# Converting second order ODE to Bessel form

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# Contents



# 1 Introduction

This gives examples of converting (when possible) a second order linear ode to Bessel form. Bessel ODE is

$$x^2y'' + xy' + \left(x^2 - n^2\right)y = 0 \tag{A}$$

Where $n$ is the order which can be integer or non-integer. This comes out when doing separation of variables for the Laplace and Helmholtz PDE in spherical and cylindrical coordinates. $n$ is integer for cylindrical coordinates and half integer values ($n = \frac{1}{2} + \mathbb{Z}$), for spherical coordinates. $n$ can also be any other real value. The case $n = \frac{1}{2} + \mathbb{Z}$ is special in that the solution of the ode is reducible to standard trigonometric functions and complex exponential function. In all other cases, the solution remains in terms of Bessel functions.

The solution to (A) is known to be

$$y(x) = c_1 J_n(x) + c_2 Y_n(x)$$

Where $J_n(x)$ is Bessel function of first kind (order $n$). And $Y_n(x)$ Bessel function of second kind (order $n$).

There is also the modified Bessel ODE which differ by a sign

$$x^2 y'' + x y' - \left(x^2 + n^2\right) y = 0 \tag{B}$$

There is however a generalized form of (A,B). Which will be used below. (Bowman 1958). This form is

$$x^2 y'' + (1 - 2\alpha) x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - \left(n^2 \gamma^2 - \alpha^2\right)\right) y = 0 \tag{C}$$

Which is obtained by applying the transformation $\eta = \frac{y}{x^\alpha}, \xi = \beta x^\gamma$ to (A). The above has the solution

$$y(x) = x^\alpha \left(c_1 J_n(\beta x^\gamma) + c_2 Y_n(\beta x^\gamma)\right) \qquad \text{integer } n \tag{C1}$$
$$y(x) = x^\alpha \left(c_1 J_n(\beta x^\gamma) + c_2 J_{-n}(\beta x^\gamma)\right) \qquad \text{noninteger } n \tag{C2}$$

# 2 Collection of transformations

This section shows number of transformations applied to second order linear ode in order to make it of the form (A) or (B) if it is not already in that form. Once the ode is in form A or B, then its solution is now known using Bowman transformation.

## 2.1 Example $x^2 y'' + x y' + (a x^2 - n^2) y = 0$

$$x^2 y'' + x y' + \left(a x^2 - n^2\right) y = 0 \tag{1}$$

Comparing (1) to (C) shows that

$$\begin{aligned} (1 - 2\alpha) &= 1 \\ 2\gamma &= 2 \\ a &= \beta^2 \gamma^2 \\ \gamma^2 &= 1 \\ \alpha &= 0 \end{aligned}$$

Solving shows that $\gamma = 1, \beta = \sqrt{a}$. Hence the solution from (C1) can now be written directly as

$$y(x) = c_1 J_n\left(\sqrt{a}\, x\right) + c_2 Y_n\left(\sqrt{a}\, x\right)$$

Another way to obtain this solution is to use the transformation

$$x = \frac{1}{\sqrt{a}} z$$

Which converts (1) to

$$z^2y'' + zy' + \left(x^2 - v^2\right) y = 0 \tag{2}$$

This is now in standard form (A) which has solution

$$y(z) = c_1 J_v(z) + c_2 Y_v(z)$$

Replacing back $z = \sqrt{a}\, x$ in the above gives

$$y(x) = c_1 J_v\left(\sqrt{a}\, x\right) + c_2 Y_v\left(\sqrt{a}\, x\right)$$

So the rule is that, the term is $\left(ax^2 - n^2\right) y$ then we can just replace $J_n(x)$ and $Y_n(x)$ in the standard solution with $J_n\left(\sqrt{a}\, x\right)$ and $Y_n\left(\sqrt{a}\, x\right)$. For example $x^2y'' + xy' + \left(4x^2 - 9\right) y = 0$ will have the solution $y(x) = c_1 J_3(2x) + c_2 Y_3(2x)$.

