Solving Abel first order differential equation

First and Second kind

$$y' = f_0 + f_1 y + f_2 y^2 + f_3 y^3$$

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Contents

1.1 introduction

This ODE has the form

$$y'(x) = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3 \tag{1}$$

Any of the following forms is called an Abel ode of first kind

$$\begin{aligned}
y' &= f_0 + f_1 y + f_2 y^2 + f_3 y^3\\
y' &= f_1 y + f_2 y^2 + f_3 y^3\\
y' &= f_2 y^2 + f_3 y^3\\
y' &= f_0 + f_2 y^2 + f_3 y^3\\
y' &= f_0 + f_3 y^3\\
y' &= f_0 + f_1 y + f_3 y^3\\
y' &= f_2 y^2 + f_3 y^3
\end{aligned}$$

The case for both $f_0(x) = 0, f_2(x) = 0$ is not allowed, else it becomes Bernoulli ode. Either $f_0 = 0$ or $f_2 = 0$ is allowed but *not both at same time.* The term $f_3(x)$ must be there in all cases.

In addition, at least one of f_i must be a function of x else it is quadrature. For example $y' = y^2 + y^3$ is not Abel ode but $y' = xy^2 + y^3$ is Abel ode.

The case when $f_2 = 0$ is important. The Abel invariant is defined only when $f_2 = 0$ as

$$\Delta = -\frac{\left(-f_0' f_3 + f_0 f_3' + 3 f_0 f_3 f_1\right)^3}{27 f_3^4 f_0^5}$$

In the above, we do not have to worry about $f_0 = 0$ because this is not possible since in Abel ode both $f_2 = 0$ and $f_0 = 0$ is not possible, else it will not be Abel ode in first place.

In the case when $f_2 \neq 0$, then f_2 is first removed from the original ode using the change of dependent variable

$$y = u(x) - \frac{f_2}{3 f_3}$$

Now the new ode in $u(x)$ will not have f_2 in it. Now we check again if the resulting ode is still Abel ode. If so, we apply transformation to convert it to separable and solve it. If not Abel ode, then it is solved using other algorithms depending on the ode type that it comes out to be.

There are two possibilities when $f_2 = 0$. Either Δ is constant (i.e. does not depend on x) or not (i.e. function of x). The constant invariant is the easy case and can always be solved. The non constant case is not fully solved and only few cases can be solved analytically. Currently if Δ is not constant, the ode is converted to Abel of second kind to try the algorithms there.

If invariant Δ is constant and $f_0 \neq 0$ (since we can not have both $f_0 = 0, f_2 = 0$) then the substitution

$$y = \left(\frac{f_0}{f_3}\right)^{\frac{1}{3}} u(x)$$

Results in a separable ode in $u(x)$ which can be easily solved. (See examples below).

There are only few cases where Abel ode is solvable when Δ not constant. When Δ is not constant, then it is converted Abel second kind to try again using the algorithms for the second kind. This is done using the substitution

$$y = \frac{1}{u}$$

See section below for how this is done with examples.

