y(x),x$2)+(4*x^2+1)*diff(y(x),x)+4*x*(x^2+1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - x^2 + \frac{1}{2}x^4\right)\left(\ln(x)\, c_2 + c_1\right) + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 40

```
AsymptoticDSolveValue[x*y''[x]+(4*x^2+1)*y'[x]+4*x*(x^2+1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^4}{2} - x^2 + 1\right) + c_2\left(\frac{x^4}{2} - x^2 + 1\right)\log(x)$$

1.8 problem Ex. 8(i), page 258

Internal problem ID [4724]

Book: A treatise on Differential Equations by A. R. Forsyth. 6th edition. 1929. Macmillan Co. ltd. New York, reprinted 1956
Section: Chapter VI. Note I. Integration of linear equations in series by the method of Frobenius. page 243
Problem number: Ex. 8(i), page 258.
ODE order: 2.
ODE degree: 1.

CAS Maple gives this as type `[[_2nd_order, _with_linear_symmetries]]`

$$x^2 y'' + 4(x+a)\, y = 0$$

With the expansion point for the power series method at $x = 0$.

✔ <u>Solution by Maple</u>

Time used: 0.0 (sec). Leaf size: 947

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+4*(x+a)*y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = \sqrt{x}\,\Bigg(c_1 x^{-\frac{\sqrt{1-16a}}{2}} &\Bigg(1 + 4\frac{1}{-1+\sqrt{1-16a}}x + 8\frac{1}{\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)}x^2 \\
&+ \frac{32}{3}\frac{1}{\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)\left(-3+\sqrt{1-16a}\right)}x^3 \\
+ \frac{32}{3}&\frac{1}{\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)\left(-3+\sqrt{1-16a}\right)\left(-4+\sqrt{1-16a}\right)}x^4 \\
+ \frac{128}{15}&\frac{1}{\left(-1+\sqrt{1-16a}\right)\left(-2+\sqrt{1-16a}\right)\left(-3+\sqrt{1-16a}\right)\left(-4+\sqrt{1-16a}\right)\left(-5+\sqrt{1-16a}\right)}x^5 \\
&+ \mathrm{O}\left(x^6\right)\Bigg) + c_2 x^{\frac{\sqrt{1-16a}}{2}}\Bigg(1 - 4\frac{1}{1+\sqrt{1-16a}}x + 8\frac{1}{\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)}x^2 \\
&- \frac{32}{3}\frac{1}{\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)\left(3+\sqrt{1-16a}\right)}x^3 \\
+ \frac{32}{3}&\frac{1}{\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)\left(3+\sqrt{1-16a}\right)\left(4+\sqrt{1-16a}\right)}x^4 \\
- \frac{128}{15}&\frac{1}{\left(1+\sqrt{1-16a}\right)\left(2+\sqrt{1-16a}\right)\left(3+\sqrt{1-16a}\right)\left(4+\sqrt{1-16a}\right)\left(5+\sqrt{1-16a}\right)}x^5 \\
&+ \mathrm{O}\left(x^6\right)\Bigg)\Bigg)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 1356

```
AsymptoticDSolveValue[x^2*y''[x]+4*(x+a)*y[x]==0,y[x],{x,0,5}]
```

$y(x)$

$$\to \left(-\frac{}{\left(\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+1\right)\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+2\right)+4a\right)\left(\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+2\right)\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)\right.\right.}\right.$$

$$+\frac{}{\left(\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+1\right)\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+2\right)+4a\right)\left(\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+2\right)\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)\right.\right.}$$

$$-\frac{64x^3}{\left(\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+1\right)\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+2\right)+4a\right)\left(\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+2\right)\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)\right.\right.}$$

$$-\frac{16x^2}{\left(\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+1\right)\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+2\right)+4a\right)\left(\frac{1}{2}\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+1\right)\left(1-\sqrt{1-16a}\right)+\right.}$$

$$+\frac{4x}{\left(\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+1\right)\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+2\right)+4a\right)\left(\frac{1}{2}\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+1\right)\left(1-\sqrt{1-16a}\right)+\right.}$$

$$\left.-\frac{4x}{\frac{1}{2}\left(\frac{1}{2}\left(1-\sqrt{1-16a}\right)+1\right)\left(1-\sqrt{1-16a}\right)+4a}+1\right)c_2 x^{\frac{1}{2}\left(1-\sqrt{1-16a}\right)}$$

$$+\left(-\frac{}{\left(\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+1\right)\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+2\right)+4a\right)\left(\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+2\right)\left(\frac{1}{2}\left(\sqrt{1-16a}+\right.\right.\right.}\right.$$

$$+\frac{}{\left(\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+1\right)\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+2\right)+4a\right)\left(\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+2\right)\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)\right.\right.}$$

$$-\frac{64x^3}{\left(\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+1\right)\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+2\right)+4a\right)\left(\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+2\right)\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)\right.\right.}$$

$$-\frac{16x^2}{\left(\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+1\right)\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+2\right)+4a\right)\left(\frac{1}{2}\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+1\right)\left(\sqrt{1-16a}+1\right)+\right.}$$

$$+\frac{4x}{\left(\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+1\right)\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+2\right)+4a\right)\left(\frac{1}{2}\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+1\right)\left(\sqrt{1-16a}+1\right)+\right.}$$

$$\left.-\frac{4x}{\frac{1}{2}\left(\frac{1}{2}\left(\sqrt{1-16a}+1\right)+1\right)\left(\sqrt{1-16a}+1\right)+4a}+1\right)c_1 x^{\frac{1}{2}\left(\sqrt{1-16a}+1\right)}$$

1.9 problem Ex. 8(ii), page 258

Internal problem ID [4725]

Book: A treatise on Differential Equations by A. R. Forsyth. 6th edition. 1929. Macmillan Co.
ltd. New York, reprinted 1956
Section: Chapter VI. Note I. Integration of linear equations in series by the method of Frobenius.
page 243
Problem number: Ex. 8(ii), page 258.
ODE order: 2.
ODE degree: 1.

CAS Maple gives this as type [[_2nd_order, _with_linear_symmetries]]

$$xy'' + \left(x^3 + 1\right) y' + bxy = 0$$

With the expansion point for the power series method at $x = 0$.

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 73

```
Order:=6;
dsolve(x*diff(y(x),x$2)+(1+x*x^2)*diff(y(x),x)+b*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(\ln\left(x\right) c_2 + c_1\right) \left(1 - \frac{1}{4}b\,x^2 + \frac{1}{64}b^2 x^4 + \frac{1}{50}b\,x^5 + O\left(x^6\right)\right)$$
$$+ \left(\frac{b}{4}x^2 - \frac{1}{9}x^3 - \frac{3}{128}b^2 x^4 - \frac{61}{4500}b\,x^5 + O\left(x^6\right)\right) c_2$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 103

```
AsymptoticDSolveValue[x*y''[x]+(1+x*x^2)*y'[x]+b*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \rightarrow c_1 \left(\frac{b^2 x^4}{64} + \frac{bx^5}{50} - \frac{bx^2}{4} + 1\right)$$
$$+ c_2 \left(-\frac{3b^2 x^4}{128} + \left(\frac{b^2 x^4}{64} + \frac{bx^5}{50} - \frac{bx^2}{4} + 1\right) \log(x) - \frac{61bx^5}{4500} + \frac{bx^2}{4} - \frac{x^3}{9}\right)$$