## 2.2 Example $x^2y'' + xy' + xy = 0$

$$x^2y'' + xy' + xy = 0 \tag{1}$$

Comparing (1) to (C) shows that

$$\begin{aligned} (1 - 2\alpha) &= 1 \qquad (2) \\ \left(\beta^2\gamma^2x^{2\gamma} - \left(n^2\gamma^2 - \alpha^2\right)\right) &= x \end{aligned}$$

Hence

$$\begin{aligned} \beta^2\gamma^2x^{2\gamma} &= x \\ \left(n^2\gamma^2 - \alpha^2\right) &= 0 \qquad (3) \end{aligned}$$

Which implies

$$2\gamma = 1 \tag{4}$$

$$\beta^2\gamma^2 = 1 \tag{5}$$

(2) gives $\alpha = 0$. (4) gives $\gamma = \frac{1}{2}$. Substituting these into (3) gives

$$n = 0$$

And (5) gives $\beta^2 = 4$ or $\beta = \pm 2$. Therefore from (C1) the solution is

$$\begin{aligned} y(x) &= x^\alpha \left(c_1 J_n\left(\beta x^\gamma\right) + c_2 Y_n\left(\beta x^\gamma\right)\right) \\ &= c_1 J_0\left(2\sqrt{x}\right) + c_2 Y_n\left(2\sqrt{x}\right) \end{aligned}$$

## 2.3 Example $x^2y'' + bxy' + (x^2 - v^2)\,y = 0$

$$x^2y'' + bxy' + \left(x^2 - v^2\right)y = 0 \tag{1}$$

Comparing (1) to the generalized form (C) $x^2y''+(1-2\alpha)\,xy'+(\beta^2\gamma^2x^{2\gamma} - (n^2\gamma^2 - \alpha^2))\,y = 0$ shows that

$$\begin{aligned}(1-2\alpha) &= b\\ 2\gamma &= 2\\ \beta^2\gamma^2 &= 1\\ \left(n^2\gamma^2 - \alpha^2\right) &= v^2\end{aligned}$$

Hence $\gamma = 1, \beta = 1$ . From first equation $\alpha = \frac{1}{2}(1-b)$. Using this in the last equation gives

$$\begin{aligned}n^2 - \frac{1}{4}(1-b)^2 &= v^2\\ n &= \sqrt{v^2 + \frac{1}{4}(1-b)^2}\end{aligned}$$

Therefore the solution (C1) is

$$\begin{aligned}y(x) &= x^{\alpha}\left(c_1J_n\left(\beta x^{\gamma}\right) + c_2Y_n\left(\beta x^{\gamma}\right)\right)\\ &= x^{\frac{1}{2}(1-b)}\left(c_1J_n(x) + c_2Y_n(x)\right)\end{aligned}$$

For example, if $b = 4$, then the ode is $x^2y'' + 4xy' + (x^2 - v^2)\,y = 0$ and the solution is

$$y(x) = x^{-\frac{3}{2}}\left(c_1J_n(x) + Y_n(x)\right)$$

Where $n = \frac{1}{2}\sqrt{\frac{4v^2+9}{2}}$ .

## 2.4 Example $xy'' + y' + Ay = 0$

$$xy'' + y' + Ay = 0 \tag{1}$$

Where $A$ is constant. Multiplying by $x$ gives

$$x^2y'' + xy' + Axy = 0$$

Comparing the above to (C) $x^2y''+(1-2\alpha)\,xy'+(\beta^2\gamma^2x^{2\gamma} - (n^2\gamma^2 - \alpha^2))\,y = 0$ shows that

$$\begin{aligned}(1-2\alpha) &= 1\\ Ax &= \beta^2\gamma^2x^{2\gamma}\\ \left(n^2\gamma^2 - \alpha^2\right) &= 0\end{aligned}$$

Which implies $\alpha = 0, 2\gamma = 1$ or $\gamma = \frac{1}{2}$. Therefore $\beta^2\gamma^2 = A$ gives $\beta^2 = 4A$ or $\beta = 2\sqrt{A}$. And $n = 0$. Hence the solution (C1) is

$$y(x) = c_1 J_0\left(2\sqrt{A}\sqrt{x}\right) + c_2 Y_0\left(2\sqrt{A}\sqrt{x}\right)$$

Alternative and longer method is the following (this is kept just for illustration, as the above method is more direct).