1.2 Algorithm

Local contents

1.2.1 Flow chart

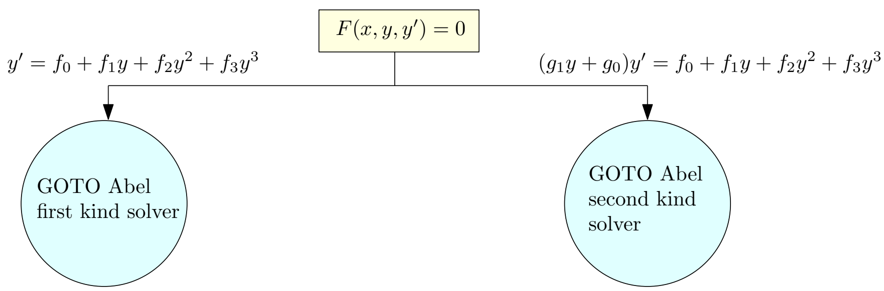

Figure 1: Top level flow chart for Abel ode

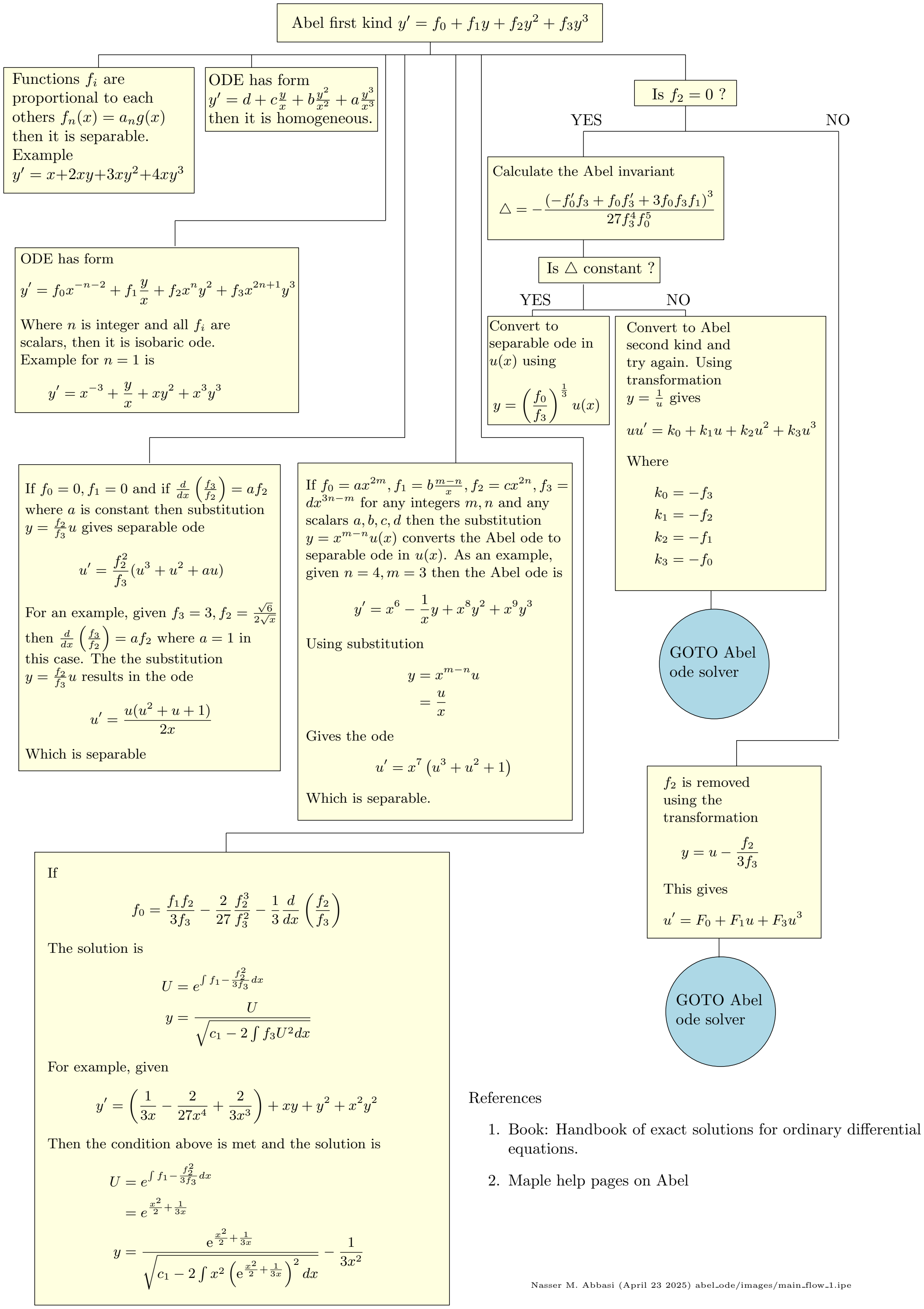

Figure 2: Flow chart for Abel first kind

1.2.2 Pseudocode for solving Abel first kind ode

The following is the algorithm for solving Abel ode.

```
FUNCTION abel_solver(ode)

INPUT: Abel ode  y'=f0 + f1 y + f2 y^2 + f3 y^3

IF f2  = 0 then -- note, f0 can not be zero now. Else not abel ode.
                -- as both f0 and f2 can not be zero at same time.

    Check if the Abel invariant DEL is constant or not.

    IF DEL not constant (i.e. depends on x) then
       RETURN can not solve.
    ELSE
       Apply transformation y= (f0/f3)^(1/3)*u(x).

       The new ode in u(x) should be separable
       Solve for u(x)
       Transform back to y(x)
       RETURN
    END IF
ELSE
    Apply transformation y=u-f2/(3*f3) to remove f2.
    This generates new_ode in u(x).

    IF new_ode happens to be anything other than Abel or Chini
      (such as separable, or quadrature) then solve it.
      Apply reverse transformation to go back from u(x) to y(x)
      using y=u - f2/(3*f3)
      RETURN
    ELSE
        IF new_ode is chini y'=f*y^n + g*y + h THEN
            IF Chini invariant is constant THEN
               Solve. See Formula in Kamke
               Aplying back transformation  to y(x) using y=u - f2/(3*f3)
               RETURN
            ELSE
               RETURN can not solve. Chini
            END IF
        ELSE
            IF new_ode is Abel THEN
               CALL abel_solver(new_ode) again recursive call.
               This will check if invariant is constant or not and
               solve it as separable if so.
               RETURN solution if any.
            ELSE
               RETURN can not solve.
            END IF
        END IF
    END IF
END IF
```

1.3 About equivalence between two Abel ode's

Given one Abel first kind ode $y'(x) = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3$, it is called equivalent to another Abel ode $u'(t) = g_0(t) + g_1(t)u + g_2(t)u^2 + g_3(t)u^3$ if there is *transformation* which converts one to the other. This transformation is given by

$$
\begin{aligned}
x &= F(t) \\
y(x) &= P(t)\, u(t) + Q(t)
\end{aligned}
\tag{1}
$$

Where $F' \neq 0, P \neq 0$. If such transformation can be found, then if given the solution of one of these ode's, the solution to the other ode can directly be fond using this transformation. In this case, we also call these two ode as belonging to same Abel equivalence class. In other words, an Abel equivalence class is the set of all Abel ode's that can be transformed to each others using the same transformation given in (1).

There are many disjoint Abel equivalence classes, each class will have all the ode that can be transformed to each others using some specific transformation (1). Here is one example below taken from paper by A.D.Roch and E.S.Cheb-Terrab called "Abel ODEs: Equivalence and integrable classes".

Given one Abel ode

$$
y'(x) = \frac{1}{2x+8}y^2 + \frac{x}{2x+8}y^3 \tag{2}
$$

Which is known to have solution

$$
c_1 + \frac{\sqrt{y^2x - 4y - 1}}{y} + 2\arctan\left(\frac{1+2y}{\sqrt{y^2x - 4y - 1}}\right) = 0 \tag{3}
$$

And now we are given a second Abel ode

$$
u'(t) = \frac{1}{t}u + \frac{f't - f}{2(f+3t)}u^2 + \frac{(f't - f)(t - f)}{2(f+3t)}u^3 \tag{4}
$$

And asked to find its solution. If we can determine if (4) is equivalent to (2) then the solution of (4) can be obtained directly. It can be found that

$$
\begin{aligned}
F(t) &= \frac{f(t)}{t} - 1 \\
Q(t) &= 0 \\
P(t) &= t
\end{aligned}
$$

Where see that $F'(t) \neq 0$ and $P(t) \neq 0$. Hence (1) becomes

$$
\begin{aligned}
x &= \frac{f(t)}{t} - 1 \\
y(x) &= tu(t)
\end{aligned}
\tag{5}
$$

Applying the transformation (5) on the solution (3) results in the solution of (4) as

$$
\begin{aligned}
A &= \sqrt{\left(\frac{f}{t} - 1\right)t^2u^2 - 4tu - 1} \\
c_1 + \frac{A}{tu} + 2\arctan\left(\frac{1+2tu}{A}\right) &= 0
\end{aligned}
\tag{6}
$$

Equation (6) above is the implicit solution to (4) obtained from the solution to (2) by using equivalence transformation as the two ode's are found to be equivalent. Finding the transformation (5) requires more calculation and not trivial. See the above paper for more information.

1.4 Examples

1.4.1 Example 1

$$y' = -xe^{-x} - y + xe^{2x}y^3$$

Comparing to

$$y' = f_0 + f_1 y + f_2 y^2 + f_3 y^3$$

Shows that

$$\begin{aligned} f_0 &= -xe^{-x} \\ f_1 &= -1 \\ f_2 &= 0 \\ f_3 &= xe^{2x} \end{aligned}$$

Since $f_2 = 0$ then we check is if the invariant depends on x or not.

$$\begin{aligned} \Delta &= -\frac{\left(-f_0' f_3 + f_0 f_3' + 3 f_0 f_3 f_1\right)^3}{27 f_3^4 f_0^5} \\ &= -\frac{\left(-(-e^{-x} + xe^{-x})\left(xe^{2x}\right) + (-xe^{-x})\left(e^{2x} + 2xe^{2x}\right) + 3(-xe^{-x})\left(xe^{2x}\right)(-1)\right)^3}{27\left(xe^{2x}\right)^4\left(-xe^{-x}\right)^5} \\ &= 0 \end{aligned}$$

Since Δ does not depend on x, then this is the easy case. We can convert the ode to separable using

$$\begin{aligned} y &= \left(\frac{f_0}{f_3}\right)^{\frac{1}{3}} u \\ &= \left(\frac{-xe^{-x}}{xe^{2x}}\right)^{\frac{1}{3}} u \\ &= \left(-e^{-3x}\right)^{\frac{1}{3}} u \\ &= -e^{-x} u \end{aligned}$$

In the above surd was used to obtain the cubic real root assuming everything is real.

Applying this change of variable to the original ode results in

$$\begin{aligned} e^{-x}(u' - u) &= xe^{-x} + xu^3 e^{-x} - e^{-x} u \\ u' - u &= x + xu^3 - u \\ u' &= x + xu^3 \\ &= x(u^3 + 1) \end{aligned}$$

Which is separable. Solving and simplifying gives

$$\ln\left(\frac{(u+1)^{\frac{1}{3}}}{(u^2 - u + 1)^{\frac{1}{6}}}\right) + \frac{\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}}{3}(2u - 1)\right) = \frac{x^2}{2} + c_1$$

But $u = -ye^x$. Hence the solution to the original Abel ode is

$$\ln\left(\frac{(-ye^x + 1)^{\frac{1}{3}}}{(y^2 e^{2x} + ye^x + 1)^{\frac{1}{6}}}\right) + \frac{\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}}{3}(-2ye^x - 1)\right) = \frac{x^2}{2} + c_1$$