Using the transformation

$$x = v^2$$

Hence

$$v = \sqrt{x} \tag{2}$$

and $\frac{dv}{dx} = \frac{1}{2\sqrt{x}}$. Therefore

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dv}\frac{dv}{dx} \\ &= \frac{dy}{dv}\frac{1}{2\sqrt{x}} \\ &= \frac{dy}{dv}\frac{1}{2v} \end{aligned} \tag{3}$$

And

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\ &= \frac{d}{dx}\left(\frac{dy}{dv}\frac{1}{2v}\right) \end{aligned}$$

But $\frac{d}{dx} = \frac{d}{dv}\frac{dv}{dx}$. The above becomes

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dv}\frac{dv}{dx}\left(\frac{dy}{dv}\frac{1}{2v}\right) \\ &= \frac{dv}{dx}\frac{d}{dv}\left(\frac{dy}{dv}\frac{1}{2v}\right) \end{aligned}$$

But $\frac{dv}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2v}$. Hence the above becomes

$$\frac{d^2y}{dx^2} = \frac{1}{2v}\frac{d}{dv}\left(\frac{dy}{dv}\frac{1}{2v}\right) \tag{4}$$

But

$$\frac{d}{dv}\left(\frac{dy}{dv}\frac{1}{2v}\right) = \frac{1}{2}\left(\frac{d^2y}{dv^2}\frac{1}{v} - \frac{dy}{dv}\frac{1}{v^2}\right)$$

Hence (4) becomes

$$\frac{d^2y}{dx^2}=\frac{1}{4v}\left(\frac{d^2y}{dv^2}\frac{1}{v}-\frac{dy}{dv}\frac{1}{v^2}\right) \tag{5}$$

Substituting (3,5) into (1) gives

$$x\frac{1}{4v}\left(\frac{d^2y}{dv^2}\frac{1}{v}-\frac{dy}{dv}\frac{1}{v^2}\right)+\frac{dy}{dv}\frac{1}{2v}+Ay=0$$

But $x=v^2$. The above becomes

$$\begin{aligned}\frac{v}{4}\left(y''\frac{1}{v}-\frac{dy}{dv}\frac{1}{v^2}\right)+y'\frac{1}{2v}+Ay&=0\\ \frac{1}{4}y''-\frac{1}{4}y'\frac{1}{v}+y'\frac{1}{2v}+Ay&=0\\ \frac{1}{4}y''+\frac{1}{4}y'\frac{1}{v}+Ay&=0\\ y''+y'\frac{1}{v}+4Ay&=0\end{aligned}$$

Multiplying through by $v^2$

$$v^2y''+vy'+4Av^2y=0$$

The above of the form

$$v^2y''+vy'+\left(a^2v^2-n^2\right)y=0$$

Where $n=0$ and $a^2=4A$ which has the standard solution

$$y(v)=c_1J_n(av)+c_2Y_n(av)$$

Where $J_n(v)$ is the Bessel function of first kind and $Y_n(v)$ is Bessel function of second kind. Since $v=\sqrt{x}$ and $a=2\sqrt{A}$ then the solution for (1) becomes (using $n=0$)

$$y(x)=c_1J_0\left(2\sqrt{A}\sqrt{x}\right)+c_2Y_0\left(2\sqrt{A}\sqrt{x}\right)$$

For example, if $A=\frac{1}{4}$. Then the ode $xy''+y'+\frac{1}{4}y=0$ and the solution above becomes

$$y(x)=c_1J_0\left(\sqrt{x}\right)+c_2Y_0\left(\sqrt{x}\right)$$

## 2.5 Example $y''-\frac{1}{x}y=0$

$$y''-\frac{1}{x}y=0 \tag{1}$$

Multiplying both sides by $x^2$ gives

$$x^2y''-xy=0$$

Comparing to (C) $x^2y'' + (1-2\alpha)xy' + \left(\beta^2\gamma^2x^{2\gamma} - (n^2\gamma^2-\alpha^2)\right)y = 0$ shows that

$$
\begin{aligned}
(1-2\alpha) &= 0\\
\beta^2\gamma^2x^{2\gamma} &= -x\\
\left(n^2\gamma^2-\alpha^2\right) &= 0
\end{aligned}
$$