1.5 Abel second kind $(g_1 y + g_0(x))\, y' = f_0 + f_1 y + f_2 y^2 + f_3 y^3$

Local contents

1.5.1 Algorithm

Local contents

1.5.1.1 Flow chart

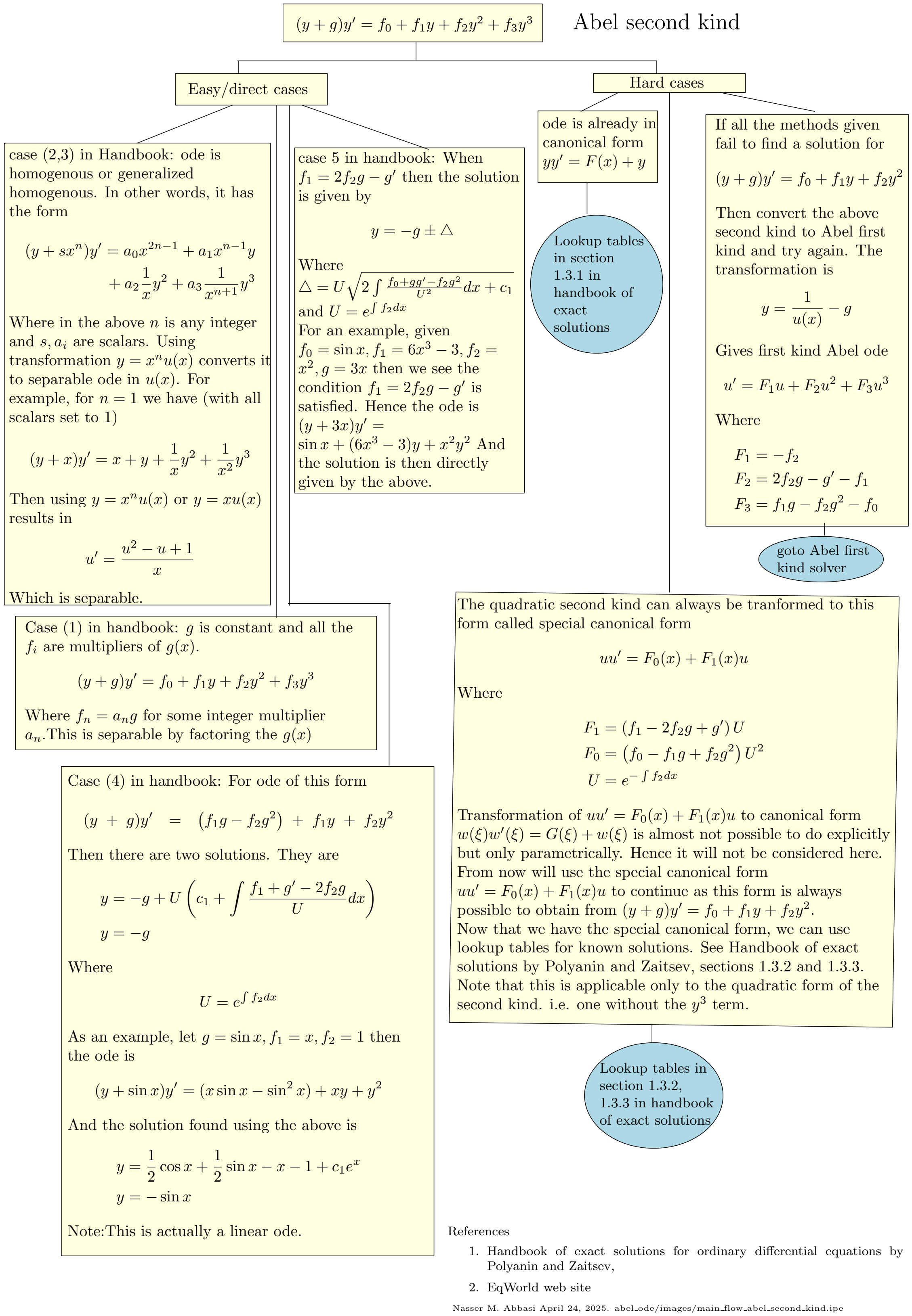

Figure 3: Flow chart for Abel second kind

1.5.2 Details

This ODE has the form

$$(g_1 y + g_0)\, y' = f_0 + f_1 y + f_2 y^2 + f_3 y^3 \tag{1A}$$

In some places, this is only shows up to quadratic term

$$(g_1 y + g_0)\, y' = f_0 + f_1 y + f_2 y^2 \tag{1B}$$

And the above second form seems more common. But it seems both can be used in everything I tried. This is transformed to first kind Abel ode in $u(x)$ using the following transformation

$$y = \frac{1}{u g_1} - \frac{g_0}{g_1} \tag{2A}$$

But a more common form is the following (used by handbook of exact solutions to ordinary differential equations) is this

$$(y + g)\, y' = f_0 + f_1 y + f_2 y^2 \tag{1C}$$

Which can be obtained from (1B) by dividing by g_1 (where the f_i are now different). Using (1C) form, then (2A) becomes

$$y = \frac{1}{u} - g \tag{2B}$$

Which is simpler than (2A) and this is what will be used from now on. The canonical form for the second kind is the following

$$y y' = F(x) + y \tag{3}$$

Here F must be function of x and can not be scalar, else the ode becomes quadrature. When the second kind is in the canonical form above there are tabulated cases of known solutions for specific cases of $f(x)$. See equation world and handbook of exact solutions.

Those tables are given for the ode in form $yy' = (sx + Ax^m) + y$. i.e. $f = sx + Ax^m$ for different values of s, m where A is arbitrary parameter. There are tables for different forms also.

Applying transformation to first kind given in (2B) results in

$$u' = k_0 + k_1 u + k_2 u^2 + k_3 u^3 \tag{4}$$

Where

$$\begin{aligned}
k_0 &= -f_3 \\
k_1 &= 3 g f_3 - f_2 \\
k_2 &= -3 g^2 f_3 + 2 g f_2 - g' - f_1 \\
k_3 &= g^3 f_3 - g^2 f_2 + f_1 g - f_0
\end{aligned} \tag{5}$$

Which is Abel of first kind.

1.5.3 Examples

Local contents

1.5.3.1 Example 1 $(y-2x)\,y' = x-y+y^2$

Given

$$(y-2x)\,y' = x-y+y^2$$

Comparing this to (1) $(y+g)\,y' = f_0+f_1y+f_2y^2+f_3y^3$ shows that

$$\begin{aligned} g &= -2x \\ f_0 &= x \\ f_1 &= -1 \\ f_2 &= 1 \\ f_3 &= 0 \end{aligned}$$

Applying the transformation in (2B) gives

$$\begin{aligned} y &= \frac{1}{u} - g \\ y &= \frac{1}{u} + 2x \end{aligned}$$

Results in the ode (4)

$$u' = k_0 + k_1u + k_2u^2 + k_3u^3 \tag{3}$$

Where, using (5) gives

$$\begin{aligned} k_0 &= -f_3 \\ &= 0 \\ k_1 &= 3gf_3 - f_2 \\ &= -1 \\ k_2 &= -3g^2f_3 + 2gf_2 - g' - f_1 \\ &= 2(-2x)\,(1) + 2 + 1 \\ &= -4x + 3 \\ k_3 &= g^3f_3 - g^2f_2 + f_1g - f_0 \\ &= -(-2x)^2 - 1(-2x) - x \\ &= x - 4x^2 \end{aligned}$$

Hence (3) becomes

$$\begin{aligned} u' &= k_0 + k_1u + k_2u^2 + k_3u^3 \\ &= -u + (-4x+3)\,u^2 + \left(-4x^2 + x\right)u^3 \end{aligned}$$

Which is Abel ode of first kind. But this can not be solved. (not all Abel odes can be solved)

1.5.3.2 Example 2 $yy' = x + y$

Given

$$yy' = x + y$$

Comparing this to $(y+g)\,y' = f_0 + f_1 y + f_2 y^2 + f_3 y^3$ shows that

$$\begin{aligned} g &= 0 \\ f_0 &= x \\ f_1 &= 1 \\ f_2 &= 0 \\ f_3 &= 0 \end{aligned}$$

Applying the transformation in (2B) gives

$$\begin{aligned} y &= \frac{1}{u} - g \\ y &= \frac{1}{u} \end{aligned}$$

Results in the ode (4)

$$u' = k_0 + k_1 u + k_2 u^2 + k_3 u^3 \tag{3}$$

Where, using (5) gives

$$\begin{aligned} k_0 &= -f_3 \\ &= 0 \\ k_1 &= 3 g f_3 - f_2 \\ &= 0 \\ k_2 &= -3 g^2 f_3 + 2 g f_2 - g' - f_1 \\ &= -1 \\ k_3 &= g^3 f_3 - g^2 f_2 + f_1 g - f_0 \\ &= -x \end{aligned}$$

Hence (3) becomes

$$\begin{aligned} u' &= k_0 + k_1 u + k_2 u^2 + k_3 u^3 \\ &= -u^2 - x u^3 \end{aligned}$$

Which is Abel ode of first kind. (it is also homogeneous class G). Solving this gives

$$\ln t - c_1 + \frac{1}{2} \ln\left(u^2 x^2 + x u - 1\right) - \frac{\sqrt{5}}{5} \operatorname{arctanh}\left(\frac{1}{5}(2 x u + 1)\sqrt{5}\right) - \ln(x u) = 0$$

But $u=\frac{1}{y}$ hence the above becomes

$$\begin{aligned}
\ln x - c_1 + \frac{1}{2}\ln\left(\frac{x^2}{y^2}+\frac{x}{y}-1\right) - \frac{\sqrt{5}}{5}\operatorname{arctanh}\left(\frac{1}{5}\left(\frac{2x}{y}+1\right)\sqrt{5}\right) - \ln\left(\frac{x}{y}\right) &= 0\\
\ln x - c_1 + \frac{1}{2}\ln\left(\frac{x^2+xy-y^2}{y^2}\right) - \ln\frac{x}{y} - \frac{\sqrt{5}}{5}\operatorname{arctanh}\left(\frac{\sqrt{5}}{5}\frac{2x+y}{y}\right) &= 0\\
\ln x - c_1 + \ln\left(\frac{\sqrt{x^2+xy-y^2}}{y}\right) - \ln\frac{x}{y} - \frac{\sqrt{5}}{5}\operatorname{arctanh}\left(\frac{\sqrt{5}}{5}\frac{2x+y}{y}\right) &= 0\\
\ln x - c_1 + \ln\left(\frac{\frac{\sqrt{x^2+xy-y^2}}{y}}{\frac{x}{y}}\right) - \frac{\sqrt{5}}{5}\operatorname{arctanh}\left(\frac{\sqrt{5}}{5}\frac{2x+y}{y}\right) &= 0\\
\ln x - c_1 + \ln\left(\frac{\sqrt{x^2+xy-y^2}}{x}\right) - \frac{\sqrt{5}}{5}\operatorname{arctanh}\left(\frac{\sqrt{5}}{5}\frac{2x+y}{y}\right) &= 0\\
\ln x - c_1 + \frac{1}{2}\ln\left(\frac{x^2+xy-y^2}{x^2}\right) - \frac{\sqrt{5}}{5}\operatorname{arctanh}\left(\frac{\sqrt{5}}{5}\frac{2x+y}{y}\right) &= 0
\end{aligned}$$