First equation gives $\alpha = \frac{1}{2}$. Second equation gives $\gamma = \frac{1}{2}$ and $\beta^2\gamma^2 = -1$. Therefore $\beta^2 = -4$ or $\beta = \pm 2i$. Last equation gives $n^2\gamma^2 = \frac{1}{4}$ or $n = 1$ since $\gamma^2 = \frac{1}{4}$. Hence the solution (C1) is

$$
\begin{aligned}
y(x) &= x^{\alpha}\left(c_1J_n(\beta x^{\gamma}) + c_2Y_n(\beta x^{\gamma})\right)\\
&= \sqrt{x}\left(c_1J_1\left(2i\sqrt{x}\right) + c_2Y_1\left(2i\sqrt{x}\right)\right)
\end{aligned}
$$

By properties of Bessel functions, where $J_n\left(ai\sqrt{x}\right) = i^nI_n\left(a\sqrt{x}\right)$, then the above becomes

$$
y(x) = \sqrt{x}\left(ic_1I_1\left(2\sqrt{x}\right) + c_2Y_1\left(2i\sqrt{x}\right)\right)
$$

Alternative longer method is the following:

Trying standard transformation $y = \sqrt{x}\,Y$. The ode becomes

$$
x^2Y'' + xY' - \left(x + \frac{1}{4}\right)Y = 0
$$

Using the transformation $x = t^2$ the above becomes

$$
t^2Y'' + tY' - \left(4t^2+1\right)Y = 0
$$

Finally applying the standard transformation $t = \frac{1}{2}z$ to fix the term $(4t^2+1)$ to standard form the above becomes

$$
z^2Y'' + zY' - \left(t^2+1\right)Y = 0
$$

This is modified Bessel ODE whose solution is known to be

$$
Y(z) = c_1I_1(z) + c_2K_1(z)
$$

Where $I_1$ is modified Bessel function of first kind and $K_1$ is modified Bessel function of second kind. But $z = 2t$. Hence the above becomes

$$
Y(t) = c_1I_1(2t) + c_2K_1(2t)
$$

But $t = \sqrt{x}$. The above becomes

$$
Y(x) = c_1I_1\left(2\sqrt{x}\right) + c_2K_1\left(2\sqrt{x}\right)
$$

But $y(x) = \sqrt{x}\,Y(z)$ hence

$$
y(x) = c_1\sqrt{x}\,I_1\left(2\sqrt{x}\right) + c_2\sqrt{x}\,K_1\left(2\sqrt{x}\right)
$$

## 2.6 Example $4x^2y'' + 4xy' + (x-4)\,y = 0$

Dividing by 4

$$x^2y'' + xy' + \left(\frac{1}{4}x - 1\right)y = 0$$

Comparing the above to (C) $x^2y'' + (1-2\alpha)\,xy' + (\beta^2\gamma^2x^{2\gamma} - (n^2\gamma^2 - \alpha^2))\,y = 0$ shows that

$$\begin{aligned}(1-2\alpha) &= 1\\ \beta^2\gamma^2x^{2\gamma} &= \frac{1}{4}x\\ \left(n^2\gamma^2 - \alpha^2\right) &= 1\end{aligned}$$

Which implies $\alpha = 0, 2\gamma = 1, \beta^2\gamma^2 = \frac{1}{4}$. Hence $\gamma = \frac{1}{2}$ and $\beta = 1$. Last equation now says $n^2\gamma^2 = 1$ or $n = 2$. Hence the solution (C1) is

$$\begin{aligned}y(x) &= x^\alpha\left(c_1J_n(\beta x^\gamma) + c_2Y_n(\beta x^\gamma)\right)\\ &= c_1J_2\left(\sqrt{x}\right) + c_2Y_2\left(\sqrt{x}\right)\end{aligned}$$