1.5.3.3 Example 3 $(xy-2x)\,y' = y - y^2 + 3x^2y^3$

Given

$$(xy-2x)\,y' = y - y^2 + 3x^2y^3$$

We start by normalizing this to form $(y+g)\,y' = f_0 + f_1y + f_2y^2 + f_3y^3$ by dividing by $x \neq 0$ which gives

$$(y-2)\,y' = \frac{1}{x}y - \frac{1}{x}y^2 + 3xy^3$$

Comparing this to (1) $(y+g)\,y' = f_0 + f_1y + f_2y^2 + f_3y^3$ shows that

$$\begin{aligned}
g &= -2\\
f_0 &= 0\\
f_1 &= \frac{1}{x}\\
f_2 &= -\frac{1}{x}\\
f_3 &= 3x
\end{aligned}$$

Applying the transformation in (2) gives

$$\begin{aligned}
y &= \frac{1}{u} - g\\
y &= \frac{1}{u} + 2
\end{aligned}$$

Results in the ode (4)

$$u' = k_0 + k_1u + k_2u^2 + k_3u^3 \tag{3}$$

Where, using (5) gives

$$\begin{aligned}
k_0 &= -f_3 \\
&= -3x \\
k_1 &= 3gf_3 - f_2 \\
&= 3(-2)(3x) + \frac{1}{x} \\
&= -18x + \frac{1}{x} \\
k_2 &= -3g^2 f_3 + 2gf_2 - g' - f_1 \\
&= -3(-2)^2 (3x) + 2(-2)\left(-\frac{1}{x}\right) - \frac{1}{x} \\
&= -\frac{3}{x}(12x^2 - 1) \\
k_3 &= g^3 f_3 - g^2 f_2 + f_1 g - f_0 \\
&= (-8)(3x) - (-2)^2\left(-\frac{1}{x}\right) + \frac{1}{x}(-2) \\
&= -\frac{2}{x}(12x^2 - 1)
\end{aligned}$$

Hence (3) becomes

$$\begin{aligned}
u' &= k_0 + k_1 u + k_2 u^2 + k_3 u^3 \\
&= -3x + \left(-18x + \frac{1}{x}\right) u + \left(-\frac{3}{x}(12x^2 - 1)\right) u^2 + \left(-\frac{2}{x}(12x^2 - 1)\right) u^3
\end{aligned}$$

Which is Abel first kind.

1.6 Converting Abel first kind ode with non constant invariant to canonical form $u'(\xi) = F(\xi)\, u^3(\xi)$

Local contents

1.6.1 Algorithm

Given Abel ode of first kind $y' = f_0 + f_1 y + f_2 y^2 + f_3 y^3$, we first remove f_2 as described above using transformation $y = u(x) - \frac{f_2}{3f_3}$ which results in $u'(x) = k_0 + k_1 u + k_3 u^3$. Now we check the Abel invariant Δ defined as

$$\Delta = -\frac{(-f_0' f_3 + f_0 f_3' + 3 f_0 f_3 f_1)^3}{27 f_3^4 f_0^5}$$

And we assume the above is not constant. Only in this case we convert the ode to canonical form (if Δ is constant, then it can be solved as shown above). So the goal now is to convert

$$u'(x) = k_0 + k_1 u + k_3 u^3$$

to

$$\eta'(\xi) = \Phi(\xi) + \eta^3(\xi)$$

This is done as follows. We start by finding

$$U = \exp\left(\int k_1 - \frac{k_2^2}{3k_3} dx\right)$$

The result will be function of x. i.e. $U(x)$. Now we apply this transformation

$$
\begin{aligned}
u(x) &= U(x)\,\eta(\xi) - \frac{k_2}{3k_3} \\
x &= \Phi(\xi)
\end{aligned}
$$

To the ode $u'(x) = k_0 + k_1 u + k_3 u^3$. This will result in

$$
\eta'(\xi) = \Phi(\xi) + \eta^3(\xi)
$$

Books say that $\Phi(\xi)$ is defined parametrically where x is the parameter. Where

$$
\begin{aligned}
\Phi(\xi) &= \frac{1}{f_3 U^3}\left(f_0 - \frac{f_1 f_2}{3f_3} + \frac{2f_2^3}{27f_3^2} - \frac{1}{3}\frac{d}{dx}\left(\frac{f_2}{f_3}\right)\right) \\
\xi &= \int k_3 U^2 dx
\end{aligned}
$$

Lets look at some examples showing how this is done.