## 2.7 Example $y'' - \frac{1}{x^{\frac{3}{2}}}y = 0$

Multiplying by $x^{\frac{3}{2}}$

$$x^{\frac{3}{2}}y'' - y = 0$$

Multiplying by $x^{\frac{1}{2}}$

$$x^2y'' - x^{\frac{1}{2}}y = 0$$

Comparing the above to (C) $x^2y'' + (1-2\alpha)\,xy' + (\beta^2\gamma^2x^{2\gamma} - (n^2\gamma^2 - \alpha^2))\,y = 0$ shows that

$$\begin{aligned}(1-2\alpha) &= 0\\ \beta^2\gamma^2x^{2\gamma} &= -x^{\frac{1}{2}}\\ \left(n^2\gamma^2 - \alpha^2\right) &= 0\end{aligned}$$

Which implies $\alpha = \frac{1}{2}, 2\gamma = \frac{1}{2}, \beta^2\gamma^2 = -1$. Hence $\gamma = \frac{1}{4}$ and $\beta^2 = -16$ or $\beta = \pm 4i$. Last equation now says $\left(n^2\frac{1}{16} - \frac{1}{4}\right) = 0$ or $n = 2$. Hence the solution (C1) is

$$\begin{aligned}y(x) &= x^\alpha\left(c_1J_n(\beta x^\gamma) + c_2Y_n(\beta x^\gamma)\right)\\ &= \sqrt{x}\left(c_1J_2\left(4ix^{\frac{1}{4}}\right) + c_2Y_2\left(4ix^{\frac{1}{4}}\right)\right)\end{aligned}$$

By properties of Bessel functions, where $J_n\left(ai\sqrt{x}\right) = i^nI_n\left(a\sqrt{x}\right)$, then the above becomes

$$y(x) = \sqrt{x}\left(-c_1I_2\left(4x^{\frac{1}{4}}\right) + c_2Y_2\left(4ix^{\frac{1}{4}}\right)\right)$$

## 2.8 Example $x^2y'' - xy + (x^2+1)\,y = 0$

$$x^2y'' - xy + \left(x^2+1\right)y = 0$$

Comparing the above to (C) $x^2y'' + (1-2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - \left(n^2\gamma^2 - \alpha^2\right)\right)y = 0$ shows that

$$\begin{aligned}(1-2\alpha) &= -1\\ \beta^2\gamma^2x^{2\gamma} &= x^2\\ -\left(n^2\gamma^2-\alpha^2\right) &= 1\end{aligned}$$

Which implies $\alpha = 1$ and $\gamma = 1$ and $\beta^2\gamma^2 = 1$ or $\beta = 1$. Last equation now becomes $-\left(n^2-1\right) = 1$ or $n^2 = 0$ or $n = 0$. Hence the solution (C1) becomes

$$\begin{aligned}y(x) &= x^\alpha\left(c_1J_n\left(\beta x^\gamma\right) + c_2Y_n\left(\beta x^\gamma\right)\right)\\ &= x\left(c_1J_0(x) + c_2Y_0(x)\right)\end{aligned}$$

## 2.9 Example $y'' - x^{-\frac{1}{4}}y = 0$

Multiplying by $x^{\frac{1}{4}}$

$$x^{\frac{1}{4}}y'' - y = 0$$

Multiplying by $x^{\frac{7}{4}}$

$$x^2y'' - x^{\frac{7}{4}}y = 0$$

Comparing the above to (C) $x^2y'' + (1-2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - \left(n^2\gamma^2 - \alpha^2\right)\right)y = 0$ shows that

$$\begin{aligned}(1-2\alpha) &= 0\\ \beta^2\gamma^2x^{2\gamma} &= -x^{\frac{7}{4}}\\ \left(n^2\gamma^2-\alpha^2\right) &= 0\end{aligned}$$

Which implies $\alpha = \frac{1}{2}$ and $2\gamma = \frac{7}{4}$ or $\gamma = \frac{7}{8}$ and $\beta^2\gamma^2 = -1$ or $\beta^2 = -\frac{1}{\left(\frac{7}{8}\right)^2} = -\frac{64}{49}$. Hence $\beta = i\frac{8}{7}$. Last equation now becomes $\left(n^2\left(\frac{49}{64}\right) - \frac{1}{4}\right) = 0$, or $n = \frac{4}{7}$. Hence the solution (C2) for non integer $n$ becomes

$$\begin{aligned}y(x) &= x^\alpha\left(c_1J_n\left(\beta x^\gamma\right) + c_2J_{-n}\left(\beta x^\gamma\right)\right)\\ &= \sqrt{x}\left(c_1J_{\frac{4}{7}}\left(i\frac{8}{7}x^{\frac{7}{8}}\right) + c_2J_{-\frac{4}{7}}\left(i\frac{8}{7}x^{\frac{7}{8}}\right)\right)\end{aligned}$$