1.6.2 Examples

Local contents

1.6.2.1 Example 1 $y' = \left(\frac{1}{x^2} - \frac{1}{x}\right) y^2 + \frac{1}{x^2}u^3$

Which is Abel first kind.

$$
\begin{aligned}
f_0 &= 0 \\
f_1 &= 0 \\
f_2 &= \left(\frac{1}{x^2} - \frac{1}{x}\right) \\
f_3 &= \frac{1}{x^2}
\end{aligned}
$$

Since f_2 is not zero, it is removed using

$$
\begin{aligned}
y(x) &= u(x) - \frac{f_2}{3f_3} \\
&= u(x) - \frac{\left(\frac{1}{x^2} - \frac{1}{x}\right)}{3\left(\frac{1}{x^2}\right)} \\
&= u(t) - \left(\frac{1}{3} - \frac{1}{3}x\right)
\end{aligned}
$$

This results in

$$
\begin{aligned}
u'(x) &= \left(-\frac{2}{27}x - \frac{1}{9} - \frac{2}{9x} + \frac{2}{27x^2}\right) + \left(\frac{-1}{3} + \frac{2}{3x} - \frac{1}{3x^2}\right)u + \frac{1}{x^2}u^3 \qquad \text{(A)} \\
&= f_0 + f_1 u + f_3 u^3
\end{aligned}
$$

Where now

$$
\begin{aligned}
f_0 &= \left(-\frac{2}{27}x - \frac{1}{9} - \frac{2}{9x} + \frac{2}{27x^2}\right)\\
f_1 &= \left(\frac{-1}{3} + \frac{2}{3x} - \frac{1}{3x^2}\right)\\
f_2 &= 0\\
f_3 &= \frac{1}{x^2}
\end{aligned}
$$

We see the ode is missing the u^2 term. The above is Abel first kind. Now we check if the invariant is constant or not. (Recall, this check is only done when the quadratic term is missing). This gives

$$
\begin{aligned}
\Delta &= -\frac{(-f_0' f_3 + f_0 f_3' + 3 f_0 f_3 f_1)^3}{27 f_3^4 f_0^5}\\
&= \frac{1458(2x^5 + 5x^4 + 8x^3 - 5x^2 + 10x - 2)^2 x^2 (x^3 + 4x^2 - 2x - 6)}{(2x^3 + 3x^2 + 6x - 2)^6}
\end{aligned}
$$

Which is clearly not constant. Hence no direct solution exist. Let now apply the transformation to convert the ode in $u(x)$ to canonical form

$$
\eta'(\xi) = \Phi(\xi) + \eta^3(\xi)
$$

We start by finding

$$
\begin{aligned}
U &= \exp\left(\int f_1(x) - \frac{f_2^2(x)}{3f_3(x)} dx\right)\\
&= \exp \int f_1 dx\\
&= \exp \int \frac{-1}{3} + \frac{2}{3x} - \frac{1}{3x^2} dx\\
&= \exp\left(-\frac{x}{3} + \frac{2}{3}\ln x + \frac{1}{3x}\right)\\
&= x^{\frac{2}{3}} e^{-\frac{x}{3}} e^{\frac{1}{3x}}
\end{aligned}
$$

Next we find

$$
\begin{aligned}
\Phi(\xi) &= \frac{1}{f_3 U^3}\left(f_0 - \frac{f_1 f_2}{3 f_3} + \frac{2 f_2^3}{27 f_3^2} - \frac{1}{3}\frac{d}{dx}\left(\frac{f_2}{f_3}\right)\right)\\
&= \frac{1}{\left(\frac{1}{x^2}\right)\left(x^{\frac{2}{3}} e^{-\frac{x}{3}} e^{\frac{1}{3x}}\right)^3}\left(-\frac{2}{27}x - \frac{1}{9} - \frac{2}{9x} + \frac{2}{27x^2}\right)\\
&= \frac{-\frac{2}{27}x - \frac{1}{9} - \frac{2}{9x} + \frac{2}{27x^2}}{\left(\frac{1}{x^2}\right)\left(x^{\frac{2}{3}} e^{-\frac{x}{3}} e^{\frac{1}{3x}}\right)^3}
\end{aligned}
$$

Where

$$
\begin{aligned}
\xi &= \int f_3 U^2 dx\\
&= \int \frac{1}{x^2}\left(x^{\frac{2}{3}} e^{-\frac{x}{3}} e^{\frac{1}{3x}}\right)^2 dx
\end{aligned}
$$

But need to find x as function of ξ in order to make RHS of $\Phi(\xi)$ all ξ. Since this integral have no closed form, can not do it. I picked not a good example or do not understand this method. Unable to find one single worked example showing how this transformation works.

1.6.2.2 Example 2 $y' = 3 + 2xy + y^2 + xy^3$

This is already an Abel first kind

$$y' = f_0 + f_1 y + f_2 y^2 + f_3 y^3$$

Where

$$\begin{aligned} f_0 &= 3 \\ f_1 &= 2x \\ f_2 &= 1 \\ f_3 &= x \end{aligned}$$

The first step is to remove f_2 using

$$\begin{aligned} y(x) &= u(x) - \frac{f_2}{3f_3} \\ &= u(x) - \frac{1}{3x} \end{aligned}$$

Which results in

$$\begin{aligned} u' &= \frac{7}{3} - \frac{7}{27x^2} + \left(2x - \frac{1}{3x}\right) u + xu^3 \qquad \text{(A)} \\ &= f_0 + f_1 u + f_3 u^3 \end{aligned}$$