## 2.10 Example $f'' + \frac{\lambda}{x}f' - \mu f = 0$

Multiplying by $x^2$

$$x^2f'' + \lambda xf' + \left(-\mu x^2\right)f = 0 \tag{1}$$

Using the generalized form of Bessel ode

$$x^2f'' + xf' + \left(x^2 - n^2\right)f = 0 \tag{A}$$

Which is given by (Bowman 1958)

$$x^2 f'' + (1-2\alpha)\, x f' + \left(\beta^2\gamma^2 x^{2\gamma} - \left(n^2\gamma^2 - \alpha^2\right)\right) f = 0 \tag{C}$$

Comparing (1) and (C) shows that

$$(1-2\alpha) = \lambda \tag{2}$$
$$\beta^2\gamma^2 x^{2\gamma} = -\mu x^2 \tag{3}$$
$$\left(n^2\gamma^2 - \alpha^2\right) = 0 \tag{4}$$

(2) gives $\alpha = \frac{1}{2} - \frac{1}{2}\lambda$. (3) gives $2\gamma = 2$ or $\gamma = 1$. And (3) also shows that $\beta^2\gamma^2 = -\mu$ or $\beta = \sqrt{-\mu}$. Now (4) gives $\left(n^2 - \left(\frac{1}{2} - \frac{1}{2}\lambda\right)^2\right) = 0$ or $n = \left(\frac{1}{2} - \frac{1}{2}\lambda\right)$. (taking positive root). But the solution to (C) is gives by

$$y(x) = x^{\alpha}\left(c_1 J_n\left(\beta x^{\gamma}\right) + c_2 Y_n\left(\beta x^{\gamma}\right)\right)$$

Therefore the solution to (1) is

$$y(x) = x^{\left(\frac{1}{2}-\frac{1}{2}\lambda\right)}\left(c_1 J_{\left(\frac{1}{2}-\frac{1}{2}\lambda\right)}\left(\sqrt{-\mu}\, x\right) + c_2 Y_{\left(\frac{1}{2}-\frac{1}{2}\lambda\right)}\left(\sqrt{-\mu}\, x\right)\right)$$

Where $J$ is the Bessel function of first kind and $Y$ is the Bessel function of the second kind.

## 2.11 Example $x^2y'' + xy' + (x^2 - 5)y = 0$

$$x^2y'' + xy' + (x^2 - 5)y = 0 \tag{1}$$

Using the generalized form of Bessel ode

$$x^2y'' + xy' + \left(x^2 - n^2\right) y = 0 \tag{A}$$

Which is given by (Bowman 1958)

$$x^2y'' + (1-2\alpha)\, xy' + \left(\beta^2\gamma^2 x^{2\gamma} - \left(n^2\gamma^2 - \alpha^2\right)\right) y = 0 \tag{C}$$

Comparing (1) and (C) shows that

$$(1-2\alpha) = 1 \tag{2}$$
$$\beta^2\gamma^2 x^{2\gamma} = x^2 \tag{3}$$
$$\left(n^2\gamma^2 - \alpha^2\right) = 5 \tag{4}$$

(2) gives $\alpha = 0$. (3) gives $\gamma = 1$ and $\beta^2\gamma^2 = 1$ or $\beta = 1$. Now (4) gives $n^2\gamma^2 = 5$ or $n = \sqrt{5}$.But the solution to (C) is given by

$$y(x) = x^{\alpha}\left(c_1 J_n\left(\beta x^{\gamma}\right) + c_2 Y_n\left(\beta x^{\gamma}\right)\right)$$

Therefore the solution to (1) is

$$y(x) = c_1 J_{\sqrt{5}}(x) + c_2 Y_{\sqrt{5}}(x)$$

Where $J$ is the Bessel function of first kind and $Y$ is the Bessel function of the second kind.

# 3 References


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