Where

$$\begin{aligned} f_0 &= \frac{7}{3} - \frac{7}{27x^2} \\ f_1 &= 2x - \frac{1}{3x} \\ f_2 &= 0 \\ f_3 &= x \end{aligned}$$

We see the ode is missing the quadratic term. Now we check the invariant

$$\begin{aligned} \Delta &= -\frac{\left(-f_0' f_3 + f_0 f_3' + 3 f_0 f_3 f_1\right)^3}{27 f_3^4 f_0^5} \\ &= -\frac{11664 \left(27x^4 - 3x^2 - 1\right)^2 x \left(9x^4 - 4x^2 + 2\right)}{49 \left(9x^2 - 1\right)^6} \end{aligned}$$

Which is not constant. Hence no direct solution exist. Let now apply the transformation to convert the ode (A) in $u(x)$ to canonical form

$$\eta'(\xi) = \Phi(\xi) + \eta^3(\xi)$$

We start by finding

$$\begin{aligned} U &= \exp\left(\int f_1(x) - \frac{f_2^2(x)}{3f_3(x)} dx\right) \\ &= \exp\left(\int f_1(x)\, dx\right) \\ &= \exp \int 2x - \frac{1}{3x} dx \\ &= \exp\left(x^2 - \frac{1}{3}\ln x\right) \\ &= \frac{e^{x^2}}{x^{\frac{1}{3}}} \end{aligned}$$

Next we find

$$
\begin{aligned}
\Phi(\xi) &= \frac{1}{f_3 U^3}\left(f_0 - \frac{f_1 f_2}{3 f_3} + \frac{2 f_2^3}{27 f_3^2} - \frac{1}{3}\frac{d}{dx}\left(\frac{f_2}{f_3}\right)\right) \\
&= \frac{1}{x\left(\frac{e^{x^2}}{x^{\frac{1}{3}}}\right)^3}\left(\frac{7}{3} - \frac{7}{27x^2}\right) \\
&= \frac{\frac{7}{3} - \frac{7}{27x^2}}{x\left(\frac{e^{x^2}}{x^{\frac{1}{3}}}\right)^3} \\
&= -\frac{1}{27x^2}\left(7e^{-3x^2} - 63x^2 e^{-3x^2}\right)
\end{aligned}
$$

And where

$$
\begin{aligned}
\xi &= \int f_3 U^2 dx \\
&= \int x \left(\frac{e^{x^2}}{x^{\frac{1}{3}}}\right)^2 dx
\end{aligned}
$$

But can't inverse this also. So stop here. Need to find how this transformation is done. This transformation is not practical to use as it can only be done parametrically. Hence will not implement. The above examples are left here just for reference

1.7 Converting Abel first kind to second kind

Local contents

1.7.1 Algorithm

It is also possible to just work with Abel second kind all the time instead of working on solving the first kind. Given first kind Abel

$$y' = f_0 + f_1 y + f_2 y^2 + f_3 y^3$$

This is converted to second kind using the transformation

$$y = \frac{1}{u}$$

Which results in

$$u'u = k_0 + k_1 u + k_2 u^2 + k_3 u^3$$

Where

$$
\begin{aligned}
k_0 &= -f_3 \\
k_1 &= -f_2 \\
k_2 &= -f_1 \\
k_3 &= -f_0
\end{aligned}
$$

1.7.2 Examples

Local contents

1.7.2.1 Example 1 $y' = -xe^{-x} - y + xe^{2x}y^3$

$$y' = -xe^{-x} - y + xe^{2x}y^3$$

Comparing to

$$y' = f_0 + f_1 y + f_2 y^2 + f_3 y^3$$

Shows that

$$\begin{aligned} f_0 &= -xe^{-x} \\ f_1 &= -1 \\ f_2 &= 0 \\ f_3 &= xe^{2x} \end{aligned}$$

Applying transformation $y = \frac{1}{u}$ gives

$$u'u = k_0 + k_1 u + k_2 u^2 + k_3 u^3 \tag{1}$$

Where

$$\begin{aligned} k_0 &= -f_3 = -xe^{2x} \\ k_1 &= -f_2 = 0 \\ k_2 &= -f_1 = 1 \\ k_3 &= -f_0 = xe^{-x} \end{aligned}$$

Hence (1) becomes

$$u'u = -xe^{2x} + u^2 + xe^{-x}u^3$$

Which is second kind Abel

1.7.2.2 Example 2 $y' = x + y + y^3$

$$y' = x + y + y^3$$

Comparing to

$$y' = f_0 + f_1 y + f_2 y^2 + f_3 y^3$$

Shows that

$$\begin{aligned} f_0 &= x \\ f_1 &= 1 \\ f_2 &= 0 \\ f_3 &= 1 \end{aligned}$$

Applying transformation $y = \frac{1}{u}$ gives

$$u'u = k_0 + k_1 u + k_2 u^2 + k_3 u^3$$

Where

$$
\begin{aligned}
k_0 &= -f_3 = -1 \\
k_1 &= -f_2 = 0 \\
k_2 &= -f_1 = -1 \\
k_3 &= -f_0 = -x
\end{aligned}
$$

Hence (1) becomes

$$
u'u = -1 - u^2 - xu^3
$$

Which is second kind Abel.